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# INCLUSION OF CONFIGURATION SPACES IN CARTESIAN PRODUCTS, AND THE VIRTUAL COHOMOLOGICAL DIMENSION OF THE BRAID GROUPS OF $\mathbb{S}^2$ AND $\mathbb{R}P^2$

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**Let $S$ be a surface, perhaps with boundary, and either compact or with a finite number of points removed from the interior of the surface. We consider the inclusion $\iota : F_n(S) \to \prod_1^n S$ of the $n$-th configuration space $F_n(S)$ of $S$ into the $n$-fold Cartesian product of $S$, as well as the induced homomorphism $\iota_\# : P_n(S) \to \prod_1^n \pi_1(S)$, where $P_n(S)$ is the $n$-string pure braid group of $S$. Both $\iota$ and $\iota_\#$ were studied initially by J. Birman, who conjectured that $\mathrm{Ker}(\iota_\#)$ is equal to the normal closure of the Artin pure braid group $P_n$ in $P_n(S)$. The conjecture was later proved by C. Goldberg for compact surfaces without boundary different from the 2-sphere $\mathbb{S}^2$ and the projective plane $\mathbb{R}P^2$. In this paper, we prove the conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$. In the case of $\mathbb{R}P^2$, we prove that $\mathrm{Ker}(\iota_\#)$ is equal to the commutator subgroup of $P_n(\mathbb{R}P^2)$, we show that it may be decomposed in a manner similar to that of $P_n(\mathbb{S}^2)$ as a direct sum of a torsion-free subgroup $L_n$ and the finite cyclic group generated by the full twist braid, and we prove that $L_n$ may be written as an iterated semidirect product of free groups. Finally, we show that the groups $B_n(\mathbb{S}^2)$ and $P_n(\mathbb{S}^2)$ (resp. $B_n(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$) have finite virtual cohomological dimension equal to $n-3$ (resp. $n-2$), where $B_n(S)$ denotes the full $n$-string braid group of $S$. This allows us to determine the virtual cohomological dimension of the mapping class groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$ with marked points, which in the case of $\mathbb{S}^2$ reproves a result due to J. Harer.**

## 1. Introduction

Let $S$ be a connected surface, perhaps with boundary, and either compact or with a finite number of points removed from the interior of the surface. The *n-th configuration space* of $S$ is defined by

$$F_n(S) = \{(x_1, \dots, x_n) \in S^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

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It is well known that $\pi_1(F_n(S)) \cong P_n(S)$, the *pure braid group* of $S$ on $n$ strings, and that $\pi_1(F_n(S)/S_n) \cong B_n(S)$, the *braid group* of $S$ on $n$ strings, where $F_n(S)/S_n$ is the quotient space of $F_n(S)$ by the free action of the symmetric group $S_n$ given by permuting coordinates [Fadell and Neuwirth 1962; Fox and Neuwirth 1962]. We compose elements of $B_n(S)$ from left to right. If $S$ is the 2-disc $\mathbb{D}^2$ then $B_n(\mathbb{D}^2)$ (resp. $P_n(\mathbb{D}^2)$) is the Artin braid group $B_n$ (resp. the Artin pure braid group $P_n$). The canonical projection $F_n(S) \to F_n(S)/S_n$ is a regular $n!$-fold covering map, and thus gives rise to the short exact sequence

$$1 \to P_n(S) \to B_n(S) \to S_n \to 1. \tag{1}$$

If $\mathbb{D}^2$ is a topological disc lying in the interior of $S$ and containing the basepoints of the braids then the inclusion $j : \mathbb{D}^2 \to S$ induces a group homomorphism $j_\# : B_n \to B_n(S)$. This homomorphism is injective if $S$ is different from the 2-sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{R}P^2$ [Birman 1969; Goldberg 1973]. Let $j_\#|_{P_n} : P_n \to P_n(S)$ denote the restriction of $j_\#$ to the corresponding pure braid groups. If $\beta \in B_n$ then we shall denote its image $j_\#(\beta)$ in $B_n(S)$ simply by $\beta$. It is well known that the centre of $B_n$ and of $P_n$ is infinite cyclic, generated by the full twist braid that we denote by $\Delta_n^2$, and that $\Delta_n^2$, considered as an element of $B_n(\mathbb{S}^2)$ or of $B_n(\mathbb{R}P^2)$, is of order 2 and generates the centre. If $G$ is a group then we denote its commutator subgroup by $\Gamma_2(G)$ and its Abelianisation by $G^{\mathrm{Ab}}$, and if $H$ is a subgroup of $G$ then we denote its normal closure in $G$ by $\langle\langle H \rangle\rangle_G$.

Let $\prod_1^n S = S \times \cdots \times S$ denote the $n$-fold Cartesian product of $S$ with itself, let $\iota_n : F_n(S) \to \prod_1^n S$ be the inclusion map, and let

$$\iota_{n\#} : \pi_1(F_n(S)) \to \pi_1\left(\prod_1^n S\right)$$

denote the induced homomorphism on the level of fundamental groups. To simplify the notation, we shall often just write $\iota$ and $\iota_\#$ if $n$ is given. The study of $\iota_\#$ was initiated by Birman [1969]. She had conjectured that $\langle\langle \mathrm{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} = \mathrm{Ker}(\iota_\#)$ if $S$ is a compact orientable surface, but states without proof that her conjecture is false if $S$ is of genus greater than or equal to 1 [Birman 1969, page 45]. However, Goldberg [1973, Theorem 1] proved the conjecture several years later in both the orientable and nonorientable cases for compact surfaces without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$. In connection with the study of Vassiliev invariants of surface braid groups, González-Meneses and Paris [2004] showed that $\mathrm{Ker}(\iota_\#)$ is also normal in $B_n(S)$, and that the resulting quotient is isomorphic to the semidirect product $\pi_1\left(\prod_1^n S\right) \rtimes S_n$, where the action is given by permuting coordinates (their work was within the framework of compact orientable surfaces without boundary, but their construction is valid for any surface $S$). In the case of $\mathbb{R}P^2$, this result was reproved using geometric methods [Tochimani 2011].

If $S = \mathbb{S}^2$, then $\operatorname{Ker}(\iota_\#)$ is clearly equal to $P_n(\mathbb{S}^2)$, and so by [Gonçalves and Guaschi 2004b, Theorem 4], it may be decomposed as

$$\operatorname{Ker}(\iota_\#) = P_n(\mathbb{S}^2) \cong P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\}) \times \mathbb{Z}_2, \tag{2}$$

where the first factor of the direct product is torsion-free, and the $\mathbb{Z}_2$-factor is generated by $\Delta_n^2$.

The aim of this paper is to resolve Birman's conjecture for surfaces without boundary in the remaining cases, namely $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, to determine the cohomological dimension of $B_n(S)$ and $P_n(S)$, where $S$ is one of these two surfaces, and to elucidate the structure of $\operatorname{Ker}(\iota_\#)$ in the case of $\mathbb{R}P^2$. In Section 2, we start by considering the case $S = \mathbb{R}P^2$, we study $\operatorname{Ker}(\iota_\#)$, which we denote by $K_n$, and we show that it admits a decomposition similar to that of (2).

**Proposition 1.** *Let $n \in \mathbb{N}$.*

(a) (i) *Up to isomorphism, the homomorphism*

$$\iota_\# : \pi_1(F_n(\mathbb{R}P^2)) \to \pi_1\Bigl(\prod_1^n \mathbb{R}P^2\Bigr)$$

*coincides with Abelianisation. In particular, $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$.*

(ii) *If $n \geq 2$ then there exists a torsion-free subgroup $L_n$ of $K_n$ such that $K_n$ is isomorphic to the direct sum of $L_n$ and the subgroup $\langle \Delta_n^2 \rangle$ generated by the full twist that is isomorphic to $\mathbb{Z}_2$.*

(b) *If $n \geq 2$ then any subgroup of $P_n(\mathbb{R}P^2)$ that is normal in $B_n(\mathbb{R}P^2)$ and that properly contains $K_n$ possesses an element of order* 4.

Note that if $n = 1$ then $B_1(\mathbb{R}P^2) = P_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ and $\Delta_1^2$ is the trivial element, so parts (a)(ii) and (b) do not hold. Part (a)(i) will be proved in Proposition 8. We shall see later on in Remark 14 that there are precisely $2^{n(n-2)}$ subgroups that satisfy the conclusions of part (a)(ii), and to prove the statement, we shall exhibit an explicit torsion-free subgroup $L_n$. We then prove Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$, using Proposition 1(a)(i) in the case of $\mathbb{R}P^2$.

**Theorem 2.** *Let $S$ be $\mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geq 1$. Then* $\langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} = \operatorname{Ker}(\iota_\#)$.

In Section 3, we analyse $L_n$ in more detail, and we show that it may be decomposed as an iterated semidirect product of free groups.

**Theorem 3.** *Let $n \geq 3$. Consider the Fadell–Neuwirth short exact sequence*

$$1 \to P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_n(\mathbb{R}P^2) \xrightarrow{q_{2\#}} P_2(\mathbb{R}P^2) \to 1, \tag{3}$$

*where $q_{2\#}$ is given geometrically by forgetting the last $n-2$ strings. Then $L_n$ may be identified with the kernel of the composition*

$$P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_n(\mathbb{R}P^2) \xrightarrow{\iota_\#} \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}},$$

*where the first homomorphism is that appearing in* (3). *The image of this composition is the product of the last $n-2$ copies of $\mathbb{Z}_2$. In particular, $L_n$ is of index $2^{n-2}$ in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Further, $L_n$ is isomorphic to an iterated semidirect product of free groups of the form $\mathbb{F}_{2n-3} \rtimes (\mathbb{F}_{2n-5} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3)\cdots))$, where for all $m \in \mathbb{N}$, $\mathbb{F}_m$ denotes the free group of rank $m$.*

In the semidirect product decomposition of $L_n$, note that every factor acts on each of the preceding factors. This is also the case for $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ (see (13)), and as we shall see in Remark 13(a), this implies an Artin combing-type result for this group. Analysing these semidirect products in more detail, we obtain the following results.

**Proposition 4.** *If $n \geq 3$ then*

(a) $\left(P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})\right)^{\mathrm{Ab}} \cong \mathbb{Z}^{2(n-2)}$,

(b) $(L_n)^{\mathrm{Ab}} \cong \mathbb{Z}^{n(n-2)}$.

In two papers in preparation, we shall analyse the homotopy fibre of $\iota$, as well as the induced homomorphism $\iota_\#$ when $S = \mathbb{S}^2$ or $\mathbb{R}P^2$ [Gonçalves and Guaschi $\geq$ 2017], and when $S$ is a space form manifold of dimension different from two [Golasiński et al. 2016]. In the first of these papers, we shall also see that $L_n$ is closely related to the fundamental group of an orbit configuration space of the open cylinder.

In Section 4, we study the virtual cohomological dimension of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. Recall from [Brown 1982, page 226] that if a group $\Gamma$ is virtually torsion-free then all finite index torsion-free subgroups of $\Gamma$ have the same cohomological dimension by Serre's theorem, and this dimension is defined to be the *virtual cohomological dimension* of $\Gamma$. Using (2) and (3), we prove the following result, namely that if $S = \mathbb{S}^2$ or $\mathbb{R}P^2$, the groups $B_n(S)$ and $P_n(S)$ have finite virtual cohomological dimension, and we compute these dimensions.

**Theorem 5.** (a) *Let $n \geq 4$. Then the virtual cohomological dimension of both $B_n(\mathbb{S}^2)$ and $P_n(\mathbb{S}^2)$ is equal to the cohomological dimension of the group $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. Furthermore, for all $m \geq 1$, the cohomological dimension of the group $P_m(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$ is equal to $m$.*

(b) *Let $n \geq 3$. Then the virtual cohomological dimension of both $B_n(\mathbb{R}P^2)$ and $P_n(\mathbb{R}P^2)$ is equal to the cohomological dimension of the group $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Furthermore, for all $m \geq 1$, the cohomological dimension of the group $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ is equal to $m$.*

The methods of the proof of Theorem 5 have recently been applied to compute the cohomological dimension of the braid groups of all other compact surfaces (orientable and nonorientable) without boundary [Gonçalves et al. 2016]. Theorem 5 also allows us to deduce the virtual cohomological dimension of the punctured mapping class groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$. If $n \geq 0$, let $\mathcal{MCG}(S, n)$ denote the mapping class group of a connected, compact surface $S$ relative to an $n$-point set. If $S$ is orientable then Harer [1986, Theorem 4.1] determined the virtual cohomological dimension of $\mathcal{MCG}(S, n)$. In the case of $\mathbb{S}^2$ and $\mathbb{D}^2$, he obtained the following results:

(a) If $n \geq 3$, the virtual cohomological dimension of $\mathcal{MCG}(\mathbb{S}^2, n)$ is equal to $n-3$.

(b) If $n \geq 2$, the cohomological dimension of $\mathcal{MCG}(\mathbb{D}^2, n)$ is equal to $n-1$ (recall that $\mathcal{MCG}(\mathbb{D}^2, n)$ is isomorphic to $B_n$ [Birman 1974]).

As a consequence of Theorem 5, we are able to compute the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ for $S = \mathbb{S}^2$ and $\mathbb{R}P^2$.

