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CONFORMALLY KÄHLER RICCI SOLITONS AND BASE METRICS FOR WARPED PRODUCT RICCI SOLITONS

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We investigate Kähler metrics conformal to gradient Ricci solitons, and base metrics of warped product gradient Ricci solitons. A slight generalization of the latter we name quasi-solitons. A main assumption that is employed is functional dependence of the soliton potential, with the conformal factor in the first case, and with the warping function in the second. The main result in the first case is a partial classification in dimension $n \geq 4$. In the second case, Kähler quasi-soliton metrics satisfying the above main assumption are shown to be, under an additional genericity hypothesis, necessarily Riemannian products. Another theorem concerns quasi-soliton metrics satisfying the above main assumption, which are also conformally Kähler. With some additional assumptions it is shown that such metrics are necessarily base metrics of Einstein warped products, that is, quasi-Einstein.

1. Introduction

The study of the Ricci flow [Hamilton 1982] has inspired the introduction of a metric type generalizing the Einstein condition. A gradient Ricci soliton is a Riemannian metric satisfying

$$\mathrm{Ric} + \nabla df = \lambda g, \quad \lambda \text{ constant}.$$

The function f is called the soliton potential. Such solitons are further referred to as shrinking, steady or expanding, depending on the sign of λ.

We consider Ricci solitons in two settings: the case where they are conformal to Kähler metrics, and the case where they are warped products. Conformal classes of Ricci solitons have been studied recently in [Jauregui and Wylie 2015; Catino et al. 2016; Maschler 2015]. Kähler metrics in such a conformal class, with nontrivial conformal factor, have been examined in [Maschler 2008; Derdziński 2012]. Warped product Ricci solitons, on the other hand, have been studied extensively when the base of the warped product is one-dimensional; see for instance [Chow et al. 2007]. The cigar soliton and the Bryant soliton are examples in this category.

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In each case we focus on an auxiliary metric which at least partially determines the soliton. In the first case that would be the associated Kähler metric in the conformal class, and in the second case it is the induced metric on the base of the warped product. The latter metric is a special case of what we call a (gradient Ricci) quasi-soliton, in analogy with how base metrics of Einstein warped products are often called quasi-Einstein metrics. We consider only quasi-soliton metrics which are Kähler, or conformally Kähler.

A common thread for these two cases of auxiliary metrics is the appearance of two Hessians in their defining equations. One is the Hessian of the soliton potential f, while the other Hessian depends on the case: it is that of the conformal factor τ in the first case, and that of the warping function ℓ in the second.

These equations are, of course, more complex than the original Ricci soliton equation, and handling them in full generality still appears beyond reach. Our strategy is thus to consider mainly the case where functional dependence of the above two functions holds, in either setting. In other words, we require

(1-1i) $d\tau \wedge df = 0$ for the associated Kähler metric,

(1-1ii) $d\ell \wedge df = 0$ for the induced metric on the base of the warped product.

In the latter case we call the metric a *special* quasi-soliton.

An example where condition (1-1i) occurs in the Kähler conformally soliton case is when the conformal factor τ is additionally a potential for a Killing vector field of the Kähler metric (a Killing potential). The latter condition was studied in [Maschler 2008] and plays a role in Theorem 7.4. It turns out that the condition (1-1i) also implies, generically, the existence of a Killing potential which, however, is of a more general kind, being only functionally dependent on τ, rather than being τ itself. An instance of this more general setting was first considered in [Derdziński 2012].

Another metric type that plays an important role in all our main theorems is the SKR metric, i.e., a metric that admits a so-called special Kähler–Ricci potential. This notion was introduced by Derdzinski and Maschler [2003; 2006] for the purpose of classifying conformally Kähler Einstein metrics. In all our main theorems the proofs involve a Ricci–Hessian equation of the form

$$\alpha \nabla d\tau + \mathrm{Ric} = \gamma g,$$

for functions α and γ. The theory of SKR metrics which is then applied is closely tied to such equations.

The main results in this article are Theorems 6.2, 7.3 and 7.4. The first of these gives a partial classification of Kähler metrics conformal to gradient Ricci solitons in dimension $n \geq 4$ satisfying condition (1-1i). Theorem 7.3 presents a reducibility result for special quasi-soliton metrics which are Kähler. The conclusion of this theorem, that the metric is locally a Riemannian product, is analogous to a similar result for quasi-Einstein metrics [Case et al. 2011]. Theorem 7.4 mixes the two main themes of this paper, as it involves special quasi-soliton metrics that are conformal to an irreducible Kähler metric. With some additional assumptions, the conclusion of the theorem is that the metric must in fact be quasi-Einstein. This is in contrast with the existence of conformally Kähler quasi-Einstein metrics [Maschler 2011; Batat et al. 2015], and it remains to be seen whether this difference holds in general, or else is the result of the added assumptions.

Examples of metrics satisfying the conditions of Theorem 6.2 appear in [Maschler 2008; Derdziński 2012]. In one of the possible outcomes of the theorem, occurring in dimension four, the Ricci soliton must be non-Einstein and steady ($\lambda = 0$). There are at this time many known examples of non-Einstein steady Ricci solitons in all dimensions. Recent examples were given by Buzano, Dancer and Wang [Buzano et al. 2015] and Stolarski [2015]. A discussion of their potential relevance to this theorem is given at the end of Section 6.

The structure of the paper is as follows. After some preliminaries in Section 2, we give several forms for the conformally soliton equation in Section 3. We then determine in Section 4, in the context of the first metric type considered, certain implications of the assumption that vector fields that occur in the conformally soliton equation are of one of several well-known classical types. One such assumption which does not occur in nontrivial cases has, nonetheless, an interesting classification, which we give in the Appendix. In Section 5 we recall the salient features of SKR metric theory. The main theorem in the conformally Kähler case is given in Section 6, and the two main theorems for special quasi-soliton metrics appear in Section 7.

2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension n, and $\tau : M \to \mathbb{R}$ a C^∞ function. We write metrics conformally related to g in the form $\hat{g} = \tau^{-2} g$.

We recall a few conformal change formulas. The covariant derivative is

$$\widehat{\nabla}_w u = \nabla_w u - (d_w \log \tau) u - (d_u \log \tau) w + \langle w, u \rangle \nabla \log \tau, \tag{2-1}$$

where d_u denotes the directional derivative of a vector field u and the angle brackets stand for g. It follows that the $\hat{g}$-Hessian and $\hat{g}$-Laplacian of a C^2 function f are given by

$$\widehat{\nabla} df = \nabla df + \tau^{-1}[2\, d\tau \odot df - g(\nabla\tau, \nabla f) g], \tag{2-2i}$$

$$\widehat{\Delta} f = \tau^2 \Delta f - (n-2)\tau g(\nabla\tau, \nabla f), \tag{2-2ii}$$

where $d\tau \odot df = \frac{1}{2}(d\tau \otimes df + df \otimes d\tau)$. Finally, the well-known formula for the

Ricci tensor of $\hat{g}$ is given by

$$\widehat{\mathrm{Ric}} = \mathrm{Ric} + (n-2)\tau^{-1}\nabla d\tau + \left[\tau^{-1}\Delta\tau - (n-1)\tau^{-2}|\nabla\tau|^2\right]g, \tag{2-3}$$

where Δ denotes the Laplacian and the norm $|\cdot|$ is with respect to g.

Recall that a (real) vector field w on a complex manifold (M, J) is holomorphic if the Lie derivative $\mathcal{L}_w J$ vanishes.

Proposition 2.1. *Let ∇ be a torsion-free connection on a complex manifold (M, J). For any vector field w,*

$$\mathcal{L}_w J = \nabla_w J + [J, \nabla w],$$

where the square brackets denote the commutator.

In fact, write $(\mathcal{L}_w J)u = \mathcal{L}_w(Ju) - J\mathcal{L}_w u$ and replace each Lie derivative by the Lie brackets, and each of these by the torsion-free condition for ∇, giving $\nabla_w Ju - \nabla_{Ju} w - J\nabla_w u + J\nabla_u w$. The first and third terms together give $(\nabla_w J)(u)$, while the second and fourth terms give $[J, \nabla w](u)$.

