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PEDRO FREJLICH AND IOAN MĂRCUȚ

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THE NORMAL FORM THEOREM AROUND POISSON TRANSVERSALS

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Dedicated to Alan Weinstein on the occasion of his 70th birthday

We prove a normal form theorem for Poisson structures around Poisson transversals (also called cosymplectic submanifolds), which simultaneously generalizes Weinstein's symplectic neighborhood theorem from symplectic geometry and Weinstein's splitting theorem. Our approach turns out to be essentially canonical, and as a byproduct, we obtain an equivariant version of the latter theorem.

1. Introduction

This paper is devoted to the study of semilocal properties of *Poisson transversals*. These are submanifolds X of a Poisson manifold (M, π) that meet each symplectic leaf of π *transversally* and *symplectically*. A Poisson transversal X carries a canonical Poisson structure, whose leaves are the intersections of leaves of π with X, and are endowed with the pullback symplectic structure.

Even though this class of submanifolds has very rarely been dealt with in full generality — much to our dismay and surprise — Poisson transversals permeate the whole theory of Poisson manifolds, often playing a quite fundamental role. This lack of specific attention is especially intriguing since they are a special case of several distinguished classes of submanifolds which have aroused interest lately: Poisson transversals are Lie–Dirac submanifolds [Xu 2003], Poisson–Dirac submanifolds

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[Crainic and Fernandes 2004], and also pre-Poisson submanifolds [Cattaneo and Zambon 2009] (see also [Zambon 2011] for a survey on submanifolds in Poisson geometry).

No wonder, then, that Poisson transversals have shown up already in the infancy of Poisson geometry, in the foundational paper of Weinstein [1983]. Namely, if L is a symplectic leaf and $x \in L$, then a submanifold X that intersects L transversally at x and has complementary dimension is a Poisson transversal, and its induced Poisson structure governs much of the geometry transverse to L. In fact, a small enough tubular neighborhood of L in M will have the property that all its fibers are Poisson transversals. Such fibrations are nowadays called *Poisson fibrations*, and were studied by Vorobjev [2001] — mostly in connection with the local structure around symplectic leaves — and also by Fernandes and Brahic [2008]. That Poisson fibrations are related to Haefliger's formalism of geometric structures described by groupoid-valued cocycles (see [Haefliger 1958] and also [Gromov 1986]) — of which the "automatic transversality" of Lemma 7 is also reminiscent — should not escape notice. In fact, in physics literature, Poisson fibrations have long been known in the guise of *second class constraints*, and motivated the introduction by P. Dirac [1950] of what we know today as the induced Dirac bracket, which in our language is the induced Poisson structure on the fibers.

The role played by Poisson transversals in Poisson geometry is similar to that played by symplectic submanifolds in symplectic geometry and by transverse submanifolds in foliation theory (see the examples in the next section). The key observation is that the transverse geometry around a Poisson transversal X is of nonsingular and *contravariant* nature: it behaves more like a 2-form than as a bivector in the directions conormal to X. This allows us to make particularly effective use of the tools of "contravariant geometry". In the core of our arguments lies the fact that the contravariant exponential map $\exp_{\mathcal{X}}$ associated to a Poisson spray $\mathcal{X}$ gives rise to a tubular neighborhood adapted to $X \subset (M, \pi)$, in complete analogy with the classical construction of a tubular neighborhood of a submanifold X in a Riemannian manifold (M, g), thus effectively reducing many problems to the symplectic case.

The main result of this paper is a local normal form theorem around Poisson transversals, which simultaneously generalizes Weinstein's splitting theorem [1983] and Weinstein's symplectic neighborhood theorem [1971]. At a Poisson transversal X of (M, π), the restriction of the Poisson bivector $\pi|_X \in \Gamma(\wedge^2 TM|_X)$ determines

- a Poisson structure on X, denoted π_X,
- a nondegenerate, fiberwise 2-form on the conormal bundle $p : N^*X \to X$, denoted

$$w_X \in \Gamma(\wedge^2 NX).$$

Let $\widetilde{\sigma}$ be a closed 2-form on N^*X that extends $\sigma := -w_X$, i.e., which restricts on $T(N^*X)|_X = TX \oplus N^*X$ to the trivial extension of σ by zero.

To such an extension we associate a Poisson structure $\pi(\widetilde{\sigma})$ on an open set $U(\widetilde{\sigma}) \subset N^*X$ around X. The symplectic leaves of $\pi(\widetilde{\sigma})$ are in one-to-one correspondence with the leaves of π_X; namely if (L, ω_L) is a leaf of π_X, the corresponding leaf of $\pi(\widetilde{\sigma})$ is an open set $\widetilde{L} \subset p^{-1}(L)$ around L endowed with the 2-form $\omega_{\widetilde{L}} := p^*(\omega_L) + \widetilde{\sigma}|_{\widetilde{L}}$. The Poisson manifold $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$ is the local model of π around X. We will provide a more conceptual description of the local model using Dirac geometry.

Theorem 1 (normal form theorem). *Let (M, π) be a Poisson manifold and $X \subset M$ be an embedded Poisson transversal. An open neighborhood of X in (M, π) is Poisson diffeomorphic to an open neighborhood of X in the local model $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$.*

Under stronger assumptions (which always hold around points in X) we can provide an even more explicit description of the normal form. Assuming *symplectic* triviality of the conormal bundle to X, the theorem implies a generalized version of the Weinstein splitting theorem, expressing the Poisson as a product, i.e., in the form (1) below. This coincides with Weinstein's setting when we look at (small) Poisson transversals of complementary dimension to a symplectic leaf.

The proof of Theorem 1 relies on the symplectic realization constructed in [Crainic and Mărcuț 2011] with the aid of global Poisson geometry, and on elementary Dirac-geometric techniques; the former is the crucial ingredient that allows us to have a good grasp of directions conormal to the Poisson transversal, and the latter furnishes the appropriate language to deal with objects which have mixed covariant-contravariant behavior. As an illustration of the strength and canonicity of our methods, we present as an application the proof of an equivariant version of Weinstein's splitting theorem. Other applications of the normal form theorem, which reveal the Poisson-topological aspects of Poisson transversals, will be treated elsewhere.

Theorem 2. *Let (M, π) be a Poisson manifold and let G be a compact Lie group acting by Poisson diffeomorphisms on M. If $x \in M$ is a fixed point of G, then there are coordinates $(p_1, \dots, p_n, q_1, \dots, q_n, y_1, \dots, y_m) \in \mathbb{R}^{2n+m}$ centered at x such that*

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{j,k=1}^{m} \varpi_{j,k}(y) \frac{\partial}{\partial y_j} \wedge \frac{\partial}{\partial y_k}, \tag{1}$$

and in these coordinates G acts linearly and keeps the subspaces $\mathbb{R}^{2n} \times \{0\}$ and $\{0\} \times \mathbb{R}^m$ invariant.

This answers in the negative a question posed by Miranda and Zung [2006] about the necessity of the "tameness" condition they assume in their proof of this result. We wish to thank Miranda for bringing this problem to our attention.

We should probably also say a few words about terminology. Poisson transversals are also referred to as *cosymplectic submanifolds* in the literature, and this is motivated by the fact that the conormal directions to such a submanifold are symplectic, i.e., the Poisson tensor is nondegenerate on the conormal bundle to the submanifold. Even though this nomenclature is perfectly reasonable, there are several reasons why we decided not to use this name. Foremost among these:

(1) There is already a widely used notion of a cosymplectic manifold, defined as a manifold of dimension $2n+1$, endowed with a closed 1-form θ and a closed 2-form ω such that $\theta \wedge \omega^n$ is a volume form.

