CLASSIFYING RESOLVING SUBCATEGORIES

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We use the theory of Auslander–Buchweitz approximations to classify certain resolving subcategories containing a semidualizing or a dualizing module. In particular, we show that if the ring has a dualizing module, then the resolving subcategories containing maximal Cohen–Macaulay modules are in bijection with grade consistent functions and thus are the precisely the dominant resolving subcategories.

1. Introduction

Classifying various types of subcategories of $\mathrm{mod}(R)$ and $D(R)$ for a commutative ring R has been the subject of much recent research. These classifications are intrinsically connected to spec R or some other topological space. For instance, the Hopkins–Neeman theorem [Hopkins 1987; Neeman 1992] and Gabriel's theorem [1962] give a bijection between the Serre subcategories of $\mathrm{mod}(R)$, the thick subcategories of perfect complexes, and the specialization closed subsets of spec R. Another example is the work regarding the classification of thick subcategories of $\mathrm{mod}(R)$ such as in [Takahashi 2010; Stevenson 2014b].

Recently, much attention has been given to classifying the resolving subcategories of $\mathrm{mod}(R)$. The study of resolving subcategories began with Auslander and

Bridger's influential work [1969] where they define the category of Gorenstein dimension zero modules, which we denote by GDZ. Also, they generalize the notion of projective dimension by defining Gorenstein dimension through approximations of Gorenstein dimension zero modules. In their paper, they also prove that GDZ has certain homological closure properties which cause Gorenstein dimension to behave similarly to projective dimension. They then take these homological closure properties of GDZ as the definition of resolving subcategories. We can take dimension with respect to a resolving subcategory, and, as in the case of GDZ, these homological closure properties force this dimension function to also behave similarly to projective dimension. See Section 2 for further exposition.

The classification of resolving subcategories was advanced by Dao and Takahashi in [2015], where they give a bijection between the set of resolving subcategories of the category of finite projective dimension modules and the set of grade consistent functions. A function $f : \operatorname{spec} R \to \mathbb{N}$ is called grade consistent if it is increasing (as a morphism of posets) and $f(p) \leq \operatorname{grade}(p)$ for all $p \in \operatorname{spec}(R)$. This result motivated the author to find other situations where a similar bijection exists, furthering the use of grade consistent functions in classifying resolving subcategories. Before the work of Dao and Takahashi, Takahashi [2013] classified, over Cohen–Macaulay rings, resolving subcategories closed under tensor products and Auslander transposes, and in [2011] he classified the contravariantly finite resolving subcategories of a Henselian local Gorenstein ring. Takahashi [2009] also studied resolving subcategories which are free on the punctured spectrum. Auslander and Reiten [1991] discovered a connection between resolving subcategories and tilting theory, and they classified all the contravariantly finite resolving subcategories using cotilting bundles. After the work of Dao and Takahashi, the resolving subcategories of the category of finite projective dimension modules were also classified in [Angeleri Hügel et al. 2014] in terms of descending sequences of specialization closed subsets of $\operatorname{spec} R$, and were also classified in [Angeleri Hügel and Saorín 2014] in terms of certain t-structures.

In this paper, we assume that R is commutative and Noetherian, and we consider only finitely generated modules. Let $\mathcal{P}$ denote the category of projective modules and Γ the set of grade consistent functions. For categories $\mathcal{M}, \mathcal{X} \subseteq \operatorname{mod}(R)$ and $f \in \Gamma$, we define

$$\Lambda_{\mathcal{M}}(f) = \{X \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{M}_p\text{-dim} X_p \leq f(p) \text{ for all } p \in \operatorname{spec} R\}$$

and

$$\Phi_{\mathcal{M}}(\mathcal{X}) : \operatorname{spec} R \to \mathbb{N},$$

$$p \mapsto \sup \{\operatorname{add} \mathcal{M}_p\text{-dim} X_p \mid X \in \mathcal{X}\},$$

where $\operatorname{add} \mathcal{M}_p$ is the smallest subcategory of $\operatorname{mod}(R_p)$ closed under direct sums and summands and containing M_p for every $M \in \mathcal{M}$, and where $\operatorname{add} \mathcal{M}_p\text{-dim} X_p$ is the smallest resolution of X_p by objects in $\operatorname{add} \mathcal{M}_p$. Let $\mathfrak{R}$ denote the collection of

resolving subcategories of $\mathrm{mod}(R)$. Set $\Delta(\mathcal{M}) = \{X \in \mathrm{mod}(R) \mid \mathcal{M}\text{-dim } X < \infty\}$ for any $\mathcal{M} \subseteq \mathrm{mod}(R)$, and let $\mathfrak{R}(\mathcal{M})$ be the collection of resolving subcategories $\mathcal{X}$ such that $\mathcal{M} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{M})$. Using our new notation, we can restate Dao and Takahashi's result [2015].
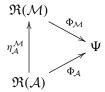
Theorem 1.1. *When R is Noetherian,*

$$\mathfrak{R}(\mathcal{P}) \underset{\Phi_{\mathcal{P}}}{\overset{\Lambda_{\mathcal{P}}}{\rightleftarrows}} \Gamma$$

is a bijection, where $\Lambda_{\mathcal{P}}$ and $\Phi_{\mathcal{P}}$ are inverses of each other.

Our first main result is Theorem 4.2, which is the following. Note that throughout this paper, all thick subcategories contain R.

Theorem A. *Let Ψ be a set of increasing functions from $\mathrm{spec}\, R$ to $\mathbb{N}$. Suppose $\mathcal{A} \subseteq \mathcal{M}$ such that $\mathcal{A}$ cogenerates $\mathcal{M}$ and add $\mathcal{A}_p$ is thick in add $\mathcal{M}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{spec}\, R$. Define $\eta_{\mathcal{A}}^{\mathcal{M}} : \mathfrak{R}(\mathcal{A}) \to \mathfrak{R}(\mathcal{M})$ by $\eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \mathrm{res}(\mathcal{X} \cup \mathcal{M})$ and $\rho_{\mathcal{A}}^{\mathcal{M}} : \mathfrak{R}(\mathcal{M}) \to \mathfrak{R}(\mathcal{A})$ by setting $\rho_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \Delta(\mathcal{A}) \cap \mathcal{X}$. If $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverses of each other giving a bijection between $\mathfrak{R}(\mathcal{A})$ and Ψ, then we have the commutative diagram*

$$\begin{array}{ccc} \mathfrak{R}(\mathcal{M}) & & \\ {\scriptstyle\eta_{\mathcal{A}}^{\mathcal{M}}}\big\uparrow & \xrightarrow{\ \Phi_{\mathcal{M}}\ } & \Psi \\ \mathfrak{R}(\mathcal{A}) & \xrightarrow[\ \Phi_{\mathcal{A}}\ ]{} & \end{array}$$

where $\Phi_{\mathcal{M}}$ is bijective with $\Lambda_{\mathcal{M}}$ its inverse. Moreover, $\rho_{\mathcal{A}}^{\mathcal{M}}$ is the inverse of $\eta_{\mathcal{A}}^{\mathcal{M}}$.

This result allows us to extend the bijection from [Dao and Takahashi 2015] to a plethora of categories. We use it to prove the following result which is essentially Theorem 8.5. Note that $\mathcal{G}_C$ is the category of totally C-reflexive modules where C is a semidualizing module: see Definition 3.1 and Definition 3.4. Define, $\rho_{\mathcal{M}}^{\mathcal{N}}$ and $\eta_{\mathcal{M}}^{\mathcal{N}}$ similarly to $\rho_{\mathcal{A}}^{\mathcal{M}}$ and $\eta_{\mathcal{A}}^{\mathcal{M}}$.

Theorem B. *For any thick subcategory $\mathcal{M}$ of $\mathcal{G}_C$ containing C, $\Lambda_{\mathcal{M}}$ and $\Phi_{\mathcal{M}}$ give a bijection between $\mathfrak{R}(\mathcal{M})$ and Γ. Furthermore, let $\mathfrak{S}$ denote the collection of thick subcategories of $\mathcal{G}_C$ containing C. The following is a bijection*:

$$\Lambda : \mathfrak{S} \times \Gamma \longrightarrow \bigcup_{\mathcal{M} \in \mathfrak{S}} \mathfrak{R}(\mathcal{M}) \subseteq \mathfrak{R}.$$
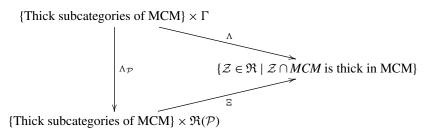
For any $\mathcal{M}, \mathcal{N} \in \mathfrak{S}$ *with* $\mathcal{M} \subseteq \mathcal{N}$, *then the following diagram commutes*:

$$
\begin{array}{ccc}
\mathfrak{R}(\mathcal{N}) & & \\
\uparrow {\scriptstyle \eta_{\mathcal{M}}^{\mathcal{N}}} & \searrow {\scriptstyle \Phi_{\mathcal{N}}} & \\
& & \Gamma \\
\mathfrak{R}(\mathcal{M}) & \nearrow {\scriptstyle \Phi_{\mathcal{M}}} &
\end{array}
$$

In particular, $\rho_{\mathcal{M}}^{\mathcal{N}}$ *and* $\eta_{\mathcal{M}}^{\mathcal{N}}$ *are inverse functions*.

These theorems show that the classification of resolving subcategories is intrinsically linked to the classification of thick subcategories of totally C-reflexive modules and hence to the classification of thick subcategories of $\mathrm{mod}(R)$, a topic of current research. See, for instance, [Takahashi 2010; Neeman 1992]. Applying these results in the Gorenstein case yields Theorem 9.1 which, letting MCM denote the category of maximal Cohen–Macaulay modules, states

Theorem C. *If R is Gorenstein, then we have the following commutative diagram of bijections*:

$$
\begin{array}{ccc}
\{\text{Thick subcategories of MCM}\} \times \Gamma & & \\
\Big\downarrow {\scriptstyle \Lambda_{\mathcal{P}}} & \searrow {\scriptstyle \Lambda} & \\
& & \{\mathcal{Z} \in \mathfrak{R} \mid \mathcal{Z} \cap MCM \text{ is thick in MCM}\} \\
\{\text{Thick subcategories of MCM}\} \times \mathfrak{R}(\mathcal{P}) & \nearrow {\scriptstyle \Xi} &
\end{array}
$$

where $\Xi(\mathcal{M}, \mathcal{X}) = \mathrm{res}(\mathcal{M} \cup \mathcal{X})$.

Of independent interest, using semidualizing modules, we generalize the famed Auslander transpose. This generalization is similar to but different from the generalizations in [Geng 2013; Huang 1999].

This paper is organized as follows: Section 2 gives general information about resolving subcategories, and Section 3 gives pertinent background regarding semidualizing modules. We prove Theorem A in Section 4. In Section 5, we generalize the Auslander transpose, which we use in Section 6 to classify resolving subcategories which are locally maximal Cohen–Macaulay. In Section 7 we prove a special case of Theorem B. We prove Theorem B in full generality in Section 8 by examining the thick subcategories of maximal Cohen–Macaulay modules containing C. In the last section, these results are applied to the Gorenstein case. Here, Theorem C and several other results are proven.

2. Resolving preliminaries

We proceed with an overview of resolving subcategories. All subcategories considered are full and closed under isomorphisms. For any collection $\mathcal{M} \subseteq \mathrm{mod}(R)$, let $\mathrm{add}(\mathcal{M})$ be the smallest subcategory of $\mathrm{mod}(R)$ containing $\mathcal{M}$ which is closed under direct sums and summands.

Definition 2.1. Given a ring R, a full subcategory $\mathcal{M} \subseteq \mathrm{mod}(R)$ is resolving if the following hold:

(1) R is in $\mathcal{M}$.

(2) $M \oplus N$ is in $\mathcal{M}$ if any only if M and N are in $\mathcal{M}$.

(3) If $0 \to M \to N \to L \to 0$ is exact and $L \in \mathcal{M}$, then $N \in \mathcal{M}$ if and only if $M \in \mathcal{M}$.

By [Yoshino 2005, Lemma 3.2], this is equivalent to saying these conditions hold:

(1) All projectives are in $\mathcal{M}$.

(2) If $M \in \mathcal{M}$, then $\mathrm{add}(M) \subseteq \mathcal{M}$.

(3) $\mathcal{M}$ is closed under extensions.

(4) $\mathcal{M}$ is closed under syzygies.

For a subset $\mathcal{M} \subseteq \mathrm{mod}(R)$, we denote by $\mathrm{res}(\mathcal{M})$ the smallest resolving subcategory containing $\mathcal{M}$. Also, $\mathrm{add}\,\mathcal{M}$ will be the smallest subcategory containing $\mathcal{M}$ which is closed under direct sums and summands. Let $\mathcal{P}$ be the category of finitely generated projective R-modules.

Example 2.2. The following categories are easily seen to be resolving.

(1) $\mathcal{P}$,

(2) $\mathrm{mod}(R)$,

(3) the set of Gorenstein dimension zero modules,

(4) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Ext}^{>n}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$,

(5) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Tor}^{>n}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$,

(6) when R is Cohen–Macaulay, the set of maximal Cohen–Macaulay modules.

A special class of resolving subcategories are thick subcategories.

Definition 2.3. Let $\mathcal{N} \subseteq \mathrm{mod}(R)$. A resolving subcategory $\mathcal{M} \subseteq \mathcal{N}$ is a thick subcategory of $\mathcal{N}$ (or $\mathcal{M}$ is thick in $\mathcal{N}$) if for any exact sequence $0 \to L \to M \to N \to 0$ with $L, M \in \mathcal{M}$, if N is in $\mathcal{N}$, then N is in $\mathcal{M}$ too. A thick subcategory refers to a thick subcategory of $\mathrm{mod}(R)$.

For any $\mathcal{M} \subseteq \mathrm{mod}(R)$, let $\mathrm{Thick}(\mathcal{M})$ be the smallest thick subcategory of $\mathrm{mod}(R)$ containing $\mathcal{M}$.

Example 2.4. The following categories are easily seen to be thick subcategories (moreover, each example is the thick closure of a resolving subcategory from Example 2.2):

(1) the set of modules with finite projective dimension,

(2) $\mathrm{mod}(R)$,

(3) the set of modules with finite Gorenstein dimension,

(4) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Ext}^{\gg 0}(M, B) = 0$ for all $B \in \mathcal{B}\}$,

(5) for any $\mathcal{B} \subseteq \mathrm{Mod}(R)$ and any $n \geq 0$, $\{M \mid \mathrm{Tor}^{\gg 0}(M, B) = 0$ for all $B \in \mathcal{B}\}$.

Resolving subcategories are studied in part because dimension with respect to a resolving subcategory has nice properties. For a subset $\mathcal{M} \subseteq \mathrm{mod}(R)$ and a module $X \in \mathrm{mod}(R)$, we say that $\mathcal{M}\text{-}\dim X = n$ if $n \in \mathbb{N}$ is the smallest number such that there is an exact sequence

$$0 \to M_n \to \cdots \to M_0 \to X \to 0$$

with $M_0, \ldots, M_n \in \mathcal{M}$. Projective dimension and Gorenstein dimension are dimensions with respect to resolving subcategories of projective modules and Gorenstein dimension zero modules respectively. The following proposition (see [Auslander and Buchweitz 1989, Proposition 3.3]) causes nice properties to hold for dimension with respect to a resolving subcategory.

Proposition 2.5. *If $\mathcal{M}$ is resolving and $\mathcal{M}\text{-}\dim(X) \leq n$, then for any exact sequence*

$$0 \to L \to M_{n-1} \to \cdots \to M_0 \to X \to 0$$

with each $M_i \in \mathcal{M}$, L is in $\mathcal{M}$.

This proposition allows us to prove the following results.

