Pacific Journal of Mathematics

BERNSTEIN-TYPE THEOREMS FOR SPACELIKE STATIONARY GRAPHS IN MINKOWSKI SPACES

XIANG MA, PENG WANG AND LING YANG

Volume 287 No. 1 March 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.287.159

BERNSTEIN-TYPE THEOREMS FOR SPACELIKE STATIONARY GRAPHS IN MINKOWSKI SPACES

XIANG MA, PENG WANG AND LING YANG

For entire spacelike stationary 2-dimensional graphs in Minkowski spaces, we establish Bernstein-type theorems under specific boundedness assumptions either on the W-function or on the total (Gaussian) curvature. These conclusions imply the classical Bernstein theorem for minimal surfaces in $\mathbb{R}^3$ and Calabi's theorem for spacelike maximal surfaces in $\mathbb{R}^3_1$.

1. Introduction

The classical Bernstein theorem [1915] says that any entire minimal graph in $\mathbb{R}^3$ has to be an affine plane. In other words, suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 0. \tag{1-1}$$

Then f has to be affine linear. This conclusion is generally not true in the higher-codimensional case. The simplest counterexample is the minimal graph $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{C}\} \subset \mathbb{R}^4$ of an arbitrary nonlinear holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

To find a suitable generalization, usually we have to add some boundedness assumptions on the growth rate of the function f. Chern and Osserman [1967] obtained one such Bernstein-type theorem as follows. Suppose that $f = (f_1, \dots, f_m)$ is a smooth vector-valued function from $\mathbb{R}^2$ to $\mathbb{R}^m$. If $M = \operatorname{graph} f$ is a minimal graph, and

$$W := \left[\det\left(\delta_{ij} + \sum_{1\le\alpha\le m} \frac{\partial f_\alpha}{\partial x_i}\frac{\partial f_\alpha}{\partial x_j}\right)\right]^{1/2} \tag{1-2}$$

is uniformly bounded, then M has to be an affine plane.

This W-function is a significant quantity for various reasons.

Ma is supported by NSFC Project 11471021; Wang is supported by NSFC Projects 11571255 and 11471078; Yang is supported by NSFC Project 11471078.
MSC2010: 30F15, 53C24, 58J05.
Keywords: Bernstein problems, spacelike, stationary, entire graph.

For any $f:\mathbb{R}^2\to\mathbb{R}^m$, denote the metric on graph(f) as $g=g_{ij}\,dx_i\,dx_j$ under the global coordinate chart $x=(x_1,x_2)\mapsto(x,\,f(x))\in$ graph f. Then the area element is given by $W\,dx_1\wedge dx_2$. Thus W is a geometric measure of the area growth of the graph of f.

Secondly, Chern and Osserman's theorem can be stated in the language of PDEs as below. Namely, the entire solution to the PDE system

$$\begin{aligned}&\sum_{1\le i\le 2}\frac{\partial}{\partial x_i}(Wg^{ij})=0,\qquad j=1,2,\\ &\sum_{1\le i,j\le 2}\frac{\partial}{\partial x_i}\left(Wg^{ij}\frac{\partial f_\alpha}{\partial x_j}\right)=0,\qquad \alpha=1,\dots,m\end{aligned}\tag{1-3}$$

has to be affine linear provided that $W\le C$ for a positive constant C, where

$$(g_{ij}):=I_2+J_f^T E J_f\tag{1-4}$$

(I_2 and E denote the identity matrices of size 2 and m separately and J_f is the Jacobian matrix of f), $(g^{ij}):=(g_{ij})^{-1}$ and $W=\det(g_{ij})^{1/2}$. A key point from the analytic viewpoint is that the boundedness of W ensures that (1-3) is a uniformly elliptic PDE system.

For more work on the generalization of Chern and Osserman's theorem in relation to the W-function, see [Barbosa 1979], [Fischer-Colbrie 1980], [Jost et al. 2014; 2015].

Now we consider entire spacelike stationary graphs in Minkowski spaces. They too correspond to solutions to (1-3), the differences being that $f=(f_1,\dots,f_m)$ is now from $\mathbb{R}^2$ to m-dimensional Minkowski space $\mathbb{R}^m_1$, and the E appearing in (1-4) should be replaced by the Minkowski inner product matrix diag$(1,1,\dots,1,-1)$. Here we need to assume that (g_{ij}) is positive definite everywhere.

When $m=1$, M becomes a spacelike maximal graph in $\mathbb{R}^3_1$, which has to be an affine plane. This is a well-known Bernstein-type result by E. Calabi [1970]. But for higher-codimensional cases, the Bernstein-type result fails to be true even if the W-function is uniformly bounded. Such a counterexample, which can be found in [Ma et al. 2013], is given by the function

$$f(x_1,x_2)=\big(2\sinh(x_1)\cos\big(-\tfrac{\sqrt{2}}{2}x_2\big),\,2\cosh(x_1)\cos\big(-\tfrac{\sqrt{2}}{2}x_2\big)\big).$$

So it is a more subtle problem about the value distribution of the W-function for entire spacelike stationary graphs in Minkowski spaces. This is the main topic of the present paper.

As the first step, we generalize Osserman's result [1969, §5] to entire spacelike stationary graphs in the Minkowski space. They are still conformally equivalent to the complex plane (see Theorem 3.1), and have an explicit simple representation formula. Based on these formulas, we establish the following results:

- Let M be an entire spacelike stationary graph in $\mathbb{R}^4_1$. Then the W-function is either constant, or takes each value in $[r^{-1}, r]$ infinitely often, where r can be any positive number strictly bigger than 1. Moreover, W is a constant if and only if M is a flat surface (see Theorem 4.1).
- For any entire spacelike stationary graph M in $\mathbb{R}^4_1$, if $W \le 1$ (or $W \ge 1$) always holds true on M, then M has to be flat (see Corollary 4.2). Note that Calabi's theorem [1970] and the classical Bernstein theorem [1915] can easily be deduced from the above two conclusions, respectively.
- For any entire spacelike stationary graph M in $\mathbb{R}^n_1$ $(n \ge 4)$, if $W \le 1$, then M must be flat (see Theorem 5.1). (On the contrary, the same conclusion does not necessarily hold true in the case $W \ge 1$; see Proposition 5.2.)