(2) The general point of view of transverse geometric structures is of great insight into Poisson transversals when we rephrase the problem in terms of Dirac structures and contravariant geometry. Moreover, the proximity between the dual pairs used in the proof of the normal form theorem, and the gadget of Morita equivalence, which is known to govern the transverse geometry to the symplectic leaves, is too obvious to ignore.

2. Some basic properties of Poisson transversals

Let (M, π) be a Poisson manifold. A *Poisson transversal* in M is an embedded submanifold $X \subset M$ that meets each symplectic leaf of π *transversally* and *symplectically*. We translate both these conditions algebraically. Let $x \in X$ and let (L, ω) be the symplectic leaf through x. Transversality translates to

$$T_x X + T_x L = T_x M.$$

Taking annihilators in this equation, we obtain that $N_x^* X \cap \ker(\pi_x^\sharp) = \{0\}$, or equivalently, that the restriction of $\pi^\sharp$ to $N_x^* X$ is injective:

$$0 \to N_x^* X \xrightarrow{\pi_x^\sharp} T_x M. \tag{2}$$

For the second condition, note that the kernel of $\omega_x|_{T_x X \cap T_x L}$ is $T_x X \cap \pi_x^\sharp(N_x^* X)$. So the condition that $T_x X \cap T_x L$ be a symplectic subspace is equivalent to

$$T_x X \cap \pi_x^\sharp(N_x^* X) = \{0\}. \tag{3}$$

Since $T_x X$ and $N_x^* X$ have complementary dimensions, (2) and (3) imply the following decomposition, which is equivalent to X being a Poisson transversal:

$$TX \oplus \pi^\sharp(N^* X) = TM|_X. \tag{4}$$

The decomposition of the tangent bundle (4) canonically gives an embedded normal bundle, denoted

$$NX := \pi^\sharp(N^*X) \subset TM|_X,$$

and a corresponding decomposition for the cotangent bundle

$$N^*X \oplus N^\circ X = T^*M|_X.$$

For $\xi \in N_x^*X$ and $\eta \in N_x^\circ X$, we have that $\pi^\sharp(\xi) \in N_xX$, hence $\pi(\xi, \eta) = 0$. This implies that $\pi|_X$ has no mixed component in the decomposition

$$\wedge^2 TM|_X = \wedge^2 TX \oplus (TX \otimes NX) \oplus \wedge^2 NX.$$

Therefore $\pi|_X$ splits as

$$\pi|_X = \pi_X + w_X, \quad \pi_X \in \Gamma(\wedge^2 TX), \quad w_X \in \Gamma(\wedge^2 NX).$$

It is well known that these two tensors satisfy the following properties, but for completeness we include a proof.

Lemma 3. *The bivector π_X is Poisson and w_X, regarded as a 2-form on N^*X, is fiberwise nondegenerate.*

Proof. To prove that π_X is Poisson, we will use Dirac-geometric techniques (for other approaches, see [Crainic and Fernandes 2004; Xu 2003]; for the basics of Dirac geometry, see [Bursztyn and Radko 2003]). It suffices to show that the pullback via the inclusion $i : X \to M$ of the Dirac structure $L_\pi := \{\pi^\sharp(\xi) + \xi : \xi \in T^*M\}$ equals the almost Dirac structure $L_{\pi_X} := \{\pi_X^\sharp(\xi) + \xi : \xi \in T^*X\}$, since this makes L_{π_X} automatically involutive, and hence π_X Poisson. But to show this it suffices to prove the following inclusion:

$$\begin{aligned} L_{\pi_X} &= \{\pi_X^\sharp(\xi) + \xi : \xi \in T^*X\} = \{\pi_X^\sharp(i^*\eta) + i^*\eta : \eta \in N^\circ X\} \\ &= \{\pi^\sharp(\eta) + i^*\eta : \eta \in N^\circ X\} \subset i^*L_\pi, \end{aligned}$$

where we used that $w_X^\sharp(\eta) = 0$, for $\eta \in N^\circ X$.

The map $w_X^\sharp : N^*X \to NX$ is just the restriction of π, which, by the decomposition (4), is a linear isomorphism. ☐

We recall three natural instances of Poisson transversals, which appear throughout Poisson geometry:

Example 4. If π is nondegenerate then X is a Poisson transversal if and only if X is a symplectic submanifold of (M, π).

Example 5. If L is the symplectic leaf of (M, π) through a point $x \in M$, a submanifold X that intersects L transversally at x and is of complementary dimension is a Poisson transversal around x.

Example 6. If (M, π) is a regular Poisson manifold with underlying foliation $\mathcal{F}$ of codimension q, then every submanifold X of dimension q that is transverse to $\mathcal{F}$ is a Poisson transversal.

A very useful — and somewhat surprising — fact about Poisson transversals is that they behave well with respect to Poisson maps:

Lemma 7. *Let $\varphi : (M_0, \pi_0) \to (M_1, \pi_1)$ be a Poisson map and $X_1 \subset M_1$ be a Poisson transversal. Then*:

(1) *φ is transverse to X_1.*

(2) *$X_0 := \varphi^{-1}(X_1)$ is also a Poisson transversal.*

(3) *φ restricts to a Poisson map $\varphi|_{X_0} : (X_0, \pi_{X_0}) \to (X_1, \pi_{X_1})$.*

(4) *The differential of φ along X_0 restricts to a fiberwise linear isomorphism between embedded normal bundles $\varphi_*|_{NX_0} : NX_0 \to NX_1$.*

(5) *The map $F : N^*X_0 \to N^*X_1,\ F(\xi) = (\varphi^*)^{-1}(\xi),\ \xi \in N^*X_0$ is a fiberwise linear symplectomorphism between the symplectic vector bundles*

$$F : (N^*X_0, w_{X_0}) \to (N^*X_1, w_{X_1}).$$

Corollary 8. *Let (M, π) be a Poisson manifold, $X \subset M$ be a Poisson transversal and $W \subset M$ be a Poisson submanifold. Then W and X intersect transversally, and $X \cap W$ is*

- *a Poisson transversal in $(W, \pi|_W)$, and*
- *a Poisson submanifold of (X, π_X).*

Proof of Lemma 7. Consider $x \in X_0$ and let $y := \varphi(x) \in X_1$. Since φ is a Poisson map we have:

$$\pi_1^\sharp(\eta) = \varphi_*\big(\pi_0^\sharp(\varphi^*\eta)\big), \quad \text{for all } \eta \in T_y^*M_1,$$

therefore $\pi_1^\sharp(T_y^*M_1) \subset \varphi_*(T_xM_0)$. But X_1 being a Poisson transversal now implies that φ is transverse to X_1:

$$T_yM_1 = T_yX_1 + \pi_1^\sharp(T_y^*M_1) = T_yX_1 + \varphi_*(T_xM_0).$$

In particular, X_0 is a submanifold of M_0. To show that X_0 is a Poisson transversal, we will prove that the decomposition $TX_0 \oplus \pi_0^\sharp(N^*X_0) = TM_0|_{X_0}$ holds. Note first that

$$T_xX_0 = (\varphi_*)^{-1}(T_yX_1) \quad \text{and} \quad N_x^*X_0 = \varphi^*(N_y^*X_1).$$

Let $v \in T_xM_0$, and decompose $\varphi_*v = u + \pi_1^\sharp(\eta)$, with $u \in T_yX_1$ and $\eta \in N_y^*X_1$. Then $\varphi^*\eta \in N_x^*X_0$ and $w := v - \pi_0^\sharp(\varphi^*\eta)$ projects to u, hence $w \in T_xX_0$. This shows that $v = w + \pi_0^\sharp(\varphi^*\eta) \in T_x X_0 + \pi_0^\sharp(N_x^* X_0)$, hence

$$T_x M_0 = T_x X_0 + \pi_0^\sharp(N_x^* X_0).$$

Counting dimensions, we conclude that this is a direct sum decomposition, and therefore X_0 is a Poisson transversal.

