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COMAN CONJECTURE FOR THE BIDISC

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We show the equality between the Lempert function and the Green function with two poles with equal weights in the bidisc, thus giving the positive answer to a conjecture of Coman in the simplest unknown case. Actually, we prove a slightly more general equality which in some sense is natural when studied from the point of view of the Nevanlinna–Pick problem in the bidisc.

1. Presentation of the problem and its history

Let D be a domain in $\mathbb{C}^n$ and let $\varnothing \neq P := \{p_1, \dots, p_N\} \subset D$ where $p_j \neq p_k$, $j \neq k$. Let also $\nu : P \to (0, \infty)$. Denote $\nu_j := \nu(p_j)$. Let $z \in D$.

Define $l_D(z; P; \nu) := l_D(z; (p_1, \nu_1), \dots, (p_N, \nu_N))$ as the infimum of the numbers

$$\sum_{j=1}^{N} \nu_j \log |\lambda_j|$$

such that there is an analytic disc $\psi : \mathbb{D} \to D$ with $\psi(0) = z$, $\psi(\lambda_j) = p_j$, $j = 1, \dots, N$.

Recall that $l_D(z; P; \nu) = \min\{l_D(z; A; \nu_{|A}) : \varnothing \neq A \subset P\}$ (see [Nikolov and Pflug 2006] for arbitrary D or [Wikström 2001] for D convex). The last equality will be of interest for us since in the case of taut domains (convex and bounded domains are taut) the infimum in the definition of $l_D(z; P; \nu)$ will be attained by some analytic disc defining $l_D(z; A; \nu_{|A})$ for some $\varnothing \neq A \subset P$.

The function $l_D(\,\cdot\,; P; \nu)$ is called *the Lempert function with the poles at P and with the weight function ν (or weights ν_j).*

Analoguously we define *the pluricomplex Green function $g_D(z; P; \nu)$ with the poles at P and the weight function ν* as the supremum of numbers $u(z)$ over all

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negative plurisubharmonic functions $u : D \to [-\infty, 0)$ with logarithmic poles at P, i.e., such that

$$u(\,\cdot\,) - \nu_j \log \|\,\cdot\, - p_j\|$$

is bounded above near p_j, $j = 1, \dots, N$.

It is trivial that $g_D(z; P; \nu) \le l_D(z; P; \nu)$. D. Coman [2000] conjectured the equality $l_D(\,\cdot\,; P; \nu) = g_D(\,\cdot\,; P; \nu)$ for all convex domains D.

The conjecture has an obvious motivation in the Lempert Theorem [1981] which implies the equality in the case $N = 1$, and in the fact that the equality in the case of the unit ball and two poles with equal weights ($D = \mathbb{B}_n,\ N = 2,\ \nu_1 = \nu_2$) holds (see [Coman 2000] and also [Edigarian and Zwonek 1998]).

The conjecture turned out to be false. The first counterexample was found in [Carlehed and Wiegerinck 2003] ($D := \mathbb{D}^2,\ N = 2$ and different weights). Later a counterexample was found in the case of the bidisc ($D = \mathbb{D}^2$) with $N = 4$ and all weights equal (see [Thomas and Trao 2003]).

The simplest nontrivial case that was not clear yet was the case of the bidisc, two poles and equal weights. Recall that a partial positive answer in this case was found in [Carlehed 1999] (see also [Edigarian and Zwonek 1998]) in the case the poles were lying on $\mathbb{D} \times \{0\}$. In [Wikström 2003] numerical computations were carried out which strongly suggested that the equality in the case $D = \mathbb{D}^2,\ N = 2,\ \nu_1 = \nu_2$ should hold. The aim of this paper is to show that actually the Coman conjecture holds in the bidisc ($D = \mathbb{D}^2$), $N = 2$, two arbitrary poles and $\nu_1 = \nu_2$. In our proof we show even more: the equality of the Carathéodory function (defined below) and the Lempert function with two poles and equal weights in the bidisc. The methods we use originated with the study of the Nevanlinna–Pick problem for the bidisc.

2. Nevanlinna–Pick problem, m-complex geodesics, formulation of the solution

As already mentioned, the aim of the paper is to show a more general result than one claimed in the Coman conjecture for the bidisc, two poles and equal weights. To formulate the main result we need to introduce a new function. Since we shall be interested in equal weights we restrict ourselves from now on to the case when $\nu \equiv 1$. To make the presentation clearer we adopt the notation

$$d_D(z, \{p_1, \dots, p_N\}) := d_D(z; \{(p_1, 1), \dots, (p_N, 1)\})$$

($d = l$ or g) where the $p_j \in D$ are pairwise disjoint, $j = 1, \dots, N$.

