Pacific Journal of Mathematics

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ALEJANDRO CABRERA AND THIAGO DRUMMOND

Volume 287 No. 2 April 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.297

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VB-groupoids define a special class of Lie groupoids which carry a compatible linear structure. We show that their differentiable cohomology admits a refinement by considering the complex of cochains which are k-homogeneous on the linear fiber. Our main result is a van Est theorem for such cochains. We also work out two applications to the general theory of representations of Lie groupoids and algebroids. The case $k = 1$ yields a van Est map for representations up to homotopy on 2-term graded vector bundles and, moreover, to a new proof of a rigidity conjecture posed by Crainic and Moerdijk. Arbitrary k-homogeneous cochains on suitable VB-groupoids lead to a novel van Est theorem for differential forms on Lie groupoids with values in a representation.

1. Introduction

The van Est theorem [1953a; 1953b; 1955a; 1955b] is a classical result relating the differentiable cohomology associated to a Lie group with the underlying Lie algebra cohomology. More precisely, given a Lie group G with Lie algebra $\mathfrak{g}$, the van Est map is a map

$$\mathrm{VE} : C^p(G) = \{f \in C^\infty(G^p) : f(g_1, \ldots, g_p) = 0 \text{ if } g_i = e\} \to \mathrm{CE}(\mathfrak{g}) = \Lambda^p \mathfrak{g}^*$$

MSC2010: 22A22, 53D17.
Keywords: Lie groupoids, van Est theorem.

taking (normalized) differentiable p-cochains on G to Lie algebra p-cochains. It is defined (up to sign) by

$$\mathrm{VE}(f)(u_1,\dots,u_p)=\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)R_{u_{\sigma(1)}}\dots R_{u_{\sigma(p)}}f, \tag{1-1}$$

where $R_u : C^p(G)\to C^{p-1}(G)$ is the operator which differentiates $f(\cdot, g_2,\dots,g_p)$ at the unit e with respect to the right-invariant vector field corresponding to u. The map VE can be seen as a model for the pullback of functions along the projection of the universal G-bundle $EG\to BG$. The van Est theorem then states that if G is (topologically) p_0-connected, the map induced by VE in cohomology is an isomorphism for $p\le p_0$ and injective for $p=p_0+1$. In the setting of Lie groupoids, the van Est theorem was first studied by A. Weinstein and P. Xu [1991] for $p_0=1$ and later generalized for arbitrary degrees by M. Crainic [2003] (see also the more recent work of D. Li-Bland and E. Meinrenken [2015]).

In this paper, we provide a refinement of this theorem for a particular class of Lie groupoids endowed with a compatible linear structure, called VB-groupoids [Pradines 1988] (see also [Bursztyn et al. 2016; Gracia-Saz and Mehta 2010; 2011]). In this case, the linear structure allows us to refine the van Est theorem by looking at *homogeneous cochains*, and we are able to derive several interesting applications from this general result.

To illustrate our approach, we examine here a simple situation involving a Lie group G and a linear representation $\rho_G : G\to \mathrm{Aut}(V)$ on a (finite-dimensional) real vector space V. The associated complex of differentiable cochains for G with values in V is $C^p(G,V)=\{f : G^p\to V : f(g_1,\dots,g_p)=0 \text{ if } g_i=e\}$ with a differential $\delta : C^p(G,V)\to C^{p+1}(G,V)$ which encodes ρ_G (see Example 2.5 below for an explicit formula). Infinitesimally, associated to the induced Lie algebra representation $\rho_{\mathfrak{g}}:\mathfrak{g}\to\mathrm{End}(V)$, we have the Chevalley–Eilenberg complex of Lie algebra cochains with values in V, namely $\mathrm{CE}^p(\mathfrak{g},V)=\Lambda^p\mathfrak{g}^*\otimes V$. In this setting, there exists a natural analogue of the van Est map

$$\Psi_\rho : C^p(G,V)\to \mathrm{CE}^p(\mathfrak{g},V). \tag{1-2}$$

How can one prove a van Est theorem for Ψ_ρ? There are two approaches: the first one is to re-prove the statement from scratch mimicking the proof of the standard case. The second one is to *deduce* the desired result from the known van Est theorem for *Lie groupoids* by relating the map (1-2) to the van Est map VE for the action groupoid $\mathcal{V}=V^*\rtimes G$. It is the second approach that we pursue in this paper.

To relate VE and Ψ_ρ, notice that both $\mathcal{V}$ and its space of p-composable arrows $B_p\mathcal{V}$ define vector bundles $\mathcal{V}\to G$ and $B_p\mathcal{V}\to G^p$, respectively. (Actually, $B_p\mathcal{V}$ is isomorphic to $V^*\times G^p$.) One can then show that the differentiable cochains

$f \in C^\infty(B_p\mathcal{V})$ which are fiberwise k-homogeneous define a subcomplex

$$C^p_{k\text{-hom}}(\mathcal{V}) \subset C^p(\mathcal{V}).$$

Analogously, the Lie algebroid $\mathfrak{v} = V^* \rtimes \mathfrak{g}$ of $\mathcal{V}$ also defines a trivial vector bundle $\mathfrak{v} \to \mathfrak{g}$ and the fiberwise k-homogeneous cochains define a subcomplex

$$\mathrm{CE}^p_{k\text{-hom}}(\mathfrak{v}) \subset \Lambda^p \mathfrak{v}^*.$$

The key point is that VE preserves k-homogeneous cochains, thus restricting to a map

$$\mathrm{VE}_{k\text{-hom}} : C^p_{k\text{-hom}}(\mathcal{V}) \to \mathrm{CE}^p_{k\text{-hom}}(\upsilon)$$

which, by a simple homological algebra argument (see page 307), is an isomorphism (resp. injective) in cohomology whenever VE is. Finally, to obtain the van Est theorem for V-valued cochains one has to verify that

$$H^p(C^\bullet_{1\text{-hom}}(\mathcal{V})) \simeq H^p(C^\bullet(G, V)),$$
$$H^p(\mathrm{CE}^\bullet_{1\text{-hom}}(\mathfrak{v})) \simeq H^p(\Lambda^\bullet \mathfrak{g}^* \otimes V),$$
$$\mathrm{VE}_{1\text{-hom}} \simeq \Psi_\rho.$$

In this paper, we follow the same reasoning but with $\mathcal{V}$ replaced by an arbitrary VB-groupoid. The main arguments follow directly as above but nontrivial computational effort needs to go into the last ingredient of the argument, namely, into relating the complexes of homogeneous (groupoid and algebroid) cochains to certain complexes already introduced in the literature from different perspectives. In particular, we obtain explicit formulas for the underlying van Est maps.

We work out two applications: in the first, we deduce a van Est theorem for representations up to homotopy in 2-term graded vector bundles [Arias Abad and Crainic 2012; 2013; Gracia-Saz and Mehta 2010; 2011] by looking at 1-homogeneous cochains and generalizing the case of ρ above, recovering results from [Arias Abad and Schätz 2011]. Moreover, we prove a cohomological vanishing result for these 1-homogeneous cochains which, in the case of the *adjoint representation*, leads to a realization of the original idea proposed in [Crainic and Moerdijk 2008] for showing a rigidity result for certain proper groupoids. (This last result was also proven in [Arias Abad and Schätz 2011] using different methods.) The second application provides a new van Est theorem for differential forms on Lie groupoids with coefficients in a representation, generalizing [Arias Abad and Crainic 2011] on the Bott–Shulmann complex and [Crainic et al. 2015b] on Spencer operators. It is interesting to notice that, in this second application, another idea is incorporated (which has its roots in *supergeometry* and was used in a Lie-theoretic context by Mehta [2009]): forms in $\Lambda^k V^*$ are k-homogeneous functions on V^k. For this application, we need the refinement of the van Est theorem in its full extent (i.e., for

k-homogeneous cochains, where k is arbitrary). Even in the particular case of differential forms with trivial coefficients, our proof of the corresponding van Est theorem is new and can be seen as illustration of the usefulness of homogeneous cochains.

Outline of the paper.

- In Section 2 we set up some notation, introduce homogeneous cochains on VB-groupoids and VB-algebroids and provide our main result: the corresponding refinement of the van Est theorem.
- In Section 3, we specialize to 1-homogeneous cochains and deduce a van Est result for representations up to homotopy. Along the way, we mention how this argument can be used to provide an alternative proof of the rigidity conjecture as originally proposed in [Crainic and Moerdijk 2008].
- In Section 4, by means of k-homogeneous cochains in suitable VB-groupoids and VB-algebroids, we prove a van Est theorem for differential forms with coefficients in a representation.

To keep the main text as simple as possible, we postpone to the Appendix some of the more technical or computational parts of the arguments in Section 4. Most of the explicit formulas contained in the Appendix follow from extensions of known lift properties of vector fields to Lie groupoids (see [Mackenzie and Xu 1994; 1998]). We would like to mention that part of this paper grew out of the project of understanding the Lie theory of multiplicative tensors on Lie groupoids [Bursztyn and Drummond $\geq$ 2017].

2. Homogeneous cochains and the van Est map for VB-groupoids

In this section, we present a refinement of groupoid and algebroid cohomology theory for VB-groupoids and VB-algebroids by considering k-homogeneous cochains. We also show that an analogue of the van Est theorem holds for such homogeneous cochains.

Homogeneous functions on vector bundles. Given any vector bundle $\pi : V \to B$, fiberwise multiplication by scalars $h : \mathbb{R} \times V \to V$ defines an action of the multiplicative monoid $\mathbb{R}$ which we shall call the *homogeneous structure of* $V \to B$. Following [Grabowski and Rotkiewicz 2009], we recall that the homogeneous structure completely characterizes the underlying vector bundle structure and that, in particular, a smooth map between the total spaces defines a vector bundle morphism if and only if it commutes with the underlying $\mathbb{R}$-actions. (See [Bursztyn et al. 2016] for applications of these ideas in a Lie-theoretic context.)

For each $k \in \mathbb{N}$, we consider

$$C^\infty_{k\text{-hom}}(V) := \{f \in C^\infty(V) : h_\lambda^* f = \lambda^k f \ \ \forall\, \lambda \in \mathbb{R}\},$$

the set of fiberwise k-homogeneous functions. Note that

$$C^\infty_{0\text{-hom}}(V) = \{f \in C^\infty(V) : \exists f_0 \in C^\infty(B) \text{ such that } f = f_0 \circ \pi\} \cong C^\infty(B).$$

Multiplication of functions gives a map $C^\infty_{k\text{-hom}}(V) \times C^\infty_{k'\text{-hom}}(V) \to C^\infty_{k+k'\text{-hom}}(V)$ and, in particular, each $C^\infty_{k\text{-hom}}(V)$ is a $C^\infty(B)$-module. In fact, $C^\infty_{k\text{-hom}}(V) \cong \Gamma(B, S^k V^*)$ for the symmetric algebra bundle $S^k V^* \to B$. The isomorphism $\Gamma(B, V^*) \cong C^\infty_{1\text{-hom}}(V)$ takes a section $\mu \in \Gamma(B, V^*)$ to the fiberwise-linear function $\ell_\mu \in C^\infty_{1\text{-hom}}(V)$ given by

$$\ell_\mu(v) = \langle \mu(b), v\rangle, \quad v \in V_b,\ b \in B.$$

The k-th derivative along the fiber defines a projection $P_{k\text{-hom}} : C^\infty(V) \to C^\infty_{k\text{-hom}}(V)$,

$$P_{k\text{-hom}}(f) = \frac{1}{k!}\frac{d^k}{d\lambda^k}(h^*_\lambda f)|_{\lambda=0}. \tag{2-1}$$

If $(x, \xi_1, \dots, \xi_n)$ are trivializing coordinates on V, then

$$P_{k\text{-hom}}(f)(x,\xi) = \sum_{k_1+\cdots+k_n=k} \frac{1}{k_1!\cdots k_n!}\frac{\partial^k f}{\partial \xi_1^{k_1}\cdots\partial\xi_n^{k_n}}(x,0)\,\xi_1^{k_1}\cdots\xi_n^{k_n}.$$

Homogeneous groupoid cochains. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with source and target maps $\mathsf{s}, \mathsf{t} : \mathcal{G} \to M$, unit $\mathbf{1} : M \to \mathcal{G}$, inversion $\iota : \mathcal{G} \to \mathcal{G}$ and multiplication $\mathsf{m} : \mathcal{G}_\mathsf{s}\times_\mathsf{t}\mathcal{G} \to \mathcal{G}$. We denote by $B_p\mathcal{G}$ the manifold of composable p-tuples ($B_0\mathcal{G} = M$). The nerve of $\mathcal{G}$ is the simplicial manifold whose space of p-simplices is $B_p\mathcal{G}$ with the simplicial structure given by the face maps $\partial_i : B_p\mathcal{G} \to B_{p-1}\mathcal{G},\ i = 0, \dots, p$, defined by

$$\partial_i(g_1,\dots,g_p) = \begin{cases} (g_2,\dots,g_p) & \text{if } i = 0,\\ (g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_p) & \text{if } 1 \le i \le p-1,\\ (g_1,\dots,g_{p-1}) & \text{if } i = p,\end{cases}$$

and the degeneracy maps $s_i : B_{p-1}\mathcal{G} \to B_p\mathcal{G},\ i = 0, \dots, p-1$, defined by

$$s_i(g_1,\dots,g_{p-1}) = (g_1,\dots,g_i,1_{\mathsf{t}(g_{i+1})},g_{i+1},\dots,g_{p-1}).$$

For $p = 1,\ \partial_0 = \mathsf{s},\ \partial_1 = \mathsf{t}$ and $s_0 = \mathbf{1}$.

The nerve defines a functor $B_\bullet$ from the category of Lie groupoids to the category of simplicial manifolds. For a groupoid morphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$, the morphism $B\phi : B\mathcal{G}_1 \to B\mathcal{G}_2$ is defined by $B_p\phi(g_1,\dots,g_p) = (\phi(g_1),\dots,\phi(g_p))$.

The space of (*normalized*) p-*cochains* $C^p(\mathcal{G})$ on $\mathcal{G}$ consists of smooth functions $f : B_p\mathcal{G} \to \mathbb{R}$ such that $s_i^* f = 0$ for $i = 0, \dots, p-1$. These define a cochain

complex with differential $\delta : C^{p-1}(\mathcal{G}) \to C^p(\mathcal{G})$ defined by

$$\delta = \sum_{i=0}^{p} (-1)^i \partial_i^*. \tag{2-2}$$

The *differentiable cohomology* of $\mathcal{G}$ is the cohomology of the complex $(C^\bullet(\mathcal{G}), \delta)$ and we denote it by $H^\bullet(\mathcal{G})$. For $f_1 \in C^p(\mathcal{G})$, $f_2 \in C^{p'}(\mathcal{G})$, the cup product $f_1 \star f_2 \in C^{p+p'}(\mathcal{G})$ is defined by

$$(f_1 \star f_2)(g_1, \ldots, g_{p+p'}) = f_1(g_1, \ldots, g_p) f_2(g_{p+1}, \ldots, g_{p+p'}). \tag{2-3}$$

It defines an algebra structure on $C^\bullet(\mathcal{G})$ which passes to cohomology due to the Leibniz formula

$$\delta(f_1 \star f_2) = \delta(f_1) \star f_2 + (-1)^p f_1 \star \delta(f_2).$$

In the following, we investigate how the differentiable cohomology of a *VB-groupoid* interacts with its underlying homogeneous structure.

Definition 2.1. A VB-groupoid is given by a commutative square

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ E & \longrightarrow & M, \end{array} \tag{2-4}$$

where the left and right sides are Lie groupoids and the top and bottom sides are vector bundles satisfying the following compatibility condition:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h_\lambda^{\mathcal{G}}} & \mathcal{V} \\ \downarrow\downarrow & & \downarrow\downarrow \\ E & \xrightarrow[h_\lambda]{} & E \end{array} \tag{2-5}$$

defines a Lie groupoid morphism for each $\lambda \in \mathbb{R}$, where $h_\lambda^{\mathcal{G}} : \mathcal{V} \to \mathcal{V}$ and $h_\lambda : E \to E$ are the homogeneous structures corresponding to $\mathcal{V} \to \mathcal{G}$ and $E \to M$, respectively. We denote the structure maps of $\mathcal{V} \rightrightarrows E$ by $\mathrm{s}_\mathcal{V}, \mathrm{t}_\mathcal{V}, \mathbf{1}_\mathcal{V}, \iota_\mathcal{V}, m_\mathcal{V}$.

Instead of looking at the homogeneous structure, VB-groupoids can be alternatively defined by focusing on the fiberwise defined sum (see [Gracia-Saz and Mehta 2011]). Our choice of definition comes from [Bursztyn et al. 2016], where the two definitions are shown to be equivalent (see Theorem 3.2.3 therein).

VB-groupoids have found several applications in recent years ([Bursztyn and Cabrera 2012; Bursztyn et al. 2016; Bursztyn and Drummond ≥ 2017; Gracia-Saz and Mehta 2011; Mackenzie and Xu 1994; 1998] to mention just a few). Natural examples of VB-groupoids are given by the tangent $T\mathcal{G} \rightrightarrows TM$ and cotangent

$T^*\mathcal{G} \rightrightarrows \mathrm{Lie}(\mathcal{G})^*$ groupoids, which provide intrinsic versions of the adjoint and coadjoint representations (up to homotopy; see Section 3 below) of a Lie groupoid $\mathcal{G}$. Ordinary representations also provide examples of VB-groupoids, as we shall see in detail in Example 2.5 below.

From now on, we focus on introducing *homogeneous cochains* on a VB-groupoid and to study their properties with respect to the van Est map, while having in mind the applications to be developed in Sections 3 and 4. The first result states that $B_\bullet$ restricts to a functor from VB-groupoids to simplicial vector bundles.

Lemma 2.2. *Let $\mathcal{V} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G} \rightrightarrows M$. The space of p-composable arrows $B_p\mathcal{V}$ is a vector bundle over $B_p\mathcal{G}$. Moreover, the face and degeneracy maps are all vector bundle maps.*

Proof. Consider $\mathcal{V}^p = \mathcal{V} \times \cdots \times \mathcal{V}$ as a vector bundle over $\mathcal{G}^p$. We shall present $B_p\mathcal{V}$ as a subbundle of $\mathcal{V}^p$ restricted to $B_p\mathcal{G} \subset \mathcal{G}^p$. It follows from the commutativity of (2-4) that $B_p\mathcal{V}$ projects onto $B_p\mathcal{G}$. As $B_p\mathcal{V}$ is a smooth submanifold of $\mathcal{V}^p$, it remains to check that it is invariant by the homogeneous structure of the vector bundle $\mathcal{V}^p \to \mathcal{G}^p$ (see [Grabowski and Rotkiewicz 2009]). This is a straightforward consequence of the fact that (2-5) is a groupoid morphism. The statement regarding the face and degeneracy maps follows now from the fact that the multiplication $m_{\mathcal{V}} : B_2\mathcal{V} \to \mathcal{V}$ is a vector bundle map (see also [Bursztyn et al. 2016]). □

Note that the homogeneous structure $h_\lambda^{B_p\mathcal{G}} : B_p\mathcal{V} \to B_p\mathcal{V}$ of the vector bundle $B_p\mathcal{V} \to B_p\mathcal{G}$ satisfies

$$B_p h_\lambda^{\mathcal{G}} = h_\lambda^{B_p\mathcal{G}}.$$

It is now a straightforward consequence of Lemma 2.2 that homogeneous cochains define a subcomplex of the differentiable cohomology of $\mathcal{V}$.