Proposition 2.2. *Let (M, J) be a complex manifold with a Hermitian metric g. Given a C^2 function q on M, set $w = \nabla q$. Then ∇dq is J-invariant if and only if $[J, \nabla w] = 0$.*

In fact, $\nabla dq(Ja, b) = g(Ja, \nabla_b w) = -g(a, J\nabla_b w) = -g(a, J(\nabla w)(b))$, while $-\nabla dq(a, Jb) = -g(a, \nabla_{Jb} w) = -g(a, (\nabla w)(Jb))$.

In the following well-known proposition $\imath_v$ denotes interior multiplication by a vector field, while δ denotes the divergence operator.

Proposition 2.3. *Let σ be a smooth function on a Kähler manifold such that $v = \nabla\sigma$ is a holomorphic gradient vector field. Then $2\imath_v\mathrm{Ric} = -dY$ and $2\delta\nabla d\sigma = dY$ for $Y = \Delta\sigma$.*

For a proof, see [Derdzinski and Maschler 2003, (5.4) and (2.9)(c)].

3. Various forms of the conformally soliton equation

Let g be a Riemannian metric and τ a smooth function on a given manifold, for which $\hat{g} = g/\tau^2$ is a gradient Ricci soliton with soliton potential f. The soliton equation for $\hat{g}$, together with its associated scalar equation, are

$$\widehat{\mathrm{Ric}} + \widehat{\nabla} df = \lambda\hat{g}, \qquad \text{with } \lambda \text{ constant}, \tag{3-1i}$$

$$\widehat{\Delta} f - \hat{g}(\widehat{\nabla} f, \widehat{\nabla} f) + 2\lambda f = a, \qquad \text{for a constant } a. \tag{3-1ii}$$

To obtain this in terms of g, we apply equations (2-3) and (2-2i) to (3-1i). The result is

$$\mathrm{Ric} + (n-2)\tau^{-1}\nabla d\tau + \nabla df + 2\tau^{-1} d\tau \odot df = \gamma g. \tag{3-2}$$

for

$$\gamma = \tau^{-2}[\lambda + (n-1)|\nabla\tau|^2] - \tau^{-1}[\Delta\tau - g(\nabla\tau, \nabla f)], \tag{3-3}$$

with $|\nabla\tau|^2 = g(\nabla\tau, \nabla\tau)$.

We will now rewrite (3-2) in a different form. Specifically, for the vector fields $v = \nabla\tau$ and $w = \tau^2\nabla f$, equation (3-2) is equivalent to

$$\mathrm{Ric} + \alpha\mathcal{L}_v g + \beta\mathcal{L}_w g = \gamma g, \tag{3-4}$$

with $\alpha = \frac{1}{2}(n-2)\tau^{-1}$, $\beta = (2\tau^2)^{-1}$, and $\mathcal{L}$ denoting the Lie derivative. To show this, recall that for any vector fields a, b,

$$(\mathcal{L}_w g)(a, b) = g(\nabla_a w, b) + g(a, \nabla_b w), \tag{3-5}$$

or $\mathcal{L}_w g = [\nabla w + (\nabla w)^*]_\flat$, where $*$ denotes the adjoint and $\flat$ is the isomorphism associated with lowering an index. Now clearly $\mathcal{L}_v g = \mathcal{L}_{\nabla\tau} g = 2\nabla d\tau$. To compute the Lie derivative term for w, write $w = h\nabla f$. Then

$$\mathcal{L}_w g = 2h\nabla df + 2\,dh \odot df.$$

Setting $h = \tau^2$ and dividing by $2\tau^2$ gives

$$\nabla df + 2\tau^{-1} d\tau \odot df = (2\tau^2)^{-1}\mathcal{L}_{\tau^2\nabla f}\, g = \beta\mathcal{L}_w g.$$

Another form for equation (3-2) is obtained as follows. It is natural to combine the two Hessian terms into one. For this, set

$$\mu = \log\tau, \quad \theta = f + (n-2)\log\tau, \quad \psi = 2\theta - (n-2)\mu.$$

Then (3-2), (3-3) and (3-1ii) read

$$\mathrm{Ric} + \nabla d\theta + d\mu \odot d\psi = \gamma g, \quad \gamma = \lambda e^{-2\mu} - \Delta\mu + g(\nabla\theta, \nabla\mu), \tag{3-6i}$$

$$e^{2\mu}[\Delta f - g(\nabla\theta, \nabla f)] + 2\lambda f = a. \tag{3-6ii}$$

To derive (3-6ii) one uses (2-2ii), which, in terms of μ, reads

$$e^{-2\mu}\widehat{\Delta} f = \Delta f - (n-2)g(\nabla\mu, \nabla f).$$

4. The Kähler condition and distinguished vector fields

Let g be a metric which is Kähler with respect to a complex structure J on a manifold M, and conformal to a gradient Ricci soliton. Equation (3-4) then holds, and the J-invariance of g and its Ricci curvature implies that

$$\alpha\mathcal{L}_v g + \beta\mathcal{L}_w g \text{ is } J\text{-invariant.} \tag{4-1}$$

Applying (3-5) to the relation $\mathcal{L}_x g(J\cdot\,,\cdot) = -\mathcal{L}_x g(\cdot\,, J\cdot)$, for both $x = v$ and $x = w$, and recalling that $J^* = -J$, we see that (4-1) is equivalent to the vanishing of a commutator: $[\alpha(\nabla v + (\nabla v)^*) + \beta(\nabla w + (\nabla w)^*), J] = 0$, or

$$[\alpha(\mathcal{L}_v g)^\sharp + \beta(\mathcal{L}_w g)^\sharp, J] = 0, \tag{4-2}$$

where $\sharp$ denotes the isomorphism acting by raising an index.

The most obvious case where (4-2) holds is when both summands vanish separately, so that, w, for example, satisfies

$$[(\mathcal{L}_w g)^\sharp, J] = 0. \tag{4-3}$$

We wish to study relations between these two vanishing conditions for v and w. We first note that (4-3) includes as special cases the following three classical types of vector fields (the first being, of course, a special case of the second):

- a Killing vector field ($\mathcal{L}_w g = 0$),
- a conformal vector field ($(\mathcal{L}_w g)^\sharp = hI$, for a function h and I the identity),
- a holomorphic vector field ($[\nabla w, J] = 0$ on a Kähler manifold).

This last type is holomorphic by Proposition 2.1 in the Kähler case, and it is indeed a special case since $[\nabla w, J]^* = [(\nabla w)^*, J]$ and $[\nabla w + (\nabla w)^*, J] = 0$, the latter equality being equivalent to (4-3).

We will see in the next theorem that the Killing case does not lead to important Kähler conformally soliton metrics. However, Kähler metrics with a Killing field of the form $w = \tau^2 \nabla f$ can be classified, as we show in the Appendix.

To state the next result, we continue to assume g is Kähler and conformal to a gradient Ricci soliton $\hat{g}$, but now on a manifold of dimension $n > 2$. With notation as above for τ, f, v and w we have:

Theorem 4.1. *The following conclusions hold for the vector fields v and w:*

(1) *If w is a conformal vector field, then $\hat{g}$ is Einstein.*

(2) *If w is a holomorphic vector field and either v is holomorphic as well, or $\widehat{\nabla} df$ is J-invariant, then $\operatorname{span}_{\mathbb{C}}\{v\} = \operatorname{span}_{\mathbb{C}}\{w\}$ away from the zero sets of v and w.*

Proof. The key to both parts is that $w = \tau^2 \nabla f$ is also the $\hat{g}$-gradient of f, i.e., $w = \widehat{\nabla} f$. Therefore $\mathcal{L}_w \hat{g} = \mathcal{L}_{\widehat{\nabla} f} \hat{g} = 2\widehat{\nabla} df$. As the condition that a vector field be conformal is conformally invariant, it follows that when w is conformal, the Ricci soliton equation (3-1i) reduces, using Schur's lemma, to the Einstein equation. This proves (1).