Let us recall the definition of *the Carathéodory function with the poles at* p_j (*with weights equal to one*)

(1) $$c_D(z, p_1, \dots, p_N) := \sup\{\log|F(z)| : F \in \mathcal{O}(D, \mathbb{D}),\ F(p_j) = 0,\ j = 1, \dots, N\}.$$

It is simple to see that

$$c_D(\,\cdot\,, p_1, \dots, p_N) \le g_D(\,\cdot\,, p_1, \dots, p_N) \le l_D(\,\cdot\,, p_1, \dots, p_N).$$

Our main result is the following:

Theorem 1. *Let $p, q \in \mathbb{D}^2$ be two distinct points. Then*

$$c_{\mathbb{D}^2}(z; p; q) = l_{\mathbb{D}^2}(z; p; q) \quad \textit{for} \quad z \in \mathbb{D}^2.$$

Note that the function F for which the supremum in the definition of the Carathéodory function is attained always exists. On the other hand, in the case where D is a taut domain, for a point $z \in D$ and pole set P there are always a set $\varnothing \neq Q = \{q_1, \dots, q_M\} \subset P$ and a mapping $f \in \mathcal{O}(\mathbb{D}, D)$, $\lambda_j \in \mathbb{D}$ such that $f(0) = z$, $f(\lambda_j) = q_j$, $j = 1, \dots, M$ and $l_D(z; P) = l_D(z; Q) = \sum_{j=1}^M \log|\lambda_j|$. Consequently, in case the equality $c_D(z; p_1, \dots, p_N) = l_D(z; p_1; \dots; p_N)$ holds, there exist $f \in \mathcal{O}(\mathbb{D}, D)$, $F \in \mathcal{O}(D, \mathbb{D})$ such that $f(0) = z$, $f(\lambda_j) = q_j$, $F(q_j) = 0$, $|F(0)| = \prod_{j=1}^M |\lambda_j|$, $j = 1, \dots, M$, and (thus) $F \circ f$ is a finite Blaschke product of degree $M \le N$. This observation leads us to introduce and consider the notions of m-extremals and m-geodesics.

First recall that given a system of m pairwise different numbers $(\lambda_1, \dots, \lambda_m)$, $\lambda_j \in \mathbb{D}$ and a domain $D \subset \mathbb{C}^n$, a holomorphic mapping $f : \mathbb{D} \to D$ is called *a (weak) m-extremal* for $(\lambda_1, \dots, \lambda_m)$ if there is no holomorphic mapping $g : \mathbb{D} \to D$ such that $g(\mathbb{D}) \Subset D$ and $g(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$. In case f is m-extremal with respect to any choice of m pairwise different arguments the mapping f is called *m-extremal.* A holomorphic mapping $f : \mathbb{D} \to D$ is called *an m-geodesic* if there is an $F \in \mathcal{O}(D, \mathbb{D})$ such that $F \circ f$ is a finite Blaschke product of degree at most $m - 1$. The function F will be called *the left inverse to f*. It is immediate to see that any m-geodesic is an m-extremal.

The notions of (weak) m-extremals and m-geodesics, which have clear origin in Nevanlinna–Pick problems for functions in the unit disk, have been recently introduced and studied in [Agler et al. 2013; 2015], [Kosiński and Zwonek 2016a], [Kosiński 2014] and [Warszawski 2015]. It is worth recalling that the description of m-extremals in the unit disc is classical and well known. The mapping $h \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ is m-extremal for $(\lambda_1, \dots, \lambda_m)$, $\lambda_j \in \mathbb{D}$ if and only if h is a finite Blaschke product of degree at most $m - 1$. Moreover, in such a case the interpolating function is uniquely determined (see [Pick 1915]).

The remark after Theorem 1 on the form of functions for which the extremum in the definition of the Lempert function may be attained may be formulated as follows. For any taut domain D, for any system of poles $P = \{p_1, \dots, p_N\} \subset D$ and any $z \in D \setminus P$ there are a subset $Q = \{q_1, \dots, q_M\} \subset P$ and $f \in \mathcal{O}(\mathbb{D}, D)$ such that $f(\lambda_j) = q_j$, $j = 1, \dots, M$, $f(0) = z$, and f is a weak $(M+1)$-extremal for

$(0, \lambda_1, \dots, \lambda_M)$. Assuming additionally the equality $c_D(z; P) = l_D(z; P)$ would then imply the existence of a special $(M+1)$-geodesic, the one having some subset $Q \subset P$ in its image but such that the left inverse F maps the whole set P to 0. Consequently a necessary (but not sufficient!) condition for having the desired equality at z for the set of poles P is the existence of some $(M+1)$-geodesic passing through a subset $Q \subset P$ and mapping 0 to z.

Below we present a result on uniqueness of left inverses for m-geodesics in convex domains in $\mathbb{C}^2$ which we shall use in a (very special) case of the bidisc. The result is a simple generalization of a similar result formulated for 2-geodesics that can be found in [Kosiński and Zwonek 2016b] (however, for the clarity of the presentation we restrict ourselves to dimension two). We also present its proof here for the sake of completeness.