Proposition 2.3. *Let $\mathcal{V} \rightrightarrows E$ be a VB-groupoid. If*

$$P_{k\text{-hom}}^{\mathcal{G},p} : C^\infty(B_p\mathcal{V}) \to C^\infty_{k\text{-hom}}(B_p\mathcal{V})$$

is the projection (2-1) induced by $h_\lambda^{B_p\mathcal{G}}$, then

$$P_{k\text{-hom}}^{\mathcal{G},p+1} \circ \delta = \delta \circ P_{k\text{-hom}}^{\mathcal{G},p}.$$

In particular,

$$\delta(C^\infty_{k\text{-hom}}(B_p\mathcal{V})) \subset C^\infty_{k\text{-hom}}(B_{p+1}\mathcal{V}).$$

Thus, for a VB-groupoid $\mathcal{V} \rightrightarrows E$, we define natural subcomplexes of $(C^\bullet(\mathcal{V}), \delta)$ by considering the set of fiberwise k-homogeneous functions:

$$C^\bullet_{k\text{-hom}}(\mathcal{V}) := C^\infty_{k\text{-hom}}(\mathcal{V}^{(\bullet)}) \quad \text{and} \quad H^\bullet_{k\text{-hom}}(\mathcal{V}) = H(C^\bullet_{k\text{-hom}}(\mathcal{V})).$$

Remark 2.4. For $k=0$, we have $C^\bullet_{0\text{-hom}}(\mathcal{V}) \simeq C^\bullet(\mathcal{G})$ and the cup product (2-3) on $C^\bullet(\mathcal{V})$ induces a right $C^\bullet(\mathcal{G})$-module (resp. $H^\bullet(\mathcal{G})$-module) structure on $C^\bullet_{k\text{-hom}}(\mathcal{V})$ (resp. $H^\bullet_{k\text{-hom}}(\mathcal{V})$).

Example 2.5. Let $C \to M$ be a (left) representation of the Lie groupoid $\mathcal{G} \rightrightarrows M$. The vector bundle $\mathcal{V} = \mathsf{t}^*C^* \to \mathcal{G}$ carries a VB-groupoid structure $\mathsf{t}^*C^* \rightrightarrows C^*$ defined by

$$\mathsf{s}_\mathcal{V}(g,\xi) = \Delta_g^*(\xi), \quad \mathsf{t}_\mathcal{V}(g,\xi) = \xi,$$

$$\iota_\mathcal{V}(g,\xi) = (g^{-1}, \Delta_g^*(\xi)), \quad \mathbf{1}_\mathcal{V}(\xi) = (\mathbf{1}_{\pi(\xi)}, \xi), \quad \mathsf{m}_\mathcal{V}((g,\xi_1),(h,\xi_2)) = (gh,\xi_1),$$

where $\Delta_g : C_{\mathsf{s}(g)} \to C_{\mathsf{t}(g)}$ is the action of $g \in \mathcal{G}$. Note that $\mathsf{t}^*C^* = C^* \rtimes \mathcal{G}$, the action groupoid for the adjoint action of $\mathcal{G}$ on C^*. As vector bundles over $B_p\mathcal{G}$, one has that $B_p(\mathsf{t}^*C^*) = \mathsf{t}_p^*C^*$, where $\mathsf{t}_p : B_p\mathcal{G} \to M$ is given by $\mathsf{t}_p(g_1,\dots,g_p) = \mathsf{t}(g_1)$ and the isomorphism is given by $((g_1,\xi_1),\dots,(g_p,\xi_p)) \mapsto ((g_1,\dots,g_p),\xi_1)$. In particular,

$$C^p_{1\text{-hom}}(\mathcal{V}) \cong \Gamma(B_p\mathcal{G}, \mathsf{t}_p^*C).$$

The right $C^\bullet(\mathcal{G})$-module structure on $C^\bullet_{1\text{-hom}}(\mathcal{V})$ corresponds to a right module structure on $\Gamma(B_\bullet\mathcal{G}, \mathsf{t}_\bullet^*C)$ given by

$$(\phi \star f)(g_1,\dots,g_{p+p'}) = \phi(g_1,\dots,g_p)\, f(g_{p+1},\dots,g_{p+p'}), \tag{2-6}$$
$$f \in C^{p'}(\mathcal{G}),\ \phi \in \Gamma(B_p\mathcal{G}, \mathsf{t}_p^*C).$$

Further, the differential on $C^\bullet_{1\text{-hom}}(\mathcal{V})$ corresponds to the differential on $\Gamma(B_\bullet\mathcal{G}, \mathsf{t}_\bullet^*C)$ given by

$$\begin{aligned}&(\delta\phi)(g_1,\dots,g_{p+1})\\ &= \Delta_{g_1}(\phi(g_2,\dots,g_p)) + \sum_{i=1}^{p-1}(-1)^i\phi(g_1,\dots,g_ig_{i+1},\dots,g_p) + (-1)^p\phi(g_1,\dots,g_{p-1}).\end{aligned}$$

Hence, as $H^\bullet(\mathcal{G})$-modules, $H^\bullet_{1\text{-hom}}(\mathcal{V}) \cong H^\bullet(\mathcal{G}, C)$, the cohomology of $\mathcal{G}$ with coefficients on the representation C (see [Crainic 2003]). More generally, $H^\bullet_{k\text{-hom}}(\mathcal{V}) \cong H^\bullet(\mathcal{G}, S^kC)$.

Homogeneous algebroid cochains. Given a VB-groupoid $\mathcal{V} \rightrightarrows E$, its Lie algebroid $\mathfrak{v} \to E$ inherits the structure of a VB-algebroid; see [Bursztyn et al. 2016]. As for VB-groupoids, we take our working definition from that paper.

Definition 2.6. A VB-algebroid is given by a commutative square

$$\begin{array}{ccc} \mathfrak{v} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ E & \longrightarrow & M, \end{array} \tag{2-7}$$

where the left and right sides are Lie algebroids and the top and bottom sides are vector bundles satisfying the following compatibility condition:

(2-8)
$$\begin{array}{ccc} \mathfrak{v} & \xrightarrow{h_\lambda^{\mathfrak{g}}} & \mathfrak{v} \\ \downarrow & & \downarrow \\ E & \xrightarrow[h_\lambda]{} & E \end{array}$$

defines a Lie algebroid morphism for each $\lambda \in \mathbb{R}$, where $h_\lambda^{\mathfrak{g}}$ and h_λ are the homogeneous structures of the vector bundles $\mathfrak{v} \to \mathfrak{g}$ and $E \to M$, respectively.

Parallel to VB-groupoids, VB-algebroids together with Lie theory for VB-objects have found several applications in recent years (again, we list just a few of the available references: [Bursztyn and Cabrera 2012; Bursztyn et al. 2016; Bursztyn and Drummond ≥ 2017; Gracia-Saz and Mehta 2010; Mackenzie and Xu 1994; 1998]). The tangent $TA \to TM$ and the cotangent lift $T^*A \to A^*$ define examples of VB-algebroids corresponding to $T\mathcal{G}$ and $T^*\mathcal{G}$ when $A = \mathrm{Lie}(\mathcal{G})$, providing intrinsic versions of the adjoint and coadjoint representations (up to homotopy; see Section 3 below) of a Lie algebroid A. Ordinary representations of A also provide examples of VB-groupoids, as explained in Example 2.9 below. We now investigate the infinitesimal version of the notion of *homogeneous cochains*.

For any Lie algebroid $A \to M$, let $\mathrm{CE}^p(A) := \Gamma(M, \Lambda^p A^*)$ and $d : \mathrm{CE}^p(A) \to \mathrm{CE}^{p+1}(A)$ be the (Chevalley–Eilenberg) differential. The Lie algebroid cohomology $H^\bullet(A)$ is the cohomology of the complex $(\mathrm{CE}^\bullet(A), d)$. The wedge product on $\Gamma(M, \Lambda^\bullet A^*)$ induces a graded commutative algebra structure on $H^\bullet(A)$.

When considering a VB-algebroid $A = \mathfrak{v}$, the dual $\mathfrak{v}^*$ is always taken with respect to the Lie algebroid side $\mathfrak{v} \to E$, so that $\mathrm{CE}^p(\mathfrak{v}) = \Gamma(E, \Lambda^p \mathfrak{v}^*)$. The space of fiberwise (with respect to $\mathfrak{v} \to \mathfrak{g}$) k-homogeneous p-forms on $\mathfrak{v} \to E$ is

(2-9)
$$\Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*) := \{\alpha \in \Gamma(E, \Lambda^q \mathfrak{v}^*) : h_\lambda^{\mathfrak{g}*}\alpha = \lambda^k \alpha \ \ \forall\, \lambda \in \mathbb{R}\}.$$

The wedge product induces a map

$$\cdot \wedge \cdot : \Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*) \times \Gamma_{k'\text{-hom}}(E, \Lambda^{p'} \mathfrak{v}^*) \to \Gamma_{k+k'\text{-hom}}(E, \Lambda^{p+p'} \mathfrak{v}^*).$$

Similarly to equation (2-1), there exists a projection $P_{k\text{-hom}}^{\mathfrak{g},p} : \Gamma(E, \Lambda^p \mathfrak{v}^*) \to \Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*)$ defined by

(2-10)
$$P_{k\text{-hom}}^{\mathfrak{g},p}\alpha = \frac{1}{k!}\frac{d^k}{d\lambda^k}(h_\lambda^{\mathfrak{g}*}\alpha)|_{\lambda=0}.$$

Proposition 2.7. *Let $\mathfrak{v} \to E$ be a VB-algebroid. For each $k \in \mathbb{N}_0$ and every $p \geq 0$,*

$$P_{k\text{-hom}}^{\mathfrak{g},p+1} \circ d = d \circ P_{k\text{-hom}}^{\mathfrak{g},p}.$$

In particular,

$$d(\Gamma_{k\text{-hom}}(E, \Lambda^p\mathfrak{v}^*)) \subset \Gamma_{k\text{-hom}}(E, \Lambda^{p+1}\mathfrak{v}^*).$$

Proof. Since the Chevalley–Eilenberg differential d is a local operator, we can assume $\mathfrak{v} \to E$ is trivial. By looking at $h_\lambda^{\mathfrak{v}*}\alpha$ as a smooth 1-parameter family of forms, one can see that d commutes with $d/d\lambda$. The statement then follows from the fact that $h_\lambda^{\mathfrak{v}}$ is a Lie algebroid morphism and, hence, $h_\lambda^{\mathfrak{v}*}$ commutes with d. □

Thus, for each $k \in \mathbb{N}_0$, the k-homogeneous forms define a subcomplex $\mathrm{CE}^\bullet_{k\text{-hom}}(\mathfrak{v})$ of $(\mathrm{CE}^\bullet(\mathfrak{v}), d)$. The notation we use is

$$\mathrm{CE}^p_{k\text{-hom}}(\mathfrak{v}) := \Gamma_{k\text{-hom}}(E, \Lambda^p\mathfrak{v}^*) \quad \text{and} \quad H^\bullet_{k\text{-hom}}(\mathfrak{v}) = H(\mathrm{CE}^\bullet_{k\text{-hom}}(\mathfrak{v})).$$

Remark 2.8. For $k = 0$, we have $\Gamma_{0\text{-hom}}(E, \Lambda^p\mathfrak{v}^*) \cong \Gamma(M, \Lambda^p\mathfrak{g}^*)$ and the wedge product turns $\Gamma_{k\text{-hom}}(E, \Lambda^\bullet\mathfrak{v}^*)$ (resp. $H^\bullet_{k\text{-hom}}(\mathfrak{v})$) into a right $\Gamma(M, \Lambda^\bullet\mathfrak{g}^*)$-module (resp. $H^\bullet(\mathfrak{g})$-module).

Example 2.9. Let $C \to M$ be a representation of the Lie algebroid $\mathfrak{g} \to M$ defined by a flat $\mathfrak{g}$-connection $\nabla : \Gamma(\mathfrak{g}) \times \Gamma(C) \to \Gamma(C)$. Consider the vector bundle $\mathfrak{v} = C^* \times_M \mathfrak{g} \to C^*$. Given $u \in \Gamma(\mathfrak{g})$, let $\chi_u : C^* \to \mathfrak{v}$ be the section given by

$$\chi_u(\xi) = (\xi, u(m)) \quad \text{for } \xi \in C^*_m. \tag{2-11}$$

The sections χ_u with u varying on $\Gamma(\mathfrak{g})$ generate $\Gamma(C^*, \mathfrak{v})$ as a $C^\infty(C^*)$-module. One can now show that the action algebroid structure $C^* \rtimes \mathfrak{g} \to C^*$, determined by

$$[\chi_{u_1}, \chi_{u_2}] = \chi_{[u_1,u_2]}, \quad u_1, u_2 \in \Gamma(\mathfrak{g}),$$

$$\rho_{\mathfrak{v}}(\chi_{u_1})(\ell_\mu) = \ell_{\nabla_{u_1}\mu}, \ \rho_{\mathfrak{v}}(\chi_{u_1})(f \circ \pi) = (\mathcal{L}_{\rho(u_1)}f) \circ \pi, \quad f \in C^\infty(M), \ \mu \in \Gamma(C),$$

endows $\mathfrak{v} \to C^*$ with a VB-algebroid structure, where $\pi : C^* \to M$ is the projection. The chain complex $\mathrm{CE}^\bullet_{1\text{-hom}}(\mathfrak{v})$ is naturally isomorphic to $\Gamma(\Lambda^\bullet\mathfrak{g}^* \otimes C)$ with the Koszul differential

$$\begin{aligned} d_\nabla\gamma(u_1, \dots, u_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\nabla_{u_i}\gamma(u_1, \dots, \widehat{u_i}, \dots, u_{p+1}) \\ &\quad + \sum_{1\le i<j\le p+1}(-1)^{i+j}\gamma([u_i, u_j], u_1, \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, u_{p+1}), \end{aligned}$$

where $\gamma \in \Gamma(\Lambda^p\mathfrak{g}^* \otimes C)$. More precisely, the evaluation map $\mathrm{ev} : \mathrm{CE}^p_{1\text{-hom}}(\mathfrak{v}) \to \Gamma(\Lambda^p\mathfrak{g}^* \otimes C)$, given by

$$\langle \mathrm{ev}(\alpha)(u_1, \dots, u_p), \xi\rangle = \alpha(\chi_{u_1}(\xi), \dots, \chi_{u_p}(\xi)) \quad \text{for } u_1, \dots, u_p \in \Gamma(\mathfrak{g}), \ \xi \in C^*,$$

defines a chain isomorphism. The induced right $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$-module structure on $\Gamma(\Lambda^\bullet\mathfrak{g}^* \otimes C)$ is wedge multiplication on the right in the $\Lambda^\bullet\mathfrak{g}^*$ factor. In particular,

as $H(\mathfrak{g})$-modules, $H^\bullet_{1\text{-hom}}(\mathfrak{v}) \cong H^\bullet(\mathfrak{g}, C)$, the cohomology of $\mathfrak{g}$ with values in the representation C. As for groupoids, $H^\bullet_{k\text{-hom}}(\mathfrak{v}) \cong H^\bullet(\mathfrak{g}, S^k C)$.

Van Est theorem for homogeneous cochains. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g}$. For every section $u \in \Gamma(\mathfrak{g})$, consider the corresponding right invariant vector field $\vec{u} \in \mathfrak{X}(\mathcal{G})$. In the following, we denote by $B_p u$ the vector field on the space of p-composable arrows $B_p\mathcal{G}$ given by

$$B_p u(g_1, \dots, g_p) = (\vec{u}(g_1), 0_{g_2}, \dots, 0_{g_p}). \tag{2-12}$$

Let us now recall the definition of the van Est map. First, using the degeneracy map $s_0 : B_{p-1}\mathcal{G} \to B_p\mathcal{G}$, we define $R_u : C^p(\mathcal{G}) \to C^{p-1}(\mathcal{G})$ by

$$R_u = s_0^* \circ \mathcal{L}_{B_p u}.$$

The van Est map $\mathrm{VE} : C^p(\mathcal{G}) \to \mathrm{CE}^p(\mathfrak{g})$ is defined (up to p-dependent sign) as follows [Crainic 2003]: for a p-cochain $f \in C^p(\mathcal{G})$,

$$\mathrm{VE}(f)(u_1, \dots, u_p) = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\, R_{u_{\sigma(1)}} \dots R_{u_{\sigma(p)}}(f). \tag{2-13}$$

In [Crainic 2003] it is shown that it induces a map in cohomology which preserves the corresponding product structures. We also need the following naturality result about VE.

Lemma 2.10. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Lie groupoids with Lie algebroids $\mathfrak{h}_1, \mathfrak{h}_2$, respectively. If $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a Lie groupoid morphism with the corresponding Lie algebroid morphism* $\mathrm{Lie}(\phi) : \mathfrak{h}_1 \to \mathfrak{h}_2$, *then*

$$\mathrm{VE}(B_p\phi^* f) = \mathrm{Lie}(\phi)^* \mathrm{VE}(f) \quad \forall f \in C^p(\mathcal{H}_2).$$

Proof. For any $\chi \in \Gamma(\mathfrak{h}_1)$ we can write

$$\mathrm{Lie}(\phi)(\chi) = \sum_i \gamma_i (\tilde{\chi}_i \circ \phi_0) \in \Gamma(\phi_0^* \mathfrak{h}_2),$$

where $\phi_0 = B_0\phi : M_1 \to M_2$ denotes the map between objects induced by ϕ, $\gamma_i \in C^\infty(M_1)$ and $\tilde{\chi}_i \in \Gamma(\mathfrak{h}_2)$. A direct computation shows that

$$R_\chi((B_p\phi)^* f) = \sum_i (\mathrm{t}_{p-1}^* \gamma_i)(B_{p-1}\phi)^*(R_{\tilde{\chi}_i} f) \quad \forall f \in C^p(\mathcal{H}_2).$$

If we apply the above formula p times, we notice that most of the terms in $R_{\chi_1} \dots R_{\chi_p}(B_p\phi)^* f$ will vanish since VE is defined on normalized cochains (namely, $s_i^* f = 0$). The only remaining terms are

$$\sum_{i_1, \dots, i_p} \gamma_{i_1} \dots \gamma_{i_p} \phi_0^*(R_{\tilde{\chi}_1} \dots R_{\tilde{\chi}_p} f),$$

and we thus get the statement of the lemma. □

The main result about the van Est map in the present context is as follows.

Theorem 2.11 [Crainic 2003]. *Let $\mathcal{G}$ be a Lie groupoid and let $\mathfrak{g}$ be its Lie algebroid. The van Est map (2-13) induces an algebra homomorphism*

$$\mathrm{VE} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g}).$$

Moreover, if $\mathcal{G}$ has p_0-connected source fibers, then VE *is an isomorphism in degrees $p \le p_0$, and it is injective for $p = p_0 + 1$.*

To get our refinement of Theorem 2.11 for homogeneous cochains on VB-groupoids and algebroids, we first state a simple homological algebra fact.