To prove (2), note first that the combination of Propositions 2.1 and 2.2 for a Kähler metric yields the result that the vector field $v = \nabla \tau$ is holomorphic exactly when $\nabla d\tau$ is J-invariant. This in turn is equivalent, by (2-3) and the fact that the metric g and its Ricci curvature are J-invariant, to $\widehat{\mathrm{Ric}}$ being J-invariant. Finally, the latter condition is equivalent to $\widehat{\nabla} df$ being J-invariant, by the soliton equation

(3-1i). The combination, again, of Propositions 2.1 and 2.2, but this time for a Hermitian metric, yields equivalence of the latter condition with $\mathcal{L}_{\widehat{\nabla} f} J = \widehat{\nabla}_{\widehat{\nabla} f} J$, or

$$\mathcal{L}_w J = \widehat{\nabla}_w J. \tag{4-4}$$

Now from (2-1), for any vector field u,

$$\begin{aligned}(\widehat{\nabla}_w J)u &= \widehat{\nabla}_w (Ju) - J\widehat{\nabla}_w u \\ &= \nabla_w (Ju) - \tau^{-1}(d_w\tau)Ju - \tau^{-1}(d_{Ju}\tau)w + \langle w, Ju\rangle\tau^{-1}v \\ &\qquad - [J\nabla_w(u) - \tau^{-1}(d_w\tau)Ju - \tau^{-1}(d_u\tau)Jw + \langle w, u\rangle\tau^{-1}Jv] \\ &= \tau^{-1}(-\langle v, Ju\rangle w + \langle w, Ju\rangle v + \langle v, u\rangle Jw - \langle w, u\rangle Jv) \\ &= \tau^{-1}(\langle Jv, u\rangle w - \langle Jw, u\rangle v + \langle v, u\rangle Jw - \langle w, u\rangle Jv),\end{aligned}$$

where we used the fact that $\nabla_w J = 0$, and the angle brackets denote g. Combining this with (4-4) we see that as w is holomorphic, the last expression vanishes for *every* vector field u. Substituting first $u = v$ and then $u = Jv$ shows that away from the zeros of v, the vector fields w and Jw are pointwise in $\operatorname{span}\{v, Jv\}$. As this reasoning is symmetric for v and w, the result follows. □

In the examples of [Maschler 2008] the manifolds on which g and $\hat{g}$ reside are locally total spaces of holomorphic line bundles over manifolds admitting a Kähler–Einstein metric, and g is an SKR metric (see Section 5), while the conformal factor τ is a Killing potential. For these examples f is an affine function in τ^{-1} (see [Maschler 2008, Proposition 3.1]), so that, in that case, v and w are holomorphic and in fact $\operatorname{span}_{\mathbb{R}} v = \operatorname{span}_{\mathbb{R}} w$, away from the zeros of these vector fields.

5. SKR metrics

We recall here some facts from [Derdzinski and Maschler 2003] and [Maschler 2008] on the notion of an SKR metric, i.e., a Kähler metric g admitting a special Kähler–Ricci potential σ. For the definition, recall that a smooth function σ on a Kähler manifold (M, J, g) is called a Killing potential if $J\nabla\sigma$ is a Killing vector field. The definition of a special Kähler–Ricci potential consists then of the requirement that σ is a Killing potential and, at each noncritical point of it, all nonzero tangent vectors orthogonal to the complex span of $\nabla\sigma$ are eigenvectors of both the Ricci tensor and the Hessian of σ, considered as operators. This rather technical definition implies that a Ricci–Hessian equation holds for σ on a suitable open set (see [Derdzinski and Maschler 2003, Remark 7.4]), namely

$$\mathrm{Ric} + \alpha\nabla d\sigma = \gamma g, \tag{5-1}$$

for some functions α, γ which are functionally dependent on σ.

We say that equation (5-1) is a *standard* Ricci–Hessian equation if $\alpha d\alpha \neq 0$ whenever $d\sigma \neq 0$. This condition will appear in all our main theorems. However, even if it does not hold over the entire set where $d\sigma \neq 0$, these theorems will hold, with the same proofs, on any open subset of $\{d\sigma \neq 0\}$ where $\alpha d\alpha \neq 0$. We have:

Proposition 5.1. *A Kähler metric on a manifold of dimension at least four is an SKR metric, provided it satisfies a standard Ricci–Hessian equation of the form (5-1) with $d\alpha \wedge d\sigma = d\gamma \wedge d\sigma = 0$.*

This result appears in [Maschler 2008, Proposition 3.5] with proof referenced from [Derdzinski and Maschler 2003], a proof that has to be interpreted with the aid of [Maschler 2008, Remark 3.6]. Note also that in dimension greater than four, if the Ricci–Hessian equation of a Kähler metric satisfies $d\alpha \wedge d\sigma = 0$ then it automatically also satisfies $d\gamma \wedge d\sigma = 0$ (see [Maschler 2008, Proposition 3.3]).

If an SKR metric is locally irreducible, the theory of such metrics (see §4 of [Maschler 2008]) implies that a pair of equations holds on the open set where the Ricci–Hessian equation (5-1) holds:

$$\begin{aligned}(\sigma - c)^2\phi'' + (\sigma - c)[m - (\sigma - c)\alpha]\phi' - m\phi &= K,\\ -(\sigma - c)\phi'' + [\alpha(\sigma - c) - (m+1)]\phi' + \alpha\phi &= \gamma.\end{aligned} \tag{5-2}$$

Here ϕ is defined pointwise as the eigenvalue of the Hessian of σ, considered as an operator, corresponding to the eigendistribution $[\mathrm{span}_{\mathbb{C}} \nabla\sigma]^{\perp}$, and c is a constant. This eigenvalue and σ are functionally dependent, so that the primes represent differentiations with respect to σ. Furthermore, K is a constant whose exact expression in terms of SKR data will not concern us, while $m = \frac{1}{2}\dim(M)$. We further have the following relations between ϕ, $\Delta\sigma$ and $Q := g(\nabla\sigma, \nabla\sigma)$:

$$\Delta\sigma = 2m\phi + 2(\tau - c)\phi', \qquad Q = 2(\tau - c)\phi. \tag{5-3}$$

Note that for an irreducible SKR metric, the function ϕ is nowhere zero on the open dense set where $d\sigma \neq 0$.

In analyzing equations such as (5-2) we will repeatedly use in Section 7 the following elementary lemma, taken from [Maschler 2008].

Lemma 5.2. *For a system*

$$\begin{aligned}A\phi'' + B\phi' + C\phi &= D,\\ \phi' + p\phi &= q,\end{aligned} \tag{5-4}$$

with rational coefficients, either $A(p^2 - p') - Bp + C = 0$ holds identically, or else the solution is given by $\phi = (D - A(q' - pq) - Bq)/(A(p^2 - p') - Bp + C)$.

We now state the local classification of SKR metrics (Theorem 18.1 in [Derdzinski and Maschler 2003]).

Theorem 5.3. *Let (M, g, σ) be a manifold with an SKR metric and a special Kähler–Ricci potential. Then every point for which $d\sigma \neq 0$ has a neighborhood where g is, up to a biholomorphic isometry, given explicitly on an open set in the following local model.*

Here is the model metric, which is obtained as a special case of the Calabi ansatz. For simplicity we only give it in the irreducible case. Let $\pi : (L, \langle\cdot\,,\cdot\rangle) \to (N, h)$ be a Hermitian holomorphic line bundle over a Kähler manifold which is also Einstein if $n > 4$, where $n-2$ is the (real) dimension of N. Assume that the curvature of the Chern connection associated to $\langle\cdot\,,\cdot\rangle$ is a multiple of the Kähler form of h. (Note that, if $n > 4$, N is compact and h is not Ricci flat, this implies that L is smoothly isomorphic to a rational power of the anticanonical bundle of N.) Consider, on the total space of L excluding the zero section, the metric g given by

$$g|_{\mathcal{H}} = 2|\sigma - c|\pi^* h, \quad g|_{\mathcal{V}} = \frac{Q(\sigma)}{(ar)^2}\,\mathrm{Re}\langle\cdot\,,\cdot\rangle, \tag{5-5}$$

where:

- $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal distributions of L, assumed to be g-orthogonal to each other and the latter being determined via the Chern connection of $\langle\cdot\,,\cdot\rangle$.
- c, $a \neq 0$ are constants.
- r is the norm induced by $\langle\cdot\,,\cdot\rangle$.
- σ is a function on $L \setminus 0$, obtained by composing with the norm r another function, denoted via abuse of notation by $\sigma(r)$, and obtained as follows: one fixes an open interval I and a positive C^∞ function $Q(\sigma)$ on I, solves the differential equation $(a/Q)d\sigma = d(\log r)$ to obtain a diffeomorphism $r(\sigma) : I \to (0, \infty)$, and defines $\sigma(r)$ as the inverse of this diffeomorphism.