Lemma 2. *Let D be a convex domain in $\mathbb{C}^2$, $\lambda_j \in \mathbb{D}$, $j = 1, \dots, m$, $m \geq 2$, be pairwise different and let $f, g : \mathbb{D} \to D$ be such that $f(\lambda_j) = g(\lambda_j) =: z_j$ and $f \not\equiv g$. Assume additionally that $F, G \in \mathcal{O}(D, \mathbb{D})$ are such that $F \circ f$ and $G \circ g$ are Blaschke products of degree at most $m-1$. Then $F \equiv G$. Moreover, for any $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{D}$ such that $\mu f(\lambda) + (1-\mu)g(\lambda) \in D$ we have the equality*

$$F(\mu f(\lambda) + (1-\mu)g(\lambda)) = F(f(\lambda)).$$

Proof. For $t \in [0, 1]$ define $h_t := tf + (1-t)f \in \mathcal{O}(\mathbb{D}, D)$. Then $h_t(\lambda_j) = z_j$, $j = 1, \dots, m$, so, due to the uniqueness of the solution of the extremal problem in the disk, we get that $F \circ h_t \equiv G \circ h_t =: B$, $t \in [0, 1]$, is a finite Blaschke product of degree $\leq m-1$. Consequently, we get the equality $F \equiv G$ on the set

$$\{tf(\lambda) + (1-t)g(\lambda) = g(\lambda) + t(f(\lambda) - g(\lambda)) : t \in [0, 1], \lambda \in \mathbb{D}\}.$$

Moreover, the identity principle (applied to the map $\mu \mapsto F(\mu f(\lambda) + (1-\mu)g(\lambda))$) implies that

$$F(\mu f(\lambda) + (1-\mu)g(\lambda)) = G(\mu f(\lambda) + (1-\mu)g(\lambda)) = B(\lambda)$$

for all $(\mu, \lambda) \in V$ where V is the set (domain) of all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{D}$ such that

$$\Phi(\mu, \lambda) := \mu f(\lambda) + (1-\mu)g(\lambda) = g(\lambda) + \mu(f(\lambda) - g(\lambda)) \in D.$$

Note that $V \supset [0, 1] \times \mathbb{D}$. The equality mentioned earlier gives, in particular, $F \equiv G$ on $\Phi(V)$.

Let $\varnothing \neq U \Subset \mathbb{D}$ be a domain such that $f(\lambda) \neq g(\lambda)$, $\lambda \in \bar{U}$, and $B_{|U}$ is injective.

Let $V \supset \Omega := U_1 \times U \supset [0, 1] \times U$ be a domain. We claim that $\Phi_{|\Omega}$ is injective, which would finish the proof as in such a case $\Phi(\Omega)$ would be open and then the application of the identity principle would imply that $F \equiv G$ on D.

To see the injectivity take $(\mu_1,\lambda_1),(\mu_2,\lambda_2)\in\Omega$ such that $\Phi(\mu_1,\lambda_1)=\Phi(\mu_2,\lambda_2)$. Then $B(\lambda_1)=B(\lambda_2)$ so the injectivity of $B_{|U}$ implies that $\lambda_1=\lambda_2$ which (since $f(\lambda_1)-g(\lambda_1)=f(\lambda_2)-g(\lambda_2)\neq 0$) gives the equality $\mu_1=\mu_2$. □

3. Properties of extremals for the Lempert function in case the Coman conjecture holds

Let us now restrict our considerations to the case of the bidisc and two poles $p,q\in\mathbb{D}^2$, $p\neq q$. Without loss of generality we may assume that $z=(0,0)$. Simple continuity properties of the Lempert and Carathéodory function allow us to reduce the Coman conjecture to the proof of the equality

$$c(p,q):=c_{\mathbb{D}^2}((0,0),p,q)=l_{\mathbb{D}^2}((0,0),p,q)=:l(p,q)$$

for (p,q) from some open, dense subset of $\mathbb{D}^2\times\mathbb{D}^2\setminus\triangle$ to be defined later ($\triangle$ denotes the diagonal in the corresponding Cartesian product $X\times X$, here $X=\mathbb{D}^2$).

Below we shall present the starting point for our considerations. The proof contains the reasoning which will lead us to the structure of the proof of the equality $c(p,q)=l(p,q)$ presented later.