**Corollary 6.** *Let $n \geq 4$ (resp. $n \geq 3$). Then the virtual cohomological dimension of $\mathcal{MCG}(\mathbb{S}^2, n)$ (resp. $\mathcal{MCG}(\mathbb{R}P^2, n)$) is finite and is equal to $n-3$ (resp. $n-2$).*

If $S = \mathbb{S}^2$ or $\mathbb{R}P^2$ then for the values of $n$ given by Theorem 5 and Corollary 6, the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ is equal to that of $B_n(S)$. If $S = \mathbb{S}^2$, we thus recover the corresponding result of Harer.

## 2. The structure of $K_n$, and Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$

Let $n \in \mathbb{N}$. As we mentioned in the Introduction, if $S$ is a surface different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, the kernel of the homomorphism $\iota_\# : P_n(S) \to \pi_1\bigl(\prod_1^n S\bigr)$ was studied in [Birman 1969; Goldberg 1973], and if $S = \mathbb{S}^2$ then $\mathrm{Ker}(\iota_\#) = P_n(\mathbb{S}^2)$. In the first part of this section, we recall a presentation of $P_n(\mathbb{R}P^2)$, and we prove Proposition 1(a)(i). The second part of this section is devoted to proving the rest of Proposition 1 and Theorem 2, the latter being Birman's conjecture for $\mathbb{S}^2$ and $\mathbb{R}P^2$.

Consider the model of $\mathbb{R}P^2$ given by identifying antipodal boundary points of $\mathbb{D}^2$. We equip $F_n(\mathbb{R}P^2)$ with a basepoint $(x_1, \dots, x_n)$. For $1 \leq i < j \leq n$ (resp. $1 \leq k \leq n$), we define the element $A_{i,j}$ (resp. $\tau_k$, $\rho_k$) of $P_n(\mathbb{R}P^2)$ by the geometric braids depicted on the left side of Figure 1. Note that the arcs represent the projections of the strings onto $\mathbb{R}P^2$, so that all of the strings of the given braid are vertical, with the exception of the $j$-th (resp. $k$-th) string that is based at the point $x_j$ (resp. $x_k$). As may be seen on the right side of Figure 1, the generator $A_{i,j}$ may also be represented by a loop based at the point $x_i$.

**Theorem 7** [Gonçalves and Guaschi 2007, Theorem 4]. *Let $n \in \mathbb{N}$. The following constitutes a presentation of the pure braid group $P_n(\mathbb{R}P^2)$:*

***Generators***: $A_{i,j}$, $1 \leq i < j \leq n$, *and* $\tau_k$, $1 \leq k \leq n$.

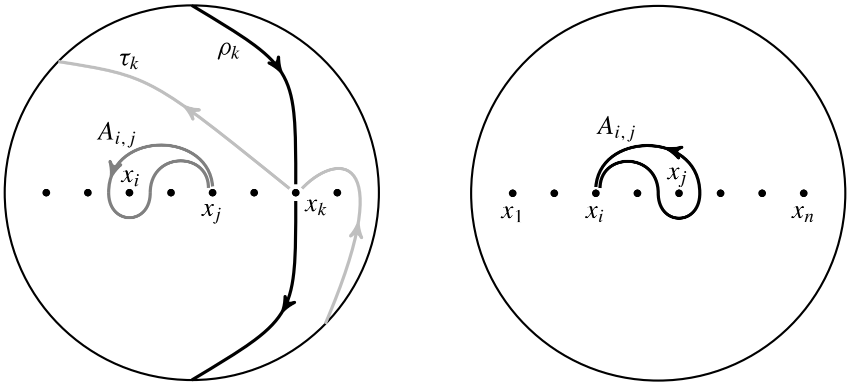


**Figure 1.** The elements $A_{i,j}$, $\tau_k$ and $\rho_k$ of $P_n(\mathbb{R}P^2)$.

***Relations***:

(a) *The Artin relations between the $A_{i,j}$ emanating from those of $P_n$*:

(4) $A_{r,s}A_{i,j}A_{r,s}^{-1} =$

$$\begin{cases} A_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j,\\ A_{i,j}^{-1}A_{r,j}^{-1}A_{i,j}A_{r,j}A_{i,j} & \text{if } r<i=s<j,\\ A_{s,j}^{-1}A_{i,j}A_{s,j} & \text{if } i=r<s<j,\\ A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{i,j}A_{r,j}^{-1}A_{s,j}^{-1}A_{r,j}A_{s,j} & \text{if } r<i<s<j. \end{cases}$$

(b) *For all* $1 \le i < j \le n$, $\tau_i\tau_j\tau_i^{-1} = \tau_j^{-1}A_{i,j}^{-1}\tau_j^2$.

(c) *For all* $1 \le i \le n$, $\tau_i^2 = A_{1,i}\cdots A_{i-1,i}A_{i,i+1}\cdots A_{i,n}$.

(d) *For all* $1 \le i < j \le n$ *and* $1 \le k \le n$ *with* $k \ne j$,

$$\tau_k A_{i,j}\tau_k^{-1} = \begin{cases} A_{i,j} & \text{if } j<k \text{ or } k<i,\\ \tau_j^{-1}A_{i,j}^{-1}\tau_j & \text{if } k=i,\\ \tau_j^{-1}A_{k,j}^{-1}\tau_j A_{k,j}^{-1}A_{i,j}A_{k,j}\tau_j^{-1}A_{k,j}\tau_j & \text{if } i<k<j. \end{cases}$$

This enables us to prove that $\iota_\#$ is in fact Abelianisation, which is part (a)(i) of Proposition 1.

**Proposition 8.** *Let $n \in \mathbb{N}$. The homomorphism $\iota_\# : P_n(\mathbb{R}P^2) \to \pi_1\left(\prod_1^n \mathbb{R}P^2\right)$ is defined on the generators of Theorem 7 by $\iota_\#(A_{i,j}) = (\overline{0}, \ldots, \overline{0})$ for all $1 \le i < j \le n$, and $\iota_\#(\tau_k) = (\overline{0}, \ldots, \overline{0}, \overline{1}, \overline{0}, \ldots, \overline{0})$, where $\overline{1}$ is in the $k$-th position, for all $1 \le k \le n$. Further, $\iota_\#$ is Abelianisation, and* $\operatorname{Ker}(\iota_\#) = K_n = \Gamma_2(P_n(\mathbb{R}P^2))$.

*Proof.* For $1 \le k \le n$, let $p_k : F_n(\mathbb{R}P^2) \to \mathbb{R}P^2$ denote projection onto the $k$-th coordinate. Observe that $\iota_\# = p_{1\#} \times \cdots \times p_{n\#}$, where $p_{k\#} : P_n(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$ is the induced homomorphism on the level of fundamental groups. Identifying

$\pi_1(\mathbb{R}P^2)$ with $\mathbb{Z}_2$ and using the geometric realisation of Figure 1 of the generators of the presentation of $P_n(\mathbb{R}P^2)$ given by Theorem 7, it is straightforward to check that for all $1 \le k, l \le n$ and $1 \le i < j \le n$, we have $p_{k\#}(A_{i,j}) = \bar{0}$, $p_{k\#}(\tau_l) = \bar{0}$ if $l \neq k$ and $p_{k\#}(\tau_k) = \bar{1}$, and this yields the first part of the proposition. The second part follows easily from the presentation of the Abelianisation $(P_n(\mathbb{R}P^2))^{\mathrm{Ab}}$ of $P_n(\mathbb{R}P^2)$ obtained from Theorem 7. More precisely, if we denote the Abelianisation of an element $x \in P_n(\mathbb{R}P^2)$ by $\bar{x}$, relations (b) and (c) imply respectively that for all $1 \le i < j \le n$ and $1 \le k \le n$, $\overline{A_{i,j}}$ and $\overline{\tau_k}^2$ represent the trivial element of $(P_n(\mathbb{R}P^2))^{\mathrm{Ab}}$. Since the remaining relations give no other information under Abelianisation, it follows that $(P_n(\mathbb{R}P^2))^{\mathrm{Ab}} \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, where $\overline{\tau_k} = (\bar{0}, \dots, \bar{0}, \bar{1}, \bar{0}, \dots, \bar{0})$ with $\bar{1}$ in the $k$-th position and $\overline{A_{i,j}} = (\bar{0}, \dots, \bar{0})$ via this isomorphism, and the Abelianisation homomorphism indeed coincides with $\iota_\#$ on $P_n(\mathbb{R}P^2)$. □

**Remark 9.** (a) Since $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$, it follows immediately that $K_n$ is normal in $B_n(\mathbb{R}P^2)$, since $\Gamma_2(P_n(\mathbb{R}P^2))$ is characteristic in $P_n(\mathbb{R}P^2)$, and $P_n(\mathbb{R}P^2)$ is normal in $B_n(\mathbb{R}P^2)$.

(b) A presentation of $K_n$ may be obtained by a long but routine computation using the Reidemeister–Schreier method, although it is not clear how to simplify the presentation. In Theorem 3, we will provide an alternative description of $K_n$ using algebraic methods.

(c) In what follows, we shall use Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$ [1966, page 83], whose generating set consists of the standard braid generators $\sigma_1, \dots, \sigma_{n-1}$ emanating from the 2-disc, as well as the surface generators $\rho_1, \dots, \rho_n$ depicted in Figure 1. We have the following relation between the elements $\tau_k$ and $\rho_k$:

$$\tau_k = \rho_k^{-1} A_{k,k+1} \cdots A_{k,n} \quad \text{for all } 1 \le k \le n,$$

where for $1 \le i < j \le n$, $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$. In particular, it follows from Proposition 8 that

$$\iota_\#(\rho_k) = \iota_\#(\tau_k) = (\bar{0}, \dots, \bar{0}, \bar{1}, \bar{0}, \dots, \bar{0}) \quad \text{for all } 1 \le k \le n, \tag{5}$$

where $\bar{1}$ is in the $k$-th position.

If $n \ge 2$, the full twist braid $\Delta_n^2$, which may be defined by $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$, is of order 2 [Van Buskirk 1966, page 95], it generates the centre of $B_n(\mathbb{R}P^2)$ [Murasugi 1982, Proposition 6.1], and it is the unique element of $B_n(\mathbb{R}P^2)$ of order 2 [Gonçalves and Guaschi 2004a, Proposition 23]. Since $\Delta_n^2 \in P_n(\mathbb{R}P^2)$, it thus belongs to the centre of $P_n(\mathbb{R}P^2)$, and just as for the Artin braid groups and the braid groups of $\mathbb{S}^2$, it generates the centre of $P_n(\mathbb{R}P^2)$:

**Proposition 10.** *Let $n \ge 2$. Then the centre $Z(P_n(\mathbb{R}P^2))$ of $P_n(\mathbb{R}P^2)$ is generated by $\Delta_n^2$.*

*Proof.* We prove the result by induction on $n$. If $n = 2$ then $P_2(\mathbb{R}P^2) \cong \mathcal{Q}_8$ [Van Buskirk 1966, page 87], the quaternion group of order 8, and the result follows since $\Delta_2^2$ is the element of $P_2(\mathbb{R}P^2)$ of order 2. So suppose that $n \geq 3$. From the preceding remarks, $\langle \Delta_n^2 \rangle \subset Z(P_n(\mathbb{R}P^2))$. Conversely, let $x \in Z(P_n(\mathbb{R}P^2))$, and consider the Fadell–Neuwirth short exact sequence

$$1 \to \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_{n-1}\}) \to P_n(\mathbb{R}P^2) \xrightarrow{q_{(n-1)\#}} P_{n-1}(\mathbb{R}P^2) \to 1,$$

where $q_{(n-1)\#}$ is the surjective homomorphism induced on the level of fundamental groups by the projection $q_{n-1} : F_n(\mathbb{R}P^2) \to F_{n-1}(\mathbb{R}P^2)$ onto the first $n-1$ coordinates. Now $q_{(n-1)\#}(x) \in Z(P_{n-1}(\mathbb{R}P^2))$ by surjectivity, and thus $q_{(n-1)\#}(x) = \Delta_{n-1}^{2\varepsilon}$ for some $\varepsilon \in \{0, 1\}$ by the induction hypothesis. Further, $q_{(n-1)\#}(\Delta_n^2) = \Delta_{n-1}^2$, hence

$$\Delta_n^{-2\varepsilon} x \in \operatorname{Ker}(q_{(n-1)\#}) \cap Z(P_n(\mathbb{R}P^2)),$$

and therefore $\Delta_n^{-2\varepsilon} x \in Z(\operatorname{Ker}(q_{(n-1)\#}))$. But $Z(\operatorname{Ker}(q_{(n-1)\#}))$ is trivial because $\operatorname{Ker}(q_{(n-1)\#})$ is a free group of rank $n-1$. This implies that $x \in \langle \Delta_n^2 \rangle$ as required. □

*Proof of Proposition 1.* Let $n \geq 3$.