Corollary 2.6. *If $\mathcal{M}$ is resolving, then $\mathcal{M}\text{-}\dim(X) = \inf\{n \mid \Omega^n X \in \mathcal{M}\}$.*

Proof. If $\Omega^n X \in \mathcal{M}$, then we have

$$0 \to \Omega^n X \to F_{n-1} \to \cdots \to F_0 \to X \to 0$$

with each F_i projective. This shows that $\mathcal{M}\text{-}\dim X \leq n$. If $\mathcal{M}\text{-}\dim X \leq n$, the same sequence and Corollary 2.6 show that $\Omega^n X$ is in $\mathcal{M}$. $\square$

Lemma 2.7. *If $\mathcal{M}$ is resolving, then $\mathcal{M}\text{-}\dim X \oplus Y = \max\{\mathcal{M}\text{-}\dim X, \mathcal{M}\text{-}\dim Y\}$.*

Proof. We have $\Omega^n(X \oplus Y) = \Omega^n X \oplus \Omega^n Y$ for a suitable choice of syzygies. Since $\Omega^n(X \oplus Y)$ is in $\mathcal{M}$ if and only if $\Omega^n X$ and $\Omega^n Y$ are in $\mathcal{M}$, the result follows from Corollary 2.6. Parts (1) and (2) are essentially proved in [Masek 1999, Theorem 18]. $\square$

Lemma 2.8. *If $\mathcal{M}$ is a resolving subcategory, and $0 \to X \to Y \to Z \to 0$ is exact, then the following inequalities hold.*
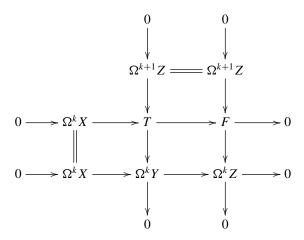
(1) $\mathcal{M}\text{-dim } X \leq \max\{\mathcal{M}\text{-dim } Y, \mathcal{M}\text{-dim } Z - 1\}$,

(2) $\mathcal{M}\text{-dim } Y \leq \max\{\mathcal{M}\text{-dim } X, \mathcal{M}\text{-dim } Z\}$,

(3) $\mathcal{M}\text{-dim } Z \leq \max\{\mathcal{M}\text{-dim } X, \mathcal{M}\text{-dim } Y\} + 1$.

Proof. For suitable choices of syzygies, we have the following.

$$0 \to \Omega^k X \to \Omega^k Y \to \Omega^k Z \to 0$$

If $k = \max\{\mathcal{M}\text{-dim } X, \mathcal{M}\text{-dim } Z\}$, then, by Corollary 2.6, $\Omega^k X$ and $\Omega^k Z$ are in $\mathcal{M}$, and thus, so is $\Omega^k Y$, giving us (2). If $k = \max\{\mathcal{M}\text{-dim } X, \mathcal{M}\text{-dim } Y\}$, then, again by Corollary 2.6, $\Omega^k X$ and $\Omega^k Y$ is in $\mathcal{M}$. Therefore $\mathcal{M}\text{-dim } \Omega^k Z \leq 1$, and so $\Omega^{k+1} Z$ is in $\mathcal{M}$. Thus by Corollary 2.6, $\mathcal{M}\text{-dim } Z \leq k + 1$, proving (3).

Now take $k = \max\{\mathcal{M}\text{-dim } Y, \mathcal{M}\text{-dim } Z - 1\}$. Then $\Omega^k Y$ and $\Omega^{k+1} Z$ are in $\mathcal{M}$. We take the pushout diagram

with F free and hence in $\mathcal{M}$. Since, by Corollary 2.6, $\Omega^{k+1} Z$ and $\Omega^k Y$ are in $\mathcal{M}$, so is T. Since $F \in \mathcal{M}$, $\Omega^k X$ has to also be in $\mathcal{M}$. Hence $\mathcal{M}\text{-dim } X \leq k$, and we have (1). $\square$

For a subset $\mathcal{M} \subseteq \mathrm{mod}(R)$, let $\Delta(\mathcal{M})$ denote the category of modules X such that $\mathcal{M}\text{-dim } X$ is finite. If $\mathcal{M}$ is resolving, then by Corollary 2.6, $\Delta(\mathcal{M}) = \{X \in \mathrm{mod}(R) \mid \Omega^{\gg 0} X \in \mathcal{M}\}$. The next result easily follows from the previous lemma.

Corollary 2.9. *Let $\mathcal{M}$ be resolving. For any n, the set $\{X \in \mathrm{mod}(R) \mid \mathcal{M}\text{-dim } X \leq n\}$ is resolving. Furthermore, $\Delta(\mathcal{M})$ is thick, and $\mathrm{Thick}(\mathcal{M}) = \Delta(\mathcal{M})$.*

Through these results, we may construct many resolving and thick subcategories. It is easy to show that the intersection of a collection of resolving subcategories and the intersection of a collection of thick subcategories are resolving and thick

respectively. The following lemma allows us to construct even more resolving subcategories. For $\mathcal{M} \subseteq \operatorname{mod}(R)$, we say $\mathcal{M}_p = \{M_p \mid M \in \mathcal{M}\}$.

Lemma 2.10. *Let R and S be rings and $F : \operatorname{mod}(R) \to \operatorname{mod}(S)$ be an exact functor with $F(R) = S$. Then for any resolving subcategory $\mathcal{M} \subseteq \operatorname{mod}(S)$, $F^{-1}(\mathcal{M})$ is a resolving subcategory of $\operatorname{mod}(R)$.*

The proof is elementary and is left to the reader. Applying this lemma to the localization functor, for any $V \subseteq \operatorname{spec} R$, the category of all $M \in \operatorname{mod}(R)$ with M_p free for all $p \in V$ is also resolving. The following lemmas give insight into the behavior of resolving categories under localization. The first lemma is from [Takahashi 2010, Lemma 4.8; Dao and Takahashi 2014, Lemma 3.2(1)], and the second is from [Dao and Takahashi 2015, Proposition 3.3].

Lemma 2.11. *If $\mathcal{M}$ is a resolving subcategory, then so is $\operatorname{add} \mathcal{M}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{spec} R$.*

Lemma 2.12. *The following are equivalent for a resolving subcategory $\mathcal{M}$ and a module $M \in \operatorname{mod}(R)$:*

(1) *$M \in \mathcal{M}$,*

(2) *$M_p \in \operatorname{add} \mathcal{M}_p$ for all $p \in \operatorname{spec} R$,*

(3) *$M_{\mathfrak{m}} \in \operatorname{add} \mathcal{M}_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$.*

Recall the definition of Λ and Γ from the introduction. These lemmas show that if $\mathcal{M}$ is resolving, then for all $f \in \Gamma$, $\Lambda_{\mathcal{M}}(f)$ is a resolving subcategory.

Corollary 2.13. *Set*

$$\Lambda_{\mathcal{M}}(f) = \{M \in \operatorname{mod}(R) \mid \operatorname{add} \mathcal{M}_p\text{-dim } M_p \leq f(p) \ \text{for all} \ p \in \operatorname{spec} R\}.$$

If $\mathcal{M}$ is resolving, then for all $f \in \Gamma$, $\Lambda_{\mathcal{M}}(f)$ is a resolving subcategory.

Let MCM denote the category of maximal Cohen–Macaulay modules. As noted earlier, when R is Cohen–Macaulay, MCM is resolving. Furthermore, letting $d = \dim R$, $\Omega^d M$ is in MCM for every $M \in \operatorname{mod}(R)$. Hence, $\Delta(\text{MCM}) = \operatorname{mod}(R)$. The following shows that dimension with respect to MCM is very computable.

Lemma 2.14. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ are resolving subcategories. Then $\mathcal{M}$ is thick in $\mathcal{N}$ if and only if for every module $X \in \Delta(\mathcal{M})$, we have $\mathcal{M}\text{-dim } X = \mathcal{N}\text{-dim } X$. Furthermore, if R is Cohen–Macaulay, $\mathcal{M}$ is a thick subcategory of MCM if and only if dimension with respect to $\mathcal{M}$ satisfies the Auslander–Buchsbaum formula, i.e., for all $X \in \Delta(\mathcal{M})$ we have*

$$\mathcal{M}\text{-dim } X + \operatorname{depth} X = \operatorname{depth} R.$$

Proof. Suppose M is thick in N and $X \in \Delta(\mathcal{M})$. Then we may write $0 \to M_d \to \cdots \to M_0 \to X \to 0$ with $M_i \in \mathcal{M}$ and $d = \mathcal{M}\text{-dim } X$. Since each M_i is also in $\mathcal{N}$,

we have $\mathcal{N}$-dim $X \leq d$. Setting $e = \mathcal{N}$-dim $X \leq d$, by Corollary 2.6, there exists a $N \in \mathcal{N}$ such that

$$0 \to M_d \to \cdots \to M_e \to N \to 0 \quad \text{and} \quad 0 \to N \to M_{e-1} \to \cdots \to M_0 \to X \to 0$$

are exact. However, since $\mathcal{M}$ is thick in $\mathcal{N}$, N is also in $\mathcal{M}$, which implies that $e = d$, proving the only if part of the statement.

Now suppose that $\mathcal{M}$-dim $X = \mathcal{N}$-dim X for all $X \in \Delta(\mathcal{M})$. Now suppose $0 \to L \to M \to N \to 0$ is exact with $L, M \in \mathcal{M}$ and $N \in \mathcal{N}$. Then $N \in \Delta(\mathcal{M})$ and $\mathcal{M}$-dim $N = \mathcal{N}$-dim $N = 0$. Therefore $N \in \mathcal{M}$, and so $\mathcal{M}$ is thick in $\mathcal{N}$.

Assume R is Cohen–Macaulay. Let $\mathcal{M}$ be a resolving subcategory whose dimension satisfies the Auslander–Buchsbaum formula. Then for any module $M \in \Delta(\mathcal{M}) \cap \text{MCM}$, we have

$$\text{depth } R = \mathcal{M}\text{-dim } M + \text{depth } M = \mathcal{M}\text{-dim } M + \text{depth } R.$$

Thus $\mathcal{M}$-dim $M = 0$ forcing M to be in $\mathcal{M}$. Hence $\mathcal{M}$ is contained in MCM.

By what we have proved so far, it suffices to show that dimension with respect to MCM satisfies the Auslander–Buchsbaum formula. But this follows from Corollary 2.6. □

Recall the definition of Φ and Γ from the introduction. If dimension with respect to add $\mathcal{M}_p$ satisfies the Auslander–Buchsbaum formula for all $p \in \text{spec } R$, then for all $\mathcal{X} \subseteq \Delta(\mathcal{M})$, $\Phi_{\mathcal{M}}(\mathcal{X})$ is in Γ. Before proceeding, we need one more definition and a result.

Definition 2.15. Let $\mathcal{A} \subseteq \mathcal{M}$. We say $\mathcal{A}$ cogenerates $\mathcal{M}$, if for every $M \in \mathcal{M}$, there exists an exact sequence $0 \to M \to A \to M' \to 0$ with $M' \in \mathcal{M}$ and $A \in \mathcal{A}$.

The following is an important theorem from [Auslander and Buchweitz 1989, Theorem 1.1].

Theorem 2.16. *Suppose $\mathcal{A}$ and $\mathcal{M}$ are resolving with $\mathcal{A} \subseteq \mathcal{M}$. If $\mathcal{A}$ cogenerates $\mathcal{M}$, then for every $X \in \Delta(\mathcal{M})$ with $\mathcal{M}$-dim $X = n$, there exists an $A \in \Delta(\mathcal{A})$ and $M \in \mathcal{M}$ such that $\mathcal{A}$-dim $A = n$ and $0 \to X \to A \to M \to 0$ is exact.*

3. Preliminaries: semidualizing modules

We fix a module $C \in \text{mod}(R)$ and write $M^\dagger = \text{Hom}(M, C)$.

Definition 3.1. A finitely generated module X is totally C-reflexive if it satisfies the following:

(1) $\text{Ext}^{>0}(X, C) = 0$,

(2) $\text{Ext}^{>0}(X^\dagger, C) = 0$,

(3) The natural homothety map $\eta_X : X \to X^{\dagger\dagger}$ defined by $\mu \mapsto (\varphi \mapsto \varphi(\mu))$ is an isomorphism.

Let $\mathcal{G}_C$ denote the category of totally C-reflexive modules.

The set $\mathcal{G}_C$ is essentially the subcategory over which † is a dualizing functor. The notion of totally C-reflexivity generalizes Gorenstein dimension zero. In fact, when $C = R$, $\mathcal{G}_R$ is simply the category of Gorenstein dimension zero modules, which are also known as totally reflexive modules. See [Masek 1999] for further information on the subject. The following proposition shows us that $\mathcal{G}_C$ is almost resolving.

Lemma 3.2. *The set $\mathcal{G}_C$ is closed under direct sums, summands, and extensions.*

Proof. It is easy to show that $\mathcal{G}_C$ is closed under direct sums and direct summands. Suppose we have

$$0 \to X \to Y \to Z \to 0$$

with $X, Z \in \mathcal{G}_C$. It is easy to check that Y satisfies condition (1) of Definition 3.1. We have

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0 \quad \text{and} \quad 0 \to X^{\dagger\dagger} \to Y^{\dagger\dagger} \to Z^{\dagger\dagger} \to 0.$$

From the first exact sequence, it is easy to see that Y satisfies condition (2) of Definition 3.1. We can then use the five lemma to show that Y satisfies condition (3) of Definition 3.1. $\qquad\square$

In general, $\mathcal{G}_C$ is not resolving. For example, if $C = R/xR$ for a regular element $x \in R$, $\mathrm{Ext}^1(R/xR, R/xR) = R/xR \neq 0$. So R cannot be in $\mathcal{G}_{R/xR}$, and thus $\mathcal{G}_{R/xR}$ cannot be resolving. It is clear from the definition that $R \in \mathcal{G}_C$ is a necessary condition for $\mathcal{G}_C$ to be resolving. In fact, this condition is sufficient.

Proposition 3.3. *The subcategory $\mathcal{G}_C$ is resolving if and only if $\mathcal{G}_C$ contains R.*

Proof. If $\mathcal{G}_C$ is resolving, by definition it contains R, so we prove the converse. So suppose R is in $\mathcal{G}_C$. In light of the last lemma, we need only to prove that if $0 \to X \to Y \to Z \to 0$ is exact with $Y, Z \in \mathcal{G}_C$, then X is in $\mathcal{G}_C$ as well. Since Y and Z satisfy condition (1) of Definition 3.1, it is easy to show that X does too. Also, since $\mathrm{Ext}^1(Z, C) = 0$, we have

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to 0.$$

Hence, we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_Z} & & \\
0 & \longrightarrow & X^{\dagger\dagger} & \longrightarrow & Y^{\dagger\dagger} & \longrightarrow & Z^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}^1(X^\dagger, C) \longrightarrow 0
\end{array}
$$

Since η_Y and η_Z are isomorphisms, the five lemma shows that η_X is too, and that $\mathrm{Ext}^1(X^\dagger, C) = 0$. Thus X satisfies condition (3) of Definition 3.1. It is easy to check using the first exact sequence that $\mathrm{Ext}^{>1}(X^\dagger, C) = 0$, showing that X satisfies condition (2) of Definition 3.1. $\square$

Motivated by this proposition, we say that a module, C, is semidualizing if R is in $\mathcal{G}_C$. This is easily seen to be equivalent to the following definition which is standard in the literature.

Definition 3.4. A module C is called *semidualizing* if both $\mathrm{Ext}^{>0}(C, C) = 0$ and $R \cong \mathrm{Hom}(C, C)$ via the map $r \mapsto (c \mapsto rc)$.