Lemma 3. *Let $p,q\in\mathbb{D}^2\setminus\Delta$ be such that $|p_1|\neq|p_2|$, $|q_1|\neq|q_2|$, $p_1\neq q_1$ and $p_2\neq q_2$. Then the equality $c(p,q)=l(p,q)$ holds if an only if one of the following conditions is satisfied*:

(1) *up to a permutation of coordinates $|p_2|<|p_1|$, $|q_2|<|q_1|$ and $m(p_2/p_1,q_2/q_1)\leq m(p_1,q_1)$, or $p_2=\omega p_1$, $q_2=\omega q_1$ for some unimodular ω, where m is the Möbius distance on the disc, see Section 4*,

(2) *there exist α,β,c in the unit disc, a unimodular constant ω, and $t\in(0,1)$ such that an analytic disc where m_α,m_β are (idempotent) Möbius maps*

$$\varphi(\lambda)=\lambda(m_\alpha(\lambda),\omega m_\beta(\lambda)),\quad \lambda\in\mathbb{D},$$

satisfies $\varphi(c)=p$ and $\varphi(m_\gamma(c))=q$, where $\gamma=t\alpha+(1-t)\beta$.

In order to prove Lemma 3 we need the following technical result:

Lemma 4. *Let $\alpha,\beta\in\mathbb{D}$, $\alpha\neq\beta$, $t\in[0,1]$, $\omega,\tau\in\mathbb{T}$. Define*

$$\varphi(\lambda):=\lambda(m_\alpha(\lambda),\omega m_\beta(\lambda))$$

and let

$$G(x):=\frac{tx_1+(1-t)\bar{\omega}x_2+\tau\bar{\omega}x_1x_2}{1+\tau((1-t)x_1+t\bar{\omega}x_2)},\quad x=(x_1,x_2)\in\mathbb{D}^2. \tag{2}$$

Set $G(\varphi(\lambda))=:\lambda f(\lambda)$, $\lambda\in\mathbb{D}$. Denote $f(0)=\gamma:=t\alpha+(1-t)\beta$. Then f is an automorphism of $\mathbb{D}$ (equal to m_γ) if and only if $\tau=\overline{(\alpha-\beta)}/(\alpha-\beta)$.

Proof. The proof of the above lemma reduces to showing that in the inequality $|f'(0)|/(1-|f(0)|^2) \leq 1$ the equality holds if and only if $\tau = \overline{(\alpha-\beta)}/(\alpha-\beta)$ which is elementary although tedious. □

Proof of necessity in Lemma 3. Assume that we have the equality for (p, q). There are two possibilities (up to a permutation of variables p and q):

(i) there exist a holomorphic $\varphi : \mathbb{D} \to \mathbb{D}^2$, $F : \mathbb{D}^2 \to \mathbb{D}$ and $c \in (0, 1)$ such that $\varphi(0) = (0, 0)$, $\varphi(c) = p$ and $F(p) = F(q) = c$, $F(0, 0) = 0$.

Then $F(\varphi(\lambda)) = \lambda$, so $\varphi(\lambda) = (\omega\lambda, \psi(\lambda))$ where $|\omega| = 1$ (up to switching coordinates). If $\psi \notin \operatorname{Aut}(\mathbb{D})$ then Lemma 2 implies that $F(z) = \bar{\omega} z_1$ so $p_1 = q_1$ and $|p_2| \leq |p_1|$.

The second subcase is when $\psi \in \operatorname{Aut}(\mathbb{D})$ and $\psi(0) = 0$. But then $|p_1| = |p_2|$.

(ii) The function φ realizing the infimum is a weak 3-extremal with respect to $(0, c, d)$ such that $\varphi(0) = (0, 0)$, $\varphi(c) = p$, $\varphi(d) = q$. The special left inverse $F : \mathbb{D}^2 \to \mathbb{D}$ would satisfy the equalities $F(p) = F(q) = 0$ and $F(0) = cd$. Consequently $F \circ \varphi = m_c m_d$. We have two possibilities:

(a) φ is a geodesic (2-extremal). This holds if either

- $|p_2| < |p_1|$, $|q_2| < |q_1|$ and $m(p_2/p_1, p_2/p_1) \leq m(p_1, q_1)$, or
- $|p_1| < |p_2|$, $|q_1| < |q_2|$ and $m(p_1/p_2, q_1/q_2) \leq m(p_2, q_2)$, or
- $p_2 = \omega p_1$ and $q_2 = \omega q_1$ for some unimodular ω.

(b) φ is not a 2-extremal. First note that $\varphi(\lambda) = \lambda\psi(\lambda)$ where ψ is a 2-extremal (geodesic). Consequently, up to a permutation of the coordinates, $\varphi(\lambda) = \lambda(m(\lambda), h(\lambda))$, where m is some Möbius map and $h \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. In the case h is not a Möbius map the mapping φ is not uniquely determined — in the sense that for the triple $(0, c, d)$ there also exists another 3-extremal mapping $\tilde{\varphi}$ which maps this triple of numbers to the same triple of points. But existence of the left inverse already gives its uniqueness (see Lemma 2); moreover, it follows from the same lemma that $F(\lambda m(\lambda), \mu) = m_c(\lambda) m_d(\lambda)$ for any $\mu \in \mathbb{D}$, which easily implies that $F(z) = a(z_1)$, where a is some Möbius map. But the last property may hold only if $p_1 = q_1$.