Note, moreover, that φ_* preserves the embedded normal bundles

$$\varphi_*(N_x X_0) = \varphi_*\big(\pi_0^\sharp(N_x^* X_0)\big) = \varphi_*\big(\pi_0^\sharp(\varphi^*(N_y^* X_1))\big) = \pi_1^\sharp(N_y^* X_1) = N_y X_1,$$

and because they have the same rank, $\varphi_*|_{N X_0}$ is a fiberwise isomorphism. Since we also have $\varphi_*(T_x X_0) \subset T_y X_1$, the Poisson condition $\varphi_*(\pi_{0,x}) = \pi_{1,y}$ implies that $\varphi_*(\pi_{X_0,x}) = \pi_{X_1,y}$ and $\varphi_*(w_{X_0,x}) = w_{X_1,y}$. This implies (3) and (4). $\square$

3. The local model

The local model around a Poisson transversal depends on an extra choice:

Definition 9. Let (E, σ) be a symplectic vector bundle over X. A *closed extension* of σ is a closed 2-form $\widetilde{\sigma}$ defined on a neighborhood of X in E, such that its restriction to $TE|_X = TX \oplus E$ equals the trivial extension of σ to $TE|_X$. We denote the space of all closed extensions by $\Upsilon(E, \sigma)$.

Closed extensions always exist, and can be constructed employing the standard de Rham homotopy operator (see, e.g., the extension theorem in [Weinstein 1977]).

In the warm-up for the construction below of the local model, let us revisit the three instances which are generalized by our main result.

Example 10 (Weinstein's symplectic neighborhood theorem [1971]). Let (M, ω) be a symplectic manifold, and $(X, \omega_X) \subset M$ be a symplectic submanifold. The symplectic orthogonal of TX, denoted by $E := TX^\omega$, is a symplectic vector bundle with bilinear form $\sigma := \omega|_E$. The local model around X is given by the closed 2-form $\widetilde{\sigma} + p^*(\omega_X)$ on E, where $p : E \to X$ is the projection and $\widetilde{\sigma} \in \Upsilon(E, \sigma)$. Weinstein's symplectic neighborhood theorem says that a neighborhood of X in (M, ω) is symplectomorphic to a neighborhood of X in $(E, \widetilde{\sigma} + p^*(\omega_X))$.

Example 11 (Weinstein's splitting theorem [1983]). Let (M, π) be a Poisson manifold and let $x \in M$. Let also (L, ω) be the symplectic leaf through $x \in M$, and (X, π_X) a Poisson transversal at x, of complementary dimension. The local model around x is given by the product of Poisson manifolds

$$(T_x L, \omega_x^{-1}) \times (X, \pi_X).$$

Weinstein's splitting theorem (or Darboux–Weinstein theorem) asserts that (M, π) is Poisson diffeomorphic around x to an open set around $(0, x)$ in the local model.

Example 12 (transversals to foliations). Let M be a manifold carrying a smooth (regular) foliation $\mathcal{F}$, and let $X \subset M$ be a submanifold transverse to $\mathcal{F}$,

$$T_xX + T_x\mathcal{F} = T_xM, \quad \text{for all } x \in X.$$

Let $\mathcal{F}_X$ be the induced foliation on X. The local model of the foliation $\mathcal{F}$ around X is $(NX, p^*\mathcal{F}_X)$, where $p : NX \to X$ is the normal bundle to X; note that the leaves of the local model are of the form $p^{-1}(L)$, for L a leaf of $\mathcal{F}_X$. To build an isomorphism between $\mathcal{F}$ and its model around X, consider a metric g on $T\mathcal{F}$ and let $\exp_g : T\mathcal{F} \supset U \to M$ denote the leafwise exponential map of g, i.e., for each leaf L, $\exp_g : (TL \cap U) \to L$ is the (Levi-Civita) exponential map of the Riemannian manifold $(L, g|_L)$. Then $T\mathcal{F}_X^{\perp} \subset T\mathcal{F}|_X$ is a complement to TX in $TM|_X$, and the composition

$$NX \xrightarrow{\sim} T\mathcal{F}_X^{\perp} \xrightarrow{\exp_g} M$$

pulls the foliation $\mathcal{F}$ to the local model.

The idea for constructing the local model around a Poisson transversal is to put the foliation in normal form in the sense of Example 12, and then perform Weinstein's construction of Example 10 along all symplectic leaves simultaneously.

Let (E, σ) be a symplectic vector bundle over a Poisson manifold (X, π_X) with projection $p : E \to X$ and consider a closed extension $\widetilde{\sigma} \in \Upsilon(E, \sigma)$. As mentioned in the introduction, the symplectic leaves of the local model are $(\widetilde{L}, \omega_{\widetilde{L}})$, for (L, ω_L) a symplectic leaf of (X, π_X), where $\widetilde{L} \subset p^{-1}(L)$ is an open set containing L and

$$\omega_{\widetilde{L}} := \widetilde{\sigma}|_{\widetilde{L}} + p^*(\omega_L).$$

To show this construction yields a smooth Poisson bivector around X, we rewrite it using the language of Dirac geometry. Let L_{π_X} be the Dirac structure corresponding to π_X. Dirac structures can be pulled back along submersions. The pullback of L_{π_X} to E, denoted by $p^*(L_{\pi_X})$, has presymplectic leaves $(p^{-1}(L), p^*(\omega_L))$, where (L, ω_L) is a symplectic leaf of π_X. Finally, the gauge transform by $\widetilde{\sigma}$, denoted by $p^*(L_{\pi_X})^{\widetilde{\sigma}}$, has the required effect: it adds to each leaf the restriction of $\widetilde{\sigma}$.

Lemma 13. *Let (E, σ) be a symplectic vector bundle over a Poisson manifold (X, π_X), and let $\widetilde{\sigma} \in \Upsilon(E, \sigma)$ be a closed extension of σ. On a neighborhood $U(\widetilde{\sigma})$ of X in E, we have that the Dirac structure*

$$L(\widetilde{\sigma}) := p^*(L_{\pi_X})^{\widetilde{\sigma}}$$

corresponds to a Poisson structure $\pi(\widetilde{\sigma})$ that decomposes along X as

$$\pi(\widetilde{\sigma})|_X = \pi_X + \sigma^{-1} \in \Gamma(\wedge^2 TX) \oplus \Gamma(\wedge^2 E).$$

Equivalently, (X, π_X) is a Poisson transversal for $\pi(\widetilde{\sigma})$, the canonical normal bundle is $E \subset TE|_X$, and the induced nondegenerate bivector is $w_X = \sigma^{-1}$.