The metric g is the model SKR metric, with special Kähler–Ricci potential $\sigma = \sigma(r)$, and $|\nabla\sigma|_g^2 = Q(\sigma(r))$.

SKR metrics on compact manifolds also admit a global classification (Theorem 16.3 of [Derdzinski and Maschler 2006]), which shows they reside only on $\mathbb{CP}^1$-bundles $\mathbb{P}(L \oplus \mathbb{C})$ over manifolds N as above, or on complex projective spaces.

6. Functional dependence

Recall equation (3-6i):

$$\mathrm{Ric} + \nabla d\theta + d\mu \odot d\psi = \gamma g, \quad \gamma = \lambda e^{-2\mu} - \Delta\mu + g(\nabla\theta, \nabla\mu), \tag{6-1}$$

with $\mu = \log\tau,\ \theta = f + (n-2)\log\tau$ and $\psi = 2\theta - (n-2)\mu$. This was one of the forms of equation (3-2) characterizing a metric g conformal to a gradient Ricci soliton. If g is also Kähler on a manifold (M, J) of real dimension at least four, constancy of θ implies that g is in fact Kähler–Einstein. This follows since, in this case, the above relation defining ψ shows that the term $d\mu \odot d\psi$ is just a constant multiple of $d\mu \otimes d\mu$, and the latter vanishes, as it is the only term in (6-1) that is not J-invariant.

Note that f cannot be constant on a nonempty open subset of M without being constant everywhere in M, by a real-analyticity argument stemming from [Ivey 1996]. Hence the same holds for θ, because we see from the previous paragraph that constancy of θ on a nonempty open set implies the same for f.

Proposition 6.1. *Assume g is Kähler and conformal to a gradient Ricci soliton in dimension $n \geq 4$ with θ nonconstant. If*

$$df \wedge d\tau = 0$$

(*equivalently*, $d\mu \wedge d\theta = 0$), *then g satisfies on an open dense set a Ricci–Hessian equation of the form*

$$\alpha \nabla d\sigma + \mathrm{Ric} = \gamma g, \tag{6-2}$$

for appropriate functionally dependent functions α and σ.

In fact, in the set where $d\theta \neq 0$, choose any function t of θ with $dt \neq 0$, so that θ and μ become functions of t, on some interval of the variable t. For the moment, t is not further specified. Denoting the derivative with respect to t by $(\dot{\ })$, we have

$$\nabla d\theta + d\mu \odot d\psi = \dot{\theta} \nabla dt + [\ddot{\theta} + 2\dot{\mu}\dot{\theta} - (n-2)\dot{\mu}^2]\, dt \odot dt. \tag{6-3}$$

Next, we choose a function σ of t such that $\dot{\sigma} > 0$ and

$$\ddot{\sigma}/\dot{\sigma} = [\ddot{\theta} + 2\dot{\mu}\dot{\theta} - (n-2)\dot{\mu}^2]/\dot{\theta} \tag{6-4}$$

on the open dense set where $\dot{\theta} \neq 0$. The right-hand side of this equation is given, so that this stipulation amounts to the requirement that an (easily solvable) ODE holds for σ.

We now fix $t = \sigma$. For this choice, (6-4) becomes

$$\ddot{\theta} + 2\dot{\mu}\dot{\theta} - (n-2)\dot{\mu}^2 = 0, \tag{6-5}$$

which holds on the image under σ of an open dense set, namely the intersection of the noncritical set of σ, with points where $\dot{\theta} \neq 0$. It follows from (6-5) and (6-3) that $\nabla d\theta + d\mu \odot d\psi = \alpha \nabla d\sigma$, with $\alpha = \dot{\theta}$. This translates the first of equations (6-1) into a Ricci–Hessian equation.

We now record some relations that will be used in the next theorem, with assumptions as in Proposition 6.1. Let $Q = g(\nabla\sigma, \nabla\sigma)$, $Y = \Delta\sigma$ and s be the scalar curvature of g. First, from (6-1),

$$\gamma = \lambda e^{-2\mu} - \dot{\mu} Y + (\alpha\dot{\mu} - \ddot{\mu}) Q, \tag{6-6}$$

as $\Delta\mu = \dot{\mu} Y + \ddot{\mu} Q$ and $g(\nabla\theta, \nabla\mu) = \alpha\dot{\mu} Q$. Next, we have

$$\alpha Y + s = n\gamma, \tag{6-7i}$$

$$\alpha dY + Y\dot{\alpha} d\sigma + ds = n d\gamma, \tag{6-7ii}$$

$$\alpha dY + \dot{\alpha} dQ + ds = 2d\gamma, \tag{6-7iii}$$

$$\alpha dQ - dY = 2\gamma d\sigma. \tag{6-7iv}$$

These equations are obtained in succession by taking the g-trace of (6-2); forming the d-image of (6-7i); finally, applying twice the divergence operator 2δ and, separately, interior multiplication by $\nabla\sigma$, i.e., $2\imath_{\nabla\sigma}$, to (6-2) and using Proposition 2.3 and the Bianchi relation $2\delta \mathrm{Ric} = ds$.

Further relations are obtained by subtracting (6-7iii) from (6-7ii), then applying $\cdots \wedge d\sigma$ to (6-8i), d to (6-7iv) and d followed by $\cdots \wedge d\sigma$ to (6-6), which yield

$$Y\dot{\alpha}\, d\sigma - \dot{\alpha}\, dQ = (n-2)\, d\gamma, \tag{6-8i}$$

$$\dot{\alpha}\, d\sigma \wedge dQ = (n-2)\, d\gamma \wedge d\sigma, \tag{6-8ii}$$

$$\dot{\alpha}\, d\sigma \wedge dQ = 2\, d\gamma \wedge d\sigma, \tag{6-8iii}$$

$$d\gamma \wedge d\sigma = (\alpha\dot{\mu} - \ddot{\mu})\, dQ \wedge d\sigma - \dot{\mu}\, dY \wedge d\sigma. \tag{6-8iv}$$

We can now state the following partial classification theorem.

Theorem 6.2. *Let g be a Kähler metric conformal to a gradient Ricci soliton $\hat{g}$ on a manifold M of dimension $n \geq 4$, so that equations (3-2) and (6-1) hold. If $df \wedge d\tau = 0$ (equivalently, $d\mu \wedge d\theta = 0$), then one of the following must occur*:

(i) *g is a Kähler–Ricci soliton.*

(ii) *g satisfies a Ricci–Hessian equation, and if it is standard, g is an SKR metric.*

(iii) *$n = 4$ and $\hat{g}$ is an Einstein metric.*

(iv) *$n = 4$ and $\hat{g}$ is a non-Einstein steady gradient Ricci soliton* ($\lambda = 0$).

The Ricci–Hessian equation in (ii) holds on an open dense set.

After proving this theorem, we address its relation to various known examples.

Proof. If θ is constant, we have seen g is Kähler–Einstein, a special case of (i). Assume from now on that θ is nonconstant. Then by Proposition 6.1, g satisfies the Ricci–Hessian equation (6-2) on an open dense set.

When α is constant, so is γ, by (6-8i) and thus (6-2) gives (i). Next, we assume in the rest of this proof that α is nonconstant.

If $n > 4$ (or, $dQ \wedge d\sigma = 0$ everywhere), then $d\gamma \wedge d\sigma = 0$, as verified by subtracting (6-8iii) from (6-8ii) (or, using (6-8iii)). If the Ricci–Hessian equation is standard, taking into consideration that $d\alpha \wedge d\sigma = 0$ because $\alpha = \dot{\theta}$, Proposition 5.1 implies that (ii) holds.

