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APPROXIMABILITY OF CONVEX BODIES AND VOLUME ENTROPY IN HILBERT GEOMETRY

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The approximability of a convex body is a number which measures the difficulty in approximating that convex body by polytopes. In the interior of a convex body one can define its Hilbert geometry. We prove on the one hand that the volume entropy is twice the approximability for a Hilbert geometry in dimension two or three, and on the other hand that in higher dimensions the approximability is a lower bound of the entropy. As a corollary we solve the volume entropy upper bound conjecture in dimension three and give a new proof in dimension two different from the one given in (*Pacific J. Math.* 245:2 (2010), 201–225). Moreover, our method allows us to prove the existence of Hilbert geometries with intermediate volume growth on the one hand, and that in general the volume entropy is not a limit on the other hand.

Introduction and statement of results

Hilbert geometries are all the metric spaces obtained by defining the so-called Hilbert distance on open bounded convex sets in $\mathbb{R}^n$. The definition of this distance uses cross ratios in the same way as in the Klein projective model of the hyperbolic geometry [Hilbert 1971]. These metric spaces are actually length spaces whose structure is defined by a Finsler metric which is Riemannian if and only if the underlying open bounded convex set is an ellipsoid [Kay 1967].

These geometries were introduced by D. Hilbert in a letter addressed to F. Klein and have attracted a lot of interest lately. The studies of the shape of spheres in [Busemann 1955, Chapter 18] and of perpendicularity in [Busemann and Kelly 1953, Chapter 28] seem to be among the first ones to appear. In the same period P. J. Kelly and E. Straus [1958], Y. Nasu [1961] and D. C. Kay [1967] were looking at characterisations of the hyperbolic geometry among them in terms of curvature, transitive actions and the ptolemaic inequality, respectively. After a break of twenty or so years, they started to be studied from the projective structure viewpoint by

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W. Goldman [1990] and by I. Kim [2005], and from the perspective of the group acting on them by P. de la Harpe [1993]. At the start of the new millennium the quest for characterisation of the Hilbert geometries being hyperbolic in the sense of Gromov began with A. Karlsson and G. Noskov [2002] and more noticeably with an equivalence between the hyperbolicity and a property of the boundary called quasisymmetric convexity discovered by Y. Benoist [2003; 2008], who also studied dynamical aspects of these geometries and clarified the fractal shape of their boundary in dimension three. At the same time the infinite-dimensional ones were studied from a functional-analytical point of view; see, for instance, [Lins and Nussbaum 2008]. Lately, understanding the analogue of geometric finiteness in the setting of projective structures has been at the centre of the works of L. Marquis [2012], M. Crampon and Marquis [2014], and D. Cooper, D. Long and S. Tillman [Cooper et al. 2015]. Other aspects of interest can be found in the recent *Handbook of Hilbert geometry* [Papadopoulos and Troyanov 2014].

The present paper focuses on the volume growth of these geometries and more specifically on the volume entropy.

Let Ω be a bounded open convex set in $\mathbb{R}$ endowed with its Hilbert geometry. If we consider the *Busemann volume* Vol_Ω and denote by $B_\Omega(p,r)$ the metric ball of radius r centred at the point $p \in \Omega$, then the *lower* and the *upper volume entropies* of Ω will be defined respectively by

$$\underline{\mathrm{Ent}}\,\Omega = \liminf_{r\to+\infty} \frac{\ln(\mathrm{Vol}_\Omega B_\Omega(p,r))}{r} \quad \text{and} \quad \overline{\mathrm{Ent}}\,\Omega = \limsup_{r\to+\infty} \frac{\ln(\mathrm{Vol}_\Omega B_\Omega(p,r))}{r}. \tag{1}$$

When the two limits coincide we denote their common limit by $\mathrm{Ent}\,\Omega$ and call it the *volume entropy* of Ω.

Let us stress that in this definition the upper and lower volume entropy of Ω do not depend on the base point p and are actually projective invariants attached to Ω.

The question we address in this paper is twofold. On the one hand it is an investigation of the existence of an analogue, for all Hilbert geometries, of the relation between the volume entropy and the Hausdorff dimension of the radial limit set on the universal cover of a compact Riemannian manifold with nonpositive curvature. On the other hand we focus on the *volume entropy upper bound conjecture*, which states that if Ω is an open and bounded convex subset of $\mathbb{R}^n$, then $\overline{\mathrm{Ent}}\,\Omega \le n-1$. To put our work into perspective let us recall the main related results.

The first one is a complete answer to the conjecture in the two-dimensional case by G. Berck, A. Bernig and C. Vernicos in [Berck et al. 2010], where the authors actually obtained an upper bound as a function of d, the upper Minkowski dimension (or *ball-box* dimension) of the set of extreme points of Ω, namely

$$\overline{\mathrm{Ent}}\,\Omega \le \frac{2}{3-d} \le 1. \tag{2}$$

The second result is a more precise statement with respect to the asymptotic volume growth of balls. It involves another projective invariant introduced by Berck, Bernig and Vernicos in the introduction of [Berck et al. 2010], called the *centroprojective* area of Ω and defined by

$$\mathcal{A}_p(\Omega) := \int_{\partial\Omega} \frac{\sqrt{k(x)}}{\langle n(x), x-p\rangle^{\frac{1}{2}(n-1)}} \left(\frac{2\alpha(x)}{1+\alpha(x)}\right)^{\frac{1}{2}(n-1)} \mathrm{d}A(x), \tag{3}$$

where for any $x \in \partial\Omega$, $k(x)$ is the Gauss curvature, $n(x)$ is the outward normal and $\alpha(x) > 0$ is the function defined by $p - \alpha(x)(x-p) \in \partial\Omega$. Let us recall here that both k and n are defined almost everywhere as Alexandroff's theorem states [Alexandroff 1939].

Now, the second main theorem in [Berck et al. 2010] — which encompasses previous results given by B. Colbois and P. Verovic [2004] — asserts that if $\partial\Omega$ is $C^{1,1}$ we have

$$\lim_{r\to+\infty} \frac{\mathrm{Vol}_\Omega B_\Omega(p,r)}{\sinh^{n-1} r} = \frac{1}{n-1}\mathcal{A}_p(\Omega) \neq 0 \tag{4}$$

and $\mathrm{Ent}\,\Omega = n-1$ is a limit. Moreover, without any assumption on Ω we have $\underline{\mathrm{Ent}}\,\Omega \geq n-1$ whenever $\mathcal{A}_p(\Omega) \neq 0$.

The third one — which is also a rigidity result — requires stronger assumptions about Ω: it has to be divisible, meaning that it admits a compact quotient, and its Hilbert metric has to be hyperbolic in the sense of Gromov, which implies its boundary is C^1 and strictly convex by [Benoist 2003]. Let us stress that the Hilbert metric on such an Ω is the hyperbolic one if and only if Ω has a $C^{1,1}$ boundary, and that its volume entropy is positive since hyperbolicity implies the nonvanishing of the Cheeger constant (see Theorem 1.5 in [Colbois and Vernicos 2007]). A result by Crampon [2009] states that for a divisible open bounded convex set Ω in $\mathbb{R}^n$ whose boundary is C^1 we have $\mathrm{Ent}\,\Omega \leq n-1$ with equality if and only if Ω is an ellipsoid.

In the present paper we link the volume entropy to another invariant associated with a convex body, called the *approximability*. This name was introduced by R. Schneider and J. A. Wieacker [1981]. The approximability measures in some sense how well a convex set can be approximated by polytopes. More precisely, let $N(\varepsilon, \Omega)$ be the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than $\varepsilon > 0$. Then the lower and upper approximability of Ω are defined by

$$\underline{a}(\Omega) := \liminf_{\varepsilon\to 0} \frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon} \quad \text{and} \quad \overline{a}(\Omega) := \limsup_{\varepsilon\to 0} \frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon}. \tag{5}$$

The key inequality which is of interest in our work — obtained by Fejes Tóth [1948] in dimension two and by E. M. Bronshteyn and L. D. Ivanov [1975] in the general case — asserts that for any bounded convex set in $\mathbb{R}^n$ the following upper bound on the upper approximability holds: $\bar{a}(\Omega) \le \frac{1}{2}(n-1)$.

Our main result is as follows.

Theorem 1 (main theorem). *Given an open bounded convex set Ω in $\mathbb{R}^n$, we have*

$$2\underline{a}(\Omega) \le \underline{\operatorname{Ent}}\,\Omega \quad \textit{and} \quad 2\bar{a}(\Omega) \le \overline{\operatorname{Ent}}\,\Omega, \tag{6}$$

with equality for $n = 2$ or $n = 3$.

The equality case in (6), together with the upper bound for the upper approximability, implies the following corollary.

Corollary 2 (volume entropy upper bound conjecture). *For any open bounded convex set Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have $\overline{\operatorname{Ent}}\,\Omega \le n-1$.*

The equality case in this main theorem heavily relies on the study of polytopal Hilbert geometries. As it happens we get an optimal control of the volume of metric balls in dimension two and three, for in those two cases the number of edges of a polytope is bounded from above by the number of its vertices up to a multiplicative and an additive constant. This does not hold in higher dimensions, following McMullen's upper bound theorem [McMullen 1971; McMullen and Shephard 1971].

The second important result concerns the two-dimensional case, where we can prove that there are Hilbert geometries with intermediate volume growth.

Theorem 3 (intermediate volume growth). *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function that satisfies*

$$\liminf_{r\to+\infty} \frac{e^r}{f(r)} > 0.$$

Then there exist an open bounded convex set Ω in $\mathbb{R}^2$ and a point o in Ω such that

$$\liminf_{r\to+\infty} \frac{\operatorname{Vol}_\Omega B_\Omega(o,r)}{f(r)} > 0 \quad \textit{and} \quad \limsup_{r\to+\infty} \frac{\operatorname{Vol}_\Omega B_\Omega(o,r)}{f(r)r^2} < +\infty, \tag{7}$$

and

$$\underline{\operatorname{Ent}}\,\Omega = \liminf_{r\to+\infty} \frac{\ln f(r)}{r} \quad \textit{and} \quad \overline{\operatorname{Ent}}\,\Omega = \limsup_{r\to+\infty} \frac{\ln f(r)}{r}. \tag{8}$$

In particular there are open bounded convex sets $\Omega \subset \mathbb{R}^2$ with

- *maximal volume entropy and zero centroprojective area,*
- *zero volume entropy which are not polytopes.*

This theorem is a consequence of our method for proving the equality in dimension two in the main theorem (see Section 2) and Schneider and Wieacker's results [1981] on the approximability in dimension two. The last statement follows from our work [Vernicos 2009], where we showed that polytopes have polynomial growth of order r^2 in dimension two.

The intermediate volume growth theorem allows us to settle in a quite definite way the question of whether the entropy is a limit or not.

Corollary 4. *The volume entropy is not a limit in general. More precisely, for any* α *and* β *with* $0 \le \alpha \le \beta \le 1$ *there exist an open bounded convex set* Ω *in* $\mathbb{R}^2$ *such that*

$$\underline{\mathrm{Ent}}\,\Omega = \alpha \quad \textit{and} \quad \overline{\mathrm{Ent}}\,\Omega = \beta.$$

The equalities and inequalities also imply the following new results:

Corollary 5. *Given an open bounded convex set* Ω *in* $\mathbb{R}^n$, *we have*

- $d_H \le \underline{\mathrm{Ent}}\,\Omega$, *where* d_H *is the Hausdorff dimension of the set of farthest points of* Ω;
- *if* $n = 2$ *or* 3 *then* $\bar{a}(\Omega)$ *is a projective invariant of* Ω *and* $\overline{\mathrm{Ent}}\,\Omega = \overline{\mathrm{Ent}}\,\Omega^*$, *where* Ω^* *is the polar dual of* Ω;
- *if* $n = 2$, *then* $\bar{a}(\Omega) \le 1/(3-d)$.