Thus the generic case for φ being a 3-extremal from the definition of the Lempert function which are not 2-extremals is the one given by the formula

$$\varphi(\lambda) = \lambda(\omega' m_\alpha(\lambda), \omega m_\beta(\lambda)), \quad \lambda \in \mathbb{D}, \tag{3}$$

where $\alpha, \beta \in \mathbb{D}$ and $\omega', \omega \in \mathbb{T}$. Multiplying α, β, c, d by a unimodular constant one may assume that $\omega' = 1$.

Our aim is now to show what the necessary form of functions $F \in \mathcal{O}(\mathbb{D}^2, \mathbb{D})$ such that $F \circ \varphi$ is a Blaschke product should be. We present below the reasoning, employing some results of McCarthy and Agler. Let us also mention that G. Knese (personal communication, 2014) let us know about another approach which leads to the same form of left inverses.

We are looking for a form of a function $F:\mathbb{D}^2\to\mathbb{D}$ such that $F\circ\varphi=m_c m_d$. Set $G:=m_{cd}\circ F$. Clearly $G\circ\varphi(0)=0$, so it suffices to consider the following situation:

$$G(\lambda m_\alpha(\lambda),\omega\lambda m_\beta(\lambda))=\lambda m_\gamma(\lambda),\quad \lambda\in\mathbb{D}.$$

We are looking for a formula for G. Note that we consider only the case when $\alpha\neq\beta$. The cases $\gamma=\alpha$ or $\beta=\gamma$ are also excluded.

Assuming that G and γ do exist consider the following Pick problem:

$$\begin{cases}(0,0)\mapsto 0\\ (\gamma m_\alpha(\gamma),\omega\gamma m_\beta(\gamma))\mapsto 0,\\ (\lambda' m_\alpha(\lambda'),\omega\lambda' m_\beta(\lambda'))\mapsto\lambda' m_\gamma(\lambda'),\end{cases}$$

where λ' is any point in $\mathbb{D}$, $\lambda'\neq\lambda$. It is quite clear that this problem is strictly 2-dimensional, extremal and nondegenerate (with the notions understood as defined in [Agler and McCarthy 2002, Chapter 12], itself drawing from [Agler and McCarthy 2000] where the terminology is slightly different). Therefore, it follows from [Agler and McCarthy 2002, Theorem 12.13, p. 201–204] that the above problem has a unique solution which is given by a rational inner function of degree 2, with no terms in x_1^2 or x_2^2. It is easily seen that the solution to this problem is a left inverse we are looking for. Therefore,

$$G(x)=\frac{Ax_1+Bx_2+Cx_1x_2}{1+Dx_1+Ex_2+Gx_1x_2}.$$

Now we proceed in a standard way: comparing multiplicities in the poles of m_α and m_β, etc. After additional calculations we get that $A+\omega B=1$ and then

$$G(x)=\frac{tx_1+(1-t)\bar{\omega}x_2-\eta x_1x_2}{1-((1-t)x_1+tx_2)\omega},\tag{4}$$

where $t\in(0,1)$ and $\eta\in\mathbb{T}$. In particular, $\gamma=t\alpha+(1-t)\beta$. It is clear that $d=m_\gamma(c)$, which finishes the proof of necessity. □

Proof of sufficiency in Lemma 3. Assume first that condition (1) is satisfied. In other words there is $\psi\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})$ is such that $\psi(p_1)=p_2/p_1,\ \psi(q_1)=q_2/q_1$. Let $F(z):=m_{p_1}(z_1)m_{q_1}(z_1),\ z\in\mathbb{D}^2$. Put

$$\varphi(\lambda):=(\lambda,\lambda\psi(\lambda)),\quad\lambda\in\mathbb{D}.$$

Observe that $\varphi(0)=(0,0),\ \varphi(p_1)=p,\ \varphi(q_1)=q,\ F(0,0)=p_1q_1$ and $F(p)=F(q)=0$ which give the equality

$$c(p;q)\le l(p;q)\le\log|p_1q_1|\le c(p;q).$$

Now suppose that (2) holds, i.e., the analytic disc $\lambda \mapsto \varphi(\lambda) = \lambda(m_\alpha(\lambda), \omega m_\beta(\lambda))$ satisfies $\varphi(c) = p$ and $\varphi(m_\gamma(c)) = q$. Let G be the function given by the formula (2) with $\tau = \overline{(\alpha - \beta)}/(\alpha - \beta)$. It follows from Lemma 4 that $G(\varphi(\lambda)) = \lambda m_\gamma(\lambda)$ for $\lambda \in \mathbb{D}$. In particular

$$F := m_{cm_{\gamma(c)}} \circ G$$

satisfies $F \circ \varphi = \tau m_c m_{m_\gamma(c)}$ for some $\tau \in \mathbb{T}$. This gives the equality

$$c(\varphi(c), \varphi(m_\gamma(c))) = l(\varphi(c), \varphi(m_\gamma(c))). \qquad \square$$

The above result is a key one — it will turn out that the set of pairs of points $(\varphi(\lambda), \varphi(m_\gamma(\lambda)))$ (parametrized by $(\alpha, \beta, c, t, \omega)$) will build an open set, which together with the one constructed with the help of extremals for the Lempert functions being 2-geodesics will be dense in $\mathbb{D}^2 \times \mathbb{D}^2$ — that will complete the proof.