Another measure of the complexity of a complete stationary surface is its total Gaussian curvature $\int_M |K|\,dM$. This is closely related with its end behavior at infinity; see the generalized Jorge–Meeks formula in [Ma et al. 2013]. Using the Weierstrass representation formula given in the same work, one can compute the integral of the Gauss curvature and the normal curvature of an arbitrary spacelike stationary surface in $\mathbb{R}^4_1$. A Bernstein-type theorem (Theorem 6.1) follows immediately, which states that an entire spacelike stationary graph in $\mathbb{R}^4_1$ has to be flat, provided that $\int_M |K|\,dM < \infty$. (This result cannot be generalized to higher-codimensional cases.)

2. Entire graphs in Minkowski spaces and the W-function

Let $\mathbb{R}^m_1$ denote the m-dimensional Minkowski space. The Minkowski inner product of any $\boldsymbol{u} = (u_1, \dots, u_{m-1}, u_m)$ and $\boldsymbol{v} = (v_1, \dots, v_{m-1}, v_m) \in \mathbb{R}^m_1$ is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \cdots + u_{m-1} v_{m-1} - u_m v_m. \tag{2-1}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^m_1$

$$(x_1, x_2) \mapsto f(x_1, x_2) = (f_1(x_1, x_2), \dots, f_m(x_1, x_2)) \tag{2-2}$$

be a smooth vector-valued function. As in §3 of [Osserman 1969], we introduce the vector notation

$$p := \frac{\partial f}{\partial x_1}, \qquad q := \frac{\partial f}{\partial x_2}. \tag{2-3}$$

Let $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ be the entire graph in $\mathbb{R}^{2+m}_1$ generated by f. Then the metric on M is

$$g = g_{ij}\, dx_i\, dx_j, \tag{2-4}$$

with

$$g_{11} = 1 + \langle p, p\rangle, \quad g_{22} = 1 + \langle q, q\rangle, \quad g_{12} = g_{21} = \langle p, q\rangle. \tag{2-5}$$

According to the properties of positive definite matrices, M is a spacelike surface if and only if $1 + \langle p, p\rangle > 0$ and $\det(g_{ij}) > 0$. Hence

$$W = \det(g_{ij})^{1/2} > 0 \tag{2-6}$$

for any spacelike graph.

Denote by $\mathcal{P}_0$ the orthogonal projection of $\mathbb{R}^{2+m}_1$ onto $\mathbb{R}^2$. Then $w := W^{-1}$ is equivalent to the Jacobian determinant of $\mathcal{P}_0|_M$. Thus $W \le 1$ (resp., $\equiv 1$, ≥ 1) is equivalent to saying that $\mathcal{P}_0|_M$ is an area-increasing (resp., area-preserving, area-decreasing) map.

For entire graphs in Euclidean space, it is well known that the orthogonal projection onto the coordinate plane is a length-decreasing map, which becomes an isometry if and only if the graph is parallel to the coordinate plane. Therefore every entire graph in Euclidean space must be complete. But the following examples show these properties cannot be generalized to entire graphs in Minkowski spaces.

Examples.

- Let $\boldsymbol{y}_0$ be a nonzero lightlike vector in $\mathbb{R}^m_1$, h be a smooth real-valued function on $\mathbb{R}^2$ and $f := h\boldsymbol{y}_0$. Then

$$p = \frac{\partial h}{\partial x_1}\boldsymbol{y}_0 \quad \text{and} \quad q = \frac{\partial h}{\partial x_2}\boldsymbol{y}_0,$$

and hence $g_{ij} = \delta_{ij}$, which implies the projection of $M = \operatorname{graph} f$ onto $\mathbb{R}^2$ is an isometry, but M cannot be an affine plane of $\mathbb{R}^{2+m}_1$ whenever h is nonlinear.

- Let $t \in \mathbb{R} \mapsto \theta(t) \in (-\pi/2, \pi/2)$ be a smooth odd function which satisfies $\lim_{t\to+\infty} \theta(t) = \pi/2$ and $\pi/2 - \theta(t) = O(t^{-2})$. Denote

$$h(t) := \int_0^t \sin\theta(t)\, dt.$$

Then h is a smooth even function on $\mathbb{R}$. Define

$$f(x_1, x_2) = (0, \dots, 0, h(r)) \qquad \left(r = \sqrt{x_1^2 + x_2^2}\right).$$

Then $p = \partial f/\partial x_1 = (0, \ldots, 0, h'(r)x_1/r)$, $q = \partial f/\partial x_2 = (0, \ldots, 0, h'(r)x_2/r)$ and hence

$$g_{11} = 1 + \langle p, p\rangle = 1 - \frac{h'(r)^2 x_1^2}{r^2} \geq 1 - h'(r)^2 = \cos^2\theta(t) > 0,$$

$$\det(g_{ij}) = \det\begin{pmatrix} 1 - \dfrac{h'(r)^2 x_1^2}{r^2} & -\dfrac{h'(r)^2 x_1 x_2}{r^2} \\ -\dfrac{h'(r)^2 x_1 x_2}{r^2} & 1 - \dfrac{h'(r)^2 x_2^2}{r^2} \end{pmatrix} = 1 - h'(r)^2 > 0.$$

Therefore $M = \operatorname{graph} f$ is an entire spacelike graph. Define $\gamma : \mathbb{R} \to \mathbb{R}^3$ by $\gamma(t) = (t, 0, f(t, 0))$. Then γ is a smooth curve in M tending to infinity. Since $f(t, 0) = (0, \ldots, 0, h(t))$,

$$L(\gamma) = \int_{-\infty}^{\infty} \sqrt{1 - h'(t)^2}\, dt = \int_{-\infty}^{\infty} \cos\theta(t)\, dt.$$

But $\cos\theta(t) \sim \pi/2 - |\theta(t)| \sim |t|^{-2}$ when $t \to \infty$. Therefore $L(\gamma) < \infty$ and hence M cannot be complete.

3. Isothermal parameters of spacelike stationary graphs

Let $\boldsymbol{x} : M \to \mathbb{R}^{2+m}_1$ be a spacelike surface in Minkowski space. If the mean curvature vector field $\boldsymbol{H}$ vanishes everywhere, then M is said to be *stationary*. M is stationary if and only if the restriction of any coordinate function on M is harmonic. Namely, $\Delta x_l \equiv 0$ for each $1 \leq l \leq 2+m$, with Δ the Laplace–Beltrami operator with respect to the induced metric on M; see [Ma et al. 2013]. Now we additionally assume M to be an entire graph over $\mathbb{R}^2$. More precisely, there exists $f : \mathbb{R}^2 \to \mathbb{R}^m_1$, such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$. The denotation of p, q, g_{ij}, W is the same as in Section 2. For an arbitrary smooth function F on M,

(3-1)
$$\Delta F = W^{-1}\partial_i(W g^{ij}\partial_j F),$$

where

(3-2)
$$(g^{ij}) = (g_{ij})^{-1} = W^{-2}\begin{pmatrix} 1 + \langle q, q\rangle & -\langle p, q\rangle \\ -\langle p, q\rangle & 1 + \langle p, p\rangle \end{pmatrix}.$$

The stationarity of M implies x_1 and x_2 are both harmonic functions on M, hence

(3-3)
$$\begin{aligned} 0 = W\Delta x_1 &= \partial_i(W g^{ij}\partial_j x_1) = \partial_i(W g^{ij}\delta_{1j}) = \partial_i(W g^{i1}) \\ &= \frac{\partial}{\partial x_1}\Big(\frac{1 + \langle q, q\rangle}{W}\Big) - \frac{\partial}{\partial x_2}\Big(\frac{\langle p, q\rangle}{W}\Big), \end{aligned}$$

and similarly,

$$0 = W\Delta x_2 = \partial_i (W g^{i2}) = -\frac{\partial}{\partial x_1}\Big(\frac{\langle p, q\rangle}{W}\Big) + \frac{\partial}{\partial x_2}\Big(\frac{1+\langle p, p\rangle}{W}\Big). \tag{3-4}$$

The above two equations imply the existence of smooth functions ξ_1 and ξ_2 such that

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1+\langle p, p\rangle}{W}, \quad \frac{\partial \xi_1}{\partial x_2} = \frac{\langle p, q\rangle}{W}, \quad \frac{\partial \xi_2}{\partial x_1} = \frac{\langle p, q\rangle}{W}, \quad \frac{\partial \xi_2}{\partial x_2} = \frac{1+\langle q, q\rangle}{W}. \tag{3-5}$$

As in §5 of [Osserman 1969], one can define the Lewy's transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L : (x_1, x_2) \mapsto (\eta_1, \eta_2)$ by

$$\eta_i = x_i + \xi_i(x_1, x_2), \quad i = 1, 2. \tag{3-6}$$

Since the Jacobi matrix of L,

$$J_L = I_2 + \left(\frac{\partial \xi_i}{\partial x_j}\right) = I_2 + W^{-1}(g_{ij}), \tag{3-7}$$

is positive definite, L is a local diffeomorphism. Again based on the fact that $\big(\partial\xi_i/\partial x_j\big)$ is positive definite, one can proceed as in [Lewy 1937] or §5 of [Osserman 1969] to show that L is length-increasing, thus L is injective. Let Ω be the image of L. Then Ω is open. If $\Omega \neq \mathbb{R}^2$, take η in the complement of Ω that is nearest to $L(0)$, and find a sequence of points $\{\eta^{(k)} : k \in \mathbb{Z}^+\}$ such that $|\eta^{(k)} - L(0)| < |\eta - L(0)|$ and $\lim_{k\to\infty} \eta^{(k)} = \eta$. Then there exists $x^{(k)} \in \mathbb{R}^2$ such that $\eta^{(k)} = L(x^{(k)})$. Since L is length-increasing, $\{x^{(k)} : k \in \mathbb{Z}^+\}$ lies in a bounded domain of $\mathbb{R}^2$, so there exists an subsequence converging to $x \in \mathbb{R}^2$, which implies $L(x) = \eta$ and causes a contradiction. Therefore $\Omega = \mathbb{R}^2$ and then L is a diffeomorphism of $\mathbb{R}^2$ onto itself.

Denote by λ_1^2, λ_2^2 (where $\lambda_1, \lambda_2 > 0$) the eigenvalues of (g_{ij}). Then $W = \det(g_{ij})^{1/2} = \lambda_1\lambda_2$, and there exists an orthogonal matrix O, such that

$$(g_{ij}) = O^T \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} O.$$

Hence,

$$J_L = I_2 + W^{-1}(g_{ij}) = O^T \begin{pmatrix} 1 + \dfrac{\lambda_1}{\lambda_2} & \\ & 1 + \dfrac{\lambda_2}{\lambda_1} \end{pmatrix} O = (\lambda_1^{-1} + \lambda_2^{-1}) O^T \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} O,$$

and furthermore,

$$\begin{aligned} d\eta_1^2 + d\eta_2^2 &= \begin{pmatrix} d\eta_1 & d\eta_2 \end{pmatrix} \begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix} = \begin{pmatrix} dx_1 & dx_2 \end{pmatrix} J_L^T J_L \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \\ &= (\lambda_1^{-1} + \lambda_2^{-1})^2 \begin{pmatrix} dx_1 & dx_2 \end{pmatrix} O^T \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{pmatrix} O \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (\lambda_1^{-1}+\lambda_2^{-1})^2 \begin{pmatrix} dx_1 & dx_2 \end{pmatrix} (g_{ij}) \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \\
&= (\lambda_1^{-1}+\lambda_2^{-1})^2 (g_{ij}\, dx_i\, dx_j),
\end{aligned}
$$

i.e.,

$$
g = g_{ij}\, dx_i\, dx_j = (\lambda_1^{-1}+\lambda_2^{-1})^{-2}(d\eta_1^2 + d\eta_2^2). \tag{3-8}
$$

This means that (η_1, η_2) are global isothermal parameters on M. Define

$$
\zeta := \eta_1 + \sqrt{-1}\eta_2 \tag{3-9}
$$

and

$$
\beta_l := \frac{\partial x_l}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial x_l}{\partial \eta_1} - \sqrt{-1}\frac{\partial x_l}{\partial \eta_2}\right) \quad \text{for} \quad l = 1, \dots, 2+m. \tag{3-10}
$$

Then the harmonicity of coordinate functions implies

$$
0 = \frac{\partial^2 x_l}{\partial \zeta \partial \bar{\zeta}} = \frac{\partial \beta_l}{\partial \bar{\zeta}},
$$

i.e., $\beta_1, \dots, \beta_{2+m}$ are all holomorphic functions on M. A straightforward calculation shows $-4\,\mathrm{Im}(\bar{\beta}_1\beta_2)$ equals the Jacobian of the inverse of Lewy's transformation, which is positive everywhere, thus $\beta_2/\beta_1 = \bar{\beta}_1\beta_2/|\beta_1|^2$ is an entire function whose imaginary part is always negative. The classical Liouville's theorem implies $\beta_2/\beta_1 \equiv c := a - b\sqrt{-1}$, where $a, b \in \mathbb{R}$ and $b > 0$. In conjunction with (3-10) we get

$$
\frac{\partial x_2}{\partial \eta_1} = a\frac{\partial x_1}{\partial \eta_1} - b\frac{\partial x_1}{\partial \eta_2} \quad \text{and} \quad \frac{\partial x_2}{\partial \eta_2} = b\frac{\partial x_1}{\partial \eta_1} + a\frac{\partial x_1}{\partial \eta_2}. \tag{3-11}
$$