Homological lemma. *Let $(C_i^\bullet, \delta_i)$ be differential complexes, $i = 1, 2$, endowed with projections $P_i : C_i^\bullet \to C_i^\bullet$ (i.e., $P_i \circ \delta_i = \delta_i \circ P_i$ and $P_i^2 = P_i$). If $F : C_1^\bullet \to C_2^\bullet$ is a morphism satisfying $F \circ P_1 = P_2 \circ F$, then for each p such that $F : H^p(C_1) \to H^p(C_2)$ is injective (resp. surjective) its restriction $F_r : H^p(S_1) \to H^p(S_2)$ is also injective (resp. surjective), where $S_i^\bullet = P_i(C_i^\bullet)$.*

We are thus left with studying the behavior of the projections onto homogeneous cochains under the van Est map. To that end, let $\mathcal{V} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G} \rightrightarrows M$ and let $\mathfrak{v} \to E$ be its Lie algebroid.

Proposition 2.12. *For each $k \in \mathbb{N}_0$ and every $p \ge 0$,*

$$\mathrm{VE} \circ P^{\mathcal{G},p}_{k\text{-hom}} = P^{\mathfrak{g},p}_{k\text{-hom}} \circ \mathrm{VE}.$$

In particular, $\mathrm{VE}(C^\infty_{k\text{-hom}}(B_p\mathcal{V})) \subset \Gamma_{k\text{-hom}}(\Lambda^p \mathfrak{v}^*)$.

Proof. Let $h^{\mathcal{G}}_\lambda : \mathcal{V} \to \mathcal{V}$ and $h^{\mathfrak{g}}_\lambda : \mathfrak{v} \to \mathfrak{v}$ be the homogeneous structures of the vector bundles $\mathcal{V} \to \mathcal{G}$ and $\mathfrak{v} \to \mathfrak{g}$, respectively. By Lemma 2.10, the fact that $h^{\mathcal{G}}_\lambda$ is a groupoid homomorphism with $\mathrm{Lie}(h^{\mathcal{G}}_\lambda) = h^{\mathfrak{g}}_\lambda$ implies that

$$\mathrm{VE} \circ h^{\mathcal{G}*}_\lambda = h^{\mathfrak{g}*}_\lambda \circ \mathrm{VE} \quad \forall\, \lambda.$$

Hence, by applying $\frac{d}{d\lambda}\big|_{\lambda=0}$ on both sides, one obtains the commutation relation between VE and the projections $P^{\cdot,p}_{\mathrm{hom},k}$. The result now follows directly. □

The restriction of the van Est map to the subcomplex of k-homogeneous cochains shall be denoted by

$$\mathrm{VE}_{k\text{-hom}} := \mathrm{VE}|_{C^p_{k\text{-hom}}(\mathcal{V})} : C^p_{k\text{-hom}}(\mathcal{V}) \to \mathrm{CE}^p_{k\text{-hom}}(\mathfrak{v}).$$

Example 2.13 (0-homogeneous cochains). For $k = 0$, using the isomorphisms $C^p_{0\text{-hom}}(\mathcal{V}) \cong C^p(\mathcal{G})$ and $\mathrm{CE}^p_{0\text{-hom}}(\mathfrak{v}) \cong \mathrm{CE}^p(\mathfrak{g})$, one can check that $\mathrm{VE}_{0\text{-hom}} \cong \mathrm{VE}_{\mathcal{G}} : C^p(\mathcal{G}) \to \mathrm{CE}^p(\mathfrak{g})$. To see this, take $f \in C^p_{0\text{-hom}}(\mathcal{V})$ and $\chi_1, \dots, \chi_p \in \mathfrak{v}$, and notice that $\mathrm{VE}_{0\text{-hom}}(f)(\chi_1, \dots, \chi_p)$ only depends on the projections $u_i \in \mathfrak{g}$ of χ_i,

$i = 1, \dots, p$. Hence, to compute $\mathrm{VE}_{0\text{-hom}}$, it suffices to take $\chi_1, \dots, \chi_p$ linear sections[1] of $\mathfrak{v}$ covering $u_1, \dots, u_p \in \Gamma(\mathfrak{g})$. In this case,

$$\mathrm{VE}_{0\text{-hom}}(f)(\chi_1, \dots, \chi_p) = \pi_E^* \mathrm{VE}_{\mathcal{G}}(f_0)(u_1, \dots, u_p),$$

where $f = \pi^*_{B_p\mathcal{G}} f_0$, $f_0 \in C^\infty(B_p\mathcal{G})$, and where $\pi_E : E \to M$, $\pi_{B_p\mathcal{V}} : B_p\mathcal{V} \to B_p\mathcal{G}$ are the vector bundle projections.

We are now ready to state and prove our main theorem.

Theorem 2.14. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g}$. For a VB-groupoid $\mathcal{V} \rightrightarrows E$ over $\mathcal{G}$ with underlying VB-algebroid $\mathfrak{v} \to E$, the van Est map on k-homogeneous cochains induces a module homomorphism*

$$\mathrm{VE}_{k\text{-hom}} : H^\bullet_{k\text{-hom}}(\mathcal{V}) \to H^\bullet_{k\text{-hom}}(\mathfrak{v})$$

covering the algebra homomorphism $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. Moreover, if $\mathcal{G}$ has p_0-connected source fibers, then $\mathrm{VE}_{k\text{-hom}}$ is an isomorphism for all $p \le p_0$ and it is injective for $p = p_0 + 1$.

Proof. The $H^\bullet(\mathcal{G})$-module structure on $H^\bullet_{k\text{-hom}}(\mathfrak{v})$ comes from the cup product of $C^\bullet_{k\text{-hom}}(\mathcal{V})$ and $C^\bullet_{0\text{-hom}}(\mathcal{V}) \cong C^\bullet(\mathcal{G})$. So, the first statement follows from the fact that $\mathrm{VE}_{k\text{-hom}}$ is the restriction of the van Est map of $\mathcal{V}$ to homogeneous cochains and that $\mathrm{VE}_{0\text{-hom}} \cong \mathrm{VE}_{\mathcal{G}}$.

Let us now assume that $\mathcal{G}$ has p_0-connected source fibers. First note that this implies that $\mathcal{V} \rightrightarrows E$ is also source p_0-connected. Indeed, a source fiber of $\mathcal{V} \rightrightarrows E$ is an affine bundle over the corresponding source fiber of $\mathcal{G} \rightrightarrows M$. So, the van Est theorem (Theorem 2.11) implies that $\mathrm{VE} : H^p(\mathcal{V}) \to H^p(\mathfrak{v})$ is an isomorphism for $p \le p_0$ and injective for $p = p_0 + 1$. The result now follows from Proposition 2.12 by applying the homological lemma to $F = \mathrm{VE}$, $(C_1^\bullet, \delta_1) = (C^\infty(B_\bullet\mathcal{V}), \delta)$ and $(C_2^\bullet, \delta_2) = (\Gamma(E, \Lambda^\bullet\mathfrak{v}^*), d)$ with projections $P_1 = P^{\mathcal{G},\bullet}_{k\text{-hom}} : C^\infty(B_\bullet\mathcal{V}) \to C^\infty_{k\text{-hom}}(B_\bullet\mathcal{V})$ and $P_2 = P^{\mathfrak{g},\bullet}_{k\text{-hom}} : \Gamma(E, \Lambda^\bullet\mathfrak{v}^*) \to \Gamma_{k\text{-hom}}(E, \Lambda^\bullet\mathfrak{v}^*)$. □

3. 1-homogeneous cochains and representations up to homotopy

In [Gracia-Saz and Mehta 2010; 2011], it was shown that VB-groupoids and VB-algebroids provide an intrinsic version of the notion of (2-term) representation up to homotopy, generalizing the example given in the introduction, as well as Examples 2.5 and 2.9 above. In this section, we show how Theorem 2.14, when applied to 1-homogeneous cochains, recovers a van Est result for the underlying 2-term representations up to homotopy [Arias Abad and Schätz 2011]. We also outline a new proof, realizing an original proposal [Crainic and Moerdijk 2008] of a rigidity conjecture involving the deformation cohomology underlying proper groupoids.

[1] A linear section χ of $\mathfrak{v}$ is a section $\chi : E \to \mathfrak{v}$ which is a vector bundle homomorphism covering a section $u : M \to \mathfrak{g}$ (see [Gracia-Saz and Mehta 2010]).

VB-groupoid and VB-algebroid cohomology. Following [Gracia-Saz and Mehta 2011], given a VB-groupoid $\pi : \mathcal{V} \to \mathcal{G}$ we define $C^p_{\mathrm{VB}}(\mathcal{V})$ to be the space of 1-homogeneous cochains $\phi \in C^\infty_{1\text{-hom}}(B_p\mathcal{V})$ satisfying the two additional conditions

(1) $\phi(0_g, \xi_1, \ldots, \xi_{p-1}) = 0$,

(2) $\phi(0_g \cdot \xi_1, \ldots, \xi_p) = \phi(\xi_1, \ldots, \xi_p)$

for all $(\xi_1, \ldots, \xi_p) \in B_p\mathcal{V}$ and $g \in \mathcal{G}$ such that $(0_g, \xi_1) \in B_2\mathcal{V}$. As observed in [Gracia-Saz and Mehta 2011], condition (1) above implies that $\phi(\xi_1, \xi_2, \ldots, \xi_p)$ only depends on ξ_1 and on the projections $g_i = \pi(\xi_i) \in \mathcal{G}$, $i = 1, \ldots, p$, while condition (2) is a left-invariance property.

It is shown in that paper that $C^\bullet_{\mathrm{VB}}(\mathcal{V})$ defines a subcomplex of $C^\bullet_{1\text{-hom}}(\mathcal{V})$. Moreover, the cup product with $C^\bullet_{0\text{-hom}}\mathcal{V} \cong C^\bullet(\mathcal{G})$ defines a right $C^\bullet(\mathcal{G})$-submodule structure on $C^\bullet_{\mathrm{VB}}(\mathcal{V})$. The next lemma relates the cohomology of the two complexes.

Lemma 3.1. *The inclusion $\iota : C^\bullet_{\mathrm{VB}}(\mathcal{V}) \hookrightarrow C^\bullet_{1\text{-hom}}(\mathcal{V})$ induces an isomorphism of right $H^\bullet(\mathcal{G})$-modules in cohomology.*

Proof. It is enough to show that for every $\phi \in C^p_{1\text{-hom}}(\mathcal{V})$ with $\delta\phi \in C^{p+1}_{\mathrm{VB}}(\mathcal{V})$ there exists a $\psi \in C^{p-1}_{1\text{-hom}}(\mathcal{V})$ so that $\phi + \delta\psi \in C^p_{\mathrm{VB}}(\mathcal{V})$. To that end, first notice that if an arbitrary ϕ is such that both ϕ and $\delta\phi$ satisfy condition (1), then ϕ satisfies condition (2). This follows directly from evaluating

$$0 = (\delta\phi)(0_g, \xi_1, \ldots, \xi_p).$$

We are thus left with showing that for each $\phi \in C^\infty_{1\text{-hom}}(B_p\mathcal{V})$ such that $\delta\phi$ satisfies (1) there exists a $\psi \in C^\infty_{1\text{-hom}}(B_{p-1}\mathcal{V})$ such that $\phi + \delta\psi$ satisfies (1). This, in turn, follows by applying recursively the following claim: if $\delta\phi$ satisfies (1) and

$$\phi(\xi_0, \ldots, \xi_{p-1}) = 0 \tag{3-1}$$

for all $(\xi_0, \ldots, \xi_{p-1}) \in B_p\mathcal{V}$ such that $\xi_i = 0_{g_i}$, $i = 0, \ldots, l \le p-1$, then there exists a $\psi \in C^\infty_{1\text{-hom}}(B_{p-1}\mathcal{V})$ such that $\phi + \delta\psi$ satisfies (3-1) for all $(\xi_0, \ldots, \xi_{p-1}) \in B_p\mathcal{V}$ such that $\xi_i = 0_{g_i}$, $i = 0, \ldots, l-1$. Notice that for $l = p-1$, (3-1) follows from ϕ being homogeneous of degree 1. To prove this claim for $l < p-1$, one chooses any $\psi \in C^\infty_{1\text{-hom}}(B_{p-1}\mathcal{V})$ such that

$$\psi(\xi_1, \ldots, \xi_{p-1}) = -\phi(0_{\pi(\xi_{p-1})^{-1}\cdots\pi(\xi_1)^{-1}}, \xi_1, \ldots, \xi_{p-1})$$

for all $(\xi_1, \ldots, \xi_{p-1}) \in B_{p-1}\mathcal{V}$ such that $\mathrm{t}_{\mathcal{V}}(\xi_1) = 0_{\mathrm{t}(\pi(\xi_1))}$. This is always possible since the subset of such elements in $B_{p-1}\mathcal{V}$ is a smooth embedded submanifold since the target map is a submersion. What needs to be shown now is

$$(\phi + \delta\psi)(\xi_0, \ldots, \xi_{p-1}) = 0 \quad \forall\, (\xi_0, \ldots, \xi_{p-1}) \in B_p\mathcal{V},\ \xi_i = 0_{g_i},\ i = 0, \ldots, l-1.$$

Finally, this last identity follows by evaluating

$$0 = (\delta\phi)(0_{\pi(\xi_{p-1})^{-1}\cdots\pi(\xi_0)^{-1}}, \xi_0, \ldots, \xi_{p-1})$$

and using the recursion hypothesis. □

For a VB-algebroid $\mathfrak{v} \to A$, the *VB-algebroid cochain complex* is defined exactly as the complex of 1-homogeneous cochains

$$\mathrm{CE}^k_{\mathrm{VB}}(\mathfrak{v}) := \mathrm{CE}^k_{\text{1-hom}}(\mathfrak{v}).$$

The restriction of the van Est map to 1-homogeneous cochains as on page 307 provides a map $\mathrm{VE}_{\text{1-hom}} : C^\bullet_{\text{1-hom}}(\mathcal{V}) \to \mathrm{CE}^\bullet_{\mathrm{VB}}(\mathfrak{v})$. Its restriction to the subcomplex $C^\bullet_{\mathrm{VB}}(\mathcal{V}) \subset C^\bullet_{\text{1-hom}}(\mathcal{V})$ will be denoted by

$$\mathrm{VE}_{\mathrm{VB}} : C^\bullet_{\mathrm{VB}}(\mathcal{V}) \to \mathrm{CE}^\bullet_{\mathrm{VB}}(\mathfrak{v}).$$

Corollary 3.2. *With the notations above, the van Est map*

$$\mathrm{VE}_{\mathrm{VB}} : H^\bullet(C_{\mathrm{VB}}(\mathcal{V})) \to H^\bullet(\mathrm{CE}_{\mathrm{VB}}(\mathfrak{v}))$$

is a right-module homomorphism over $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. *Moreover, if* $\mathcal{G}$ *is source* p_0*-connected, then* $\mathrm{VE}_{\mathrm{VB}}$ *is an isomorphism in degree* p *for all* $p \le p_0$ *and it is injective for* $p = p_0 + 1$.

Cohomological vanishing for proper groupoids. The VB-groupoid cohomology can be shown to be trivial in several cases as shown by the following proposition.

Proposition 3.3. *When* $\mathcal{G}$ *is a proper groupoid or, more generally, admits a Haar system* $d\mu$ *together with a cutoff function* $c \in C^\infty(M)$ (see, e.g., [Arias Abad and Crainic 2013] and the proof below), *then*

$$H^p(C^\bullet_{\mathrm{VB}}(\mathcal{V})) = 0, \quad p \ge 2.$$

Proof. The idea is to define a map $C^p_{\mathrm{VB}}(\mathcal{V}) \ni \phi \mapsto \kappa(\phi) \in C^{p-1}_{\mathrm{VB}}(\mathcal{V})$ for $p \ge 2$ by the formula

$$\kappa(\phi)(\xi_1, \ldots, \xi_{p-1}) = \int_{\mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))} \phi\big(\xi_1, \ldots, \xi_{p-1}, \sigma(h, \mathsf{s}_{\mathcal{V}}(\xi_{p-1}))\big) c(s(h))\, d\mu(h),$$

where $g_i = \pi(\xi_i) \in \mathcal{G}$, $i = 1, \ldots, p-1$, as before and $\sigma : \mathsf{t}^*E \to \mathcal{V}$ is any linear splitting of the epimorphism $\mathsf{t}_{\mathcal{V}} : \mathcal{V} \to \mathsf{t}^*E$. Notice that the right-hand side in the formula above is independent of the choice of σ since ϕ only depends on $(g_1, \ldots, g_{p-1}, h)$ and ξ_1. The key point is that, for $\delta\phi = 0$, $\phi \in C^p_{\mathrm{VB}}(\mathcal{V})$, $p \ge 2$, we have $\delta\kappa(\phi) = (-1)^p\phi$, hence leading to the above cohomological vanishing. This

statement can be checked by direct computation: let us write $\xi_{p+1}(h) = \sigma(h, \mathsf{s}_{\mathcal{V}}(\xi_p))$ for $h \in \mathsf{t}^{-1}(\mathsf{s}(g_p))$ and $\eta_p(k) = \sigma(k, \mathsf{s}_{\mathcal{V}}(\xi_{p-1}))$ for $k \in \mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))$. Then

$$
\begin{aligned}
&\delta\kappa(\phi)(\xi_1, \dots, \xi_p)\\
&= \int_{\mathsf{t}^{-1}(\mathsf{s}(g_p))} \Big[\phi(\xi_2, \dots, \xi_p, \xi_{p+1}(h))\\
&\qquad\qquad + \sum_{i=1}^{p-1} (-1)^i \phi(\xi_1, \dots, \xi_i \xi_{i+1}, \dots, \xi_p, \xi_{p+1}(h)) \Big] c(s(h))\, d\mu(h)\\
&\quad + (-1)^p \int_{\mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))} \phi(\xi_1, \dots, \xi_{p-1}, \eta_p(k))\, c(s(k))\, d\mu(k)\\
&= (-1)^p \int_{\mathsf{t}^{-1}(\mathsf{s}(g_p))} \big[-\phi(\xi_1, \dots, \xi_{p-1}, \xi_p \xi_{p+1}(h)) + \phi(\xi_1, \dots, \xi_p) \big] c(s(h))\, d\mu(h)\\
&\quad + (-1)^p \int_{\mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))} \phi(\xi_1, \dots, \xi_{p-1}, \eta_p(k))\, c(s(k))\, d\mu(k)\\
&= (-1)^p \phi(\xi_1, \dots, \xi_p).
\end{aligned}
$$

Above, the first equality follows from the definitions of δ and κ, the second equality follows by applying $\delta\phi = 0$ inside the square brackets and, finally, the third equality follows by the normalization condition $\int_{\mathsf{t}^{-1}(x)} c(s(h))\, d\mu(h) = 1$ and by the left invariance of the measure $\int_{\mathsf{t}^{-1}(s(g))} f(gh)\, d\mu(h) = \int_{\mathsf{t}^{-1}(\mathsf{t}(g))} f(k)\, d\mu(k)$ together with the independence of $\phi(\xi_1, \dots, \xi_p)$ on the ξ_j for $j > 1$, as was mentioned before. □

Let us now mention an application of the above general vanishing result, following [Crainic and Moerdijk 2008]. Given a Lie algebroid $\mathfrak{g} \to M$, there exists a complex $C^\bullet_{\mathrm{def}}(\mathfrak{g})$ controlling the deformations of $\mathfrak{g}$ and which is related to VB-cohomology as follows. Consider the induced linear Poisson structure on $\mathfrak{g}^*$, $\pi \in \Gamma(\Lambda^2 T\mathfrak{g}^*)$. The cotangent Lie algebroid $T^*\mathfrak{g} \to \mathfrak{g}^*$ has the property that its Chevalley–Eilenberg complex $(\mathrm{CE}(T^*\mathfrak{g}), d)$ is isomorphic to the Poisson complex $(\mathfrak{X}(\mathfrak{g}^*), [\pi, \cdot])$; see [Mackenzie and Xu 1994]. Under this isomorphism, the subcomplex $\mathrm{CE}^\bullet_{\mathrm{VB}}(T^*\mathfrak{g}) \subset \mathrm{CE}^\bullet(T^*\mathfrak{g})$ corresponds to the so-called *linear Poisson complex* $\mathfrak{X}_{\mathrm{lin}}(\mathfrak{g}^*)$ of $\mathfrak{g}^*$. On the other hand, Proposition 7 in [Crainic and Moerdijk 2008] shows that $\mathfrak{X}^\bullet_{\mathrm{lin}}(\mathfrak{g}^*) \cong C^\bullet_{\mathrm{def}}(\mathfrak{g})$, so that

$$\mathrm{CE}^\bullet_{\mathrm{VB}}(T^*\mathfrak{g}) \cong \mathfrak{X}^\bullet_{\mathrm{lin}}(\mathfrak{g}^*) \cong C^\bullet_{\mathrm{def}}(\mathfrak{g}).$$

On the groupoid side, for a Lie groupoid $\mathcal{G} \rightrightarrows M$, the complex $C_{\mathrm{VB}}(T^*\mathcal{G})$ was shown in [Crainic et al. 2015a] to be isomorphic to the complex $C_{\mathrm{def}}(\mathcal{G})$ controlling deformations of the Lie groupoid structure.

In this context, Corollary 3.2 recovers a result from [Crainic et al. 2015a]: the map

$$\mathrm{VE}_{\mathrm{def}} : H^\bullet_{\mathrm{def}}(\mathcal{G}) \to H^\bullet_{\mathrm{def}}(\mathfrak{g})$$

defines a (graded) module homomorphism covering $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$ which induces isomorphisms in degrees $p \leq p_0$ and a monomorphism in degree $p = p_0 + 1$ when $\mathcal{G}$ is source p_0-connected.

By combining this result with our general vanishing criteria (Proposition 3.3 above), we further obtain an independent proof of the (cohomological) rigidity conjecture of [Crainic and Moerdijk 2008]: if $\mathcal{G}$ is proper and source 2-connected, then $H^2_{\mathrm{def}}(\mathfrak{g}) = 0$. Note that the map $\mathrm{VE}_{\mathrm{def}}$ is the "lin version" of the van Est map which was assumed to exist by Crainic and Moerdijk [2008] as a step towards proving their conjecture.

Remark 3.4. The conjecture was originally proved in [Arias Abad and Schätz 2011] using a van Est result for representations up to homotopy. In particular, they used a vanishing result for cohomologies with coefficients in representations up to homotopy established in [Arias Abad and Crainic 2013]. Our vanishing result should be considered as a geometric counterpart to theirs in the 2-term case (see below).

Splittings and representations up to homotopy. VB-groupoids and VB-algebroids can be (noncanonically) *split* into the base Lie groupoid and Lie algebroid data and representation-like information on the fibers (recall Examples 2.5 and 2.9). It turns out that the correct notion encoding this split data is that of (2-term) *representations up to homotopy* [Arias Abad and Crainic 2012; 2013; Gracia-Saz and Mehta 2010; 2011], which we now recall.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g} \to M$ and $\mathcal{E} = C[1] \oplus E$ a graded vector bundle over M with C in degree -1 and E in degree 0. The associated space of $\mathcal{E}$-valued (normalized) p-cochains is defined as

$$C(\mathcal{G}, \mathcal{E})^p := \{\mu := (\mu_E, \mu_C) \in \Gamma(B_p\mathcal{G}; \mathrm{t}_p^* E) \oplus \Gamma(B_{p+1}\mathcal{G}; \mathrm{t}_{p+1}^* C) \mid s_i^* \mu_E = 0,\ s_i^* \mu_C = 0\},$$

where $s_i : B_\bullet\mathcal{G} \to B_{\bullet+1}\mathcal{G}$ is the i-th degeneracy map. There is a (right) $C^\bullet(\mathcal{G})$-module structure on $C(\mathcal{G}, \mathcal{E})^\bullet$ defined by $\mu \star f = (\mu_E \star f, \mu_C \star f)$, where each component is given by formula (2-6). A representation up to homotopy of $\mathcal{G}$ on $\mathcal{E}$ is an $\mathbb{R}$-linear map $\mathcal{D}_{\mathcal{G}} : C(\mathcal{G}, \mathcal{E})^\bullet \to C(\mathcal{G}, \mathcal{E})^{\bullet+1}$ satisfying $\mathcal{D}^2_{\mathcal{G}} = 0$ and

$$\mathcal{D}_{\mathcal{G}}(\mu \star f) = \mathcal{D}_{\mathcal{G}}(\mu) \star f + (-1)^p \mu \star (\delta f), \quad \mu \in C(\mathcal{G}, \mathcal{E})^p,\ f \in C^{p'}(\mathcal{G}).$$

The resulting cohomology is denoted by $H(\mathcal{G}, \mathcal{E})$. Note that $\star$ defines a right $H(\mathcal{G})$-module structure on $H(\mathcal{G}, \mathcal{E})$.

A representation up to homotopy on $\mathcal{E}$ can be alternatively given by quasiactions Δ^E and Δ^C of $\mathcal{G}$ on E and C, respectively, a bundle map $\partial : C \to E$ and a smooth correspondence which, for each $(g_1, g_2) \in B_2\mathcal{G}$, gives a linear map $\Omega_{(g_1,g_2)} : E|_{\mathrm{s}(g_2)} \to C|_{\mathrm{t}(g_1)}$ satisfying certain structural equations (see [Arias Abad and Crainic 2013; Gracia-Saz and Mehta 2011]). Moreover, in analogy with the case of an

ordinary representation (cf. Example 2.5), a representation up to homotopy of $\mathcal{G}$ on $\mathcal{E}$ endows $\mathcal{V} = \mathsf{s}^*E^* \oplus_{\mathcal{G}} \mathsf{t}^*C^* \rightrightarrows C^*$ with a VB-groupoid structure [Gracia-Saz and Mehta 2011]. The structure maps are given by

$$\begin{gathered}\mathsf{s}_{\mathcal{V}}(\xi, g, \eta) = (\Delta_g^C)^*\xi - \partial^*\eta, \ \ \mathsf{t}_{\mathcal{V}}(\xi, g, \eta) = \xi, \quad \xi \in C^*|_{\mathsf{t}(g)}, \ \eta \in E^*|_{\mathsf{s}(g)}, \\ (\xi_1, g_1, \eta_1) \cdot (\xi_2, g_2, \eta_2) = \big(\xi_1, \ g_1 g_2, \ \Omega^*_{(g_1,g_2)}\xi_1 + (\Delta^E_{g_2})^*\eta_1 + \eta_2\big)\end{gathered} \tag{3-2}$$

for compatible arrows and $\mathbf{1}_{\mathcal{V}}(\xi) = (\xi, 1_m, 0)$ for $\xi \in C^*|_m$. Finally, in [Gracia-Saz and Mehta 2011] the authors show that every VB-groupoid can be presented (noncanonically) in this form, thus establishing a correspondence between VB-groupoids and 2-term representations up to homotopy of G.

The above correspondence between VB-groupoid structures and representations up to homotopy can be understood from the following relation between the cochain complex associated to $\mathcal{E}$ and that of 1-homogeneous cochains on $\mathcal{V}$. Consider the map $\Psi : C(\mathcal{G}, \mathcal{E})^p \to C^\infty_{\text{1-hom}}(B_{p+1}\mathcal{V})$ defined by

$$\begin{aligned}&\Psi(\mu)((\xi_1, g_1, \eta_1), \dots, (\xi_{p+1}, g_{p+1}, \eta_{p+1})) \\ &\qquad = \langle \eta_1, \mu_E(g_2, \dots, g_{p+1})\rangle + \langle \xi_1, \mu_C(g_1, \dots, g_{p+1})\rangle.\end{aligned} \tag{3-3}$$

In [Gracia-Saz and Mehta 2011] (see Theorem 5.6), it is proven that $\Psi : C(\mathcal{G}, \mathcal{E})^\bullet \to C^{\bullet+1}_{\text{1-hom}}(\mathcal{V})$ is a monomorphism of graded $C(\mathcal{G})$-modules satisfying

$$\Psi \circ (-\mathcal{D}_{\mathcal{G}}) = \delta \circ \Psi$$

whose image coincides with the VB-groupoid cochain complex $C^\bullet_{\text{VB}}(\mathcal{V}) \subset C^\bullet_{\text{1-hom}}(\mathcal{V})$ (shifted by one, hence the minus sign in the equation above). We then obtain the next lemma as a direct consequence of Lemma 3.1.

Lemma 3.5. *The map $\Psi : H^\bullet(\mathcal{G}, \mathcal{E}) \to H^{\bullet+1}_{\text{1-hom}}(\mathcal{V})$ induced in cohomology is an isomorphism of right $H^\bullet(\mathcal{G})$-modules.*

Infinitesimal counterpart. Let $\mathfrak{g}$ be a Lie algebroid and $\mathcal{E}$ be as before, and consider

$$\Omega(\mathfrak{g}, \mathcal{E})^p = \Gamma(\Lambda^p\mathfrak{g}^* \otimes E) \oplus \Gamma(\Lambda^{p+1}\mathfrak{g}^* \otimes C).$$

The space $\Omega(\mathfrak{g}, \mathcal{E})$ is a right $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$-module with multiplication defined by wedge product on the right on the $\Lambda^\bullet\mathfrak{g}^*$ factor. A representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$ is an $\mathbb{R}$-linear map $\mathcal{D}_{\mathfrak{g}} : \Omega(\mathfrak{g}, \mathcal{E})^\bullet \to \Omega(\mathfrak{g}, \mathcal{E})^{\bullet+1}$ satisfying $\mathcal{D}^2_{\mathfrak{g}} = 0$ and

$$\mathcal{D}_{\mathfrak{g}}(\omega \wedge \beta) = \mathcal{D}_{\mathfrak{g}}(\omega) \wedge \beta + (-1)^p \omega \wedge d\beta, \quad \omega \in \Omega(\mathfrak{g}, \mathcal{E})^p, \ \beta \in \Gamma(\Lambda\mathfrak{g}^*).$$

We denote the cohomology of $(\Omega(\mathfrak{g}, \mathcal{E}), \mathcal{D}_{\mathfrak{g}})$ by $H(\mathfrak{g}, \mathcal{E})$.

As in the VB-groupoid case, VB-algebroid structures on $\mathfrak{v} = C^* \times_M \mathfrak{g} \times_M E^* \to C^*$ are in one-to-one correspondence with representations up to homotopy of $\mathfrak{g}$ on

$\mathcal{E} = C[1] \oplus E$ (see [Gracia-Saz and Mehta 2010]). We recall here how this correspondence can be seen from the cohomological perspective. The space of sections $\Gamma(C^*, \mathfrak{v})$ is generated, as a $C^\infty(C^*)$-module, by sections:

$$\chi_u(\xi) = (\xi, u(m), 0), \quad \Upsilon_\eta(\xi) = (\xi, 0, \eta(m))$$

for $\xi \in C^*|_m$, $u \in \Gamma(\mathfrak{g})$, $\eta \in \Gamma(E^*)$. Define a map

$$\mathrm{ev} : \mathrm{CE}^{p+1}_{1\text{-hom}}(\mathfrak{v}) \to \Omega(\mathfrak{g}, \mathcal{E})^p, \quad \mathrm{ev}(\alpha) = (\hat{\alpha}_E, \hat{\alpha}_C), \tag{3-4}$$

where $\hat{\alpha}_E \in \Gamma(\Lambda^p \mathfrak{g}^* \otimes E)$ and $\hat{\alpha}_C \in \Gamma(\Lambda^{p+1} \mathfrak{g}^* \otimes C)$, by

$$\langle \hat{\alpha}_E(u_1, \dots, u_p), \eta \rangle = \alpha(\Upsilon_\eta, \chi_{u_1}, \dots, \chi_{u_p}) \in C^\infty_{0\text{-hom}}(C^*) \cong C^\infty(M),$$
$$\hat{\alpha}_C(u_1, \dots, u_{p+1}) = \alpha(\chi_{u_1}, \dots, \chi_{u_{p+1}}) \in C^\infty_{1\text{-hom}}(C^*) \cong \Gamma(C)$$

for $u_1, \dots, u_{p+1} \in \Gamma(\mathfrak{g})$, $\eta \in \Gamma(E^*)$.

Lemma 3.6. *Under the identification* $\Gamma(\Lambda^\bullet \mathfrak{g}^*) \cong \mathrm{CE}_{hom,0}(\mathfrak{v})$, *the map* ev *is a* (*right*) $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$*-module isomorphism.*

Proof. Let $\{\xi^k\}_{k=1}^{\mathrm{rank}(C^*)}$, $\{\gamma^j\}_{j=1}^{\mathrm{rank}(\mathfrak{g}^*)}$ and $\{e_i\}_{i=1}^{\mathrm{rank}(E)}$ be local frames for C^*, $\mathfrak{g}^*$ and E respectively. We identify e_i (resp. γ^j) with the corresponding section of $\mathfrak{v}^*$: $C^*|_m \ni \xi \mapsto (\xi, 0, e_i(m))$ (resp. $\xi \mapsto (\xi, \gamma^j(m), 0)$). Locally, any element $\alpha \in \mathrm{CE}^{p+1}_{1\text{-hom}}(\mathfrak{v})$ is written as

$$\alpha(m, \xi) = a_k A^k_{j_1 \dots j_{p+1}}(m) \gamma^{j_1} \wedge \cdots \wedge \gamma^{j_{p+1}} + B^i_{j_1 \dots j_p}(m) e_i \wedge \gamma^{j_1} \wedge \cdots \wedge \gamma^{j_p},$$

where $\xi = a_k \xi^k(m)$. From the definition, one sees that

$$A^k_{j_1 \dots j_{p+1}}(m) = \langle \hat{\alpha}_C(u_{j_1}, \dots, u_{j_{p+1}}), \xi^k(m) \rangle,$$
$$B^i_{j_1 \dots j_p}(m) = \langle \hat{\alpha}_E(u_{j_1, \dots, u_{j_p}}), \eta^i(m) \rangle,$$

where $\{u_j\}$, $\{\eta^i\}$ are local frames for $\mathfrak{g}$ and E^* dual to $\{\gamma^j\}$, $\{e_i\}$, respectively. It is now straightforward to prove the statement. □

Hence, the operator $\mathcal{D}_\mathfrak{g}$ defined by $\mathcal{D}_\mathfrak{g} \circ \mathrm{ev} = \mathrm{ev} \circ (-d)$, where d is the Chevalley–Eilenberg differential of $\mathfrak{v}$, defines a representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$. (Note that ev shifts degree by minus one, hence the sign in the definition of $\mathcal{D}_\mathfrak{g}$.) It is shown in [Gracia-Saz and Mehta 2010] that, moreover, every VB-algebroid can be split as $\mathfrak{v} \simeq C^{\infty *} \times_M \mathfrak{g} \times_M E^* \to C^*$, thus establishing a correspondence between VB-algebroids and 2-term representations up to homotopy of $\mathfrak{g}$.

Given a representation up to homotopy $\mathcal{D}_\mathcal{G} : C(\mathcal{G}, \mathcal{E}) \to C(\mathcal{G}, \mathcal{E})$ of $\mathcal{G}$ on $\mathcal{E}$, the VB-groupoid $\mathcal{V} \rightrightarrows C^*$ defined by (3-2), seen as a Lie groupoid over C^*, has a Lie algebroid whose underlying bundle is precisely $\mathfrak{v} = C^* \times_M \mathfrak{g} \times_M E^* \to C^*$. In this case, the above construction of $\mathcal{D}_\mathfrak{g}$ can understood as the differentiation of the representation $\mathcal{D}_\mathcal{G}$, namely, $\mathcal{D}_\mathfrak{g} = \mathrm{Lie}(\mathcal{D}_\mathcal{G})$. (See also [Arias Abad and Schätz 2011].)

Remark 3.7. A representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$ can be alternatively described by a map $\partial : C \to E$, $\mathfrak{g}$-connections ∇^E and ∇^C on E and C, respectively, and a curvature term $R \in \Gamma(\Lambda^2\mathfrak{g}^* \otimes \mathrm{Hom}(E, C))$ satisfying some compatibility equations (see [Arias Abad and Crainic 2012; Gracia-Saz and Mehta 2010]). We refer to [Brahic et al. 2014] for the formulas of the operators $(\partial, \nabla^E, \nabla^C, R)$ corresponding to $\mathrm{Lie}(\mathcal{D}_\mathcal{G})$ in terms of the data defining $\mathcal{D}_\mathcal{G}$.

Van Est theorem for representations up to homotopy. Define $\mathrm{VE}_{\mathrm{rep}} : C(\mathcal{G}, \mathcal{E})^p \to \Omega(\mathfrak{g}, \mathcal{E})^p$ by $\mathrm{VE}_{\mathrm{rep}} := \mathrm{ev} \circ \mathrm{VE}_{1\text{-hom}} \circ \Psi$. Diagrammatically,

$$\begin{array}{ccc} C(G,\mathcal{E})^k & \xrightarrow{\Psi} & C^\infty_{1\text{-hom}}(\mathcal{V}^{(k+1)}) \\ \Big\downarrow{\scriptstyle \mathrm{VE}_{\mathrm{rep}}} & & \Big\downarrow{\scriptstyle \mathrm{VE}_{1\text{-hom}}} \\ \Omega(\mathfrak{g},\mathcal{E})^k & \xleftarrow{\mathrm{ev}} & \mathrm{CE}^{k+1}_{1\text{-hom}}(\mathfrak{v}) \end{array} \tag{3-5}$$

It is clear from the previous discussion that $\mathrm{VE}_{\mathrm{rep}}$ induces a map in cohomology.

Theorem 3.8. *The van Est map* $\mathrm{VE}_{\mathrm{rep}} : H^\bullet(\mathcal{G}, \mathcal{E}) \to H^\bullet(\mathfrak{g}, \mathcal{E})$ *is a right module homomorphism over* $\mathrm{VE}_\mathcal{G} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. *Moreover, if* $\mathcal{G}$ *is source* p_0*-connected, then the induced map in cohomology* $\mathrm{VE}_{\mathrm{rep}} : H^p(\mathcal{G}, \mathcal{E}) \to H^p(\mathfrak{g}, \mathcal{E})$ *is an isomorphism for* $-1 \le p \le p_0 - 1$ *and it is injective for* $p = p_0$.

Proof. This is a straightforward consequence of Theorem 2.14 and Lemmas 3.5 and 3.6. Notice the shift in grading for which one has isomorphisms. This arises because one has to apply Theorem 2.14 to $C^\infty_{1\text{-hom}}(B_{k+1}\mathcal{V}) \to \mathrm{CE}^{k+1}_{1\text{-hom}}(\mathfrak{v})$ in order to analyze $C(\mathcal{G}, \mathcal{E})^k \to \Omega^k(\mathfrak{g}, \mathcal{E})$. □

The fact that the above cohomology groups are isomorphic was also proven in [Arias Abad and Schätz 2011] using different techniques (in the more general setting of representations on arbitrarily graded vector bundles). Notice that, from our perspective, it just arises as a refinement of the usual van Est map for $\mathcal{V}$ for 1-homogeneous cochains.

Remark 3.9 (formulas for $\mathrm{VE}_{\mathrm{rep}}$). For $u \in \Gamma(\mathfrak{g})$, define the map $R_u : C^p(\mathcal{G}, \mathcal{E}) \to C^{p-1}(\mathcal{G}, \mathcal{E})$ by

$$(R_u\mu_C)(g_1, \ldots, g_p) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Delta^C_{\phi^u_\epsilon(\mathfrak{t}(g_1))^{-1}} \mu_C(\phi^u_\epsilon(\mathfrak{t}(g_1)), g_1, \ldots, g_p),$$

where $\phi^u_\epsilon : M \to \mathcal{G}$ is the flow of the right-invariant vector field $\vec{u}$ and the definition $R_u\mu_E$ is analogous. Note that our conventions are different from those in [Arias Abad and Schätz 2011]. One can now check the identities

$$R_{\chi_u}\Psi(\mu) = \Psi(R_u\mu_C, 0), \qquad R_{\Upsilon_\eta}\Psi(\mu) = q^*\langle \mu_E, \eta\rangle,$$
$$R_{\chi_v}R_{\Upsilon_\eta}\Psi(\mu) = q^*\langle R_v\mu_E, \eta\rangle, \quad R_{\Upsilon_\eta}R_{\chi_v}\Psi(\mu) = 0,$$

where $q : B_\bullet\mathcal{V} \to B_\bullet\mathcal{G}$ is the projection map. Using these identities, it is now straightforward to check that

$$\mathrm{VE}_{\mathrm{rep}}(\mu) = (\hat{\mu}_E, \hat{\mu}_C) \in \Gamma(\Lambda^p\mathfrak{g}^* \otimes E) \oplus \Gamma(\Lambda^{p+1}\mathfrak{g}^* \otimes C)$$

is given by

$$\hat{\mu}_E(u_1, \dots, u_p) = (-1)^p \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) R_{u_{\sigma(1)}} \dots R_{u_{\sigma(p)}} \mu_0,$$

$$\hat{\mu}_C(u_1, \dots, u_{p+1}) = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}(\sigma) R_{u_{\sigma(1)}} \dots R_{u_{\sigma(p+1)}} \mu_C.$$

4. Differential forms with values in a representation

In this section, we study differential forms on a Lie groupoid $\mathcal{G}$ with values in a representation $C \to M$. These objects were introduced in [Crainic et al. 2015b] together with their infinitesimal counterparts, the Spencer operators. We here provide a van Est theorem for them as an application of our main result. The key idea is to reinterpret forms as homogeneous functions.

Van Est theorem for differential forms with coefficients. We start this section by formally defining the ingredients entering the van Est theorem for forms with coefficients (Theorem 4.4 below) without any reference to the VB-groupoids and algebroids. Later, we show how VB-groupoids and VB-algebroids provide a useful framework for interpreting many of the definitions and for giving a proof of Theorem 4.4.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $C \to M$ be a representation of $\mathcal{G}$ and consider the map $\mathtt{t}_p : B_p\mathcal{G} \to M$, $\mathtt{t}_p(g_1, \dots, g_p) = \mathtt{t}(g_1)$. When no confusion arises, we omit the reference to p and simply denote $\mathtt{t}_p$ by $\mathtt{t}$. The space of q-differential forms on the nerve of $\mathcal{G}$ with coefficients in C is $\Omega^q(B_\bullet\mathcal{G}, \mathtt{t}^*C)$. It carries a differential $\delta : \Omega^q(B_{p-1}\mathcal{G}, \mathtt{t}^*C) \to \Omega^q(B_p\mathcal{G}, \mathtt{t}^*C)$ defined by

$$\delta\omega|_{(g_1,\dots,g_p)} = \Delta_{g_1} \circ \partial_0^*\omega + \sum_{i=1}^{p} (-1)^i \partial_i^*\omega \quad \text{for } p \geq 2,$$

$$\delta\omega|_g = \Delta_g \circ \mathtt{s}^*\omega - \mathtt{t}^*\omega \quad \text{for } p = 1.$$

It is straightforward to check that $\delta^2 = 0$.

Note that, for $\omega \in \Omega^q(\mathcal{G}, \mathtt{t}^*C)$,

$$\delta\omega|_{(g_1,g_2)} = \Delta_{g_1} \circ \mathrm{pr}_2^*\omega - m^*\omega + \mathrm{pr}_1^*\omega,$$

where $\mathrm{pr}_i(g_1, g_2) = g_i$ for $i = 1, 2$. In this case, a form $\omega \in \Omega^q(\mathcal{G}, \mathtt{t}^*C)$ which satisfies $\delta\omega = 0$ is called *multiplicative* (see [Crainic et al. 2015b]). Note that

$\Omega^q(B_\bullet\mathcal{G}, \mathrm{t}^*C)$ is a right dg-module for $C^\bullet(\mathcal{G})$ with the module structure defined as usual by

$$(\omega\star f)|_{(g_1,\dots,g_{p+p'})} = \omega|_{(g_1,\dots,g_p)} f(g_{p+1},\dots,g_{p+p'}),\ \ \omega\in\Omega^q(B_p\mathcal{G},\mathrm{t}^*C),\ f\in C^{p'}(\mathcal{G}).$$

Remark 4.1. In the case of trivial coefficients (i.e., when C is the trivial line bundle), the de Rham differential turns $\Omega^q(B_p\mathcal{G}, t^*C) = \Omega^q(B_p\mathcal{G})$ into a double complex known as the Bott–Shulman double complex associated to $\mathcal{G}$ (see [Arias Abad and Crainic 2011]). In the remainder of this paper, we focus on the cohomology of δ alone and leave the investigation of compatible double complex structures (corresponding to "multiplicative linear flat connections") for future work.

Let $\mathfrak{g}\to M$ be the Lie algebroid of $\mathcal{G}$. Similarly to [Arias Abad and Crainic 2011], we define the Weil complex $W^{p,q}(\mathfrak{g}, C)$ to be the space of sequences $c = (c_0, c_1, \dots)$, where each

$$c_k : \underbrace{\Gamma(\mathfrak{g})\times\cdots\times\Gamma(\mathfrak{g})}_{p-k \text{ times}} \to \Omega^{q-k}(M, S^k\mathfrak{g}^*\otimes C)$$

is an $\mathbb{R}$-linear skew-symmetric map whose failure at being $C^\infty(M)$-linear is controlled by

$$\begin{aligned}(4\text{-}1)\quad & c_k(fu_1,\dots,u_{p-k}|\cdot)\\ &= fc_k(u_1,\dots,u_{p-k}|\cdot)+df\wedge c_{k+1}(u_2,\dots,u_{p-k}|u_1,\cdot)\quad \forall f\in C^\infty(M).\end{aligned}$$

For each q, the complex $W^{\bullet,q}(\mathfrak{g}, C)$ carries a differential $\mathrm{d}_W : W^{p,q}(\mathfrak{g}, C)\to W^{p+1,q}(\mathfrak{g}, C)$, which we now define. First, note that $\Omega^i(M, S^j\mathfrak{g}^*\otimes C)$ is a module for the Lie algebra $\Gamma(\mathfrak{g})$. Indeed, for $\alpha\in\Omega^i(M)$ and $P\in\Gamma(S^j\mathfrak{g}^*\otimes C)$,

$$u\cdot(\alpha\otimes P) = (\mathcal{L}_{\rho(u)}\alpha)\otimes P + \alpha\otimes(u\cdot P),\quad u\in\Gamma(\mathfrak{g}),$$

defines an action of $\Gamma(\mathfrak{g})$ on $\Omega^i(M, S^j\mathfrak{g}^*\otimes C)$, where

$$(u\cdot P)(v_1,\dots,v_k) = \nabla_u P(v_1,\dots,v_k) - \sum_{i=1}^{k} P(v_1,\dots,[u,v_i],\dots,v_k),$$

and $\nabla : \Gamma(\mathfrak{g})\times\Gamma(C)\to\Gamma(C)$ is the $\mathfrak{g}$-connection giving the representation C. Now, d_W is defined by

$$\begin{aligned}(4\text{-}2)\quad & \mathrm{d}_W(c)_k(u_1,\dots,u_{p-k+1}|v_1,\dots,v_k)\\ &= (-1)^k\Bigg(d_{\mathrm{CE}}(c_k)(u_1,\dots,u_{p-k+1}|v_1,\dots,v_k)\\ &\qquad - \sum_{j=1}^{k} i_{\rho(v_j)}c_{k-1}(u_1,\dots,u_{p-k+1}|v_1,\dots,\widehat{v_j},\dots,v_k)\Bigg),\end{aligned}$$

where d_{CE} is the Chevalley–Eilenberg differential on $C^\bullet(\Gamma(\mathfrak{g}), \Omega^{q-k}(M, S^k\mathfrak{g}^*\otimes C))$. There is a right $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$-module structure on $W^{\bullet,q}(\mathfrak{g}, C)$. It is defined, for $\beta \in \Gamma(\Lambda^{p'}\mathfrak{g}^*)$ and $c \in W^{p,q}(\mathfrak{g}, C)$, by

$$(c \wedge \beta)_k(u_1, \dots, u_{p+p'-k} | \cdot) = \sum_{\sigma \in S(p-k,p')} \operatorname{sgn}(\sigma) c_k(u_{\sigma(1)}, \dots, u_{\sigma(p-k)}) \beta(u_{\sigma(p-k+1)}, \dots, u_{\sigma(p+p'-k)}),$$

where $S(p-k, p')$ is the space of $(p-k, p')$-unshuffles.

Proposition 4.2. *$W^{\bullet,q}(\mathfrak{g}, C)$ is a right dg-module for $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$.*

This result will follow from an evaluation isomorphism similar to (3-4) (see Proposition 4.12 below) between $W^{\bullet,q}(\mathfrak{g}, C)$ and another right dg-module for $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$. It is important to remark that all the signs appearing in the above formula for d_W, as well as in formula (4-4) below, are natural consequences of a simple ordering convention in the definition of this evaluation isomorphism.

Remark 4.3. For $p = 0$ we have $W^{0,q}(\mathfrak{g}, C) = \Omega^q(M, C)$. In this case, for $c \in W^{0,q}(\mathfrak{g}, C)$ we have $\mathrm{d}_W(c) = (\mathrm{d}_W(c)_0, \mathrm{d}_W(c)_1)$, where $\mathrm{d}_W(c)_0 : \Gamma(\mathfrak{g}) \to \Omega^q(M, C)$ and $\mathrm{d}_W(c)_1 \in \Omega^{q-1}(M, \mathfrak{g}^* \otimes C)$ are given by

$$\mathrm{d}_W(c)_0(u) = u \cdot c \quad \text{and} \quad \mathrm{d}_W(c)_1(v) = i_{\rho(v)}c.$$

For $W^{1,q}(\mathfrak{g}, C)$, its elements are $c = (c_0, c_1)$, where $c_0 : \Gamma(g) \to \Omega^q(M, C)$ and $c_1 \in \Omega^{q-1}(M, \mathfrak{g}^* \otimes C) \cong \operatorname{Hom}(\mathfrak{g}, \Lambda^{q-1}T^*M \otimes C)$. In this case,

$$\begin{aligned}
\mathrm{d}_W(c)_0(u_1, u_2) &= u_1 \cdot c_0(u_2) - u_2 \cdot c_0(u_1) - c_0([u_1, u_2]),\\
\mathrm{d}_W(c)_1(u|v) &= i_{\rho(v)}c_0(u) - u \cdot c_1(v) + c_1([u, v]),\\
\mathrm{d}_W(c)_2(v_1, v_2) &= -i_{\rho(v_1)}c_1(v_2) - i_{\rho(v_2)}c_1(v_2).
\end{aligned}$$

Note that, in the case $p = 1$, the equation $\mathrm{d}(c) = 0$ is equivalent to (c_0, c_1) being a C-valued Spencer operator on $\mathfrak{g}$ [Crainic et al. 2015b] and, thus, in particular, to (c_0, c_1) being an infinitesimally multiplicative form [Arias Abad and Crainic 2011] when $C = \mathbb{R}$, with the trivial representation.

Van Est map. Given $u \in \Gamma(\mathfrak{g})$, let $\phi^u_\epsilon : \mathcal{G} \to \mathcal{G}$ be the flow of the right-invariant vector field $\vec{u}$. The flow of the corresponding vector field $B_p u \in \mathfrak{X}(B_p\mathcal{G})$ is given by

$$\psi^u_\epsilon(g_1, \dots, g_p) = (\phi^u_\epsilon(g_1), g_2, \dots, g_p).$$

Define operators $R_u : \Omega^q(B_p\mathcal{G}, \mathfrak{t}^*C) \to \Omega^q(B_{p-1}\mathcal{G}, \mathfrak{t}^*C)$ and $J_u : \Omega^q(B_p\mathcal{G}, \mathfrak{t}^*C) \to \Omega^{q-1}(B_{p-1}\mathcal{G}, \mathfrak{t}^*C)$ by

$$\begin{aligned}
R_u\omega|_{(g_1,\dots,g_{p-1})} &= s_0^*\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Delta_{\phi^u_\epsilon(\mathfrak{t}(g_1)))^{-1}} \circ {\psi^u_\epsilon}^*\omega\right),\\
J_u\omega &= s_0^* i_{B_p u}\omega.
\end{aligned} \tag{4-3}$$

The van Est map $\mathrm{VE}_\Omega : \Omega^q(B_p\mathcal{G}, \mathfrak{t}^*C) \to W^{p,q}(\mathfrak{g}, C)$, defined by $\mathrm{VE}_\Omega(\omega) = (c_0(\omega), c_1(\omega), \dots)$, has each $c_k(\omega)$ given by

$$c_k(\omega)(u_1, \dots, u_{p-k}|v_1, \dots, v_k) = (-1)^{k(k-1)/2} \sum_{\sigma \in S(p)} \mathrm{sgn}(\sigma)(-1)^{\epsilon(\sigma,k)} D_{\sigma(1)} \dots D_{\sigma(p)}\omega, \tag{4-4}$$

where

$$D_j = \begin{cases} J_{v_j} & \text{if } j \in \{1, \dots, k\}, \\ R_{u_{j-k}} & \text{if } j \in \{k+1, \dots, p\}, \end{cases} \tag{4-5}$$

and

$$\epsilon(\sigma, k) = \#\{(i, j) \in \{1, \dots, k\} \times \{1, \dots, k\} \mid i < j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

Theorem 4.4. VE_Ω *induces a map on cohomology* $\mathrm{VE}_\Omega : H^\bullet(\Omega^q(B_\bullet\mathcal{G}, \mathfrak{t}^*C)) \to H^\bullet(W^{\bullet,q}(\mathfrak{g}, C))$ *which is a right module homomorphism over* $\mathrm{VE}_\mathcal{G} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. *Moreover, if* $\mathcal{G}$ *is source* p_0*-connected, then*

$$VE_\Omega : H^p(\Omega^q(B_\bullet\mathcal{G}, \mathfrak{t}^*C)) \to H^p(W^{\bullet,q}(\mathfrak{g}, C))$$

is an isomorphism for $p \le p_0$ *and it is injective for* $p = p_0 + 1$, *for each fixed* q.

In the remainder of the paper, we prove Theorem 4.4 by showing how it can be framed as a van Est result for a class of VB-groupoids. Notice that the above theorem recovers Theorem 5.1 of [Arias Abad and Crainic 2011] (up to some sign conventions) when $C = M \times \mathbb{R}$ with the trivial representation. It is interesting that, even in this particular case, our proof is independent of the one given in that paper.

Forms as functions. The key idea in the proof of Theorem 4.4 is that differential forms can be seen as homogeneous functions on an appropriate space. In this subsection, we elaborate on this classical viewpoint.

Let $V_1, \dots, V_{q+1}$ be vector bundles over B and consider the fiber product $\prod_{j=1}^{q+1} V_j = V_1 \times_B \cdots \times_B V_{q+1}$ with the natural vector bundle structure over B (the Whitney sum $V_1 \oplus \cdots \oplus V_{q+1} \to B$).

Simple functions. For $i = 1, \dots, q+1$, let $0_i : \prod_{j \neq i} V_j \to \prod_j V_j$ be the inclusion which puts a zero in the i-th coordinate. Then a function $f \in C^\infty\big(\prod_j V_j\big)$ is said to be *simple* if

$$0_i^* f = 0 \quad \forall\, i = 1, \dots, q+1.$$

For a subset $I \subset \{1, \dots, q+1\}$, denote by $|I|$ its cardinality and by $0_I : \prod_{j \notin I} V_j \to \prod_j V_j$ the inclusion which puts a zero in the entries indicated by the elements of I.

Define $P_{(l)} : C^\infty\big(\prod_j V_j\big) \to C^\infty\big(\prod_j V_j\big)$, $l = -1, 0, 1, \dots, q$, by

$$\begin{aligned} P_{(-1)}(f) &= f, \\ P_{(l)}(f) &= P_{(l-1)}(f) - \sum_{|I|=q+1-l} 0_I^* P_{(l-1)}(f) \quad \text{for } l = 0, \dots, q. \end{aligned} \tag{4-6}$$

Each $P_{(l)}$, $l = 0, \dots, q$, is a projection onto the space of functions of $\prod_j V_j$ which vanishes whenever $q+1-l$ entries are zero. In particular, $P_{\mathrm{spl}} := P_{(q)}$ is a projection onto the space of simple functions.

Multilinearity and skew-symmetry. The map

$$\begin{aligned} \Gamma(B, V_1^* \otimes \cdots \otimes V_{q+1}^*) &\to C^\infty\big(\textstyle\prod_{j=1}^{q+1} V_j\big), \\ \mu_1 \otimes \cdots \otimes \mu_{q+1} &\mapsto (\ell_{\mu_1} \circ \mathrm{pr}_1) \cdots (\ell_{\mu_{q+1}} \circ \mathrm{pr}_{q+1}), \end{aligned}$$

is a monomorphism of $C^\infty(B)$-modules, where $\mathrm{pr}_i : \prod_{j=1}^{q+1} V_j \mapsto V_i$ is the projection onto the i-th summand. It follows from Taylor's theorem that its image is the space of simple $(q+1)$-homogeneous functions.

We are mainly interested in the case $V_1 = \cdots = V_q = V$ and $V_{q+1} = W^*$ and we denote the q-fold fiber product $V \times_B \cdots \times_B V$ by $\bigtimes_B^q V$. A function $f \in C^\infty\big(\bigtimes_B^q V \times_B W^*\big)$ is said to be *skew-symmetric* if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(q)}, \xi) = \mathrm{sgn}(\sigma) f(v_1, \dots, v_q, \xi) \quad \forall\, v_i \in V,\ \xi \in W^*,\ \sigma \in S_q.$$

The map $P_{\mathrm{sk}} : C^\infty\big(\bigtimes_B^q V \times_B W^*\big) \to C^\infty\big(\bigtimes_B^q V \times_B W^*\big)$, defined by

$$P_{\mathrm{sk}}(f) = \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma) f \circ \sigma, \tag{4-7}$$

is a projection onto the space of skew-symmetric functions, where S_q is the symmetric group and $\sigma : \bigtimes_B^q V \times_B W^* \to \bigtimes_B^q V \times_B W^*$ is the permutation of the first q entries belonging to V according to σ. Let us define

$$\begin{aligned} \mathfrak{F} : \Gamma(B, \Lambda^q V^* \otimes W) &\to C^\infty\big(\textstyle\bigtimes_B^q V \times W^*\big), \\ \omega = (\mu_1 \wedge \cdots \wedge \mu_q) \otimes \xi &\mapsto q!\, P_{\mathrm{sk}}\big((\ell_{\mu_1} \circ \mathrm{pr}_1) \cdots (\ell_{\mu_q} \circ \mathrm{pr}_q)(\ell_\xi \circ \mathrm{pr}_{q+1})\big). \end{aligned} \tag{4-8}$$

It is straightforward to check that $\mathfrak{F}$ is a monomorphism of $C^\infty(B)$-modules whose image is the space of simple, skew-symmetric $(q+1)$-homogeneous functions. We denote the image of $\mathfrak{F}$ by $C^\infty_{\mathrm{ext}}\big(\bigtimes_B^q V \times W^*\big)$. The projections P_{sk}, P_{spl} and $P_{q+1\text{-hom}}$ commute with each other, and so

$$P_{\mathrm{ext}} := P_{\mathrm{sk}} \circ P_{\mathrm{spl}} \circ P_{q+1\text{-hom}} : C^\infty\big(\textstyle\bigtimes_B^q V \times_B W^*\big) \to C^\infty_{\mathrm{ext}}\big(\bigtimes_B^q V \times_B W^*\big) \tag{4-9}$$

is a projection onto $C^\infty_{\mathrm{ext}}\big(\bigtimes_B^q V \times_B W^*\big)$.

Example 4.5. For $V = B \times \mathbb{R}^n$, let $\{\theta_1, \dots, \theta_m\}$ be a local frame for W and $\{e^1, \dots, e^n\}$ be a global frame for V^*. A point $p \in \bigtimes_B^q V \times_B W^*$, $q \le n$, has coordinates

$$p = (x, \underline{y_1}, \dots, \underline{y_q}, \xi_1, \dots, \xi_m), \quad x \in B,\ \underline{y_j} = (y_{1,j}, \dots, y_{n,j}) \in \mathbb{R}^n,\ \xi_l \in \mathbb{R}.$$

For a function $f \in C^\infty\big(\bigtimes_B^q V \times_B W^*\big)$, we have $P_{\mathrm{ext}} f = \frac{1}{q!}\mathfrak{F}(\omega_f)$, where $\omega_f \in \Gamma(B, \Lambda^q V^* \otimes W)$ is given by

$$\omega_f(p) = \sum_{1 \le k_1 < \cdots < k_q \le n} \sum_{i=1}^{m} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \frac{\partial^{q+1} f}{\partial y_{k_{\sigma(1)},1} \cdots \partial y_{k_{\sigma(q)},q}\, \partial \xi_i}(x, 0)\, e^{k_1} \wedge \cdots \wedge e^{k_q} \otimes \theta_i(x).$$

The VB-groupoid behind the curtains. We define here the VB-groupoid whose differentiable cochain complex contains the complex of differential forms with coefficients. Later on, we show how the Weil complex is embedded in the Chevalley–Eilenberg complex of its Lie algebroid.

Differential forms with coefficients. Let $T\mathcal{G} \rightrightarrows TM$ be the tangent groupoid, obtained by taking the derivative of all the structure maps defining $\mathcal{G}$. Let us introduce the VB-groupoid $\mathbb{G}_q \rightrightarrows \mathbb{M}_q$ defined by

$$\begin{array}{ccc} \mathbb{G}_q = \underbrace{T\mathcal{G} \times_{\mathcal{G}} \cdots \times_{\mathcal{G}} T\mathcal{G}}_{q \text{ times}} \times_{\mathcal{G}} \mathsf{t}^* C^* & \longrightarrow & \mathcal{G} \\ \Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow \\ \mathbb{M}_q = \underbrace{TM \times_M \cdots \times_M TM}_{q \text{ times}} \times_M C^* & \longrightarrow & M \end{array} \tag{4-10}$$

where the structure maps are defined[2] componentwise and $\mathsf{t}^* C^* \rightrightarrows C^*$ is the action groupoid corresponding to the right action of $\mathcal{G}$ (see Example 2.5) on C^* obtained by taking adjoints. We frequently omit the subscript q when no confusion arises. The q-fold fiber products on (4-10) are also denoted as $\bigtimes_{\mathcal{G}}^q T\mathcal{G} \times_{\mathcal{G}} \mathsf{t}^* C^*$ and $\bigtimes_M^q TM \times_M C^*$.

Lemma 4.6. *The space of p-composable arrows $B_p\mathbb{G}$ is isomorphic as a vector bundle over $B_p\mathcal{G}$ to the q-fold fiber product $TB_p\mathcal{G} \times_{B_p\mathcal{G}} \cdots \times_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{B_p\mathcal{G}} \mathsf{t}^* C^*$. More concisely,*

$$B_p\mathbb{G} = B_p\big(\bigtimes_{\mathcal{G}}^q T\mathcal{G} \times_{\mathcal{G}} \mathsf{t}^* C^*\big) \cong \big(\bigtimes_{B_p\mathcal{G}}^q TB_p\mathcal{G}\big) \times_{B_p\mathcal{G}} \mathsf{t}^* C^*. \tag{4-11}$$

[2]There is a more general fact playing a role here: Whitney sums of VB-groupoids yield VB-groupoids (see [Bursztyn and Cabrera 2012]).

The proof consists in simply defining the isomorphism

$$B_p\mathbb{G} \ni (\underline{U}^{(1)}, \dots, \underline{U}^{(p)}) \mapsto \big((U_1^{(1)}, \dots, U_1^{(p)}), \dots, (U_q^{(1)}, \dots, U_q^{(p)}), (g_1, \dots, g_p, \xi_1)\big),$$

where each $\underline{U}^{(i)} = (U_1^{(i)}, \dots, U_q^{(i)}, (g_i, \xi_i)) \in \mathbb{G}$.

One important consequence of the isomorphism (4-11) is that the space of differential forms $\Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$ can be identified with a subspace of $C^\infty(B_p\mathbb{G})$, which we denote by $C^\infty_{\mathrm{ext}}(B_p\mathbb{G})$. It is the image of the map (4-8):

$$\mathfrak{F} : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to C^\infty\Big(\Big(\bigtimes^q_{B_p\mathcal{G}} TB_p\mathcal{G}\Big) \times_{B_p\mathcal{G}} \mathrm{t}^*C^*\Big) \cong C^\infty(B_p\mathbb{G}). \tag{4-12}$$

In order to characterize $C^\infty_{\mathrm{ext}}(B_p\mathbb{G})$ more explicitly, note that, given a permutation $\sigma \in S_q$, the permutation map

$$\sigma_{\mathcal{G}} : \bigtimes^q_{\mathcal{G}} T\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^*C \to \bigtimes^q_{\mathcal{G}} T\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^*C$$

is a groupoid morphism and, under the isomorphism (4-11),

$$B_p\sigma_{\mathcal{G}} \cong \sigma_{B_p\mathcal{G}} \tag{4-13}$$

for the corresponding permutation map

$$\sigma_{B_p\mathcal{G}} : \bigtimes^q_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{B_p\mathcal{G}} \mathrm{t}^*C \to \bigtimes^q_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^*C.$$

Similarly, the zero maps $0^{\mathcal{G}}_i : \mathbb{G}_{q-1} \to \mathbb{G}_q$ $(i = 1, \dots, q)$ and $0^{\mathcal{G}}_{q+1} : \bigtimes^q_{\mathcal{G}} T\mathcal{G} \to \mathbb{G}_q$ are groupoid morphisms and

$$B_p 0^{\mathcal{G}}_i \cong 0^{B_p\mathcal{G}}_i \quad \forall\, i = 1, \dots, q+1. \tag{4-14}$$

Hence,

$$C^\infty_{\mathrm{ext}}(B_p\mathbb{G}) = \big\{ f \in C^\infty_{(q+1)\text{-hom}}(B_p\mathbb{G}) \mid (B_p\sigma_{\mathcal{G}})^* f = \mathrm{sgn}(\sigma)\, f,\ (B_p 0^{\mathcal{G}}_i)^* f = 0 \ \forall\, \sigma \in S_q,\ i = 1, \dots, q+1 \big\}.$$

Note that the projection (4-9) gives here, under the isomorphism (4-11), a projection $P_{\mathrm{ext},\mathcal{G}} : C^\infty(B_p\mathbb{G}) \to C^\infty_{\mathrm{ext}}(B_p\mathbb{G})$.

Proposition 4.7. *The projection $P_{\mathrm{ext},\mathcal{G}}$ satisfies*

$$P_{\mathrm{ext},\mathcal{G}} \circ \delta = \delta \circ P_{\mathrm{ext},\mathcal{G}}.$$

In particular, $C^\infty_{\mathrm{ext}}(B_\bullet\mathbb{G})$ is a subcomplex.

Proof. The result follows directly from (4-13), (4-14) and the fact that

$$(B_{p+1}\phi)^* \delta f = \delta (B_p\phi)^* f$$

for an arbitrary groupoid morphism $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ and $f \in C^p(\mathcal{H}_2)$. □

In the following, we denote by $C^\bullet_{\mathrm{ext}}(\mathbb{G})$ and $H^\bullet_{\mathrm{ext}}(\mathbb{G})$ the complex $(C^\infty_{\mathrm{ext}}(B_\bullet\mathbb{G}), \delta)$ and its cohomology, respectively.

Proposition 4.8. *The map $\mathfrak{F} : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to C^p_{\mathrm{ext}}(\mathbb{G})$ is a dg-module isomorphism.*

Proof. Let $\partial_i : B_{p+1}\mathcal{G} \to B_p\mathcal{G}$ and $\partial\!\!\!/_i : B_{p+1}\mathbb{G} \to B_p\mathbb{G}$, $i = 0, \dots, p+1$, be the face maps and let $s_j : B_{p-1}\mathcal{G} \to B_p\mathcal{G}$ and $\mathbb{s}_j : B_{p-1}\mathbb{G} \to B_p\mathbb{G}$, $j = 0, \dots, p-1$, be the degeneracy maps for $\mathcal{G}$ and $\mathbb{G}$, respectively. The result follows from the fact that

$$\partial\!\!\!/_0^*\mathfrak{F}_\omega = \mathfrak{F}_{g_1\cdot\partial_0^*\omega}, \quad \partial\!\!\!/_i^*\mathfrak{F}_\omega = \mathfrak{F}_{\partial_i^*\omega}, \quad \mathbb{s}_j^*\mathfrak{F}_\omega = \mathfrak{F}_{s_j^*\omega} \qquad \forall\, \omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C),$$

when restricted to the fiber over $(g_1, \dots, g_p) \in B_p\mathcal{G}$. □

Remark 4.9. The framework presented here can be used to define multiplicativity for differential forms on a Lie groupoid with values in a 2-term representation up to homotopy. This was done in [Egea 2016] by simply changing t^*C^* to $\mathcal{V} = \mathrm{s}^*E^* \oplus \mathrm{t}^*C^*$ with the VB-groupoid structure defined by (3-2).

Weil complex. The Lie algebroid $\mathbb{A}_q \to \mathbb{M}$ of the Lie groupoid (4-10) $\mathbb{G}_q \rightrightarrows \mathbb{M}_q$ is the q-fold fiber-product[3] $\bigtimes^q_{\mathfrak{g}} T\mathfrak{g} \times_{\mathfrak{g}} \pi^*C^* \to \bigtimes^q_M TM \times_M C^*$, where $\pi : \mathfrak{g} \to M$ denotes the projection map of the Lie algebroid of $\mathcal{G}$.

Definition 4.10. Let $\alpha \in \Gamma(\mathbb{M}, \Lambda^\bullet\mathbb{A}^*)$. We say that α is skew-symmetric with respect to $\mathbb{A} \to \mathfrak{g}$ if

$$\sigma_{\mathfrak{g}}^*\alpha = \mathrm{sgn}(\sigma)\alpha \quad \forall\, \sigma \in S_q, \tag{4-15}$$

where $\sigma_{\mathfrak{g}} : \mathbb{A}_q \to \mathbb{A}_q$ permutes the q-coordinates on $\bigtimes^q_{\mathfrak{g}} T\mathfrak{g}$ according to σ. Similarly, α is multilinear with respect to $\mathbb{A} \to \mathfrak{g}$ if

$$h_\lambda^{\mathfrak{g}\,*}\alpha = \lambda^{q+1}\alpha, \tag{4-16}$$

$$(0_i^{\mathfrak{g}})^*\alpha = 0 \quad \forall\, i = 1, \dots, q+1, \tag{4-17}$$

where $\mathfrak{h}^{\mathfrak{g}}_\lambda : \mathbb{A}_q \to \mathbb{A}_q$ is the homogeneous structure of the vector bundle $\mathbb{A}_q \to \mathfrak{g}$, and $0_i^{\mathfrak{g}} : \mathbb{A}_{q-1} \to \mathbb{A}_q$ and $0^{\mathfrak{g}}_{q+1} : \bigtimes^q_{\mathfrak{g}} T\mathfrak{g} \to \mathbb{A}_q$, $i = 1, \dots, q$, are the zero maps.

Let $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*_q)$ be the subspace of $\Gamma(\mathbb{M}, \Lambda^p\mathbb{A}^*_q)$ of skew-symmetric multilinear forms with respect to $\mathbb{A} \to \mathfrak{g}$. In particular, $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*_q)$ is a subset of $\Gamma_{(q+1)\text{-hom}}(\mathbb{M}, \Lambda^p\mathbb{A}_q)$. In the following, we frequently omit the reference to q on the Lie algebroid $\mathbb{A}_q$. There exists a projection $P_{\mathrm{ext},\mathfrak{g}} : \Gamma(\mathbb{M}, \Lambda^p\mathbb{A}) \to \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*)$ obtained exactly as (4-9) composing the projection $P^{\mathfrak{g},\,p}_{(q+1)\text{-hom}}$ (2-10) with the ones constructed from the zero maps $0_i^{\mathfrak{g}}$ and permutations $\sigma^{\mathfrak{g}}$ exactly as in (4-6) and (4-7), respectively.

[3] As with VB-groupoids, Whitney sums of VB-algebroids yield VB-algebroids. Moreover, Whitney sums are preserved by the Lie functor (see [Bursztyn and Cabrera 2012]).

Proposition 4.11. *The projection* $P_{\mathrm{ext},\mathfrak{g}}$ *satisfies*

$$P_{\mathrm{ext},\mathfrak{g}} \circ d = d \circ P_{\mathrm{ext},\mathfrak{g}}.$$

In particular, $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*)$ *is a subcomplex of* $\mathrm{CE}^\bullet(\mathbb{A})$.

Proof. The result follows from the fact that the maps $h^{\mathfrak{g}}_\lambda$, $0^{\mathfrak{g}}_i$, $\sigma_{\mathfrak{g}}$ are all Lie algebroid morphisms. In fact,

$$h^{\mathfrak{g}}_\lambda = \mathrm{Lie}(\mathfrak{h}^{\mathcal{G}}_\lambda), \quad 0^{\mathfrak{g}}_i = \mathrm{Lie}(0^{\mathcal{G}}_i), \quad \sigma_{\mathfrak{g}} = \mathrm{Lie}(\sigma_{\mathcal{G}}) \tag{4-18}$$

for the corresponding maps $\mathfrak{h}^{\mathcal{G}}_\lambda$, $0^{\mathcal{G}}_i$, $\sigma_{\mathcal{G}}$ on the Lie groupoid $\mathbb{G}$. □

In the following, we shall denote by $\mathrm{CE}^\bullet_{\mathrm{ext}}(\mathbb{A})$ and by $H^\bullet_{\mathrm{ext}}(\mathbb{A})$ the complex $(\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*), d)$ and its cohomology, respectively. Note that $\mathrm{CE}^\bullet_{\mathrm{ext}}(\mathbb{A})$ is a right dg-module for $\Gamma(\Lambda^\bullet \mathfrak{g}) \cong \Gamma_{0\text{-hom}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*)$ by considering the wedge product.

Proposition 4.12. *There exists a right* $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$*-module isomorphism* $\mathrm{ev} : \mathrm{CE}^\bullet_{\mathrm{ext}}(\mathbb{A}) \to W^{\bullet,q}(\mathfrak{g}, C)$ *satisfying*

$$\mathrm{ev} \circ d = \mathrm{d}_W \circ \mathrm{ev}.$$

We refer to the Appendix (see Proposition A.3) for a proof. It is important to note that Proposition 4.12 implies that $W^{\bullet,q}(\mathfrak{g}, C)$ is a right dg-module for $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$ as stated in Proposition 4.2. It is also worth noting that ev is a map defined similarly to (3-4) (i.e., it evaluates an element $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ on a set of generators of $\Gamma(\mathbb{M}, \mathbb{A})$ to give the sequence $(c_0, c_1, \dots) \in W^{p,q}(\mathfrak{g}, C)$).

Remark 4.13. An alternative characterization of $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ can be given by seeing vector bundles as Lie groupoids (with multiplication given by addition on the fibers). Set $\mathbb{A}^{(p)} = \bigtimes^p_{\mathbb{M}} \mathbb{A}$ and $\mathfrak{g}^{(p)} = \bigtimes^p_M \mathfrak{g}$. One has $\mathbb{A}^{(p)} = B_p \mathbb{A}$ and $\mathfrak{g}^{(p)} = B_p \mathfrak{g}$. In particular, the isomorphism (4-11) implies that

$$\mathbb{A}^{(p)} \cong \bigtimes^q_{\mathfrak{g}^{(p)}} T\mathfrak{g}^{(p)} \times_{\mathfrak{g}^{(p)}} \pi^* C^* \tag{4-19}$$

as vector bundles over $\mathfrak{g}^{(p)}$, where $\pi : \mathfrak{g}^{(p)} \to M$ is defined (following the previous convention for $t : \mathcal{G}^{(p)} \to M$) as $\pi(u_1, \dots, u_p) = \pi(u_1)$. Hence, $\Omega^q(\mathfrak{g}^{(p)}, \pi^* C)$, the space of differential forms on $\mathfrak{g}^{(p)}$ with values on C, can be embedded as a subspace of $C^\infty(\mathbb{A}^{(p)})$ via (4-8). Similarly, $\Gamma(\mathbb{M}, \Lambda^p \mathbb{A})$ can also be embedded as a subspace of $C^\infty(\mathbb{A}^{(p)})$. One can now check that

$$\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*) = \Gamma(\mathbb{M}, \Lambda^p \mathbb{A}) \cap \Omega^q(\mathfrak{g}^{(p)}, \pi^* C).$$

In the case where $C = \mathbb{R}$, with the trivial representation, Li-Bland and Meinrenken [2015] gave a similar characterization of the Weil algebra as a subspace of differential forms on $\mathfrak{g}$. In this context, the case $p = 1$ was already studied by Bursztyn, Cabrera and Ortiz [Bursztyn and Cabrera 2012; Bursztyn et al. 2009].

Proof of the van Est theorem for differential forms with coefficients. Let $\mathrm{VE} : (C^\bullet(\mathbb{G}), \delta_{\mathbb{G}}) \to (\Gamma(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*), d)$ be the van Est map (2-13) for the groupoid $\mathbb{G} \rightrightarrows \mathbb{M}$.

Proposition 4.14. *We have*

$$\mathrm{VE} \circ P_{\mathrm{ext}}^{\mathcal{G},p} = P_{\mathrm{ext}}^{\mathfrak{g},p} \circ \mathrm{VE}.$$

In particular, $\mathrm{VE}(C^\infty_{\mathrm{ext}}(B_p\mathbb{G})) \subset \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$.

Proof. From Proposition 2.12, one already has that VE satisfies $\mathrm{VE} \circ P^{\mathcal{G},p}_{q+1\text{-hom}} = P^{\mathfrak{g},p}_{q+1\text{-hom}} \circ \mathrm{VE}$. It remains to show that VE commutes with the projections associated to the skew-symmetry and the simplicity properties. But this follows from Lemma 2.10 together with the relations (4-13), (4-14) and (4-18). □

Let $\mathrm{VE}_{\mathrm{ext}} : C^\infty_{\mathrm{ext}}(B_p\mathbb{G}) \to \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ be the restriction of the van Est map.

Lemma 4.15. *The following diagram commutes*:

$$\begin{array}{ccc} \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) & \xrightarrow{\ \mathfrak{F}\ } & C^\infty_{\mathrm{ext}}(B_p\mathbb{G}) \\ \Big\downarrow{\scriptstyle \mathrm{VE}_\Omega} & & \Big\downarrow{\scriptstyle \mathrm{VE}_{\mathrm{ext}}} \\ W^{p,q}(\mathfrak{g}, C) & \xleftarrow{\ \mathrm{ev}\ } & \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*) \end{array} \tag{4-20}$$

The proof of Lemma 4.15 consists of a direct but technical verification that we postpone until the Appendix (see page 328). Finally, we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. As ev and $\mathfrak{F}$ are dg-module isomorphisms, it remains to show that $\mathrm{VE}_{\mathrm{ext}}$ induces isomorphisms on the cohomology $H^p(C^\infty_{\mathrm{ext}}(B_\bullet\mathbb{G})) \to H^p(\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^\bullet\mathbb{A}^*))$ for $p \leq p_0$ and a monomorphism for $p = p_0 + 1$. Since the ordinary van Est map $\mathrm{VE}_{\mathbb{G}}$ for $\mathbb{G}$ satisfies the above, the theorem then follows from the homological lemma by means of the underlying projections exactly as in the proof of Theorem 2.14. □

Remark 4.16. The space $\Omega^\bullet(B_\bullet\mathcal{G}, \mathrm{t}^*C)$ is a bigraded right module for the bigraded algebra $\Omega^\bullet(B_\bullet\mathcal{G})$ with the cup product [Dupont 1978]. The multiplication is given by

$$\omega \cup \eta = (-1)^{qp'} \mathrm{pr}^*\omega \wedge \mathrm{pr}'^*\eta, \quad \omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C),\ \eta \in \Omega^{q'}(B_{p'}\mathcal{G}),$$

where $\mathrm{pr} : B_{p+p'}\mathcal{G} \to B_p\mathcal{G}$ (resp. $\mathrm{pr}' : B_{p+p'}\mathcal{G} \to B_{p'}\mathcal{G}$) is the projection onto the first p arrows (resp. last p' arrows). It is interesting to note that such module structure can also be described within the VB-groupoid context. Indeed, by considering the projections $\widetilde{\mathrm{pr}} : \mathbb{G}^{q+q'} \to \mathbb{G}^q$ and $\widetilde{\mathrm{pr}}' : \mathbb{G}^{q+q'} \to \bigtimes_{\mathcal{G}}^{q'} T\mathcal{G}$, one can check that $\mathfrak{F}_{\omega\cup\eta} \in C^\infty(B_{p+p'}\mathbb{G}^{q+q'})$ can be obtained from $(B_p\widetilde{\mathrm{pr}})^*\mathfrak{F}_\omega \in C^\infty(B_p\mathbb{G}^{q+q'})$ and $(B_{p'}\widetilde{\mathrm{pr}}')^*\mathfrak{F}_\eta \in C^\infty(B_{p'}\mathbb{G}^{q+q'})$ by skew-symmetrizing their cup product

$$(B_p\widetilde{\mathrm{pr}})^*\mathfrak{F}_\omega \star (B_{p'}\widetilde{\mathrm{pr}}')^*\mathfrak{F}_\eta \in C^\infty(B_{p+p'}\mathbb{G}^{q+q'}).$$

Similarly, one can define a bigraded module structure on $W^{\bullet,\bullet}(\mathfrak{g}, C)$ for the Weil algebra $W^{\bullet,\bullet}(\mathfrak{g})$ [Arias Abad and Crainic 2011] using the wedge product for their models as subcomplexes of the Chevalley–Eilenberg complexes. These bigraded module structures should be useful for studying "multiplicative linear flat" connections on C.

Appendix: Formulas for the evaluation map

We turn to the proof of Lemma 4.15 relating the formula for VE_Ω with the standard van Est map for $\mathbb{G}$ and $\mathbb{A}$. In the process, we also give a detailed description (see (A-8) below) of the map $\mathrm{ev} : \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}_q^*) \to W^{p,q}(\mathfrak{g}, C)$, making use of special sections of $\mathbb{A}_q$.

Special sections. Let $TB \to B$ be the tangent bundle of B. Given a vector field $X \in \mathfrak{X}(B)$, let X^T, $X^{\mathrm{v}} \in \mathfrak{X}(TB)$ be its *tangent* and *vertical lift* respectively.[4] Define vector fields $X^{T,q}$ and $X^{\mathrm{v},q}_{(j)}$, $j = 1, \dots, q$, on the manifold $\bigtimes^q_B TB$ as follows:

$$X^{T,q}(v_1, \dots, v_q) = (X^T(v_1), \dots, X^T(v_q)), \tag{A-1}$$

$$X^{\mathrm{v},q}_{(j)}(v_1, \dots, v_q) = (0_{v_1}, \dots, X^{\mathrm{v}}(v_j), \dots, 0_{v_q}). \tag{A-2}$$

Let now $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\pi : \mathfrak{g} \to M$. For a representation $C \to M$ of $\mathcal{G}$, consider the Lie groupoid (4-10), $\bigtimes^q_{\mathcal{G}} T\mathcal{G} \times_{\mathcal{G}} \times \mathfrak{t}^* C^* \rightrightarrows \bigtimes^q_M TM \times_M C^*$, with corresponding Lie algebroid $\bigtimes^q_{\mathfrak{g}} T\mathfrak{g} \times_{\mathfrak{g}} \pi^* C^*$. For a section $u : M \to \mathfrak{g}$, let $Tu : TM \to T\mathfrak{g}$ be its derivative and $\chi_u : C^* \to \pi^* C^* = C^* \times_M \mathfrak{g}$ the section defined by (2-11). The expressions

$$\mathbb{T}u(x_1, \dots, x_q, \xi) = (Tu(x_1), \dots, Tu(x_q), \chi_u(\xi)),$$

$$\mathbb{Z}_i u(x_1, \dots, x_q, \xi) = \left(T0(x_1), \dots, T0(x_i) + \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\epsilon u(m)), \dots, T0(x_q), 0_\xi \right),$$

for $i = 1, \dots, q$, $x_1, \dots, x_q \in T_m M$, $\xi \in C^*_m$ and $m \in M$, define sections of the Lie algebroid $\mathbb{A} = \bigtimes^q_{\mathfrak{g}} T\mathfrak{g} \times_{\mathfrak{g}} \pi^* C^* \to \mathbb{M} = \bigtimes^q_M TM \times_M C^*$. It is known that $\mathbb{T}u$ and $\mathbb{Z}_i u$, $i = 1, \dots, q$, generate $\Gamma(\mathbb{M}, \mathbb{A})$ as a $C^\infty(\mathbb{M})$ module.[5]

Lemma A.1. *As vector fields on* $B_p(\bigtimes^q_{\mathcal{G}} T\mathcal{G} \times_{\mathcal{G}} \mathfrak{t}^* C^*) \cong \bigtimes^q_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{B_p\mathcal{G}} \mathfrak{t}^* C^*$, *the following identities hold*:

$$B_p(\mathbb{T}u) = ((B_p u)^{T,q}, X_u), \tag{A-3}$$

$$B_p(\mathbb{Z}_i v) = ((B_p v)^{\mathrm{v},q}_{(i)}, 0), \tag{A-4}$$

[4] The flow at time ϵ of X^T (resp. X^{v}) is the derivative of the flow at time ϵ of X (resp. translation by ϵX).

[5] This follows from a general result regarding *core* and *linear sections* of double vector bundles (see [Mackenzie 2011]).

where $X_u \in \mathfrak{X}(\mathrm{t}^*C^*)$ is the vector field whose time-ϵ flow is given by

$$(g_1, \dots, g_p, \xi) \mapsto (\psi_u^\epsilon(g_1, \dots, g_p), \Delta^*_{\phi_u^\epsilon(\mathrm{t}(g_1))^{-1}}(\xi)).$$

In particular, for $\omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$,

$$\mathbb{s}_0^*\mathcal{L}_{B_p(\mathbb{T}u)}\mathfrak{F}_\omega = \mathfrak{F}_{R_u\omega}, \tag{A-5}$$

$$\mathbb{s}_0^*\mathcal{L}_{B_p(\mathbb{Z}_i u)}\mathfrak{F}_\omega = (-1)^{i-1}\mathfrak{F}_{J_u\omega} \circ \mathrm{pr}_{(i)}^{p-1,q}, \tag{A-6}$$

where R_u and J_u were defined in (4-3),

$$\mathrm{pr}_{(i)}^{\cdot,q} : \bigtimes_{B.\mathcal{G}}^q TB.\mathcal{G} \times_{B.\mathcal{G}} \mathrm{t}^*C^* \to \bigtimes_{B.\mathcal{G}}^{q-1} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathrm{t}^*C^*$$

is the projection which forgets the i-th component and $\mathbb{s}_0$ is the first degeneracy map for $\mathbb{G}$.

Proof. For $u \in \Gamma(\mathfrak{g})$, consider the sections $\mathbb{Z}u$, Tu of $T\mathfrak{g} \to TM$, where

$$\mathbb{Z}u(x) = T0(x) + \frac{d}{d\epsilon}\bigg|_{\epsilon=0}(\epsilon u(m)), \quad x \in T_mM.$$

One has that $\overrightarrow{Tu} = \vec{u}^T$ and $\overrightarrow{\mathbb{Z}u} = \vec{u}^{\mathrm{v}}$ as vector fields on $T\mathcal{G} \rightrightarrows TM$ (see [Mackenzie and Xu 1994]). Also, the flow of the right invariant vector field $\overrightarrow{\chi_u} \in \mathfrak{X}(\mathrm{t}^*C^*)$ is given by

$$(g, \xi) \mapsto (\phi_u^\epsilon(g), \phi_u^\epsilon(\mathrm{t}(g))^{-1} \cdot \xi).$$

The identities (A-3) and (A-4) now follow from analyzing the flows together with the rearrangement isomorphism (4-11). Hence, for $\omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$,

$$\begin{aligned}
&(\mathcal{L}_{B_p(\mathbb{T}u)}\mathfrak{F}_\omega)|_{(\bar{U}_1,\dots,\bar{U}_q,(g_1,\dots,g_p,\xi))} \\
&\quad = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathfrak{F}_\omega\big(T\psi_u^\epsilon(\bar{U}_1), \dots, T\psi_u^\epsilon(\bar{U}_q), (\psi_u^\epsilon(g_1, \dots, g_p), \phi_u^\epsilon(\mathrm{t}(g_1))^{-1} \cdot \xi)\big) \\
&\quad = \big\langle \xi, \phi(\mathrm{t}(g_1))^{-1} \cdot (\psi_u^\epsilon)^*\omega(\bar{U}_1, \dots, \bar{U}_q)\big\rangle.
\end{aligned}$$

Now, (A-5) follows from the commutation relations on Proposition 4.8. The identity (A-6) follows similarly. □

The evaluation map. We now describe the chain isomorphism $\mathrm{ev}: \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*) \to W^{p,q}(\mathfrak{g}, C)$. First, for $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*) \subset \Gamma(\mathbb{M}, \Lambda^p\mathbb{A}^*)$, define

$$\tilde{c}_k(\alpha) : \big(\bigtimes^{p-k} \Gamma(\mathfrak{g})\big) \times \big(\bigtimes^k \Gamma(\mathfrak{g})\big) \to C^\infty\big(\bigtimes_M^q TM \times_M C^*\big)$$

as

$$\tilde{c}_k(\alpha)(u_1, \dots, u_{p-k}|v_1, \dots, v_k) = \alpha(\mathbb{Z}_1v_1, \dots, \mathbb{Z}_kv_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}).$$

Lemma A.2. *There exists a map* $c_k(\alpha) : \bigtimes^p \Gamma(\mathfrak{g}) \to \Omega^{q-k}(M, C)$ *such that*

$$\tilde{c}_k(\alpha) = \mathfrak{F}_{c_k(\alpha)} \circ \mathrm{pr}_{[1,k]} \quad \forall\, \alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}_q^*), \tag{A-7}$$

where $\mathrm{pr}_{[1,k]} : \bigtimes_M^q TM \times_M C^* \to \bigtimes_M^{q-k} TM \times_M C^*$ *is the projection which forgets the first* k *entries.*

Proof. The multilinearity with respect to both vector bundle structures, $\mathbb{A} \to \mathbb{M}$ and $\mathbb{A} \to \mathfrak{g}$, implies that $\tilde{c}_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k) = F_\alpha \circ \mathrm{pr}_{[1,k]}$, where $F_\alpha \in C^\infty\big(\bigtimes^{q-k} TM \times_M C^*\big)$ is given by

$$F_\alpha(y_1, \dots, y_{q-k}, \xi) = \tilde{c}_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k)(\underbrace{0_m, \dots, 0_m}_{k \text{ times}}, y_1, \dots, y_{q-k}, \xi).$$

We now have to check that $F_\alpha \in C^\infty_{\mathrm{ext}}\big(\bigtimes^{q-k} TM \times_M C^*\big)$, i.e., F is $(q-k+1)$-homogeneous, simple and skew-symmetric. The homogeneity of F_α follows from the homogeneity of α together with the linearity of the sections $\mathbb{T}u$ and the properties of the section $\mathbb{Z}_j v$:

$$\mathbb{Z}_j(v)|_{(0_m,\dots,0_m,\lambda y_1,\dots,\lambda y_{q-k},\lambda\xi)} = h^{\mathfrak{g}}_\lambda\Big(\mathbb{Z}_j\Big(\frac{1}{\lambda} v\Big)\Big|_{(0_m,\dots,0_m,y_1,\dots,y_{q-k},\xi)}\Big),$$
$$\mathbb{Z}_j(\lambda v) = \lambda \cdot \mathbb{Z}_j(v),$$

where $\lambda > 0$ and $\cdot$ stands for the multiplication for $\mathbb{A} \to \mathbb{M}$. The simplicity of F_α follows from the identity

$$(F_\alpha \circ 0_i) \circ \mathrm{pr}_{[1,k]} = ((0^{\mathfrak{g}}_{k+i})^* \alpha)(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) = 0$$

for $i = 1, \dots, q-k+1$. Finally, let $\sigma \in S_{q-k} \subset S_q$, seen as the subgroup acting as the identity on $\{1, \dots, k\}$. One can check that

$$\begin{aligned}(F_\alpha \circ \sigma_M) \circ \mathrm{pr}_{[1,k]} &= (\sigma^*_{\mathfrak{g}} \alpha)(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) \\ &= \mathrm{sgn}(\sigma)\, \alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) \\ &= \mathrm{sgn}(\sigma)\, F_\alpha \circ \mathrm{pr}_{[1,k]}.\end{aligned}$$

This shows that $F_\alpha \in C^\infty_{\mathrm{ext}}\big(\bigtimes^{q-k} TM \times_M C^*\big)$ and, therefore, there exists $c_k(\alpha) : \bigtimes^p \Gamma(\mathfrak{g}) \to \Omega^{q-k}(M, C)$ such that $F_\alpha = \mathfrak{F}_{c_k(\alpha)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)}$. □

Our aim is to prove that

$$\mathrm{ev}(\alpha) = (c_0(\alpha), c_1(\alpha), \dots) \tag{A-8}$$

defines a map from $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ into $W^{p,q}(\mathfrak{g}, C)$. First note that the sequence $(c_0(\alpha), c_1(\alpha), \dots)$ completely determines $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$. Indeed, as $\Gamma(\mathbb{M}, \mathbb{A})$ is generated as a $C^\infty(\mathbb{M})$-module by sections of the type $\mathbb{T}u$, $\mathbb{Z}_i v$, any element of

$\Gamma(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ is determined by its values on these sections. Now, one can check that, for $\alpha \in \Gamma_{\text{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$,

$$i_{\mathbb{Z}_j v} i_{\mathbb{Z}_j w} \alpha = 0 \quad \text{for } j = 1, \dots, q, \tag{A-9}$$

and, for a permutation $\sigma \in S_q$,

$$\begin{aligned} \alpha(\mathbb{Z}_{\sigma(1)} v_1, \dots, \mathbb{Z}_{\sigma(k)} v_k, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}) \\ = \operatorname{sgn}(\sigma) \sigma_M^* (\alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k})). \end{aligned} \tag{A-10}$$

Hence, to recover α from its values on the sections $\mathbb{T}u, \mathbb{Z}_i v$, it suffices to know the values of α encoded on the sequence $(c_0(\alpha), c_1(\alpha), \dots)$. The next result gives the desired proof of Proposition 4.12.

Proposition A.3. *Given $\alpha \in \Gamma_{\text{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$, one has that*

(1) *$c_k(\alpha)$ is skew-symmetric on the u entries*;

(2) *$c_k(\alpha)$ is symmetric on the v entries*;

(3) *given $f \in C^\infty(M)$,*

$$\begin{aligned} c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, f v_k) &= f c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k), \\ c_k(\alpha)(f u_1, \dots, u_{p-k} | v_1, \dots, v_k) &= f c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k) \\ &\quad + df \wedge c_{k+1}(\alpha)(u_2, \dots, u_{p-k} | v_1, \dots, v_k, u_1). \end{aligned}$$

In particular, each c_k can be viewed as an $\mathbb{R}$-linear skew-symmetric map $c_k : \bigtimes^{p-k} \Gamma(\mathfrak{g}) \to \Omega^{q-k}(M, S^k \mathfrak{g}^* \otimes C)$. Moreover, the map $\text{ev} : \Gamma_{\text{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*) \to W^{p,q}(\mathfrak{g}, C)$ defined by (A-8) is a right $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$-module isomorphism satisfying

$$\text{ev} \circ \mathrm{d}_{\text{ext}} = \mathrm{d}_W \circ \text{ev}.$$

Proof.

(1) This follows directly from the skew-symmetry of α with respect to $\mathbb{A} \to \mathbb{M}$.

(2) Let $\sigma \in S_k \subset S_q$, seen as the subgroup acting as the identity on $\{k+1, \dots, q\}$. From (A-10) and the skew-symmetry of α with respect to $\mathbb{A} \to \mathbb{M}$,

$$\begin{aligned} &\alpha(\mathbb{Z}_1 v_{\sigma(1)}, \dots, \mathbb{Z}_k v_{\sigma(k)}, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}) \\ &\quad = \operatorname{sgn}(\sigma) \alpha(\mathbb{Z}_{\sigma(1)} v_{\sigma(1)}, \dots, \mathbb{Z}_{\sigma(k)} v_{\sigma(k)}, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}) \\ &\quad = (\operatorname{sgn}(\sigma))^2 \alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}). \end{aligned}$$

In the second equality we have used the fact that

$$\alpha(\mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}, \mathbb{Z}_{\sigma(1)} v_{\sigma(1)}, \dots, \mathbb{Z}_{\sigma(k)} v_{\sigma(k)}) \in C^\infty\big(\bigtimes^q TM \times_M C^*\big)$$

does not depend on the first k coordinates.

(3) One can check that

$$\mathbb{Z}_i(fv) = (f\circ\pi)\cdot\mathbb{Z}_i v,$$
$$\mathbb{T}(fu) = (f\circ\pi)\cdot\mathbb{T}u + \sum_{j=1}^{q}(\ell_{df}\circ \mathrm{pr}_j)\cdot\mathbb{Z}_j u,$$

where all the sums and scalar multiplications are with respect to $\pi : \mathbb{A}\to\mathbb{M}$, $\mathrm{pr}_j : \bigtimes_M^q TM\times_M C^*\to TM$ is the projection onto the j-th factor and $\ell_{df}\in C^\infty(TM)$ is the linear function corresponding to $df\in\Omega^1(M)$. To simplify notation, we identify $\Omega^{q-k}(M,C)$ with its image on $C^\infty\big(\bigtimes^{q-k} TM\times_M C^*\big)$ under $\mathfrak{F}$ in the following. The first equation of (3) is now straightforward to check. As for the second, it follows from (A-9) and (A-10) that

$$\begin{aligned}
&c_k(\alpha)(fu_1,\dots,u_{p-k}|v_1,\dots,v_k)\circ\mathrm{pr}_{[1,k]}\\
&= (f\circ\pi)\alpha(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p-k})\\
&\quad+\sum_{j=k+1}^{q}(\ell_{df}\circ\mathrm{pr}_j)\alpha(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{Z}_ju_1,\mathbb{T}u_2,\dots,\mathbb{T}u_{p-k})\\
&= (f\circ\pi)c_k(\alpha)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)\circ\mathrm{pr}_{[1,k]}\\
&\quad+\underbrace{\sum_{j=k+1}^{q}(-1)^{j-k-1}(\ell_{df}\circ\mathrm{pr}_j)\alpha(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{Z}_{k+1}u_1,\mathbb{T}u_2,\dots,\mathbb{T}u_{p-k})\circ\sigma^j_M}_{(*)},
\end{aligned}$$

where $\sigma^j\in S_q$ is the cycle $(j\ j{-}1\cdots k{+}2\ k{+}1)$, for $k+1\le j\le q$, which has sign equal to $(-1)^{j-k-1}$. It is now straightforward to check that $(*)$ equals $df\wedge c_{k+1}(u_2,\dots,u_{p-k}|v_1,\dots,v_k,u_1)\circ\mathrm{pr}_{[1,k]}$.

It remains to prove that ev is a dg-module isomorphism. Let us first prove that ev commutes with the multiplication. Let $\beta\in\Gamma(\Lambda^{p'}\mathfrak{g}^*)\cong\Gamma_{0\text{-hom}}(\mathbb{M},\Lambda^{p'}\mathbb{A}^*)$ and consider $\mathrm{ev}(\alpha\wedge\beta)=(c_0(\alpha\wedge\beta),c_1(\alpha\wedge\beta),\dots)$. By definition,

$$\begin{aligned}
&c_k(\alpha\wedge\beta)(u_1,\dots,u_{p+p'-k}|v_1,\dots,v_k)\circ\mathrm{pr}_{[1,k]}\\
&\quad=(\alpha\wedge\beta)(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p+p'-k})\\
&\quad=\sum_{\sigma\in S(p-k,p')}\mathrm{sgn}(\sigma)\alpha(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{T}u_{\sigma(1)},\dots,\mathbb{T}u_{\sigma(p-k)})\\
&\qquad\qquad\times\beta(\mathbb{T}u_{\sigma(p-k+1)},\dots,\mathbb{T}u_{\sigma(p+p'-k)}),
\end{aligned}$$

where $S(p{-}k,\,p')$ is the space of $(p{-}k,\,p')$-unshuffles and the last equality follows from the fact that the contraction of β with any section of type $\mathbb{Z}.v.$ is zero. The result now follows easily.

Finally, to prove that ev intertwines the differential, consider

$$\mathrm{ev}(d\alpha)=(c_0(d\alpha),\dots,c_k(d\alpha),\dots),$$

where

$$
\begin{aligned}
&c_k(d\alpha)(u_1,\dots,u_{p+1-k}|v_1,\dots,v_k)\circ \mathrm{pr}_{[1,k]}\\
&\quad = d\alpha(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p+1-k})\\
&\quad = \underbrace{\sum_{j=1}^{k}(-1)^{j+1}\mathcal{L}_{\rho(\mathbb{Z}_jv_j)}\alpha(\mathbb{Z}_1v_1,\dots,\widehat{\mathbb{Z}_jv_j},\dots)}_{(A)}\\
&\qquad + \underbrace{\sum_{i=1}^{p-k+1}(-1)^{i+k+1}\mathcal{L}_{\rho(\mathbb{T}u_i)}\alpha(\mathbb{Z}_1v_1,\dots,\widehat{\mathbb{T}u_i},\dots)}_{(B)}\\
&\qquad\quad + \underbrace{\sum_{1\le i<j\le p+1-k}(-1)^{i+j}\alpha([\mathbb{T}u_i,\mathbb{T}u_j],\dots,\widehat{Tu_i},\dots,\widehat{Tu_j},\dots)}_{(C)}\\
&\qquad\quad\quad + \underbrace{\sum_{i=1}^{p+1-k}\sum_{j=1}^{k}(-1)^{j+(k+i)}\alpha([\mathbb{Z}_jv_j,\mathbb{T}u_i],\dots,\widehat{\mathbb{Z}_jv_j},\dots,\widehat{\mathbb{T}u_i},\dots)}_{(D)}.
\end{aligned}
$$

Notice that there are no terms containing $[\mathbb{Z}_{j_1}v_{j_1},\mathbb{Z}_{j_2}v_{j_2}]$ since these brackets are all zero. To study the remaining terms, we use some properties of the tangent Lie algebroid $T\mathfrak{g}\to TM$ (see [Mackenzie and Xu 1994]) and the action algebroid $C^*\times_M\mathfrak{g}\to C^*$.

(A): From (A-10),

$$
\begin{aligned}
&\alpha(\mathbb{Z}_1v_1,\dots,\widehat{\mathbb{Z}_jv_j},\dots,\mathbb{Z}_kv_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p+k-1})\\
&\qquad = (-1)^{k-j}\sigma_M^*(\mathfrak{F}_{c_{k-1}(\alpha)(u_1,\dots,u_{p+1-k}|v_1,\dots,v_{j-1},v_{j+1},\dots,v_k)}\circ\mathrm{pr}_{[1,k-1]}),
\end{aligned}
$$

where $\sigma=(j\ k)(j\ k{-}1)\cdots(j\ j{+}1)\in S_q$. Now,

$$
\begin{aligned}
\mathrm{pr}_{[1,k-1]}\circ\sigma_M(x_1,\dots,x_q,\xi) &= (x_j,x_{k+1},\dots,x_q,\xi),\\
\rho(\mathbb{Z}_jv_j) &= (\rho(v_j)^{\mathrm{v},q}_{(j)},0)
\end{aligned}
$$

and

$$
\mathcal{L}_{(X^{\mathrm{v},q}_{(1)},0)}\mathfrak{F}_\omega=\mathfrak{F}_{i_X\omega}\circ\mathrm{pr}_{(1)}\quad\forall\,X\in\mathfrak{X}(M),\ \omega\in\Omega^q(M,C),
$$

where $\mathrm{pr}_{(1)}:\bigtimes^q TM\times_M C^*\to\bigtimes^{q-1}TM\times_M C^*$ is the projection which forgets the first component. These facts imply that

$$
(A)=(-1)^{k+1}\sum_{j=1}^{k}\big(i_{\rho(v_j)}c_{k-1}(\alpha)(u_1,\dots,u_{p+1-k}|v_1,\dots,v_{j-1},v_{j+1},\dots,v_k)\big)\circ\mathrm{pr}_{[1,k]}.
$$

(B): The fact that $\rho(\mathbb{T}u_i) = (\rho(u_i)^T, \rho(\chi_u))$, where $(\cdot)^T$ stands for tangent lift, implies that, for $\omega \in \Omega^q(M, C)$, $\mathcal{L}_{\rho(\mathbb{T}u_i)}\mathfrak{F}_\omega = \mathfrak{F}_{u_i\cdot\omega}$, where $u\cdot(\beta\otimes\mu) = \mathcal{L}_{\rho(u)}\beta\otimes\mu + \beta\otimes\nabla_u\mu$, $\beta\in\Omega^q(M)$, $\mu\in\Gamma(C)$ and ∇ is the $\mathfrak{g}$-connection on C defining the representation of $\mathfrak{g}$ on C. Hence,

$$(B) = (-1)^k \sum_{i=1}^{p-k+1} (-1)^{i+1}\big(u_i\cdot c_k(u_1,\dots,\widehat{u_i},\dots,u_{p+1-k}|v_1,\dots,v_k)\big)\circ \mathrm{pr}_{[1,k]}.$$

(C) and (D): From the identities $[\mathbb{T}u_i, \mathbb{T}u_j] = \mathbb{T}[u_i,u_j]$, $[\mathbb{T}u_i, \mathbb{Z}_j v_j] = \mathbb{Z}_j[u_i,v_j]$, it is straightforward to check that

$$(C) = (-1)^k \sum_{1\le i_1<i_j\le p-k+1} (-1)^{i_1+i_2}\big(c_k(\alpha)([u_{i_1},u_{i_2}],u_1,\dots,\widehat{u_{i_1}},\dots,\widehat{u_{i_2}},\dots,u_{p-k+1}|v_1,\dots,v_k)\big)\circ\mathrm{pr}_{[1,k]},$$

$$(D) = (-1)^k \sum_{i=1}^{p-k+1}\sum_{j=1}^{k}(-1)^i\big(c_k(\alpha)(u_1,\dots,\widehat{u_i},\dots,u_{p+1-k}|v_1,\dots,[u_i,v_j],\dots,v_k)\big)\circ\mathrm{pr}_{[1,k]}.$$

Hence,

$$(A)+(B)+(C)+(D) = \mathrm{d}_W(c(\alpha))_k(u_1,\dots,u_{p-k+1}|v_1,\dots,v_k)\circ\mathrm{pr}_{[1,k]}$$
$$\Longrightarrow\quad c(d\alpha)_k = \mathrm{d}_W(c(\alpha))_k,$$

as we wanted. □

Proof of Lemma 4.15.

Lemma 4.15 rephrased. *Let ω be an element of $\Omega^q(B_p\mathcal{G}, \mathfrak{t}^*C)$ and consider $\mathrm{VE}_\Omega(\omega) = (c_0(\omega), c_1(\omega),\dots)$ as defined in (4-4). Also, let $\alpha = \mathrm{VE}_{\mathrm{ext}}(\mathfrak{F}_\omega) \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*)$ and consider $\mathrm{ev}(\alpha) = (c_0(\alpha), c_1(\alpha),\dots)$ defined by (A-7). Then*

$$c_k(\omega) = c_k(\alpha) \quad \forall\, k\ge 0.$$

Proof. From (2-13),

$$\mathfrak{F}_{c_k(\alpha)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)}\circ\mathrm{pr}_{[1,k]} = \mathrm{VE}_{\mathrm{ext}}(\mathfrak{F}_\omega)(\mathbb{Z}_1v_1,\dots,\mathbb{Z}_kv_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p-k})$$
$$= \sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)R_{\chi_{\sigma(1)}}\dots R_{\chi_{\sigma(p)}}\mathfrak{F}_\omega,$$

where $\chi_i = \mathbb{Z}_iv_i$ (resp. $\mathbb{T}u_{i-k}$) if $i\in\{1,\dots,k\}$ (resp. if $i\in\{k+1,\dots,p\}$). The main ingredients of the proof are the identities from Lemma A.1:

$$R_{\mathbb{T}u_i}\mathfrak{F}_\omega = \mathbb{s}_0^*\mathcal{L}_{B_p\mathbb{T}u_i}\mathfrak{F}_\omega = \mathfrak{F}_{R_{u_i}\omega},$$
$$R_{\mathbb{Z}_iv_i}\mathfrak{F}_\omega = (-1)^{i-1}\mathfrak{F}_{J_{v_i}\omega}\circ\mathrm{pr}_{(i)}^{p-1,q},$$

where $\mathrm{pr}_{(i)}^{\cdot,q} : \bigtimes_{B.\mathcal{G}}^{q} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathfrak{t}^*C^* \to \bigtimes_{B.\mathcal{G}}^{q-1} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathfrak{t}^*C^*$ is the projection which forgets the i-th component. In the rest of the proof, the difficulty lies in the combinatorics needed to count the number of -1's appearing due to the presence of the sections $\mathbb{Z}_i v_i$.

Let $0 \le r \le p$, $1 \le s \le q$ and $1 \le i, j \le s$. For $\eta \in \Omega^{s-1}(B_r\mathcal{G}, \mathfrak{t}^*C)$, one can check that

$$R_{\mathbb{T}u}(\mathfrak{F}_\eta \circ \mathrm{pr}_{(i)}^{r,s}) = \mathfrak{F}_{R_u\eta} \circ \mathrm{pr}_{(i)}^{r-1,s}$$

and

$$R_{\mathbb{Z}_j v}(\mathfrak{F}_\eta \circ \mathrm{pr}_{(i)}^{r,s}) = \begin{cases} (-1)^{j-1}\mathfrak{F}_{J_v\eta} \circ \mathrm{pr}_{(j)}^{r-1,s-1} \circ \mathrm{pr}_{(i)}^{r-1,s} & \text{if } i > j, \\ -(-1)^{j-1}\mathfrak{F}_{J_v\eta} \circ \mathrm{pr}_{(i)}^{r-1,s-1} \circ \mathrm{pr}_{(j)}^{r-1,s} & \text{if } i < j, \\ 0 & \text{if } i = j. \end{cases}$$

Let us now fix a permutation $\sigma \in S_q$. For $1 \le l \le k$, let $j_l = \sigma^{-1}(l)$ for $l \ge 1$ and set $j_0 = 0$. Denote by τ the permutation of $\{0, 1, \dots, k\}$ such that $j_{\tau(0)} < \cdots < j_{\tau(k)}$. One can now prove by induction that, for $j_{\tau(l)} \le r < j_{\tau(l+1)}$,

$$R_{\chi_{\sigma(r+1)}} \dots R_{\chi_{\sigma(p)}}\mathfrak{F}_\omega = \delta(k,l)\big(\mathfrak{F}_{D_{\sigma(r+1)}\dots D_{\sigma(p)}\omega} \circ \mathrm{pr}_{(i_1)}^{r,q-k+l} \circ \cdots \circ \mathrm{pr}_{(i_{k-l})}^{r,q}\big), \tag{A-11}$$

where the D_j are the operators (4-5), $\{i_1 < \cdots < i_{k-l}\} = \{\tau(l+1), \dots, \tau(k)\}$ and

$$\delta(k,l) = (-1)^{k-l}(-1)^{\tau(l+1)+\cdots+\tau(k)}(-1)^{N(\tau,l)}$$

with

$$N(\tau,l) = \#\big\{(i,j) \in \{l+1,\dots,k\} \times \{l+1,\dots,k\} \mid i < j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\big\}.$$

Note that, for $l = 0$,

$$\delta(k,0) = (-1)^k(-1)^{1+\cdots+k}(-1)^{\epsilon(\sigma,k)} = (-1)^{k(k-1)/2}(-1)^{\epsilon(\sigma,k)}.$$

In particular, when $r = 0$, we have

$$\begin{aligned} &\mathfrak{F}_{c_k(\alpha)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)} \circ \mathrm{pr}_{[1,k]} \\ &\quad = \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) R_{\chi_{\sigma(1)}} \dots R_{\chi_{\sigma(p)}}\mathfrak{F}_\omega \\ &\quad = (-1)^{k(k-1)/2} \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma)(-1)^{\epsilon(\sigma,k)}\mathfrak{F}_{D_{\sigma(1)}\dots D_{\sigma(p)}\omega} \circ \mathrm{pr}_{(1)}^{0,q-k} \circ \cdots \circ \mathrm{pr}_{(k)}^{0,q} \\ &\quad = \mathfrak{F}_{c_k(\omega)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)} \circ \mathrm{pr}_{[1,k]}. \end{aligned}$$

□

Acknowledgments

We would like to thank H. Bursztyn and O. Brahic for useful discussions. Cabrera would also like to thank R. Mehta for his insightful ideas in the early stages of this

work. The authors are also especially grateful to Matias del Hoyo for pointing out an incompleteness in the proof of Lemma 3.1 and for helping to complete it.

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Received May 14, 2016. Revised June 28, 2016.

ALEJANDRO CABRERA
DEPARTAMENTO DE MATEMATICA APLICADA, INSTITUTO DE MATEMATICA
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CAIXA POSTAL 68530
21941-909 RIO DE JANEIRO-RJ
BRAZIL

acabrera@labma.ufrj.br

THIAGO DRUMMOND
DEPARTAMENTO DE MATEMATICA, INSTITUTO DE MATEMATICA
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CAIXA POSTAL 68530
21941-909 RIO DE JANEIRO-RJ
BRAZIL

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Riverside, CA 92521-0135
chari@math.ucr.edu

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Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

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University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

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Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

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Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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PACIFIC JOURNAL OF MATHEMATICS

Volume 287 No. 2 April 2017