(a) Recall that part (a)(i) of Proposition 1 was proved in Proposition 8, so let us prove part (ii). The projection $q_2 : F_n(\mathbb{R}P^2) \to F_2(\mathbb{R}P^2)$ onto the first two coordinates gives rise to the Fadell–Neuwirth short exact sequence (3). Since $K_n = \Gamma_2(P_n(\mathbb{R}P^2))$ by Proposition 8, the image of the restriction $q_{2\#}|_{K_n}$ of $q_{2\#}$ to $K_n$ is the subgroup $\Gamma_2(P_2(\mathbb{R}P^2)) = \langle \Delta_2^2 \rangle$, and so we obtain the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K_n \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] \arrow[d] & K_n \arrow[r, "q_{2\#}|_{K_n}"] \arrow[d] & \langle \Delta_2^2 \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] & P_n(\mathbb{R}P^2) \arrow[r, "q_{2\#}"] & P_2(\mathbb{R}P^2) \arrow[r] & 1,
\end{tikzcd} \tag{6}$$

where the vertical arrows are inclusions. Now $\langle \Delta_2^2 \rangle \cong \mathbb{Z}_2$, so $K_n$ is an extension of the group $\operatorname{Ker}(q_{2\#}|_{K_n}) = K_n \cap P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ by $\mathbb{Z}_2$. The fact that $q_{2\#}(\Delta_n^2) = \Delta_2^2$ implies that the upper short exact sequence splits, a section being defined by the correspondence $\Delta_2^2 \mapsto \Delta_n^2$, and since $\Delta_n^2 \in Z(P_n(\mathbb{R}P^2))$, the action by conjugation on $\operatorname{Ker}(q_{2\#}|_{K_n})$ is trivial. Part (a) of the proposition follows by taking $L_n = \operatorname{Ker}(q_{2\#}|_{K_n})$ and by noting that $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ is torsion-free.

(b) Recall first that any torsion element in $P_n(\mathbb{R}P^2) \setminus \langle \Delta_n^2 \rangle$ is of order 4 [Gonçalves and Guaschi 2004a, Corollary 19 and Proposition 23], and is conjugate in $B_n(\mathbb{R}P^2)$ to one of $a^n$ or $b^{n-1}$, where $a = \rho_n \sigma_{n-1} \cdots \sigma_1$ and $b = \rho_{n-1} \sigma_{n-2} \cdots \sigma_1$ satisfy

$$a^n = \rho_n \cdots \rho_1 \quad \text{and} \quad b^{n-1} = \rho_{n-1} \cdots \rho_1 \tag{7}$$

by [Gonçalves and Guaschi 2010b, Proposition 10]. Let $N$ be a normal subgroup of $B_n(\mathbb{R}P^2)$ that satisfies $K_n \subsetneqq N \subset P_n(\mathbb{R}P^2)$. We claim that for all $u \in \pi_1\big(\prod_1^n \mathbb{R}P^2\big)$ (which we identify henceforth with $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$), exactly one of the following two conditions holds:

(i) $N \cap \iota_\#^{-1}(\{u\})$ is empty.

(ii) $\iota_\#^{-1}(\{u\})$ is contained in $N$.

To prove the claim, suppose that $x \in N \cap \iota_\#^{-1}(\{u\}) \neq \varnothing$, and let $y \in \iota_\#^{-1}(\{u\})$. Now $\iota_\#(x) = \iota_\#(y) = u$, so there exists $k \in K_n$ such that $x^{-1}y = k$. Since $K_n \subset N$, it follows that $y = xk \in N$, which proves the claim. Further, $\iota_\#(a^n) = (\bar{1}, \dots, \bar{1})$ and $\iota_\#(b^{n-1}) = (\bar{1}, \dots, \bar{1}, \bar{0})$ by Proposition 8 and equations (5) and (7), so by the claim it suffices to prove that there exists $z \in N$ such that $\iota_\#(z) \in \{(\bar{1}, \dots, \bar{1}), (\bar{1}, \dots, \bar{1}, \bar{0})\}$, for then we are in case (ii) above, and it follows that one of $a^n$ or $b^{n-1}$ belongs to $N$.

It thus remains to prove the existence of such a $z$. Let $x \in N \setminus K_n$. Then $\iota_\#(x)$ contains an entry equal to $\bar{1}$ because $K_n = \operatorname{Ker}(\iota_\#)$. If $\iota_\#(x) = (\bar{1}, \dots, \bar{1})$ then we are done. So assume that $\iota_\#(x)$ also contains an entry that is equal to $\bar{0}$. By (5), there exist $1 \leq r < n$ and $1 \leq i_1 < \cdots < i_r \leq n$ such that $\iota_\#(\rho_{i_1} \cdots \rho_{i_r}) = \iota_\#(x)$. It follows from the claim and the fact that $x \in N$ that $\rho_{i_1} \cdots \rho_{i_r} \in N$ also, and so without loss of generality, we may suppose that $x = \rho_{i_1} \cdots \rho_{i_r}$. Further, since $\iota_\#(x)$ contains both a $\bar{0}$ and a $\bar{1}$, there exists $1 \leq j \leq r$ such that $p_{i_j \#}(x) = \bar{1}$ and $p_{(i_j+1)\#}(x) = \bar{0}$, the homomorphisms $p_{k\#}$ being those defined in the proof of Proposition 8. Note that we consider the indices modulo $n$, so if $i_j = n$ (so $j = r$) then we set $i_j + 1 = 1$. By [Gonçalves and Guaschi 2004a, page 777], conjugation by $a^{-1}$ permutes cyclically the elements $\rho_1, \dots, \rho_n, \rho_1^{-1}, \dots, \rho_n^{-1}$ of $P_n(\mathbb{R}P^2)$, so the $(n-1)$-st (resp. $n$-th) entry of $x' = a^{-(n-1-i_j)} x a^{(n-1-i_j)}$ is equal to $\bar{1}$ (resp. $\bar{0}$), and $x' \in N$ because $N$ is normal in $B_n(\mathbb{R}P^2)$. Using the relation $b = \sigma_{n-1} a$, we determine the conjugates of the $\rho_i$ by $b^{-1}$:

$$b^{-1}\rho_i b = a^{-1}\sigma_{n-1}^{-1}\rho_i\sigma_{n-1}a = a^{-1}\rho_i a = \rho_{i+1} \quad \text{for all } 1 \leq i \leq n-2,$$

$$\begin{aligned} b^{-1}\rho_{n-1} b &= a^{-1}\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}a = a^{-1}\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}^{-1}.\,\sigma_{n-1}^{2}a \\ &= a^{-1}\rho_n a.\,a^{-1}\sigma_{n-1}^2 a = \rho_1^{-1}.\,a^{-1}\sigma_{n-1}^2 a, \end{aligned}$$

where we used the relations $\rho_i\sigma_{n-1} = \sigma_{n-1}\rho_i$ if $1 \leq i \leq n-2$ and $\sigma_{n-1}^{-1}\rho_{n-1}\sigma_{n-1}^{-1} = \rho_n$ of Van Buskirk's presentation of $B_n(\mathbb{R}P^2)$, as well as the effect of conjugation by $a^{-1}$ on the $\rho_j$. Now $\sigma_{n-1}^2 = A_{n-1,n} \in K_n$ by Proposition 8, so $a^{-1}\sigma_{n-1}^2 a \in K_n$ by Remark 9(a), and hence $\iota_\#(b^{-1}\rho_{n-1}b) = (\bar{1}, \bar{0}, \dots, \bar{0})$. It then follows that $\iota_\#(a^{-1}x'a)$ and $\iota_\#(b^{-1}x'b)$ have the same entries except in the first and last positions, so if $x'' = a^{-1}x'a.\,b^{-1}x'b$, we have $\iota_\#(x'') = (\bar{1}, \bar{0}, \dots, \bar{0}, \bar{1})$. Further, $x'' \in N$ since $N$ is normal in $B_n(\mathbb{R}P^2)$. Let $n = 2m + \varepsilon$, where $m \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$. Then setting

$$z = a^{-\varepsilon}x''a^{\varepsilon} \cdot a^{-(2+\varepsilon)}x''a^{2+\varepsilon} \cdots a^{-(2(m-1)+\varepsilon)}x''a^{2(m-1)+\varepsilon},$$

we see once more that $z \in N$, and $\iota_\#(z) = (\bar{1}, \ldots, \bar{1})$ if $n$ is even and $\iota_\#(z) = (\bar{1}, \ldots, \bar{1}, \bar{0})$ if $n$ is odd, which completes the proof of the existence of $z$, and thus that of Proposition 1(b). $\square$

*Proof of Theorem 2.* Let $S = \mathbb{S}^2$ or $\mathbb{R}P^2$. If $n = 1$ then $\iota_\#$ is an isomorphism and $\operatorname{Im}(j_\#|_{P_n})$ is trivial so the result holds. If $n = 2$ and $S = \mathbb{S}^2$ then $P_n(\mathbb{S}^2)$ is trivial, and there is nothing to prove. Now suppose that $S = \mathbb{S}^2$ and $n \geq 3$. As we mentioned in the Introduction, $\operatorname{Ker}(\iota_\#) = P_n(\mathbb{S}^2)$. Let $(A_{i,j})_{1 \leq i < j \leq n}$ be the generating set of $P_n$, where $A_{i,j}$ has a geometric representative similar to that given in Figure 1. It is well known that the image of this set by $j_\#$ yields a generating set for $P_n(\mathbb{S}^2)$ (see [Scott 1970, page 616]), so $j_\#|_{P_n}$ is surjective, and the statement of the theorem follows. Finally, assume that $S = \mathbb{R}P^2$ and $n \geq 2$. Once more, $\operatorname{Im}(j_\#|_{P_n}) = \langle A_{i,j} \mid 1 \leq i < j \leq n \rangle$, and since $A_{i,j} \in \operatorname{Ker}(\iota_\#)$ by Proposition 8, we conclude that $\langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)} \subset \operatorname{Ker}(\iota_\#)$. To prove the converse, first recall from Proposition 8 that $\operatorname{Ker}(\iota_\#) = \Gamma_2(P_n(\mathbb{R}P^2))$. Using the standard commutator identities

$$[x, yz] = [x, y]\big[y, [x, z]\big][x, z]$$

and

$$[xy, z] = \big[x, [y, z]\big][y, z][x, z],$$

$\Gamma_2(P_n(\mathbb{R}P^2))$ is equal to the normal closure in $P_n(\mathbb{R}P^2)$ of the set

$$\big\{[x, y] \,\big|\, x, y \in \{A_{i,j},\ \rho_k \mid 1 \leq i < j \leq n \text{ and } 1 \leq k \leq n\}\big\}.$$

It then follows using the relations of Theorem 7 that the commutators $[x, y]$ belonging to this set also belong to $\langle\langle A_{i,j} \mid 1 \leq i < j \leq n \rangle\rangle_{P_n(\mathbb{R}P^2)}$, which is nothing other than $\langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)}$. We conclude by normality that $\operatorname{Ker}(\iota_\#) \subset \langle\langle \operatorname{Im}(j_\#|_{P_n}) \rangle\rangle_{P_n(S)}$. $\square$

## 3. Some properties of the subgroup $L_n$

Let $S = \mathbb{S}^2$ or $S = \mathbb{R}P^2$, and for all $m, n \geq 1$, let $\Gamma_{m,n}(S) = P_m(S \setminus \{x_1, \ldots, x_n\})$ denote the $m$-string pure braid group of $S$ with $n$ points removed. In this section, we study $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, which is $\Gamma_{n-2,2}(\mathbb{R}P^2)$, in more detail, and we prove Theorem 3 and Proposition 4, which enable us to understand better the structure of the subgroup $L_n$ defined in the proof of Proposition 1(a)(ii).

We start by exhibiting a presentation of the group $\Gamma_{m,n}(\mathbb{R}P^2)$ in terms of the generators of $P_{m+n}(\mathbb{R}P^2)$ described at the beginning of Section 2. A presentation for $\Gamma_{m,n}(\mathbb{S}^2)$ is given in [Gonçalves and Guaschi 2005, Proposition 7] and will be recalled later in Proposition 15, when we come to proving Theorem 5. For $1 \leq i < j \leq m+n$, let

$$C_{i,j} = A_{j-1,j}^{-1} \cdots A_{i+1,j}^{-1} A_{i,j} A_{i+1,j} \cdots A_{j-1,j} \tag{8}$$

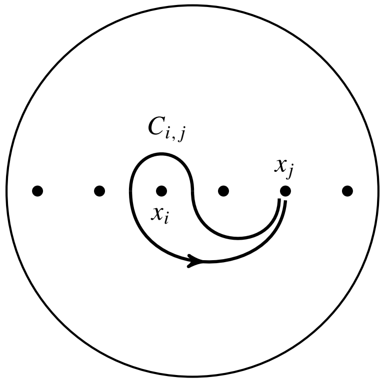


**Figure 2.** The element $C_{i,j}$ in $P_{m+n}(\mathbb{R}P^2)$.

in $P_{m+n}(\mathbb{R}P^2)$ (see Figure 2). In what follows, any element of the form $A_{i,j}$ or $C_{i,j}$, where $i \geq j$, should be interpreted as the trivial element. The proof of the following proposition is similar in nature to that for $\mathbb{S}^2$, but is a little more involved due to the presence of extra generators that emanate from the fundamental group of $\mathbb{R}P^2$.

**Proposition 11.** *Let $n, m \geq 1$. The following constitutes a presentation of the group $\Gamma_{m,n}(\mathbb{R}P^2)$:*

***Generators***: *$A_{i,j}$, $\rho_j$, where $1 \leq i < j$ and $n+1 \leq j \leq m+n$.*

***Relations***:

(I) *The Artin relations described by* (4) *among the generators $A_{i,j}$ of $\Gamma_{m,n}(\mathbb{R}P^2)$.*

(II) *For all $1 \leq i < j$ and $n+1 \leq j < k \leq m+n$, $A_{i,j}\rho_k A_{i,j}^{-1} = \rho_k$.*

(III) *For all $1 \leq i < j$ and $n+1 \leq k < j \leq m+n$,*

$$\rho_k A_{i,j} \rho_k^{-1} = \begin{cases} A_{i,j} & \text{if } k < i, \\ \rho_j^{-1} C_{i,j}^{-1} \rho_j & \text{if } k = i, \\ \rho_j^{-1} C_{k,j}^{-1} \rho_j A_{i,j} \rho_j^{-1} C_{k,j} \rho_j & \text{if } k > i. \end{cases}$$

(IV) *For all $n+1 \leq k < j \leq m+n$, $\rho_k \rho_j \rho_k^{-1} = C_{k,j}\rho_j$.*

(V) *For all $n+1 \leq j \leq m+n$,*

$$\rho_j \Big( \prod_{i=1}^{j-1} A_{i,j} \Big) \rho_j = \Big( \prod_{l=j+1}^{m+n} A_{j,l} \Big).$$

*The elements $C_{i,j}$ and $C_{k,j}$ appearing in relations (III) and (IV) should be rewritten using* (8).

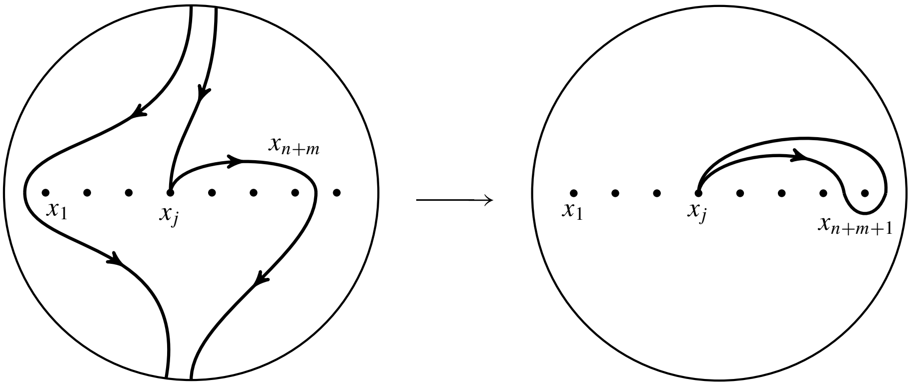


**Figure 3.** The relation $\left(\prod_{l=j+1}^{m+n} A_{j,l}\right)^{-1} \rho_j \left(\prod_{i=1}^{j-1} A_{i,j}\right) \rho_j = A_{j,n+m+1}$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $n+1 \leq j \leq n+m+1$.

*Proof.* If $m, n \geq 1$, we have the following Fadell–Neuwirth short exact sequence of pure braid groups of $\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}$:

$$
\begin{aligned}
(9) \quad 1 \to P_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}\}) \\
\to \Gamma_{m+1,n}(\mathbb{R}P^2) \xrightarrow{q} \Gamma_{m,n}(\mathbb{R}P^2) \to 1,
\end{aligned}
$$

where the homomorphism $q$ is given geometrically by forgetting the last string. The generators $A_{i,j}$ and $\rho_j$ of $\Gamma_{m,n}(\mathbb{R}P^2)$ given in the statement of the proposition are represented geometrically as in Figure 1, and the basepoints of the $m$ strings of $\Gamma_{m,n}(\mathbb{R}P^2)$ are the points $x_{n+1}, \ldots, x_{n+m}$. Using induction on $m$, we apply standard methods to obtain a group presentation of an extension from presentations of the kernel and the quotient [Johnson 1997, Proposition 1, Chapter 10], using the geometric representations of Figure 1 to derive some of the relations.

Let $n \geq 1$. If $m = 1$ then $\Gamma_{1,n}(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2 \setminus \{x_1, \ldots, x_n\}, x_{n+1})$ is generated by $\{A_{i,n+1}, \rho_{n+1} \mid 1 \leq i \leq n\}$ subject to the surface relation $\prod_{i=1}^{n} A_{i,n+1} = \rho_{n+1}^{-2}$, which is equivalent to the single relation given by (V). Since the remaining relations (I)–(IV) are empty, the given presentation of $\Gamma_{1,n}(\mathbb{R}P^2)$ is correct.

Now suppose that the given presentation of $\Gamma_{m,n}(\mathbb{R}P^2)$ is correct for some $m \geq 1$. We shall show that we obtain the presentation of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ by applying the above-mentioned methods to the short exact sequence (9). Although $\mathrm{Ker}(q)$ is a free group, it shall be convenient to consider it as the group with generating set

$$Y_{n+m+1} = \{A_{i,n+m+1}, \ \rho_{n+m+1} \mid 1 \leq i \leq n+m\},$$

subject to the single relation $\rho_{n+m+1}\left(\prod_{i=1}^{n+m} A_{i,n+m+1}\right)\rho_{n+m+1} = 1$ (this may be seen by taking $j = n+m+1$ in Figure 3). According to [Johnson 1997, Proposition 1, Chapter 10], $\Gamma_{m+1,n}(\mathbb{R}P^2)$ is generated by the union of $Y_{n+m+1}$ with the set of coset representatives

$$X_{m,n} = \{A_{i,j}, \ \rho_j \mid 1 \leq i < j \text{ and } n+1 \leq j \leq m+n\}$$

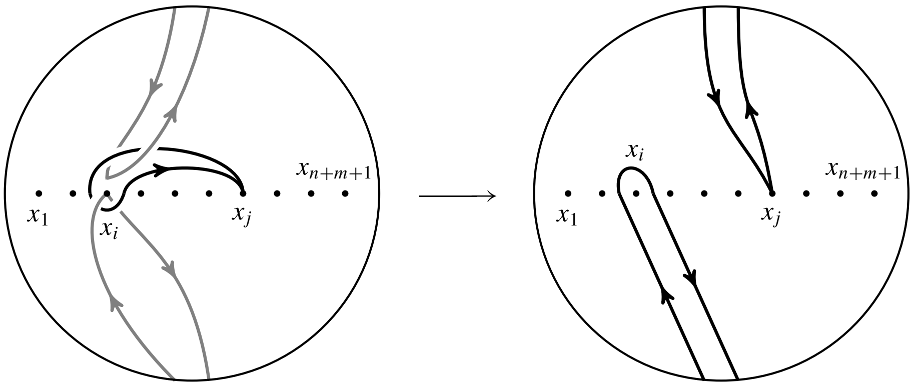


**Figure 4.** The relation $\rho_i A_{i,j} \rho_i^{-1} = \rho_j^{-1} C_{i,j}^{-1} \rho_j$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $1 \leq i < j$ and $n+1 \leq j \leq n+m+1$.

in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ of the given set of generators of $\Gamma_{m,n}(\mathbb{R}P^2)$. This yields the required set of generators of $\Gamma_{m+1,n}(\mathbb{R}P^2)$. Once more by [Johnson 1997, Proposition 1, Chapter 10], there are three types of relation in $\Gamma_{m+1,n}(\mathbb{R}P^2)$:

(1) the (single) given relation of $\mathrm{Ker}(q)$, which yields the surface relation (V) with $j = n+m+1$;

(2) the relators of $\Gamma_{m,n}(\mathbb{R}P^2)$, rewritten in terms of the elements of $Y_{n+m+1}$;

(3) the conjugates of the elements of $Y_{n+m+1}$ by the elements of $X_{m,n}$, also rewritten in terms of the elements of $Y_{n+m+1}$.

Let us study the relations of type (2) using the geometric representatives given in Figure 1. The Artin relations (I) of $\Gamma_{m,n}(\mathbb{R}P^2)$ lift directly to relations in $\Gamma_{m+1,n}(\mathbb{R}P^2)$, and yield the relations (I) of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for all $n+1 \leq j \leq n+m$. The relations (II) (resp. relations (III) with $k < i$) of $\Gamma_{m,n}(\mathbb{R}P^2)$ involve elements that are represented geometrically by disjoint loops. They also lift directly to relations in $\Gamma_{m+1,n}(\mathbb{R}P^2)$, and yield the relations (II) (resp. relations (III) with $k < i$) of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for all $k \leq n+m$ (resp. for all $n+1 \leq j \leq n+m$). The relations (III) with $k = i$ or $k > i$ (resp. relations (IV)) of $\Gamma_{m,n}(\mathbb{R}P^2)$ are represented in Figures 4, 5 and 6 respectively (in $\Gamma_{m,n}(\mathbb{R}P^2)$, the point $x_{n+m+1}$ is unmarked), and from these figures, we see that each of the relations also lifts directly to $\Gamma_{m+1,n}(\mathbb{R}P^2)$. We thus obtain all of the relations (I)–(IV) of $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for all $n+1 \leq j \leq n+m$ in relations (I), (III) and (IV), and for all $k \leq n+m$ in relation (II). From Figure 3, we observe that $\left(\prod_{l=j+1}^{m+n} A_{j,l}\right)^{-1} \rho_j \left(\prod_{i=1}^{j-1} A_{i,j}\right) \rho_j = A_{j,n+m+1}$ for all $n+1 \leq j \leq n+m$. Together with the relation of type (1), this yields all of the relations (V) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$. It remains to determine the relations of type (3).

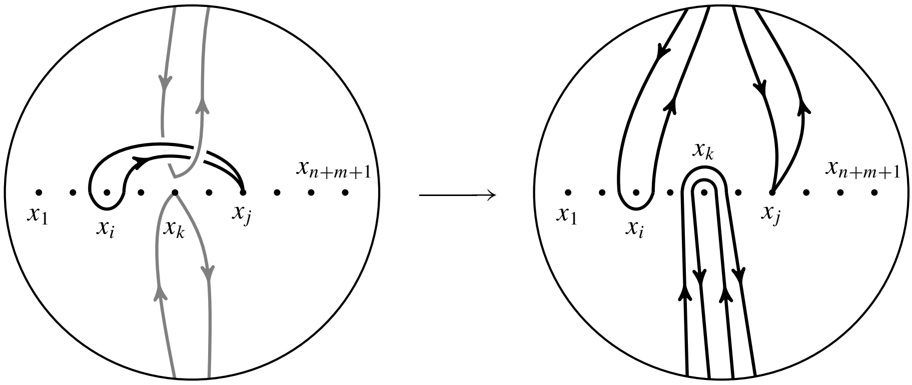


**Figure 5.** The relation $\rho_k A_{i,j} \rho_k^{-1} = \rho_j^{-1} C_{k,j}^{-1} \rho_j A_{i,j} \rho_j^{-1} C_{k,j} \rho_j$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $1 \le i < k < j$ and $n+1 \le k < j \le n+m+1$.

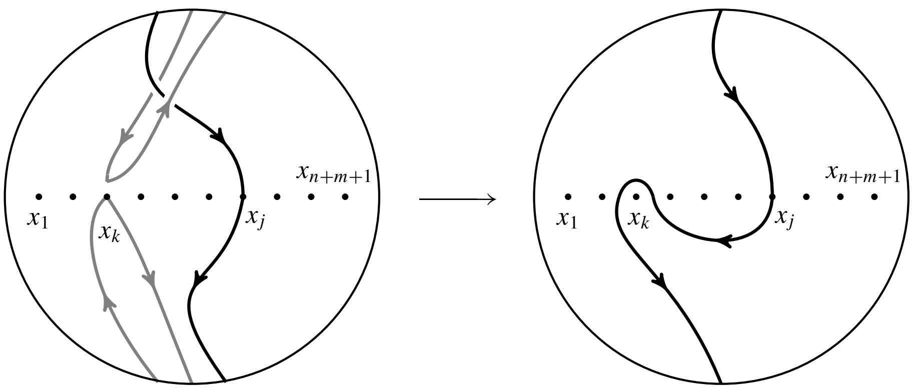


**Figure 6.** The relation $\rho_k \rho_j \rho_k^{-1} = C_{k,j} \rho_j$ in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ for $n+1 \le k < j \le n+m+1$.

• If $A_{i,j} \in X_{m,n}$ and $A_{k,n+m+1} \in Y_{n+m+1}$ then $A_{i,j} A_{k,n+m+1} A_{i,j}^{-1}$ is given by the Artin relations (4), and together with the Artin relations of type (2), we obtain all of the relations (I) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$.

• If $A_{i,j} \in X_{m,n}$ and $\rho_{n+m+1} \in Y_{n+m+1}$, then since $j \le n+m$, $A_{i,j}$ and $\rho_{n+m+1}$ commute since they are represented geometrically by disjoint loops. This yields relations (II) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$ with $k = n+m+1$. Together with the corresponding relations of type (2), we obtain all of the relations (II) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$.

• If $\rho_k \in X_{m,n}$ and $A_{i,n+m+1} \in Y_{n+m+1}$, we consider three cases:

(a) If $k < i$, then $\rho_k$ and $A_{i,n+m+1}$ commute since they are represented geometrically by disjoint loops.

(b) If $k=i$, we obtain $\rho_i A_{i,n+m+1}\rho_i^{-1} = \rho_{n+m+1}^{-1} C_{i,n+m+1}^{-1} \rho_{n+m+1}$ by taking $j = n+m+1$ in Figure 4.

(c) If $k>i$, by taking $j=n+m+1$ in Figure 5, we see that $\rho_k A_{i,n+m+1}\rho_k^{-1} = \rho_{n+m+1}^{-1} C_{k,n+m+1}^{-1} \rho_{n+m+1} A_{i,n+m+1} \rho_{n+m+1}^{-1} C_{k,n+m+1} \rho_{n+m+1}$.

Together with the corresponding relations of type (2), we obtain all of the relations (III) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$.

• If $\rho_k \in X_{m,n}$ and $\rho_{n+m+1} \in Y_{n+m+1}$ then by taking $j=n+m+1$ in Figure 6, we see that $\rho_k \rho_{n+m+1} \rho_k^{-1} = C_{k,n+m+1}\rho_{n+m+1}$, which yields relations (IV) with $j=n+m+1$. Together with the corresponding relations of type (2), we obtain all of the relations (IV) in $\Gamma_{m+1,n}(\mathbb{R}P^2)$. □

In the rest of this section, we shall assume that $n=2$, and we shall focus our attention on the groups $\Gamma_{m,2}(\mathbb{R}P^2)$, where $m \geq 1$, which we interpret as subgroups of $P_{m+2}(\mathbb{R}P^2)$ via the short exact sequence (3). Before proving Theorem 3 and Proposition 4, we introduce some notation that will be used to study the subgroups $K_n$ and $L_n$. Let $m \geq 2$, and consider the Fadell–Neuwirth short exact sequence

$$1 \to \Omega_{m+1} \to P_m(\mathbb{R}P^2 \setminus \{x_1,x_2\}) \xrightarrow{r_{m+1}} P_{m-1}(\mathbb{R}P^2 \setminus \{x_1,x_2\}) \to 1, \tag{10}$$

where $r_{m+1}$ is given geometrically by forgetting the last string, and where $\Omega_{m+1} = \pi_1(\mathbb{R}P^2 \setminus \{x_1,\ldots,x_{m+1}\}, x_{m+2})$. From the Fadell–Neuwirth short exact sequences of the form of (3), $r_{m+1}$ is the restriction of $q_{(m+1)\#} : P_{m+2}(\mathbb{R}P^2) \to P_{m+1}(\mathbb{R}P^2)$ to $\mathrm{Ker}(q_{2\#})$. The kernel $\Omega_{m+1}$ of $r_{m+1}$ is a free group of rank $m+1$ with a basis $\mathcal{B}_{m+1}$ being given by

$$\mathcal{B}_{m+1} = \{A_{k,m+2}, \rho_{m+2} \mid 1 \leq k \leq m\}. \tag{11}$$

The group $\Omega_{m+1}$ may also be described as the subgroup of $P_m(\mathbb{R}P^2 \setminus \{x_1,x_2\})$ generated by $\{A_{1,m+2},\ldots,A_{m+1,m+2},\rho_{m+2}\}$ subject to the relation

$$A_{m+1,m+2} = A_{m,m+2}^{-1}\cdots A_{1,m+2}^{-1}\rho_{m+2}^{-2}, \tag{12}$$

obtained from relation (V) of Proposition 11. Equations (8) and (12) imply notably that $A_{l,m+2}$ and $C_{l,m+2}$ belong to $\Omega_{m+1}$ for all $1 \leq l \leq m+1$. Using geometric methods, for $m \geq 2$, we proved the existence of a section

$$s_{m+1} : P_{m-1}(\mathbb{R}P^2 \setminus \{x_1,x_2\}) \to P_m(\mathbb{R}P^2 \setminus \{x_1,x_2\})$$

for $r_{m+1}$ in [Gonçalves and Guaschi 2010a, Theorem 2(a)]. Applying induction to (10), it follows that for all $m \geq 1$,

$$P_m(\mathbb{R}P^2 \setminus \{x_1,x_2\}) \cong \Omega_{m+1} \rtimes (\Omega_m \rtimes (\cdots \rtimes (\Omega_3 \rtimes \Omega_2)\cdots)). \tag{13}$$

So $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \mathbb{F}_{m+1} \rtimes (\mathbb{F}_m \rtimes (\cdots \rtimes (\mathbb{F}_3 \rtimes \mathbb{F}_2)\cdots))$, which may be interpreted as the Artin combing operation for $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. It follows from this and (11) that $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ admits $\mathcal{X}_{m+2}$ as a generating set, where

$$\mathcal{X}_{m+2} = \{A_{i,j},\ \rho_j \mid 3 \le j \le m+2,\ 1 \le i \le j-2\}. \tag{14}$$

**Remark 12.** For what follows, we will need to know an explicit section $s_{m+1}$ for $r_{m+1}$. Such a section may be obtained as follows: for $m \ge 2$, consider the homomorphism $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by forgetting the string based at $x_3$. By [Gonçalves and Guaschi 2010a, Theorem 2(a)]), a geometric section is obtained by doubling the second (vertical) string, so that there is a new third string, and renumbering the following strings, which gives rise to an algebraic section for the given homomorphism of the form

$$A_{i,j} \mapsto \begin{cases} A_{1,j+1} & \text{if } i = 1, \\ A_{2,j+1}A_{3,j+1} & \text{if } i = 2, \\ A_{i+1,j+1} & \text{if } 3 \le i < j, \end{cases}$$
$$\rho_j \mapsto \rho_{j+1}$$

for all $3 \le j \le m+1$. However, in view of the nature of $r_{m+1}$, we would like this new string to be in the $(m+2)$-nd position. We achieve this by composing the above algebraic section with conjugation by $\sigma_{m+1}\cdots\sigma_3$, which gives rise to a section

$$s_{m+1} : P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$$

for $r_{m+1}$ that is defined by

$$\begin{cases} s_{m+1}(A_{i,j}) = \begin{cases} A_{j,m+2}A_{1,j}A_{j,m+2}^{-1} & \text{if } i = 1, \\ A_{j,m+2}A_{2,j} & \text{if } i = 2, \\ A_{i,j} & \text{if } 3 \le i < j, \end{cases} \\ s_{m+1}(\rho_j) = \rho_j A_{j,m+2}^{-1} \end{cases} \tag{15}$$

for all $1 \le i < j$ and $3 \le j \le m+1$. A long but straightforward calculation using the presentation of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ given by Proposition 11 shows that $s_{m+1}$ does indeed define a section for $r_{m+1}$.

We now prove Theorem 3, which lets us give a more explicit description of $L_n$.

*Proof of Theorem 3.* Let $n \ge 3$. By the commutative diagram (6) of short exact sequences, the restriction of the homomorphism $q_{2\#} : P_n(\mathbb{R}P^2) \to P_2(\mathbb{R}P^2)$ to $K_n$ factors through the inclusion $\langle \Delta_2^2 \rangle \to P_2(\mathbb{R}P^2)$, and the kernel $L_n$ of $q_{2\#}|_{K_n}$ is contained in $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. We may then add a third row to this diagram:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & L_n \arrow[r] \arrow[d] & K_n \arrow[r, "q_{2\#}|_{K_n}"] \arrow[d] & \langle \Delta_2^2 \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] \arrow[d, dashed, "\hat{\iota}_{n-2}"] & P_n(\mathbb{R}P^2) \arrow[r, "q_{2\#}"] \arrow[d, "\iota_{n\#}"] & P_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "\iota_{2\#}"] & 1 \\
1 \arrow[r] & \mathbb{Z}_2^{n-2} \arrow[r, "\ell"] \arrow[d, dashed] & \mathbb{Z}_2^n \arrow[r, "\hat{q}_2"] \arrow[d] & \mathbb{Z}_2^2 \arrow[r] \arrow[d] & 1, \\
 & 1 & 1 & 1 &
\end{tikzcd}
\tag{16}
$$

where $\hat{q}_2 : \mathbb{Z}_2^n \to \mathbb{Z}_2^2$ is projection onto the first two factors, and $\ell : \mathbb{Z}_2^{n-2} \to \mathbb{Z}_2^n$ is the monomorphism defined by

$$\ell(\overline{\varepsilon_1}, \dots, \overline{\varepsilon_{n-2}}) = (\bar{0}, \bar{0}, \overline{\varepsilon_1}, \dots, \overline{\varepsilon_{n-2}}).$$

The commutativity of diagram (16) thus induces a homomorphism

$$\hat{\iota}_{n-2} : P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to \mathbb{Z}_2^{n-2}$$

that is the restriction of $\iota_{n\#}$ to $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ that makes the bottom left-hand square commute. To see that $\hat{\iota}_{n-2}$ is surjective, notice that if $x \in \mathbb{Z}_2^{n-2}$ then the first two entries of $\ell(x)$ are equal to $\bar{0}$, and using (5), it follows that there exist $3 \leq i_1 < \cdots < i_r \leq n$ such that $\iota_{n\#}(\rho_{i_1} \cdots \rho_{i_r}) = \ell(x)$. Furthermore, $\rho_{i_1} \cdots \rho_{i_r} \in \mathrm{Ker}(q_{2\#})$, and by the commutativity of the diagram, we also have $\iota_{n\#}(\rho_{i_1} \cdots \rho_{i_r}) = \ell \circ \hat{\iota}_{n-2}(\rho_{i_1} \cdots \rho_{i_r})$, whence $x = \hat{\iota}_{n-2}(\rho_{i_1} \cdots \rho_{i_r})$ by the injectivity of $\ell$. It remains to prove the exactness of the first column. The fact that $L_n \subset \mathrm{Ker}(\hat{\iota}_{n-2})$ follows easily. Conversely, if $x \in \mathrm{Ker}(\hat{\iota}_{n-2})$ then $x \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, and $x \in K_n$ by the commutativity of the diagram, so $x \in L_n$. This proves the first two assertions of the theorem.

To prove the last part of the theorem, let $m \geq 1$, and consider (10). Since $\hat{\iota}_m$ is the restriction of $\iota_{(m+2)\#}$ to $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, we have $\hat{\iota}_m(\rho_j) = (\bar{0}, \dots, \bar{0}, \bar{1}, \bar{0}, \dots, \bar{0})$, where $\bar{1}$ is the in the $(j-2)$-nd position, and $\hat{\iota}_m(A_{i,j}) = (\bar{0}, \dots, \bar{0})$ for all $1 \leq i < j$ and $3 \leq j \leq m+2$. So for each $2 \leq l \leq m+1$, $\hat{\iota}_m$ restricts to a surjective homomorphism $\hat{\iota}_m|_{\Omega_l} : \Omega_l \to \mathbb{Z}_2$ of each of the factors of (13), with $\mathbb{Z}_2$ being the $(l-1)$-st factor of $\mathbb{Z}_2^m$, and using (11), we see that $\mathrm{Ker}(\hat{\iota}_m|_{\Omega_l})$ is a free group of rank $2l-1$ with basis $\widehat{\mathcal{B}}_l$ given by

$$\widehat{\mathcal{B}}_l = \{A_{k,l+1}, \rho_{l+1} A_{k,l+1} \rho_{l+1}^{-1}, \rho_{l+1}^2 \mid 1 \leq k \leq l-1\}. \tag{17}$$

As we shall now explain, for all $m \geq 2$, the short exact sequence (10) may be extended to a commutative diagram of short exact sequences as follows:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathrm{Ker}(\hat{\iota}_m|_{\Omega_{m+1}}) \arrow[r] \arrow[d] & L_{m+2} \arrow[r, "r_{m+1}|_{L_{m+2}}"] \arrow[d] & L_{m+1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Omega_{m+1} \arrow[r] \arrow[d, "\hat{\iota}_m|_{\Omega_{m+1}}"] & P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r, shift left, "r_{m+1}"] \arrow[d, "\hat{\iota}_m"] & P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \arrow[r] \arrow[l, shift left, dashed, "s_{m+1}"] \arrow[d, "\hat{\iota}_{m-1}"] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathbb{Z}_2^m \arrow[r] \arrow[d] & \mathbb{Z}_2^{m-1} \arrow[r] \arrow[d] & 1. \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{18}$$

To obtain this diagram, we start with the commutative diagram that consists of the second and third rows and the three columns (so a priori, the arrows of the first row are missing). The commutativity implies that $r_{m+1}$ restricts to the homomorphism $r_{m+1}|_{L_{m+2}} : L_{m+2} \to L_{m+1}$, which is surjective, since if $w \in L_{m+1}$ is written in terms of the elements of $\mathcal{X}_{m+1}$ then the same word $w$, considered as an element of $P_m(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, belongs to $L_{m+2}$, and satisfies $r_{m+1}(w) = w$. Then the kernel of $r_{m+1}|_{L_{m+2}}$, which is also the kernel of $\hat{\iota}_m|_{\Omega_{m+1}}$, is equal to $L_{m+2} \cap \Omega_{m+1}$. This establishes the existence of the complete commutative diagram (18) of short exact sequences. By induction, it follows from (17) and (18) that for all $m \geq 1$, $L_{m+2}$ is generated by

$$\widehat{\mathcal{X}}_{m+2} = \bigcup_{j=3}^{m+2} \widehat{\mathcal{B}}_{j-1} = \{A_{i,j},\ \rho_j A_{i,j} \rho_j^{-1},\ \rho_j^2 \mid 3 \leq j \leq m+2,\ 1 \leq i \leq j-2\}. \tag{19}$$

By (15), for each $x \in \widehat{\mathcal{X}}_{m+1}$, $\hat{\iota}_m \circ s_{m+1}(x)$ is the trivial element of $\mathbb{Z}_2^m$, and thus $s_{m+1}(x) \in L_{m+2}$. Hence $s_{m+1}$ restricts to a section $s_{m+1}|_{L_{m+1}} : L_{m+1} \to L_{m+2}$ for $r_{m+1}|_{L_{m+2}}$. We conclude by induction on the first row of (18) that

$$L_{m+2} \cong \mathrm{Ker}(\hat{\iota}_m|_{\Omega_{m+1}}) \rtimes L_{m+1} \tag{20}$$

$$\cong \mathrm{Ker}(\hat{\iota}_m|_{\Omega_{m+1}}) \rtimes \big(\mathrm{Ker}(\hat{\iota}_m|_{\Omega_m}) \rtimes \big(\cdots \rtimes \big(\mathrm{Ker}(\hat{\iota}_m|_{\Omega_3}) \rtimes \mathrm{Ker}(\hat{\iota}_m|_{\Omega_2})\big) \cdots\big)\big), \tag{21}$$

the actions being induced by those of (13), so by (17), $L_{m+2}$ is isomorphic to a repeated semidirect product of the form $\mathbb{F}_{2m+1} \rtimes (\mathbb{F}_{2m-1} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots))$. The last part of the statement of Theorem 3 follows by taking $m = n-2$. □

A finer analysis of the actions that appear in (13) and (21) now allows us to determine the Abelianisations of $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and $L_n$.

*Proof of Proposition 4.* If $n = 3$ then the two assertions are clear. So assume by induction that they hold for some $n \geq 3$. From the split short exact sequence (10) and (20) with $m = n-1$, we have

$$\begin{cases} P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \cong \Omega_n \rtimes_\psi P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\}), \\ \qquad\qquad L_{n+1} \cong \operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n}) \rtimes_\psi L_n, \end{cases} \tag{22}$$

where $\psi$ denotes the action given by the section $s_n$, as well as the action induced by the restriction of the section $s_n$ to $L_n$.

Before going any further, we recall some general considerations from [Gonçalves and Guaschi 2009, pages 3387–88] concerning the Abelianisation of semidirect products. If $H$ and $K$ are groups, and if $\varphi : H \to \operatorname{Aut}(K)$ is an action of $H$ on $K$, then one may deduce easily from Proposition 3.3 of that paper that

$$(K \rtimes_\varphi H)^{\mathrm{Ab}} \cong \Delta(K) \oplus H^{\mathrm{Ab}}, \tag{23}$$

where

$$\Delta(K) = K/K_1, \quad K_1 = \langle \Gamma_2(K) \cup \widehat{K} \rangle \text{ and } \widehat{K} = \langle \varphi(h)(k) \cdot k^{-1} \mid h \in H \text{ and } k \in K \rangle.$$

Recall that $\widehat{K}$ is normal in $K$ (see [Gonçalves and Guaschi 2009, lines 1–4, page 3388]), so $K_1$ is normal in $K$, $K_1 = \Gamma_2(K).\widehat{K} = \widehat{K}.\Gamma_2(K)$, and $\Delta(K) \cong K^{\mathrm{Ab}}/p(\widehat{K})$, where $p : K \to K^{\mathrm{Ab}}$ is the canonical projection. If $k \in K$, let $\bar{k} = p(k)$. For all $k, k' \in K$ and $h, h' \in H$, we have

$$\varphi(h^{-1})(k) \cdot k^{-1} = \big(\varphi(h)(\varphi(h^{-1})(k)) \cdot (\varphi(h^{-1})(k))^{-1}\big)^{-1}, \tag{24}$$

$$\varphi(h)(k^{-1}) \cdot k = \big(k^{-1}(\varphi(h)(k) \cdot k^{-1})k\big)^{-1}, \tag{25}$$

$$\begin{aligned} \varphi(hh')(k) \cdot k^{-1} &= \varphi(h)(\varphi(h')(k)) \cdot \varphi(h')(k^{-1}) \cdot \varphi(h')(k) \cdot k^{-1} \\ &= \varphi(h)(k'') \cdot k''^{-1} \cdot \varphi(h')(k) \cdot k^{-1}, \end{aligned} \tag{26}$$

$$\varphi(h)(kk') \cdot (kk')^{-1} = \big(\varphi(h)(k) \cdot k^{-1}\big) \cdot k\big(\varphi(h)(k') \cdot k'^{-1}\big)k^{-1}, \tag{27}$$

where $k'' = \varphi(h')(k)$ belongs to $K$. Let $\mathcal{H}$ and $\mathcal{K}$ be generating sets for $H$ and $K$, respectively. By induction on word length relative to the elements of $\mathcal{H}$, (24) and (26) imply that $\widehat{K}$ is generated by elements of the form $\varphi(h)(k) \cdot k^{-1}$, where $h \in \mathcal{H}$ and $k \in K$. A second induction on word length relative to the elements of $\mathcal{K}$ and (25) and (27) imply that $\widehat{K}$ is normally generated by the elements of the form $\varphi(h)(k) \cdot k^{-1}$, where $h \in \mathcal{H}$ and $k \in \mathcal{K}$. It follows that the subgroup $p(\widehat{K})$ of $K^{\mathrm{Ab}}$ is generated by the elements of the form $\overline{\varphi(h)(k) \cdot k^{-1}}$, where $h \in \mathcal{H}$ and $k \in \mathcal{K}$, and that a presentation of $\Delta(K)$ may be obtained from a presentation of $K^{\mathrm{Ab}}$ by adjoining these elements as relators.

We now take $K = \Omega_n$ (resp. $K = \operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$), $H = P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ (resp. $H = L_n$) and $\varphi = \psi$. Applying the induction hypothesis and (23) to (22), to prove parts (a) and (b), it thus suffices to show that

$$\Delta(\Omega_n) \cong \mathbb{Z}^2, \tag{28}$$

$$\Delta\big(\operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})\big) \cong \mathbb{Z}^{2n-1}, \tag{29}$$

respectively. We first establish the isomorphism (28). As we saw above, to obtain a presentation of $\Delta(\Omega_n)$, we add the relators of the form $\overline{\psi(\tau)(\omega)\cdot\omega^{-1}}$ to a presentation of $(\Omega_n)^{\mathrm{Ab}}$, where $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$, with $\mathcal{X}_n$ and $\mathcal{B}_n$ as defined in (14) and (11), respectively. In $(\Omega_n)^{\mathrm{Ab}}$, these relators may be written as $\overline{s_n(\tau)\omega(s_n(\tau))^{-1}\omega^{-1}}$, or equivalently in the form

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}}\;\overline{\omega^{-1}}. \tag{30}$$

We claim that it is not necessary to know explicitly the section $s_n$ in order to determine these relators. Indeed, for all $\tau \in \mathcal{X}_n$, we have $r_n(\tau) = \tau$; note that we abuse notation here by letting $\tau$ also denote the corresponding element of $\mathcal{X}_{n+1}$ in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$. Thus $s_n(\tau)\tau^{-1} \in \operatorname{Ker}(r_n)$, and hence there exists $\omega_\tau \in \Omega_n$ such that $s_n(\tau) = \omega_\tau\tau$. Therefore

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}} = \overline{\omega_\tau\tau\omega\tau^{-1}\omega_\tau^{-1}} = \overline{\omega_\tau}\;\overline{\tau\omega\tau^{-1}}\;\overline{\omega_\tau^{-1}} = \overline{\tau\omega\tau^{-1}}$$

in $(\Omega_n)^{\mathrm{Ab}}$, and thus the relators of (30) become

$$\overline{s_n(\tau)\omega(s_n(\tau))^{-1}\omega^{-1}} = \overline{\tau\omega\tau^{-1}}\;\overline{\omega^{-1}}. \tag{31}$$

This proves the claim. Hence the subgroup $p(\widehat{\Omega}_n)$ of $(\Omega_n)^{\mathrm{Ab}}$ is generated by the elements of the form given by (31), where $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$. In what follows, the relations (I)–(V) refer to those of the presentation of $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ described by Proposition 11. Using this presentation, we see immediately that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$ in $(\Omega_n)^{\mathrm{Ab}}$ for all $\tau \in \mathcal{X}_n$ and $\omega \in \mathcal{B}_n$, with the following exceptions:

(i) $\tau = \rho_j$ and $\omega = A_{j,n+1}$ for all $3 \le j \le n-1$. Then

$$\overline{\rho_j A_{j,n+1}\rho_j^{-1}} = \overline{C_{j,n+1}^{-1}} = \overline{A_{j,n+1}^{-1}},$$

using relation (III) and (8), which yields the element $(\overline{A_{j,n+1}})^2$ of $p(\widehat{\Omega}_n)$.

(ii) $\tau = \rho_j$ and $\omega = \rho_{n+1}$ for all $3 \le j \le n$. Then

$$\overline{\rho_j\rho_{n+1}\rho_j^{-1}} = \overline{C_{j,n+1}\rho_{n+1}} = \overline{A_{j,n+1}}\;\overline{\rho_{n+1}}$$

by relation (IV) and (8), which yields the element $\overline{A_{j,n+1}}$ of $p(\widehat{\Omega}_n)$.

The relators of (ii) above clearly give rise to those of (i), and so $p(\widehat{\Omega}_n)$ is the subgroup of $(\Omega_n)^{\mathrm{Ab}}$ generated by the elements $\overline{A_{j,n+1}}$, where $3 \le j \le n$. Since by (11), $(\Omega_n)^{\mathrm{Ab}}$ is the free Abelian group with basis

$$\{\overline{A_{j,n+1}}, \overline{\rho_{n+1}} \mid 1 \le j \le n-1\},$$

$\Delta(\Omega_n)$ is the Abelian group generated by this set, subject to the condition that $\overline{A_{j,n+1}}$ is trivial for all $3 \le j \le n$. So in $\Delta(\Omega_n)$, the elements $\overline{A_{j,n+1}}$ are trivial for all $j = 3, \dots, n-1$. Further, $\overline{A_{n,n+1}}$ is also trivial, hence by relation (12), one of the remaining generators $\overline{A_{j,n+1}}$ may be deleted, where $j \in \{1, 2\}$, say $\overline{A_{2,n+1}}$, from which we see that $\Delta(\Omega_n)$ is a free Abelian group of rank 2 with $\{\overline{A_{1,n+1}}, \overline{\rho_{n+1}}\}$ as a basis. This establishes the isomorphism (28), and so proves part (a).

We now prove part (b) by establishing the isomorphism (29). We equip $K = \operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$ (resp. $H = L_n$) with the basis $\widehat{\mathcal{B}}_n$ (resp. the generating set $\widehat{\mathcal{X}}_n$) of (17) (resp. of (19)). Since $K$ is a free group of rank $2n-1$, it suffices to show that $p(\widehat{K})$ is the trivial subgroup of $K^{\mathrm{Ab}}$. The fact that $K$ is normal in $\Omega_n$ implies that $A_{l,n+1}$, $\rho_{n+1}A_{l,n+1}\rho_{n+1}^{-1}$, $C_{l,n+1}$ and $\rho_{n+1}C_{l,n+1}\rho_{n+1}^{-1}$ belong to $K$ for all $1 \le l \le n$ by (8) and (12). Repeating the argument given between (30) and (31), we see that (31) holds for all $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$, where $\bar{k}$ denotes the element $p(k)$ of $K^{\mathrm{Ab}}$ for all $k \in K$. For $\alpha \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, let $c_\alpha$ denote conjugation in $K$ by $\alpha$ (which we consider to be an element of $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$). Since $K = \Omega_n \cap L_{n+1}$ by the commutative diagram (18), $K$ is normal in $P_{n-1}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, and hence the automorphism $c_\alpha$ is well defined. The fact that $\Gamma_2(K)$ is a characteristic subgroup of $K$ implies that $c_\alpha$ induces an automorphism $\hat{c}_\alpha$ of $K^{\mathrm{Ab}}$ (the inverse of $\hat{c}_\alpha$ is $\hat{c}_{\alpha^{-1}}$). In particular, if $\alpha, \alpha' \in P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$ and $\omega \in K$ then

$$\hat{c}_{\alpha\alpha'}(\overline{\omega}) = \overline{\alpha\alpha'\omega\alpha'^{-1}\alpha^{-1}} = \overline{c_{\alpha\alpha'}(\omega)} = \hat{c}_\alpha(\hat{c}_{\alpha'}(\overline{\omega})).$$

From the first part of the proof, $p(\widehat{K})$ is generated by the elements $\hat{c}_\tau(\overline{\omega})\overline{\omega^{-1}}$, where $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$. To complete the proof of part (b), it suffices to prove that these elements are trivial in $K^{\mathrm{Ab}}$, or equivalently, that $\hat{c}_\tau(\overline{\omega}) = \overline{\omega}$ for all $\tau \in \widehat{\mathcal{X}}_n$ and $\omega \in \widehat{\mathcal{B}}_n$.

(1) First suppose that $\tau = A_{i,j}$, where $3 \le j \le n$ and $1 \le i \le j-2$.

(i) Let $\omega = A_{l,n+1}$, for $1 \le l \le n-1$. Then

$$\tau\omega\tau^{-1} = \begin{cases} A_{l,n+1} & \text{if } j<l \text{ or if } l<i, \\ A_{l,n+1}^{-1}A_{i,n+1}^{-1}A_{l,n+1}A_{i,n+1}A_{l,n+1} & \text{if } j=l, \\ A_{j,n+1}^{-1}A_{l,n+1}A_{j,n+1} & \text{if } i=l, \\ A_{j,n+1}^{-1}A_{i,n+1}^{-1}A_{j,n+1}A_{i,n+1}A_{l,n+1}A_{i,n+1}^{-1}A_{j,n+1}^{-1}A_{i,n+1}A_{j,n+1} & \text{if } i<l<j \end{cases}$$

by the Artin relations (4), from which we conclude that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

(ii) If $\omega = \rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}$, where $1 \le l \le n-1$, then

$$\tau\omega\tau^{-1} = \rho_{n+1}(A_{i,j} A_{l,n+1} A_{i,j}^{-1})\rho_{n+1}^{-1},$$

and from case (i), we deduce also that $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

(iii) Let $\omega = \rho_{n+1}^2$. Then $\tau\omega\tau^{-1} = \omega$, hence $\overline{\tau\omega\tau^{-1}} = \overline{\omega}$.