Proof. The condition that $L(\widetilde{\sigma})$ be Poisson is open, thus it suffices to show that $L(\widetilde{\sigma})$ has the expected form along X. This can be easily checked, since

$$p^*(L_{\pi_X})|_X = \{\pi_X^\sharp(\xi) + Y + \xi : \xi \in T^*X,\ Y \in E\},$$

and therefore

$$\begin{aligned} L(\widetilde{\sigma})|_X &= \{\pi_X^\sharp(\xi) + Y + \xi + \iota_Y\sigma : \xi \in T^*X,\ Y \in E\} \\ &= \{\pi_X^\sharp(\xi) + (\sigma^{-1})^\sharp(\eta) + \xi + \eta : \xi \in T^*X,\ \eta \in E^*\} \\ &= \{(\pi_X + \sigma^{-1})^\sharp(\theta) + \theta : \theta \in T^*E|_X\}. \end{aligned}$$

□

Definition 14. The Poisson manifold $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$ from the lemma is called the *local model* associated to (E, σ) and (X, π_X).

If X is a Poisson transversal of a Poisson manifold (M, π), π_X is the induced Poisson structure on X, $E = N^*X$ is the conormal bundle to X and

$$\sigma = -w_X = -(\pi|_{N^*X}),$$

then $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$ is called the *local model of* π *around* X.

Remark 15. We point out that there is a choice in having the local models of π around X live in the *conormal* bundle to X, as opposed to its normal bundle NX, as is typically the case for normal form theorems. In fact, since

$$w_X : (N^*X, -w_X) \to (NX, w_X^{-1})$$

is an isomorphism of symplectic vector bundles, we can translate canonically all our constructions to NX via w_X.

That we chose N^*X instead of NX is meant to emphasize that we regard the conormal N^*X as the more appropriate notion of "contravariant normal", an opinion which is corroborated by the scheme of proof of Theorem 1, where we spread out a tubular neighborhood of X by following contravariant geodesics starting in directions conormal to X.

The construction of the local model depends on the choice of a closed extension. A Poisson version of the Moser argument, which first appeared in [Alekseev and Meinrenken 2007] (see also [Alekseev and Meinrenken 2016]) will be later employed to prove that different extensions induce isomorphic local models.

Lemma 16 (Moser lemma). *Suppose we are given a path of Poisson structures of the form* $t \mapsto \pi_t := \pi^{t\,d\alpha}$, *where* π *is a Poisson structure and* $\alpha \in \Omega^1(M)$. *Then the isotopy* $\phi_{\mathcal{V}}^{t,s}$ *generated by the time-dependent vector field* $\mathcal{V}_t := -\pi_t^\sharp(\alpha)$ *stabilizes* π_t:

$$\phi_{\mathcal{V}*}^{t,s}\pi_s = \pi_t,$$

whenever this is defined.

Proof. Recall that Poisson cohomology is computed by the complex $(\mathfrak{X}^\bullet(M), d_\pi)$, where $d_\pi : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$ is defined by $d_\pi := [\pi, \cdot]$ and $[\cdot,\cdot]$ stands for the Schouten bracket on multivector fields. Moreover, π, regarded as a map $\pi^\sharp : T^*M \to TM$, induces a chain map

$$(-1)^{\bullet+1} \wedge^\bullet \pi^\sharp : (\Omega^\bullet(M), d) \to (\mathfrak{X}^\bullet(M), d_\pi),$$

from the de Rham complex of differential forms, see, e.g., [Dufour and Zung 2005]. In particular,

$$L_{\mathcal{V}_t}\pi_t = \left[\pi_t, \pi_t^\sharp(\alpha)\right] = d_{\pi_t}\pi_t^\sharp(\alpha) = -\wedge^2 \pi_t^\sharp(d\alpha).$$

As maps, this can be written as

$$(L_{\mathcal{V}_t}\pi_t)^\sharp = \pi_t^\sharp \circ (d\alpha)^\flat \circ \pi_t^\sharp,$$

where $(d\alpha)^\flat : TM \to T^*M$ stands for $d\alpha$ regarded as a map. Also, by the very definition of gauge transformation, we have the identity $\pi^\sharp = \pi_t^\sharp \circ \left(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\right)$, whence

$$\begin{aligned}
0 = \frac{d\pi^\sharp}{dt} &= \frac{d\pi_t^\sharp}{dt} \circ \left(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\right) + \pi_t^\sharp \circ (d\alpha)^\flat \circ \pi^\sharp \\
&= \left(\frac{d\pi_t^\sharp}{dt} + \pi_t^\sharp \circ (d\alpha)^\flat \circ \pi_t^\sharp\right) \circ \left(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\right) \\
&= \left(\frac{d\pi_t}{dt} + L_{\mathcal{V}_t}\pi_t\right)^\sharp \circ \left(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\right).
\end{aligned}$$

Finally, we obtain

$$\frac{d}{dt}(\phi_{\mathcal{V}}^{t,s})^*\pi_t = (\phi_{\mathcal{V}}^{t,s})^*\left(L_{\mathcal{V}_t}\pi_t + \frac{d\pi_t}{dt}\right) = 0,$$

showing that $(\phi_{\mathcal{V}}^{t,s})^*\pi_t = \pi_s$. $\square$

Next, we show that different choices of closed extensions yield isomorphic local models.

Lemma 17. *If (E,σ) is a symplectic vector bundle over a Poisson manifold (X,π_X), then all corresponding local models are isomorphic around X by diffeomorphisms that fix X up to first order.*

Proof. If $\widetilde{\sigma}_1 \in \Upsilon(E,\sigma)$ is a second extension, $\widetilde{\sigma}_1 - \widetilde{\sigma}$ is a closed 2-form on E that vanishes on $TE|_X$. Since the inclusion $X \subset E$ is a homotopy equivalence, $\widetilde{\sigma}_1 - \widetilde{\sigma}$ is exact, and one can choose a primitive $\eta \in \Omega^1(E)$ that vanishes on $TE|_X$. Actually, by the relative Poincaré lemma in [Weinstein 1977], one may choose η with vanishing first derivatives along X. Denote $\pi(\widetilde{\sigma})$ and $\pi(\widetilde{\sigma} + d\eta)$ by π_0 and π_1, respectively. Then π_1 is the gauge transform by $d\eta$ of π_0, denoted $\pi_1 = \pi_0^{d\eta}$. These bivectors can be interpolated by the family of Poisson structures

$$\pi_t := \pi_0^{t\,d\eta}, \quad t \in [0,1].$$

Now, π_t corresponds to the smooth family of Dirac structures $L_t := p^*(L_{\pi_X})^{\widetilde{\sigma}+t\,d\eta}$, and the set $U \subset \mathbb{R} \times E$ of those points (t,x) where $L_{t,x}$ is Poisson is open. Since $[0,1] \times X \subset U$, there is an open neighborhood V of X in E such that $[0,1] \times V \subset U$. Thus, π_t is defined on V for all $t \in [0,1]$. By the Moser lemma (Lemma 16), we see that the flow of the time-dependent vector field

$$Y_t := -\pi_t^\sharp(\eta)$$

trivializes the family, i.e., $(\phi_Y^{t,s})^*(\pi_t) = \pi_s$ whenever it is defined. Since η and its first derivatives vanish along X, it follows that $\phi_Y^{t,s}$ fixes X and that its differential is the identity on $TE|_X$. Arguing as before, the set where $\phi_Y^{t,0}$ is defined up to $t = 1$ contains an open neighborhood $V' \subset V$ of X, so we obtain a Poisson diffeomorphism

$$\phi_Y^{1,0} : (V', \pi_0) \xrightarrow{\sim} \big(\phi_Y^{1,0}(V'), \pi_1\big). \qquad \square$$

4. The normal form theorem

The normal form theorem (Theorem 1) for a Poisson structure (M,π) around a Poisson transversal X states that π and its local model (built out of $\pi|_X$) are isomorphic around X. In the symplectic case, this follows from the Moser argument in a straightforward manner. For general Poisson manifolds, the proof is more involved. The main difficulty is to put the foliation in normal form; namely, to find a tubular neighborhood of X along the leaves of π. If the foliation is regular, such a construction can be performed by restricting a metric to the leaves and taking leafwise the Riemannian exponential (cf. Example 12). If π is not regular, it is not a priori clear if these maps glue to a smooth tubular neighborhood of X in M. We will use instead a contravariant version of this argument in which we replace the classical exponential from Riemannian geometry by its Poisson-geometric analog: the contravariant exponential. The more surprising outcome is that a contravariant exponential not only puts the foliation in normal form, but also provides a closed extension *and* the required isomorphism to the local model. A funny consequence is that a choice of Poisson spray $\mathcal{X}$ for (M,π) puts *all* of its Poisson transversals in normal form canonically and simultaneously!