4. Proof of the equality $c(p; q) = l(p; q)$

To prove the Coman conjecture for the bidisc we consider open sets in $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$ whose union forms a dense subset of $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$ and on each part the desired equality holds. Let us denote $\sigma(p, q) := ((p_2, p_1), (q_2, q_1)),\ p, q \in \mathbb{D}^2$. Define U as the set of points $(p; q) \in \mathbb{D}^2 \times \mathbb{D}^2$ satisfying the following inequalities

$$|p_2| < |p_1|, |q_2| < |q_1| \quad \text{and} \quad m(p_2/p_1, q_2/q_1) < m(p_1, q_1), \tag{5}$$

where m is the Möbius distance on the unit disc given by the formula $m(\lambda_1, \lambda_2) := \big|(\lambda_1 - \lambda_2)/(1 - \bar{\lambda}_1\lambda_2)\big|$.

Denote

$$\Omega_1 := U \cup \sigma(U).$$

The equality on Ω_1 was proved in Lemma 3.

We shall consider now the set given by 3-geodesics that are not 2-geodesics and that appeared in Lemma 3.

Consider a real-analytic mapping

$$\Phi : \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{T} \times (0, 1) \to \mathbb{D}^2 \times \mathbb{D}^2$$

given by the formula (below and in the sequel $\gamma := t\alpha + (1 - t)\beta$)

$$(\alpha, \beta, c, \omega, t) \mapsto \big(\varphi_{\alpha,\beta,\omega}(c), \varphi_{\alpha,\beta,\omega}(m_\gamma(c))\big),$$

where

$$\varphi_{\alpha,\beta,\omega}(\zeta) := (\omega\zeta m_\alpha(\zeta), \zeta m_\beta(\zeta)), \quad \zeta \in \mathbb{D}.$$

Motivated by the considerations in Section 3 we define open sets.

Denote $\mathcal{A} := \{(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2 : p_1 = q_1 \quad \text{or} \quad p_2 = q_2\}$ and

$$F_1 := \{(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2 : |p_2| > |p_1| \quad \text{and} \quad |q_2| < |q_1|\}. \tag{6}$$

We also define the set F_2 as the set of points $(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2$ satisfying the following inequalities:

$$|p_2| < |p_1| \quad \text{and} \quad |q_2| < |q_1| \quad \text{and} \quad m\Big(\frac{p_2}{p_1}, \frac{q_2}{q_1}\Big) > m(p_1, q_1). \tag{7}$$

Let $F_3 = \sigma(F_1)$, and $F_4 = \sigma(F_2)$. Let $E_j := F_j \setminus \mathcal{A}$.

Define

$$\Omega_2 := E_1 \cup E_2 \cup E_3 \cup E_4.$$

Certainly the sets E_j are disjoint and open. Moreover, they are connected. Actually, $\mathcal{A}$ is an analytic set so it is sufficient to show the connectivity of F_j. But F_1 is the image of $\mathbb{D}\times\mathbb{D}_*\times\mathbb{D}_*\times\mathbb{D}$ under the mapping $\lambda \mapsto (\lambda_1\lambda_2, \lambda_2, \lambda_4, \lambda_3\lambda_4)$. On the other hand the set F_2 is the image, under the mapping $\lambda \mapsto (\lambda_1, \lambda_1\lambda_2, \lambda_3, \lambda_3\lambda_4)$ of the set $B := \{\lambda \in \mathbb{D}_* \times \mathbb{D} \times \mathbb{D}_* \times \mathbb{D} : m(\lambda_1, \lambda_3) < m(\lambda_2, \lambda_4)\}$. To show connectedness of the last set it suffices to show that $\tilde{B} := \{\lambda \in \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{D} : m(\lambda_1, \lambda_3) < m(\lambda_2, \lambda_4)\}$ is connected, as B is obtained from $\tilde{B}$ by removing an analytic set. This is the case because any point $\lambda \in \tilde{B}$ may be joined by the curve $[0, 1] \ni t \mapsto (t\lambda_1, \lambda_2, t\lambda_3, \lambda_4)$ with $(0, \lambda_2, 0, \lambda_4)$. And now it is sufficient to see that the set $\{0\} \times \mathbb{D}_* \times \{0\} \times \mathbb{D}_*$ is arc-connected.