For the remainder of the paper, we let C denote a semidualizing module. Semidualizing modules were first discovered by Foxby [1972] and were later rediscovered in different guises by various authors, including Vasconceles [1974], who called them spherical modules, and Golod, who called them suitable modules. For an excellent treatment of the general theory of semidualizing modules, see [Sather-Wagstaff 2009]. Examples of semidualizing modules include R and dualizing modules. If R is Cohen–Macaulay and D is a dualizing module, then $\mathcal{G}_D$ is simply MCM. Dimension with respect to $\mathcal{G}_C$ is often called Gorenstein C-dimension, or G_C-dimension for short, since it is a generalization of Gorenstein dimension. We would expect G_C and Gorenstein dimension to have similar properties. Thus we have the following lemma, which is an easy exercise, and proposition, which is from [Gerko 2001, Theorem 1.22].

Lemma 3.5. *If $X \in \Delta(\mathcal{G}_C)$, then $\mathcal{G}_C$-dim $X = \min\{n \mid \mathrm{Ext}^{>n}(X, C) = 0\}$.*

Proposition 3.6. *For any semidualizing module C, G_C-dimension satisfies the Auslander–Buchsbaum formula, i.e., for any module $X \in \Delta(\mathcal{G}_C)$, we have*

$$\mathcal{G}_C\text{-dim } X + \mathrm{depth } X = \mathrm{depth } R.$$

In light of Lemma 2.14, when R is Cohen–Macaulay this means that $\mathcal{G}_C$ is a thick subcategory of MCM. Interest in understanding G_C-dimension and the structure of $\mathcal{G}_C$ is not new. The following conjecture by Gerko [2001, Conjecture 1.23] is equivalent to saying that $\mathcal{G}_R$ is a thick subcategory of $\mathcal{G}_C$.

Conjecture 3.7. *If C is semidualizing, then for any module X, $\mathcal{G}_C$-dim $X \leq \mathcal{G}_R$-dim X, and equality holds when both are finite.*

We give one more construction in this section. Take any $X \in \mathcal{G}_C$. Then we have $0 \to \Omega X^\dagger \to R^n \to X^\dagger \to 0$ is exact. Since $R^\dagger \cong C$, applying † yields the exact sequence

$$0 \to X \to C^n \to (\Omega X^\dagger)^\dagger \to 0.$$

Hence $\mathcal{G}_C$ is cogenerated by add C. Furthermore, if $F_\bullet$ is a projective resolution of $X^\dagger$ with $X \in \mathcal{G}_C$, then $F_\bullet^\dagger$ is an add C coresolution of X. Splicing this together with a free resolution $G_\bullet$ of X, we get what is called a complete PP_C or a complete P_C-resolution of X. See [White 2010; Sather-Wagstaff 2009] for more on the matter.

Before proceeding, we summarize the notations of this paper.

(1) R is a commutative noetherian ring.

(2) $\mathcal{P}$ is the subcategory of projective R-modules.

(3) Γ is the set of grade consistent functions.

(4) $\mathcal{M}$-dim X is the dimension of X with respect to the category $\mathcal{M} \subseteq \mathrm{mod}(R)$.

(5) add $\mathcal{M}$ is the smallest category closed under direct sums and summands containing $\mathcal{M} \subseteq \mathrm{mod}(R)$.

(6) $\Lambda_\mathcal{M}(f) = \{X \in \mathrm{mod}(R) \mid \text{add } \mathcal{M}_p\text{-dim } X_p \le f(p) \text{ for all } p \in \mathrm{spec}\, R\}$ with $f \in \Gamma$.

(7) $\Phi_\mathcal{M}(\mathcal{X})(p) = \sup\{\text{add } \mathcal{M}_p\text{-dim } X_p \mid X \in \mathcal{X}\}$ with $\mathcal{M}, \mathcal{X} \subseteq \mathrm{mod}(R)$ subcategories.

(8) $\Delta(\mathcal{M}) = \{X \in \mathrm{mod}(R) \mid \mathcal{M}\text{-dim } X < \infty\}$ with $\mathcal{M} \subseteq \mathrm{mod}(R)$ a category.

(9) $\mathfrak{R}(\mathcal{M}) = \{\mathcal{X} \subseteq \mathrm{mod}(R) \mid \mathcal{M} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{M}) \mathcal{X} \text{ is resolving}\}$.

(10) $\mathfrak{R}$ the collection of resolving subcategories.

(11) $\mathrm{Thick}_\mathcal{N}(\mathcal{M})$ the smallest thick subcategory of $\mathcal{N}$ containing $\mathcal{M}$ with $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathrm{mod}(R)$ subcategories.

(12) C is a semidualizing module .

(13) $\mathcal{G}_C$ the collection of totally C-reflexive modules.

(14) $X^\dagger = \mathrm{Hom}(X, C)$.

(15) For a resolving subcategory $\mathcal{A}$ and a module $M \in \mathrm{mod}(R)$, set $\mathrm{res}_\mathcal{A} M = \mathrm{res}(\mathcal{A} \cup \{M\})$.

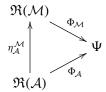## 4. Comparing resolving subcategories

For the entirety of this section, let $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{N}$ be resolving subcategories. Recall that $\mathfrak{R}(\mathcal{A})$ is the collection of resolving subcategories $\mathcal{X}$ such that $\mathcal{A} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{A})$. In this section, we compare $\mathfrak{R}(\mathcal{A})$ and $\mathfrak{R}(\mathcal{M})$ when $\mathcal{A}$ is contained in $\mathcal{M}$. If $\mathcal{A} \subseteq \mathcal{M}$, we may define $\eta_\mathcal{A}^\mathcal{M} : \mathfrak{R}(\mathcal{A}) \to \mathfrak{R}(\mathcal{M})$ by $\mathcal{X} \mapsto \mathrm{res}(\mathcal{X} \cup \mathcal{M})$ and $\rho_\mathcal{A}^\mathcal{M} : \mathfrak{R}(\mathcal{M}) \to \mathfrak{R}(\mathcal{A})$ by $\mathcal{X} \mapsto \mathcal{X} \cap \Delta(\mathcal{A})$. Note that if $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$, then $\eta_\mathcal{A}^\mathcal{N} = \eta_\mathcal{M}^\mathcal{N} \eta_\mathcal{A}^\mathcal{M}$ and $\rho_\mathcal{A}^\mathcal{N} = \rho_\mathcal{A}^\mathcal{M} \rho_\mathcal{M}^\mathcal{N}$.

Proposition 4.1. *If $\mathcal{A}$ cogenerates $\mathcal{M}$, then the map $\rho_\mathcal{A}^\mathcal{M}$ is injective.*

Proof. Suppose that $\rho_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \rho_{\mathcal{A}}^{\mathcal{M}}(\mathcal{Y})$ for $\mathcal{X}, \mathcal{Y} \in \mathfrak{R}(\mathcal{M})$, i.e., $\mathcal{X} \cap \Delta(\mathcal{A}) = \mathcal{Y} \cap \Delta(\mathcal{A})$. Take any $X \in \mathcal{X}$. Since $X \in \Delta(\mathcal{M})$ and $\mathcal{A}$ cogenerates $\mathcal{M}$, by Theorem 2.16, there exists $A \in \Delta(\mathcal{A})$ and $M \in \mathcal{M}$ such that $0 \to X \to A \to M \to 0$ is exact. Since $M \in \mathcal{M} \subseteq \mathcal{X}$ and $X \in \mathcal{X}$, we know that A is also in $\mathcal{X}$. But then A is in $\mathcal{X} \cap \Delta(\mathcal{A}) = \mathcal{Y} \cap \Delta(\mathcal{A})$ and thus also in $\mathcal{Y}$. Since $M \in \mathcal{M} \subseteq \mathcal{Y}$, we know that X must also be in $\mathcal{Y}$. Hence $\mathcal{X} \subseteq \mathcal{Y}$, and, by symmetry, we have equality. Therefore, $\rho_{\mathcal{A}}^{\mathcal{M}}$ is injective. $\square$

In certain circumstances, this map is a bijection. The following is Theorem A from the introduction.

Theorem 4.2. *Let* Ψ *be a set of increasing functions from* spec R *to* $\mathbb{N}$. *Suppose,* $\mathcal{A} \subseteq \mathcal{M}$ *such that* $\mathcal{A}$ *cogenerates* $\mathcal{M}$ *and* add $\mathcal{A}_p$ *is thick in* add $\mathcal{M}_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in$ spec R. *If* $\Phi_{\mathcal{A}}$ *and* $\Lambda_{\mathcal{A}}$ *are inverse functions giving a bijection between* $\mathfrak{R}(\mathcal{A})$ *and* Ψ, *then the following diagram commutes*:

$$
\begin{array}{ccc}
\mathfrak{R}(\mathcal{M}) & & \\
\uparrow & \searrow^{\Phi_{\mathcal{M}}} & \\
\eta_{\mathcal{A}}^{\mathcal{M}} \Big\uparrow & & \Psi \\
\Big\uparrow & \nearrow_{\Phi_{\mathcal{A}}} & \\
\mathfrak{R}(\mathcal{A}) & &
\end{array}
$$

Furthermore, $\Lambda_{\mathcal{M}}$ *and* $\rho_{\mathcal{A}}^{\mathcal{M}}$ *are the respective inverses of* $\Phi_{\mathcal{M}}$ *and* $\eta_{\mathcal{A}}^{\mathcal{M}}$.

The proof of this result will be given after this brief lemma.

Lemma 4.3. *If* $\mathcal{X}$ *and* $\mathcal{Y}$ *are subcategories and* $\mathcal{M}$ *is resolving*, *then*

$$\Phi_{\mathcal{M}}(\mathrm{res}(\mathcal{X} \cup \mathcal{Y})) = \Phi_{\mathcal{M}}(\mathcal{X}) \vee \Phi_{\mathcal{M}}(\mathcal{Y}).$$

Proof. Since every element in $\mathrm{res}(\mathcal{X} \cup \mathcal{Y})$ is obtained by taking extensions, syzygies, and direct summands a finite number of times, and since these operations never increase the $\mathcal{M}$ dimension, we have $\Phi_{\mathcal{M}}(\mathrm{res}(\mathcal{X} \cup \mathcal{Y})) \leq \Phi_{\mathcal{M}}(\mathcal{X}) \vee \Phi_{\mathcal{M}}(\mathcal{Y})$. However, since $\mathcal{X}, \mathcal{Y} \subseteq \mathrm{res}(\mathcal{X} \cup \mathcal{Y})$, we actually have equality. $\square$

Proof of Theorem 4.2. First, we show that $\rho_{\mathcal{A}}^{\mathcal{M}}$ and $\eta_{\mathcal{A}}^{\mathcal{M}}$ are inverse functions and are thus both bijections. Proposition 4.1 shows that $\rho_{\mathcal{A}}^{\mathcal{M}}$ is injective. Fix $\mathcal{X} \in \mathfrak{R}(\mathcal{A})$ and let $\mathcal{Z} = \rho_{\mathcal{A}}^{\mathcal{M}} \eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}) = \mathrm{res}(\mathcal{X} \cup \mathcal{M}) \cap \Delta(\mathcal{A})$. It suffices to show that $\mathcal{Z} = \mathcal{X}$. Setting $f = \Phi_{\mathcal{A}}(\mathcal{X})$, this is equivalent to showing that $\Phi_{\mathcal{A}}(\mathcal{Z}) = f$, since $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverse functions. Since $\mathcal{X} \subseteq \mathcal{Z}$, we know that $\Phi_{\mathcal{A}}(\mathcal{Z}) \geq f$. From Lemma 4.3,

$$\Phi_{\mathcal{M}}(\mathrm{res}(\mathcal{X} \cup \mathcal{M})) = \Phi_{\mathcal{M}}(\mathcal{X}) \vee \Phi_{\mathcal{M}}(\mathcal{M}) = \Phi_{\mathcal{M}}(\mathcal{X}).$$

Furthermore, since add $\mathcal{A}_p$ is thick in add $\mathcal{M}_{\mathfrak{p}}$ for all $p \in$ spec R, Lemma 2.14 implies that add $\mathcal{A}_p$-dim A and add $\mathcal{M}_p$-dim A are the same for all $p \in$ spec R and

$A \in \Delta(\mathcal{A})$. Hence $\Phi_{\mathcal{A}}(\mathcal{X}) = \Phi_{\mathcal{M}}(\mathcal{X})$ and $\Phi_{\mathcal{A}}(\mathcal{Z}) = \Phi_{\mathcal{M}}(\mathcal{Z})$. Therefore,

$$f \leq \Phi_{\mathcal{A}}(\mathcal{Z}) = \Phi_{\mathcal{M}}(\mathcal{Z}) \leq \Phi_{\mathcal{M}}(\mathrm{res}(\mathcal{X} \cup \mathcal{M})) = \Phi_{\mathcal{M}}(\mathcal{X}) = \Phi_{\mathcal{A}}(\mathcal{X}) = f$$

and so, $\Phi_{\mathcal{A}}(\mathcal{Z}) = f$. Hence, $\rho_{\mathcal{A}}^{\mathcal{M}}$ and $\eta_{\mathcal{A}}^{\mathcal{M}}$ are inverse functions. Also, this argument shows that $\Phi_{\mathcal{A}}(\mathcal{X}) = \Phi_{\mathcal{M}}(\mathrm{res}(\mathcal{X} \cup \mathcal{M})) = \Phi_{\mathcal{M}}(\eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{X}))$, showing that the diagram commutes and that $\Phi_{\mathcal{M}}$ is also a bijection.

It remains to show that $\Lambda_{\mathcal{M}} = \Phi_{\mathcal{M}}^{-1}$. For any $f \in \Psi$, $\eta_{\mathcal{A}}^{\mathcal{M}}(\Lambda_{\mathcal{A}}(f))$ is contained in $\Lambda_{\mathcal{M}}(f)$. Because $\Phi_{\mathcal{M}}$ is an increasing function and both $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverse functions, we have

$$f = \Phi_{\mathcal{A}}\Lambda_{\mathcal{A}}(f) = \Phi_{\mathcal{M}}(\eta_{\mathcal{A}}^{\mathcal{M}}(\Lambda_{\mathcal{A}}(f))) \leq \Phi_{\mathcal{M}}\Lambda_{\mathcal{M}}(f) \leq f.$$

Thus we have $\Phi_{\mathcal{M}}\Lambda_{\mathcal{M}}(f) = f$, and we are done. $\qquad \square$

For a resolving subcategory $\mathcal{A}$, let $\mathfrak{S}(\mathcal{A})$ be the collection of resolving subcategories $\mathcal{M}$ such that $\mathcal{M}$ and $\mathcal{A}$ satisfy the hypotheses of Theorem 4.2, i.e., $\mathcal{A}$ cogenerates $\mathcal{M}$ and add $\mathcal{A}_{\mathfrak{p}}$ is thick in add $\mathcal{M}_{\mathfrak{p}}$ for all $p \in \mathrm{spec}\, R$. The following theorem shows that we can patch together the bijections in Theorem 4.2.

Theorem 4.4. *Let $\mathcal{A}$ be a resolving subcategory and Ψ be a set of increasing functions from $\mathrm{spec}\, R$ to $\mathbb{N}$. If $\Phi_{\mathcal{A}}$ and $\Lambda_{\mathcal{A}}$ are inverse functions giving a bijection between $\mathfrak{R}(\mathcal{A})$ and Ψ, then*

$$\Lambda : \mathfrak{S}(\mathcal{A}) \times \Psi \longrightarrow \bigcup_{\mathcal{M} \in \mathfrak{S}(\mathcal{A})} \mathfrak{R}(\mathcal{M}) \subseteq \mathfrak{R}.$$

is a bijection. Furthermore, for any $\mathcal{M}, \mathcal{N} \in \mathfrak{S}(\mathcal{A})$ with $\mathcal{M} \subseteq \mathcal{N}$, the map $\rho_{\mathcal{M}}^{\mathcal{N}}$ is the inverse of $\eta_{\mathcal{M}}^{\mathcal{N}}$, and the following diagram commutes:

(1)

$$
\begin{array}{c}
\mathfrak{R}(\mathcal{N}) \\
\eta_{\mathcal{M}}^{\mathcal{N}} \uparrow \quad \searrow^{\Phi_{\mathcal{N}}} \\
\mathfrak{R}(\mathcal{M}) \xrightarrow{\;\Phi_{\mathcal{M}}\;} \Psi \\
\eta_{\mathcal{A}}^{\mathcal{M}} \uparrow \quad \nearrow_{\Phi_{\mathcal{A}}} \\
\mathfrak{R}(\mathcal{A})
\end{array}
$$

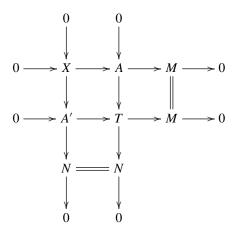Before we proceed with the proof of Theorem 4.4, we need a lemma.