So assume $n = 4$ and $dQ \wedge d\sigma \neq 0$ somewhere in M (and, consequently, almost everywhere, by an argument involving real-analyticity, valid in dimension four). By (6-7iv), (6-8iii) and (6-8iv),

$$(\dot{\alpha} + 2\alpha\dot{\mu} - 2\ddot{\mu})\, dQ - 2\dot{\mu} dY \quad \text{and} \quad 2\dot{\mu}(dY - \alpha dQ) \tag{6-9}$$

are both functional multiples of $d\sigma$. Adding these two relations, we conclude that $(\dot{\alpha} - 2\ddot{\mu})\, dQ \wedge d\sigma = 0$, so that (6-5) with $n = 4$ gives $\dot{\alpha} = 2\ddot{\mu}$ and

$$\alpha = 2(\dot{\mu} + p), \tag{6-10i}$$

$$2\dot{\alpha} + \alpha^2 = 4p^2, \tag{6-10ii}$$

$$4(\alpha\dot{\mu} - \ddot{\mu}) = (3\alpha + 2p)(\alpha - 2p), \tag{6-10iii}$$

for a constant p, where (6-10i) is obtained by integration, (6-10ii) using (6-10i) and (6-5) with $n = 4$, while (6-10iii) follows from (6-10i) and (6-10ii) by algebraic manipulations that use again $\dot{\alpha} = 2\ddot{\mu}$. Also, as $\dot{\theta} = \alpha$,

$$\dot{f} = 2p, \tag{6-11i}$$

$$p[e^{2\mu}(Y - \alpha Q) + 2\lambda\sigma] = \text{constant}. \tag{6-11ii}$$

In fact, differentiating the relation $\theta = f + (n-2)\mu$ with $n = 4$ and (6-10i) give (6-11i). Thus, f equals $2p\sigma$ plus a constant. Hence $\Delta f = 2pY$, and (6-11ii) follows from (3-6ii) and (6-10i). If $p = 0$ then f is constant, and this, by the soliton equation (3-1i), implies (iii).

Suppose, finally, that $p \neq 0$ while $n = 4$ and $dQ \wedge d\sigma \neq 0$ somewhere. As a consequence of (6-8i) and (6-10ii),

$$4\, d\gamma = (4p^2 - \alpha^2)(Y d\sigma - dQ). \tag{6-12}$$

On the other hand, (6-6), (6-10i) and (6-10iii) give

$$4\gamma = 4\lambda e^{-2\mu} + (\alpha - 2p)[(\alpha + 2p)Q + 2(\alpha Q - Y)]. \tag{6-13}$$

Since $p \neq 0$, (6-11ii) yields $\alpha Q - Y = e^{-2\mu}(2\lambda\sigma - b)$ for some constant b, so that (6-13) and (6-12) become

$$4\gamma = e^{-2\mu}[4\lambda + (2\lambda\sigma - b)(2\alpha - 4p)] + (\alpha^2 - 4p^2)Q, \tag{6-14i}$$

$$4\, d\gamma = (4p^2 - \alpha^2)[\alpha Q d\sigma - e^{-2\mu}(2\lambda\sigma - b)d\sigma - dQ]. \tag{6-14ii}$$

Thus $(4p^2-\alpha^2)[\alpha Q d\sigma - e^{-2\mu}(2\lambda\sigma - b)d\sigma]$ equals the sum of $Qd(\alpha^2-4p^2)$ and $d[e^{-2\mu}(4\lambda+(2\lambda\sigma-b)(2\alpha-4p))]$, since both expressions coincide with $4d\gamma+(4p^2-\alpha^2)\,dQ$, which for the former is clear from (6-14ii), and for the latter follows if one applies d to (6-14i). This equation yields $4e^{-2\mu}(2\lambda\sigma-b)(2p-\alpha)\alpha=0$, as seen by evaluating these expressions via the first two parts of (6-10), and subtracting the former expression from the latter. As we are assuming α is not constant, it follows necessarily that λ (and b) must be zero. This gives (iv), completing the proof. ☐

We remark on the relation of the four possible outcomes in this theorem to known examples. Many examples of Ricci solitons that are Kähler have been described in the literature (see for instance [Koiso 1990; Cao 1996; Pedersen et al. 1999; Wang and Zhu 2004; Dancer and Wang 2011]), and they are all examples of outcome (i), with constant conformal factor τ. A glance at the proof of Theorem 6.2 shows that the door is left open for another possibility. Namely, when θ is nonconstant but $\alpha=\dot\theta$ is constant, equation (6-5) yields that either $\mu=\log\tau$ is constant or $\dot\mu$ is constant, so that τ is an exponential in an expression affine in σ. We do not know if there exists a corresponding example of a gradient Kähler–Ricci soliton nontrivially conformal to a gradient Ricci soliton. The case of Einstein metrics conformal to other Einstein metrics is classical. On non-Einstein gradient Ricci solitons conformal to other such solitons, see [Jauregui and Wylie 2015; Maschler 2015].

Concerning the SKR metric option in possibility (ii), there are known examples of SKR metrics nontrivially conformal to Ricci solitons. Such metrics include, up to biholomorphic isometry, all Kähler conformally Einstein ones in dimension $n>4$ [Derdzinski and Maschler 2003; 2006; 2005]. Regarding SKR metrics conformal to non-Einstein gradient Ricci solitons, examples were constructed in [Maschler 2008] and [Derdziński 2012]. The former examples are special among those of the latter, as for them the conformal factor τ, rather than some function σ of it, is a Killing potential, and, more importantly, the soliton is itself Kähler with respect to another complex structure.

Note that the characteristics of the spaces that admit SKR metrics are fairly restrictive, in that they are quite specific holomorphic line bundles (if the base is not Ricci flat) over a base that is Kähler–Einstein (if $n>4$, see Section 5). Thus many of the later examples of Ricci solitons, Kähler or not, such as the cohomogeneity one metrics on vector bundles over a product of Fano Kähler–Einstein manifolds [Dancer and Wang 2011; 2009], are not SKR or conformal to SKR metrics. On the other hand, the recent examples of Stolarski [2015] live on exactly the right type of space, and it is an interesting question whether his metrics are conformal to SKR metrics.

Note that the examples in [Maschler 2008], are of a type first considered by Koiso [1990] and Cao [1996]. They, along with those in [Derdziński 2012], are also irreducible. Although the development of reducible SKR metrics runs in parallel to that of the irreducible ones, with somewhat simpler formulas, and a simpler classification of conformally Einstein such metrics, the theory of reducible SKR metrics conformal to non-Einstein Ricci solitons is currently underdeveloped, compared with the irreducible case, perhaps because the eigenfunction ϕ of the Hessian in the Ricci–Hessian equation is then identically zero, and so equations (5-2) do not hold. But see also Remark 7.2.

A Kähler conformally Einstein metric of the type given in possibility (iii) is not an SKR metric, since the latter must satisfy $dQ \wedge d\sigma = 0$, a relation that, as the proof of Theorem 6.2 shows, does not hold in this case. Instead, one has a relevant example on the two-point blowup of $\mathbb{CP}^2$, namely the Chen–LeBrun–Weber metric [2008].

Possibility (iv) is perhaps the least expected. We do not know if there are metrics of this type, and this constitutes an interesting question. The stipulation that the soliton $\hat{g}$ is steady brings to mind the four-dimensional version of the examples in [Buzano et al. 2015], which was already considered in [Ivey 1994]. However, the condition $dQ \wedge d\sigma \neq 0$ that must be satisfied points more towards a metric like that of (iii) rather than to a bundle-like metric. But unlike case (iii), such a metric cannot occur on a compact manifold, as it is well known that compact manifolds do not admit non-Einstein steady gradient Ricci solitons (see [Ivey 1993]).

7. Quasi-solitons

Many of the original examples of gradient Ricci solitons arise as warped products over a one-dimensional base (see for instance [Chow et al. 2007]). We consider here the case of an arbitrary base.