Let $G_j := \Phi^{-1}(E_j)$. To finish the proof of the assertion it suffices to show that

$$\Phi|_{G_j} : G_j \to E_j$$

is surjective. In fact, in such a case $\Phi(G_j) = E_j$ so the equality $l = c$ holds on Ω_2, which together with Ω_1 builds a dense subset of $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$.

Therefore, to finish the proof of the theorem we go to the proof of the surjectivity of the mappings defined above.

Without loss of generality we may restrict to the cases $j = 1, 2$.

First note that the sets G_j are nonempty. Therefore, to finish the proof it is sufficient to show that $\Phi(G_j)$ is open and closed in E_j.

First we show that $\Phi(G_j)$ is closed. The proof may be conducted with the standard sequence procedure; however, we shall make use of considerations that were given in Section 3.

Take (p, q) in the closure of $\Phi(G_j)$ with respect to E_j. The continuity property implies that $c(p, q) = l(p, q)$. It follows immediately from Lemma 3 that (p, q) lies in $\Phi(G_j)$.

To show that the image is open it suffices to prove that Φ is locally injective.

So assume that $\Phi(\alpha, \beta, c, \omega, t) = \Phi(\tilde{\alpha}, \tilde{\beta}, \tilde{c}, \tilde{\omega}, \tilde{t})$.

Let $\varphi := \varphi_{\alpha,\beta,\omega}$, $\tilde{\varphi} := \varphi_{\tilde{\alpha},\tilde{\beta},\tilde{\omega}}$.

Let $F(x) = (\bar{\omega}tx_1 + (1-t)x_2 + \eta x_1x_2)/(1-((1-t)\bar{\omega}x_1 + tx_2)\eta)$, where η is properly chosen. It simply follows from the previous discussion that F is a left inverse to both φ and $\tilde{\varphi}$. Therefore, $F = \tilde{F}$, where $\tilde{F}$ denotes the appropriate left inverse to $\tilde{\varphi}$. Thus $t = \tilde{t}$ and $\omega = \tilde{\omega}$. Moreover, $cm_\gamma(c) = \tilde{c}m_{\tilde{\gamma}}(\tilde{c}) =: l \neq 0$. Therefore, it suffices to show the local injectivity of the function

$$\Psi : (\alpha, \beta, c) \mapsto \left(cm_\alpha(c), \frac{l}{c}m_\alpha\left(\frac{l}{c}\right), cm_\beta(c), \frac{l}{c}m_\beta\left(\frac{l}{c}\right)\right)$$

defined for $(\alpha, \beta, c) \in \mathbb{D}^3$ such that $(z, w) = \Phi(\alpha, \beta, c)$ satisfies $|z_1| \neq |z_2|$, $|w_1| \neq |w_2|$, $z_1 \neq w_1$ and $z_2 \neq w_2$ (in particular, $\alpha \neq \beta$, $c \neq 0$).

Proposition 5. *Ψ is locally injective. Moreover, Ψ is two-to-one.*

Proof. Observe first that $\Psi(\alpha, \beta, c) = \Psi(-\alpha, -\beta, -c)$. Therefore, to get the assertion, it suffices to show that for fixed points $z := (z_1, z_2)$, $w := (w_1, w_2)$ such that $z_1 \neq z_2$, $w_1 \neq w_2$, $z_1 \neq w_1$ and $z_2 \neq w_2$ the equation $\Psi(\alpha, \beta, c) = (z_1, z_2, w_1, w_2)$ has at most two solutions.

From the equation we deduce that

$$\alpha = c\frac{z_2(1-z_1/l)}{z_2 - z_1} + \frac{1}{c}\frac{z_1(z_2-l)}{z_2-z_1}, \qquad \text{and} \quad \bar{\alpha} = c\frac{1-z_2/l}{z_1-z_2} + \frac{1}{c}\frac{z_1-l}{z_1-z_2},$$
$$\beta = c\frac{w_2(1-w_1/l)}{w_2 - w_1} + \frac{1}{c}\frac{w_1(w_2-l)}{w_2-w_1}, \quad \text{and} \quad \bar{\beta} = c\frac{1-w_2/l}{w_1-w_2} + \frac{1}{c}\frac{w_1-l}{w_1-w_2}.$$

We can write the above equations in the form

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = M\begin{pmatrix}c\\ 1/c\end{pmatrix}, \quad \begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix} = N\begin{pmatrix}c\\ 1/c\end{pmatrix},$$

where $M, N \in \mathbb{C}^{2\times 2}$. Set $v := \binom{c}{1/c}$. The equations imply that $Mv = \bar{N}\bar{v}$.