Section 1 presents the various lemmas and notions needed in Section 2 to prove the main theorem, and in Section 3 we present the proof of the intermediate volume growth theorem.

1. Preliminaries on Hilbert geometries and convex bodies

1.1. ***Notations and definitions.*** A *proper* open set in $\mathbb{R}^n$ is a set that does not contain a whole line. A nonempty proper open convex set in $\mathbb{R}^n$ will be called a *proper convex domain.* The closure of a bounded convex domain is usually called a *convex body.*

A Hilbert geometry (Ω, d_Ω) is a proper convex domain Ω in $\mathbb{R}^n$ endowed with its Hilbert distance d_Ω defined as follows: for any two distinct points p and q in Ω, the line passing through p and q meets the boundary $\partial\Omega$ of Ω at two points a and b such that a, p, q, b appear in that order on the line. We denote by $[a, p, q, b]$ the cross ratio of (a, p, q, b), i.e.,

$$[a, p, q, b] = \frac{qa}{pa} \times \frac{pb}{qb} > 1,$$

where for any two points x, y in $\mathbb{R}^n$, xy is their distance with respect to the standard Euclidean norm $\|\cdot\|$. Should a or b be at infinity, the corresponding ratio will be considered equal to 1. Then we define

$$d_\Omega(p, q) = \tfrac{1}{2} \ln[a, p, q, b].$$

Note that the invariance of the cross ratio by a projective map implies the invariance of d_Ω by such a map.

The proper convex domain Ω is also naturally endowed with the C^0 Finsler metric F_Ω defined as follows: given $p \in \Omega$ and $v \in T_p\Omega = \mathbb{R}^n$ with $v \neq 0$, the straight line passing through p with direction vector v meets $\partial\Omega$ at two points p_Ω^+ and p_Ω^- such that $p_\Omega^+ - p_\Omega^-$ and v have the same direction. Then let t^+ and t^- be the two positive numbers such that $p + t^+ v = p_\Omega^+$ and $p - t^- v = p_\Omega^-$ (in other words, these numbers correspond to the amount of time needed to reach the boundary of Ω when starting at p with the velocities v and $-v$, respectively). Then we define

$$F_\Omega(p, v) = \frac{1}{2}\left(\frac{1}{t^+} + \frac{1}{t^-}\right) \quad \text{and} \quad F_\Omega(p, 0) = 0.$$

Should p_Ω^+ or p_Ω^- be at infinity, then the corresponding ratio will be taken to be equal to 0.

The Hilbert distance d_Ω is the length distance associated to F_Ω. We shall denote by $B_\Omega(p, r)$ the metric ball of radius r centred at the point $p \in \Omega$ and by $S_\Omega(p, r)$ the corresponding metric sphere.

Thanks to that Finsler metric, we can make use of two important Borel measures on Ω. The first one, which coincides with the Hausdorff measure associated to the metric space (Ω, d_Ω) (see Example 5.5.13 in [Burago et al. 2001]), is the *Busemann* volume, denoted by Vol_Ω and defined as follows. Given any point p in Ω, let $\beta_\Omega(p) = \{v \in \mathbb{R}^n \mid F_\Omega(p, v) < 1\}$ be the open unit ball in $T_p\Omega = \mathbb{R}^n$ with respect to the norm $F_\Omega(p, \cdot)$ and let ω_n be the Euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^n$. Then given any Borel set A in Ω, its Busemann volume Vol_Ω is defined by

$$\mathrm{Vol}_\Omega A = \int_A \frac{\omega_n}{\lambda(\beta_\Omega(p))}\, \mathrm{d}\lambda(p),$$

where λ denotes the standard Lebesgue measure on $\mathbb{R}^n$.

The second one is the *Holmes–Thompson* volume on Ω, which we will denote by $\mu_{HT,\Omega}$. Given any Borel set A in Ω, its Holmes–Thompson volume is defined by

$$\mu_{HT,\Omega}(A) = \int_A \frac{\lambda(\beta_\Omega^*(p))}{\omega_n}\, \mathrm{d}\lambda(p),$$

where $\beta_\Omega^*(p)$ is the polar dual of $\beta_\Omega(p)$.

We can actually consider a whole family of measures as follows. Let $\mathcal{E}_n$ be the set of pointed proper open convex sets in $\mathbb{R}^n$. These are the pairs (ω, x) such that ω is a proper open convex set and x is a point in ω. We shall say that a function $f : \mathcal{E}_n \to \mathbb{R}$ is a *proper density* if it is positive and satisfies the three following properties:

- *Continuity* with respect to the Hausdorff pointed topology on $\mathcal{E}_n$.
- *Monotone decreasing* with respect to the inclusion; i.e., if $x \in \omega \subset \Omega$ then $f(\Omega, x) \leq f(\omega, x)$.
- *Chain rule compatibility*: for any projective transformation T one has

$$f\big(T(\omega), T(x)\big)\,\mathrm{Jac}(T, x) = f(\omega, x).$$

We will say that f is a *normalised proper density* if $f(\omega, x)\,d\lambda(x)$ is the Riemannian volume when ω is an ellipsoid. Let us denote by $\mathcal{PD}_n$ the set of proper densities over $\mathcal{E}_n$.

A result of Benzécri [1960] states that the action of the group of projective transformations on $\mathcal{E}_n$ is cocompact. Therefore, for any pair f, g in $\mathcal{PD}_n$, there exists a constant $C > 0$ ($C \geq 1$ for the normalised ones) such that for any $(\omega, x) \in \mathcal{E}_n$ one has

$$\frac{1}{C} \leq \frac{f(\omega, x)}{g(\omega, x)} \leq C. \tag{9}$$

Given a density f in $\mathcal{PD}_n$ there is a natural Borel measure associated to any open bounded convex set Ω, denoted by $\mu_{f,\Omega}$ and defined as follows: for any Borel subset A of Ω we let

$$\mu_{f,\Omega}(A) = \int_A f(\Omega, p)\,\mathrm{d}\lambda(p).$$

Integrating the inequalities (9) we obtain that for any two proper densities f, g in $\mathcal{PD}_n$, there exists a constant $C > 0$ such that for any Borel set $A \subset \Omega$ we have

$$\frac{1}{C}\mu_{g,\Omega}(A) \leq \mu_{f,\Omega}(A) \leq C\mu_{g,\Omega}(A). \tag{10}$$

We call the family of measures obtained in this way *proper measures with density*.

To a proper density $g \in \mathcal{PD}_{n-1}$ we can also associate an $(n-1)$-dimensional measure, denoted by $\mu_{\cdot,g,\Omega}$, on hypersurfaces in Ω as follows. Let Σ be a smooth hypersurface, and consider for a point p in the hypersurface Σ its tangent hyperplane $H(p)$. Then the measure will be given by

$$\frac{d\mu_{\Sigma,g,\Omega}}{d\sigma}(p) = \frac{d\mu_{g,\Omega\cap H(p)}}{d\sigma}(p), \tag{11}$$

where σ denotes the Hausdorff $(n-1)$-dimensional measure associated with the standard Euclidean distance. In Section 2 we will simply denote by $\mathrm{Vol}_{n-1,\Omega}$ and Area_Ω the $(n-1)$-dimensional measures associated with the Holmes–Thompson and the Busemann measures, respectively.

Let now $\mu_{f,\Omega}$ be a proper measure with density over Ω. Then the volume entropies of Ω are defined by

$$\underline{\operatorname{Ent}}\,\Omega = \liminf_{r\to+\infty} \frac{\ln \mu_{f,\Omega}(B_\Omega(p,r))}{r}, \qquad \overline{\operatorname{Ent}}\,\Omega = \limsup_{r\to+\infty} \frac{\ln \mu_{f,\Omega}(B_\Omega(p,r))}{r}. \tag{12}$$

These numbers do not depend on either f nor p, and are equal to the spherical entropies (see Theorem 2.14 of [Berck et al. 2010]):

$$\underline{\operatorname{Ent}}\,\Omega = \liminf_{r\to+\infty} \frac{\ln(\operatorname{Area}_\Omega S_\Omega(p,r))}{r}, \qquad \overline{\operatorname{Ent}}\,\Omega = \limsup_{r\to+\infty} \frac{\ln(\operatorname{Area}_\Omega S_\Omega(p,r))}{r}. \tag{13}$$

1.2. ***Properties of the Holmes–Thompson and the Busemann measures.***

Lemma 6 (monotonicity of the Holmes–Thompson measure). *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^n$. The Holmes–Thompson area measure is monotonic on the set of convex bodies in Ω; that is, for any pair of convex bodies K_1 and K_2 in Ω such that $K_1 \subset K_2$ one has*

$$\operatorname{Vol}_{n-1,\Omega} \partial K_1 \le \operatorname{Vol}_{n-1,\Omega} \partial K_2. \tag{14}$$

Proof. If $\partial\Omega$ is C^2 with everywhere-positive Gaussian curvature then the tangent unit spheres of the Finsler metric are quadratically convex.

According to Álvarez Paiva and Fernandes [1998, Theorem 1.1 and Remark 2] there exists a Crofton formula for the Holmes–Thompson area, from which the inequality (14) follows.

Such smooth convex bodies are dense in the set of all convex bodies for the Hausdorff topology. By approximation, it follows that (14) is valid for any Ω. □

Lemma 6 associated with the Blaschke–Santaló inequality and the inequality (10) immediately implies the following result (see also [Berck et al. 2010, Lemma 2.12]).

Lemma 7 (rough monotonicity of the Busemann measure). *Let (Ω, d_Ω) be a Hilbert geometry, and let p be a point in Ω. There exists a monotonic function f_Ω and a constant $C_n < 1$ such that for all $r > 0$*

$$C_n f_\Omega(r) \le \operatorname{Area}_\Omega S_\Omega(p,r) \le f_\Omega(r), \tag{15}$$

where $f_\Omega(r)$ is the Holmes–Thompson area of the sphere $S_\Omega(p,r)$.

Let us finish by recalling one last statement also proved in [Berck et al. 2010, Lemma 2.13].

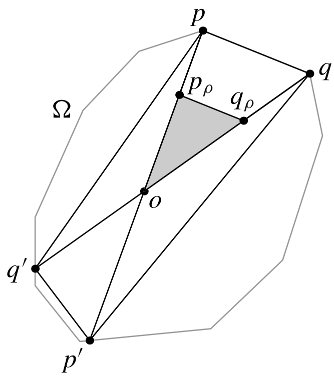

Figure 1. The area of the triangle (op_ρ, q_ρ) is bounded by $C\rho^2$.

Lemma 8 (coarea inequalities). *For all $r > 0$*

$$\frac{1}{2}\frac{\omega_n}{\omega_{n-1}}\operatorname{Area}_\Omega S_\Omega(p,r) \le \frac{\partial}{\partial r}\operatorname{Vol}_\Omega B_\Omega(p,r) \le \frac{n}{2}\frac{\omega_n}{\omega_{n-1}}\operatorname{Area}_\Omega S_\Omega(p,r).$$

1.3. ***Upper bound on the area of triangles.*** In this section we bound from above independently of the two-dimensional Hilbert geometries the area of affine triangles which are subset of a metric ball, when one of the vertices is the centre of that ball. We also give a lower bound on the length of some metric segments, when their vertices go to the boundary of the Hilbert geometry.

Lemma 9. *Let (Ω, d_Ω) be a two-dimensional Hilbert geometry. Then there exists a constant C independent of Ω such that for any point o in Ω and any pair of points p_ρ and q_ρ in the metric ball $B_\Omega(o,\rho)$, the area of the affine triangle $(op_\rho q_\rho)$ is less than $C\rho^2$.*

Proof. Given p_ρ and q_ρ in $B_\Omega(o,\rho)$, let p and q be the intersections of the boundary $\partial\Omega$ with the half-lines $[o, p_\rho)$ and $[o, q_\rho)$ respectively. Let p' and q' be, respectively, the intersections of the half-lines $[p_\rho, o)$ and $[q_\rho, o)$ with the boundary $\partial\Omega$. (See Figure 1.)

Then the volume of the triangle $(op_\rho q_\rho)$ with respect to the Hilbert geometry of Ω is less than or equal to its volume with respect to the Hilbert geometry of the quadrilateral $(pqp'q')$. However, the distances of p_ρ and q_ρ from o remain the same in both Hilbert geometries.

Up to a change of chart, we can suppose that this quadrilateral is actually a square. This allows us to use Theorem 1 from [Vernicos 2015], which states that the Hilbert geometry of the square is bi-Lipschitz to the product of the Hilbert geometries of its sides, using the identity as a map. In other words it is bi-Lipschitz to the Euclidean plane, with a Lipschitz constant equal to $C_0 > 1$, independent of our initial conditions.

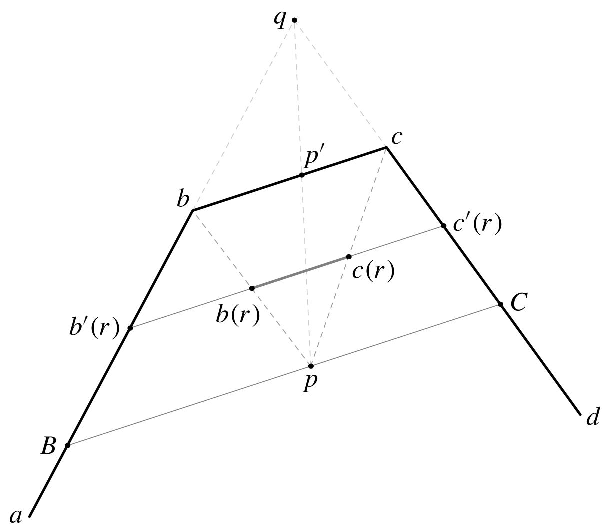

Figure 2. Distance estimate of Claim 10.

Thus our affine triangle is inside a Euclidean disc of radius $C_0\rho$, which implies that its area with respect to the Hilbert geometry of Ω is less than $C_0^4 \times \pi \times \rho^2$. $\square$

To prove that the volume entropy is bounded from below by the approximability we will need to bound from below the length of certain segments in a given Hilbert geometry Ω. To do so we will compare their length in the initial convex domain with their length in a convex domain projectively equivalent to a triangle, and containing the initial convex domain Ω.

Let us make this precise. Consider four points a, b, c and d in the Euclidean plane $(\mathbb{R}^2, \langle\cdot\rangle)$ such that $\mathcal{Q} = (abcd)$ is a convex quadrilateral. We assume that the scalar products $\langle\overrightarrow{ab}, \overrightarrow{bc}\rangle$ and $\langle\overrightarrow{bc}, \overrightarrow{cd}\rangle$ are positive and we let q be the intersection point between the straight lines (ab) and (cd).

Suppose that Ω is a convex domain such that the segments $[a, b]$, $[b, c]$ and $[c, d]$ belong to its boundary. Given p a point in the convex domain Ω we denote by p' the intersection between the straight line (pq) and the segment $[b, c]$, and we define $s = bp'/bc$.

We then denote by $[b(r), c(r)]$ the image of the segment $[b, c]$ under the dilation centred at p with ratio $0 < \tanh r < 1$. The image of the segment $[b, c]$ under the dilation centred at q sending p' to p will be denoted by $[B, C]$.

Claim 10. *Under the above assumption*,

$$d_\Omega\big(b(r), c(r)\big) \geq \frac{1}{2}\ln\left(\frac{bc}{s\cdot BC}\frac{\tanh r}{1-\tanh r}+1\right) + \frac{1}{2}\ln\left(\frac{bc}{(1-s)\cdot BC}\frac{\tanh r}{1-\tanh r}+1\right). \tag{16}$$

Proof. Straightforward computation, using the fact that the convex domain Ω is inside the convex domain Q obtained as the intersection of the half-planes defined by the lines (ab), (bc) and (cd), and therefore

$$d_\Omega\big(b(r), c(r)\big) \geq d_Q\big(b(r), c(r)\big).$$

Let $b'(r)$ be the intersection of the lines (ab) and $\big(b(r)c(r)\big)$, and let $c'(r)$ be the intersection of the lines (cd) and $\big(b(r)c(r)\big)$. (See Figure 2.) Then we have

$$d_Q\big(b(r), c(r)\big) = \tfrac{1}{2}\ln\left(\frac{b(r)c'(r)}{c(r)c'(r)} \cdot \frac{c(r)b'(r)}{b(r)b'(r)}\right).$$

Let us focus on the first ratio. On the one hand $b(r)c'(r) = b(r)c(r) + c(r)c'(r)$, and on the other hand following Thales' theorem

$$\begin{aligned} b(r)c(r) &= \tanh(r)bc, \\ c(r)c'(r) &= (1-\tanh r)pC. \end{aligned} \tag{17}$$

But $pC = BC \cdot (p'c/bc) = (1-s)BC$, and therefore we obtain

$$\ln\left(\frac{b(r)c'(r)}{c(r)c'(r)}\right) = \ln\left(\frac{bc}{(1-s)\cdot BC}\,\frac{\tanh r}{1-\tanh r} + 1\right).$$

The second ratio is treated in the same way. $\square$

1.4. ***Intrinsic and extrinsic Hausdorff topologies of Hilbert geometries.*** We describe the link between the Hausdorff topology induced by a Euclidean metric with the Hausdorff topology induced by the Hilbert metric on a compact subset of an open convex set.

We recall that the Löwner ellipsoid of a compact set is the ellipsoid with least volume containing that set. In this section we will suppose, without loss of generality, that Ω is a bounded open convex set whose Löwner ellipsoid $\mathcal{E}$ is the Euclidean unit ball with centre o. It is a standard result that $(1/n)\mathcal{E}$ is then contained in Ω; i.e.,

$$\frac{1}{n}\mathcal{E} \subset \Omega \subset \mathcal{E}. \tag{18}$$

Definition 11 (asymptotic ball and sphere). The *asymptotic ball* of radius R centred at o is the image of Ω by the dilation of ratio $\tanh R$ centred at o, and we denote it by $\mathrm{AsB}(o, R)$. The image of the boundary $\partial\Omega$ by the same dilation will be called the *asymptotic sphere* of radius R centred at o and denoted by $\mathrm{AsS}(o, R)$.

Recall that the Hausdorff distance is the distance between nonempty compact subsets in a metric space. We shall use both the Euclidean and Hilbert distance and we will use the terminology *Hausdorff–Euclidean* and *Hausdorff–Hilbert* to distinguish both cases.

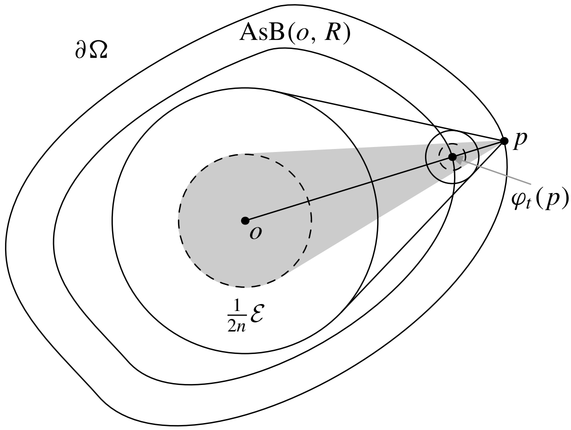

Figure 3. Illustration of Proposition 12's proof.

We would like to relate the Hausdorff–Hilbert neighbourhoods of the asymptotic ball $\mathrm{AsB}(o, R)$ with its Hausdorff–Euclidean neighbourhoods.

Proposition 12. *Let Ω be a convex domain and let o be the centre of its Löwner ellipsoid, which we assume to be the unit Euclidean ball.*

(1) *The $(1-\tanh R)/(2n)$-Hausdorff–Euclidean neighbourhood of the asymptotic ball $\mathrm{AsB}(o, R)$ is contained in its $\left(\frac{1}{2}\ln 3\right)$-Hausdorff–Hilbert neighbourhood.*

(2) *For any $K > 0$, the K-Hausdorff–Hilbert neighbourhood of the asymptotic ball $\mathrm{AsB}(o, R)$ is contained in its $(1-\tanh R)$-Hausdorff–Euclidean neighbourhood.*

Proof. For any point $p \in \partial\Omega$ on the boundary of Ω and for $0 < t < 1$ let $\varphi_t(p) = o + t \cdot \overrightarrow{op}$. This map sends $\partial\Omega$ bijectively to the asymptotic sphere $\mathrm{AsS}(o, \operatorname{arctanh} t)$ centred at o with radius $\operatorname{arctanh} t$. (See Figure 3.)

Proof of part (1). Any point of a compact set in the $(1-\tanh R)/(2n)$-Hausdorff–Euclidean neighbourhood of $\mathrm{AsB}(o, R)$, either lies inside $\mathrm{AsB}(o, R)$ or is contained in a Euclidean ball of radius $(1-\tanh R)/(2n)$ centred on a point of $\mathrm{AsB}(o, R)$.

We recall that the ball of radius $1/n$ is a subset of Ω, and thus so is the ball of radius $1/(2n)$; that is,

$$\frac{1}{2n}\mathcal{E} \subset \frac{1}{n}\mathcal{E} \subset \Omega.$$

Let $p \in \partial\Omega$ be a point on the boundary. By convexity, the interior of $K(p)$, the convex hull of p and $(1/n)\mathcal{E}$, is a subset of Ω — it is the projection of a cone of basis $(1/n)\mathcal{E}$. Hence $\mathcal{E}_{p,\alpha}$, the image of $(1/n)\mathcal{E}$ by the dilation of ratio $0 < \alpha < 1$ centred at p, lies in the "cone" $K(p)$. The set $\mathcal{E}_{p,\alpha}$ is therefore a Euclidean ball of radius α/n centred at $\varphi_{1-\alpha}(p)$, and it is a subset of Ω.

A point in the Euclidean ball of radius $\alpha/(2n)$ centred at $\varphi_{1-\alpha}(p)$ is at a distance less than or equal to $\frac{1}{2}\ln 3$ from $\varphi_{1-\alpha}(p)$ with respect to the Hilbert distance of $\mathcal{E}_{p,\alpha}$.

Now a standard comparison argument states that for any two points x and y in $\mathcal{E}_{p,\alpha} \subset \Omega$,

$$d_\Omega(x, y) \leq d_{\mathcal{E}_{p,\alpha}}(x, y).$$

From this inequality it follows that any point in the Euclidean ball of radius $\alpha/(2n)$ centred at $\varphi_{1-\alpha}(p)$ is in the Hilbert metric ball centred at $\varphi_{1-\alpha}(p)$ of radius $\frac{1}{2}\ln 3$.

Now for any $1 \geq \alpha > 1 - \tanh R$, the Euclidean ball of radius $\alpha/(2n)$ contains the Euclidean ball of radius $(1 - \tanh R)/(2n)$.

This implies that for any point x in the asymptotic ball $\mathrm{AsB}(o, R)$, the Euclidean ball of radius $(1 - \tanh R)/(2n)$ centred at x is contained in the Hilbert ball of radius $\frac{1}{2}\ln 3$ centred at x, which allows us to obtain the first part of our claim.

Proof of part (2). This follows from the fact that under our assumptions, Ω itself is in the $(1-\tanh R)$-Hausdorff–Euclidean neighbourhood of the asymptotic ball $\mathrm{AsB}(o, R)$. □

Corollary 13. *Let Ω be a convex domain and let o be the centre of its Löwner ellipsoid, which we assume to be the unit Euclidean ball.*

(1) *The* $(1-\tanh(R+\ln 2))/(2n)$*-Hausdorff–Euclidean neighbourhood of* $B(o, R)$ *is contained in its* $\ln(3(n+1))$*-Hausdorff–Hilbert neighbourhood.*

(2) *For any* $K > 0$, *the K-Hausdorff–Hilbert neighbourhood of* $B(o, R)$ *is contained in its* $(1-\tanh(R+K-\ln(n+1)))$*-Hausdorff–Euclidean neighbourhood.*

The proof of this corollary is a straightforward consequence of the following lemma applied to the conclusion of the Proposition 12.

Lemma 14. *Let Ω be a convex domain, and suppose that o is a point in the interior of Ω such that the unit Euclidean open ball centred at o contains Ω, and Ω contains the Euclidean closed ball centred at o of radius* $1/(2n)$. *Then we have*

$$B(o, R) \subset \mathrm{AsB}(o, R + \ln 2) \quad \textit{and} \quad \mathrm{AsB}(o, R) \subset B(o, R + \ln(n + 1)). \tag{19}$$

This lemma is a refinement of a result of [Colbois and Verovic 2004] in our case.

Proof of Lemma 14. Let x be a point on the boundary $\partial\Omega$ of Ω, and let x^* be the second intersection of the straight line (ox) with $\partial\Omega$. Then our assumption implies

$$\frac{1}{2n} < xo \leq 1 \quad \text{and} \quad \frac{1}{2n} < ox^* \leq 1. \tag{20}$$

Actually the first inclusion is always true. Indeed suppose y is on the half-line $[ox)$ such that $d_\Omega(o, y) \leq R$, which in other words implies that we have

$$\frac{ox}{yx}\frac{yx^*}{ox^*} \leq e^{2R};$$

therefore

$$ox \leq e^{2R}\frac{ox^*}{yx^*}(ox - oy) \leq e^{2R}(ox - oy),$$

which implies in turn that

$$oy \leq \frac{e^{2R}-1}{e^{2R}}ox \leq (1-e^{-2R})ox \leq \tanh(R+\ln 2)ox.$$

Now regarding the second inclusion: consider a point y on the half-line $[ox)$ such that $oy \leq \tanh(R)ox$. On the one hand we have

$$\frac{ox}{yx} = \frac{ox}{ox-oy} \leq \frac{1}{1-\tanh R} = \frac{e^{2R}+1}{2},$$

and, on the other hand, thanks to the inequalities (20) we get

$$\frac{yx^*}{ox^*} \leq \frac{ox+ox^*}{ox^*} \leq 1+\frac{ox}{ox^*} \leq 1+2n, \tag{21}$$

which implies that

$$\frac{ox}{yx}\frac{yx^*}{ox^*} \leq \frac{e^{2R}+1}{2}(1+2n) \leq (1+2n)e^{2R} \leq (1+n)^2e^{2R}. \tag{22}$$

The conclusion follows. □

1.5. ***Distance function to a sphere in a Hilbert geometry.*** This section is an adaptation in the realm of Hilbert geometries of a result concerning the spheres in a Minkowski space provided to the author by A. Thompson [2012].

Let us first start by recalling the following important fact regarding the distance of a point to a geodesic in a Hilbert geometry (see [Busemann 1955, Chapter II, Section 18, page 109]):

Proposition 15. *Let (Ω, d_Ω) be a Hilbert geometry. The distance function of a straight geodesic (that is, given by an affine line) to a point is a peakless function; i.e., if $\gamma : [t_1, t_2] \to \Omega$ is a geodesic segment, then for any $x \in \Omega$ and $t_1 \leq s \leq t_2$ one has*

$$d_\Omega\big(x, \gamma(s)\big) \leq \max\big\{d_\Omega\big(x, \gamma(t_1)\big), d_\Omega\big(x, \gamma(t_2)\big)\big\}.$$

Let us now turn our attention to metric spheres in a two-dimensional Hilbert geometry.

Proposition 16. *Let (Ω, d_Ω) be a two-dimensional Hilbert geometry. Suppose o is a point of Ω, and p and q are two points on the intersection of the metric sphere $S(o, R)$ centred at o and of radius R with a line passing by o. If C denotes one of the arcs of the sphere $S(o, R)$ from p to q, then for any point p' on the half-line $[o, p)$, the function $\varphi(x) = d_\Omega(p', x)$ is monotonic on C.*

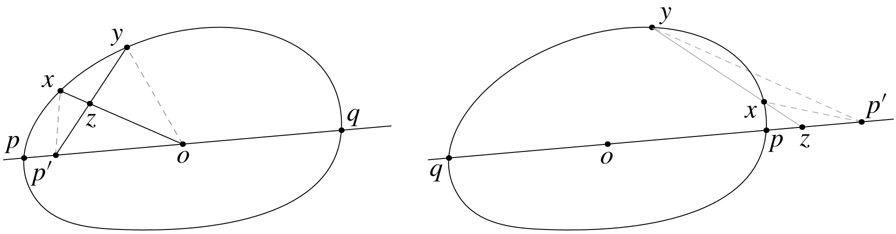

Figure 4. Monotonicity of the distance of a point to a sphere.

Proof. Let p, x, y, q be points in that order on C. We have to show that

$$d_\Omega(p', x) \le d_\Omega(p', y).$$

Suppose first that the line segments $[o, x]$ and $[p', y]$ intersect at a point z. (See Figure 4.) Hence we have

$$\begin{aligned} d_\Omega(o, x) + d_\Omega(p', y) &= \big(d_\Omega(o, z) + d_\Omega(z, x)\big) + \big(d_\Omega(p', z) + d_\Omega(z, y)\big) \\ &= \big(d_\Omega(p', z) + d_\Omega(z, x)\big) + \big(d_\Omega(o, z) + d_\Omega(z, y)\big) \\ &\ge d_\Omega(p', x) + d_\Omega(o, y). \end{aligned}$$

Now, as $d_\Omega(o, y) = d_\Omega(o, x) = R$, the result follows.

Suppose now that $[o, x]$ and $[p', y]$ do not intersect, which implies that p' is outside the ball $B(o, R)$. Then the line (yx) intersects (op) at z. Because x and y lie on the sphere of radius R, we have $d_\Omega(o, z) > R$. Also, as p is one of the nearest points to p' on C, we have $d_\Omega(p', z) \le d_\Omega(p', p) \le d_\Omega(p', y)$. Hence if we apply Proposition 15 to the segment $[z, y]$ and p', as $x \in [z, y]$ we get

$$d_\Omega(p', x) \le \max\{d_\Omega(p', z), d_\Omega(p', y)\} = d_\Omega(p', y). \qquad \square$$

2. Volume entropy and approximability

This section is devoted to the proof of the main theorem. This is done in two steps. The first step consists in bounding the entropy from above in dimension two and three by the approximability thanks to the study of the volume growth in polytopes. The second step is to bound the entropy from below. This is done by exhibiting a separated subset of the Hilbert geometry whose growth is bigger than the approximability. We conclude this section with the various corollaries implied.

Theorem 17. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. The doubles of the approximabilities of Ω are bigger than the volume entropies*; *i.e.*,

$$\underline{\mathrm{Ent}}\,\Omega \le 2\underline{a}(\Omega) \quad \textit{and} \quad \overline{\mathrm{Ent}}\,\Omega \le 2\overline{a}(\Omega).$$

The proof of this theorem relies on the following stronger statement which is a sort of uniform bound on the volume of metric balls and metric spheres in a polytopal Hilbert geometry. The key fact is that this bound depends, in a coarse sense, linearly on the number of vertices of the polytope.

Theorem 18. *Let $n = 2$ or $n = 3$. There are affine maps a_n, b_n from $\mathbb{R}$ to $\mathbb{R}$ and polynomials q_n, p_{n-1} of degree n and $n-1$ such that for any open convex polytope $\mathcal{P}_N$ with N vertices inside the unit Euclidean ball of $\mathbb{R}^n$ and containing the ball of radius $1/(2n)$, one has*

$$\begin{aligned} \mathrm{Vol}_{n-1,\mathcal{P}_N} S_{\mathcal{P}_N}(o, R) &\le a_n(N) p_{n-1}(R), \\ \mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, R) &\le b_n(N) q_n(R). \end{aligned} \tag{23}$$

The same result holds for the asymptotic balls.

Let us stress that our method also yields a control in terms of the vertices in higher dimensions as well, using the so-called upper bound conjecture proved by McMullen [McMullen 1971; McMullen and Shephard 1971], but alas a polynomial of degree strictly bigger than 1 replaces the affine functions a_n and b_n. This is why we can't state the equality in the main theorem in higher dimensions.

Notice that this theorem is still valid if we replace the Hausdorff measures by any measures defined by a pair of proper densities $f \in \mathcal{PD}_n$ and $g \in \mathcal{PD}_{n-1}$. The change of measures will only impact the values of the constants.

Proof of Theorem 18. We will have to deal with dimension two and dimension three separately, even though both cases follow the same main steps.

The first step of our proof consists in proving the first inequality of (23) for the Holmes–Thompson measure and for an asymptotic sphere. The uniform inclusion of metric balls into asymptotic balls (19) then implies the result for the metric spheres thanks to the monotonicity of the Holmes–Thompson measure (Lemma 6).

The second step is an integration using the coarea inequality (25), which allows us to get the second inequality of (23) for metric balls with respect to the Busemann measure.

Let us now make all this more precise. We fix a polytope $\mathcal{P}_N$ with N vertices and for any real $R > 0$ we let P_R be the asymptotic ball of radius R centred at o, and let ∂P_R be the associated asymptotic sphere. We also introduce the constant $c_n = \ln(n+1)$.

Two-dimensional case. The idea is to find an upper bound on the length of each edge of the asymptotic sphere ∂P_R, depending only on R.

To do so, we can use the fact that each edge belongs to the triangle defined by joining its extremities to the point o. Hence, thanks to the triangle inequality its length is less than the sum of these two other segments. However, using the second inclusion (19) of Lemma 14, we know that the asymptotic ball P_R is inside the

Hilbert ball of radius $R + c_2$ centred at o of the convex polygon $\mathcal{P}_N$. Hence the length of each edge is less than $2 \cdot (R + c_2)$. Therefore the length of the polygon ∂P_R is less than $N \cdot 2 \cdot (R + c_2)$.

Following the first inclusion (19) of Lemma 14, the metric ball of radius r centred at o is a subset of the asymptotic ball of radius $r + \ln 2$ centred at o. Therefore, we can use the monotonicity of the Holmes–Thompson length (see Lemma 6) to get for all $r > 0$,

$$\text{Length}_{\mathcal{P}_N} S_{\mathcal{P}_N}(o, r) \leq \text{Length}_{\mathcal{P}_N} \partial P_{r+\ln 2} \leq N \cdot 2(r + \ln 2 + c_2). \tag{24}$$

Now using the coarea inequality of Lemma 8, taking into account that the Busemann length is equal to the Holmes–Thompson length one gets

$$\frac{\partial}{\partial r} \text{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, r) \leq \frac{\pi}{4} \cdot N \cdot 2(r + \ln 2 + c_2). \tag{25}$$

Hence, integrating the inequality (25) over the interval $[0, R]$, we finally obtain the following inequality for the ball of radius $R > 0$:

$$\text{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o, R) \leq \frac{\pi}{4} \cdot N \cdot (R^2 + 2(\ln 2 + c_2)R). \tag{26}$$

The inequalities (24) and (26) are the expected results in dimension two.

Three-dimensional case. Once again the idea is to find an upper bound on the area of faces of the asymptotic sphere ∂P_R. Alas, contrary to the two-dimensional case, there is not a unique type of faces, and it is therefore pointless to look for an upper bound depending only on the radius R.

However, each face can be seen as the basis of a pyramid with apex the point o. All other faces are then triangles, whose areas can be bounded thanks to Lemma 9. An analogue of the triangle inequality is available in the form of the minimality of the Holmes–Thompson area (see Berck [2009]). In other words, the Holmes–Thompson area of each face of ∂P_R is less than the sum of the Holmes–Thompson areas of the triangles obtained as the convex hull of o and an edge of the given face of ∂P_R. Let us call $\mathcal{T}_o$ such a triangle (the subscript o is to stress the fact that the point o is one of its vertices).

To bound the area of the triangle $\mathcal{T}_o$ it suffices to focus on the intersection of the polytope $\mathcal{P}_N$ with the affine plane containing the triangle $\mathcal{T}_o$. This is a polygon $\widetilde{P}$, to which we can apply Lemma 9, which bounds from above the area of a two-dimensional triangle inside a metric ball centred on one of its vertices. This is exactly the situation of our triangle $\mathcal{T}_o$ with respect to the Hilbert geometry associated to the polygon $\widetilde{P}$. Indeed it is included in the asymptotic ball of radius R, and again thanks to Lemma 14 we know that it is inside the metric ball of radius $R + c_3$ with respect to the Hilbert geometry of $\mathcal{P}_N \in \mathbb{R}^3$. As $\widetilde{P}$ is a plane section of $\mathcal{P}_N \in \mathbb{R}^3$, this still holds for $\mathcal{T}_o$ seen as a subset of $\widetilde{P}$. Hence Lemma 9 implies that the area of the triangle $\mathcal{T}_o$ is less than $C(R+c_3)^2$, for some constant $C>1$ independent of R.

Therefore, if $e(N)$ is the number of edges of $\mathcal{P}_N$, the area of the asymptotic sphere ∂P_R is less than $2e(N)C(R+c_3)^2$.

Let $f(N)$ be the number of faces of $\mathcal{P}_N$ and let us recall Euler's formula:

$$N-e(N)+f(N)=2.$$

Each face being surrounded by at least three edges and each edge belonging to two faces, one has the classical inequality (where equality is obtained in a simplex)

$$3f(N)\le 2e(N).$$

Combining the previous two inequalities we get a linear upper bound on the number of edges by the number of vertices:

$$2\le N-\tfrac{1}{3}e(N) \quad\Rightarrow\quad e(N)\le 3N-6.$$

Hence the area of the asymptotic sphere ∂P_R is less than $(3N-6)\cdot 2C\cdot(R+c_3)^2$.

We can now conclude almost as in the two-dimensional case. Following the first inclusion (19) of Lemma 14, the metric ball of radius r centred at o is a subset of the asymptotic ball of radius $r+\ln 2$ centred at o. Therefore, we can use the monotonicity of the Holmes–Thompson area measure (see Lemma 6) to get for all $r>0$,

$$\mathrm{Vol}_{2,\mathcal{P}_N} S_{\mathcal{P}_N}(o,r)\le \mathrm{Vol}_{2,\mathcal{P}_N}\partial P_{r+\ln 2}\le(3N-6)\cdot 2C\cdot(r+\ln 2+c_3)^2. \tag{27}$$

Notice that this inequality (27) corresponds to the first part of the inequality (23).

The rough monotonicity of the Busemann measure (see the right-hand side of the inequality (15) in Lemma 7) states that the Busemann area is smaller than the Holmes–Thompson one, hence combined with the inequality (27) above, we get that for all $r>0$

$$\mathrm{Area}_{\mathcal{P}_N} S_{\mathcal{P}_N}(o,r)\le(3N-6)\cdot 2C\cdot(r+\ln 2+c_3)^2. \tag{28}$$

Taking into account the coarea inequality (see Lemma 8) in conjunction with the inequality (28) leads to the differential inequality

$$\frac{\partial}{\partial r}\mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o,r)\le 2\cdot(3N-6)\cdot 2C\cdot(r+\ln 2+c_3)^2, \tag{29}$$

which we can integrate over the interval $[0,R]$ to finally obtain that for all $R>0$

$$\mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(o,R)\le 2\cdot(N-2)\cdot 2C\cdot\big((r+\ln 2+c_3)^3-c_3^3\big). \tag{30}$$

This concludes our proof in the three-dimensional case. □

Let us remark that if we link this to our study of the asymptotic volume of the Hilbert geometry of polytopes [Vernicos 2013] we obtain the following corollary:

Corollary 19. *Let $\mathcal{P}_N$ be an open convex polytope with N vertices in $\mathbb{R}^n$, for $n = 2$ or 3. Then there are three constants α_n, β_n and γ_n such that for any point $p \in \mathcal{P}_N$ one has*

$$\alpha_n \cdot N \leq \liminf_{R \to +\infty} \frac{\operatorname{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(p, R)}{R^n} \leq \beta_n \cdot N + \gamma_n.$$

Now let us come back to our initial problem and see how Theorem 18 implies Theorem 17.

Proof of Theorem 17. We remind the reader that $\operatorname{Vol}_{n-1,\Omega}$ stands for the $(n-1)$-dimensional Holmes–Thompson measure. Let o be the centre of the Löwner ellipsoid of Ω, which we assume to be the unit Euclidean ball. We consider R large enough in order to have the Euclidean ball of radius $1/(2n)$ inside all the asymptotic balls involved in the sequel.

The idea of the proof consists in replacing for all R large enough the convex set Ω by a convex polytope $\mathcal{P}_R$ such that

- $\mathcal{P}_R$ is a subset of Ω;
- the asymptotic ball P_R of the polytope $\mathcal{P}_R$ is inside the $(1-\tanh R)/(2n)$-Euclidean neighbourhood of the corresponding asymptotic ball $\operatorname{AsB}_\Omega(o, R)$ of Ω;
- the exponential volume growth, with respect to the geometry of Ω, of the two families of asymptotic balls $(P_R)_{R\in\mathbb{R}}$ and $(\operatorname{AsB}_\Omega(o, R))_{R\in\mathbb{R}}$ is the same.

Let us insist on the fact that the convex polytope $\mathcal{P}_R$ depends on R.

Then using Theorem 18 we will bound from above the area in dimension three or the perimeter in dimension two of the convex polytope P_R by a function depending linearly on the number of vertices of P_R and polynomially on R. This will allow us to conclude.

Fix R. Among all polytopes included in both the asymptotic ball $\operatorname{AsB}_\Omega(o, R)$ and its $(1-\tanh R)/(2n)$-Hausdorff–Euclidean neighbourhood pick a polytope P_R with the minimal number of vertices $N(R)$. Notice that we have

$$N(R) = N\left(\frac{1-\tanh R}{2n \tanh R}, \Omega\right). \tag{31}$$

Claim. *There exists a constant $C > 0$ such that for all R,*

$$\operatorname{AsB}_\Omega(o, R - C) \subset P_R \subset \operatorname{AsB}_\Omega(o, R). \tag{32}$$

To prove this claim, on the one hand we deduce from the first inclusion of Lemma 14 that

$$B_\Omega(o, R - \ln 2) \subset \operatorname{AsB}_\Omega(o, R).$$

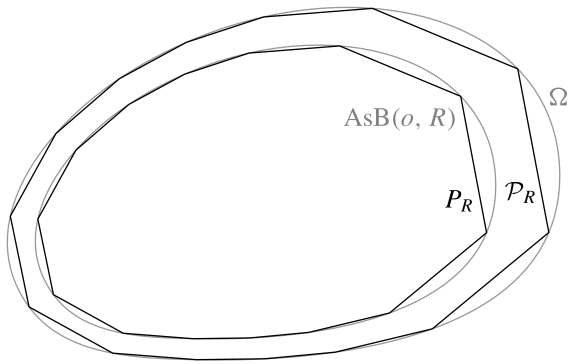

Figure 5. The asymptotic ball and an approximating polytope.

On the other hand the comparison of both Hausdorff–Hilbert and Hausdorff–Euclidean neighbourhoods, as stated in Proposition 12, implies that the convex polytope P_R lies in the $\left(\frac{1}{2}\ln 3\right)$-Hausdorff–Hilbert neighbourhood of the asymptotic ball $\mathrm{AsB}_\Omega(o, R)$. From these we deduce the inclusion

$$B_\Omega(o, R-\ln 6) \subset P_R \subset \mathrm{AsB}_\Omega(o, R). \tag{33}$$

Taking into account the second inclusion of Lemma 14 we get

$$\mathrm{AsB}_\Omega(o, R-\ln 6-\ln(n+1)) \subset P_R \subset \mathrm{AsB}_\Omega(o, R), \tag{34}$$

which proves our claim with $C = \ln 6 + \ln(n+1)$.

Thanks to the monotonicity of the Holmes–Thompson measure (see Lemma 6) we know that the area of the boundary ∂P_R is less than the area of the asymptotic sphere $\mathrm{AsS}_\Omega(o, R)$, but larger than the area of the asymptotic sphere of radius $R-C$; that is,

$$\mathrm{Vol}_{n-1,\Omega}\,\mathrm{AsS}_\Omega(o, R-C) \le \mathrm{Vol}_{n-1,\Omega}\,\partial P_R \le \mathrm{Vol}_{n-1,\Omega}\,\mathrm{AsS}_\Omega(o, R). \tag{35}$$

From (35) we deduce that the logarithms of the areas of ∂P_R and $\mathrm{AsS}_\Omega(o, R)$ are asymptotically the same in the following sense:

$$\lim_{R\to+\infty} \frac{\ln(\mathrm{Vol}_{n-1,\Omega}\,\mathrm{AsS}_\Omega(o, R))}{\ln(\mathrm{Vol}_{n-1,\Omega}\,\partial P_R)} = 1. \tag{36}$$

Let us denote by $\mathcal{P}_R$ the image of P_R by the dilation of ratio $1/\tanh R$. This is the dilation sending $\mathrm{AsB}_\Omega(o, R)$ to Ω. (See Figure 5.) Hence, by construction, $\mathcal{P}_R \subset \Omega$ and therefore we have

$$\mathrm{Vol}_{n-1,\Omega}\,\partial P_R \le \mathrm{Vol}_{n-1,\mathcal{P}_R}\,\partial P_R. \tag{37}$$

Now thanks to Theorem 18, for $n = 2$ or $n = 3$ and $R > 0$ such that $\tanh R > \frac{3}{4}$, there are two constants a_n, b_n and a polynomial Q_n of degree n such that

$$\operatorname{Vol}_{n-1,\Omega} \partial P_R \leq (a_n N(R) + b_n) Q_n(R). \tag{38}$$

To conclude we remark that

$$\liminf_{R\to+\infty} \frac{\ln N(R)}{R} = 2\underline{a}(\Omega) \quad \text{and} \quad \limsup_{R\to+\infty} \frac{\ln N(R)}{R} = 2\overline{a}(\Omega),$$

and use it with the inequality (38) to get for instance

$$\limsup_{R\to+\infty} \frac{\ln(\operatorname{Vol}_{n-1,\Omega} \partial P_R)}{R} \leq 2\overline{a}(\Omega).$$

Finally the limit (36) implies that

$$\limsup_{R\to+\infty} \frac{\ln(\operatorname{Vol}_{n-1,\Omega} \operatorname{AsS}_\Omega(o, R))}{R} \leq 2\overline{a}(\Omega).$$

The left-hand side of this last inequality is easily seen to be the spherical entropy (see (13)), which ends our proof. □

The next corollary follows from a result of Bronshteyn and Ivanov (Theorem 31) which states that $2\overline{a} \leq n - 1$.

Corollary 20. *Let Ω be an open bounded convex set in $\mathbb{R}^n$ for $n = 2$ or 3. Then*

$$\overline{\operatorname{Ent}}\,\Omega \leq n - 1.$$

We are now going to study the reverse inequality.

Theorem 21. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. The volume entropies of Ω are greater than or equal to twice the approximabilities of Ω; i.e.,*

$$2\underline{a}(\Omega) \leq \underline{\operatorname{Ent}}\,\Omega \quad \textit{and} \quad 2\overline{a}(\Omega) \leq \overline{\operatorname{Ent}}\,\Omega.$$

Proof. Without loss of generality we suppose that the Euclidean unit ball is the Löwner ellipsoid of Ω and that o is the centre of that ball.

The idea of the proof is the following:

- We will show that for a good positive δ and any positive real number R there exists a δ-separated set $\mathcal{S}_R$ in the metric ball of radius $B(o, R + 2\delta)$ such that the convex closure P_R of that set contains the ball $B(o, R)$.
- We will then use the fact that the cardinality of this δ-separated set will be larger than the cardinality of the set of vertices of a vertex-minimising convex polytope included in the annulus $B(o, R + 2\delta) \setminus B(o, R)$.

In other words, the number of points in the δ-separated set will be bounded from below by the number $N(\varepsilon(R), \Omega)$ from the introduction. Here ε will be a function of R.

- To conclude we will take into account that the union of the open metric balls of radius $\frac{1}{2}\delta$ centred at the point of the δ-separated set $\mathcal{S}_R$ is disjoint and is in the ball $B(o, R+3\delta)$. Thus we get a lower bound on the volume of the ball $B(o, R+3\delta)$ in terms of $N(\varepsilon(R), \Omega)$ times a constant depending on the dimension.

Let us now start the proof. Consider the $\left(\frac{1}{2}\ln 3\right)$-Hilbert neighbourhood of the metric ball $B(o, R)$, that is,

$$V(R) = B\left(o, R + \tfrac{1}{2}\ln 3\right),$$

and take a maximal ($\delta = \frac{1}{4}\ln 3$)-separated set $\mathcal{S}_R$ on its boundary. This set contains $\#\mathcal{S}_R$ points. Now let us take the convex hull $\mathcal{C}_R$ of these points. This is a polytope with $N_2(R) \leq \#\mathcal{S}_R$ vertices.

Claim 22. *The polytope $\mathcal{C}_R$ is included in the 2δ-Hilbert neighbourhood of $B(o, R)$ and contains $B(o, R)$.*

Notice that if the claim holds, then for some real constant c independent of R (see Corollary 13 once again), we have

$$\#\mathcal{S}_R \geq N_2(R) \geq \widetilde{N}(R-c) := N\left(\tfrac{1}{4}(1-\tanh(R-c)), \mathrm{AsB}(o, R-c)\right). \tag{39}$$

Proof of Claim 22. First notice that $V(R)$ is a convex set (see Busemann [1955, Chapter II, Section 18, page 105]). Therefore the convex hull is inside the 2δ-Hilbert neighbourhood of $B(o, R)$, that is, $V(R)$.

Now let us suppose by contradiction that $\mathcal{C}_R$ does not contain $B(o, R)$. Hence there exists some point q in $B(o, R)$ which is not in $\mathcal{C}_R$. We will show that we can find a point on the sphere $S(o, R+2\delta)$ which is at a distance bigger than δ from all points of $\mathcal{S}_R$, which will contradict its maximality.

Under our assumption, the Hahn–Banach separation theorem asserts that there exists a linear form a, some constant c and a hyperplane $H = \{x \mid a(x) = c\}$ which separates q and $\mathcal{C}_R$, i.e., $a(q) > c$ and $a(x) < c$ for all $x \in \mathcal{C}_R$. Consider then $H_q = \{x \mid a(x) = a(q)\}$, the hyperplane parallel to H containing q. Let us say that a point x such that $a(x) \geq a(q)$ is *above* the hyperplane H_q.

Then let us define by $V'_o = \{x \in \partial V(R) \mid a(x) \geq a(q)\}$ the part of the boundary of $V(R)$ which is above H_q. Now we want to metrically project each point of V'_o onto H_q, that is to say that to each point of V'_o we associate its closest point on H_q. However if Ω is not strictly convex, the projection might not be unique (see the Appendix); that is why we are going to distinguish two cases.

First case: the convex set Ω is strictly convex. Then the metric projection is a map from V_o' to H_q and it is continuous; furthermore the points on $H_q \cap V_o'$ are fixed and by convexity $H_q \cap V_o'$ is homeomorphic to an $(n-2)$-dimensional sphere. Therefore by the Borsuk–Ulam theorem (or a version of it known as the *antipodal map theorem*), there is a point p on V_o' whose metric projection is q.

Now as p is on the boundary of $V(R)$, that is, the sphere $B(o, R+2\delta)$, and q is in $B(o, R)$ we necessarily have

$$d_\Omega(p, q) \geq \tfrac{1}{2} \ln 3.$$

Hence for all points x in $H_q \cap V_o'$, we have

$$d_\Omega(p, x) \geq d_\Omega(p, q) \geq \tfrac{1}{2} \ln 3.$$

Second case: the convex set Ω is not strictly convex. Then let us approximate it by a smooth and strictly convex set Ω' such that $\Omega \subset \Omega'$, and for all pairs of points $x, y \in V(R)$,

$$\tfrac{2}{3} \times d_{\Omega'}(x, y) \geq d_\Omega(x, y) \geq d_{\Omega'}(x, y). \tag{40}$$

Then metrically project V_o' onto H_q with respect to Ω'. By the same argument as in the first case, we obtain a point p such that for all x in $H_q \cap V_o'$ we have

$$d_{\Omega'}(p, x) \geq d_{\Omega'}(p, q) \geq \tfrac{3}{2} d_\Omega(p, q) \geq \tfrac{3}{4} \ln 3,$$

which also implies by the inequalities (40) that for all x in $H_q \cap V_o'$ we have

$$d_\Omega(p, x) \geq \tfrac{3}{4} \ln 3.$$

In either case, using Proposition 16 of Section 1.5, we deduce that all points on ∂V_R at distance less than or equal to $\frac{1}{4} \ln 3$ from p are above H_q and are therefore contained in V_o'. We then infer that there are no points of $\mathcal{S}_R$ at distance less than or equal to $\frac{1}{4} \ln 3$ from p, which contradicts the maximality of the set $\mathcal{S}_R$. □

Now consider the union of the balls of radius $\frac{1}{2}\delta$ centred at the points of $\mathcal{S}_R$. This union is a subset of the ball $B(o, R+3\delta)$ and the balls are mutually disjoint. Now following our paper [Vernicos 2013], there exists a constant a_n such that for any open proper convex set Ω and $x \in \Omega$, the volume of the ball of radius r centred at x is at least $a_n r^n$. Hence from this fact and the inequality (39) we get that for all $R > 0$,

$$\begin{aligned} \operatorname{Vol}_\Omega B(o, R+3\delta) &\geq \#\mathcal{S}_R \cdot a_n \delta^n \\ &\geq N\big(\tfrac{1}{4}(1-\tanh(R-c)), \operatorname{AsB}(o, R-c)\big) \cdot a_n \delta^n. \end{aligned} \tag{41}$$

Now if we take the logarithm of the previous inequalities, divide by R and take either the lim inf or the lim sup we conclude the proof of Theorem 21. □

The proof of the main theorem (Theorem 1) is now complete, and we turn to its corollaries.

A point x of a convex body K is called a *farthest point* of K if and only if, for some point $y \in \mathbb{R}^n$, x is farthest from y among the points of K. The set of farthest points of K, which are special exposed points, will be denoted by $\exp^* K$. Thus a point $x \in K$ belongs to $\exp^* K$ if and only if there exists a ball which circumscribes K and contains x in its boundary.

In dimension two we get the following corollary:

Corollary 23. *Let Ω be a plane Hilbert geometry, and let d_M be the Minkowski dimension of extremal points and d_H the Hausdorff dimension of the set $\exp^* \Omega$ of farthest points. Then we have*

$$d_H \leq \underline{\mathrm{Ent}}\,\Omega \leq \overline{\mathrm{Ent}}\,\Omega \leq \frac{2}{3-d_M}. \tag{42}$$

The inequality on the left remains valid for higher-dimensional Hilbert geometries.

Proof. The inequality on the left of (42) comes from [Schneider and Wieacker 1981], whereas the one on the right is the first main theorem in [Berck et al. 2010]. □

Remark 24. Inequality (42) induces a new result concerning the approximability in dimension two, as it implies that

$$\bar{a}(\Omega) \leq \frac{1}{3-d}.$$

Lastly we are also able to prove the following result, which relates the entropy of a convex set and the entropy of its polar body.

Corollary 25. *Let Ω be a Hilbert geometry of dimension two or three. Then*

$$\underline{\mathrm{Ent}}\,\Omega = \underline{\mathrm{Ent}}\,\Omega^* \quad \textit{and} \quad \overline{\mathrm{Ent}}\,\Omega = \overline{\mathrm{Ent}}\,\Omega^*.$$

Proof. It suffices to prove that the approximability of a convex body Ω containing the origin and its polar Ω^* are equal. Without loss of generality we can assume that the unit ball is Ω's John ellipsoid. Hence Ω is contained in the ball of radius the dimension and its polar contains the ball of radius the inverse of the dimension and is included in the unit ball. Now, notice that for ε small enough, if P_k is a polytope with k vertices inside the ε-Hausdorff neighbourhood of Ω, then its polar P_k^* is a polytope with k faces containing Ω^* and contained in its $(\varepsilon \cdot C)$-Hausdorff neighbourhood for some constant C depending only on the dimension. A known fact (see Gruber [2007, Section 11.2]) states that the approximability can be computed by minimising either the vertices or the faces. Hence $\underline{a}(\Omega) = \underline{a}(\Omega^*)$ and $\bar{a}(\Omega) = \bar{a}(\Omega^*)$. The statement therefore follows from the main theorem. □

3. Intermediate growth

In this section we focus on the two-dimensional case. The intermediate volume growth will follow from Theorem 18 and the following proposition, which allows us to control both the length of sphere and its volume in dimension two from below, thanks to the number of vertices of an ad hoc approximating polytope, in the fashion of Theorem 18, except that here the lower bounds depend on Ω.

Proposition 26. *Let Ω be an open bounded convex set in $\mathbb{R}^2$ whose Löwner ellipsoid is the Euclidean unit ball centred at $o \in \Omega$. Let $N(\varepsilon, \Omega)$ be the minimal number of vertices of a polygon containing Ω at Hausdorff–Euclidean distance less than ε from Ω, and for any positive real number R let*

$$N(R) := N\left(\frac{1-\tanh R}{4\tanh R}, \Omega\right).$$

Then there exist three constants R_2, K_2 and C_2 independent of Ω such that for all real numbers $R > R_2$ we have

$$\begin{aligned} \text{Length}_\Omega\, S_\Omega(o, R) &\geq \big(N\big(R - \tfrac{3}{2}\ln 3\big) - 2\big)K_2, \\ \text{Vol}_\Omega\, B_\Omega\big(o, R + \tfrac{1}{2}K_2\big) &\geq \big(N\big(R - \tfrac{3}{2}\ln 3\big) - 2\big)C_2(K_2)^2. \end{aligned} \tag{43}$$

The same result holds for the asymptotic balls with $R > R_2 + \ln 2$.

We want to stress once again that there is actually no loss in generality in supposing the Euclidean unit ball to be the Löwner ellipsoid of Ω.

Proof. For any positive real number R let $\varepsilon(R) = \frac{1}{4}(1-\tanh R)$. The idea is to build a convex polygon in the $\varepsilon(R)$-neighbourhood of an asymptotic ball of radius R in such a way that we can control uniformly from below the length of the edges. More precisely we have the following.

Claim 27. *There exists a convex polygon $\mathcal{P}_R$ such that*

- *$\mathcal{P}_R$ contains the asymptotic ball* $\text{AsB}(o, R)$ *and is in its $\varepsilon(R)$-Hausdorff–Euclidean neighbourhood*;
- *all the edges of $\mathcal{P}_R$ but one are tangent to* $\text{AsB}(o, R)$ *and all its vertices belong to the boundary* $\partial_R\,\text{AsB}$ *of the $\varepsilon(R)$-Hausdorff neighbourhood of the asymptotic ball* $\text{AsB}_\Omega(o, R)$.

This claim is a consequence of the following algorithm:

Step 1 Draw one tangent to $\text{AsB}_\Omega(o, R)$. It will meet the boundary $\partial_R\,\text{AsB}$ of its $\varepsilon(R)$-Hausdorff neighbourhood at two points x_1 and x_2, where $\overrightarrow{ox_1}$ and $\overrightarrow{ox_2}$ are positively oriented.

Step 2 We start from x_2 and draw the second tangent to $\text{AsB}_\Omega(0, R)$ passing by x_2. This second tangent will meet the boundary $\partial_R\,\text{AsB}$ at a second point x_3.

Step 3 For $k > 2$, if the second tangent t_{k+1} to $\mathrm{AsB}_\Omega(0, R)$ passing by x_k has its second intersection with $\partial_R\,\mathrm{AsB}$ on the arc from x_1 to x_k (in the orientation of the construction), we stop and consider for $\mathcal{P}_R$ the convex hull of $x_1, \ldots, x_k$; otherwise we take for x_{k+1} that second intersection of the tangent t_{k+1} with $\partial_R\,\mathrm{AsB}$ and start that step again.

This algorithm will necessarily finish, because by convexity the arclength of $x_i x_{i+1}$ on $\partial_R\,\mathrm{AsB}$ built this way is bigger than $2\varepsilon(R)$. At the end of this algorithm we obtain, by minimality, a polygon which has at least $N(R) = N\big(\varepsilon(R), \mathrm{AsB}_\Omega(o, R)\big) = N\big(\varepsilon(R)/\tanh R, \Omega\big)$ edges.

Recall that Proposition 12 guarantees us that the $\varepsilon(R)$-Euclidean neighbourhood of the asymptotic ball $\mathrm{AsB}_\Omega(o, R)$ is included in its $\big(\frac{1}{2}\ln 3\big)$-Hausdorff–Hilbert neighbourhood and therefore, taking into account the inclusions (19), we obtain

$$B_\Omega(o, R - \ln 2) \subset \mathrm{AsB}_\Omega(o, R) \subset \mathcal{P}_R \subset B_\Omega\big(o, R + \tfrac{3}{2}\ln 3\big).$$

Moreover, the length coincides with the Holmes–Thompson one-dimensional measure. Therefore, the monotonicity of the latter, as seen in Lemma 6, implies the following inequalities:

$$\begin{aligned} \text{Length}_\Omega\, S_\Omega(o, R - \ln 2) &\le \text{Length}_\Omega\, \partial\mathrm{AsB}_\Omega(o, R) \\ &\le \text{Length}_\Omega\, \partial\mathcal{P}_R \\ &\le \text{Length}_\Omega\, S_\Omega\big(o, R + \tfrac{3}{2}\ln 3\big). \end{aligned} \tag{44}$$

Now let $\mathfrak{P}_R$ be the image of $\mathcal{P}_R$ under the dilation of ratio $1/\tanh R$ centred at o. By construction $\mathfrak{P}_R$ contains Ω, which implies

$$\text{Length}_{\mathfrak{P}_R}\, \partial\mathcal{P}_R \le \text{Length}_\Omega\, \partial\mathcal{P}_R.$$

Therefore it suffices to prove the following claim:

Claim 28. *Let $I(R) \in \partial_R\,\mathrm{AsB}$ be a vertex of $\mathcal{P}_R$ such that the two edges containing $I(R)$ are tangent to $\mathrm{AsB}_\Omega(o, R)$ at $b(R)$ and $c(R)$. Then for any $R > \tanh^{-1}\big(\frac{1}{2}\big) = R_2$,*

$$d_\Omega\big(b(R), c(R)\big) \ge d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \ge \ln\tfrac{6}{5} = K_2.$$

Indeed, let us assume that Claim 28 is true, and for $R > r_2$ consider a vertex v of $\mathcal{P}_R$ whose incident edges are tangent to $\mathrm{AsB}(o, R)$. Let b and c be the two points of tangency. Then by the triangle inequality,

$$d_\Omega(b, v) + d_\Omega(c, v) \ge d_\Omega(b, c) \ge K_2.$$

Therefore the length of $\mathcal{P}_R$ is bigger than $(\widetilde{N}(R) - 2)K_2$, where $\widetilde{N}(R)$ is the number of edges of $\mathcal{P}_R$ (because of the possible exception at x_1 and the last point of the construction above). Hence taking $R_2 = r_2 + \frac{3}{2}\ln 3$, thanks to (44), we get for $R > R_2$

$$\text{Length}_\Omega\, S_\Omega(o, R) \geq \big(\widetilde{N}\big(R - \tfrac{3}{2}\ln 3\big) - 2\big)K_2, \tag{45}$$

and as $\widetilde{N}\big(R - \frac{3}{2}\ln 3\big) \geq N\big(R - \frac{3}{2}\ln 3\big)$ the first inequality in (43) is proved.

Now concerning the volume of the ball, Claim 28 and Proposition 16 imply that the contact points of the edges of $\mathcal{P}_R$ with $\text{AsB}_\Omega(o, R)$ form a K_2-separated set. Hence we can conclude in the same way as we did during the proof of Theorem 21; i.e., the balls of radius $\frac{1}{2}K_2$ centred at those points are disjoint and included in the metric ball $B_\Omega\big(o, R + \frac{3}{2}\ln 3 + \frac{1}{2}K_2\big)$. Now following [Vernicos 2013], there exists a constant C depending only on the dimension such that the volume of the ball of radius r is at least $C \cdot r^2$. Hence we obtain that

$$\text{Vol}_\Omega\, B_\Omega\big(o, R + \tfrac{3}{2}\ln 3 + \tfrac{1}{2}K_2\big) \geq \big(\widetilde{N}(R) - 2\big) \cdot C \cdot \big(\tfrac{1}{2}K_2\big)^2, \tag{46}$$

and the last inequality (43) follows once again from the inequality $\widetilde{N}(R) \geq N(R)$.

Proof of Claim 28. Let $a(R)$ (respectively $d(R)$) be the vertex opposite $I(R)$ on the edge containing $b(R)$ (respectively $c(R)$).

Now let us consider the images I, a, b, c and d of the five points $I(R)$, $a(R)$, $b(R)$, $c(R)$ and $d(R)$ by the dilation of ratio $1/\tanh R$ centred at o. Then we are in the same configuration as in Claim 10, with $\mathfrak{P}_R$ instead of Ω. Let

$$u(R) = \frac{bc}{BC}\frac{\tanh R}{1 - \tanh R};$$

then following (16) we have

$$d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \geq \tfrac{1}{2}\ln\left(1 + \frac{u(R) + u(R)^2}{s(1-s)}\right).$$

Therefore we need to obtain a lower bound for $u(R)$. To do this, let p be the intersection of the line oI with the lines (bc). Then thanks to Thales' theorem we have

$$\frac{BC}{bc} = \frac{oI}{pI} = \frac{op + pI}{pI} = 1 + \frac{op}{pI}.$$

Concerning the distance op, recall that the unit ball centred at o is the Löwner ellipsoid of Ω and therefore we get $op \leq 1/\tanh R$, because by convexity p is in Ω. Regarding the distance pI, as $I(R)$ is on the boundary of the $\frac{1}{4}(1-\tanh R)$-Euclidean neighbourhood of $\text{AsB}(o, R)$, we have that I is on the boundary of the $(1 - \tanh R)/(4\tanh R)$-neighbourhood of Ω. Hence we obtain

$$pI \geq \frac{1 - \tanh R}{4\tanh R},$$

because the segment $[p, I]$ intersects Ω. In this way we obtain

$$\frac{BC}{bc} \leq 1 + \frac{4}{1 - \tanh R},$$

which in turn implies that

$$1 \leq \frac{5 - \tanh R}{1 - \tanh R} \frac{bc}{BC} \leq \frac{5}{1 - \tanh R} \frac{bc}{BC}.$$

Hence

$$\tfrac{1}{5} \tanh R \leq u(R). \tag{47}$$

Therefore if $\tanh R_2 = \frac{1}{2}$ then for all $R > R_2$ we get $10u(R) > 1$.

Finally, using the fact that $s(1-s) \leq \frac{1}{4}$ and taking $R > R_2$ we get

$$d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \geq \tfrac{1}{2} \ln\big(1 + \tfrac{2}{5} + \tfrac{1}{25}\big) = \ln \tfrac{6}{5} > 0.18. \qquad \square$$

Proof of Theorem 3 (*intermediate volume growth theorem*). Following Theorem 4 of [Schneider and Wieacker 1981, page 154] and its proof, for any increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\liminf_{r \to +\infty} \frac{e^r}{f(r)} > 0$$

there exists a convex set Ω_f such that

$$0 < \liminf_{r \to +\infty} \frac{N(1 - \tanh r, \Omega_f)}{f(r)} \leq \limsup_{r \to +\infty} \frac{N(1 - \tanh r, \Omega_f)}{f(r)} < +\infty. \tag{48}$$

In the sequel we will write $N(r) = N(1 - \tanh r, \Omega_f)$ and drop the subscript Ω_f in the notation of metric and asymptotic balls.

Now let o be the centre of the Löwner ellipsoid of Ω_f. Following Proposition 26 for $K_2 = \ln \frac{6}{5}$ and $r > 0$ satisfying

$$\tanh\big(r - \tfrac{3}{2} \ln 3 - \tfrac{1}{2} K_2\big) \geq \tfrac{1}{2},$$

we have that

$$\operatorname{Vol}_{\Omega_f} B(o, r) \geq \Big(N\big(r - \tfrac{3}{2} \ln 3 - \tfrac{1}{2} K_2\big) - 2\Big) C(K_2)^2. \tag{49}$$

This inequality implies that

$$\liminf_{r \to +\infty} \frac{\operatorname{Vol}_{\Omega_f} B(o, r)}{f(r)} \geq C(K_2)^2 \liminf_{r \to +\infty} \frac{N\big(r - \frac{3}{2} \ln 3 - \frac{1}{2} K_2\big) - 2}{f(r)}. \tag{50}$$

Now using inequalities (35) to (38) from the proof of Theorem 17 we get the existence of three constants a, b and c such that if $K = \ln 18$ and $r > 0$ is a real number satisfying $\tanh(r - C) > \frac{3}{4}$ then

$$\operatorname{Vol}_{\Omega_f} \operatorname{AsB}(o, r - C) \leq N\left(\frac{1 - \tanh r}{4 \tanh r}, \Omega_f\right)(ar^2 + br + c). \tag{51}$$

The inclusion $B(o, r-\ln 2-C)\subset \mathrm{AsB}(o, r-C)$ given by (19) in Corollary 13's proof allows us to obtain

$$\mathrm{Vol}_{\Omega_f} B(o, r-C-\ln 2) \le N\left(\frac{1-\tanh r}{4\tanh r}, \Omega_f\right)(ar^2+br+c), \tag{52}$$

which in turn implies that

$$\limsup_{r\to+\infty} \frac{\mathrm{Vol}_{\Omega_f} B(o,r)}{r^2 f(r)} \le a\times \limsup_{r\to+\infty} \frac{N\left(\frac{1-\tanh r}{4\tanh r}, \Omega_f\right)}{f(r)}. \tag{53}$$

Combining inequalities (49) and (51) and using the asymptotic comparison (48) we finally conclude that

$$\liminf_{r\to+\infty} \frac{\ln(\mathrm{Vol}_{\Omega_f} B(o,r))}{r} = \liminf_{r\to+\infty} \frac{\ln f(r)}{r}.$$

In the above proofs we can replace lim inf by lim sup.

To obtain the penultimate statement consider $f(r)=e^r/r^3$, and apply our result to get a convex set Ω_f whose entropy is 1. However, by the definition of the centroprojective area and our result in the two-dimensional case [Berck et al. 2010] we have

$$\begin{aligned}\mathcal{A}_o(\Omega_f) = \lim \frac{\mathrm{Vol}_{\Omega_f} B(o,r)}{\sinh r} &= \limsup \frac{\mathrm{Vol}_{\Omega_f} B(o,r)}{\sinh r}\\ &= \limsup \frac{\mathrm{Vol}_{\Omega_f} B(o,r)}{e^r r^{-1}} \times \frac{e^r}{r\sinh r} = 0.\end{aligned} \tag{54}$$

For the last statement take $f(r)=r^3$ and apply our result to get a convex set Ω_f such that

$$\limsup \frac{\mathrm{Vol}_{\Omega_f} B(o,r)}{r^2} = \limsup \frac{r\,\mathrm{Vol}_{\Omega_f} B(o,r)}{r^3} = +\infty;$$

hence, following our paper [Vernicos 2013], Ω_f is not a polytope. Furthermore the entropy of such a convex set is zero as we have $\limsup_\infty \ln(r^3)/r = 0$. □

To conclude this section let us show how Corollary 4, related to the values attained by the lower and upper volume entropies, easily follows: Suppose first that $0<\alpha\le\beta\le 1$, and start by considering a sequence $(U_n)_{n\in\mathbb{N}}$ defined for some $x>0$ by $U_0=e^{bx}$, and for all $k\ge 0$ by

$$U_{2k+1} = e^{\alpha U_{2k}} \quad\text{and}\quad U_{2k+2} = e^{\beta U_{2k+1}}.$$

Then take an increasing function $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $r\in\mathbb{R}$,

$$e^{\alpha r} \le f(r) \le e^{\beta r},$$

and $f(U_n)=U_{n+1}$ for all $n\ge 0$. We can define such a function piecewise linearly.

If $\alpha=0$, replace $r\mapsto e^{\alpha r}$ by $r\mapsto 2r$ above and take $U_{2k+1}=2U_{2k}$ for all $k\ge 0$.

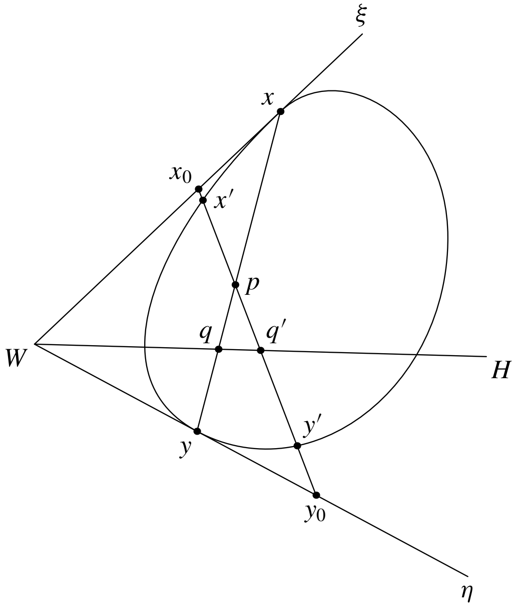

Figure 6. Metric projection of p on H.

Appendix: Metric projection in a Hilbert geometry

The following is a reformulation and a detailed proof of a statement found in [Busemann and Kelly 1953, Sections 21 and 28] in any dimension.

Proposition 29. *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^n$. Let p be a point of Ω and H a hyperplane intersecting Ω. Then $q \in H \cap \Omega$ is a metric projection of p onto H, i.e.,*

$$d_\Omega(p, H) = d_\Omega(p, q),$$

if and only if $\partial\Omega$ has, at its intersection with the straight line (pq), supporting hyperplanes concurrent with H (the intersection of these three hyperplanes is an $(n-2)$-dimensional affine space).

Proof. Let us suppose first that such concurrent support hyperplanes exist. Let x and y be the intersections of the line (pq) with $\partial\Omega$. Assume that ξ and η are supporting hyperplanes of $\partial\Omega$ respectively at x and y whose intersection with H is the $(n-2)$-dimensional affine space W. (See Figure 6.) Let us show that for any $p' \in (pq)$ and any $q' \in H$ we have

$$d_\Omega(p', q') \geq d_\Omega(p', q). \tag{55}$$

Let us suppose that x is on the half-line $[qp')$ and y on the half-line $[p'q)$ and denote by x' and y' the intersections of $\partial\Omega$ with the half-lines $[q'p')$ and $[p'q')$

respectively. Then let x_0 be the intersection of ξ with the line $(p'q')$ and y_0 the intersection of $(p'q')$ with η. By Thales' theorem, the cross ratio of $[x_0, p', q', y_0]$ is equal to the cross ratio of $[x, p', q, y]$ and standard computation shows that

$$[x_0, p', q', y_0] \leq [x', p', q', y'],$$

with equality if and only if $x_0 = x'$ and $y' = y_0$. Hence the inequality (55) holds, and if the convex set is strictly convex, this inequality is always strict, for $q' \neq q$.

Reciprocally: recall that when a point q' of Ω goes to the boundary, its distance to p goes to infinity. Hence by continuity of the distance and compactness there exists a point q on $H \cap \Omega$ such that $d_\Omega(p, H) = d_\Omega(p, q)$. Now consider the Hilbert ball $B_\Omega(p, r)$ of radius $r = d_\Omega(p, H)$ centred at p. Let once more x, y, ξ and η be defined as before, and let H' be the hyperplane passing by q and $\xi \cap \eta = W$. Then this hyperplane has to be tangent to the ball $B_\Omega(p, r)$; otherwise one can find a point q' on H' inside the open ball (i.e., $d(p, q') < r$). However, by the reasoning done in our first step we would conclude that this point is at a distance greater than or equal to r, which would be a contradiction. By minimality of the point q, H is also a supporting hyperplane of $B_\Omega(p, r)$ at q. Hence we have to distinguish between two cases. If Ω is C^1, then by the uniqueness of the tangent hyperplanes at every point, $H = H'$. Otherwise, Ω is not C^1 at x or y. In that case it is possible to replace one of the hyperplanes, say ξ, with ξ' passing by x and $H \cap \eta$ (which might be at infinity, which would mean that we consider parallel hyperplanes). □

Notice that there is no uniqueness of the metric projections (also called "foot" by Busemann). However, if the convex set is strictly convex, then we will have a unique projection, and if furthermore the convex set is C^1, this projection will be given by a unique pair of supporting hyperplanes.

A.1. ***Approximability of convex bodies seen as a dimension.*** In this section we relate our definition of approximability with the definition given in [Schneider and Wieacker 1981].

Recall that for a convex body Ω and $\varepsilon > 0$, $N(\varepsilon, \Omega)$ denotes the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than ε.

Theorem 30 [Schneider and Wieacker 1981]. *Let* $\underline{a}_s := \liminf_{\varepsilon\to 0^+} N(\varepsilon, \Omega)\varepsilon^s$. *Then* $s \to \underline{a}_s$ *admits a critical value* $\underline{a}(\Omega)$, *called the* approximability number *of* Ω, *such that if* $s > \underline{a}(\Omega)$ *then* $\underline{a}_s(\Omega) = 0$, *and if* $s < \underline{a}(\Omega)$ *then* $\underline{a}_s(\Omega) = \infty$.

In the same way, we can introduce the *upper approximability number* of Ω, denoted by $\bar{a}(\Omega)$, as the critical value of $s \mapsto \bar{a}_s(\Omega)$, where

$$\bar{a}_s(\Omega) := \limsup_{\varepsilon\to 0^+} N(\varepsilon, \Omega)\varepsilon^s.$$

The reader familiar with the definition of the ball-box dimension (also known as the Minkowski dimension) will have no difficulty seeing that this definition coincides with the one given in the Introduction.

Now the main result in [Bronshteyn and Ivanov 1975] asserts that for any convex set Ω inscribed in the unit Euclidean ball, there are no more than $c(n)\varepsilon^{(1-n)/2}$ points whose convex hull is no more than ε away from Ω in the Hausdorff topology, which gives the next result.

Theorem 31 [Bronshteyn and Ivanov 1975]. *Let Ω be a convex body in $\mathbb{R}^n$. Then*

$$\bar{a}(\Omega) \leq \tfrac{1}{2}(n-1).$$

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