Lemma 4.5. *The set $\mathfrak{S}(\mathcal{A})$ is closed under intersections.*

Proof. Let $\mathcal{M}, \mathcal{N} \in \mathfrak{S}(\mathcal{A})$. Take any $p \in \mathrm{spec}\, R$. Suppose $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence of R_p-modules with $A_1, A_2 \in \mathrm{add}\, \mathcal{A}_p$ and $A_3 \in \mathrm{add}(\mathcal{M} \cap \mathcal{N})_p$. Then A_3 is in add $\mathcal{M}_p$. Therefore, since add $\mathcal{A}_{\mathfrak{p}}$ is thick in add $\mathcal{M}_p$ by assumption, A_3 is in add $\mathcal{A}_{\mathfrak{p}}$. Since add $\mathcal{A}_{\mathfrak{p}}$ is resolving and contained in add$(\mathcal{M} \cap \mathcal{N})_p$, add $\mathcal{A}_{\mathfrak{p}}$ is thick in add$(\mathcal{M} \cap \mathcal{N})_p$.

It remains to show that $\mathcal{A}$ cogenerates $\mathcal{M} \cap \mathcal{N}$. Take $X \in \mathcal{M} \cap \mathcal{N}$. We have

$$0 \to X \to A \to M \to 0 \quad \text{and} \quad 0 \to X \to A' \to N \to 0,$$

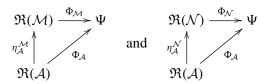with $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $A, A' \in \mathcal{A}$. Consider the following pushout diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A' & \longrightarrow & T & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

It is easy to see $T \in \mathcal{M} \cap \mathcal{N}$. We also have the exact sequence

$$0 \to X \to A \oplus A' \to T \to 0.$$

Since $A \oplus A' \in \mathcal{A}$, this completes the proof. $\square$

Proof of Theorem 4.4. Suppose $\mathcal{M}, \mathcal{N} \in \mathfrak{S}$ with $\mathcal{M} \subseteq \mathcal{N}$. From Theorem 4.2, the following diagrams commute:

$$
\begin{array}{ccc}
\mathfrak{R}(\mathcal{M}) & \xrightarrow{\Phi_{\mathcal{M}}} & \Psi \\
\eta^{\mathcal{M}}_{\mathcal{A}} \uparrow & \nearrow \Phi_{\mathcal{A}} & \\
\mathfrak{R}(\mathcal{A}) & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathfrak{R}(\mathcal{N}) & \xrightarrow{\Phi_{\mathcal{N}}} & \Psi \\
\eta^{\mathcal{N}}_{\mathcal{A}} \uparrow & \nearrow \Phi_{\mathcal{A}} & \\
\mathfrak{R}(\mathcal{A}) & &
\end{array}
$$

From here, it is easy to show that diagram (1) commutes and $\Phi_{\mathcal{N}}$ and $\eta^{\mathcal{N}}_{\mathcal{M}}$ are bijections with $(\eta^{\mathcal{N}}_{\mathcal{M}})^{-1} = \rho^{\mathcal{N}}_{\mathcal{M}}$.

Also, Theorem 4.2 shows that $\mathrm{Im}(\Lambda) = \bigcup_{\mathcal{M} \in \mathfrak{S}} \mathfrak{R}(\mathcal{M})$. It remains to show that Λ is injective. Suppose $\mathcal{X} = \Lambda_{\mathcal{M}}(f) = \Lambda_{\mathcal{N}}(g)$. Then $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$; hence, $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{X}$. For any $X \in \mathcal{X}$ and any n greater than $\mathcal{M}$-dim X and $\mathcal{N}$-dim X, $\Omega^n X$ is in $\mathcal{M} \cap \mathcal{N}$ by Corollary 2.6. Therefore, $\mathcal{X}$ is contained in $\Delta(\mathcal{M} \cap \mathcal{N})$ and thus $\mathcal{X} \in \mathfrak{R}(\mathcal{M} \cap \mathcal{N})$. By the previous lemma, $\mathcal{M} \cap \mathcal{N}$ is in $\mathfrak{S}(\mathcal{A})$, so $\Lambda_{\mathcal{M} \cap \mathcal{N}} : \Psi \to \mathfrak{R}(\mathcal{M} \cap \mathcal{N})$ is a bijection, by Theorem 4.2. So there exists an $h \in \Psi$ such that $\Lambda_{\mathcal{M} \cap \mathcal{N}}(h) = \mathcal{Z} = \Lambda_{\mathcal{M}}(f) = \Lambda_{\mathcal{N}}(g)$. Therefore, we may assume that $\mathcal{M}$ is contained in $\mathcal{N}$.
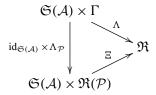
Since $\mathcal{X} \in \mathfrak{R}(M)$ and $\mathcal{X} \in \mathfrak{R}(\mathcal{N})$, we have $\mathcal{N} \subseteq \mathcal{X} \subseteq \Delta(\mathcal{M})$. Thus, because $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$,

$$\mathcal{N} = \mathcal{N} \cap \Delta(\mathcal{M}) = \rho_{\mathcal{M}}^{\mathcal{N}}(\mathcal{N}) = \eta_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{M}}^{\mathcal{N}}(\mathcal{N}) = \eta_{\mathcal{A}}^{\mathcal{M}} \rho_{\mathcal{A}}^{\mathcal{N}}(\mathcal{N}) = \eta_{\mathcal{A}}^{\mathcal{M}}(\mathcal{A}) = \mathcal{M}.$$

Since $\Lambda_{\mathcal{M}}$ is injective, we then also have $f = g$.   □

As mentioned earlier, it is shown in [Dao and Takahashi 2015] that we have $\Lambda_{\mathcal{P}}$ is a bijection from Γ to $\mathfrak{R}(\mathcal{P})$. In Sections 8 and 9 we apply Theorem 4.4 when $\mathcal{A} = \mathcal{P}$, and show that $\mathfrak{S}(\mathcal{P})$ contains the collection of thick subcategories of $\mathcal{G}_R$. The following results gives an alternative way of viewing Theorem 4.4.

Proposition 4.6. *In the situation of Theorem 4.4, if $\Psi = \Gamma$ and $\mathcal{P}$ is thick in $\mathcal{M}$, then the following diagram commutes*:

where $\Xi(\mathcal{M}, \mathcal{X}) = \mathrm{res}(\mathcal{M} \cup \mathcal{X})$. Furthermore, $\mathrm{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda_{\mathcal{P}}$ is bijective and Ξ is injective.

Proof. Since $\Lambda_{\mathcal{P}}$ is bijective, $\mathrm{id}_{\mathfrak{S}(\mathcal{A})} \times \Lambda_{\mathcal{P}}$ is too. It suffices to show that for any $(\mathcal{M}, f) \in \mathfrak{S}(\mathcal{A}) \times \Gamma$ we have $\Xi(\mathcal{M}, \Lambda_{\mathcal{P}}(f)) = \Lambda_{\mathcal{M}}(f)$. Set $\mathcal{Z} = \Xi(\mathcal{M}, \Lambda_{\mathcal{P}}(f))$. First note that $\mathcal{Z}$ is in $\mathfrak{R}(\mathcal{M})$. Since $\mathcal{P}$ is thick in $\mathcal{M}$ and hence in $\mathcal{M}$, by Lemma 4.3,

$$\Phi_{\mathcal{M}}(\mathcal{Z}) = \Phi_{\mathcal{M}}(\mathrm{res}(\mathcal{M} \cup \Lambda_{\mathcal{P}}(f))) = \Phi_{\mathcal{M}}(\mathcal{M}) \vee \Phi_{\mathcal{M}}(\Lambda_{\mathcal{P}}(f)) = \Phi_{\mathcal{P}}(\Lambda(\mathcal{P})(f)) = f$$

and thus $\Lambda_{\mathcal{M}}(f) = \mathcal{Z}$, proving the claim.   □

5. A generalization of the Auslander transpose

Let C be a semidualizing module, and set $-^{\dagger} = \mathrm{Hom}(-, C)$. For the entirety of this section, $\mathcal{A}$ denotes a thick subcategory of $\mathcal{G}_C$ that is closed under †. Recalling Proposition 3.6, $\mathcal{A}$-dim satisfies the Auslander–Buchsbaum formula. We set $\mathrm{res}_{\mathcal{A}} M = \mathrm{res}(\{M\} \cup \mathcal{A})$.

The Auslander transpose has been an invaluable tool in both representation theory and commutative algebra. In this section, we generalize the notion of the Auslander transpose using semidualizing modules and list some properties which we will use. The Auslander transpose has previously been generalized in [Geng 2013; Huang 1999], but the construction here is different.

Definition 5.1. An $\mathcal{A}$-*presentation* of X is an exact sequence $A_1 \xrightarrow{\varphi} A_0 \to X \to 0$ with $A_1, A_0 \in \mathcal{A}$. Set $\mathrm{Tr}_{\mathcal{A}} X = \mathrm{coker}\, \varphi^{\dagger}$.

When $C = R$, we get the usual Auslander transpose of X which we denote by $\operatorname{Tr} X$. The "functor" $\operatorname{Tr}_\mathcal{A}$ is not well defined up to isomorphism or even stable isomorphism, motivating a new equivalence relation. Finding the correct equivalence relation is actually a subtle affair. The equivalence relation must make $\operatorname{Tr}_\mathcal{A} X$ be well defined, but it must also detect resolving subcategories. For modules X and Y, we write $X \sim' Y$ and $Y \sim' X$ if there exists an $A \in \mathcal{A}$ such that $0 \to X \to Y \to A \to 0$ is exact. Let $\mathcal{A}$-equivalence, denoted by $\sim$, be the transitive closure of the relation $\sim'$. Since $\sim'$ is symmetric and reflexive, $\sim$ is an equivalence relation. Stable equivalence implies $\mathcal{A}$-equivalence, and when $\mathcal{A} = \mathcal{P}$, they are the same. We will see in a moment that $\operatorname{Tr}_\mathcal{A} X$ has the desired properties.

Remark 5.2. We would like to think of $\operatorname{Tr}_\mathcal{A}$ as a functor. However, $\operatorname{mod}(R)$ modulo $\mathcal{A}$-equivalence does not form a sensible category. However, a very similar construction is functorial. Let $P \xrightarrow{\varphi} Q \to X \to 0$ be a projective presentation. Set $\operatorname{Tr}_C X = \operatorname{coker} \varphi^\dagger$. A similar construction is given in [Geng 2013; Huang 1999]. We will briefly show that $\operatorname{Tr}_C : \operatorname{mod}(R)/\mathcal{A} \to \operatorname{mod}(R)/\mathcal{A}$ is a functor. We thank the referee for bringing the following construction to our attention.

We give some definitions first.

(1) $\mathcal{X}/\mathcal{Y}$ is the category whose objects are $\mathcal{X}$, and whose morphisms are

$$\operatorname{Hom}_{\mathcal{X}/\mathcal{Y}}(X_1, X_2) := \operatorname{Hom}_\mathcal{X}(X_1, X_2)/F_\mathcal{Y}(X_1, X_2),$$

where $X_1, X_2 \in \mathcal{X}$ and $F_\mathcal{Y}(X_1, X_2)$ is the subgroup of morphisms in $\mathcal{X}$ which factor through an object in $\mathcal{Y}$.

(2) Morph $\mathcal{X}$ is the category whose objects are morphisms $f : X_1 \to X_2$. A morphism (g_1, g_2) between objects $f : X_1 \to X_2$ and $f' : X_1' \to X_2'$ in Morph $\mathcal{X}$ is a pair of morphisms $g_1 : X \to X'$ and $g_2 : Y \to Y'$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{g_1} & X_1' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ X_2 & \xrightarrow{g_2} & X_2' \end{array}$$

(3) For $f, f' \in \operatorname{Morph} \mathcal{X}$, a morphism $(g_1, g_2) : f \to f'$ is homotopically trivial if there exists an $h : X_2 \to X_1'$ such that $f'hf = g_2 f = f' g_1$. Let $H_\mathcal{X}(f, f')$ denote the subgroup of homotopically trivial maps.

(4) H-Morph $\mathcal{X}$ is the category whose objects are the same as Morph $\mathcal{X}$ but whose morphisms are

$$\operatorname{Hom}_{H\text{-Morph}\,\mathcal{X}}(f, f') = \operatorname{Hom}_{\operatorname{Morph}\,\mathcal{X}}(f, f')/H_\mathcal{X}(f, f').$$

We now mimic the construction of the Auslander transpose [1971, Chapter 3, Section 1]. Set $\mathcal{C} = \text{add } C$. The functor † restricts to a functor $^\dagger : \mathcal{P} \to \mathcal{C}$ which induces a contravariant functor

$$^\dagger : \text{Morph } \mathcal{P} \to \text{Morph } \mathcal{C},$$
$$f \mapsto f^\dagger.$$

It is easy to check that the group homomorphism

$$^\dagger : \text{Hom}_{\text{Morph } \mathcal{P}}(f, f') \to \text{Hom}_{\text{Morph } \mathcal{C}}(f'^\dagger, f^\dagger),$$
$$(g_1, g_2) \mapsto (g_2^\dagger, g_1^\dagger),$$

maps the subgroup $H_{\mathcal{P}}(f, f')$ to $H_{\mathcal{C}}(f'^\dagger, f^\dagger)$. Therefore, † induces a functor H-Morph $\mathcal{P} \to H$-Morph $\mathcal{C}$. Furthermore, it is easy to check that the map

$$\text{coker} : H\text{-Morph } \mathcal{C} \to \text{mod}(R)/\mathcal{C},$$
$$f \mapsto \text{coker } f,$$

is a well defined functor. The discussion in [loc. cit.] indicates that there is a functor $\rho : \text{mod}(R)/\mathcal{P} \to H$-Morph $\mathcal{P}$ which sends a module to a projective presentation. We summarize these discussions with the following commutative diagram:

$$
\begin{array}{ccccccc}
& \text{Morph } \mathcal{P} & \xrightarrow{\ \dagger\ } & \text{Morph } \mathcal{C} & & \\
& \downarrow & & \downarrow & & \\
\text{mod}(R)/\mathcal{P} \xrightarrow{\ \rho\ } & H\text{-Morph } \mathcal{P} & \xrightarrow{\ \dagger\ } & H\text{-Morph } \mathcal{C} & \xrightarrow{\ \text{coker}\ } & \text{mod}(R)/\mathcal{C}
\end{array}
$$

The composition of the bottom row is Tr_C. Since $\mathcal{A}$ is closed under † and is thick in MCM, Tr_C fixes $\mathcal{A}$. Thus, since $\mathcal{P}, \mathcal{C} \subseteq \mathcal{A}$, it follows that Tr_C induces a functor $\text{mod}(R)/\mathcal{A} \to \text{mod}(R)/\mathcal{A}$, as desired.

This approach has two deficiencies. First of all, we cannot compute Tr_C using $\mathcal{A}$-resolutions. We will use $\mathcal{A}$-resolutions, for example in Lemma 5.4(4), to show that $\text{Tr}_{\mathcal{A}} \text{Tr}_{\mathcal{A}} X \sim X$. Second of all, if X and Y are isomorphic in $\text{mod}(R)/\mathcal{A}$, it is not clear if $\text{res}_{\mathcal{A}} X = \text{res}_{\mathcal{A}} Y$. Because of these issues, Tr_C cannot take the place of $\text{Tr}_{\mathcal{A}}$ in this work.

We proceed to show that $\mathcal{A}$-equivalence is sufficient for our purposes.
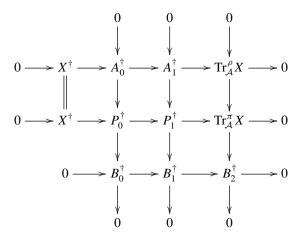
Proposition 5.3. *For a module X, the module $\text{Tr}_{\mathcal{A}} X$ is unique up to $\mathcal{A}$-equivalence.*

Proof. Let π be the projective presentation $P_1 \to P_0 \to X \to 0$, and let ρ be the $\mathcal{A}$-presentation $A_1 \to A_0 \to X \to 0$. Suppose there is an epimorphism $\pi \to \rho$.

Then there exists the commutative diagram

$$(2)$$

whose columns are exact, with B_0, B_1, B_2 in $\mathcal{A}$. Applying † to the diagram yields

where $\mathrm{Tr}_{\mathcal{A}}^{\rho} X$ and $\mathrm{Tr}_{\mathcal{A}}^{\pi} X$ denote $\mathrm{Tr}_{\mathcal{A}} X$ computed using ρ and π, respectively. Since the rows are exact, and the middle two columns are exact, the snake lemma shows the last column is exact. Since $B_2^{\dagger} \in \mathcal{A}$, we see that $\mathrm{Tr}_{\mathcal{A}}^{\rho} X \sim \mathrm{Tr}_{\mathcal{A}}^{\pi} X$.

Consider any two $\mathcal{A}$-presentations, ρ and ρ'. It is easy to construct projective presentations ψ and ψ' with epimorphisms $\psi \to \rho$ and $\psi' \to \rho'$. In the proof of [Masek 1999, Proposition 4], it is shown that there is a projective presentation of π and epimorphisms $\pi \to \psi$ and $\pi \to \psi'$. Using our work so far, we know that $\mathrm{Tr}_{\mathcal{A}}^{\rho} X \sim \mathrm{Tr}_{\mathcal{A}}^{\pi} X \sim \mathrm{Tr}_{\mathcal{A}}^{\rho'}$. $\qquad\square$

Lemma 5.4. *For any $X, Y \in \mathrm{mod}(R)$ such that $X \sim Y$, the following are true*:

(1) $\mathrm{res}_{\mathcal{A}} X = \mathrm{res}_{\mathcal{A}} Y$,

(2) $\Omega X \sim \Omega Y$,

(3) $\mathrm{Tr}_{\mathcal{A}} X \sim \mathrm{Tr}_{\mathcal{A}} Y$,

(4) $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X \sim X$.

Proof. It suffices to assume that $0 \to X \to Y \to A \to 0$ with $A \in \mathcal{A}$. Proving (1) is trivial. For suitable choices of syzygies, we have $0 \to \Omega X \to \Omega Y \to \Omega A \to 0$. Since ΩA is in $\mathcal{A}$, and since syzygies are unique up to stable, and hence $\mathcal{A}$-equivalence, this proves (2).
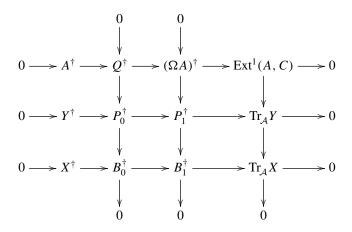
Now we show (3). Consider the diagram with exact rows

$$
\begin{array}{ccccccc}
P_1 & \longrightarrow & P_0 & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \Omega A & \longrightarrow & Q & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

with Q, P_0, P_1 projective and surjective vertical arrows. Using the snake lemma, we can extend this to the diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
B_1 & \longrightarrow & B_0 & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
P_1 & \longrightarrow & P_0 & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \Omega A & \longrightarrow & Q & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

such that B_1, B_0 are in $\mathcal{A}$. Applying † to this diagram gives the following:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 \longrightarrow A^\dagger & \longrightarrow & Q^\dagger & \longrightarrow & (\Omega A)^\dagger & \longrightarrow & \operatorname{Ext}^1(A, C) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow Y^\dagger & \longrightarrow & P_0^\dagger & \longrightarrow & P_1^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} Y & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow X^\dagger & \longrightarrow & B_0^\dagger & \longrightarrow & B_1^\dagger & \longrightarrow & \operatorname{Tr}_{\mathcal{A}} X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Ext}^1(A, C) = 0$, applying the snake lemma to the middle two columns yields $\mathrm{Tr}_{\mathcal{A}} X \cong \mathrm{Tr}_{\mathcal{A}} Y$. This proves (3).

To see (4), consider the projective presentation

$$P_1 \to P_0 \to X \to 0.$$

Then

$$P_0^\dagger \to P_1^\dagger \to \mathrm{Tr}_{\mathcal{A}} X \to 0$$

is an $\mathcal{A}$-presentation, which we use to compute $\mathrm{Tr}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}} X$, giving the result. $\qquad\square$

We close this section with an example of a property shared by $\mathrm{Tr}_{\mathcal{A}}$ and Tr.

Lemma 5.5. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence in* $\mathrm{mod}(R)$. *For suitable choices of* $\mathrm{Tr}_{\mathcal{A}}$, *we have the exact sequence*

$$0 \to Z^\dagger \to Y^\dagger \to X^\dagger \to \mathrm{Tr}_{\mathcal{A}} Z \to \mathrm{Tr}_{\mathcal{A}} Y \to \mathrm{Tr}_{\mathcal{A}} X \to 0.$$

Furthermore, if $\mathrm{Ext}^i(X, C) = 0$, *then*

$$0 \to \mathrm{Tr}_{\mathcal{A}} \, \Omega^i Z \to \mathrm{Tr}_{\mathcal{A}} \, \Omega^i Y \to \mathrm{Tr}_{\mathcal{A}} \, \Omega^i X \to 0.$$

Proof. Let θ denote the map from Y to Z. We have the short exact sequence

$$0 \to \Omega^i X \to \Omega^i Y \xrightarrow{\Omega^i \theta} \Omega^i Z \to 0$$

for all $i \geq 0$. We can construct the following short exact sequence of $\mathcal{A}$ presentations.

Applying † and also the snake lemma yields

$$0 \to (\Omega^i Z)^\dagger \xrightarrow{(\Omega^i \theta)^\dagger} \Omega^i Y \xrightarrow{\lambda} \Omega^i X^\dagger \xrightarrow{\varepsilon} \mathrm{Tr}_{\mathcal{A}} \, \Omega^i Z \xrightarrow{\eta} \mathrm{Tr}_{\mathcal{A}} \, \Omega^i Y \to \mathrm{Tr}_{\mathcal{A}} \, \Omega^i X \to 0.$$

Setting $i = 0$ at this stage gives us the first claim. The short exact sequence $0 \to \Omega^i X \to \Omega^i Y \to \Omega^i Z \to 0$ gives the following long exact sequence of Ext

modules:

$$0 \to (\Omega^i Z)^\dagger \xrightarrow{(\Omega^i \theta)^\dagger} \Omega^i Y \xrightarrow{\lambda} \Omega^i X^\dagger \xrightarrow{\delta} \text{Ext}^1(\Omega^i Z, C) \xrightarrow{\text{Ext}^1(\Omega^i \theta, C)} \text{Ext}^1(\Omega^i Y, C) \to \cdots.$$

We also have

$$\cdots \to \text{Ext}^i(X, C) \to \text{Ext}^{i+1}(Z, C) \xrightarrow{\text{Ext}^{i+1}(\theta, C)} \text{Ext}^{i+1}(Y, C) \to \cdots.$$

Since $\text{Ext}^i(X, C) = 0$ by assumption, $\text{Ext}^{i+1}(\theta, C)$ and $\text{Ext}^1(\Omega^i \theta, C)$ are injective, forcing δ to be zero. Thus λ is surjective. Then the first long exact sequence shows that ε is zero, and so η is injective, giving the desired result. $\qquad\square$

6. Resolving subcategories which are maximal Cohen–Macaulay on the punctured spectrum

We keep the same conventions used in the previous section, except we also assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring. Recall that since $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$, according to Proposition 3.6, dimension with respect to $\mathcal{A}$ satisfies the Auslander–Buchsbaum formula. Set

$$\text{res}_{\mathcal{A}} M = \text{res}(\{M\} \cup \mathcal{A}),$$
$$\Delta(\mathcal{A})_0 = \{M \in \Delta(\mathcal{A}) \mid M_p \in \text{add } \mathcal{A}_{\mathfrak{p}} \text{ for all } p \in \text{spec } R \setminus \mathfrak{m}\},$$
$$\Delta(\mathcal{A})_0^i = \{M \in \Delta(\mathcal{A})_0 \mid \mathcal{A}\text{-dim } M \le i\}.$$

This section is devoted to proving the following:

Theorem 6.1. *If $(R, \mathfrak{m}, k)$ is a local ring with $\dim R = d$, the filtration*

$$\mathcal{A} = \Delta(\mathcal{A})_0^0 \subsetneq \Delta(\mathcal{A})_0^1 \subsetneq \cdots \subsetneq \Delta(\mathcal{A})_0^d = \Delta(\mathcal{A})_0$$

is a complete list of the resolving subcategories of $\Delta(\mathcal{A})_0$ containing $\mathcal{A}$.

This theorem and its proof is a generalization of [Dao and Takahashi 2015, Theorem 2.1]. We now use results from the previous section to make the building blocks of the proof of Theorem 6.1.

Lemma 6.2. *For any module $X \in \text{mod}(R)$ and for suitable choices of $\text{Tr}_{\mathcal{A}} X$ and $\Omega \text{Tr}_{\mathcal{A}} \Omega X$,*

$$0 \to \text{Ext}^1(X, C) \to \text{Tr}_{\mathcal{A}} X \to \Omega \text{Tr}_{\mathcal{A}} \Omega X \to 0.$$

Proof. With F_0, F_1, F_2 projective, consider the sequence

$$F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0 \to X \to 0.$$

We have coker $g^\dagger = \mathrm{Tr}_{\mathcal{A}} X$. By the universal property of kernel and cokernel, we have the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}\, g^\dagger & \longrightarrow & F_1^\dagger & \longrightarrow & \mathrm{Tr}_{\mathcal{A}} X & \longrightarrow & 0 \\
& & \downarrow{\iota} & & \| & & \downarrow{\varepsilon} & & \\
0 & \longrightarrow & \ker f^\dagger & \longrightarrow & F_1^\dagger & \longrightarrow & \mathrm{Im}\, f^\dagger & \longrightarrow & 0
\end{array}
$$

The snake lemma yields the exact sequence

$$0 \to \ker \iota \to 0 \to \ker \varepsilon \to \mathrm{Ext}^1(X, C) \to 0 \to \mathrm{coker}\,\varepsilon \to 0.$$

Thus ε is surjective and $\ker \varepsilon \cong \mathrm{Ext}^1(X, C)$, giving the exact sequence

$$0 \to \mathrm{Ext}^1(X, C) \to \mathrm{Tr}_{\mathcal{A}} X \to \mathrm{Im}\, f^\dagger \to 0.$$

It remains to show that $\mathrm{Im}\, f^\dagger \sim \Omega\, \mathrm{Tr}_{\mathcal{A}}\, \Omega X$.

We have the short exact sequence $0 \to \mathrm{Im}\, f^\dagger \to F_2^\dagger \to \mathrm{Tr}_{\mathcal{A}}\, \Omega X \to 0$. Consider the pushout diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega\, \mathrm{Tr}_{\mathcal{A}}\, \Omega X & = & \Omega\, \mathrm{Tr}_{\mathcal{A}}\, \Omega X \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}\, f^\dagger & \longrightarrow & T & \longrightarrow & G & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Im}\, f^\dagger & \longrightarrow & F_2^\dagger & \longrightarrow & \mathrm{Tr}_{\mathcal{A}}\, \Omega X & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

with G projective. We have $\mathrm{Im}\, f^\dagger \sim T \sim \Omega\, \mathrm{Tr}_{\mathcal{A}}\, \Omega X$, as desired. $\qquad\square$

Lemma 6.3. *If* $X \in \Delta(\mathcal{A})_0$, *for all* $0 \le i < \mathrm{depth}\, C$, *for suitable choices of* $\mathrm{Tr}_{\mathcal{A}}$, *the following is exact*:

$$0 \to \mathrm{Tr}_{\mathcal{A}}\, \Omega^{i+1}\, \mathrm{Tr}_{\mathcal{A}}\, \Omega^{i+1} X \to \mathrm{Tr}_{\mathcal{A}}\, \Omega^i\, \mathrm{Tr}_{\mathcal{A}}\, \Omega^i X \to \mathrm{Tr}_{\mathcal{A}}\, \Omega^i\, \mathrm{Ext}^{i+1}(X, C) \to 0.$$

Proof. Using Lemma 6.2, we have

$$0 \to \mathrm{Ext}^{i+1}(X, C) \to \mathrm{Tr}_{\mathcal{A}}\, \Omega^i X \to \Omega\, \mathrm{Tr}_{\mathcal{A}}\, \Omega^{i+1} X \to 0.$$

Since $X \in \Delta(\mathcal{A})_0$, we have $\mathrm{Ext}^{i+1}(X, C)_p = 0$ for every nonmaximal prime p. Thus $\mathrm{Ext}^{i+1}(X, C)$ has finite length, and so

$$\mathrm{Ext}^i\, (\mathrm{Ext}^{i+1}(X, C), C) = 0$$

for all $0 \leq i < \operatorname{depth} C$. Thus, we can apply Lemma 5.5.   □

Lemma 6.4. *Let* $X \in \Delta(\mathcal{A})_0$ *and* $0 < n \leq \operatorname{depth} C$. *Then*

$$\operatorname{res}_{\mathcal{A}}\big(X, \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C), \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C), \cdots, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\big)$$
$$= \operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n X, \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C),$$
$$\operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C), \cdots, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\big)$$

Proof. The previous lemma tells us that

$$\operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X, \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C)\big)$$
$$= \operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X, \operatorname{Tr}_{\mathcal{A}} \operatorname{Ext}^1(X, C)\big),$$
$$\operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Tr}_{\mathcal{A}} \Omega X, \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C)\big)$$
$$= \operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega^2 \operatorname{Tr}_{\mathcal{A}} \Omega^2 X, \operatorname{Tr}_{\mathcal{A}} \Omega \operatorname{Ext}^2(X, C)\big),$$
$$\vdots$$
$$\operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} X, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\big)$$
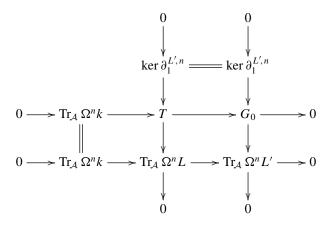$$= \operatorname{res}_{\mathcal{A}}\big(\operatorname{Tr}_{\mathcal{A}} \Omega^n \operatorname{Tr}_{\mathcal{A}} \Omega^n X, \operatorname{Tr}_{\mathcal{A}} \Omega^{n-1} \operatorname{Ext}^n(X, C)\big).$$

Since $\operatorname{Tr}_{\mathcal{A}} \operatorname{Tr}_{\mathcal{A}} X \sim X$, the result is now clear.   □

Lemma 6.5. *Let* $0 \leq n < \operatorname{depth} R$ *and* L *a nonzero finite length module. There exists an* $\mathcal{A}$-*resolution* $(G_\bullet, \partial^{L,n})$ *of* $\operatorname{Tr}_{\mathcal{A}} \Omega^n L$ *such that* $G_i = 0$ *for all* $i > n + 1$ *and*

$$\ker \partial_i^{L,n} = \operatorname{Tr}_{\mathcal{A}} \Omega^{n-i} L$$

for all $1 \leq i \leq n$. *In particular,* $\operatorname{Tr}_{\mathcal{A}} \Omega^i L \in \operatorname{res}_{\mathcal{A}}(\operatorname{Tr}_{\mathcal{A}} \Omega^n L)$ *for all* $0 \leq i \leq n$, $\mathcal{A}$-$\dim(\operatorname{Tr}_{\mathcal{A}} \Omega^n L) = n + 1$, *and* $\operatorname{Tr}_{\mathcal{A}} \Omega^n L \in \Delta(\mathcal{A})_0^{n+1}$.