Let $\bar{g}$ be a warped product (gradient Ricci) soliton metric on a manifold $M = B \times F$, so that

$$\bar{g} = g_B + \ell^2 g_F := g + \ell^2 g_F, \quad \overline{\mathrm{Ric}} + \bar{\nabla} df = \lambda \bar{g}, \tag{7-1}$$

where ℓ is (the pullback of) a function on the base B and λ is constant. When $\bar{g}$ is Einstein, the base metric $g = g_B$ is often called quasi-Einstein. In the setting of (7-1), g_B will be a special case of what we call a *quasi-soliton* metric. The latter is defined as a metric g satisfying (7-2i) below, for some functions f and ℓ and some constant λ. There, and in what follows, we drop the subscript B in the notation for g_B-dependent quantities.

Proposition 7.1. *With notation as above, the soliton equation for* $\bar{g}$ (see (7-1)) *is equivalent to the system*

$$\mathrm{Ric} - \frac{k}{\ell}\nabla d\ell + \nabla df = \lambda g, \quad k = \dim(F), \tag{7-2i}$$

$$\mathrm{Ric}_F = \nu g_F, \tag{7-2ii}$$

(7-2iii) *where* $\nu = -\ell d_{\nabla f}\ell + \ell^2 \ell^{\#} + \lambda \ell^2$, *for* $\ell^{\#} = \ell^{-1}\Delta\ell + (k-1)\ell^{-2}|\nabla\ell|^2$.

In particular the fiber metric is Einstein if $\dim(F) > 2$, and f turns out to be a function with vanishing fiber covariant derivative (see below), so that we regard it as a function on B. Unlike the quasi-Einstein case [Kim and Kim 2003], the scalar equation on the left in (7-2iii), with ν a constant, does not follow from (7-2i).

Proof. To derive the equations, we need the well-known Ricci curvature formulas for warped products (see [O'Neill 1983]), and additionally, similar equations for the Hessian of f. For the latter we use the covariant derivative formulas for warped products, together with the known fact that for a C^1 function defined on the base, the gradient of its pullback equals the pullback of its base gradient.

Let, x, y denote lifts of vector fields on B, and u, v lifts of vector fields on F. Then

$$\begin{aligned} &\bar\nabla_x y \text{ is the lift of } \nabla_x y \text{ on } B,\\ &\bar\nabla_x v = \bar\nabla_v x = d_x \log(\ell) v,\\ &[\bar\nabla_v w]^F \text{ is the lift of } \nabla^F_v w \text{ on } F,\\ &[\bar\nabla_v w]^B = -\bar g(v, w)\nabla \log \ell. \end{aligned} \tag{7-3}$$

Hence,

$$\begin{aligned} \bar\nabla df(x, y) &= \bar g\big(\bar\nabla_x(\bar\nabla f)^B, y\big) + \bar g\big(\bar\nabla_x(\bar\nabla f)^F, y\big)\\ &= g\big(\nabla_x(\bar\nabla f)^B, y\big) + \bar g\big(d_x \log \ell(\bar\nabla f)^F, y\big) = g\big(\nabla_x(\bar\nabla f)^B, y\big),\\ \bar\nabla df(x, v) &= \bar g\big(\bar\nabla_x(\bar\nabla f)^B, v\big) + \bar g\big(\bar\nabla_x(\bar\nabla f)^F, v\big) = \ell d_x \ell g_F\big((\bar\nabla f)^F, v\big),\\ \bar\nabla df(v,w) &= \bar g\big(\bar\nabla_v(\bar\nabla f)^B, w\big) + \bar g\big(\bar\nabla_v(\bar\nabla f)^F, w\big)\\ &= d_{(\bar\nabla f)^B}(\log \ell)\bar g\big(v,w\big) + \bar g\big(\nabla^F_v(\bar\nabla f)^F, w\big) - \bar g\big(v, (\bar\nabla f)^F\big)\bar g\big(\nabla \log \ell, w\big)\\ &= \ell d_{(\bar\nabla f)^B}(\ell) g_F\big(v,w\big) + \ell^2 g_F\big(\nabla^F_v(\bar\nabla f)^F, w\big). \end{aligned}$$

We combine these with the Ricci curvature formulas

$$\begin{aligned} &\mathrm{Ric}(x, y) = \mathrm{Ric}_B(x, y) - \Big(\frac{k}{\ell}\Big)\nabla d\ell(x, y),\\ &\mathrm{Ric}(x, v) = 0,\\ &\mathrm{Ric}(v, w) = \mathrm{Ric}_F(v, w) - \ell^\# g(v, w). \end{aligned} \tag{7-4}$$

We now notice that the soliton equation applied to x and v implies that $(\bar\nabla f)^F = 0$ so that f can be regarded as the pullback of a function on B. This readily gives equations (7-2). ☐

Remark 7.2. The structure (7-1) above can at times give an example of a Ricci soliton which is conformally Kähler. Namely, $\tilde g = \ell^{-2}\bar g$ is clearly a product metric, and if, for example, $\dim B = 2$, so that the quasi-Einstein metric g_B is Kähler with respect to some complex structure on B, while g_F is chosen to be Kähler–Einstein on F, then $\tilde g$ is Kähler, reducible and conformal to a Ricci soliton.

In analogy with the previous section, we will be considering quasi-soliton metrics for which f and ℓ are functionally dependent, that is,

$$df \wedge d\ell = 0. \tag{7-5}$$

We call such metrics *special quasi-soliton* metrics.

It is known that Kähler quasi-Einstein metrics which are not Einstein do not exist on a compact manifold, and in general must be certain Riemannian product metrics [Case et al. 2011]. Similarly we show:

Theorem 7.3. *Let g be a Kähler special quasi-soliton metric on a manifold M of dimension at least four. Then g satisfies a Ricci–Hessian equation on an open set. If this equation is standard, then g is a Riemannian product there. If the dimension is greater than four, then one of the factors in this product is a Kähler–Einstein manifold of codimension two.*

Proof. As the quasi-soliton metric is special, we have $\nabla df = f'\nabla d\ell + f'' d\ell \otimes d\ell$, where the prime denotes differentiation with respect to ℓ. Then (7-2i) becomes

$$\mathrm{Ric} + \Big(f' - \frac{k}{\ell}\Big)\nabla d\ell + f'' d\ell \otimes d\ell = \lambda g. \tag{7-6}$$

In analogy with Proposition 6.1, we introduce a function σ with $d\ell \wedge d\sigma = 0$ and rewrite the special quasi-Einstein equation (7-6) as

$$\mathrm{Ric} + \tilde{\alpha}\ell' \nabla d\sigma + (\tilde{\alpha}\ell'' + f''\ell'^2)\, d\sigma \otimes d\sigma = \lambda g, \tag{7-7}$$

for $\tilde{\alpha} = f'(\ell) - k/\ell$, with the convention that primes on ℓ represent differentiations with respect to σ, while primes on f still represent differentiations with respect to ℓ. The restriction on the open set where an ODE analogous to (6-4) holds is $\alpha := \tilde{\alpha}\ell' \neq 0$ (corresponding to $\dot{\theta} \neq 0$ in Proposition 6.1). On that set, equation (7-7) becomes a Ricci–Hessian equation of the form

$$\mathrm{Ric} + \alpha \nabla d\sigma = \lambda g, \quad \alpha = \tilde{\alpha}\ell',$$

provided we choose σ so that the differential equation

$$\tilde{\alpha}\ell'' + f''\ell'^2 = 0 \tag{7-8}$$

also holds.

Assuming the Ricci–Hessian equation is standard, Proposition 5.1 now shows that g is an SKR metric on the open set described above. If g is irreducible, the theory of SKR metrics gives the two equations (5-2), which now take the form

$$(\sigma - c)^2\phi'' + (\sigma - c)[m - (\sigma - c)\alpha]\phi' - m\phi = K, \tag{7-9i}$$

$$-(\sigma - c)\phi'' + [\alpha(\sigma - c) - (m+1)]\phi' + \alpha\phi = \lambda, \tag{7-9ii}$$

where ϕ is defined pointwise as the eigenvalue of the Hessian of σ, mentioned in Section 5.

Divide (7-9i) by $\sigma - c$, add to (7-9ii) and multiply both sides of the resulting equality by -1, to obtain

$$\phi' + \left(\frac{m}{\sigma - c} - \alpha\right)\phi = -\lambda - \frac{K}{\sigma - c}. \tag{7-10}$$

We will apply Lemma 5.2 to the system consisting of (7-9i) (whose coefficients we now call A, B, C, D) and (7-10) (with the obvious p and q). According to the lemma, the solution ϕ is the ratio $(D - A(q' - pq) - Bq)/(A(p^2 - p') - Bp + C)$, if the denominator is nonzero. But one easily computes that $D - A(q' - pq) - Bq = 0$. However, as mentioned in Section 5, the function ϕ is nowhere zero on the set where $d\sigma \neq 0$ when g is irreducible. Hence the only possibility is that $A(p^2 - p') - Bp + C$ vanishes identically. But it is easily seen from the definitions of A, B, C, p that

$$A(p^2 - p') - Bp + C = \alpha'(\sigma - c)^2.$$

We conclude that α is constant, so g is additionally a gradient Ricci soliton. Writing this condition explicitly we get, with primes now denoting solely differentiation with respect to σ,

$$(f \circ \ell)' - k\frac{\ell'}{\ell} = b,$$

where b is constant. But equation (7-8) can also be written as

$$(f \circ \ell)'' - k\frac{\ell''}{\ell} = 0.$$

Differentiating the first of these two equations and combining it with the second shows that ℓ is constant, hence g is Einstein. But this means $\alpha \equiv 0$, contradicting that the Ricci–Hessian equation for g is standard. Hence g must be reducible. The structure of the Riemannian product constituting g follows from SKR theory. □

Next we consider the problem of whether quasi-soliton metrics can be conformally Kähler. This is certainly possible for quasi-Einstein metrics (see [Maschler 2011; Batat et al. 2015]). We have the following result, analogous in form and in proof to the previous one, though it requires more assumptions and is computationally more difficult.

Theorem 7.4. *Let M be a manifold of dimension $n = 2m > 4$ and g an irreducible Kähler metric on M conformal to a special quasi-soliton $\hat{g} = g/\tau^2$ having warping function ℓ, potential f and appropriate constants k and λ. Assume τ is a Killing potential for g and $d\ell \wedge d\tau = 0$. Then g satisfies a Ricci–Hessian equation. If the latter is standard, then $\hat{g}$ is quasi-Einstein.*

Proof. Being a special quasi-soliton, $\hat{g}$ satisfies equation (7-6), i.e.,

$$\widehat{\mathrm{Ric}} + \mu \widehat{\nabla} d\ell + \chi\, d\ell \otimes d\ell = \lambda \hat{g}, \tag{7-11}$$

for $\mu = f'(\ell) - k/\ell$ and $\chi = f''(\ell)$.

Using (2-3) and (2-2i), we see that g satisfies

$$\begin{aligned} \mathrm{Ric} + (n-2)\tau^{-1}\nabla d\tau + (\tau^{-1}\Delta\tau - (n-1)\tau^{-2}Q)g \\ + \mu(\nabla d\ell + 2\tau^{-1} d\tau \odot d\ell - \tau^{-1} g(\nabla\tau, \nabla\ell)g) + \chi\, d\ell \otimes d\ell = \lambda\tau^{-2} g, \end{aligned} \tag{7-12}$$

with $Q = g(\nabla\tau, \nabla\tau)$. Since $d\ell \wedge d\tau = 0$, writing $d\ell = \ell'(\tau)d\tau$ and rearranging terms, we rewrite this equation as

$$\begin{aligned} \mathrm{Ric} + \alpha\nabla d\tau + (\mu(\ell'' + 2\tau^{-1}\ell') + (\ell')^2\chi)\, d\tau \otimes d\tau \\ = (\lambda\tau^{-2} - \tau^{-1}\Delta\tau + (\alpha + \tau^{-1})\tau^{-1}Q)g \quad \text{for } \alpha = (n-2)\tau^{-1} + \mu\ell'. \end{aligned} \tag{7-13}$$

As g is Kähler and τ is a Killing potential, the term with $d\tau \otimes d\tau$ is the only one which is not J-invariant. Hence its coefficient must vanish:

$$\mu(\ell'' + 2\tau^{-1}\ell') + (\ell')^2\chi = 0. \tag{7-14}$$

As a result, equation (7-13) is Ricci–Hessian:

$$\mathrm{Ric} + \alpha\nabla d\tau = \gamma g, \quad \text{where } \gamma = \lambda\tau^{-2} - \tau^{-1}\Delta\tau + (\alpha + \tau^{-1})\tau^{-1}Q. \tag{7-15}$$

Since clearly $d\alpha \wedge d\tau = 0$, and $n > 4$, as mentioned in Section 5, we also have $d\gamma \wedge d\tau = 0$. Under the assumption that the Ricci–Hessian equation is standard, we conclude from Proposition 5.1 that (g, τ) is an SKR metric with τ the special Kähler–Ricci potential. As in the previous theorem, irreducibility of g again implies that two ODEs hold for the horizontal Hessian eigenvalue function ϕ. They are

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K, \tag{7-16i}$$

$$\begin{aligned} -(\tau - c)\phi'' + (\alpha(\tau - c) - (m+1))\phi' + \alpha\phi = \gamma \\ = \lambda\tau^{-2} - \tau^{-1}(2m\phi + 2(\tau - c)\phi') + (\alpha + \tau^{-1})\tau^{-1}2(\tau - c)\phi, \end{aligned} \tag{7-16ii}$$

where K, c are constants, and we have used formulas (5-3) giving $\Delta\tau$ and Q in terms of ϕ.

Simplifying the second equation, we then replace it by a first-order equation as in the previous theorem, to obtain the equivalent system

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = K, \tag{7-17i}$$

$$\begin{aligned} \frac{(\tau - c)(\tau - 2c)}{\tau}\phi' - \Big(\frac{(\tau - c)(\tau - 2c)}{\tau}\alpha + \frac{2(\tau - c)^2 - m\tau(\tau - 2c)}{\tau^2}\Big)\phi \\ = \frac{K\tau^2 + \lambda\tau - \lambda c}{\tau^2}. \end{aligned} \tag{7-17ii}$$

Naming the coefficients A, B, C, D, p, q as before, we now apply Lemma 5.2 to the system (7-17). This time the computation of the two quantities used in the lemma is quite laborious, though still elementary. A symbolic computational program simplifies the result to the following.

$$\begin{aligned} &A(p^2-p')-Bp+C=\frac{(\tau-c)^2((\tau-2c)\tau\alpha'+2(\tau-c)\alpha+2-2m)}{\tau(\tau-2c)},\\ &D-A(q'-pq)-Bq=0. \end{aligned} \tag{7-18}$$

By the lemma and the fact that ϕ is nowhere zero, solutions are only possible if the first expression vanishes identically, so that α solves

$$(\tau-2c)\tau\alpha'+2(\tau-c)\alpha+2-2m=0.$$

The solutions of this equation take the form

$$\alpha=\frac{n-2}{\tau}+\frac{C}{\tau(\tau-2c)}, \tag{7-19}$$

where C is a constant. As (7-15) and (7-16ii) imply that the form of α determines that of γ, we have the following outcome. If $c=0$, the metric g is conformal to a gradient Ricci soliton [Maschler 2008, Proposition 2.4], while if $c\neq 0$ then g is conformal to a quasi-Einstein metric [Maschler 2011][1]. (The case $C=0$ is a special case of both these types, where g is conformally Einstein [Derdzinski and Maschler 2003].)

To rule out the case that $\hat{g}$ is a nontrivial gradient Ricci soliton, we note first that the expression defining α in (7-13), when compared to that in (7-19), results in

$$(f\circ\ell)'-k\frac{\ell'}{\ell}=\frac{C}{\tau(\tau-2c)}.$$

Additionally, equation (7-14) can also be written as

$$(f\circ l)''-k\frac{\ell''}{\ell}+2\Big((f\circ\ell)'-k\frac{\ell'}{\ell}\Big)\tau^{-1}=0.$$

Substituting the first of these equations in the last term of the second, and combining the result with the derivative of the first equation gives, after eliminating $(f\circ\ell)''-k\ell''/\ell$ and rearranging terms,

$$k\frac{\ell'^2}{\ell^2}=\frac{2C}{\tau^2(\tau-2c)}+\Big(\frac{C}{\tau(\tau-2c)}\Big)'=-\frac{2cC}{\tau^2(\tau-2c)^2}.$$

Hence the Ricci soliton case $c=0$ implies that ℓ is constant, so that comparing the two expressions for α again yields $C=0$, i.e., that $\hat{g}$ is Einstein, which is, of course, a special case of the quasi-Einstein condition. □

[1] See (2.3) in that paper, where the quasi-Einstein case is given by $\alpha=(n-2)/\tau+a/(\tau(1+k\tau))$, where a is a constant and $k=-\frac{1}{2c}$. This corresponds to formula (7-19) with $C=-2ac$.

We comment here on the assumption in this theorem that τ is a Killing potential, which is the same assumption that singles out the examples in [Maschler 2008] among those of [Derdziński 2012]. In analogy with the previous theorems, it is possible instead to replace equation (7-13) by a similar equation involving $\nabla d\sigma$ and $d\sigma \otimes d\sigma$, for a function σ of τ that will serve, after choosing it appropriately, as the special Kähler–Ricci potential instead of τ. One can obtain then two differential equations analogous to (7-16) and (7-17) with independent variable σ. However, these equations will involve τ and its derivatives with respect to σ, and this unknown dependency hinders the determination of solutions and the corresponding α. Even if one knew this α as a function of σ, this will not easily shed light on what metric g is conformal to (with conformal factor τ). Finally, without the Killing assumption on τ, it is not clear that a similar result should be expected, as existence of more general conformally Kähler quasi-solitons may occur. This is in analogy with the fact mentioned above, that there do exist conformally Kähler quasi-Einstein metrics.

Appendix: Killing vector fields of the form $w = \tau^2 \nabla f$

We consider here the classification problem for Killing fields of the form $w = \tau^2 \nabla f$, a form that played an important role in Section 4. In the following τ and f will denote smooth functions on a given manifold.

Proposition A.1. *On a compact manifold, a Killing field of the form $w = \tau^2 \nabla f$ must be trivial.*

Proof. First, on a compact manifold ∇f has zeros, hence so does w. Let p be a zero of $w = \tau^2 \nabla f$. Since $\nabla w = 2\tau d\tau \otimes \nabla f + \tau^2 \nabla df$, and at a zero either $\tau = 0$ or $\nabla f = 0$, we see that at a zero ∇w equals either zero or $\tau^2 \nabla df$. But in the latter case ∇w is symmetric, yet it is also skew-symmetric as w is a Killing field, hence ∇w must be zero in this case as well. However, a Killing field w is uniquely determined by the values of w and ∇w at one point. As those values are zero at p, we see that w must be the zero vector field. □

Without compactness, we have the following classification for such vector fields.

Theorem A.2. *A Riemannian metric g with a Killing vector field of the form $w = \tau^2 \nabla f$ is, near generic points, a warped product with a one-dimensional fiber. If g is also Kähler, it is, near such points, a Riemannian product of a Kähler metric with a surface metric admitting a nontrivial Killing vector field.*

We note here that a surface with a nontrivial Killing vector field can be presented as a warped product with a one-dimensional fiber and base.

Proof. First, the orthogonal complement $\mathcal{H}$ to $\operatorname{span}(w)$ is generically $[\nabla f]^{\perp}$, which is obviously integrable. Next, $\mathcal{H}$ is totally geodesic. This follows immediately since $g(\dot{x}, w)$ is constant for any geodesic $x(t)$ and Killing field w.

By a result going back to [Hiepko 1979] and [Ponge and Reckziegel 1993] (see especially Theorem 3.1 in the survey [Zeghib 2011]), a metric is a warped product if and only if it admits two orthogonal foliations, one totally geodesic and the other spherical. In our case we have just shown the foliation orthogonal to w is totally geodesic. The fibers tangent to $\mathrm{span}(w)$, on the other hand, are certainly totally umbilic, as they are one-dimensional. This is part of the definition of spherical. The other part is that the mean curvature vector is parallel with respect to the normal connection. We now check this.

Let $w' = w/|w|$ be a unit vector parallel to w, defined away from its zeros. The mean curvature vector to the fibers is then, by definition, $n = \nabla_{w'} w'$, which takes values in $\mathcal{H}$. The requirement that $\mathrm{span}(w)$ be spherical amounts to showing that for any $x \in \mathcal{H}$, we have $g(\nabla_w n, x) = 0$. The flow of w certainly preserves itself (as $[w, w] = 0$) and also g and ∇ (as w is Killing). Therefore the flow also preserves $w' = w/\sqrt{g(w, w)}$ and thus also $n = \nabla_{w'} w'$. Hence $[w, n] = 0$, so that

$$\begin{aligned} 2g(\nabla_w n, x) &= 2g(\nabla_n w, x) = g(\nabla_n w, x) - g(n, \nabla_x w) \\ &= -g(w, \nabla_n x) + g(w, \nabla_x n) \\ &= g(w, [x, n]) = 0, \end{aligned}$$

as $\mathcal{H}$ is integrable. This concludes the first part of the proof.

What remains is to classify Kähler warped products with a one-dimensional fiber. Suppose the manifold is given by $M = B \times F$, with F the fiber (an interval). Since the base foliation corresponding to B is totally geodesic, parallel transport along one of its leaves with respect to g is the same as parallel transport with respect to the induced metric on this leaf, and therefore it preserves the tangent spaces to these leaves. It is well known that it also preserves the normal spaces to the leaves; for completeness, we show explicitly that the unit vector field w' perpendicular to the leaves is preserved. If x and y are, as usual, vector fields tangent to the leaves, then $g(w', y) = 0$, so $0 = d_x g(w', y) = g(\nabla_x w', y) + g(w', \nabla_x y) = g(\nabla_x w', y)$ because the leaves are totally geodesic, and similarly $0 = d_x g(w', w') = 2g(\nabla_x w', w')$. So $\nabla_x w'$, being orthogonal to a basis, is zero, i.e., w' is parallel in directions tangent to the leaves.

As g is Kähler, the complex structure J commutes with any ∇_x, so that Jw' is also parallel in leaf directions. But Jw' is itself tangent to leaves of the base foliation. Therefore, by the local de Rham theorem, the induced metric on any leaf splits locally into a Riemannian product so that $B = N \times I$, where the one-dimensional factor I is tangent to Jw', and N is J-invariant, hence has holomorphic (and totally geodesic) leaves in M.

Armed with this information it remains to show that, near generic points,

g is a product of a Kähler metric on N and a local metric of revolution on $I \times F$.

For this we turn to a computation that is based on the formulas (see for example [O'Neill 1983]) for the connection of the warped product metric $g = g_B + l^2 g_F$, where the function l is a (lift of) a function on B. Let t be a nontrivial vector field tangent to F which is projectable onto F. Let $s = Jt$, a vector field tangent to I. Then standard formulas for warped products give

$$\nabla_t t = (\nabla_t t)^B + (\nabla_t t)^F = -|t|^2 \nabla(\log l) + ct, \tag{A-1}$$

with c some function, and the last term takes that form because the fiber is one-dimensional. Next, as s is tangent to I, there is some function h on M such that the vector field hs is projectable onto I. Therefore, again by warped product formulas,

$$\nabla_t(hs) = hs(\log l)t. \tag{A-2}$$

But $\nabla_t(hs) = (d_t h)s + h\nabla_t s = (d_t h)s + hJ\nabla_t t = (d_t h)s - h|t|^2 J\nabla(\log l) + hcs$, by (A-1). Equating this expression with the right-hand side of (A-2) and taking components tangent to N gives $h|t|^2[J\nabla(\log l)]^N = 0$, so that, away from the zeros of h and t, $[J\nabla(\log l)]^N = 0$. Now each tangent space T_pN is J-invariant, so J commutes with the projection to N. Hence $\nabla(\log l)^N = 0$ and so $\nabla(\log l)$ is parallel to s, which means that the warping function l is constant on the leaves of N, and only changes along the fibers associated with I. Thus g is a Riemannian product of the type claimed above. □

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