We conclude that $\hat{c}_{A_{i,j}} = \mathrm{Id}_{K^{\mathrm{Ab}}}$.

(2) Let $\tau = \rho_j A_{i,j} \rho_j^{-1}$, where $3 \le j \le n$ and $1 \le i \le j-2$. Then for all $\omega \in \widehat{\mathcal{B}}_n$, we have

$$\overline{\tau\omega\tau^{-1}} = \overline{c_\tau(\omega)} = \hat{c}_{\rho_j} \circ \hat{c}_{A_{i,j}} \circ \hat{c}_{\rho_j^{-1}}(\overline{\omega}) = \overline{\omega},$$

since $\hat{c}_{A_{i,j}} = \mathrm{Id}_{K^{\mathrm{Ab}}}$, so $\hat{c}_{\rho_j A_{i,j} \rho_j^{-1}} = \mathrm{Id}_{K^{\mathrm{Ab}}}$.

(3) By (19), it remains to study the elements of the form $\overline{\tau\omega\tau^{-1}}$, where $\tau = \rho_j^2$, $3 \le j \le n$, and $\omega \in \widehat{\mathcal{B}}_n$. Since $\hat{c}_{\rho_j^2}(\overline{\omega}) = \overline{\rho_j^2 \omega \rho_j^{-2}} = \hat{c}_{\rho_j}^2(\overline{\omega})$, we first analyse $\hat{c}_{\rho_j}$.

(i) If $\omega = A_{l,n+1}$, where $1 \le l \le n-1$, then by relation (III) and (8) and (12), we have

(32)
$$\begin{aligned}
&\hat{c}_{\rho_j}(\overline{\omega})\\
&\quad = \hat{c}_{\rho_j}(\overline{A_{l,n+1}}) = \overline{\rho_j A_{l,n+1} \rho_j^{-1}}\\
&\quad = \begin{cases} \overline{A_{l,n+1}} & \text{if } j < l,\\ \overline{\rho_{n+1}^{-2} \cdot \rho_{n+1} C_{l,n+1}^{-1} \rho_{n+1}^{-1} \cdot \rho_{n+1}^2} & \text{if } j = l,\\ \overline{\rho_{n+1}^{-2} \cdot \rho_{n+1} C_{j,n+1}^{-1} \rho_{n+1}^{-1} \cdot \rho_{n+1}^2 \cdot A_{l,n+1} \cdot \rho_{n+1}^{-2} \cdot \rho_{n+1} C_{j,n+1} \rho_{n+1}^{-1} \cdot \rho_{n+1}^2} & \text{if } j > l \end{cases}\\
&\quad = \begin{cases} \overline{A_{l,n+1}} & \text{if } j \ne l,\\ \overline{\rho_{n+1} C_{j,n+1}^{-1} \rho_{n+1}^{-1}} = \left(\overline{\rho_{n+1} A_{j,n+1} \rho_{n+1}^{-1}}\right)^{-1} & \text{if } j = l. \end{cases}
\end{aligned}$$

(ii) Let $\omega = \rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}$, where $1 \le l \le n-1$. Relation (IV) implies that $\rho_j \rho_{n+1} \rho_j^{-1} = C_{j,n+1} \rho_{n+1}$, and so by case (i) above, we have

$$\hat{c}_{\rho_j}(\overline{\omega}) = \hat{c}_{\rho_j}\left(\overline{\rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}}\right) = \begin{cases} \overline{\rho_{n+1} A_{l,n+1} \rho_{n+1}^{-1}} & \text{if } j \ne l,\\ \overline{C_{j,n+1}^{-1}} = \left(\overline{A_{j,n+1}}\right)^{-1} & \text{if } j = l. \end{cases} \tag{33}$$

Combining (32) and (33), we see that

$$\hat{c}_{\rho_j^2}(\overline{\omega}) = \overline{\omega} \quad \text{for all } \omega \in \widehat{\mathcal{B}}_n \setminus \{\rho_{n+1}^2\}. \tag{34}$$

(iii) Let $\omega = \rho_{n+1}^2$. By relation (IV) and (8), (12), (32) and (33), we have

$$\begin{aligned}\hat{c}_{\rho_j}(\overline{\omega}) = \hat{c}_{\rho_j}(\overline{\rho_{n+1}^2}) = \overline{(\rho_j\rho_{n+1}\rho_j^{-1})^2} = \overline{C_{j,n+1}\cdot\rho_{n+1}C_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2}\\ = \hat{c}_{\rho_j}\big(\overline{\rho_{n+1}A_{j,n+1}^{-1}\rho_{n+1}^{-1}}\big)\cdot\hat{c}_{\rho_j}(\overline{A_{j,n+1}^{-1}})\cdot\overline{\rho_{n+1}^2},\end{aligned}$$

from which we obtain

$$\hat{c}_{\rho_j^2}\Big(\overline{\rho_{n+1}^2}\Big) = \overline{\rho_{n+1}A_{j,n+1}^{-1}\rho_{n+1}^{-1}}\cdot\overline{A_{j,n+1}^{-1}}\cdot\overline{C_{j,n+1}\cdot\rho_{n+1}C_{j,n+1}\rho_{n+1}^{-1}\cdot\rho_{n+1}^2} = \overline{\rho_{n+1}^2}$$

using (34). So by (17), we also have $\hat{c}_{\rho_j^2} = \mathrm{Id}_{K^{\mathrm{Ab}}}$.

Hence for all $\tau \in \widehat{\mathfrak{X}}_n$ and $\omega \in \widehat{\mathfrak{B}}_n$, it follows that $\hat{c}_\tau(\overline{\omega}) = \overline{\omega}$, and thus $p(\widehat{K})$ is the trivial subgroup of $K^{\mathrm{Ab}}$. We conclude that $\Delta(K) \cong K^{\mathrm{Ab}} \cong \mathbb{Z}^{2n-1}$, and this completes the proof of part (b). □

**Remark 13.** (a) An alternative description of $P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})$, similar to that of (13), but with the parentheses in the opposite order, may be obtained as follows. Let $m \geq 2$ and $q \geq 1$, and consider the Fadell–Neuwirth short exact sequence

$$(35)\quad \begin{aligned}1 \to P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{q+1}\}) \to P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_q\})\\ \to P_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_q\}) \to 1,\end{aligned}$$

given geometrically by forgetting the last $m-1$ strings. Since the quotient is a free group $\mathbb{F}_q$ of rank $q$, the above short exact sequence splits, and so

$$P_m(\mathbb{R}P^2 \setminus \{x_1, \dots, x_q\}) \cong P_{m-1}(\mathbb{R}P^2 \setminus \{x_1, \dots, x_{q+1}\}) \rtimes \mathbb{F}_q,$$

and thus

$$(36)\qquad P_{n-2}(\mathbb{R}P^2 \setminus \{x_1, x_2\})) \cong (\cdots((\mathbb{F}_{n-1} \rtimes \mathbb{F}_{n-2}) \rtimes \mathbb{F}_{n-3}) \rtimes \cdots \rtimes \mathbb{F}_3) \rtimes \mathbb{F}_2$$

by induction. The ordering of the parentheses thus occurs from the left, in contrast with that of (13). The decomposition given by (13) is in some sense stronger than that of (36), since in the first case, every factor acts on each of the preceding factors, which is not necessarily the case in (36), so (13) gives rise to a decomposition of the form (36). This is a manifestation of the fact that the splitting of the corresponding Fadell–Neuwirth sequence (10) is nontrivial, while that of (35) is obvious.

(b) Note that $L_4$, the kernel of the homomorphism $\hat{\iota}_2 : P_2(\mathbb{R}P^2 \setminus \{x_1, x_2\}) \to \mathbb{Z}_2^2$, is also the subgroup of index 4 of the group $(B_4(\mathbb{R}P^2))^{(3)}$ that appears in [Gonçalves and Guaschi 2011, Theorem 3(d)]. Indeed, by equation (127) of that paper, this subgroup of index 4 is isomorphic to the semidirect product

$$\mathbb{F}_5(A_{1,4}, A_{2,4}, \rho_4^2, \rho_4 A_{1,4}\rho_4^{-1}, \rho_4 A_{2,4}\rho_4^{-1}) \rtimes \mathbb{F}_3(A_{2,3}, \rho_3^2, \rho_3 A_{2,3}\rho_3^{-1}),$$

the action being given by equations (129)–(131) of the same paper (the element $B_{i,j}$ of [Gonçalves and Guaschi 2011] is the element $A_{i,j}$ of this paper).

(c) It follows from the proof of Proposition 4(b) that the induced action of $L_n$ on the Abelianisation of $\operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$ is trivial. Since $\operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$ is a free group, its higher homology groups are trivial, and so $L_n$ acts trivially on the homology of $\operatorname{Ker}(\hat{\iota}_{n-1}|_{\Omega_n})$.

**Remark 14.** Using the ideas of the last paragraph of the proof of Proposition 1(b), one may show that $L_n$ is not normal in $B_n(\mathbb{R}P^2)$. Although the subgroup $L_n$ is not unique with respect to the properties of the statement of Proposition 1(a)(ii), there are only a finite number of subgroups, $2^{n(n-2)}$ to be precise, that satisfy these properties. To prove this, we claim that the set of torsion-free subgroups $L'_n$ of $K_n$ such that $K_n = L'_n \oplus \langle \Delta_n^2 \rangle$ is in bijection with the set $\{\operatorname{Ker}(f) \mid f \in \operatorname{Hom}(L_n, \mathbb{Z}_2)\}$. To prove the claim, let $K = K_n$, $L = L_n$, and $q : K \to K/L$ be the canonical surjection, and set

$$\Delta = \{L' \mid L' < K,\ L' \text{ is torsion-free, and } K = L' \oplus \langle \Delta_n^2 \rangle\}.$$

Clearly $L \in \Delta$, so $\Delta \neq \varnothing$. Consider the map $\varphi : \Delta \to \{\operatorname{Ker}(f) \mid f \in \operatorname{Hom}(L, \mathbb{Z}_2)\}$ defined by $\varphi(L') = L \cap L'$. This map is well defined, since if $L' = L$ then $\varphi(L') = L$ is the kernel of the trivial homomorphism of $\operatorname{Hom}(L, \mathbb{Z}_2)$, and if $L' \neq L$ then $L' \not\subset L$ since $[K : L'] = [K : L] = 2$, and so $q|_{L'}$ is surjective as $K/L \cong \mathbb{Z}_2$. Thus $\operatorname{Ker}(q|_{L'}) = \varphi(L')$ is of index 2 in $L$, and in particular, $\varphi(L')$ is the kernel of some nontrivial element of $\operatorname{Hom}(L, \mathbb{Z}_2)$.

We now prove that $\varphi$ is surjective. Let $f \in \operatorname{Hom}(L, \mathbb{Z}_2)$, and set $L'' = \operatorname{Ker}(f)$. If $f = 0$ then $L'' = L$, and $\varphi(L) = L''$. So suppose that $f \neq 0$. Then $f$ is surjective, and $L'' = \operatorname{Ker}(f)$ is of index 2 in $L$. Let $x \in L \setminus L''$. Then

$$L = L'' \amalg xL'', \tag{37}$$

where $\amalg$ denotes the disjoint union. Since $K = L \amalg \Delta_n^2 L$, it follows that

$$K = L'' \amalg xL'' \amalg \Delta_n^2 L'' \amalg x\Delta_n^2 L''. \tag{38}$$

Set $L' = L'' \amalg x\Delta_n^2 L''$. By (37), $x^2 \Delta_n^2 L'' = \Delta_n^2 x^2 L'' = \Delta_n^2 L''$ because $\Delta_n^2$ is central and of order 2, and hence $K = L' \amalg xL'$. Using once more (37), we see that $L'$ is a group, and so the equality $K = L' \amalg xL'$ implies that $[K : L'] = 2$. Further, since the only nontrivial torsion element of $K$ is $\Delta_n^2$, $L'$ is torsion-free by (38), and so the short exact sequence $1 \to L' \to K \to \mathbb{Z}_2 \to 1$ splits. Thus $L' \in \Delta$, and $\varphi(L') = L''$ using (37) and (38).

It remains to prove that $\varphi$ is injective. Let $L'_1, L'_2 \in \Delta$ be such that $L'_1 \cap L = \varphi(L'_1) = \varphi(L'_2) = L'_2 \cap L$. If one of the $L'_i$, say $L'_1$, is equal to $L$ then we must also have $L'_2 = L$ because $L \subset L'_2$ and $L$ and $L'_2$ have the same index in $K$. So suppose that $L'_i \neq L$ for all $i \in \{1, 2\}$. If $i \in \{1, 2\}$ then $L'' = \varphi(L'_i) = L \cap L'_i = \operatorname{Ker}(f_i)$ for some nontrivial $f_i \in \operatorname{Hom}(L, \mathbb{Z}_2)$, and thus $[L : L''] = 2$. Let us show that $L'_1 \subset L'_2$.

Let $x \in L'_1$. If $x \in L$ then $x \in L''$, so $x \in L'_2$, and we are done. So assume that $x \notin L$, and suppose that $x \notin L'_2$. Then $q(x)$ is equal to the nontrivial element of $K/L$, and since $K/L \cong \mathbb{Z}_2$ and $\Delta_n^2 \notin L$, we see that $x\Delta_n^2 \in L$. Further, $K = L'_2 \amalg xL'_2$ since $[K : L'_2] = 2$, and so $x\Delta_n^2 \in L'_2$ (for otherwise $x\Delta_n^2 \in xL'_2$, which implies that $\Delta_n^2 \in L'_2$, which is impossible because $L'_2$ is torsion-free). Then $x\Delta_n^2 \in L \cap L'_2 = L''$, and hence $x\Delta_n^2 \in L'_1$. But this would imply that $\Delta_n^2 \in L'_1$, which contradicts the fact that $L'_1$ is torsion-free. We conclude that $L'_1 \subset L'_2$, and exchanging the rôles of $L'_1$ and $L'_2$, we see that $L'_1 = L'_2$, which proves that $\varphi$ is injective, so is bijective, which proves the claim. Therefore the cardinality of $\Delta$ is equal to the order of the group $H^1(L, \mathbb{Z}_2)$, which is equal in turn to that of $H_1(L, \mathbb{Z}_2)$. By Proposition 4(b), we have $L^{\mathrm{Ab}} = H_1(L, \mathbb{Z}) \cong \mathbb{Z}^{n(n-2)}$, so $H_1(L, \mathbb{Z}_2) \cong H_1(L, \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2^{n(n-2)}$, and the number of subgroups of $K$ that satisfy the properties of Proposition 1(a) is equal to $2^{n(n-2)}$ as asserted.

## 4. The virtual cohomological dimension of $B_n(S)$ and $P_n(S)$ for $S = \mathbb{S}^2, \mathbb{R}P^2$

Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), and for all $m, n \geq 1$, let $\Gamma_{n,m}(S) = P_n(S \setminus \{x_1, \ldots, x_m\})$ denote the $n$-string pure braid group of $S$ with $m$ points removed. In order to study various cohomological properties of the braid groups of $S$ and prove Theorem 5, we shall study $\Gamma_{n,m}(S)$. To prove Theorem 5 in the case $S = \mathbb{S}^2$, by (2), it will suffice to compute the cohomological dimension of $P_{n-3}(\mathbb{S}^2 \setminus \{x_1, x_2, x_3\})$. We recall the following presentation of $\Gamma_{n,m}(\mathbb{S}^2)$ from [Gonçalves and Guaschi 2005]. The result was stated for $m \geq 3$, but it also holds for $m \leq 2$.

**Proposition 15** [Gonçalves and Guaschi 2005, Proposition 7]. *Let $n, m \geq 1$. The following constitutes a presentation of the group $\Gamma_{n,m}(\mathbb{S}^2)$:*

***Generators***: *$A_{i,j}$, where $1 \leq i < j$ and $m+1 \leq j \leq m+n$.*

***Relations***:

(i) *The Artin relations described by* (4) *among the generators $A_{i,j}$ of $\Gamma_{n,m}(\mathbb{S}^2)$.*

(ii) *For all $m+1 \leq j \leq m+n$, $\left(\prod_{i=1}^{j-1} A_{i,j}\right)\left(\prod_{k=j+1}^{m+n} A_{j,k}\right) = 1$.*

Let $N$ denote the kernel of the homomorphism $\Gamma_{n,m}(S) \to \Gamma_{n-1,m}(S)$ obtained geometrically by forgetting the last string. If $S = \mathbb{S}^2$ then $N$ is a free group of rank $m+n-2$ and equals $\langle A_{1,m+n}, A_{2,m+n}, \ldots, A_{m+n-1,m+n}\rangle$. If $S = \mathbb{R}P^2$ then $N$ is a free group of rank $m+n-1$ and equals $\langle A_{1,m+n}, A_{2,m+n}, \ldots, A_{m+n-1,m+n}, \rho_{m+n}\rangle$. Clearly $N$ is normal in $\Gamma_{n,m}(S)$. Further, if $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$), it follows from relations (i) of Proposition 15 (resp. relations (III) and (IV) of Proposition 11) that the action by conjugation of $\Gamma_{n,m}(S)$ on $N$ induces (resp. does not induce) the trivial action on the Abelianisation of $N$. In order to determine the virtual cohomological dimension of the braid groups of $S$ and prove Theorem 5, we shall compute the cohomological dimension of a torsion-free finite-index subgroup. In the case of $\mathbb{S}^2$

(resp. $\mathbb{R}P^2$), we choose the subgroup $\Gamma_{n-3,3}(\mathbb{S}^2)$ that appears in the decomposition given in (2) (resp. the subgroup $\Gamma_{n-2,2}(\mathbb{R}P^2)$ that appears in (3)).

*Proof of Theorem 5.* Let $S=\mathbb{S}^2$ (resp. $S=\mathbb{R}P^2$), let $n>3$ and $k=3$ (resp. $n>2$ and $k=2$), and let $k\leq m<n$. Then by (2) (resp. (3)) and (1), $\Gamma_{n-m,m}(S)$ is a subgroup of finite index of both $P_n(S)$ and $B_n(S)$. Further, since $F_{n-m}(S\setminus\{x_1,\ldots,x_m\})$ is a finite-dimensional CW-complex and an Eilenberg–Mac Lane space of type $K(\pi,1)$ [Fadell and Neuwirth 1962], the cohomological dimension of $\Gamma_{n-m,m}(S)$ is finite, and the first part follows by taking $m=k$.

We now prove the second part, namely that the cohomological dimension of $\Gamma_{n-k,k}(S)$ is equal to $n-k$ for all $n>k$. We first claim that $\mathrm{cd}(\Gamma_{m,l}(S))\leq m$ for all $m\geq 1$ and $l\geq k-1$. The result holds if $m=1$ since $F_1(S\setminus\{x_1,\ldots,x_l\})$ has the homotopy type of a bouquet of circles; therefore $H^i(F_1(S\setminus\{x_1,\ldots,x_l\}),A)$ is trivial for all $i\geq 2$ and for any local coefficients $A$, and $H^1(F_1(S\setminus\{x_1,\ldots,x_l\}),\mathbb{Z})\neq 0$. Suppose by induction that the result holds for some $m\geq 1$, and consider the Fadell–Neuwirth short exact sequence

$$1\to\Gamma_{1,l+m}(S)\to\Gamma_{m+1,l}(S)\to\Gamma_{m,l}(S)\to 1$$

that emanates from the fibration

$$F_1(S\setminus\{x_1,\ldots,x_l,z_1,\ldots,z_m\})\to F_{m+1}(S\setminus\{x_1,\ldots,x_l\})\to F_m(S\setminus\{x_1,\ldots,x_l\}) \tag{39}$$

obtained by forgetting the last coordinate. By [Brown 1982, Chapter VIII], it follows that

$$\mathrm{cd}(\Gamma_{m+1,l}(S))\leq\mathrm{cd}(\Gamma_{m,l}(S))+\mathrm{cd}(\Gamma_{1,l+m}(S))\leq m+1,$$

which proves the claim. In particular, taking $l=k$, we have $\mathrm{cd}(\Gamma_{m,k}(S))\leq m$.

To conclude the proof of the theorem, it suffices to show that for each $m\geq 1$ there are local coefficients $A$ such that $H^m(\Gamma_{m,l}(S),A)\neq 0$ for all $l\geq k-1$. We will show that this is the case for $A=\mathbb{Z}$. Again by induction suppose that $H^m(\Gamma_{m,l}(S),\mathbb{Z})\neq 0$ for all $l\geq k-1$ and for some $m\geq 1$ (we saw above that this is true for $m=1$). Consider the Serre spectral sequence with integral coefficients associated to the fibration (39). Then we have that

$$E_2^{p,q}=H^p\big(\Gamma_{m,l}(S),H^q(F_1(S\setminus\{x_1,\ldots,x_l,z_1,\ldots,z_m\}),\mathbb{Z})\big).$$

Since $\mathrm{cd}(\Gamma_{m,l}(S))\leq m$ and $\mathrm{cd}(F_1(S\setminus\{x_1,\ldots,x_l,z_1,\ldots,z_m\}))\leq 1$ from above, it follows that this spectral sequence has two horizontal lines whose possible nonvanishing terms occur for $0\leq p\leq m$ and $0\leq q\leq 1$. We claim that the group $E_2^{m,1}$ is nontrivial. To see this, first note that $H^1(F_1(S\setminus\{x_1,\ldots,x_l,z_1,\ldots,z_m\}),\mathbb{Z})$ is isomorphic to the free Abelian group of rank $r=m+l-k+2$, so $r\geq m+1$, and hence $E_2^{m,1} = H^m(\Gamma_{m,l}(S), \mathbb{Z}^r)$, where we identify $\mathbb{Z}^r$ with (the dual of) $N^{\mathrm{Ab}}$. The action of $\Gamma_{m,l}(S)$ on $N$ by conjugation induces an action of $\Gamma_{m,l}(S)$ on $N^{\mathrm{Ab}}$. Let $H$ be the subgroup of $N^{\mathrm{Ab}}$ generated by the elements of the form $\alpha(x)x^{-1}$, where $\alpha \in \Gamma_{m,l}(S)$, $x \in N^{\mathrm{Ab}}$, and $\alpha(x)$ represents the action of $\alpha$ on $x$. Then we obtain a short exact sequence $0 \to H \to N^{\mathrm{Ab}} \to N^{\mathrm{Ab}}/H \to 0$ of Abelian groups, and the long exact sequence in cohomology applied to $\Gamma_{m,l}(S)$ yields

(40)
$$\cdots \to H^m(\Gamma_{m,l}(S), N^{\mathrm{Ab}}) \to H^m(\Gamma_{m,l}(S), N^{\mathrm{Ab}}/H) \to H^{m+1}(\Gamma_{m,l}(S), H) \to \cdots .$$

The last term is zero since $\mathrm{cd}(\Gamma_{m,l}(S)) \leq m$, and so the map between the two remaining terms is surjective. Let us determine $N^{\mathrm{Ab}}/H$. If $S = \mathbb{S}^2$ then from the comments following Proposition 15, the action of $\Gamma_{m,l}(S)$ on $N^{\mathrm{Ab}}$ is trivial, so $H$ is trivial, and $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^r$. So suppose that $S = \mathbb{R}P^2$. Choosing the basis

$$\{A_{1,m+l+1}, A_{2,m+l+1}, \ldots, A_{m+l-1,m+l+1}, \rho_{m+l+1}\}$$

of $N$ and using Proposition 11, one sees that the action by conjugation of the generators of $\Gamma_{m,l}(S)$ on the corresponding basis elements of $N^{\mathrm{Ab}}$ is trivial, with the exception of that of $\rho_i$ on $A_{i,m+l+1}$ for $l+1 \leq i \leq m+l-1$, which yields elements $A^2_{i,m+l+1} \in H$ (by abuse of notation, we denote the elements of $N^{\mathrm{Ab}}$ in the same way as those of $N$), and that of $\rho_i$ on $\rho_{m+l+1}$, where $l+1 \leq i \leq m+l$, which yields elements $A_{i,m+l+1} \in H$. In the quotient $N^{\mathrm{Ab}}/H$ the basis elements $A_{l+1,m+l+1}, \ldots, A_{m+l-1,m+l+1}$ thus become zero, and additionally, we have also that $A_{m+l,m+l+1}$ (which is not in the given basis) becomes zero. Hence the relation $\prod_{i=1}^{m+l} A_{i,m+l+1} = \rho^{-2}_{m+l+1}$ is sent to the relation $\prod_{i=1}^{l} A_{i,m+l+1} = \rho^{-2}_{m+l+1}$, and so $N^{\mathrm{Ab}}/H$ is generated by (the images of) the elements $A_{1,m+l+1}, \ldots, A_{l,m+l+1}$, $\rho_{m+l+1}$, subject to this relation (as well as the fact that the elements commute pairwise). It thus follows that $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^l$. Since the induced action of $\Gamma_{m,l}(S)$ on $N^{\mathrm{Ab}}/H$ is trivial, we conclude that

$$H^m\big(\Gamma_{m,l}(S), N^{\mathrm{Ab}}/H\big) = \big(H^m(\Gamma_{m,l}(S), \mathbb{Z})\big)^s,$$

where $s = m+l$ if $S = \mathbb{S}^2$ and $s = l$ if $S = \mathbb{R}P^2$. It then follows from (40) that $E_2^{m,1} = H^m(\Gamma_{m,l}(S), N^{\mathrm{Ab}}) \neq 0$. Since $E_2^{p,q} = 0$ for all $p > m$ and $q > 1$, we have $E_2^{m,1} = E_\infty^{m,1}$, thus $E_\infty^{m,1}$ is nontrivial, and hence $H^{m+1}(\Gamma_{m+1,l}(S), \mathbb{Z}) \neq 0$. □

*Proof of Corollary 6.* Let $S = \mathbb{S}^2$ (resp. $S = \mathbb{R}P^2$). If $n \geq 3$ (resp. $n \geq 2$) then $B_n(S)$ and $\mathcal{MCG}(S, n)$ are closely related by the following short exact sequence [Scott 1970]:

$$1 \to \langle \Delta_n^2 \rangle \to B_n(S) \xrightarrow{\beta} \mathcal{MCG}(S, n) \to 1,$$

where the kernel is isomorphic to $\mathbb{Z}_2$. Now assume that $n \geq 4$ (resp. $n \geq 3$), so that $B_n(S)$ is infinite. If $\Gamma$ is a torsion-free subgroup of $B_n(S)$ of finite index then $\beta(\Gamma)$, which is isomorphic to $\Gamma$, is a torsion-free subgroup of $\mathcal{MCG}(S, n)$ of finite index, and hence the virtual cohomological dimension of $\mathcal{MCG}(S, n)$ is equal to that of $B_n(S)$. The result then follows by Theorem 5. $\square$

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