Proof. Let $(F_\bullet, \partial)$ be a free resolution of L. Then we have

$$F_{n+1} \to F_n \to \Omega^n L \to 0$$

and

$$0 \to \Omega^n L \to F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to L \to 0.$$

Because L has finite length, and since $\operatorname{depth} C = \operatorname{depth} R$ by Proposition 3.6, we have $\operatorname{Ext}^i(L, C) = 0$ for all $0 \leq i \leq n$, and so we have the exact sequence

$$0 \to L^\dagger \to F_0^\dagger \xrightarrow{\partial_1^\dagger} F_1^\dagger \xrightarrow{\partial_2^\dagger} \cdots \xrightarrow{\partial_{n-1}^\dagger} F_{n-1}^\dagger \to (\Omega^n L)^\dagger \to 0.$$

Note that $L^\dagger = 0$ since L has finite length. Thus, splicing this exact sequence with

$$0 \to (\Omega^n L)^\dagger \to F_n^\dagger \xrightarrow{\partial_{n+1}^\dagger} F_{n+1}^\dagger \to \operatorname{Tr}_{\mathcal{A}} \Omega^n L \to 0,$$

we create an $\mathcal{A}$-resolution of $\mathrm{Tr}_{\mathcal{A}}\,\Omega^n L$. So we set $G_i = F^{\dagger}_{n+1-i}$ for $0 \le i \le n+1$ and $G_i = 0$ for $i > n+1$. Set $\partial_i^{L,n} = \partial^{\dagger}_{n+2-i}$ for $1 \le i \le n+1$ and $\partial_i^{L,n} = 0$ for all $i > n+1$. Using our previous arguments for values less that n, we see that $\ker \partial_i^{L,n} = \mathrm{Tr}_{\mathcal{A}}\,\Omega^{n-i} L$ for $0 \le i \le n$. Showing the first two claims.

It is now apparent that $\mathcal{A}$-$\dim \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L \le n+1$. If $\ker \partial_n^{L,n} = \mathrm{Tr}_{\mathcal{A}} L$ is in $\mathcal{A}$, then so is L since $\mathrm{Tr}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}} L \sim L$. However, this is impossible since $L^{\dagger} = \mathrm{Ext}^0(L, C) = 0$. Therefore we have $\mathcal{A}$-$\dim \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L = n+1$. $\qquad\square$

Lemma 6.6. *For all $0 \le n < \mathrm{depth}\,R$ and all nonzero finite length modules L, $\mathrm{res}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L = \mathrm{res}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}}\,\Omega^n k$.*

Proof. Let λ denote the length function for modules. If $L \ne 0$, then we can write $0 \to L' \to L \to k \to 0$ with $\lambda(L') < \lambda(L)$. Since by Proposition 3.6 $n < \mathrm{depth}\,R = \mathrm{depth}\,C$, we have $\mathrm{Ext}^n(L', C) = 0$, and so from Lemma 5.5,

$$0 \to \mathrm{Tr}_{\mathcal{A}}\,\Omega^n k \to \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L \to \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L' \to 0.$$

Thus, by induction, $\mathrm{res}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L \subseteq \mathrm{res}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}}\,\Omega^n k$.

Now we wish to show that $\mathrm{Tr}_{\mathcal{A}}\,\Omega^n k \in \mathrm{res}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L$. We proceed by double induction, first on $\lambda(L)$ and then on n. The case $L = k$ is trivial, so suppose $\lambda(L) > 1$. Write $0 \to L' \to L \to k \to 0$ again. Since L' has depth zero, we can use Lemma 6.5 to get the resolution $(G_{\bullet}, \partial^{L'})$. Thus we have the exact sequence

$$0 \to \ker \partial_1^{L',n} \to G_0 \to \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L \to 0.$$

Taking the pullback diagram with our last exact sequence yields the following:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & \ker \partial_1^{L',n} & = & \ker \partial_1^{L',n} & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & \mathrm{Tr}_{\mathcal{A}}\,\Omega^n k & \to & T & \to & G_0 & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Tr}_{\mathcal{A}}\,\Omega^n k & \to & \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L & \to & \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L' & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

It is now easy to see that it suffices to show that $\ker \partial_1^{L',n}$ is in $\mathrm{res}_{\mathcal{A}} \mathrm{Tr}_{\mathcal{A}}\,\Omega^n L$. When $n = 0$, $(G_{\bullet}, \partial^{L',n})$ is the resolution

$$0 \to G_1 \xrightarrow{\partial_1^{L',0}} G_0 \to \mathrm{Tr}_{\mathcal{A}} L' \to 0,$$

and we are done since $\ker \partial_1^{L',0} = G_1 \in \mathcal{A} \subseteq \operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} L$. So suppose $n > 0$. We have $\ker \partial_1^{L',n} = \operatorname{Tr}_\mathcal{A} \Omega^{n-1} L'$, by Lemma 6.5. By induction, $\operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} L$ and $\operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} L'$ are the same as $\operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} k$. So we have

$$\ker \partial_1^{L',n} \in \operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} L \subseteq \operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^n L,$$

where the inclusion follows from Lemma 6.5, and we are done. □

These next proofs are similar to those in [Dao and Takahashi 2015] with the appropriate changes. They are included here for the sake of completeness.

Proposition 6.7. *For every $0 < n \le \operatorname{depth} R$, we have $\Delta(\mathcal{A})_0^n = \operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} L$ for every nonzero finite length module L.*

Proof. By Lemma 6.6, we may assume that $L = k$. By Lemma 6.5, we know that $\mathcal{A}\text{-dim}(\operatorname{Tr} \Omega^{n-1} k) = n$. Since localization commutes with cokernels, duals and syzygies, we have $\operatorname{Tr} \Omega^n k$ is in $\Delta(\mathcal{A})_0$ and hence in $\Delta(\mathcal{A})_0^n$. Suppose now that $X \in \Delta(\mathcal{A})_0^n$. Then $\Omega^n X \in \mathcal{A}$, and so $\operatorname{Tr}_\mathcal{A} \Omega^n \operatorname{Tr}_\mathcal{A} \Omega^n X \in \mathcal{A}$. Furthermore, for each $i \ge 0$, the module $\operatorname{Ext}^{i+1}(X, C)$ has finite length. Hence, Lemma 6.6 implies that $\operatorname{Tr}_\mathcal{A} \Omega^i \operatorname{Ext}^{i+1}(X, C)$ is in $\operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^i k \subseteq \operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} k$, where the inclusion follows from Lemma 6.5. By Lemma 6.4, we therefore have

$$X \in \operatorname{res}_\mathcal{A}\big(\operatorname{Tr}_\mathcal{A} \Omega^n \operatorname{Tr}_\mathcal{A} \Omega^n X, \ \operatorname{Tr}_\mathcal{A} \operatorname{Ext}^1(X, C),$$
$$\operatorname{Tr}_\mathcal{A} \Omega \operatorname{Ext}^2(X, C), \ \cdots, \ \operatorname{Tr}_\mathcal{A} \Omega^{n-1} \operatorname{Ext}^n(X, C) \big) \subseteq \operatorname{res}_\mathcal{A} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} k$$

which concludes the proof. □

We now prove the main result of this section.

Proof of Theorem 6.1. We clearly have the chain

$$\mathcal{A} = \Delta(\mathcal{A})_0^0 \subsetneq \Delta(\mathcal{A})_0^1 \subsetneq \cdots \subsetneq \Delta(\mathcal{A})_0^d = \Delta(\mathcal{A})_0.$$

Take $X \in \Delta(\mathcal{A})_0^n \setminus \Delta(\mathcal{A})_0^{n-1}$ for $d \ge n \ge 1$. We need to show that $\operatorname{res}_\mathcal{A} X = \Delta(\mathcal{A})_0^n$, and we have $\operatorname{res}_\mathcal{A} X \subseteq \Delta(\mathcal{A})_0^n$. We proceed by induction. When $n = 0$, the statement is trivial. So assume that $n > 0$ and $\operatorname{res}_\mathcal{A} \Omega X = \Delta(\mathcal{A})_0^{n-1}$. Since $\operatorname{Ext}^n(X, C)$ has finite length, it suffices to show $\operatorname{Tr}_\mathcal{A} \Omega^{n-1} \operatorname{Ext}^n(X, C) \in \operatorname{res}_\mathcal{A} X$, by Proposition 6.7.

Since $\Omega^n X \in \mathcal{A}$, the short exact sequence $0 \to \Omega^n X \to P \to \Omega^{n-1} X \to 0$, with P projective, is an $\mathcal{A}$ presentation of $\Omega^{n-1} X$. Using this presentation to compute $\operatorname{Tr}_\mathcal{A}$, we see that $\operatorname{Tr}_\mathcal{A} \Omega^{n-1} X \sim \operatorname{Ext}^1(\Omega^{n-1} X, C) \cong \operatorname{Ext}^n(X, C)$. Therefore, $\operatorname{Tr}_\mathcal{A} \Omega^{n-1} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} X \sim \operatorname{Tr}_\mathcal{A} \Omega^{n-1} \operatorname{Ext}^n(X, C)$ by Lemma 5.4. Thus, it suffices to show that $\operatorname{Tr}_\mathcal{A} \Omega^{n-1} \operatorname{Tr}_\mathcal{A} \Omega^{n-1} X \in \operatorname{res}_\mathcal{A} X$, again by Lemma 5.4.

Let $0 < i \le n - 1$. Since $\operatorname{Ext}^i(X, C)$ has finite length, Lemma 6.5 implies

$$\operatorname{Tr}_\mathcal{A} \Omega^{i-1} \operatorname{Ext}^i(X, C) \in \Delta(\mathcal{A})_0^i \subseteq \Delta(\mathcal{A})_0^{n-1} = \operatorname{res}_\mathcal{A} \Omega X \subseteq \operatorname{res}_\mathcal{A} X.$$

Therefore, Lemma 6.4 implies that

$$\mathrm{Tr}_\mathcal{A}\,\Omega^{n-1}\,\mathrm{Tr}_\mathcal{A}\,\Omega^{n-1}X \in \mathrm{res}_\mathcal{A}\big(X,\,\mathrm{Tr}_\mathcal{A}\,\mathrm{Ext}^1(X,C),\,\mathrm{Tr}_\mathcal{A}\,\Omega\,\mathrm{Ext}^2(X,C),$$
$$\cdots,\,\mathrm{Tr}_\mathcal{A}\,\Omega^{n-2}\,\mathrm{Ext}^{n-1}(X,C)\big) = \mathrm{res}_\mathcal{A}(X)$$

as claimed. □

The following corollary is immediate from Theorem 6.1

Corollary 6.8. *If* $X \in \Delta(\mathcal{A})_0^n \setminus \Delta(\mathcal{A})_0^{n-1}$, *then* $\mathrm{res}_\mathcal{A} X = \Delta(\mathcal{A})_0^n$.

7. Resolving subcategories and semidualizing modules

In this section, we keep the same notations and conventions as the previous sections, except we do not assume that R is local. In this section, we classify the resolving subcategories of $\Delta(\mathcal{A})$ which contain $\mathcal{A}$. Note that it is easy to check that C_p is a semidualizing R_p-module for all $p \in \mathrm{spec}\,R$. In Corollary 8.2, we will see that for all $p \in \mathrm{spec}\,R$, add $\mathcal{A}_p$ is a thick subcategory of $\mathcal{G}_{C_p}$ closed under $\mathrm{Hom}_{R_p}(-, C_p)$. The following is a modified version of [Dao and Takahashi 2014, Lemma 4.6], which is a generalization of [Takahashi 2009, Proposition 4.2]. For a module X, let $\mathrm{NA}(X) = \{p \in \mathrm{spec}\,R \mid X_p \notin \mathrm{add}\,\mathcal{A}_p\}$.

Proposition 7.1. *Suppose* $X \in \Delta(\mathcal{A})$. *For every* $p \in \mathrm{NA}(X)$, *there is a* $Y \in \mathrm{res}_\mathcal{A} X$ *such that* $\mathrm{NA}(Y) = V(p)$ *and* add $\mathcal{A}_\pi$*-dim* $Y_\pi =$ add $\mathcal{A}_\pi$*-dim* X_π *for all* $\pi \in V(p)$.

Proof. If $\mathrm{NA}(X) = V(p)$ we are done. So fix a $q \in \mathrm{NA}(X)\setminus V(p)$. As in the proof of [Dao and Takahashi 2014, Lemma 4.6], choose an $x \in p\setminus q$ and consider the following pushout diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega X & \longrightarrow & F & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle x} & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega X & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

with F projective. Immediately, we have $Y \in \mathrm{res}_\mathcal{A} X$. Furthermore, $Y_{q'} \in \mathrm{res}\,X_{q'}$ for all $q' \in \mathrm{spec}\,R$. Therefore, we have $\mathrm{NA}(Y) \subseteq \mathrm{NA}(X)$. The proof of [Dao and Takahashi 2014, Lemma 4.6] tells us that

$$\mathrm{depth}(Y_\pi) = \min\{\mathrm{depth}(X_\pi), \mathrm{depth}(R_\pi)\}$$

for all $\pi \in V(p)$. Thus, by Proposition 3.6, add $\mathcal{A}_\pi$-dim $Y_\pi =$ add $\mathcal{A}_\pi$-dim X_π, for all $\pi \in V(p)$. In particular, this shows that $V(p)$ is contained in $\mathrm{NA}(Y)$.

Localizing at q yields the following:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega X_q & \longrightarrow & F_q & \longrightarrow & X_q & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle x} & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega X_q & \longrightarrow & Y_q & \longrightarrow & X_q & \longrightarrow & 0
\end{array}$$

Note x is a unit in R_q. Thus, by the five lemma, Y_q is isomorphic to F_p and therefore is projective. So we have $q \notin \mathrm{NA}(Y)$ and hence $\mathrm{NA}(Y) \subsetneq \mathrm{NA}(X)$.

If $\mathrm{NA}(Y) \neq \mathrm{V}(p)$, then we may repeat this process and construct a Y' that, like Y, satisfies all the desired properties except $\mathrm{V}(p) \subseteq \mathrm{NA}(Y') \subsetneq \mathrm{NA}(Y) \subsetneq \mathrm{NA}(X)$. Since spec R is Noetherian, this process must stabilize after some iteration, producing the desired module. $\qquad\square$

Lemma 7.2. *Let V be a nonempty finite subset of* spec R. *Let M be a module and $\mathcal{X}$ a resolving subcategory such that $M_p \in \mathrm{add}\,\mathcal{X}_p$ for some $p \in$ spec R. Then there exist exact sequences*

$$0 \to K \to X \to M \to 0 \quad and \quad 0 \to L \to M \oplus K \oplus R^t \to X \to 0$$

with $X \in \mathcal{X}$, $\mathrm{NA}(L) \subseteq \mathrm{NA}(M)$, and $\mathrm{NA}(L) \cap V = \varnothing$.

Proof. The result is essentially contained in the proof of [Takahashi 2010, Proposition 4.7]. It shows the existence of the exact sequences and shows that V is contained in the free locus of L and thus $\mathrm{NA}(L) \cap V = \varnothing$. Furthermore, the last exact sequence in the proof shows that for any $p \in$ spec R, L_p is in res M_p. Hence, if L_p is not in a resolving subcategory, then M_p cannot be in that category either, giving us $\mathrm{NA}(L) \subseteq \mathrm{NA}(M)$. $\qquad\square$

These lemmas help to prove the following proposition which is a key component of the proof of Theorem 7.4. This next result is also where we use Corollary 6.8 of the last section.