We start by recalling some notions and results from contravariant geometry.

Definition 18. A *Poisson spray* $\mathcal{X} \in \mathfrak{X}^1(T^*M)$ on a Poisson manifold (M,π) is a vector field on T^*M such that

(1) $p_*\mathcal{X}(\xi) = \pi^\sharp(\xi)$, for all $\xi \in T^*M$,

(2) $m_t^*\mathcal{X} = t\mathcal{X}$, for all $t > 0$,

where $p : T^*M \to M$ is the projection and $m_t : T^*M \to T^*M$ is the multiplication by t. The flow $\phi^t_{\mathcal{X}}$ of $\mathcal{X}$ is called the *geodesic flow*.

The *contravariant exponential* of $\mathcal{X}$ is the map

$$\exp_{\mathcal{X}} : U \to M, \qquad \xi \mapsto p \circ \phi^1_{\mathcal{X}}(\xi),$$

on an open set $U \subset T^*M$ where the geodesic flow is defined up to time 1. By abuse of notation, we will write $\exp_{\mathcal{X}} : T^*M \to M$, as if it were defined on T^*M.

Poisson sprays exist on every Poisson manifold. For example, if ∇ is a connection on T^*M, then the map that associates to $\xi \in T^*M$ the horizontal lift of $\pi^\sharp(\xi)$ is a Poisson spray.

The main feature of Poisson sprays is that they produce symplectic realizations.

Theorem 19 [Crainic and Mărcuț 2011]. *Given (M, π) a Poisson manifold and $\mathcal{X}$ a Poisson spray, there exists an open neighborhood $\Sigma \subset T^*M$ of the zero section, on which the average of the canonical symplectic structure $\omega_{\text{can}} \in \Omega^2(T^*M)$ under the geodesic flow*

$$\Omega_{\mathcal{X}} := \int_0^1 (\phi^t_{\mathcal{X}})^* \omega_{\text{can}}\, dt, \tag{5}$$

is a symplectic structure on Σ, and the projection $p : (\Sigma, \Omega_{\mathcal{X}}) \to (M, \pi)$ is a symplectic realization (i.e., a surjective Poisson submersion).

Let $X \subset (M, \pi)$ be a Poisson transversal. As before, we denote by π_X the induced Poisson structure on X, and by $w_X := \pi|_{N^*X}$. We are ready to state the main result of this paper.

Theorem 20 (detailed version of Theorem 1). *Let (M, π) be a Poisson manifold and let $X \subset M$ be a Poisson transversal. A Poisson spray $\mathcal{X}$ induces a closed extension of $\sigma := -w_X$ in a neighborhood of X in N^*X, given by*

$$\widetilde{\sigma}_{\mathcal{X}} := -\Omega_{\mathcal{X}}|_{N^*X} \in \Upsilon(N^*X, \sigma).$$

The corresponding local model $\pi(\widetilde{\sigma}_{\mathcal{X}})$ is isomorphic to π around X. Explicitly, a Poisson diffeomorphism between open sets around X is given by the map

$$\exp_{\mathcal{X}}|_{N^*X} : (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}})) \xrightarrow{\sim} (M, \pi).$$

For the proof of Theorem 20, we need some properties of dual pairs. Recall from [Weinstein 1971]:

Definition 21. A *dual pair* consists of a symplectic manifold (Σ, Ω), two Poisson manifolds (M_0, π_0) and (M_1, π_1), and two Poisson submersions

$$(M_0, \pi_0) \xleftarrow{s} (\Sigma, \Omega) \xrightarrow{t} (M_1, \pi_1)$$

with symplectically orthogonal fibers:

$$\ker ds^{\Omega} = \ker dt.$$

The pair is called a *full dual pair*, if s and t are surjective.

Dual pairs and Poisson transversals interact pretty well, as the following shows:

Lemma 22. *Let $(M_0,\pi_0) \xleftarrow{s} (\Sigma,\Omega) \xrightarrow{t} (M_1,\pi_1)$ be a dual pair, and let $X_0 \subset M_0$ and $X_1 \subset M_1$ be Poisson transversals. Then $\overline{\Sigma} := s^{-1}(X_0) \cap t^{-1}(X_1)$ is a symplectic submanifold that fits into the dual pair*

$$(X_0,\pi_{X_0}) \xleftarrow{s} (\overline{\Sigma},\Omega|_{\overline{\Sigma}}) \xrightarrow{t} (X_1,\pi_{X_1}).$$

Proof. First note that $\overline{\Sigma}$ is the inverse image of the Poisson transversal $X_0 \times X_1$ under the Poisson map

$$(s,t) : (\Sigma,\Omega) \to (M_0,\pi_0) \times (M_1,\pi_1).$$

By Lemma 7, (s,t) is transverse to $X_0 \times X_1$, $\overline{\Sigma}$ is a symplectic manifold and (s,t) restricts to a Poisson map

$$(s,t) : (\overline{\Sigma},\Omega|_{\overline{\Sigma}}) \to (X_0,\pi_{X_0}) \times (X_1,\pi_{X_1}).$$

It remains to show that the maps

$$\overline{s} := s|_{\overline{\Sigma}} : \overline{\Sigma} \to X_0 \quad \text{and} \quad \overline{t} := t|_{\overline{\Sigma}} : \overline{\Sigma} \to X_1$$

are submersions with symplectically orthogonal fibers. Let $m_i := \dim(M_i)$ and $x_i := \dim(X_i)$. The fact that s and t are submersions with orthogonal fibers implies that $\dim(\Sigma) = m_0 + m_1$. By transversality of (s,t) and $X_0 \times X_1$, we have that $\operatorname{codim}(\overline{\Sigma}) = \operatorname{codim}(X_0 \times X_1)$; thus $\dim(\overline{\Sigma}) = x_0 + x_1$. Now, for a point $p \in \overline{\Sigma}$, one clearly has $\ker d_p\overline{t} \subset (\ker d_p\overline{s})^{\Omega|_{\overline{\Sigma}}}$, and since $\overline{\Sigma}$ is symplectic, it follows that

$$\dim(\ker d_p\overline{s}) + \dim(\ker d_p\overline{t}) \leq \dim(\overline{\Sigma}) = x_0 + x_1.$$

On the other hand, we have that $\dim(\ker d_p\overline{s}) \geq \dim(\overline{\Sigma}) - \dim(X_0) = x_1$, and similarly $\dim(\ker d_p\overline{t}) \geq x_0$, so we obtain $\dim(\ker d_p\overline{s}) = x_1$ and $\dim(\ker d_p\overline{t}) = x_0$. This implies that $d_p\overline{s}$ and $d_p\overline{t}$ are surjective, and that $\ker d_p\overline{s}$ and $\ker d_p\overline{t}$ are symplectically orthogonal. $\square$

Lemma 23 shows how π_0, π_1 and Ω are related.

Lemma 23. *Let $(M_0,\pi_0) \xleftarrow{s} (\Sigma,\Omega) \xrightarrow{t} (M_1,\pi_1)$ be a dual pair. Then the Dirac structures L_{π_i} corresponding to π_i satisfy the following relation*:

$$s^*(L_{\pi_0})^{-\Omega} = t^*(L_{-\pi_1}).$$

Proof. An element $\chi \in s^*(L_{\pi_0})^{-\Omega}$ is of the form

$$\chi = Y + s^*\xi - \iota_Y\Omega, \quad \text{where } \xi \in T^*M_0,\ s_*Y = \pi_0^\sharp(\xi).$$

Then, since $s_*\Omega^{-1}(s^*\xi) = \pi_0^\sharp(\xi)$, we have that

$$Y - (\Omega^{-1})^\sharp(s^*\xi) \in \ker ds = (\Omega^{-1})^\sharp(t^*T^*M_1).$$

Hence there is $\eta \in T^*M_1$ such that

$$Y = (\Omega^{-1})^\sharp(s^*\xi) - (\Omega^{-1})^\sharp(t^*\eta).$$

Applying t_* and Ω (separately) to both sides, we find that

$$t_*Y = -t_*(\Omega^{-1})^\sharp(t^*\eta) = -\pi_1^\sharp(\eta) \quad \text{and} \quad s^*\xi - \iota_Y\Omega = t^*\eta,$$

and hence

$$\chi = Y + s^*\xi - \iota_Y\Omega = Y + t^*\eta \in t^*(L_{-\pi_1}).$$

This shows one inclusion; the other follows by symmetry. □

As a first step towards the proof of Theorem 20, we analyze what happens infinitesimally.

Lemma 24. *We have that $\widetilde{\sigma}_{\mathcal{X}}$ extends σ, $\widetilde{\sigma}_{\mathcal{X}} \in \Upsilon(N^*X, \sigma)$, and that $\exp_{\mathcal{X}}$ is a diffeomorphism between open sets around X.*

Proof. We identify the zero section of T^*M with M, and for $x \in M$, we identify $T_x(T^*M) = T_xM \oplus T_x^*M$. The properties of the Poisson spray imply that the geodesic flow fixes M, and that its differential along M is given (see [Crainic and Mărcuț 2011]) by

$$d_x\phi_{\mathcal{X}}^t : T_xM \oplus T_x^*M \to T_xM \oplus T_x^*M, \quad (Y, \xi) \mapsto (Y + t\pi^\sharp(\xi), \xi).$$

In particular, $\exp_{\mathcal{X}} = p \circ \phi_{\mathcal{X}}^1$ is a diffeomorphism around X, restricting to the identity along X, and the following formula for $\Omega_{\mathcal{X}}$ holds along M:

$$\Omega_{\mathcal{X}}((Y_1, \xi_1), (Y_2, \xi_2)) = \xi_2(Y_1) - \xi_1(Y_2) + \pi(\xi_1, \xi_2).$$

Taking $(Y_i, \xi_i) \in T_xX \oplus N_x^*X = T_x(N^*X)$, for $x \in X$, we obtain

$$\Omega_{\mathcal{X}}((Y_1, \xi_1), (Y_2, \xi_2)) = \pi(\xi_1, \xi_2) = w_X(\xi_1, \xi_2),$$

showing that $\widetilde{\sigma}_{\mathcal{X}} \in \Upsilon(N^*X, -w_X)$. □

Next, we observe that Theorem 19 implies the existence of self-dual pairs.

Lemma 25. *Let $\mathcal{X}$ be a Poisson spray on the Poisson manifold (M, π), and denote by $\Omega_{\mathcal{X}}$ the symplectic form from Theorem 19. On an open neighborhood of the zero* *section $\Sigma \subset T^*M$ we have a full dual pair*:

$$(M,\pi) \xleftarrow{\ p\ } (\Sigma, \Omega_{\mathcal{X}}) \xrightarrow{\exp_{\mathcal{X}}} (M,-\pi).$$

Proof. Let Σ be an open neighborhood of the zero section on which the geodesic flow $\phi^t_{\mathcal{X}}$ is defined for all $t \in [0,1]$, and on which $\Omega_{\mathcal{X}}$ is nondegenerate. In the proof of Theorem 19 from [Crainic and Mărcuț 2011] it is shown that the symplectic orthogonals of the fibers p are the fibers of $\exp_{\mathcal{X}}$. To show that $\exp_{\mathcal{X}}$ pushes $\Omega^{-1}_{\mathcal{X}}$ down to a bivector on M, one could invoke Libermann's theorem, and then, using the formulas from the proof of Lemma 24, one could check that along the zero section this bivector is indeed $-\pi$. We adopt a more direct approach. First, note that $-\mathcal{X}$ is a Poisson spray for $-\pi$, and that on $\Sigma_- := \phi^1_{\mathcal{X}}(\Sigma)$, the geodesic flow of $-\mathcal{X}$ is defined up to time 1. Moreover, $\Omega_{-\mathcal{X}}$ is nondegenerate on Σ_-, because

$$\begin{aligned}(\phi^1_{\mathcal{X}})^*\Omega_{-\mathcal{X}} &= \int_0^1 (\phi^1_{\mathcal{X}})^*(\phi^t_{-\mathcal{X}})^*\omega_{\mathrm{can}}\,dt = \int_0^1 (\phi^{1-t}_{\mathcal{X}})^*\omega_{\mathrm{can}}\,dt = \int_0^1 (\phi^t_{\mathcal{X}})^*\omega_{\mathrm{can}}\,dt \\ &= \Omega_{\mathcal{X}}.\end{aligned}$$

This also finishes the proof, since $\exp_{\mathcal{X}}$ is the composition of Poisson maps:

$$(\Sigma, \Omega_{\mathcal{X}}) \xrightarrow{\phi^1_{\mathcal{X}}} (\Sigma_-, \Omega_{-\mathcal{X}}) \xrightarrow{\ p\ } (M,-\pi). \qquad \square$$

We are ready to conclude the proof.

Proof of Theorem 20. We use the self-dual pair from Lemma 25, which, by abuse of notation, we write as if it were defined on the entire T^*M:

$$(M,\pi) \xleftarrow{\ p\ } (T^*M, \Omega_{\mathcal{X}}) \xrightarrow{\exp_{\mathcal{X}}} (M,-\pi).$$

Using Lemma 22, we take the preimage under $(p, \exp_{\mathcal{X}})$ of $X \times M$ to obtain a new dual pair (again, the maps are defined only around X),

$$(X,\pi_X) \xleftarrow{\ p\ } (T^*M|_X, \Omega_{\mathcal{X}}|_{T^*M|_X}) \xrightarrow{\exp_{\mathcal{X}}|_{T^*M|_X}} (M,-\pi).$$

By Lemma 23, we have the following equality of Dirac structures:

$$p^*(L_{\pi_X})^{-\Omega_{\mathcal{X}}|_{T^*M|_X}} = (\exp_{\mathcal{X}}|_{T^*M|_X})^*(L_\pi).$$

Since the left-hand side restricts along N^*X to the Dirac structure of the local model $\pi(\widetilde{\sigma}_{\mathcal{X}})$, we have

$$L_{\pi(\widetilde{\sigma}_{\mathcal{X}})} = (\exp_{\mathcal{X}}|_{N^*X})^*(L_\pi).$$

Since $\exp_{\mathcal{X}}|_{N^*X}$ is a diffeomorphism around X (Lemma 24), we see that it is a Poisson diffeomorphism around X:

$$\exp_{\mathcal{X}}|_{N^*X} : (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}})) \xrightarrow{\ \sim\ } (M,\pi). \qquad \square$$

5. Application: equivariant Weinstein splitting theorem

As an application of the normal form theorem (or rather of its proof), we obtain an equivariant version of Weinstein's splitting theorem around fixed points. A version of this result with extra assumptions was obtained in [Miranda and Zung 2006].