Let (u_1, u_2) be global parameters of M, satisfying $x_1 = u_1$ and $x_2 = au_1 + bu_2$. Then (3-11) tells us

$$
\frac{\partial u_2}{\partial \eta_1} = -\frac{\partial u_1}{\partial \eta_2} \quad \text{and} \quad \frac{\partial u_2}{\partial \eta_2} = \frac{\partial u_1}{\partial \eta_1}. \tag{3-12}
$$

This means the one-to-one map $(\eta_1, \eta_2) \in \mathbb{R}^2 \mapsto (u_1, u_2) \in \mathbb{R}^2$ is biholomorphic. Thereby we arrive at the following conclusion:

Theorem 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^m_1$ be a smooth vector-valued function such that $M = \mathrm{graph}\, f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a spacelike stationary surface. Then there exists a nonsingular linear transformation*

$$
\begin{aligned}
x_1 &= u_1, \\
x_2 &= au_1 + bu_2 \quad (b > 0),
\end{aligned} \tag{3-13}
$$

such that (u_1, u_2) are global isothermal parameters for M.

Now we introduce the complex coordinate $z := u_1 + \sqrt{-1}u_2$ and define

$$\alpha = (\alpha_1, \dots, \alpha_{2+m}) := \frac{\partial \boldsymbol{x}}{\partial z} = \frac{1}{2}\Big(\frac{\partial \boldsymbol{x}}{\partial u_1} - \sqrt{-1}\frac{\partial \boldsymbol{x}}{\partial u_2}\Big). \tag{3-14}$$

Then α is a holomorphic vector-valued function. The induced metric on M can be written as

$$\begin{aligned} g &= \Big\langle \frac{\partial \boldsymbol{x}}{\partial z}, \frac{\partial \boldsymbol{x}}{\partial z}\Big\rangle dz^2 + \Big\langle \frac{\partial \boldsymbol{x}}{\partial \bar{z}}, \frac{\partial \boldsymbol{x}}{\partial \bar{z}}\Big\rangle d\bar{z}^2 + 2\Big\langle \frac{\partial \boldsymbol{x}}{\partial z}, \frac{\partial \boldsymbol{x}}{\partial \bar{z}}\Big\rangle |dz|^2 \\ &= 2\,\mathrm{Re}\big(\langle \alpha, \alpha\rangle\, dz^2\big) + 2\langle \alpha, \bar{\alpha}\rangle |dz|^2. \end{aligned}$$

Here $|dz|^2 := \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = du_1^2 + du_2^2$. Since (u_1, u_2) are isothermal parameters for M,

$$\langle \alpha, \alpha\rangle = 0, \tag{3-15}$$

and hence

$$g = 2\langle \alpha, \bar{\alpha}\rangle |dz|^2. \tag{3-16}$$

Noting that $\alpha_1 = \partial x_1/\partial z = \frac{1}{2}$, $\alpha_2 = \partial x_2/\partial z = \frac{1}{2}(a - b\sqrt{-1}) = \frac{1}{2}c$, (3-15) is equivalent to

$$\alpha_{2+m}^2 = \alpha_1^2 + \cdots + \alpha_{1+m}^2 = \frac{1+c^2}{4} + \alpha_3^2 + \cdots + \alpha_{1+m}^2. \tag{3-17}$$

Thus

$$\begin{aligned} \langle \alpha, \bar{\alpha}\rangle &= |\alpha_1|^2 + \cdots + |\alpha_{1+m}|^2 - |\alpha_{2+m}|^2 \\ &= \frac{1+|c|^2}{4} + |\alpha_3|^2 + \cdots + |\alpha_{1+m}|^2 - \Big|\frac{1+c^2}{4} + \alpha_3^2 + \cdots + \alpha_{1+m}^2\Big| \\ &\geq \frac{1+|c|^2 - |1+c^2|}{4}, \end{aligned}$$

and moreover,

$$g \geq \frac{1+|c|^2 - |1+c^2|}{2}|dz|^2. \tag{3-18}$$

Observing that $1 + |c|^2 - |1 + c^2| > 0$ is a direct corollary of $b > 0$, we get a conclusion as follows.

Corollary 3.2. *Let $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ be a spacelike stationary graph generated by $f : \mathbb{R}^2 \to \mathbb{R}_1^m$. Then the induced metric on M is complete.*

Remark. As shown in [Cheng and Yau 1976], if M is a spacelike hypersurface in $\mathbb{R}_1^{n+1}$ with constant mean curvature, so that M is a closed subset of $\mathbb{R}_1^{n+1}$ with respect to the Euclidean topology, then M is complete with respect to the induced

Lorentz metric. It is natural to raise the following problem. *Let M be an n-dimensional spacelike submanifold in $\mathbb{R}^{n+m}_1$ with parallel mean curvature, so that M is a closed subset of $\mathbb{R}^{n+m}_1$. Is M a complete Riemannian manifold?* Corollary 3.2 gives a partial positive answer to the above problem.

Equation (3-13) implies $dx_1 \wedge dx_2 = b\, du_1 \wedge du_2$, and hence

$$\begin{aligned} dM &= 2\langle \alpha, \bar{\alpha}\rangle\, du_1 \wedge du_2 \\ &= 2b^{-1}\langle \alpha, \bar{\alpha}\rangle\, dx_1 \wedge dx_2 \\ &= \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}\, dx_1 \wedge dx_2, \end{aligned}$$

with dM the area element of M. In other words,

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}. \tag{3-19}$$

4. On W-functions for entire stationary graphs in $\mathbb{R}^4_1$

Theorem 4.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2_1$ be a smooth function, such that $M = \operatorname{graph} f$ is a spacelike stationary graph. Then one and only one of the following three cases occurs*:

(i) *f is affine linear and $W \equiv r$, where r is an arbitrary positive constant.*

(ii) *$f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$ with h a nonlinear harmonic function on $\mathbb{R}^2$, $\boldsymbol{y}_0$ a nonzero lightlike vector in $\mathbb{R}^2_1$ and $\boldsymbol{y}_1$ a constant vector, and $W \equiv 1$.*

(iii) *W takes each value in $[r^{-1}, r]$ infinitely often, where r is an arbitrary number in $(1, \infty)$.*

Proof. Equation (3-15) is equivalent to

$$\alpha_3^2 - \alpha_4^2 = -(\alpha_1^2 + \alpha_2^2) = -\frac{1+c^2}{4}, \tag{4-1}$$

and (3-19) gives

$$W = \frac{1+|c|^2+4(|\alpha_3|^2-|\alpha_4|^2)}{2b}. \tag{4-2}$$

If α_3 is a constant function, then (4-1) shows α_4 is also constant, and

$$x_a(z) = \operatorname{Re}\int_0^z \alpha_a\, dz + x_a(0) \qquad \text{for all } a = 3, 4$$

is affine linear. Hence f is affine linear and $W \equiv r$, where r can be taken to be any value in $(0, \infty)$. This is case (i).

Now we assume α_3 is nonconstant. Then (4-1) implies α_4 is also nonconstant.

If $c = -\sqrt{-1}$, then (4-1) gives

$$0 = \alpha_3^2 - \alpha_4^2 = (\alpha_3 + \alpha_4)(\alpha_3 - \alpha_4).$$

Noting that the zeros of a nonconstant holomorphic function have to be isolated, we get $\alpha_3 + \alpha_4 = 0$ or $\alpha_3 - \alpha_4 = 0$. Thus $|\alpha_3| = |\alpha_4|$ and then (4-2) shows $W \equiv 1$. Let β be the unique holomorphic function such that $\beta' = \alpha_3$ and $\beta(0) = 0$. Then $\alpha_3 \pm \alpha_4 = 0$ implies

$$\begin{aligned} f(x_1, x_2) &= (x_3(u_1, u_2), x_4(u_1, u_2)) = (x_3(z), x_4(z)) \\ &= \operatorname{Re} \int_0^z (\alpha_3, \alpha_4)\, dz + (x_3(0), x_4(0)) \\ &= \operatorname{Re} \beta(z)(1, \mp 1) + f(0, 0). \end{aligned}$$

Now we put $h := \operatorname{Re} \beta(z)$, $\boldsymbol{y}_0 := (1, \mp 1)$ and $\boldsymbol{y}_1 := f(0, 0)$. Then h is a nonlinear harmonic function, $\boldsymbol{y}_0$ is a lightlike vector and $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$. This is case (ii).

Otherwise $c \neq -\sqrt{-1}$ and hence $-(1 + c^2)/4 \neq 0$. Let $\mu \neq 0$ such that $\mu^2 = -(1 + c^2)/4$, and h_1, h_2 be holomorphic functions such that $\alpha_3 = \mu h_1$, $\alpha_4 = \mu h_2$. Then $\mu^2(h_1^2 - h_2^2) = \alpha_3^2 - \alpha_4^2 = \mu^2$ gives

$$1 = h_1^2 - h_2^2 = (h_1 + h_2)(h_1 - h_2),$$

which implies $h_1 + h_2$ is an entire function containing no zero. Hence there exists an entire function β, such that $h_1 + h_2 = e^{\beta}$, then $h_1 - h_2 = e^{-\beta}$ and hence

$$h_1 = \cosh \beta, \qquad h_2 = \sinh \beta. \tag{4-3}$$

By computing,

$$\begin{aligned} |h_1|^2 - |h_2|^2 &= |\cosh \beta|^2 - |\sinh \beta|^2 \\ &= \tfrac{1}{2}(e^{\beta - \bar\beta} + e^{-\beta + \bar\beta}) = \tfrac{1}{2}\big(e^{2 \operatorname{Im} \beta \sqrt{-1}} + e^{-2 \operatorname{Im} \beta \sqrt{-1}}\big) \\ &= \cos(2 \operatorname{Im} \beta), \end{aligned}$$

and hence

$$\begin{aligned} W &= \frac{1 + |c|^2 + 4(|\alpha_3|^2 - |\alpha_4|^2)}{2b} = \frac{1 + |c|^2 + 4|\mu|^2(|h_1|^2 - |h_2|^2)}{2b} \\ &= \frac{1 + |c|^2 + |1 + c^2| \cos(2 \operatorname{Im} \beta)}{2b}. \end{aligned} \tag{4-4}$$

Set

$$r_1 := \inf W = \frac{1 + |c|^2 - |1 + c^2|}{2b} \quad \text{and} \quad r_2 := \sup W = \frac{1 + |c|^2 + |1 + c^2|}{2b}.$$

Due to Picard's theorem, W takes each value in $[r_1, r_2]$ infinitely often. Noting that

$c = a - b\sqrt{-1}$, one computes

$$r_1 r_2 = \frac{(1+|c|^2)^2 - |1+c^2|^2}{4b^2} = \frac{1+2|c|^2+|c|^4-(1+c^2+\bar{c}^{\,2}+|c|^4)}{4b^2} = \frac{4b^2}{4b^2} = 1.$$

Hence $r_1 \in (0, 1)$ and $r_2 \in (1, \infty)$.

Now we take $b := 1$. Then $c = a - \sqrt{-1}$ and $r_2 = \frac{1}{2}(2 + a^2 + |a|\sqrt{a^2+4})$. Denote $\mu : t \in \mathbb{R}^+ \mapsto \mu(t) = \frac{1}{2}(2+t^2+|t|\sqrt{t^2+4})$. Then μ is a strictly increasing function and $\lim_{t\to 0}\mu(t) = 1$, $\lim_{t\to+\infty}\mu(t) = +\infty$. Hence for an arbitrary number $r \in (1, \infty)$, one can find $a \in \mathbb{R}^+$, such that $r_2 = r$ and then W takes each value in $[r^{-1}, r]$ infinitely often. This is case (iii). $\square$

Corollary 4.2. *Let M be an entire spacelike stationary graph in $\mathbb{R}^4_1$ generated by a smooth function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2_1$. If $W \le 1$ (or $W \ge 1$), then f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, with h a nonlinear harmonic function, $\boldsymbol{y}_0$ a nonzero lightlike vector and $\boldsymbol{y}_1$ a constant vector. Moreover, $W > 1$ (or $W < 1$) forces f to be affine linear, representing an affine plane in $\mathbb{R}^4_1$.*

Remark. If $f_2 \equiv 0$, then $M = \operatorname{graph} f$ is a minimal entire graph in $\mathbb{R}^3$ and $W \ge 1$. By Corollary 4.2, f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, where h is a nonlinear harmonic function and $\boldsymbol{y}_0$ is a nonzero lightlike vector. But $f_2 \equiv 0$ precludes the latter case. Hence f is an affine linear function and so the classical Bernstein theorem [1915] can be derived from Corollary 4.2. Similarly, Corollary 4.2 implies any spacelike maximal entire graph in $\mathbb{R}^3_1$ has to be affine linear. This is Calabi's theorem [1970].

5. Bernstein-type theorems for entire stationary graphs in $\mathbb{R}^{2+m}_1$

It is natural to ask whether one can generalize the conclusion of Corollary 4.2 to higher-codimensional cases.

For the first statement, i.e., $W \le 1$, the answer is "yes":

Theorem 5.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^m_1$ be a smooth function, such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a spacelike stationary graph in $\mathbb{R}^{2+m}_1$. If the orthogonal projection $\mathcal{P}_0$ of M onto the coordinate plane $\mathbb{R}^2$ is area-increasing (i.e., $W \le 1$), then f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, with h a nonlinear harmonic function, $\boldsymbol{y}_0$ a nonzero lightlike vector and $\boldsymbol{y}_1$ a constant vector. Moreover, if $\mathcal{P}_0$ is strictly area-increasing (i.e., $W < 1$), then f has to be affine linear and M is an affine plane.*

Proof. We shall consider the problem in the following four cases.

Case I. $\alpha_3, \dots, \alpha_{2+m}$ are all constant functions. As in the proof of Theorem 4.1, one can show f is an affine linear function.

Case II. α_{2+m} is a constant function, but α_l is nonconstant for some $3 \le l \le 1+m$. By the classical Liouville Theorem, there exists a point in $\mathbb{C}$, such that

$$|\alpha_l|^2 \ge |\alpha_{2+m}|^2 + b - \tfrac{1}{4}(1+|c|^2)$$

at this point. Combing with (3-19) gives

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$\geq \frac{1+|c|^2+4(|\alpha_l|^2-|\alpha_{2+m}|^2)}{2b} \geq 2.$$

This gives a contradiction to the assumption that $W \leq 1$ everywhere. Hence this case cannot occur.

Case III. α_{2+m} *is nonconstant and* $c \neq -\sqrt{-1}$. Then $c \neq \sqrt{-1}$ implies

$$\frac{1+|c|^2}{2b} = \frac{1+b^2+a^2}{2b} > 1.$$

Denote $\delta := (1+|c|^2)/(2b) - 1$. Again the classical Liouville theorem implies the existence of a point such that $|\alpha_{2+m}|^2 < \frac{1}{2}b\delta$ at this point. Hence

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$\geq \frac{1+|c|^2-4|\alpha_{2+m}|^2}{2b} > 1+\delta - \frac{4\cdot\frac{1}{2}b\delta}{2b} = 1,$$

which causes a contradiction and therefore this case cannot happen.

Case IV. α_{2+m} *is nonconstant and* $c = -\sqrt{-1}$. Let $h_3, \ldots, h_{1+m}$ be meromorphic functions such that

$$\alpha_3^2 = h_3\alpha_{2+m}^2, \ldots, \alpha_{1+m}^2 = h_{1+m}\alpha_{2+m}^2.$$

Then (3-17) tells us

$$\alpha_{2+m}^2 = \frac{1+c^2}{4} + \alpha_3^2 + \cdots + \alpha_{1+m}^2 = \alpha_3^2 + \cdots + \alpha_{1+m}^2$$
$$= (h_3 + \cdots + h_{1+m})\alpha_{2+m}^2.$$

Since α_{2+m} is a nonconstant function, we have

$$h_3 + \cdots + h_{1+m} \equiv 1.$$

Due to the triangle inequality,

$$W = \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b}$$
$$= 1 + 2(|\alpha_3^2| + \cdots + |\alpha_{1+m}^2| - |\alpha_{2+m}^2|)$$
$$= 1 + 2(|h_3| + \cdots + |h_{1+m}| - 1)|\alpha_{2+m}|^2 \geq 1,$$

and the equality holds if and only if the functions $h_3, \ldots, h_{1+m}$ all take values in $\mathbb{R}^+ \cup \{0, \infty\}$. Again using the Liouville Theorem, we know that $h_3, \ldots, h_{1+m}$

are all constant real functions. Therefore, there exist $v_3, \dots, v_{1+m} \in \mathbb{R}$, such that $v_3^2 + \cdots + v_{1+m}^2 = 1$ and

$$(\alpha_3, \dots, \alpha_{1+m}, \alpha_{2+m}) = (v_3, \dots, v_{1+m}, 1)\alpha_{2+m}.$$

Let β be the unique holomorphic function such that $\beta' = \alpha_{2+m}$ and $\beta(0) = 0$. Denote $h := \operatorname{Re}\beta$, $\boldsymbol{y}_0 := (v_3, \cdots, v_{1+m}, 1)$ and $\boldsymbol{y}_1 := f(0,0)$. Then h is a nonlinear harmonic function and $\boldsymbol{y}_0$ is a lightlike vector. We can proceed as in the proof of Theorem 4.1 to show $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$. Note that in this case $W \equiv 1$. □

But our answer is "no" for the second statement, i.e., $W \geq 1$. In fact, we have the following result:

Proposition 5.2. *For any real number $C \geq 1$ and $\varepsilon > 0$, there exists an entire spacelike stationary graph in $\mathbb{R}^{2+m}_1$ $(m \geq 3)$ generated by $f : \mathbb{R}^2 \to \mathbb{R}^m_1$ such that $\inf W \cdot \sup W = C$ and $0 < \sup W - \inf W < \varepsilon$.*

Proof. Now we put $c := -b\sqrt{-1}$ and let d be a real number to be chosen. Let μ be a complex number such that

$$\mu^2 = -\frac{1+c^2+d^2}{4} = -\frac{1-b^2+d^2}{4}.$$

Denote

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{c}{2} = -\frac{b}{2}\sqrt{-1}, \quad \alpha_3 = \cdots = \alpha_{m-1} = 0,$$
$$\alpha_m = \frac{d}{2}, \qquad \alpha_{1+m} = \mu\cosh z, \qquad \alpha_{2+m} = \mu\sinh z.$$

Since

$$\langle\alpha, \alpha\rangle = \alpha_1^2 + \alpha_2^2 + \alpha_m^2 + \alpha_{1+m}^2 - \alpha_{2+m}^2 = 0$$

and $\langle\alpha, \bar{\alpha}\rangle$ is positive, $z \mapsto \boldsymbol{x}(z) = \int_0^z \alpha(z)$ gives an entire spacelike stationary graph in $\mathbb{R}^{2+m}_1$.

As in the proof of Theorem 4.1, a similar calculation shows

$$\begin{aligned} W &= \frac{1+|c|^2+4(|\alpha_3|^2+\cdots+|\alpha_{1+m}|^2-|\alpha_{2+m}|^2)}{2b} \\ &= \frac{1+b^2+d^2+|1-b^2+d^2|\cos(2\operatorname{Im} z)}{2b}. \end{aligned}$$

Denote $r_1 := \inf W$, $r_2 := \sup W$. Then $r_1 = (1 + b^2 + d^2 - |1 - b^2 + d^2|)/(2b)$, $r_2 = (1 + b^2 + d^2 + |1 - b^2 + d^2|)/(2b)$ and

$$r_1 r_2 = \frac{(1+b^2+d^2)^2-(1-b^2+d^2)^2}{4b^2} = 1 + d^2, \quad r_2 - r_1 = \frac{|1-b^2+d^2|}{b}.$$

Now we put $d := \sqrt{C-1}$. Then $r_1 r_2 = C$, and one can choose b sufficiently close to $\sqrt{C}$, such that $r_2 - r_1 = |1 - b^2 + d^2|/b = |C - b^2|/b \in (0, \varepsilon)$. □

Remark. Calabi's theorem has been generalized to higher-dimensional cases. Namely, if f is a smooth real function on $\mathbb{R}^n$, so that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^n\}$ is an entire maximal hypersurface in $\mathbb{R}^{n+1}_1$, then f has to be affine linear. This is a well-known Bernstein-type result by Cheng and Yau [1976]. Observing that any maximal n-dimensional graph in $\mathbb{R}^{n+1}_1$ can be regarded as a stationary graph in $\mathbb{R}^{n+m}_1$ which satisfies $W \le 1$, we raise a conjecture:

Conjecture 5.3. Let $f : \mathbb{R}^n \to \mathbb{R}^m_1$ be a smooth function, such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^n\}$ is a spacelike stationary graph in $\mathbb{R}^{n+m}_1$. If $W \le 1$, then M has to be a flat manifold. Moreover, $W < 1$ forces f to be affine linear and hence M has to be an affine n-plane.

6. Stationary graphs with finite total curvature

As demonstrated in [Ma et al. 2013], the Bernstein theorem can not be generalized directly to stationary graphs in $\mathbb{R}^4_1$, because one can easily construct complete stationary graphs in $\mathbb{R}^4_1$ which are not flat. Interestingly, these examples have infinite total curvature.

On the other hand, examples of complete stationary surfaces with finite total curvature are abundant, and there holds a generalized Jorge–Meeks formula about their total Gaussian curvature (and the total normal curvature) provided that they are algebraic [Ma et al. 2013]. Thus one is naturally interested to know whether there could be a stationary graph with finite total curvature. The answer to this question is the following Bernstein type theorem. (Note that here we do not need the algebraic assumption.)

Theorem 6.1. *Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2_1$ be a smooth function, such that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a spacelike stationary graph in $\mathbb{R}^4_1$ whose curvature integral $\int_M |K|\, dM$ converges absolutely. Then f is affine linear or $f = h\boldsymbol{y}_0 + \boldsymbol{y}_1$, with h a nonlinear harmonic function, $\boldsymbol{y}_0$ a nonzero lightlike vector and $\boldsymbol{y}_1$ a constant vector. In both cases, M is flat, i.e., $K \equiv 0$.*

Proof. Denote $z = u_1 + \sqrt{-1}u_2$ as before. As in the proof of Theorem 4.1, if M is not a flat surface as we claimed, then the holomorphic differential $\partial \boldsymbol{x}/\partial z$ can be expressed as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\tfrac{1}{2}, \tfrac{1}{2}c, \mu \cosh \beta, \mu \sinh \beta\right), \tag{6-1}$$

where $c = a - b\sqrt{-1}$ is a complex constant with $b > 0$, $\mu^2 = -\frac{1}{4}(1 + c^2)$, and $\beta = \beta(z)$ is a nonconstant holomorphic function defined on $\mathbb{C}$. We will derive a contradiction from this assumption.

By the Weierstrass representation formula given in [Ma et al. 2013], $\partial \boldsymbol{x}/\partial z$ can be expressed in terms of a pair of meromorphic functions ϕ, ψ (the *Gauss maps*) and a holomorphic differential $dh = h'(z)\,dz$ (the *height differential*) as below:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\phi + \psi, -\sqrt{-1}(\phi - \psi), 1 - \phi\psi, 1 + \phi\psi)h'. \tag{6-2}$$

Comparing these two formulas, we obtain

$$h' = \frac{\mu}{2}e^{\beta}, \quad \phi = \frac{1+c\sqrt{-1}}{2\mu}e^{-\beta}, \quad \psi = \frac{1-c\sqrt{-1}}{2\mu}e^{-\beta}.$$

Note that $\dfrac{1+c\sqrt{-1}}{2\mu}\cdot\dfrac{1-c\sqrt{-1}}{2\mu} = -1$, and $b > 0$ implies

$$\left|\frac{1+c\sqrt{-1}}{2\mu}\right| > \left|\frac{1+\bar{c}\sqrt{-1}}{2\mu}\right|.$$

Denote $(1 + c\sqrt{-1})/(2\mu) := re^{\sqrt{-1}\theta}$ with $r > 1$ and $\theta \in \mathbb{R}$. Then

$$\frac{1 - c\sqrt{-1}}{2\mu} = -r^{-1}e^{-\sqrt{-1}\theta}.$$

In [Ma et al. 2013] the Gaussian curvature and the normal curvature of a stationary surface were unified in a single formula in terms of ϕ, ψ and the Laplacian with respect to the induced metric $g := e^{2\omega}|dz|^2$ as follows:

$$-K + \sqrt{-1}K^{\perp} = \Delta \ln(\phi - \bar{\psi}) = 4e^{-2\omega}\frac{\phi_z\bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2}. \tag{6-3}$$

Set $\beta := v_1 + \sqrt{-1}v_2$, where v_1, v_2 are both real functions on $\mathbb{C}$. Then

$$\begin{aligned} |K|e^{2\omega} &= 4\left|\mathrm{Re}\,\frac{\phi_z\bar{\psi}_{\bar{z}}}{(\phi-\bar{\psi})^2}\right| = 4\left|\mathrm{Re}\,\frac{e^{2\sqrt{-1}\theta}e^{-\beta-\bar{\beta}}}{(re^{\sqrt{-1}\theta}e^{-\beta}+r^{-1}e^{\sqrt{-1}\theta}e^{-\bar{\beta}})^2}\right||\beta'(z)|^2 \\ &= 4\left|\mathrm{Re}\left(\frac{1}{(re^{(\bar{\beta}-\beta)/2}+r^{-1}e^{(\beta-\bar{\beta})/2})^2}\right)\right||\beta'(z)|^2 \\ &= \frac{4\big(2+(r^2+r^{-2})\cos 2v_2\big)}{|re^{-\sqrt{-1}v_2}+r^{-1}e^{\sqrt{-1}v_2}|^4}|\beta'(z)|^2 \geq \frac{4\big(2+(r^2+r^{-2})\cos 2v_2\big)}{|r+r^{-1}|^4}|\beta'(z)|^2. \end{aligned} \tag{6-4}$$