Proposition 7.3. *Consider a module $M \in \mathrm{mod}(R)$ and a resolving subcategory $\mathcal{X} \in \mathfrak{R}(\mathcal{A})$. If for every $p \in$ spec R, there exists an $X \in \mathcal{X}$ such that*

$$\mathrm{add}\,\mathcal{A}_\mathfrak{p}\text{-dim}\,M_p \leq \mathrm{add}\,\mathcal{A}_\mathfrak{p}\text{-dim}\,X_p,$$

then M is in $\mathcal{X}$.

Proof. Because of Lemma 2.12, we may assume $(R, \mathfrak{m}, k)$ is local. We proceed by induction on $\dim \mathrm{NA}(M)$. If $\dim \mathrm{NA}(M) = -\infty$, then M is in $\mathcal{A}$ and we are done. Suppose $\dim \mathrm{NA}(M) = 0$. Then M is in $\Delta(\mathcal{A})_0^t$ where $t = \mathcal{A}\text{-dim}\,X$. By Proposition 7.1, there exists a $Y \in \mathrm{res}_\mathcal{A} X \subseteq \mathcal{X}$ with $\mathcal{A}\text{-dim}\,Y = t$ and $Y \in \Delta(\mathcal{A})_0$, and thus $Y \in \Delta(\mathcal{A})_0^t \setminus \Delta(\mathcal{A})_0^{t-1}$. By Corollary 6.8, $\mathrm{res}_\mathcal{A} Y = \Delta(\mathcal{A})_0^t$, and thus $M \in \mathrm{res}_\mathcal{A}(Y) \subseteq \mathcal{X}$.

The rest of the proof uses Lemma 7.2 and is identical to [Dao and Takahashi 2015, Theorem 3.5], except one replaces the nonfree locus of M by $\mathrm{NA}(M)$ and replaces projective dimension by $\mathcal{A}$-dim. $\qquad\square$

We come to the main theorem of this section. Recall that Γ is the set of grade consistent functions.

Theorem 7.4. *Assume R is Noetherian. If $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ which is closed under †, then $\Lambda_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$ are inverse functions giving a bijection between Γ and $\mathfrak{R}(\mathcal{A})$.*

Proof. The previous proposition shows that $\Lambda_{\mathcal{A}}\Phi_{\mathcal{A}}$ is the identity on $\mathfrak{R}(\mathcal{A})$. Let $f \in \Gamma$ and $p \in \operatorname{spec} R$. Since $\operatorname{add}\mathcal{A}_p$-$\dim X_p \le f(p)$ for every $X \in \Lambda_{\mathcal{A}}(f)$, we have $\Phi_{\mathcal{A}}(\Lambda_{\mathcal{A}}(f))(p) \le f(p)$. However, by [Dao and Takahashi 2015, Lemma 5.1] there is an $M \in \Delta(\mathcal{P}) \subseteq \Delta(\mathcal{A})$ such that $\operatorname{pd}_{R_p} M_p = f(p)$ and $\operatorname{pd}_{R_q} M_q \le f(q)$ for all $q \in \operatorname{spec} R$. Since for all $q \in \operatorname{spec} R \operatorname{pd}_q M_q = \operatorname{add}\mathcal{A}_q$-$\dim M_q$, M is in $\Lambda_{\mathcal{A}}(f)$, and we have $\Phi_{\mathcal{A}}(\Lambda_{\mathcal{A}}(f))(p) = f(p)$. Thus $\Phi_{\mathcal{A}}\Lambda_{\mathcal{A}}$ is the identity on Γ. $\square$

8. Resolving subcategories that are closed under †

We wish to expand upon Theorem 7.4 using the results in Section 4. However, to use Theorem 7.4, we need to understand which thick subcategories of $\mathcal{G}_C$ containing C are closed under duals. In this section, C is a semidualizing module. Since $\mathcal{G}_C$ is cogenerated by $\operatorname{add} C$, as seen at the end of Section 3, it stands to reason that the results of Section 4 are applicable.

Lemma 8.1. *Suppose $\mathcal{M} \subseteq \mathcal{G}_C$ is resolving with $C \in \mathcal{M}$. Then $\mathcal{M}$ is thick in $\mathcal{G}_C$ if and only if for every $M \in \mathcal{M}$, $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$. In particular, $\mathcal{M}$ is thick in $\mathcal{G}_C$ if any only if it is cogenerated by $\operatorname{add} C$.*

Since syzygies are unique up to projective summands, $(\Omega M^\dagger)^\dagger$ is unique up to $\operatorname{add} C$ summands. Thus, for our purposes, our choice of syzygy is inconsequential. When $R = C$, $(\Omega M^\dagger)^\dagger$ is the classical cosyzygy of a Gorenstein dimension zero module M. Thus in this case, the lemma is equivalent to saying that a resolving subcategory $\mathcal{M}$ of $\mathcal{G}_R$ is thick if and only if it is closed under cosyzygies.

Proof. Assume $\mathcal{M}$ is thick, and let $M \in \mathcal{M}$. We have the following exact sequence.

$$0 \to \Omega M^\dagger \to R^n \to M^\dagger \to 0$$

Applying † yields

$$0 \to M \to C^n \to (\Omega M^\dagger)^\dagger \to 0.$$

Since $C \in \mathcal{M}$, if $\mathcal{M}$ is thick in $\mathcal{G}_C$, $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$.

Conversely, suppose for every $M \in \mathcal{M}$, $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$. We wish to show that $\mathcal{M}$ is thick in $\mathcal{G}_C$. Since $\mathcal{M}$ is resolving, it suffices to check that $\mathcal{M}^\dagger$ is also resolving, since † is a duality on $\mathcal{G}_C$. It is also clear that $\mathcal{M}^\dagger$ is extension closed. Since $C \in \mathcal{M}$, we have $R \in \mathcal{M}^\dagger$. Therefore it suffices to check that $\mathcal{M}^\dagger$ is closed under syzygies. Take $Z = M^\dagger \in \mathcal{M}^\dagger$. Then since $(\Omega M^\dagger)^\dagger$ is in $\mathcal{M}$, $(\Omega M^\dagger)^{\dagger\dagger} \cong \Omega M^\dagger = \Omega Z$ is in $\mathcal{M}^\dagger$, as desired. $\square$

The following corollary, although intuitive, is not obvious, and it is not clear if it holds for other subcategories besides $\mathcal{G}_C$.

Corollary 8.2. *If $\mathcal{M}$ is thick in $\mathcal{G}_C$, then* add $\mathcal{M}_p$ *is thick in $\mathcal{G}_{C_p}$ for all $p \in$ spec R.*

Proof. Take $p \in$ spec R. From [Lemma 2.11](), we know that add $\mathcal{M}_p$ is resolving. By the previous lemma, it suffices to show that for all $M \in$ add $\mathcal{M}_p$, $(\Omega_{R_p} M^\dagger)^\dagger =$ Hom$(\Omega_{R_p}$ Hom$(M, C_p), C_p)$ is in add $\mathcal{M}_p$. For every $M \in$ add $\mathcal{M}_p$, there exists an N such that $M \oplus N = L_p$ for some $L \in \mathcal{M}$. Consider the following:

$$
\begin{aligned}
(\Omega L^\dagger)^\dagger{}_p &= \text{Hom}(\Omega_R \text{ Hom}(L, C), \, C)_p \\
&= \text{Hom}(\Omega_{R_p} \text{Hom}(L_p, C_p), \, C_p) \\
&= \text{Hom}(\Omega_{R_p} \text{Hom}(M \oplus N, C_p), \, C_p) \\
&= \text{Hom}(\Omega_{R_p} \text{Hom}(M, C_p), \, C_p) \oplus \text{Hom}(\Omega_{R_p} \text{Hom}(N, C_p), \, C_p)
\end{aligned}
$$

By the previous lemma, $(\Omega L^\dagger)^\dagger$ is in $\mathcal{M}$, and so $(\Omega_{R_p} M^\dagger)^\dagger$ is in add $\mathcal{M}_p$.   □

Proposition 8.3. *Let $\mathcal{A}$ be the smallest thick subcategory of $\mathcal{G}_C$ containing C. Then $\mathcal{A}$ is closed under †.*

Since the intersection of thick subcategories of $\mathcal{G}_C$ is thick, it is clear that $\mathcal{A}$ exists.

Proof. First, let $\mathcal{W}$ be the set of modules obtained by applying † and Ω to R successive times. Suppose for a moment that res $\mathcal{W} = \mathcal{A}$. Let $A \in \mathcal{A}$. We will show that $A^\dagger \in \mathcal{A}$ by inducting on the number of steps needed to construct A from $\mathcal{W}$. See [Takahashi 2009]() for a precise definition of the notion of steps with regards to a resolving subcategory. If A takes 0 steps to construct, then A is either R or in $\mathcal{W}$, and the claim is clear. Suppose A is constructed in $n > 0$ steps. Then there exists B_1 and B_0 which can be constructed in $n-1$ steps and satisfy one of the following situations.

(1) $0 \to A \to B_0 \to B_1 \to 0$

(2) $0 \to B_0 \to A \to B_1 \to 0$

(3) $B_0 = A \oplus B_1$

Therefore one of the following is true:

(a) $0 \to B_1^\dagger \to B_0^\dagger \to A^\dagger \to 0$

(b) $0 \to B_1^\dagger \to A^\dagger \to B_0^\dagger \to 0$

(c) $B_0^\dagger = A^\dagger \oplus B_1^\dagger$

By induction, $B_0^\dagger$ and $B_1^\dagger$ are in $\mathcal{A}$. Since $\mathcal{A}$ is thick, each of these situations implies that $A^\dagger$ is in $\mathcal{A}$.

Therefore, it suffices to show that res $\mathcal{W} = \mathcal{A}$. First, we show that res $\mathcal{W}$ is a thick subcategory containing C. In light of [Lemma 8.1](), it suffices to show that for every $A \in$ res $\mathcal{W}$, we have $(\Omega A^\dagger)^\dagger \in$ res $\mathcal{W}$. We work as we did in the previous paragraph, and we proceed by induction on the number of steps needed to construct A from $\mathcal{W}$. When it takes 0 steps, then A is either R or in $\mathcal{W}$, in which case the claim is clear.

Suppose A needs $n > 0$ steps to be constructed. Working as we did in the previous paragraph, there exists modules B_1 and B_0 which can be constructed in $n-1$ steps and satisfy one of (1), (2), or (3) above. Therefore, one of the following is true:

(a) $0 \to (\Omega A^\dagger)^\dagger \to (\Omega B_0^\dagger)^\dagger \to (\Omega B_1^\dagger)^\dagger \to 0$

(b) $0 \to (\Omega B_0^\dagger)^\dagger \to (\Omega A^\dagger)^\dagger \to (\Omega B_1^\dagger)^\dagger \to 0$

(c) $(\Omega B_1^\dagger)^\dagger = (\Omega A^\dagger)^\dagger \oplus (\Omega B_0^\dagger)^\dagger$

By induction $(\Omega B_0^\dagger)^\dagger$ and $(\Omega B_1^\dagger)^\dagger$ are in res $\mathcal{W}$. Since $\mathcal{W}$ is resolving, then so is $(\Omega A^\dagger)^\dagger$ as desired.

It suffices now to show that res $\mathcal{W} \subseteq \mathcal{A}$. To do this, we show that each $W \in \mathcal{W}$ is in $\mathcal{A}$. We induct on $c(W)$, the smallest number of times it takes to apply Ω and † to R to obtain W. If $c(W) = 0$, then $W = R$, and we are done. If $c(W) = 1$, then W is either 0 or C which are both in $\mathcal{A}$. Therefore, we may assume that $c(W) > 1$. Then one of the following situations must occur.

(1) $A = \Omega^2 B$

(2) $A = B^{\dagger\dagger}$

(3) $A = \Omega(B^\dagger)$

(4) $A = (\Omega B)^\dagger$

where $c(B) = c(A) - 2$. By induction, B is in $\mathcal{A}$. In cases (1) and (2), it is clear that A is in $\mathcal{A}$ too. We have $c(B^\dagger) \le c(B) + 1 < c(A)$, and so $B^\dagger$ is in $\mathcal{A}$ by induction. Now in case (3), the result is clear. So we assume that we are in case (4). By Lemma 8.1, $(\Omega(B^{\dagger\dagger}))^\dagger \cong (\Omega B)^\dagger = A$ must be in $\mathcal{A}$. □

For the rest of this section, $\mathcal{A}$ will continue to be the smallest thick subcategory of $\mathcal{G}_C$ containing C. It is immediate that $\mathcal{A}$ satisfies the assumptions of Theorem 7.4. We wish to apply the results from the beginning of the paper. Using the notation of Section 4, set $\mathfrak{S}(C) = \mathfrak{S}(\mathcal{A})$, i.e., let $\mathfrak{S}(C)$ be the collection resolving subcategories $\mathcal{M} \subseteq \mathrm{mod}(R)$ such that $\mathcal{A}$ cogenerates $\mathcal{M}$ and add $\mathcal{A}_p$ is thick in add $\mathcal{M}_p$ for every $p \in \mathrm{spec}\, R$.

Lemma 8.4. *Every thick subcategory of $\mathcal{G}_C$ which contains C is in $\mathfrak{S}(C)$. Furthermore, when R is Cohen–Macaulay, every element in $\mathfrak{S}(C)$ is contained in MCM. In particular, when $C = D$ is a dualizing module, $\mathfrak{S}(D)$ is the collection of thick subcategories of MCM containing D.*

Proof. Let $\mathcal{M}$ be a thick subcategory $\mathcal{G}_C$ containing C. It is clear from the definition of $\mathcal{A}$ that $\mathcal{M}$ contains $\mathcal{A}$. By Lemma 8.1, $\mathcal{M}$ is cogenerated by add C, thus also by $\mathcal{A}$. By Corollary 8.2, add $\mathcal{M}_p$ and add $\mathcal{A}_p$ are thick in $\mathcal{G}_{C_p}$ for all primes $p \in \mathrm{spec}\, R$. Therefore, dimension with respect to each of these subcategories satisfies the

Auslander–Buchsbaum formula. It follows from Lemma 2.14 that add $\mathcal{A}_p$ is thick in add $\mathcal{M}_p$

Now suppose that R is Cohen–Macaulay and $\mathcal{X} \in \mathfrak{S}(C)$. Since $\mathcal{A}$ cogenerates $\mathcal{X}$, for any $X \in \mathcal{X}$ there exists $0 \to X \to A_0 \to \cdots \to A_d \to X' \to 0$ with each $A_i \in \mathcal{A}$ and $d = \operatorname{depth} R$. Since $\mathcal{A} \subseteq \operatorname{MCM}$, X is in MCM. The last statement is now clear, since in that case $\mathcal{G}_D = \operatorname{MCM}$.   □

We now come to the main results of the paper.

Theorem 8.5. *Let $\mathcal{A}$ denote the smallest thick subcategory of $\mathcal{G}_C$ containing C. For any $\mathcal{M} \in \mathfrak{S}(C)$ (e.g., $\mathcal{M}$ is a thick subcategory of $\mathcal{G}_C$ containing C), $\Lambda_{\mathcal{M}}$ and $\Phi_{\mathcal{M}}$ give a bijection between $\mathfrak{R}(\mathcal{M})$ and Γ.*

Furthermore, the following is a bijection:

$$\Lambda : \mathfrak{S}(C) \times \Gamma \longrightarrow \bigcup_{\mathcal{M} \in \mathfrak{S}(C)} \mathfrak{R}(\mathcal{M}) \subseteq \mathfrak{R}$$

For any $\mathcal{M}, \mathcal{N} \in \mathfrak{S}(C)$ with $\mathcal{M} \subseteq \mathcal{N}$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathfrak{R}(\mathcal{N}) & & \\
\eta_{\mathcal{M}}^{\mathcal{N}} \uparrow & \searrow^{\Phi_{\mathcal{N}}} & \\
\mathfrak{R}(\mathcal{M}) & \xrightarrow{\;\Phi_{\mathcal{M}}\;} & \Gamma \\
\eta_{\mathcal{A}}^{\mathcal{M}} \uparrow & \nearrow_{\Phi_{\mathcal{A}}} & \\
\mathfrak{R}(\mathcal{A}) & &
\end{array}
$$

In particular, $\rho_{\mathcal{M}}^{\mathcal{N}}$ and $\eta_{\mathcal{M}}^{\mathcal{N}}$ are inverse functions.

Proof. Proposition 8.3 states that $\mathcal{A}$ is a thick subcategory of $\mathcal{G}_C$ which contains C and is closed under †. Therefore, by Theorem 7.4, $\Lambda_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$ give a bijection between $\mathfrak{R}(\mathcal{A})$ and Γ. The first statement is an application of Theorem 4.2 and Lemma 8.4. The rest follows from Theorem 4.4.   □

A resolving subcategory $\mathcal{X}$ is dominant if for every $p \in \operatorname{spec} R$, there is an $n \in \mathbb{N}$ such that $\Omega_{R_p}^n R_p / p R_p \in \operatorname{add} \mathcal{X}_p$.

Corollary 8.6. *Suppose R is Cohen–Macaulay and has a dualizing module. Then there is a bijection between resolving subcategories containing MCM and grade consistent functions. Furthermore, the following are equivalent for a resolving subcategory $\mathcal{X}$.*

(1) *$\mathcal{X}$ is dominant*

(2) *$\operatorname{MCM} \subseteq \mathcal{X}$*

(3) *$\Delta(\mathcal{X}) = \operatorname{mod}(R)$*

Proof. Letting D be the dualizing module of R, MCM is the same as $\mathcal{G}_D$. Hence, by the previous theorem, $\Lambda_{\mathrm{MCM}} : \Gamma \to \mathfrak{R}(\mathrm{MCM})$ is a bijection, showing the first statement. From [Dao and Takahashi 2015, Theorem 1.3], the following is a bijection:

$$\xi : \Gamma \to \{\text{Dominant resolving subcategories of } \mathrm{mod}(R)\}$$

$$f \mapsto \{X \in \mathrm{mod}(R) \mid \mathrm{depth}\, X\mathfrak{p} \geq \mathrm{ht}\, \mathfrak{p} - f(\mathfrak{p})\}$$

It is clear that $\xi(0) = \mathrm{MCM}$; hence, every dominant subcategory contains MCM. Furthermore, we have $\mathrm{mod}(R) = \Delta(\mathrm{MCM})$, and hence every dominant resolving subcategory is an element of $\mathfrak{R}(\mathrm{MCM})$. Then for any $f \in \Gamma$,

$$\xi(f) = \{X \in \mathrm{mod}(R) \mid \mathrm{depth}\, X_{\mathfrak{p}} \geq \mathrm{ht}\, \mathfrak{p} - f(\mathfrak{p})\}$$
$$= \{X \in \mathrm{mod}(R) \mid \mathrm{add}\, \mathrm{MCM}_{\mathfrak{p}}\text{-}\dim X_{\mathfrak{p}} \leq f(\mathfrak{p})\} = \Lambda_{\mathrm{MCM}}(f).$$

Thus ξ equals Λ_{MCM}, showing the equivalence of (1) and (2).

It is clear that (2) implies (3). Now assume (3) and take a $p \in \mathrm{spec}\, R$. Then $\mathcal{X}\text{-}\dim R/p < \infty$, and this implies that $\Omega^n R/p \in \mathcal{X}$ for some n. Therefore, $\Omega^n_{R_p} R_p/p R_p \in \mathrm{add}\, \mathcal{X}_p$, so $\mathcal{X}$ is dominant. $\qquad\square$

9. Gorenstein rings and vanishing of Ext

In this section, $(R, \mathfrak{m}, k)$ is a local Gorenstein ring. In this case, MCM is the same as $\mathcal{G}_R$, and Lemma 8.4 implies that $\mathfrak{S}(R)$ is merely the collection of thick subcategories of MCM. This gives us the following which recovers [Dao and Takahashi 2015, Theorem 7.4].

Theorem 9.1. *If R is Gorenstein*, *then we have the following commutative diagram of bijections*:

{Thick subcategories of MCM} $\times\ \Gamma$

$\Lambda_{\mathcal{P}}$ $\qquad\qquad$ Λ $\qquad\qquad$ $\{\mathcal{Z} \in \mathfrak{R} \mid \mathcal{Z} \cap MCM \text{ is thick in MCM}\}$

Ξ

{Thick subcategories of MCM} $\times\ \mathfrak{R}(\mathcal{P})$

where $\Xi(\mathcal{M}, \mathcal{X}) = \mathrm{res}(\mathcal{M} \cup \mathcal{X})$.

Proof. Let $\mathfrak{Z}$ be the collection of resolving subcategories whose intersection with MCM is thick in MCM. As observed before the Theorem, $\mathfrak{S}(R)$ is simply the thick subcategories of MCM. Since for any $\mathcal{M} \in \mathfrak{S}(R)$, $\Delta(\mathcal{M}) \cap \mathrm{MCM}$ is $\mathcal{M}$, the image of Λ lies in $\mathfrak{Z}$. Furthermore, for any $\mathcal{Z} \in \mathfrak{Z}$, $\mathcal{Z}$ is in $\mathfrak{R}(\mathcal{Z} \cap \mathrm{MCM})$, thus the result follows from Proposition 4.6 and Theorem 8.5. $\qquad\square$

It is natural to ask when the image Λ is all of $\mathfrak{R}$. This happens precisely when every resolving subcategory of MCM is thick. This occurs, by [op. cit., Theorem 6.4], when R is a complete intersection. We will give a necessary condition for Im $\Lambda = \mathfrak{R}$ by examining the resolving subcategories of the form

$$\mathcal{M}_{\mathcal{B}} = \{M \in \text{mod}(R) \mid \text{Ext}^{>0}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$$

where $\mathcal{B} \subseteq \text{mod}(R)$. Dimension with respect to this category can be calculated in the following manner.

Lemma 9.2. *For all* $\mathcal{B} \subseteq \text{mod}(R)$,

$$\mathcal{M}_{\mathcal{B}}\text{-dim } M = \inf\{n \mid \text{Ext}^{>n}(M, B) = 0 \text{ for all } B \in \mathcal{B}\}$$

Proof. Let $M \in \text{mod}(R)$. For all $i > 0$ and $j \geq 0$ and each $B \in \mathcal{B}$, we have $\text{Ext}^{i+j}(M, B) = \text{Ext}^i(\Omega^j M, B)$. So $\text{Ext}^{i+n}(M, B) = 0$ for all $i \geq 0$ if and only if $\Omega^n M$ is in $\mathcal{M}_{\mathcal{B}}$. $\qquad\square$

Lemma 9.3. *For any* $\mathcal{B} \subseteq \text{mod}(R)$, *we have* $\mathcal{M}_{\mathcal{B}} \cap \Delta(\mathcal{P}) = \mathcal{P}$.

Proof. To prove this, it suffices to show that if $\text{pd}(X) = n > 0$, then $\text{Ext}^n(X, B) \neq 0$. Take a minimal free resolution

$$0 \to F_n \xrightarrow{d} F_{n-1} \to \cdots \to F_0 \to X \to 0.$$

Note that $\text{Im}(d) \subseteq \mathfrak{m}F_{n-1}$. We then get the complex

$$0 \to \text{Hom}(X, B) \to \text{Hom}(F_0, B) \to \cdots \to \text{Hom}(F_{n-1}, B) \xrightarrow{d^*} \text{Hom}(F_n, B) \to 0.$$

Now $\text{Im}(d^*)$ still lies in $\mathfrak{m}\,\text{Hom}(F_n, B)$, and thus by Nakayama, d^* cannot be surjective. Hence we have $\text{Ext}^n(X, B) = \text{coker } d^* \neq 0$. $\qquad\square$

Araya [2012] defined AB dimension by AB-dim $M = \max\{b_M, \mathcal{G}_R\text{-dim } M\}$, where

$$b_M = \min\{n \mid \text{Ext}^{\gg 0}(M, B) = 0 \Rightarrow \text{Ext}^{>n}(M, B) = 0\}.$$

Note that AB dimension satisfies the Auslander–Buchsbaum formula. Also, a ring is AB if and only if every module has finite AB dimension.

Lemma 9.4. *Taking* $\mathcal{B} \subseteq \text{mod}(R)$, *if AB-dim* $M < \infty$ *for all* $M \in \Delta(\mathcal{M}_{\mathcal{B}})$, *then* $\mathcal{M}_{\mathcal{B}}$ *is a thick subcategory of MCM.*

Proof. Suppose AB-dim $\Delta(\mathcal{M}_{\mathcal{B}}) < \infty$. First, we show that $\mathcal{M}_{\mathcal{B}}$ is contained in MCM. Take any $M \in \mathcal{M}_{\mathcal{B}}$. There is an exact sequence $0 \to M \to Y \to X \to 0$ with $\text{pd}(Y) < \infty$ and $X \in$ MCM. We claim that X has AB dimension zero. Suppose $\text{Ext}^{\gg 0}(X, Z) = 0$. Then $\text{Ext}^{\gg 0}(Y, Z) = 0$ and since $\text{pd } Y = $ AB-dim Y, $\text{Ext}^{>\text{pd } Y}(Y, Z)$ is zero. Then we have $\text{Ext}^{\gg 0}(M, Z) = 0$ and thus $\text{Ext}^{>b_M}(M, Z) = 0$. Therefore $\text{Ext}^i(X, Z) = 0$ for all $i > \max\{\text{pd}(Y), b_M\} + 1$. Since R is Gorenstein, that means that X has finite $\mathcal{G}_R$ dimension, and thus X has finite AB dimension.

But since AB dimension satisfies the Auslander Buchsbaum formula, AB-dim X must be zero.

Since $Y \in \Delta(\mathcal{M}_\mathcal{B})$, we have $X \in \Delta(\mathcal{M}_\mathcal{B})$. So $\mathrm{Ext}^{\gg 0}(X, B) = 0$ for all $B \in \mathcal{B}$, and we have $\mathrm{Ext}^{>0}(X, B) = 0$ for all $B \in \mathcal{B}$. Hence X is in $\mathcal{M}_\mathcal{B}$. Therefore, Y is also in $\mathcal{M}_\mathcal{B}$, which, by Lemma 9.3, means that Y is projective and hence in MCM, forcing M to be in MCM as well.

Now to show that $\mathcal{M}_\mathcal{B}$ is thick in MCM, it suffices to show that $\mathcal{M}_\mathcal{B}$ is closed under cokernels of surjections in MCM. So take $0 \to L \to M \to N \to 0$ with $L, M, N \in \mathrm{MCM}$ and $L, M \in \mathcal{M}_\mathcal{B}$. Then $N \in \Delta(\mathcal{M}_\mathcal{B})$ and so $\mathrm{Ext}^{\gg 0}(N, B) = 0$ for all $B \in \mathcal{B}$. But then N has finite AB dimension by assumption. Since AB dimension satisfies the Auslander–Buchsbaum formula, AB-dim N is zero. So we have $\mathrm{Ext}^{>0}(N, B) = 0$ for all $B \in \mathcal{B}$, and hence, N is in $\mathcal{M}_\mathcal{B}$. $\qquad\square$

Now let $d = \dim R$.

Theorem 9.5. *If R is Gorenstein, then the following are equivalent.*

(1) *R is AB.*

(2) *$\mathcal{M}_\mathcal{B}$ is a thick subcategory of MCM for all $\mathcal{B} \subseteq \mathrm{mod}(R)$.*

(3) *$\mathrm{MCM} \cap \mathcal{M}_\mathcal{B}$ is thick in MCM for every $\mathcal{B} \subseteq \mathrm{mod}(R)$.*

(4) *$\Lambda_{\mathcal{M}_\mathcal{B}}$ gives a bijection between $\mathfrak{R}(\mathcal{M}_\mathcal{B})$ and Γ for every $\mathcal{B} \subseteq \mathrm{mod}(R)$.*

(5) *For all $\mathcal{B} \subseteq \mathrm{mod}(R)$ and $M \in \mathcal{M}_\mathcal{B}$, Γ contains the function $f : \mathrm{spec}\, R \to \mathbb{N}$ defined by*

$$f(p) = \min\{n \mid \mathrm{Ext}^{>n}(M_p, B_p) = 0 \text{ for all } B \in \mathcal{B}\}.$$

Proof. The previous lemma shows that (1) implies (2), and (2) implies (3) is trivial. Assuming (3), we will show (1). Suppose $\mathrm{Ext}^{\gg 0}(M, B) = 0$. Then M is in $\Delta(\mathcal{M}_\mathcal{B})$. Letting $\dim R = d$, we have $\Omega^d M \in \Delta(\mathcal{M}_\mathcal{B}) \cap \mathrm{MCM}$. For some $n \geq d$ we have $\Omega^n M \in \mathcal{M}_\mathcal{B} \cap \mathrm{MCM}$. But then we have

$$0 \to \Omega^n M \to F_{n-1} \to \cdots \to F_d \to \Omega^d M \to 0,$$

where each F_i is projective. By (3), $\Omega^d M$ is in $\mathcal{M}_\mathcal{B}$. So we have $\mathcal{M}_B$-dim $M \leq d$, and so $\mathrm{Ext}^{>d}(M, B) = 0$.

Theorem 4.2 shows that (2) implies (4). Lemma 9.2 shows that (4) implies (5). Since R is local, evaluating f at the maximal ideal shows that (5) implies (1). $\quad\square$

Corollary 9.6. *Set $r = d - \mathrm{depth}\, M$. If R is AB and $\mathrm{Ext}^{\gg 0}(M, B) = 0$, then $\mathrm{Ext}^r(M, B) \neq 0$. Furthermore, if $\mathrm{Ext}^r(M, B) = 0$ or $\mathrm{Ext}^i(M, B) \neq 0$ for $i > r$, then $\mathrm{Ext}^j(M, B) \neq 0$ for arbitrarily large j.*

Proof. Suppose R is AB. Then (2) holds and so $\mathcal{M}_B$-dim satisfies the Auslander–Buchsbaum formula. If $\mathrm{Ext}^{\gg 0}(M, B) = 0$ then

$$r = \mathcal{M}_B\text{-dim}\, M = \max\{n \mid \mathrm{Ext}^n(M, B) \neq 0\}.$$

The second statement is just the contrapositive of the first statement. $\quad\square$

Corollary 9.7. *If R is Gorenstein and every resolving subcategory of MCM is thick, then R is AB.*

Proof. The assumption implies (2) in Theorem 9.5. $\quad\square$

Thus if Λ in Theorem 8.5 is a bijection from $\mathfrak{S}(R) \times \Gamma$ to $\mathfrak{R}$, then R is AB. Stevenson [2014a] shows that when R is a complete intersection, every resolving subcategory of MCM is closed under duals. The following gives a necessary condition for this property.

Corollary 9.8. *If R is Gorenstein and every resolving subcategory of MCM is closed under duals, then R is AB.*

Proof. Suppose every resolving subcategory of MCM is closed under duals. Let $\mathcal{M} \subseteq \mathrm{MCM}$ be resolving. Let $-^* = \mathrm{Hom}(-, R)$. Then for every $M \in \mathcal{X}$, $(\Omega M^*)^*$ is in $\mathcal{M}$. By Lemma 8.1, $\mathcal{M}$ is thick. The result follows from the previous corollary.
$\quad\square$

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