Theorem 26 (detailed version of Theorem 2). *Let (M,π) be a Poisson manifold and G a compact Lie group acting by Poisson diffeomorphisms on M. If $x\in M$ is a fixed point of G, then there are coordinates $(p_1,\dots,p_n,q_1,\dots,q_n,y_1,\dots,y_m)\in\mathbb{R}^{2n+m}$ centered at x such that*

$$\pi=\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}+\frac{1}{2}\sum_{j,k=1}^{m}\varpi_{j,k}(y)\frac{\partial}{\partial y_j}\wedge\frac{\partial}{\partial y_k},\quad \varpi_{j,k}(0)=0,$$

and in these coordinates G acts linearly and keeps the subspaces $\mathbb{R}^{2n}\times\{0\}$ and $\{0\}\times\mathbb{R}^m$ invariant.

In other words, (M,π) is G-equivariantly Poisson diffeomorphic around x to an open set around $(0,x)$ in the product

$$(T_xL,\omega_x^{-1})\times(X,\pi_X),\tag{6}$$

where (L,ω) is the symplectic leaf through x, X is a G-invariant Poisson transversal of complementary dimension, and G acts diagonally on (6).

On equivariant symplectic trivializations. In the proof of Theorem 26 we will use a lemma on equivariant trivializations of symplectic vector bundles, which we present here. We start with a result about symplectic vector spaces:

Lemma 27. *Let (V,ω_0) be a symplectic vector space. There exist an open neighborhood $\mathcal{U}(\omega_0)$ of ω_0 in $\wedge^2V^*$, invariant under the group $\mathrm{Sp}(V,\omega_0)$ of linear symplectomorphisms of ω_0, and a smooth map*

$$b:\mathcal{U}(\omega_0)\to \mathrm{Gl}(V),\quad \omega\mapsto b_\omega,$$

satisfying

$$b_\omega^*(\omega_0)=\omega,\quad b_{\omega_0}=\mathrm{id},\quad s^{-1}\circ b_\omega\circ s=b_{s^*(\omega)},$$

for all $\omega\in\mathcal{U}(\omega_0)$ and all $s\in\mathrm{Sp}(V,\omega_0)$.

Proof. On the open set $\mathbb{O}:=\mathbb{C}\setminus(-\infty,0]$ consider the holomorphic square root,

$$\sqrt{(\cdot)}:\mathbb{O}\to\mathbb{C},\quad \sqrt{e^{a+i\theta}}:=e^{a/2+i\theta/2},\quad a\in\mathbb{R},\ \theta\in(-\pi,\pi).$$

Denote the set of linear isomorphisms of V with eigenvalues in $\mathbb{O}$ by $\mathbb{O}(V)\subset\mathrm{Gl}(V)$. By holomorphic functional calculus [Wikipedia 2013], there is an "extension" of the square root to $\mathbb{O}(V)$, which satisfies

$$\left(\sqrt{x}\right)^2 = x,\quad \sqrt{x^{-1}} = \left(\sqrt{x}\right)^{-1},\quad \sqrt{(y\circ x\circ y^{-1})} = y\circ\sqrt{x}\circ y^{-1},\quad \sqrt{x^*} = \left(\sqrt{x}\right)^*,$$

for every $x \in \mathbb{O}(V)$ and every linear isomorphism $y : V \to W$.

Consider $\mathcal{U}(\omega_0) := \{\omega_0 \circ x \mid x \in \mathbb{O}(V)\}$, and define the map

$$b : \mathcal{U}(\omega_0) \to \mathrm{Gl}(V), \quad b_\omega := \sqrt{\omega_0^{-1} \circ \omega}.$$

Note that via the identification $\wedge^2 V^* \subset \mathrm{Hom}(V, V^*)$, the action of $\mathrm{Gl}(V)$ on $\wedge^2 V^*$ becomes $y^*(\omega) = y^* \circ \omega \circ y$. Let $\omega = \omega_0 \circ x \in \mathcal{U}(\omega_0)$, with $x \in \mathbb{O}(V)$ and $s \in \mathrm{Sp}(V, \omega_0)$. The following shows that $\mathcal{U}(\omega_0)$ is $\mathrm{Sp}(V, \omega_0)$-invariant:

$$s^*(\omega) = s^* \circ \omega_0 \circ x \circ s = (s^* \circ \omega_0 \circ s) \circ (s^{-1} \circ x \circ s) = \omega_0 \circ s^{-1} \circ x \circ s \in \mathcal{U}(\omega_0).$$

For the next condition, note first that

$$b_\omega^* = \left(\sqrt{\omega_0^{-1} \circ \omega}\right)^* = \sqrt{\omega \circ \omega_0^{-1}} = \omega_0 \circ b_\omega \circ \omega_0^{-1},$$

therefore

$$b_\omega^*(\omega_0) = b_\omega^* \circ \omega_0 \circ b_\omega = \omega_0 \circ b_\omega^2 = \omega.$$

Finally, for $s \in \mathrm{Sp}(V, \omega_0)$, we have that

$$\begin{aligned} s^{-1} \circ b_\omega \circ s &= \sqrt{s^{-1} \circ \omega_0^{-1} \circ \omega \circ s} = \sqrt{s^{-1} \circ \omega_0^{-1} \circ (s^*)^{-1} \circ s^* \circ \omega \circ s} \\ &= \sqrt{(s^*(\omega_0))^{-1} \circ s^*(\omega)} = b_{s^*(\omega)}. \end{aligned}$$ □

Remark 28. The lemma can also be proved using the Moser argument. First note that $\mathcal{U}(\omega_0)$ can be described as the set of 2-forms $\omega \in \wedge^2 V^*$ for which $\omega_t := t\omega_0 + (1-t)\omega$ is nondegenerate for all $t \in [0, 1]$. The 2-form $\omega - \omega_0$ has a canonical primitive given by $\eta := \frac{1}{2}\iota_\xi(\omega - \omega_0)$, where ξ is the Euler vector field of V. Let $X_t(\omega)$ be the time-dependent vector field defined by the equation $\iota_{X_t(\omega)}\omega_t = \eta$. The Moser argument shows that the time t flow of $X_t(\omega)$ pulls $t\omega_0 + (1-t)\omega$ to ω, and one can easily check that b_ω is the time-one flow of $X_t(\omega)$.

Lemma 29. *Let $(E, \sigma) \to X$ be a symplectic vector bundle, and let G be a compact group acting on E by symplectic vector bundle automorphisms. If $x \in X$ is a fixed point, there exist an invariant open set $U \subset X$ around x and a G-equivariant symplectic vector bundle isomorphism,*

$$(E, \sigma|_U) \xrightarrow{\sim} (E_x \times U, \sigma_x),$$

where the action of G on $E_x \times U$ is the product one.