Notice that

$$\det M = \frac{z_2(1-z_1/l)w_1(w_2-l) - w_2(1-w_1/l)z_1(z_2-l)}{(z_2-z_1)(w_2-w_1)},$$
$$\det N = \frac{(1-z_2/l)(w_1-l) - (1-w_2/l)(z_1-l)}{(z_2-z_1)(w_2-w_1)}.$$

The hypotheses made on z and w ensure that $(1-z_2/l)(w_1-l)$ and $(1-w_2/l)(z_1-l)$ cannot vanish simultaneously, so if $\det N = 0$, we see that the equation $\det M = 0$ reduces to $z_2w_1 - z_1w_2 = 0$. Since $l \neq 0$, this together with $\det N = 0$ would imply $z_1 = z_2$ or $z_1 = w_1$, which is excluded. Therefore at least one of the matrices M or N is invertible. Suppose for now that M is invertible, we have $v = P\bar{v}$, with $P := M^{-1}\bar{N}$. Since $\bar{v} = \bar{P}v$, we see that $v = P\bar{P}v$.

Since $M\begin{pmatrix}l\\ 1\end{pmatrix} = \begin{pmatrix}l\\ l\end{pmatrix}$ and $N\begin{pmatrix}l\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$, then $\bar{P}P\begin{pmatrix}\bar{l}\\ 1\end{pmatrix} = |l|^{-2}\begin{pmatrix}\bar{l}\\ 1\end{pmatrix}$, so that we have an eigenvalue $|l|^{-2} > 1$ of $\bar{P}P$, and $P\bar{P} \neq I$. So $\dim\ker(I - P\bar{P}) \leq 1$, which means, since v cannot be 0, that there is a nonzero vector $w \in \mathbb{C}^2$, depending only on z, w, l, such that v is collinear to w, which implies $c^2 = w_1/w_2$. So we have at most two possible values for (α, β, c).

If $\det M = 0$, then N is invertible and we reason in the same way starting from $v = N^{-1}\bar{M}v$. $\square$

References

[Agler and McCarthy 2000] J. Agler and J. E. McCarthy, "The three point Pick problem on the bidisk", *New York J. Math.* **6** (2000), 227–236. MR Zbl

[Agler and McCarthy 2002] J. Agler and J. E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics **44**, American Mathematical Society, Providence, RI, 2002. MR Zbl

[Agler et al. 2013] J. Agler, Z. A. Lykova, and N. J. Young, "Extremal holomorphic maps and the symmetrized bidisc", *Proc. Lond. Math. Soc.* (3) **106**:4 (2013), 781–818. MR Zbl

[Agler et al. 2015] J. Agler, Z. A. Lykova, and N. J. Young, "3-extremal holomorphic maps and the symmetrized bidisc", *J. Geom. Anal.* **25**:3 (2015), 2060–2102. MR Zbl

[Carlehed 1999] M. Carlehed, "Potentials in pluripotential theory", *Ann. Fac. Sci. Toulouse Math.* (6) **8**:3 (1999), 439–469. MR Zbl

[Carlehed and Wiegerinck 2003] M. Carlehed and J. Wiegerinck, "Le cône des fonctions plurisousharmoniques négatives et une conjecture de Coman", *Ann. Polon. Math.* **80** (2003), 93–108. MR Zbl

[Coman 2000] D. Coman, "The pluricomplex Green function with two poles of the unit ball of $\mathbb{C}^n$", *Pacific J. Math.* **194**:2 (2000), 257–283. MR Zbl

[Edigarian and Zwonek 1998] A. Edigarian and W. Zwonek, "Invariance of the pluricomplex Green function under proper mappings with applications", *Complex Variables Theory Appl.* **35**:4 (1998), 367–380. MR Zbl

[Kosiński 2014] L. Kosiński, "Spectral Nevanlinna–Pick problem and weak extremals in the symmetrized bidisc", preprint, 2014. arXiv

[Kosiński and Zwonek 2016a] Ł. Kosiński and W. Zwonek, "Extremal holomorphic maps in special classes of domains", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **16**:1 (2016), 159–182. MR Zbl

[Kosiński and Zwonek 2016b] Ł. Kosiński and W. Zwonek, "Nevanlinna–Pick problem and uniqueness of left inverses in convex domains, symmetrized bidisc and tetrablock", *J. Geom. Analysis* **26**:3 (2016), 1863–1890. MR Zbl

[Lempert 1981] L. Lempert, "La métrique de Kobayashi et la représentation des domaines sur la boule", *Bull. Soc. Math. France* **109**:4 (1981), 427–474. MR Zbl

[Nikolov and Pflug 2006] N. Nikolov and P. Pflug, "The multipole Lempert function is monotone under inclusion of pole sets", *Michigan Math. J.* **54**:1 (2006), 111–116. MR Zbl

[Pick 1915] G. Pick, "Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden", *Math. Ann.* **77**:1 (1915), 7–23. MR

[Thomas and Trao 2003] P. J. Thomas and N. V. Trao, "Pluricomplex Green and Lempert functions for equally weighted poles", *Ark. