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MAXIMAL OPERATORS FOR THE p-LAPLACIAN FAMILY

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We prove existence and uniqueness of viscosity solutions for the problem

$$\max\{-\Delta_{p_1} u(x),\ -\Delta_{p_2} u(x)\} = f(x)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with $u = g$ on $\partial\Omega$. Here $-\Delta_p u = (N+p)^{-1}|Du|^{2-p}\operatorname{div}\big(|Du|^{p-2}Du\big)$ is the 1-homogeneous p-Laplacian and we assume that $2 \le p_1, p_2 \le \infty$. This equation appears naturally when one considers a tug-of-war game in which one of the players (the one who seeks to maximize the payoff) can choose at every step which are the parameters of the game that regulate the probability of playing a usual tug-of-war game (without noise) or playing at random. Moreover, the operator $\max\{-\Delta_{p_1} u(x),\ -\Delta_{p_2} u(x)\}$ provides a natural analogue with respect to p-Laplacians to the Pucci maximal operator for uniformly elliptic operators.

We provide two different proofs of existence and uniqueness for this problem. The first one is based in pure PDE methods (in the framework of viscosity solutions) while the second one is more connected to probability and uses game theory.

1. Introduction

In this paper our goal is to show existence and uniqueness of viscosity solutions to the Dirichlet problem for the maximal operator associated with the family of p-Laplacian operators, $-\Delta_p u = -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ with $2 \le p \le \infty$.

When one considers the family of uniformly elliptic second-order operators of the form $-\operatorname{tr}(AD^2u)$ and looks for maximal operators, one finds the so-called Pucci maximal operator, $P^+_{\lambda,\Lambda}(D^2u) = \max_{A\in\mathcal{A}} -\operatorname{tr}(AD^2u)$, where $\mathcal{A}$ is the set of uniformly elliptic matrices with ellipticity constant between λ and Λ. This maximal operator plays a crucial role in the regularity theory for uniformly elliptic second-order operators and has the following properties; see [Caffarelli and Cabré 1995]:

(1) (monotonicity) If $\lambda_1 \le \lambda_2 \le \Lambda_2 \le \Lambda_1$, then $P^+_{\lambda_2,\Lambda_2}(D^2u) \le P^+_{\lambda_1,\Lambda_1}(D^2u)$.

(2) (positive homogeneity) If $\alpha \ge 0$, then $P^+_{\lambda,\Lambda}(\alpha D^2u) = \alpha P^+_{\lambda,\Lambda}(D^2u)$.

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(3) (subsolutions) If u verifies $P^+_{\lambda,\Lambda}(D^2u) \le 0$ in the viscosity sense, then $-\operatorname{tr}(AD^2u) \le 0$ for every matrix A with ellipticity constants λ and Λ (that is, a subsolution to the maximal operator is a subsolution for every elliptic operator in the class). Therefore, from the comparison principle we get that a solution to $P^+_{\lambda,\Lambda}(D^2u) \le 0$ provides a lower bound for every solution of any elliptic operator in the class with the same boundary values.

If we try to reproduce these properties for the family of p-Laplacians, we are led to consider the operator $\max_{p_1\le p\le p_2} -\Delta_p u(x)$. As we will show in this paper, this operator has similar properties to the ones that hold for the Pucci maximal operator, but with respect to the p-Laplacian family.

Hence, it is natural to consider the Dirichlet problem for the partial differential equation

$$\max_{p_1\le p\le p_2} -\Delta_p u(x) = f(x) \tag{1-1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ for $2 \le p_1, p_2 \le \infty$. Here we have normalized the p-Laplacian and considered the operator

$$\Delta_p u = \frac{\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)}{(N+p)|\nabla u|^{p-2}},$$

which is called the 1-homogeneous p-Laplacian. We will assume that $f \equiv 0$ or that f is strictly positive or negative in Ω. We will consider solutions u (along the whole paper we consider solutions in the viscosity sense, see [Crandall et al. 1992]) to this problem with $f \equiv 0$, as p_1-p_2-harmonic functions.

Note that, formally, the 1-homogeneous p-Laplacian can be written as

$$\Delta_p u = \frac{p-2}{N+p}\Delta_\infty u + \frac{1}{N+p}\Delta u,$$

where Δu is the usual Laplacian and $\Delta_\infty u$ is the normalized ∞-Laplacian, that is,

$$\Delta u = \sum_{i=1}^N u_{x_i x_i} \quad \text{and} \quad \Delta_\infty u = \frac{1}{|\nabla u|^2}\sum_{i,j=1}^N u_{x_i} u_{x_i x_i} u_{x_j}.$$

Therefore, we can think about the 1-homogeneous p-Laplacian as a convex combination of the Laplacian divided by $N+2$ and the ∞-Laplacian, in fact,

$$\Delta_p u = \frac{p-2}{N+p}\Delta_\infty u + \frac{N+2}{N+p}\frac{\Delta u}{N+2} = \alpha\Delta_\infty u + \theta\Delta u$$

with $\alpha = (p-2)/(N+p)$ and $\theta = 1/(N+p)$ (we reserve β for a different constant) for $2 \le p < \infty$, and $\alpha = 1$ and $\theta = 0$ for $p = \infty$.

Since we are dealing with convex combinations, equation (1-1) becomes

$$\max_{p_1 \le p \le p_2} -\Delta_p u(x) = \max\{-\Delta_{p_1} u(x), -\Delta_{p_2} u(x)\} = f(x), \tag{1-2}$$

with $2 \le p_1, p_2 \le \infty$.

Our main result concerning viscosity solutions to (1-2) reads as follows:

Theorem 1.1. *Let Ω be a bounded domain such that the exterior ball condition holds when $p_1 \le N$ or $p_2 \le N$. Assume that $\inf_\Omega f > 0$, $\sup_\Omega f < 0$ or $f \equiv 0$. Then, given g a continuous function defined on $\partial\Omega$, there exists a unique viscosity solution $u \in C(\overline{\Omega})$ of (1-2) with $u = g$ in $\partial\Omega$.*

Moreover, a comparison principle holds: if $u, v \in C(\overline{\Omega})$ are such that

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f \quad \textit{and} \quad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge f$$

are in Ω and $v \ge u$ on $\partial\Omega$, then $v \ge u$ in Ω.

In addition, we have a Hopf's lemma: let u be a supersolution to (1-2) and $x_0 \in \partial\Omega$ be such that $u(x_0) > u(x)$ for all $x \in \Omega$, then we have

$$\limsup_{t \to 0^+} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0,$$

where ν is exterior normal to $\partial\Omega$.

Remark 1.2. An analogous result holds for the equation $\min_{p_1 \le p \le p_2} -\Delta_p u(x) = f$.

Remark 1.3. For the homogeneous case, $f \equiv 0$, we have that viscosity sub- and supersolutions to the 1-homogeneous p-Laplacian,

$$-\frac{p-2}{N+p}\Delta_\infty u - \frac{1}{N+p}\Delta u = 0,$$

coincide with viscosity sub and supersolutions to the usual ($(p-1)$-homogeneous) p-Laplacian $-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0$; see [Manfredi et al. 2012b].

Therefore, for $f \equiv 0$ we are providing existence and uniqueness of viscosity solutions to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) = 0$, with $\Delta_p u$ being the usual p-Laplacian that comes from calculus of variations.

Remark 1.4. This maximal operator for the p-Laplacian family has the following properties that are analogous to the ones described above for Pucci's operator:

(1) (monotonicity) If $p_{1,1} \le p_{2,1} \le p_{2,2} \le p_{1,2}$ then

$$\max_{p_{2,1} \le p \le p_{2,2}} -\Delta_p u \le \max_{p_{1,1} \le p \le p_{1,2}} -\Delta_p u.$$

(2) (positive homogeneity) If $\alpha \ge 0$, then

$$\max_{p_1 \le p \le p_2} -\Delta_p (\alpha u) = \alpha \max_{p_1 \le p \le p_2} -\Delta_p u.$$

(3) (subsolutions) A viscosity solution u to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) \le 0$, is a viscosity solution to $-\Delta_p u(x) \le 0$ for every $p_1 \le p \le p_2$. Hence, from the comparison principle, we get that a solution to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) \le 0$ provides a lower bound for every solution of any elliptic operator in the class with the same boundary values.

We have two different approaches for this problem. The first one is based on PDE tools in the framework of viscosity solutions. The second one is related to probability theory (game theory) using the game that we describe below.

Let us introduce a game that we call *unbalanced tug-of-war game with noise*. It is a two-player (Players I and II) zero-sum stochastic game. The game is played in a bounded open set $\Omega \subset \mathbb{R}^N$. Fix an $\varepsilon > 0$. At the initial time, the players place a token at a point $x_0 \in \Omega$ and Player I chooses a coin between two possible ones. They toss the chosen coin which is biased with probabilities α_i and β_i, $\alpha_i + \beta_i = 1$ and $0 \le \alpha_i, \beta_i \le 1$, $i = 1, 2$. Now, they play the tug-of-war with noise game described in [Manfredi et al. 2012b] with probabilities α_i, β_i. If they get heads (probability α_i), they toss a fair coin (with equal probability of heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$. Once the game position leaves Ω, let's say at the τ-th step, the game ends. The payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ (note that we only use the values of g in a strip of width ε around $\partial\Omega$). At the end Player II pays to Player I the amount given by the formula

$$g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n).$$

Note that the positions of the game depend on the strategies adopted by Players I and II. From this procedure we get two extreme functions, $u_{\mathrm{I}}(x_0)$ (the value of the game for Player I) and $u_{\mathrm{II}}(x_0)$ (the value of the game for Player II), that are in a sense the best expected outcomes that each player may expect choosing a strategy when the game starts at x_0. When $u_{\mathrm{I}}(x_0)$ and $u_{\mathrm{II}}(x_0)$ coincide at every $x_0 \in \Omega$ this function $u_\varepsilon := u_{\mathrm{I}} = u_{\mathrm{II}}$ is called the *value of the game*.

Theorem 1.5. *Assume that f is a Lipschitz function with* $\sup_\Omega f < 0$ *or* $\inf_\Omega f > 0$ *or $f \equiv 0$. The unbalanced tug-of-war game with noise with $\{\alpha_1, \alpha_2\} \ne \{0, 1\}$ when $f \equiv 0$ has a value and that value satisfies the dynamic programming principle, given by*

$$u_\varepsilon(x) = \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{y \in B_\varepsilon(x)} u_\varepsilon(y) + \inf_{y \in B_\varepsilon(x)} u_\varepsilon(y) \Big\} + \beta_i \fint_{B_\varepsilon(x)} u_\varepsilon(y)\, dy \right)$$

for $x \in \Omega$, *with* $u_\varepsilon(x) = g(x)$ *for* $x \notin \Omega$.

Moreover, if g is Lipschitz and Ω satisfies the exterior ball condition, then there exists a uniformly continuous function u such that

$$u_\varepsilon \to u \quad \textit{uniformly in} \quad \overline{\Omega}.$$

This limit u is a viscosity solution to

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = \overline{f} & \textit{on } \Omega, \\ u = g & \textit{on } \partial\Omega, \end{cases}$$

where $\overline{f} = 2f$ *and* p_1, p_2 *are given by*

$$\alpha_i = \frac{p_i - 2}{p_i + N}, \qquad \beta_i = \frac{2+N}{p_i+N}, \qquad i = 1, 2.$$

Remark 1.6. When f is strictly positive or negative, we have that the game ends almost surely (a.s.). The same is true (regardless of the strategies adopted by the players) when they play with some noise at every turn, that is, when the two β_i are positive. This fact simplifies the arguments used in the proofs.

When one of the α_i is 1 (and therefore the corresponding β_i is 0) the argument is more delicate; see Section 4.

Remark 1.7. The proof of Theorem 1.5 follows from the results in Sections 4 and 5. In Section 4 we establish that the game has a value and that the value is the unique function that satisfies the dynamic programming principle (DPP). In Section 5 we prove the convergence part of the theorem. In Proposition 4.4 we establish the existence of a function satisfying the DPP. In Theorem 4.6 we prove that the function satisfying the DPP is unique and coincides with the game value, in the case $\beta_1, \beta_2 > 0$, $\sup f < 0$ or $\inf f > 0$. The same result is obtained in the remaining cases in Theorems 4.8 and 4.9. Here is where we had to assume that $\{\alpha_1, \alpha_2\} \neq \{0, 1\}$. Finally, the convergence is established in Corollaries 5.8 and 5.9.

Remark 1.8. Note that in the limit problem one only considers the values of g on $\partial\Omega$ while in the game one needs g to be defined in a bigger set. Given a Lipschitz function defined on $\partial\Omega$ we can just extend it to this larger set without affecting the Lipschitz constant. For simplicity but making an abuse of notation we also call such an extension g.

Remark 1.9. We also prove uniqueness of solutions to the DPP; see Section 4. That is, there exists a unique function verifying

$$v(x) = \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{y \in B_\varepsilon(x)} v(y) + \inf_{y \in B_\varepsilon(x)} v(y) \Big\} + \beta_i ⨍_{B_\varepsilon(x)} v(y)\, dy \right),$$

for $x \in \Omega$, with $v(x) = g(x)$ for $x \notin \Omega$.

Remark 1.10. When Player II (the player who wants to minimize the expected outcome) has the choice of the probabilities α and β we end up with a solution to

$$\begin{cases} \min\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Let us make some brief comments on related work. First, let us recall that Pucci operators are crucial in regularity theory for uniformly elliptic operators, due to their natural comparison with a nondivergence linear operator with measurable coefficients. We refer to [Busca et al. 2005; Caffarelli and Cabré 1995; Felmer et al. 2006; Quaas and Sirakov 2006].

On the other hand, concerning probabilistic ideas for PDEs, the fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown the profound and powerful connection between the classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common origin in mean value properties. This approach turns out to be useful in the nonlinear theory as well, since p-harmonic functions verify an asymptotic mean value property; see, for example, [Manfredi et al. 2010; Hartenstine and Rudd 2013; Kawohl et al. 2012; Llorente 2014; 2015]. Concerning tug-of-war games and PDEs the story begins with [Peres et al. 2009; Peres and Sheffield 2008] and was extended in [Atar and Budhiraja 2010; Bjorland et al. 2012a; 2012b; Nyström and Parviainen 2014], etc. For the p-Laplacian the equivalence between viscosity and weak solutions was proved in [Julin and Juutinen 2012; Juutinen et al. 2001]. This probability approach was used to obtain regularity properties of solutions; we refer to [Armstrong and Smart 2010; Luiro and Parviainen 2015; Luiro et al. 2013; Ruosteenoja 2016].

We finish the introduction with a comment on the main technical novelties contained in this manuscript. To obtain existence and uniqueness for our maximal PDE we first use ideas and techniques from viscosity solutions theory. This part follows the usual steps (the first one shows a comparison principle and then applies Perron's method, including the construction of barriers near the boundary), but here some extra care is needed to deal with points at which the gradient of a test function vanishes. Concerning the game theoretical approach we want to emphasize that when $p_2 = \infty$ we don't know a priori that the game terminates almost surely and this fact introduces some extra difficulties. The argument that shows that there is a unique solution to the dynamic programming principle in this case is delicate; see Theorem 4.8. The proof of convergence of the values of the game as the size of the steps goes to zero is also different from previous results in the literature since here one has to take care of the strategy of the player who chooses the parameters of the game. In particular, the proof that when any of the two players pull in a fixed direction the expectation of the exit time is bounded above $C\varepsilon^2$ is more involved; see Lemma 5.2.

The paper is organized as follows: In Section 2 we prove the comparison principle and then existence and uniqueness for our problem using Perron's method; in Section 3 we introduce a precise description of the game; in Section 4 we show that the game has a value and that this value is the solution to the dynamic programming principle; and finally, in Section 5 we collect some properties of the value function of the game and show that these values converge to the unique viscosity solution of our problem.

2. Existence and uniqueness

First, let us state the definition of a viscosity solution. We have to handle some technical difficulties as the 1-homogeneous ∞-Laplacian is not well-defined when the gradient vanishes. Observing that

$$\Delta u = \operatorname{tr}(D^2u) \quad \text{and} \quad \Delta_\infty u = \frac{\nabla u}{|\nabla u|} D^2 u \frac{\nabla u}{|\nabla u|},$$

we can write (1-2) as $F(\nabla u, D^2u) = f$, where

$$F(v, X) = \max_{i\in\{1,2\}} \Big\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} - \theta_i \operatorname{tr}(X) \Big\}.$$

Note that F is degenerate elliptic, that is,

$$F(v, X) \le F(v, Y) \text{ for } v \in \mathbb{R}^N \setminus \{0\} \text{ and } X, Y \in S^N \text{ provided } X \ge Y,$$

as is generally requested to work in the context of viscosity solutions.

This function $F : \mathbb{R}^N \times S^N \mapsto \mathbb{R}$ is not well-defined at $v = 0$ (here S^N denotes the set of real symmetric $N \times N$ matrices). Therefore, we need to consider the lower semicontinuous F_* and upper semicontinuous F^* envelopes of F. These functions coincide with F for $v \neq 0$, and for $v = 0$ are given by

$$F^*(0, X) = \max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\min}(X) - \theta_i \operatorname{tr}(X)\},$$
$$F_*(0, X) = \max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\max}(X) - \theta_i \operatorname{tr}(X)\},$$

where $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ are the minimum and maximum eigenvalues of X, respectively.

Now we are ready to give the definition for a viscosity solution to our equation.

Definition 2.1. For $2 \le p_1, p_2 \le \infty$ consider the equation

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f$$

in Ω. Then we have the following definitions:

(1) A lower semicontinuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below, we have

$$F^*\big(\nabla\phi(x), D^2\phi(x)\big) \geq f(x).$$

(2) An upper semicontinuous function u is a subsolution if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above, we have

$$F_*\big(\nabla\psi(x), D^2\psi(x)\big) \leq f(x).$$

(3) Finally, u is a viscosity solution if it is both a sub- and supersolution.

In the case $f \equiv 0$, the comparison holds for our equation as a consequence of the main result of [Koike and Kosugi 2015]. See also [Barles and Busca 2001]. Note that the comparison principle obtained in the former is slightly more general than the one obtained in the latter. We need this more general result here as our F is not necessarily continuous when the gradient vanishes. In [Koike and Kosugi 2015] a different notion of viscosity solution is considered. We remark that when a function is a viscosity sub- or supersolution in the sense of Definition 2.1 it is also that in the sense considered in [Koike and Kosugi 2015]. Therefore we can use the comparison result established there once we check their hypotheses.

Proposition 2.2. *Let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be, respectively, a viscosity subsolution and a viscosity supersolution of (1-2) with $f \equiv 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. We just apply the main result in [Koike and Kosugi 2015], referring to notations and details therein. To this end we need to check some conditions. First, let us show that F is elliptic. In fact, we have

$$\begin{aligned} F(v, X - \mu v \otimes v) &= \max_{i\in\{1,2\}} \Big\{ -\alpha_i \frac{v}{|v|}(X - \mu v \otimes v)\frac{v}{|v|} - \theta_i \operatorname{tr}(X - \mu v \otimes v) \Big\} \\ &= \max_{i\in\{1,2\}} \Big\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} + \alpha_i \mu |v|^2 - \theta_i \operatorname{tr}(X) + \theta_i \mu |v|^2 \Big\} \\ &= \max_{i\in\{1,2\}} \Big\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} - \theta_i \operatorname{tr}(X) + \theta_i \Big\} + \mu |v|^2 \\ &= F(v, X) + \mu |v|^2. \end{aligned}$$

Moreover, F is invariant by rescaling in v and 1-homogeneous in X.

So, using the notation from [Koike and Kosugi 2015], we can take $\sigma_0(v) = |v|^2$, $\sigma_1(t) = t$ and $\rho \equiv 0$ that satisfy the conditions imposed in that paper, to obtain the comparison result. □

Now we deal with the case where f is assumed to be nontrivial and does not change sign. In fact, we assume that $\inf f > 0$ or $\sup f < 0$. We follow similar ideas to the ones in [Lu and Wang 2008].

Lemma 2.3. *If we have $u, v \in C(\overline{\Omega})$ such that*

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f \quad \textit{and} \quad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge g,$$

where $g > f$ and $v \ge u$ in $\partial\Omega$, then we have $v \ge u$ in Ω.

Proof. By adding a constant if necessary we can assume that $u, v > 0$. Arguing by contradiction we assume that

$$\max_{\overline{\Omega}}(u - v) > 0 \ge \max_{\partial\Omega}(u - v).$$

Now we double the variables and consider

$$\sup_{x,y\in\Omega} \{u(x) - v(y) - (j/2)|x - y|^2\}.$$

For large j the supremum is attained at interior points x_j, y_j such that $x_j \to \hat{x}$, $y_j \to \hat{x}$, where $\hat{x}$ is an interior point (that $\hat{x}$ cannot be on the boundary can be obtained as in [Lindqvist and Lukkari 2010]).

Now, we observe that there exists a constant C such that $j|x_j - y_j| \le C$. The theorem of sums (see Theorem 3.2 from [Crandall et al. 1992]) implies that there are symmetric matrices $\mathbb{X}_j$, $\mathbb{Y}_j$, with $\mathbb{X}_j \le \mathbb{Y}_j$ such that $(j|x_j - y_j|, \mathbb{X}_j) \in \overline{J^{2,+}}(u)(x_j)$ and $(j|x_j - y_j|, \mathbb{Y}_j) \in \overline{J^{2,-}}(v)(y_j)$, where $\overline{J^{2,+}}(u)(x_j)$ and $\overline{J^{2,-}}(v)(y_j)$ are the closures of the super- and subjets of u and v respectively. Using the equations, assuming that $x_j \ne y_j$, we have

$$\max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{X}_j) \right\} \le f(y_j)$$

and

$$\max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{Y}_j) \right\} \ge g(y_j).$$

Now we observe that, since $\mathbb{X}_j \le \mathbb{Y}_j$ we get

$$-\operatorname{tr}(\mathbb{X}_j) \ge -\operatorname{tr}(\mathbb{Y}_j)$$

and

$$-\left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle \ge -\left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle.$$

Hence

$$\begin{aligned} f(y_j) &\ge \max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{X}_j) \right\} \\ &\ge \max_{i\in\{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{Y}_j) \right\} \ge g(x_j). \end{aligned}$$

This gives a contradiction passing to the limit as $j \to \infty$.

When $x_j = y_j$ we obtain

$$\max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\max}(\mathbb{Y}_j) - \theta_i \operatorname{tr}(\mathbb{Y}_j)\} \le f(y_j)$$

and

$$\max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\min}(\mathbb{X}_j) - \theta_i \operatorname{tr}(\mathbb{X}_j)\} \ge g(x_j),$$

which also lead to a contradiction since $\lambda_{\max}(\mathbb{Y}_j) \ge \lambda_{\max}(\mathbb{X}_j) \ge \lambda_{\min}(\mathbb{X}_j)$.

Hence we have obtained that $u \le v$, as we wanted to prove. □

Lemma 2.4. *If $u, v \in C(\overline{\Omega})$ are such that*

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f, \quad \text{and} \quad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge f$$

in Ω with $\inf_\Omega f > 0$ and $v \ge u$ on $\partial\Omega$, then we have $v \ge u$ in Ω.

Proof. By adding a constant if necessary we can assume that $u, v > 0$. Let's consider $v_\delta = (1+\delta)v$, then

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f < (1+\delta) f \le \max\{-\Delta_{p_1} v_\delta, -\Delta_{p_2} v_\delta\}$$

and $v_\delta \ge v \ge u$ in $\partial\Omega$. Then by the preceding lemma we conclude that $v_\delta \ge u$ in Ω for all $\delta > 0$. Making $\delta \to 0$, we get $v \ge u$ in Ω as we wanted to show. □

Remark 2.5. The above lemma is also true when $\sup_\Omega f < 0$. So, we have comparisons for the cases $\inf_\Omega f > 0$, $\sup_\Omega f < 0$ and $f \equiv 0$. From this comparison result we get uniqueness of solutions.

Now we deal with the existence of solutions. In the proof of this result we are only using that the exterior ball condition holds for Ω when $p_1 \le N$ or $p_2 \le N$.

Theorem 2.6. *Assume that $\inf f > 0$, $\sup f < 0$ or $f \equiv 0$. Then, given g a continuous function defined on $\partial\Omega$, there exists $u \in C(\overline{\Omega})$ which is a viscosity solution of (1-2) such that $u = g$ in $\partial\Omega$.*

Proof. We consider the set

$$\mathcal{A} = \big\{ v \in C(\overline{\Omega}) : \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge f \text{ in } \Omega \text{ and } v \ge g \text{ on } \partial\Omega \big\},$$

where the inequality for the equation inside Ω is verified in the viscosity sense and the inequality on $\partial\Omega$ in the pointwise sense. Since $\Delta|x|^2 = 2n$ and $\Delta_\infty |x|^2 = 2$, we have that $\max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} > 0$ for $v(x) = -|x|^2$. Hence we can choose K_1 such that the operator applied to $-K_1|x|^2$ is greater than $\sup f$ and then we can choose K_2 such that $K_2 - K_1|x|^2 \ge g(x)$ in $\partial\Omega$. We conclude that the function $K_2 - K_1|x|^2$ is in $\mathcal{A}$ for suitable K_1, K_2. Therefore the set $\mathcal{A}$ is not empty.

We define

$$u(x) = \inf_{v\in\mathcal{A}} v(x), \quad x \in \overline{\Omega}.$$

This infimum is finite since, as the comparison holds, we have $u(x) \geq -L_2 + L_1|x|^2$ for all $u \in \mathcal{A}$ for large L_1, L_2. The function u, being the infimum of supersolutions, is a supersolution. We already know that u is upper semicontinuous, as it is the infimum of continuous functions. Let us see that it is indeed a solution. Suppose not, then there exists $\phi \in C^2$ such that ϕ touches u at $x_0 \in \Omega$ strictly from above, but

$$\max\{-\Delta_{p_1}\phi(x_0), -\Delta_{p_2}\phi(x_0)\} > f(x_0).$$

Let us write

$$\phi(x) = \phi(x_0) + \nabla\phi(x_0)\cdot(x-x_0) + \tfrac{1}{2}\big\langle D^2\phi(x_0)(x-x_0), x-x_0\big\rangle + o\big(|x-x_0|^2\big).$$

We define $\widehat{\phi}(x) = \phi(x) - \delta$ for a small positive number δ. Then $\widehat{\phi} < u$ in a small neighborhood of x_0, contained in the set $\big\{x : \max\{-\Delta_{p_1}\phi(x), -\Delta_{p_2}\phi(x)\} > f(x)\big\}$, but $\widehat{\phi} \geq u$ outside this neighborhood, if we take δ small enough.

Now we can consider $v = \min\{\widehat{\phi}, u\}$. Since u is a viscosity supersolution in Ω and $\widehat{\phi}$ also is a viscosity supersolution in the small neighborhood of x_0, it follows that v is a viscosity supersolution. Moreover, on $\partial\Omega$, $v = u \geq g$. This implies $v \in \mathcal{A}$, but $v = \widehat{\phi} < u$ near x_0, which is a contradiction with the definition of u as the infimum in $\mathcal{A}$.

Finally, we want to prove that $u = g$ on $\partial\Omega$ and that boundary values are attained with continuity. To this end, we have to construct barriers for our operator. It is enough to prove that for every $x_0 \in \partial\Omega$ and $\varepsilon > 0$ there exists a supersolution such that $v \geq g$ on $\partial\Omega$ and $v(x_0) \leq g(x_0) + \varepsilon$, and that there exists a subsolution such that $v \leq g$ on $\partial\Omega$ and $v(x_0) \geq g(x_0) - \varepsilon$. We prove now the existence of the supersolution, and the subsolution can be obtained in a similar way.

Let us consider ϕ a radial function, $\phi(x) = \psi(r)$ with $\psi'(r) > 0$. Then

$$\Delta_\infty\phi = \psi'' \quad \text{and} \quad \Delta\phi = \psi'' + \frac{N-1}{r}\psi'$$

and we get

$$\begin{aligned}
\max_{i\in\{1,2\}}\big\{-\Delta_{p_i}\phi\big\} &= \max_{i\in\{1,2\}}\big\{-\alpha_i\,\Delta_\infty\phi - \theta_i\,\Delta\phi\big\} \\
&= \max_{i\in\{1,2\}}\Big\{-\alpha_i\,\psi'' - \theta_i\Big(\psi'' + \frac{N-1}{r}\psi'\Big)\Big\} \\
&= \max_{i\in\{1,2\}}\Big\{-\frac{p_i-2}{N+p_i}\psi'' - \frac{1}{N+p_i}\Big(\psi'' + \frac{N-1}{r}\psi'\Big)\Big\} \\
&= \max_{i\in\{1,2\}}\Big\{-\frac{p_i-1}{N+p_i}\psi'' - \frac{1}{N+p_i}\frac{N-1}{r}\psi'\Big\}.
\end{aligned}$$

We want this last expression to be greater than a positive constant.

To have a function of the form $\psi(r) = r^\gamma$ with $\gamma > 0$ that fulfills this, we need

$$\max_{i\in\{1,2\}}\Big\{-\frac{p_i-1}{N+p_i}\gamma(\gamma-1) - \frac{N-1}{N+p_i}\,\gamma\Big\}r^{\gamma-2} \geq c > 0.$$

Hence we have to choose γ according to

$$0<\gamma<1-\frac{N-1}{p_i-1}.$$

We have that such γ exists if $N<p_1$ or $N<p_2$. We will require that $\min\{p_1,p_2\}>N$, that is, $N<p_1,p_2$.

In this case we can consider $v(x)=K\phi(x-x_0)+g(x_0)+\varepsilon$ with K big enough. If $Kc>\sup f$, then v is a supersolution. We have that $v(x_0)=g(x_0)+\varepsilon$, it remains to prove that $v\geq g$ on $\partial\Omega$. Since g is continuous at x_0, there exists $\delta>0$ such that $|g(x)-g(x_0)|<\varepsilon$ for every $x\in B_\delta(x_0)$. Then we have that $v\geq g$ on $\partial\Omega\cap B_\delta(x_0)$. Finally we can pick K such that $K\delta^\gamma+g(x_0)+\varepsilon>\sup g$, and we obtain $v\geq g$ on $\partial\Omega\cap B_\delta(x_0)^c$.

When $N\geq p_1$ or $N\geq p_2$, we can find (with similar computations) a barrier of the form $\psi(r)=-r^\gamma$ with $\gamma<0$. Note that this function is not well-defined at 0. In this case, we have a barrier if the exterior ball condition holds. Given $x_0\in\partial\Omega$, there exist $\lambda>0$ and $y_0\in\Omega^c$ such that $|x_0-y_0|=\lambda$ and $B_\lambda(y_0)\subset\Omega^c$. We can consider $v(x)=K(\phi(x-y_0)-\phi(x_0-y_0))+g(x_0)+\varepsilon$ and pick K in a similar way to above. □

Now, we prove a version of the Hopf lemma for our equation. Note that since we deal with viscosity solutions, the normal derivative may not exist in a classical sense.

Lemma 2.7. *Let $\Omega\subset\mathbb{R}^N$ be a domain with the interior ball condition and u a subsolution to (1-2) with $f\equiv 0$. Given $x_0\in\partial\Omega$ such that $u(x_0)>u(x)$ for all $x\in\Omega$, we have*

$$\limsup_{t\to 0+}\frac{u(x_0-t\nu)-u(x_0)}{t}<0.$$

where ν is exterior normal to $\partial\Omega$.

Proof. As the interior ball condition holds, we can assume there exists a ball centered at 0, contained in Ω that has x_0 in its boundary; that is, we have $B_r(0)\subset\Omega$ and $x_0\in\partial B_r(0)$. Let us consider $\phi(x)=1/\big(|x|^{N-2}\big)-1/\big(r^{N-2}\big)$ if $N>2$ and $\phi(x)=-\ln|x|+\ln(r)$ for $N=2$. It is easy to check that

$$\Delta\phi=0,\qquad \Delta_\infty\phi\geq 0,\qquad \text{in } B_r(0)\setminus\{0\}.$$

So we have

$$\begin{aligned}\max\{-\Delta_{p_1}\phi,-\Delta_{p_2}\phi\}\leq 0 &\qquad \text{in } B_r(0)\setminus\{0\},\\ \phi\equiv 0 &\qquad \text{on } \partial B_r(0).\end{aligned}$$

As $u(x_0)>u(x)$ for all $x\in\Omega$, in particular on $\partial B_{r/2}(0)$, then there exists $\varepsilon>0$ such that $u(x_0)-\varepsilon\phi\geq u$ on $\partial B_{r/2}(0)$. Therefore, by the comparison principle, we get $u(x_0)-\varepsilon\phi\geq u$ in $B_r(0)\setminus B_{r/2}(0)$ and the result follows. □

3. Unbalanced tug-of-war games with noise

In this section we introduce the game that we call *unbalanced tug-of-war game with noise*. First, let us describe the game without entering in mathematical details. It is a two-player zero-sum stochastic game. The game is played over a bounded open set $\Omega \subset \mathbb{R}^N$. An $\varepsilon > 0$ is given. Players I and II play as follows. At an initial time, they place a token at a point $x_0 \in \Omega$ and Player I chooses a coin between two possible ones (each of the two coins have different probabilities of getting a head). We think of this as choosing $i \in \{1, 2\}$. Now they play the *tug-of-war with noise* introduced in [Manfredi et al. 2012b] starting with the chosen coin. They toss the chosen coin, which is biased with probabilities α_i and β_i, where $\alpha_i + \beta_i = 1$ and $0 \le \alpha_i, \beta_i \le 1$. If they get heads (probability α_i), they toss a fair coin (with the same probability for heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$. Note that Player I chooses the probability of playing the usual tug-of-war game or moving at random with the choice of the first coin between two possibilities. Then they continue playing from x_1. At each turn Player I may change the choice of coin.

This procedure yields a sequence of game states $x_0, x_1, \ldots$. Once the game position leaves Ω, let's say at the τ-th step, the game ends. At that time the token will be on the compact boundary strip around Ω of width ε that we denote

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}.$$

The payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \Gamma_\varepsilon \to \mathbb{R}$. At the end, Player II pays Player I the amount given by $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$, that is, Player I will have earned

$$g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$$

while Player II will have earned

$$-g(x_\tau) - \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n).$$

We can think of this as Player II paying Player I $\varepsilon^2 f(x_i)$ when the token leaves x_i, and $g(x_\tau)$ when the game ends.

A strategy S_{I} for Player I is a pair of collections of measurable mappings

$$S_{\mathrm{I}} = \big(\{\gamma^k\}_{k=0}^\infty, \{S_{\mathrm{I}}^k\}_{k=0}^\infty\big),$$

such that, given a partial history $(x_0, x_1, \dots, x_k)$, Player I chooses coin 1 with probability

$$\gamma^k(x_0, x_1, \dots, x_k) = \gamma \in [0, 1]$$

and the next game position is

$$S_{\mathrm{I}}^k(x_0, x_1, \dots, x_k) = x_{k+1} \in B_\varepsilon(x_k)$$

if Player I wins the toss. Similarly, Player II plays according to a strategy

$$S_{\mathrm{II}} = \{S_{\mathrm{II}}^k\}_{k=0}^\infty.$$

Then, the next game position $x_{k+1} \in B_\varepsilon(x_k)$, given a partial history $(x_0, x_1, \dots, x_k)$, is distributed according to the probability

$$\pi_{S_{\mathrm{I}},S_{\mathrm{II}}}(x_0, x_1, \dots, x_k, A) = \frac{\beta|A \cap B_\varepsilon(x_k)|}{|B_\varepsilon(x_k)|} + \frac{\alpha}{2}\delta_{S_{\mathrm{I}}^k(x_0,x_1,\dots,x_k)}(A) + \frac{\alpha}{2}\delta_{S_{\mathrm{II}}^k(x_0,x_1,\dots,x_k)}(A),$$

where $\gamma = \gamma^k(x_0, x_1 \dots, x_k)$, $\alpha = \alpha_1\gamma + \alpha_2(1-\gamma)$, $\beta = \beta_1\gamma + \beta_2(1-\gamma)$ and A is any measurable set (note that α and β depend on S_{I} and $(x_0, x_1, \dots, x_k)$; we do not make this explicit to avoid overloading the notation). From now on, we shall omit k and simply denote the strategies by γ, S_{I} and S_{II}.

Let $\Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon \subset \mathbb{R}^n$ be equipped with the natural topology, and the σ-algebra $\mathcal{B}$ of the Lebesgue measurable sets. The space of all game sequences

$$H^\infty = \{x_0\} \times \Omega_\varepsilon \times \Omega_\varepsilon \times \cdots ,$$

is a product space endowed with the product topology.

Let $\{\mathcal{F}_k\}_{k=0}^\infty$ denote the filtration of σ-algebras, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ which are defined as follows: $\mathcal{F}_k$ is the product σ-algebra generated by cylinder sets of the form $\{x_0\} \times A_1 \times \cdots \times A_k \times \Omega_\varepsilon \times \Omega_\varepsilon \cdots$ with $A_i \in \mathcal{B}$. For

$$\omega = (x_0, \omega_1, \dots) \in H^\infty,$$

we define the coordinate processes

$$X_k(\omega) = \omega_k, \quad X_k : H^\infty \to \mathbb{R}^n, \quad k = 0, 1, \dots$$

so that X_k is an $\mathcal{F}_k$-measurable random variable. Moreover, $\mathcal{F}_\infty = \sigma\left(\bigcup \mathcal{F}_k\right)$ is the smallest σ-algebra so that all X_k are $\mathcal{F}_\infty$-measurable. To denote the time when the game state reaches Γ_ε, we define a random variable

$$\tau(\omega) = \inf\{k : X_k(\omega) \in \Gamma_\varepsilon, k = 0, 1, \dots\},$$

which is a stopping time relative to the filtration $\{\mathcal{F}_k\}_{k=0}^\infty$.

A starting point x_0 and the strategies $S_{\rm I}$ and $S_{\rm II}$ define (by Kolmogorov's extension theorem) a unique probability measure $\mathbb{P}^{x_0}_{S_{\rm I},S_{\rm II}}$ in H^∞ relative to the σ-algebra $\mathcal{F}^\infty$. We denote by $\mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}$ the corresponding expectation.

Then, if $S_{\rm I}$ and $S_{\rm II}$ denote the strategies adopted by Player I and II respectively, we define the expected payoff for Player I as

$$V_{x_0,\rm I}(S_{\rm I},S_{\rm II}) = \begin{cases} \mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}\big[g(X_\tau)+\varepsilon^2\sum_{n=1}^{\tau-1}f(x_n)\big] & \text{if the game ends a.s.,} \\ -\infty & \text{otherwise,} \end{cases}$$

and then the expected payoff for Player II as

$$V_{x_0,\rm II}(S_{\rm I},S_{\rm II}) = \begin{cases} \mathbb{E}^{x_0}_{S_{\rm I},S_{\rm II}}\big[g(X_\tau)+\varepsilon^2\sum_{n=1}^{\tau-1}f(x_n)\big] & \text{if the game ends a.s.,} \\ +\infty & \text{otherwise.} \end{cases}$$

Note that we penalize both players when the game doesn't end almost surely.

The *value of the game for Player I* is given by

$$u_{\rm I}(x_0) = \sup_{S_{\rm I}}\inf_{S_{\rm II}} V_{x_0,\rm I}(S_{\rm I},S_{\rm II}),$$

while the *value of the game for Player II* is given by

$$u_{\rm II}(x_0) = \inf_{S_{\rm II}}\sup_{S_{\rm I}} V_{x_0,\rm II}(S_{\rm I},S_{\rm II}).$$

When $u_{\rm I} = u_{\rm II}$ we say the game has a value $u := u_{\rm I} = u_{\rm II}$. The values $u_{\rm I}(x_0)$ and $u_{\rm II}(x_0)$ are in a sense the best outcomes each player can expect when the game starts at x_0. For the measurability of the value functions we refer to [Maitra and Sudderth 1993; 1996].

Comment. It seems natural to consider a more general protocol to determine α in a prescribed closed set. It is clear that there are only two possible scenarios: At each turn, Player I wants to maximize the value of α and Player II wants to minimize it, or the converse. An expected value for α is obtained in each case assuming each player plays optimally. Depending on the value of α in each case, we are considering a game equivalent to the one that we described previously or another one where Player II gets the choice of the first coin, for certain values of α_i.

4. The game value function and the dynamic programming principle

In this section, we prove that the game has a value, that is, $u_{\rm I} = u_{\rm II}$ and that this value function satisfies the dynamic programming principle (DPP) given by

$$u(x) = \begin{cases} \varepsilon^2 f(x) + \max\limits_{i\in\{1,2\}}\left(\dfrac{\alpha_i}{2}\Big\{\sup\limits_{B_\varepsilon(x)} u + \inf\limits_{B_\varepsilon(x)} u\Big\} + \beta_i \displaystyle\fint_{B_\varepsilon(x)} u(y)\,dy\right), & x\in\Omega, \\ g(x), & x\in\Gamma_\varepsilon. \end{cases}$$

Let's see intuitively why this holds. At each step we have that Player I chooses $i \in \{1,2\}$ and then we have three possibilities:

- With probability $\alpha_i/2$, Player I moves the token, trying to maximize the expected outcome.
- With probability $\alpha_i/2$, Player II moves the token, trying to minimize the expected outcome.
- With probability β_i, the token moves at random.

Since Player I chooses i trying to maximize the expected outcome we obtain a $\max_{i\in\{1,2\}}$ in the DPP. Finally, the expected payoff at x is given by $\varepsilon^2 f(x)$ plus the expected payoff for the rest of the game.

Similar results are proved in [Antunović et al. 2012; Liu and Schikorra 2013; Luiro et al. 2013; Manfredi et al. 2012a; Peres et al. 2009; Ruosteenoja 2016]. Note that when $\alpha_1 = \alpha_2$ (and hence $\beta_1 = \beta_2$) Player I has no choice to make and we recover known results for tug-of-war games (with or without noise); see [Peres et al. 2009; Manfredi et al. 2012b]. We follow [Ruosteenoja 2016] where the idea is to prove the existence of a function satisfying the DPP and then that this function gives the game value. For the existence of a solution to the DPP we borrow some ideas from [Antunović et al. 2012], and for the uniqueness of such a solution and the existence of the value of the game we use martingales as in [Manfredi et al. 2012a]. However we will have two different cases: One, where the noise or the strict positivity (or negativity) of f assures us that the game ends almost surely, independently of the strategies adopted by the players. And another one where we have to handle the problem of getting strategies for the players to play almost optimally and to make sure that the game ends almost surely.

In what follows, $\Omega \subset \mathbb{R}^N$ is a bounded open set and $\varepsilon > 0$, $g : \Gamma_\varepsilon \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are bounded Borel functions such that $f \equiv 0$, $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$.

Definition 4.1. A function u is sub-p_1-p_2-harmonious if

$$u(x) \le \begin{cases} \varepsilon^2 f(x) + \displaystyle\max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \Big\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\,dy \right) & x \in \Omega, \\ g(x) & x \in \Gamma_\varepsilon. \end{cases}$$

Analogously, a function u is super-p_1-p_2-harmonious if

$$u(x) \ge \begin{cases} \varepsilon^2 f(x) + \displaystyle\max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \Big\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\,dy \right), & x \in \Omega, \\ g(x), & x \in \Gamma_\varepsilon. \end{cases}$$

Finally, u is p_1-p_2-harmonious if it is both sub- and super-p_1-p_2-harmonious (i.e., the equality holds).

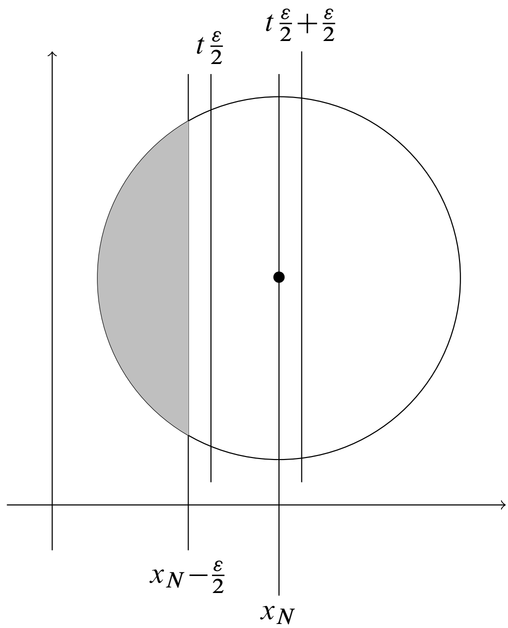

Figure 1. The partition considered in the proof of Lemma 4.2.

Here α_i and β_i are given by

$$\alpha_i = \frac{p_i - 2}{p_i + N} \quad \text{and} \quad \beta_i = \frac{N+2}{p_i + N}, \qquad i = 1, 2.$$

Our next task is to prove uniform bounds for these functions.

Lemma 4.2. *Sub-p_1-p_2-harmonious functions are uniformly bounded from above.*

Proof. We will consider the space partitioned along the x_N axis in strips of width $\varepsilon/2$. To this end we define

$$D = \frac{|\{y \in B_\varepsilon : y_N < -\varepsilon/2\}|}{|B_\varepsilon|} = \frac{|\{y \in B_1 : y_N < -1/2\}|}{|B_1|} \quad \text{and} \quad C = 1 - D.$$

The constant D gives the fraction of the ball $B_\varepsilon(x)$ covered by the shadowed section in Figure 1, $\{y \in B_\varepsilon : y_N < x_N - \varepsilon/2\}$, and C the fraction occupied by its complement.

Given $x \in \Omega$, let us consider $t \in \mathbb{R}$ such that $x_N < t\varepsilon/2 + \varepsilon/2$. We get

$$\Big\{y \in B_\varepsilon(x) : y_N < x_N - \frac{\varepsilon}{2}\Big\} \subset \Big\{z \in \mathbb{R}^N : z_N < t\frac{\varepsilon}{2}\Big\}.$$

Now, given u a sub-p_1-p_2-subharmonious function, we have that

$$u(x) \le \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \bigg(\frac{\alpha_i}{2} \Big\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \Big\} + \beta_i ⨍_{B_\varepsilon(x)} u(y)\, dy \bigg).$$

Now we can bound the terms in the right-hand side considering the partition given above, see Figure 1. We have

$$\begin{aligned}&\sup_{B_\varepsilon(x)} u \le \sup_{\Omega_\varepsilon} u,\\ &\inf_{B_\varepsilon(x)} u \le \sup_{\{y\in B_\varepsilon(x): y_N<x_N-\varepsilon/2\}} u \le \sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u,\end{aligned}$$

and

$$\begin{aligned}\fint_{B_\varepsilon(x)} u(y)\,dy \le{}& \left|\left\{y\in B_\varepsilon(x): y_N\ge x_N-\frac{\varepsilon}{2}\right\}\right| \sup_{\{y\in B_\varepsilon(x): y_N\ge x_N-\varepsilon/2\}} u\\ &+\left|\left\{y\in B_\varepsilon(x): y_N< x_N-\frac{\varepsilon}{2}\right\}\right| \sup_{\{y\in B_\varepsilon(x): y_N< x_N-\varepsilon/2\}} u\\ \le{}& C\sup_{\Omega_\varepsilon} u + D\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}u(x) \le{}& \varepsilon^2\sup_\Omega f + \max_{i\in\{1,2\}}\Big(\frac{\alpha_i}{2}\{\sup_{\Omega_\varepsilon} u + \sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u\}\\ &\qquad + \beta_i\big\{C\sup_{\Omega_\varepsilon} u + D\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u\big\}\Big)\\ ={}& \varepsilon^2\sup_\Omega f + \max_{i\in\{1,2\}}\left(\left\{\frac{\alpha_i}{2}+\beta_i C\right\}\sup_{\Omega_\varepsilon} u\right.\\ &\qquad \left. + \left\{\frac{\alpha_i}{2}+\beta_i D\right\}\sup_{\Omega_\varepsilon\cap\{z_n<t\varepsilon/2\}} u\right)\\ ={}& \varepsilon^2\sup_\Omega f + \max_{i\in\{1,2\}}\left\{\frac{\alpha_i}{2}+\beta_i C\right\}\sup_{\Omega_\varepsilon} u\\ &\qquad + \min_{i\in\{1,2\}}\left\{\frac{\alpha_i}{2}+\beta_i D\right\}\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u\\ ={}& \varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u + (1-K)\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u,\end{aligned}$$

where $K=\max_{i\in\{1,2\}}\{\alpha_i/2+\beta_i C\}$. We conclude that

$$\sup_{\Omega_\varepsilon\cap\{z_N<(t+1)\varepsilon/2\}} u_k \le \varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u_k + (1-K)\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u_k.$$

Then, inductively, we get

$$\begin{aligned}\sup_{\Omega_\varepsilon\cap\{z_N<(t+n)\varepsilon/2\}} u \le{}& \big(\varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u\big)\\ &\times\sum_{i=0}^{n-1}(1-K)^i + (1-K)^n\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u.\end{aligned}$$

We assume without loss of generality that $\Omega\subset\{x\in\mathbb{R}^N: 0<x_N<R\}$ for some $R>0$. Now, we apply the formula for $t=0$ and n such that $n\varepsilon/2>R$, and get

$$\begin{aligned}\sup_{\Omega_\varepsilon} u \le{}& \big(\varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u\big)\sum_{i=0}^{n-1}(1-K)^i + (1-K)^n\sup_{\Gamma_\varepsilon} g\\ ={}& \big(\varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u\big)\frac{1-(1-K)^n}{1-(1-K)} + (1-K)^n\sup_{\Gamma_\varepsilon} g\\ ={}& \frac{1-(1-K)^n}{K}\varepsilon^2\sup_\Omega f + \big(1-(1-K)^n\big)\sup_{\Omega_\varepsilon} u + (1-K)^n\sup_{\Gamma_\varepsilon} g.\end{aligned}$$

Hence, we obtain

$$(1-K)^n \sup_{\Omega_\varepsilon} u \le \frac{1-(1-K)^n}{K}\varepsilon^2 \sup_\Omega f + (1-K)^n \sup_{\Gamma_\varepsilon} g,$$

that gives the desired upper bound,

$$\sup_{\Omega_\varepsilon} u \le \frac{1-(1-K)^n}{K(1-K)^n}\varepsilon^2 \sup_\Omega f + \sup_{\Gamma_\varepsilon} g. \qquad \square$$

Analogously, there holds that super-p_1-p_2-harmonious functions are uniformly bounded from below.

Now with these results we can show that there exists a p_1-p_2-harmonious function as in [Liu and Schikorra 2015] applying Perron's Method. Remark that when f and g are bounded we can easily obtain the existence of sub-p_1-p_2-harmonious and super-p_1-p_2-harmonious functions.

We prefer a constructive argument (since we will use this construction again in what follows). Let $u_k : \Omega_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $u_k = g$ on Γ_ε for all $k \in \mathbb{N}$, then u_0 is sub-p_1-p_2-harmonious and

$$u_{k+1}(x) = \varepsilon^2 f(x) + \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_k + \inf_{B_\varepsilon(x)} u_k\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_k(y)\,dy\right),$$

for $x \in \Omega$.

Now, our main task is to show that this sequence converges uniformly. To this end, let us prove an auxiliary lemma where we borrow some ideas from [Antunović et al. 2012].

Lemma 4.3. *Let $x \in \Omega$, $n \in \mathbb{N}$ and fix λ_i for $i = 1, \dots, 4$, such that*

$$u_{n+1}(x) - u_n(x) \ge \lambda_1, \qquad \|u_n - u_{n-1}\|_\infty \le \lambda_2, \qquad \fint_{B_\varepsilon(x)} u_n - u_{n-1} \le \lambda_3,$$

$\lambda_3 < \lambda_1$, *and* $\lambda_4 > 0$. *Then, for* $\alpha := \max\{\alpha_1, \alpha_2\} > 0$, *there exists* $y \in B_\varepsilon(x)$ *such that*

$$\inf_{B_\varepsilon(x)} u_n \ge u_{n-1}(y) + \frac{2\lambda_1}{\alpha} - \lambda_2 - \frac{2(1-\alpha)\lambda_3}{\alpha} - \lambda_4.$$

Proof. Given $u_{n+1}(x) - u_n(x) \ge \lambda_1$, by the recursive definition, we have

$$\begin{aligned}\lambda_1 \le \varepsilon^2 f(x) + \max_{i\in\{1,2\}}&\left(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_n(y)\,dy\right)\\ &-\varepsilon^2 f(x) - \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_{n-1} + \inf_{B_\varepsilon(x)} u_{n-1}\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_{n-1}(y)\,dy\right).\end{aligned}$$

Since $\max\{a,b\}-\max\{c,d\}\le\max\{a-c,b-d\}$, we get

$$\lambda_1 \le \max_{i\in\{1,2\}}\bigg(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_n+\inf_{B_\varepsilon(x)} u_n-\sup_{B_\varepsilon(x)} u_{n-1}-\inf_{B_\varepsilon(x)} u_{n-1}\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_n(y)-u_{n-1}(y)\,dy\bigg).$$

Using that $\fint_{B_\varepsilon(x)} u_n-u_{n-1}\le\lambda_3$ we get

$$\max_{i\in\{1,2\}}\bigg(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_n+\inf_{B_\varepsilon(x)} u_n-\sup_{B_\varepsilon(x)} u_{n-1}-\inf_{B_\varepsilon(x)} u_{n-1}\Big\}+\beta_i\lambda_3\bigg)\ge\lambda_1.$$

Now $\lambda_3<\lambda_1$ implies

$$\frac{\alpha}{2}\Big\{\sup_{B_\varepsilon(x)} u_n+\inf_{B_\varepsilon(x)} u_n-\sup_{B_\varepsilon(x)} u_{n-1}-\inf_{B_\varepsilon(x)} u_{n-1}\Big\}+(1-\alpha)\lambda_3\ge\lambda_1.$$

We bound the difference between the suprema using $\|u_n-u_{n-1}\|_\infty\le\lambda_2$ and we obtain

$$\frac{\alpha}{2}\Big\{\inf_{B_\varepsilon(x)} u_n-\inf_{B_\varepsilon(x)} u_{n-1}\Big\}+\frac{\alpha\lambda_2}{2}+(1-\alpha)\lambda_3\ge\lambda_1,$$

that is,

$$\inf_{B_\varepsilon(x)} u_n\ge\inf_{B_\varepsilon(x)} u_{n-1}+\frac{2\lambda_1}{\alpha}-\lambda_2-\frac{2(1-\alpha)\lambda_3}{\alpha}.$$

Finally we can choose $y\in B_\varepsilon(x)$ such that

$$u_{n-1}(y)\le\inf_{B_\varepsilon(x)} u_{n-1}+\lambda_4,$$

which gives the desired inequality. □

Now we are ready to prove the uniform convergence and, therefore, the existence of a p_1-p_2-harmonious function.

Proposition 4.4. *The sequence u_k converges uniformly and the limit is a solution to the DPP.*

Proof. Since u_0 is sub-p_1-p_2-harmonious we have $u_1\ge u_0$. In addition, if $u_k\ge u_{k-1}$, by the recursive definition, we have $u_{k+1}\ge u_k$. Then, by induction, we obtain that the sequence of functions is an increasing sequence. Replacing $u_k\le u_{k+1}$ in the recursive definition we can see that u_k is a sub-p_1-p_2-harmonious function for all k. This gives us a uniform bound for u_k (independent of k). Hence, the u_k converge pointwise to a bounded Borel function u.

In the case $\alpha_1=\alpha_2=0$ we can pass to the limit on the recursion because of Fatou's lemma. Hence we assume $\alpha:=\max\{\alpha_1,\alpha_2\}>0$.

Now we show that the convergence is uniform. Suppose not. Observe that if $\|u_{n+1}-u_n\|_\infty\to 0$ we can extract a uniformly Cauchy subsequence, thus this subsequence converges uniformly to a limit u. This implies that the u_k converge uniformly to u, because of the monotonicity. By the recursive definition we have $\|u_{n+1}-u_n\|_\infty \geq \|u_n-u_{n-1}\|_\infty \geq 0$. Then, as we are assuming the convergence is not uniform, we have

$$\|u_{n+1}-u_n\|_\infty \to M \quad \text{and} \quad \|u_{n+1}-u_n\|_\infty \geq M$$

for some $M>0$.

Let us observe that by Fatou's lemma it follows that

$$\lim_{n\to\infty}\int_\Omega u(y)-u_n(y)\,dy = 0,$$

so we can bound $⨍_{B_\varepsilon(x)} u_{n+1}-u_n$ uniformly on x.

Given $\delta>0$, let $n_0\in\mathbb{N}$ such that for all $n\geq n_0$,

$$\|u_{n+1}-u_n\|_\infty \leq M+\delta \quad \text{and} \quad ⨍_{B_\varepsilon(x)} u_{n+1}-u_n < \delta,$$

for all $x\in\Omega$. We fix $k\geq 0$. Let $x_0\in\Omega$ such that

$$u_{n_0+k}(x_0)-u_{n_0+k-1}(x_0)\geq M-\delta.$$

Now we apply Lemma 4.3 for $\lambda_1=M-\delta,\ \lambda_2=M+\delta,\ \lambda_3=\delta$ and $\lambda_4=\delta$ and we get

$$\begin{aligned}
u_{n_0+k-1}(x_0), u_{n_0+k-1}(x_1) &\geq \inf_{B_\varepsilon(x_0)} u_{n_0+k-1}\\
&\geq u_{n_0+k-2}(x_1)+\frac{2(M-\delta)}{\alpha}-(M+\delta)-\frac{2(1-\alpha)}{\alpha}-\delta\\
&= u_{n_0+k-2}(x_1)+M\Big(\frac{2}{\alpha}-1\Big)-\delta\frac{4}{\alpha}\\
&\geq u_{n_0+k-2}(x_1)+M-\delta\frac{4}{\alpha},
\end{aligned}$$

for some $x_1\in B_\varepsilon(x_0)$. Let us define $\xi=4/\alpha$. If we repeat the argument for x_1, but now with $\lambda_1=M-\delta\xi$, we obtain

$$u_{n_0+k-2}(x_1), u_{n_0+k-2}(x_2)\geq u_{n_0+k-3}(x_2)+M-\delta\big(\xi^2+\xi\big).$$

Inductively, we obtain a sequence $x_l,\ 1\leq l\leq k-1$ such that

$$u_{n_0+k-l}(x_{l-1}), u_{n_0+k-l}(x_l)\geq u_{n_0+k-l-1}(x_l)+M-\delta\textstyle\sum_{t=1}^{l}\xi^t.$$

In Lemma 4.3 we require $\lambda_3<\lambda_1$, so we need $k(\delta)$ to satisfy

$$M-\delta\sum_{t=1}^{l}\xi^t>\delta,$$

that is,

$$M > \delta \sum_{t=0}^{l} \xi^t$$

for $1 \le l \le k-1$. As the right-hand side term grows with l, it is enough to check it for $l = k-1$. Since

$$\sum_{t=1}^{l} \xi^t = \xi \frac{\xi^l - 1}{\xi - 1} \le \xi^{l+1} - 1 \le \xi^{l+1},$$

we obtain

$$u_{n_0+k-l}(x_{l-1}) \ge u_{n_0+k-l-1}(x_l) + M - \delta \xi^{l+1}.$$

Adding these inequalities for $1 \le l \le k-1$, and $u_{n_0+k}(x_0) - u_{n_0+k-1}(x_0) \ge M - \delta$ we get

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + kM - \delta \sum\nolimits_{l=0}^{k-1} \xi^{l+1}.$$

From the last inequality and the condition for $k(\delta)$, since

$$\sum_{l=0}^{k-1} \xi^{l+1} = \sum_{l=1}^{k} \xi^l \le \xi^{k+1},$$

we have

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + kM - \delta \xi^{k+1}$$

for all k such that $M > \delta \xi^{k+1}$. For $k+1 = \lfloor \log(M/\delta)/\log \xi \rfloor$ this gives

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + \left(\frac{\log(M/\delta)}{\log \xi} - 3 \right) M,$$

which is a contradiction since

$$\lim_{\delta \to 0^+} \frac{\log(M/\delta)}{\log \xi} = \infty$$

and the sequence u_n is bounded. We have that $u_n \to u$ uniformly, therefore the result follows by passing to the limit in the recursive definition of u_n. In fact, that the uniform limit of the sequence u_n is a solution to the DPP is immediate since from the uniform convergence we can pass to the limit as $n \to \infty$ in all the terms of the DPP formula. □

Now we want to prove that this solution to the DPP, u, is unique and that it gives the value of the game. To this end we have to take special care of the fact that the game ends (or not) almost surely. First, we deal with the case $\beta_1, \beta_2 > 0$, $\sup_\Omega f < 0$ or $\inf_\Omega f > 0$. We apply a martingale argument to handle these cases. In other cases we also use the construction of the sequence u_k.

Lemma 4.5. *Assume that* $\beta_1, \beta_2 > 0$, $\sup f < 0$ *or* $\inf f > 0$. *Then, if the function* v *is a* p_1-p_2-*harmonious function for* g_v *and* f_v *such that* $g_v \le g_{u_{\mathrm{I}}}$ *and* $f_v \le f_{u_{\mathrm{I}}}$, *then* $v \le u_{\mathrm{I}}$.

Proof. By choosing a strategy according to the points where the maximal values of v are attained, we show that Player I can obtain a certain process which is a submartingale. The optional stopping theorem then implies that the expectation of the process under this strategy is bounded by v. Moreover, this process provides a lower bound for u_{I}.

Player II follows any strategy and Player I follows a strategy S_{I}^0 such that at $x_{k-1} \in \Omega$ he chooses γ to be 1 if

$$\frac{\alpha_1}{2}\Big\{\sup_{y\in B_\varepsilon(x)} u(y) + \inf_{y\in B_\varepsilon(x)} u(y)\Big\} + \beta_1 \fint_{B_\varepsilon(x)} u(y)\,dy \\ > \frac{\alpha_2}{2}\Big\{\sup_{y\in B_\varepsilon(x)} u(y) + \inf_{y\in B_\varepsilon(x)} u(y)\Big\} + \beta_2 \fint_{B_\varepsilon(x)} u(y)\,dy$$

and 0 otherwise, and he chooses to step to a point that almost maximizes v, that is, to a point $x_k \in B_\varepsilon(x_{k-1})$ such that

$$v(x_k) \ge \sup\nolimits_{B_\varepsilon(x_{k-1})} v - \eta 2^{-k}$$

for some fixed $\eta > 0$. We start from the point x_0. It follows that

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}\Big[v(x_k) + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \,:\, x_0,\dots,x_{k-1}\Big] \\
&\qquad \ge \max_{i\in\{1,2\}}\Big(\frac{\alpha_i}{2}\Big\{\inf_{B_\varepsilon(x_{k-1})} v - \eta 2^{-k} + \sup_{B_\varepsilon(x_{k-1})} v\Big\} + \beta_i \fint_{B_\varepsilon(x_{k-1})} v\,dy\Big) \\
&\qquad\qquad + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \\
&\qquad \ge v(x_{k-1}) - \varepsilon^2 f(x_{k-1}) - \eta 2^{-k} + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \\
&\qquad = v(x_{k-1}) + \varepsilon^2 \sum_{n=0}^{k-2} f(x_n) - \eta 2^{-k+1},
\end{aligned}$$

where we have estimated the strategy of Player II by inf and used the fact that v is p_1-p_2-harmonious. Thus

$$M_k = v(x_k) + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k}$$

is a submartingale.

Now we observe the following: if $\beta_1, \beta_2 > 0$ then the game ends almost surely and we can continue (see below). If $\sup f < 0$ the fact that M_k is a submartingale implies that the game ends in a finite number of moves (that can be estimated). In the case $\inf f > 0$ if the game does not end in a finite number of moves then we have to play until the accumulated payoff (recall that f gives the running payoff) is greater than v and then choose a strategy that ends the game almost surely (for example pointing to some prescribed point x_0 outside Ω).

Since $g_v \le g_{u_{\mathrm{I}}}$ and $f_v \le f_{u_{\mathrm{I}}}$, we deduce

$$\begin{aligned} u_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[g_{u_{\mathrm{I}}^{\varepsilon}}(x_\tau) + \varepsilon^2 \sum\nolimits_{n=0}^{\tau-1} f(x_n)\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}\Big[g_v(x_\tau) + \varepsilon^2 \sum\nolimits_{n=0}^{\tau-1} f(x_n) - \eta 2^{-\tau}\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \liminf_{k\to\infty} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}\Big[v(x_{\tau\wedge k}) + \varepsilon^2 \sum\nolimits_{n=0}^{(\tau-1)\wedge k} f(x_n) - \eta 2^{-(\tau\wedge k)}\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[M_0] = v(x_0) - \eta, \end{aligned}$$

where $(\tau - 1) \wedge k = \min(\tau - 1, k)$, and we used Fatou's lemma as well as the optional stopping theorem for M_k. Since η is arbitrary, this proves the claim. □

A symmetric result can be proved for u_{II}, hence we obtain the following result:

Theorem 4.6. *Assume that* $\beta_1, \beta_2 > 0$, $\sup f < 0$ *or* $\inf f > 0$. *Then there exists a unique* p_1-p_2*-harmonious function. Even more the game has a value, that is* $u_{\mathrm{I}} = u_{\mathrm{II}}$, *which coincides with the unique* p_1-p_2*-harmonious function.*

Proof. Let u be a p_1-p_2-harmonious function, which exits, as we know from Proposition 4.4. From the definition of the game values we know that $u_{\mathrm{I}} \le u_{\mathrm{II}}$. Then by Lemma 4.5 we have that

$$u_{\mathrm{I}} \le u_{\mathrm{II}} \le u \le u_{\mathrm{I}}.$$

Thus $u_{\mathrm{I}} = u_{\mathrm{II}} = u$. Since we can repeat the argument for any p_1-p_2-harmonious function, uniqueness follows. □

Remark 4.7. Note that if we have a sub-p_1-p_2-harmonious function u, then v given by $v = u - C$ in Ω and $v = u$ in Γ_ε is sub-p_1-p_2-harmonious for every constant $C > 0$. In this way we can obtain a sub-p_1-p_2-harmonious function smaller than any super-p_1-p_2-harmonious function, and then if we start the above construction with this function we get the smallest p_1-p_2-harmonious function. That is, there exists a minimal p_1-p_2-harmonious function. We can use the analogous construction to get the largest p_1-p_2-harmonious function (the maximal p_1-p_2-harmonious function).

We now tackle the remaining case in which $f \equiv 0$ and one of the β_i is 0 (that is the same as saying that one of the α_i is equal to 1).

Theorem 4.8. *There exists a unique p_1-p_2-harmonious function when $\alpha_1 = 1$, $\alpha_2 > 0$ and $f \equiv 0$.*

Proof. Suppose not, then we have u and v such that

$$v(x) = \max\left\{\frac{1}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big), \frac{\alpha}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big) + \beta \fint_{B_\varepsilon(x)} v\right\}$$

$$u(x) = \max\left\{\frac{1}{2}\Big(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\Big), \frac{\alpha}{2}\Big(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\Big) + \beta \fint_{B_\varepsilon(x)} u\right\}$$

in Ω and

$$u = v = g$$

on Γ_ε with

$$\|u - v\|_\infty = M > 0.$$

As we observed in Remark 4.7 we can assume $u \geq v$ (just take v as the minimal solution to the DPP). Now we want to build a point where the difference between u and v is almost attained and v has a large variation in the ball of radius ε around this point (all this has to be carefully quantified). First, we apply a compactness argument. We know that

$$\overline{\Omega}_{\varepsilon/4} \subset \bigcup_{x\in\Omega} B_{\varepsilon/2}(x).$$

As $\overline{\Omega}_{\varepsilon/4}$ is compact, there exists y_i such that

$$\overline{\Omega}_{\varepsilon/4} \subset \bigcup_{i=1}^{k} B_{\varepsilon/2}(y_i).$$

Let $A = \big\{i \in \{1,\dots,k\} : u \text{ or } v \text{ are not constant on } B_{\varepsilon/2}(y_i)\big\}$ and let $\lambda > 0$ such that, for every $i \in A$,

$$\sup_{B_\varepsilon(y_i)} u - \inf_{B_\varepsilon(y_i)} u > \left(4 + \frac{4\beta}{\alpha}\right)\lambda \quad \text{or} \quad \sup_{B_\varepsilon(y_i)} v - \inf_{B_\varepsilon(y_i)} v > 2\lambda.$$

We fix this λ. Now, for every $\delta > 0$ such that $\lambda > \delta$ and $M > \delta$, let $z \in \Omega$ such that $M - \delta < u(z) - v(z)$. Let

$$O = \{x \in \Omega : u(x) = u(z) \text{ and } v(x) = v(z)\} \subset \Omega.$$

Take $\bar{z} \in \partial O \subset \overline{\Omega}$. Letting i_0 be such that $\bar{z} \in B_{\varepsilon/2}(y_{i_0})$, we have

$$B_{\varepsilon/2}(y_{i_0}) \cap O \neq \varnothing \quad \text{and} \quad B_{\varepsilon/2}(y_{i_0}) \cap O^c \neq \varnothing,$$

hence $i_0 \in A$. Let $x_0 \in B_{\varepsilon/2}(y_{i_0}) \cap O$. In this way we have obtained x_0 such that $u(x_0) - v(x_0) > M - \delta$ and one of the following holds:

$$\sup_{B_\varepsilon(x_0)} u - \inf_{B_\varepsilon(x_0)} u > \left(4 + \frac{4\beta}{\alpha}\right)\lambda, \tag{1}$$

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda. \tag{2}$$

Let us show that in fact the second statement must hold. Suppose not, then the first holds and we have

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v \le 2\lambda.$$

Given that

$$v(x_0) \ge \tfrac{1}{2}\big(\sup_{B_\varepsilon(x_0)} v + \inf_{B_\varepsilon(x_0)} v\big),$$

we get

$$v(x_0) + \lambda \ge \sup_{B_\varepsilon(x_0)} v.$$

Hence

$$v(x_0) + \lambda + M \ge \sup_{B_\varepsilon(x_0)} v + M \ge \sup_{B_\varepsilon(x_0)} u.$$

Further, since

$$u(x_0) - v(x_0) > M - \delta > M - \lambda,$$

we get

$$u(x_0) + 2\lambda > \sup_{B_\varepsilon(x_0)} u,$$

and

$$\sup_{B_\varepsilon(x_0)} u > \inf_{B_\varepsilon(x_0)} u + \left(4 + \frac{4\beta}{\alpha}\right)\lambda.$$

Hence

$$u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda > \inf_{B_\varepsilon(x_0)} u.$$

If we bound the integral by the value of the supremum we can control all the terms in the DPP in terms of $u(x_0)$. We have

$$\begin{aligned}
u(x_0) &= \max\left\{ \frac{1}{2}\big(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u\big), \frac{\alpha}{2}\big(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u\big) + \beta \fint_{B_\varepsilon(x_0)} u \right\} \\
&< \max\left\{ \frac{1}{2}\left(u(x_0) + 2\lambda + u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda \right), \right. \\
&\qquad\qquad \left. \frac{\alpha}{2}\left(u(x_0) + 2\lambda + u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda \right) + \beta\big(u(x_0) + 2\lambda\big) \right\} \\
&< \max\left\{ u(x_0) - \frac{4\beta}{\alpha}\lambda, u(x_0) \right\} = u(x_0),
\end{aligned}$$

which is a contradiction. Hence, the second condition must hold, that is, we have

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda.$$

Applying the DPP we get

$$v(x_0) \geq \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x_0)} v + \inf_{B_\varepsilon(x_0)} v\Big)$$

together with the fact that

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda,$$

and then we conclude that

$$v(x_0) > \inf_{B_\varepsilon(x_0)} v + \lambda.$$

We have proved that there exists x_0 such that

$$v(x_0) > \inf_{B_\varepsilon(x_0)} v + \lambda \quad \text{and} \quad u(x_0) - v(x_0) > M - \delta.$$

Now we are going to build a sequence of points where the difference between u and v is almost maximal and where the value of v decreases by at least λ in every step. Applying the DPP to $M - \delta < u(x_0) - v(x_0)$ and bounding the difference of the suprema by M we get:

$$M - \frac{2}{\alpha}\delta + \inf_{B_\varepsilon(x_0)} v < \inf_{B_\varepsilon(x_0)} u.$$

Let x_1 be such that $v(x_0) > v(x_1) + \lambda$ and $\inf_{B_\varepsilon(x_0)} v + \delta > v(x_1)$. We get

$$M - \Big(1 + \frac{2}{\alpha}\Big)\delta + v(x_1) < u(x_1).$$

To repeat this construction we need the following two results:

- In the last inequality, if δ is small enough $u(x_1) \neq v(x_1)$, hence $x_1 \in \Omega$.
- We know that $2v(x_1) \geq \inf_{B_\varepsilon(x_1)} v + \sup_{B_\varepsilon(x_1)} v > v(x_0) + \inf_{B_\varepsilon(x_1)} v$. Hence, since $v(x_0) > v(x_1) + \lambda$, we get $v(x_1) > \inf_{B_\varepsilon(x_1)} v + \lambda$.

Then we get

$$v(x_{n-1}) > v(x_n) + \lambda$$

and

$$M - \Big(\sum_{k=0}^{n} \Big(\frac{2}{\alpha}\Big)^k \Big)\delta + v(x_n) < u(x_n).$$

We can repeat this argument as long as

$$M - \Big(\sum_{k=0}^{n} \Big(\frac{2}{\alpha}\Big)^k \Big) \delta > 0,$$

which is a contradiction with the fact that v is bounded. □

Now we want to show that this unique function that satisfies the DPP is the game value. The key point of the proof is to construct a strategy based on the approximating sequence that we used to construct the solution.

Theorem 4.9. *Given $f \equiv 0$, the game has a value, that is $u_{\mathrm{I}} = u_{\mathrm{II}}$, which coincides with the unique p_1-p_2-harmonious function.*

Proof. Let u be the unique p_1-p_2-harmonious function (the uniqueness is given by Theorems 4.6 and 4.8). We will show that $u \le u_{\mathrm{I}}$. The analogous result can be proved for u_{II}, completing the proof.

Let us consider a sub-p_1-p_2-harmonious function u_0 which is smaller than $\inf_\Omega g$ at every point in Ω. Starting with this u_0 we build the corresponding u_k as in Proposition 4.4. We have that $u_k \to u$ as $k \to \infty$.

Now, given $\delta > 0$, let $n > 0$ be such that $u_n(x_0) > u(x_0) - \delta/2$. We build a strategy S_{I}^0 for Player I: in the first n moves, given x_{k-1} he will choose to move to a point that almost maximizes u_{n-k}, that is, he chooses $x_k \in B_\varepsilon(x_{k-1})$ such that

$$u_{n-k}(x_k) > \sup_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n}$$

and chooses γ in order to maximize

$$\frac{\alpha_i}{2} \Big\{ \inf_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n} + \sup_{B_\varepsilon(x_{k-1})} u_{n-k} \Big\} + \beta_i ⨍_{B_\varepsilon(x_{k-1})} u_{n-k}\, dy.$$

After the first n moves he will follow a strategy that ends the game almost surely (for example pointing in a fix direction).

We have

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^0, S_{\mathrm{II}}} \Big[u_{n-k}(x_k) + \frac{k\delta}{2n} \,:\, x_0, \dots, x_{k-1} \Big] \\
&\quad \ge \max_{i \in \{1,2\}} \Big(\frac{\alpha_i}{2} \Big\{ \inf_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n} + \sup_{B_\varepsilon(x_{k-1})} u_{n-k} \Big\} + \beta_i ⨍_{B_\varepsilon(x_{k-1})} u_{n-k}\, dy \Big) + \frac{k\delta}{2n} \\
&\quad \ge u_{n-k+1}(x_{k-1}) + \frac{(k-1)\delta}{2n},
\end{aligned}$$

where we have estimated the strategy of Player II by inf and used the construction for the u_k. Thus

$$M_k = \begin{cases} u_{n-k}(x_k) + \dfrac{k\delta}{2n} - \dfrac{\delta}{2} & \text{for } 0 \le k \le n, \\ \inf_\Omega g & \text{for } k > n, \end{cases}$$

is a submartingale.

Now we have

$$\begin{aligned} u_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[g(x_\tau)] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}}[g(x_\tau)] \\ &\ge \inf_{S_{\mathrm{II}}} \liminf_{k\to\infty} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}}[M_k] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}}[M_0] = u_n(x_0) - \frac{\delta}{2} > u(x_0) - \delta, \end{aligned}$$

where $\tau \wedge k = \min(\tau, k)$, and we used the optional stopping theorem for M_k. Since δ is arbitrary, this proves the claim. □

As an immediate corollary of our results in this section we obtain a comparison result for solutions to the DPP.

Corollary 4.10. *If v and u are p_1-p_2-harmonious functions for g_v, f_v and g_u, f_u, respectively such that $g_v \ge g_u$ and $f_v \ge f_u$, then $v \ge u$.*

5. Properties of harmonious functions and convergence

First, we show some properties of p_1-p_2-harmonious functions that we need to prove convergence as $\varepsilon \to 0$. We want to apply the following Arzelà–Ascoli-type lemma. For its proof, see [Manfredi et al. 2012b, Lemma 4.2].

Lemma 5.1. *Let $\{u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that*

(1) *there exists $C > 0$ such that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*

(2) *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_0$,*

$$|u_\varepsilon(x) - u_\varepsilon(y)| < \eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \quad \text{uniformly in} \quad \overline{\Omega},$$

as $\varepsilon \to 0$.

So our task now is to show that the family u_ε satisfies the hypotheses of the previous lemma. To this end we need some bounds on the expected exit time in the case of a player choose a certain strategy.

Let us start showing that u_ε are uniformly bounded. In Lemma 4.2 we obtained a bound for the value of the game for a fixed ε; here we need a bound independent of ε. To this end, let us define what we understand by pulling in one direction: we fix a direction, that is, a unitary vector v and at each turn of the game the player strategy is given as $S(x_{k-1}) = x_{k-1} + (\varepsilon - \varepsilon^3/2^k)v$.

Lemma 5.2. *In a game where a player pulls in a fixed direction the expectation of the exit time is bounded above by*

$$\mathbb{E}[\tau] \leq C\varepsilon^{-2}$$

for some $C > 0$ independent of ε.

Proof. First, let us assume without loss of generality that

$$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_n < R\}$$

and that the direction that the player is pulling to is $-e_n$. Then

$$M_k = (x_k)_n + \frac{\varepsilon^3}{2^k}$$

is a supermartingale. Indeed, if the random move occurs, then we know that the expectation of $(x_{k+1})_n$ is equal to $(x_k)_n$. If the tug-of-war game is played we know that with probability one half, $(x_{k+1})_n = (x_k)_n - \varepsilon + \varepsilon^3/2^k$ and if the other player moves $(x_{k+1})_n \leq (x_k)_n + \varepsilon$, so the expectation is less than or equal to $(x_k)_n + \varepsilon^3/2^{k+1}$.

Let us consider the expectation for $(M_{k+1} - M_k)^2$. If the random walk occurs, then the expectation is $\varepsilon^2/(n+2) + o(\varepsilon^2)$. Indeed,

$$\fint_{B_\varepsilon} x_n^2 = \frac{1}{n}\fint_{B_\varepsilon} |x|^2 = \frac{1}{\varepsilon^n n|B_1|}\int_0^\varepsilon r^2|\partial B_r|\,dr = \frac{|\partial B_1|}{\varepsilon^n n|B_1|}\int_0^\varepsilon r^{n+1}\,dr = \frac{\varepsilon^2}{n+2}.$$

If the tug-of-war occurs we know that with probability one half $(x_{k+1})_n = (x_k)_n - \varepsilon + \varepsilon^3/2^k$, so the expectation is greater than or equal to $\varepsilon^2/3$.

Let us consider the expectation for $M_k^2 - M_{k+1}^2$. We have

$$\mathbb{E}[M_k^2 - M_{k+1}^2] = \mathbb{E}[(M_{k+1} - M_k)^2] + 2\mathbb{E}[(M_k - M_{k+1})M_{k+1}].$$

As $(x_k)_n$ is positive, we have $2\mathbb{E}[(M_k - M_{k+1})M_{k+1}] \geq 0$. Then

$$\mathbb{E}[M_k^2 - M_{k+1}^2] \geq \varepsilon^2/(n+2),$$

so $M_k^2 + k\varepsilon^2/(n+2)$ is a supermartingale. According to the optional stopping theorem for supermartingales,

$$\mathbb{E}\left[M_{\tau\wedge k}^2 + \frac{(\tau\wedge k)\varepsilon^2}{n+2}\right] \le M_0^2.$$

We have

$$\mathbb{E}[(\tau\wedge k)]\frac{\varepsilon^2}{n+2} \le M_0^2 - E[M_{\tau\wedge k}^2] \le M_0^2.$$

Taking the limit in k, we get a bound for the expected exit time,

$$\mathbb{E}[\tau] \le (n+2)M_0^2\varepsilon^{-2},$$

so the statement holds for $C = (n+2)R^2$. □

Lemma 5.3. *An f-p_1-p_2-harmonious function u_ε with boundary values g satisfies*

(5-1) $$\inf_{y\in\Gamma_\varepsilon} g(y) + C \inf_{y\in\Omega} f(y) \le u_\varepsilon(x) \le \sup_{y\in\Gamma_\varepsilon} g(y) + C \sup_{y\in\Omega} f(y).$$

Proof. We use the connection to games. Let one of the players choose a strategy of pulling in a fixed direction. Then

$$\mathbb{E}[\tau] \le C\varepsilon^{-2},$$

and this gives the upper bound

$$\begin{aligned}\mathbb{E}\left[g(X_\tau) + \varepsilon^2\sum_{n=0}^{\tau-1} f(X_n)\right] &\le \sup_{y\in\Gamma_\varepsilon} g(y) + E[\tau]\varepsilon^2 \sup_{y\in\Omega} f(y)\\ &\le \sup_{y\in\Gamma_\varepsilon} g(y) + C \sup_{y\in\Omega} f(y).\end{aligned}$$

The lower bound follows analogously. □

Let us show now that the u_ε are asymptotically uniformly continuous. First we need a lemma that bounds the expectation for the exit time when one player is pulling towards a fixed point.

Let us consider an annular domain $B_R(y)\setminus \overline{B}_\delta(y)$ and a game played inside. In each round the token starts at a certain point x, an ε-step tug-of-war is played inside $B_R(y)$ or the token moves at random with uniform probability in $B_R(y)\cap B_\varepsilon(x)$. If an ε-step tug-of-war is played, there is a probability of one half for either player to move the token to a point of his choosing in $B_R(y)\cap B_\varepsilon(x)$. We can think there is a third player choosing whether the ε-step tug-of-war or the random move occurs. The game ends when the position reaches $\overline{B}_\delta(y)$, that is, when $x_{\tau^*} \in \overline{B}_\delta(y)$.

Lemma 5.4. *Assume that one of the players pulls towards y in the game described above. Then, no mater how many times the tug-of-war is played or the random move is done, the exit time verifies*

$$\mathbb{E}^{x_0}(\tau^*) \le \big(C(R/\delta)\operatorname{dist}(\partial B_\delta(y), x_0) + o(1)\big)/\varepsilon^2, \tag{5-2}$$

for $x_0 \in B_R(y) \setminus \overline{B}_\delta(y)$. Here τ^* is the exit time in the previously described game and $o(1) \to 0$ as $\varepsilon \to 0$ can be taken as depending only on δ and R.

Proof. Let us denote

$$h_\varepsilon(x) = \mathbb{E}^x(\tau).$$

By symmetry, we know that h_ε is radial and it is easy to see that it is increasing in $r = |x-y|$. If we assume that the other player wants to maximize the expectation for the exit time and that the random move or tug-of-war is chosen in the same way, we have that the function h_ε satisfies a dynamic programming principle,

$$h_\varepsilon(x) = \max\left\{\frac{1}{2}\Big(\max_{B_\varepsilon(x)\cap B_R(y)} h_\varepsilon + \min_{B_\varepsilon(x)\cap B_R(y)} h_\varepsilon\Big), \fint_{B_\varepsilon(x)\cap B_R(y)} h_\varepsilon \, dz\right\} + 1,$$

by the above assumptions and that the number of steps always increases by 1 when making a step. Further, we denote $v_\varepsilon(x) = \varepsilon^2 h_\varepsilon(x)$ and obtain

$$v_\varepsilon(x) = \max\left\{\frac{1}{2}\Big(\sup_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon + \inf_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon\Big), \fint_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon \, dz\right\} + \varepsilon^2.$$

This induces us to look for a function v such that

$$v(x) \ge \fint_{B_\varepsilon(x)} v\,dz + \varepsilon^2 \quad \text{and} \quad v(x) \ge \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big) + \varepsilon^2. \tag{5-3}$$

Note that for small ε this is a sort of discrete version of the following inequalities:

$$\begin{cases} \Delta v(x) \le -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y), \\ \Delta_\infty v(x) \le -2, & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y). \end{cases} \tag{5-4}$$

This leads us to consider the problem

$$\begin{cases} \Delta v(x) = -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_\delta(y), \\ v(x) = 0, & x \in \partial B_\delta(y), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial B_{R+\varepsilon}(y), \end{cases} \tag{5-5}$$

where $\partial v/\partial \nu$ refers to the normal derivative. The solution to this problem is radially symmetric and strictly increasing in $r = |x-y|$. It takes the form

$$v(r) = \begin{cases} -ar^2 - br^{2-N} + c & \text{if } N > 2, \text{ and} \\ -ar^2 - b\log(r) + c & \text{if } N = 2. \end{cases}$$

If we extend this v to $B_\delta(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$, it satisfies $\Delta v(x)=-2(N+2)$ in $B_{R+\varepsilon}(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$. We know that

$$\Delta_\infty v = v_{rr} \le v_{rr} + \frac{N-1}{r} v_r = \Delta v.$$

Thus, v satisfies the inequalities (5-4). Then, the classical calculation shows that v satisfies (5-3) for each $B_\varepsilon(x) \subset B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$.

In addition, as v is increasing in r, it holds for each $x \in B_R(y) \setminus \overline{B}_\delta(y)$ that

$$\fint_{B_\varepsilon(x)\cap B_R(y)} v\,dz \le \fint_{B_\varepsilon(x)} v\,dz \le v(x) - \varepsilon^2$$

and

$$\tfrac{1}{2}\Big(\sup_{B_\varepsilon(x)\cap B_R(y)} v + \inf_{B_\varepsilon(x)\cap B_R(y)} v \Big) \le \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v \Big) \le v(x) - \varepsilon^2.$$

It follows that

$$\begin{aligned}
&\mathbb{E}[v(x_k) + k\varepsilon^2 : x_0, \dots, x_{k-1}] \\
&\qquad \le \max\Big\{ \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x_{k-1})\cap B_R(y)} v + \inf_{B_\varepsilon(x_{k-1})\cap B_R(y)} v \Big), \fint_{B_\varepsilon(x_{k-1})\cap B_R(y)} v\,dz \Big\} \\
&\qquad \le v(x_{k-1}) + (k-1)\varepsilon^2,
\end{aligned}$$

if $x_{k-1} \in B_R(y) \setminus \overline{B}_\delta(y)$. Thus $v(x_k) + k\varepsilon^2$ is a supermartingale, and the optional stopping theorem yields

$$\mathbb{E}^{x_0}[v(x_{\tau^*\wedge k}) + (\tau^* \wedge k)\varepsilon^2] \le v(x_0). \tag{5-6}$$

Because $x_{\tau^*} \in \overline{B}_\delta(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$, we have

$$0 \le -\mathbb{E}^{x_0}[v(x_{\tau^*})] \le o(1).$$

Furthermore, the estimate

$$0 \le v(x_0) \le C(R/\delta)\,\mathrm{dist}\big(\partial B_\delta(y), x_0\big)$$

holds for the solutions of (5-5). Thus, by passing to the limit as $k \to \infty$, we obtain

$$\varepsilon^2 \mathbb{E}^{x_0}[\tau^*] \le v(x_0) - \mathbb{E}[u(x_{\tau^*})] \le C(R/\delta)\big(\mathrm{dist}(\partial B_\delta(y), x_0) + o(1)\big).$$

This completes the proof. □

Next we derive a uniform bound and estimate for the asymptotic continuity of the family of p_1-p_2-harmonious functions.

We assume here that Ω satisfies an exterior sphere condition: for each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$.

Lemma 5.5. *Let g be Lipschitz continuous in Γ_ε and f Lipschitz continuous in Ω such that $f \equiv 0$, $\inf f > 0$ or $\sup f < 0$. The p_1-p_2-harmonious function u_ε with data g and f satisfies*

$$|u_\varepsilon(x)-u_\varepsilon(y)| \le \operatorname{Lip}(g)\big(|x-y|+\delta\big) + C(R/\delta)\big(|x-y|+o(1)\big)(1+\|f\|_\infty)+\widetilde{C}\operatorname{Lip}(f)|x-y|, \tag{5-7}$$

for every small enough $\delta > 0$ and for every two points $x, y \in \Omega \cup \Gamma_\varepsilon$. Here $o(1)$ can be taken depending only on δ and R.

Proof. The case $x, y \in \Gamma_\varepsilon$ is clear. Thus, we can concentrate on the cases $x \in \Omega$ and $y \in \Gamma_\varepsilon$ as well as $x, y \in \Omega$.

We use the connection to games. Suppose first that $x \in \Omega$ and $y \in \Gamma_\varepsilon$. By the exterior sphere condition, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. Now Player I chooses a strategy of pulling towards z, denoted by S_{I}^z. Then

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for a sufficiently large constant C, independent of ε. Indeed,

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\big[|x_k - z| : x_0, \dots, x_{k-1}\big] \\
&\quad\le \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\{|x_{k-1}-z|+\varepsilon-\varepsilon^3+|x_{k-1}-z|-\varepsilon\}+\beta_i \fint_{B_\varepsilon(x_{k-1})}|x-z|\,dx\right) \\
&\quad\le |x_{k-1}-z| + C\varepsilon^2.
\end{aligned}$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x_{k-1})}|x-z|\,dx \le |x_{k-1}-z| + C\varepsilon^2.$$

By the optional stopping theorem, this implies that

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\big[|x_\tau - z|\big] \le |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]. \tag{5-8}$$

Next we can estimate $\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]$ by the stopping time of Lemma 5.4. Let $R > 0$ be such that $\Omega \subset B_R(z)$. Thus, by (5-2),

$$\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau] \le \varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau^*] \le C(R/\delta)\big(\operatorname{dist}(\partial B_\delta(z), x_0) + o(1)\big).$$

Since $y \in \partial B_\delta(z)$,

$$\operatorname{dist}\big(\partial B_\delta(z), x_0\big) \le |y - x_0|,$$

and thus, (5-8) implies

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\big[|x_\tau - z|\big] \le C(R/\delta)\big(|x_0 - y| + o(1)\big).$$

We get

$$g(z) - C(R/\delta)\big(|x-y| + o(1)\big) \le \mathbb{E}^{x_0}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}[\,g(x_\tau)\,].$$

Thus, we obtain

$$\begin{aligned}\sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[\, g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}\Big[\, g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)\Big] \\ &\ge g(z) - C(R/\delta)\big(|x_0-y| + o(1)\big) - \varepsilon^2 \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}[\tau]\|f\|_\infty \\ &\ge g(y) - \mathrm{Lip}(g)\delta - C(R/\delta)\big(|x_0-y| + o(1)\big)(1 + \|f\|_\infty).\end{aligned}$$

The upper bound can be obtained by choosing for Player II a strategy where he points to z, and thus, (5-7) follows.

Finally, let $x, y \in \Omega$ and fix the strategies S_{I}, S_{II} for the game starting at x. We define a virtual game starting at y: we use the same coin tosses and random steps as the usual game starting at x. Furthermore, the players adopt their strategies S^v_{I}, S^v_{II} from the game starting at x, that is, when the game position is y_{k-1} a player chooses the step that would be taken at x_{k-1} in the game starting at x. We proceed in this way until for the first time $x_k \in \Gamma_\varepsilon$ or $y_k \in \Gamma_\varepsilon$. At that point we have

$$|x_k - y_k| = |x - y|,$$

and we may apply the previous steps that work for $x_k \in \Omega,\ \ y_k \in \Gamma_\varepsilon$ or for $x_k, y_k \in \Gamma_\varepsilon$.

If we are in the case $f \equiv 0$ we are done. In the case $\inf_{y\in\Omega}|f(y)| > 0$, as we know that the u_ε are uniformly bounded according to Lemma 5.3, we have that the expected exit time is bounded by

$$\widetilde{C} = \frac{\max_{y\in\Gamma_\varepsilon}|g(y)| + C\max_{y\in\Omega}|f(y)|}{\inf_{y\in\Omega}|f(y)|}.$$

So the expected difference in the running payoff in the game starting at x and the virtual one is bounded by $\widetilde{C}\,\mathrm{Lip}(f)|x-y|$, because $|x_i - y_i| = |x-y|$ for all $0 \le i \le k$. □

Corollary 5.6. *Let $\{u_\varepsilon\}$ be a family of p_1-p_2-harmonious functions. Then there exists a uniformly continuous u and a subsequence still denoted by $\{u_\varepsilon\}$ such that*

$$u_\varepsilon \to u \quad \textit{uniformly in} \quad \overline{\Omega}.$$

Proof. Using Lemmas 5.3 and 5.5 we get that the family u_ε satisfies the hypothesis of compactness in Lemma 5.1. □

Theorem 5.7. *The function u obtained as a limit in Corollary 5.6 is a viscosity solution to (1-2) when we consider the game with $f/2$ as the running payoff function.*

Proof. First, we observe that $u = g$ on $\partial\Omega$ since $u_\varepsilon = g$ on $\partial\Omega$ for all $\varepsilon > 0$. Hence, we can focus our attention on showing that u is p_1-p_2-harmonic inside Ω in the viscosity sense. To this end, we recall from [Manfredi et al. 2010] an estimate that involves the regular Laplacian ($p = 2$) and an approximation for the infinity Laplacian ($p = \infty$). Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Note that since ϕ is continuous we have

$$\min_{y\in\overline{B}_\varepsilon(x)} \phi(y) = \inf_{y\in B_\varepsilon(x)} \phi(y)$$

for all $x \in \Omega$. Let x_1^ε be the point at which ϕ attains its minimum in $\overline{B}_\varepsilon(x)$,

$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B}_\varepsilon(x)} \phi(y).$$

It follows from the Taylor expansions in [Manfredi et al. 2010] that

$$\begin{aligned}(5\text{-}9)\quad &\frac{\alpha}{2}\Big(\max_{y\in\overline{B}_\varepsilon(x)} \phi(y) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y)\Big) + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy - \phi(x) \\ &\geq \frac{\varepsilon^2}{2(n+p)}\left\{(p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right\} + o(\varepsilon^2).\end{aligned}$$

Suppose that ϕ touches u at x strictly from below. We want to prove that $F^*(\nabla\phi(x), D^2\phi(x)) \geq f(x)$. By the uniform convergence, there exists a sequence $\{x_\varepsilon\}$ converging to x such that $u_\varepsilon - \phi$ has an approximate minimum at x_ε, that is, for $\eta_\varepsilon > 0$, there exists x_ε such that

$$u_\varepsilon(x) - \phi(x) \geq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon.$$

Moreover, considering $\tilde{\phi} = \phi - u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$, we can assume that $\phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon)$. Thus, by recalling the fact that u_ε is p_1-p_2-harmonious, we obtain

$$\eta_\varepsilon \geq \varepsilon^2 \frac{f(x_\varepsilon)}{2} - \phi(x_\varepsilon) + \max_{i\in\{1,2\}}\left\{\frac{\alpha_i}{2}\Big(\max_{\overline{B}_\varepsilon(x_\varepsilon)} \phi + \min_{\overline{B}_\varepsilon(x_\varepsilon)} \phi\Big) + \beta_i \fint_{B_\varepsilon(x_\varepsilon)} \phi(y)\,dy\right\},$$

and thus, by (5-9), and choosing $\eta_\varepsilon = o(\varepsilon^2)$, we have

$$\begin{aligned}0 \geq \frac{\varepsilon^2}{2} \max_{i\in\{1,2\}}&\left\{\alpha_i\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle + \theta_i\,\Delta\phi(x_\varepsilon)\right\} \\ &+ \varepsilon^2 \frac{f(x_\varepsilon)}{2} + o(\varepsilon^2).\end{aligned}$$

Next we need to observe that

$$\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle$$

converges to $\Delta_\infty\phi(x)$ when $\nabla\phi(x) \neq 0$ and is always bounded in the limit by $\lambda_{\min}(D^2\phi(x))$ and $\lambda_{\max}(D^2\phi(x))$. Dividing by ε^2 and letting $\varepsilon \to 0$, we get

$$F^*(\nabla\phi(x), D^2\phi(x)) \geq f(x).$$

Therefore u is a viscosity supersolution.

To prove that u is a viscosity subsolution, we use a reverse inequality to (5-9) by considering the maximum point of the test function and choose a smooth test function that touches u from above. □

Now, we just observe that this probabilistic approach provides an alternative existence proof of viscosity solutions to our PDE problem.

Corollary 5.8. *Any limit function obtained as in Corollary 5.6 is a viscosity solution to the problem*

$$\begin{cases} \max\{-\Delta_{p_1}u, -\Delta_{p_2}u\} = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

In particular, the problem has a solution.

We proved that the problem has an unique solution using PDE methods, therefore we conclude that we have convergence as $\varepsilon \to 0$ of u_ε (not only along subsequences).

Corollary 5.9. *It holds that*

$$u_\varepsilon \to u \quad \text{uniformly in} \quad \overline{\Omega},$$

being u the unique solution to the problem

$$\begin{cases} \max\{-\Delta_{p_1}u, -\Delta_{p_2}u\} = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

References

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