Proof. We first construct a G-equivariant product decomposition. Let U be a G-invariant open set over which E trivializes, and fix a trivialization $E|_U \cong E_x \times U$. The action of G on $E_x \times U$ is of the form $g(e,y) = (\rho_y(g)e, gy)$. To make the action diagonal, we apply the vector bundle isomorphism,

$$\alpha : E_x \times U \xrightarrow{\sim} E_x \times U, \quad (e,y) \mapsto (A_y(e), y), \quad A_y := \int_G \rho_x(g)^{-1}\rho_y(g)d\mu(g),$$

where μ is the Haar measure on G. Note that A_y is a linear isomorphism for y near x, and that it satisfies

$$A_{gy} \circ \rho_y(g) = \rho_x(g) \circ A_y.$$

Thus, by shrinking U, we may assume that the action on $E_x \times U$ is the product action, which we simply denote by $g(e,y) = (ge, gy)$.

The symplectic structures are given by a smooth family $\{\sigma_y\}_{y\in U}$ of bilinear forms on E_x. This family is G-invariant, in the sense that it satisfies

$$\sigma_{gy} = (g^{-1})^*(\sigma_y), \quad g \in G,\ y \in U.$$

Consider the open set $\mathcal{U}(\sigma_x) \subset \wedge^2 E_x^*$ and the map $b : \mathcal{U}(\sigma_x) \to \mathrm{Gl}(E_x)$ from the previous lemma. By shrinking U, we may assume that $\sigma_y \in \mathcal{U}(\sigma_x)$, for all $y \in U$. Since $b^*_{\sigma_y}(\sigma_x) = \sigma_y$, we have a "canonical" symplectic trivialization:

$$\beta : E_x \times U \xrightarrow{\sim} E_x \times U, \quad (e,y) \mapsto (b_{\sigma_y} e, y).$$

Now $g^{-1} : E_x \to E_x$ preserves σ_x, so

$$b_{\sigma_{gy}} = b_{(g^{-1})^*\sigma_y} = g \circ b_{\sigma_y} \circ g^{-1}.$$

Equivalently, the map β is G-equivariant:

$$\beta(ge, gy) = (b_{\sigma_{gy}} ge, gy) = (g b_{\sigma_y} e, gy) = g\beta(e,y).$$

Thus, $\beta \circ \alpha$ is an isomorphism of symplectic vector bundles that trivializes the symplectic structure, and turns the G-action into the product one. □

Proof of Theorem 26. We split the proof into four steps.

Step 1: a G-invariant transversal. Let (L,ω) denote the leaf through x. Since x is a fixed point, it follows that G preserves L. Thus G acts by symplectomorphisms on (L,ω).

We fix $X \subset M$, a G-invariant transversal through x such that $\dim(L)+\dim(X) = \dim(M)$. The existence of such a transversal follows from Bochner's linearization theorem: the action around x is isomorphic to the linear action of G on T_xM; by choosing a G-invariant inner product on T_xM, we let X be an invariant ball around the origin in the orthogonal complement of T_xL.

Let $\pi|_X = \pi_X + w_X$ denote the decomposition of π along X. Then G acts by Poisson diffeomorphisms on (X, π_X), and by symplectic vector bundle automorphisms on $(N^*X, -w_X)$.

Step 2: the G-invariant spray. Let $\mathcal{X}$ be a G-invariant Poisson spray. Such a vector field can be constructed by averaging any Poisson spray; the conditions that a vector field on T^*M be a Poisson spray are affine. The flow of $\mathcal{X}$ is therefore G-equivariant. By Theorem 20, and with the notations used there, we obtain a G-equivariant Poisson diffeomorphism around X,

$$\exp_{\mathcal{X}} : (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}})) \to (M, \pi),$$

where $\widetilde{\sigma}_{\mathcal{X}} \in \Upsilon(N^*X, -w_X)$ is automatically G-invariant.

Step 3: a G-equivariant symplectic trivialization. Note first that w_X, regarded as a map $N^*X \to TM|_X$, yields a symplectic isomorphism,

$$w_{X,x} : (N^*_x X, -w_{X,x}) \xrightarrow{\sim} (T_x L, \omega_x).$$

This remark and Lemma 29 imply that around the fixed point x, by shrinking X if necessary, we can simultaneously trivialize the bundle $(N^*X, -w_X)$ symplectically and turn the action to a product action, hence, we obtain a G-equivariant symplectic vector bundle isomorphism

$$\Psi : \big(\mathrm{pr}_2 : (T_x L, \omega_x) \times X \to X\big) \xrightarrow{\sim} \big(p : (N^*X, -w_X) \to X\big),$$

where the action on $T_x L \times X$ is the product action. Therefore, $\widetilde{\omega}_{\mathcal{X}} := \Psi^*(\widetilde{\sigma}_{\mathcal{X}})$ is a closed G-invariant extension of ω_x, i.e., $\widetilde{\omega}_{\mathcal{X}} \in \Upsilon(T_x L \times X, \omega_x)$. Moreover, the map

$$\Psi : (T_x L \times X, \pi(\widetilde{\omega}_{\mathcal{X}})) \xrightarrow{\sim} (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}}))$$

is a G-equivariant Poisson diffeomorphism, where $\pi(\widetilde{\omega}_{\mathcal{X}})$ denotes the Poisson structure around X corresponding to the Dirac structure $\mathrm{pr}_2^*(L_{\pi_X})^{\widetilde{\sigma}_{\mathcal{X}}}$.

Step 4: the G-equivariant Moser argument. Note that ω_x has a second extension to $T_x L \times X$ given by $\overline{\omega}_x := \mathrm{pr}_1^*(\omega_x)$. The corresponding local model is the Poisson structure from the statement

$$(T_x L \times X, \pi(\overline{\omega}_x)) = (T_x L, \omega_x^{-1}) \times (X, \pi_X).$$

By Steps 2 and 3, we are left to find a G-equivariant diffeomorphism around X that sends $\pi(\widetilde{\omega}_{\mathcal{X}})$ to $\pi(\overline{\omega}_x)$. For this we need the equivariant version of Lemma 17, whose proof can be easily adapted to this setting: first, note that the 2-form $\overline{\omega}_x - \widetilde{\omega}_{\mathcal{X}}$ has a primitive

$$\eta \in \Omega^1(T_x L \times X)$$

such that $\eta_{(0,y)} = 0$ for all $y \in X$. Since both $\overline{\omega}_X$ and $\widetilde{\omega}_{\mathcal{X}}$ are G-invariant, by averaging, we can make η G-invariant as well. Consider the time-dependent vector field,

$$Y_t := -\pi_t^\sharp(\eta),$$

where $\pi_t := \pi(\widetilde{\omega}_{\mathcal{X}})^{t\,d\eta}$. The time-one flow $\phi_Y^{1,0}$ sends $\pi_0 = \pi(\widetilde{\omega}_{\mathcal{X}})$ to $\pi_1 = \pi(\overline{\omega}_X)$. Since both π_t and η are G-invariant, it follows that $\phi_Y^{1,0}$ is G-equivariant as well. This concludes the proof. $\square$

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PEDRO FREJLICH
DEPARTAMENTO DE MATEMÁTICA
PUC RIO DE JANEIRO
RUA MARQUÊS DE SÃO VICENTE 225
GÁVEA
22451-900 RIO DE JANEIRO
BRAZIL
frejlich.math@gmail.com

IOAN MĂRCUȚ
INSTITUTE FOR MATHEMATICS, ASTROPHYSICS AND PARTICLE PHYSICS
RADBOUD UNIVERSITY NIJMEGEN
6500 GL
NIJMEGEN
THE NETHERLANDS
i.marcut@math.ru.nl

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