Thus the assumption of finite total curvature is equivalent to saying that

$$\begin{aligned} \infty > \int_M |K|\,dM &= \int_{\mathbb{C}} |K|e^{2\omega}\,du_1 \wedge du_2 \\ &\geq \int_{\mathbb{C}} \frac{4[2+(r^2+r^{-2})\cos(2v_2)]}{|r+r^{-1}|^4}|\beta'(z)|^2\,du_1 \wedge du_2 \\ &\geq \int_{\mathbb{C}} \frac{4[2+(r^2+r^{-2})\cos(2v_2)]}{|r+r^{-1}|^4}\,dv_1 \wedge dv_2, \end{aligned} \tag{6-5}$$

where the final inequality follows from the assumption that β is a nonconstant entire function over $\mathbb{C}$, which takes almost every value of $\mathbb{C}$ at least one time. It is easily seen that the right-hand side of (6-5) is divergent, contradicting the finiteness of the total curvature. □

Remarks.

- Taking the imaginary part of (6-3), one can proceed as in (6-4)–(6-5) to get a contradiction when the condition "$\int_M |K|\, dM < \infty$" is replaced by "$\int_M |K^\perp|\, dM < \infty$". Therefore, if $M \subset \mathbb{R}^4_1$ is an entire spacelike stationary graph over $\mathbb{R}^2$, whose normal curvature integral converges absolutely, then M has to be a flat surface.
- Let M be a noncompact surface with a complete metric. If $\int_M |K|\, dM < \infty$, then there is a compact Riemann surface $\overline{M}$, such that M is conformally equivalent to $\overline{M} \setminus \{p_1, p_2, \dots, p_r\}$, with $p_1, \dots, p_r \in \overline{M}$. This is a purely intrinsic result, discovered by A. Huber [1957]. Moreover, if we additionally assume M to be a minimal surface in $\mathbb{R}^{2+m}$ (m is arbitrary), then the Gauss map of M is algebraic, and vice versa; see Theorem 1 of [Chern and Osserman 1967]. But this conclusion is no longer true for spacelike stationary surfaces in $\mathbb{R}^4_1$, due to the examples with finite total curvature and essential singularities; see [Ma et al. 2013]. Hence, unlike the $\mathbb{R}^4$ case [Osserman 1969], the conclusion of Theorem 6.1 cannot be deduced directly from (6-1).
- Combining Theorem 1 of [Chern and Osserman 1967] and §5 of [Osserman 1969], it is easy to conclude that $M = \operatorname{graph} f := \{(x, f(x)) : x \in \mathbb{R}^2\}$ is a minimal surface in $\mathbb{R}^4$ with finite total curvature if and only if $f = p(z)$ or $p(\bar{z})$, with p an arbitrary polynomial. Noting that any minimal graph in $\mathbb{R}^4$ over $\mathbb{R}^2$ can be regarded as a spacelike stationary graph in $\mathbb{R}^n_1$ ($n \geq 5$), the conclusion of Theorem 6.1 can not be generalized to spacelike stationary graphs in higher-dimensional Minkowski spaces.

References

[Barbosa 1979] J. L. M. Barbosa, "An extrinsic rigidity theorem for minimal immersions from S^2 into S^n", *J. Differential Geom.* **14**:3 (1979), 355–368. MR Zbl

[Bernstein 1915] S. Bernstein, "Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique", *Comm. Soc. Math. Kharkow* (2) **15**:1 (1915), 38–45. JFM

[Calabi 1970] E. Calabi, "Examples of Bernstein problems for some nonlinear equations", pp. 223–230 in *Global analysis* (Berkeley, CA, 1968), edited by S.-S. Chern and S. Smale, Proc. Sympos. Pure Math. **15**, American Mathematical Society, Providence, RI, 1970. MR Zbl

[Cheng and Yau 1976] S. Y. Cheng and S. T. Yau, "Maximal space-like hypersurfaces in the Lorentz–Minkowski spaces", *Ann. of Math.* (2) **104**:3 (1976), 407–419. MR Zbl

[Chern and Osserman 1967] S.-S. Chern and R. Osserman, "Complete minimal surfaces in Euclidean n-space", *J. Analyse Math.* **19** (1967), 15–34. MR Zbl

[Fischer-Colbrie 1980] D. Fischer-Colbrie, "Some rigidity theorems for minimal submanifolds of the sphere", *Acta Math.* **145**:1-2 (1980), 29–46. MR Zbl

[Huber 1957] A. Huber, "On subharmonic functions and differential geometry in the large", *Comment. Math. Helv.* **32** (1957), 13–72. MR Zbl

[Jost et al. 2014] J. Jost, Y. L. Xin, and L. Yang, "A spherical Bernstein theorem for minimal submanifolds of higher codimension", preprint, 2014. arXiv

[Jost et al. 2015] J. Jost, Y. L. Xin, and L. Yang, "Curvature estimates for minimal submanifolds of higher codimension and small G-rank", *Trans. Amer. Math. Soc.* **367**:12 (2015), 8301–8323. MR Zbl

[Lewy 1937] H. Lewy, "A priori limitations for solutions of Monge–Ampère equations, II", *Trans. Amer. Math. Soc.* **41**:3 (1937), 365–374. MR Zbl

[Ma et al. 2013] X. Ma, C. Wang, and P. Wang, "Global geometry and topology of spacelike stationary surfaces in the 4-dimensional Lorentz space", *Adv. Math.* **249** (2013), 311–347. MR Zbl

[Osserman 1969] R. Osserman, *A survey of minimal surfaces*, Van Nostrand, New York, 1969. Reprinted Dover, New York, 1986. MR

Received May 7, 2015. Revised February 2, 2016.

XIANG MA
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
CHINA
maxiang@math.pku.edu.cn

PENG WANG
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI, 200092
CHINA
netwangpeng@tongji.edu.cn

LING YANG
SCHOOL OF MATHEMATICAL SCIENCES
FUDAN UNIVERSITY
SHANGHAI, 200433
CHINA
yanglingfd@fudan.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 287 No. 1 March 2017

0030-8730(201703)287:1;1-W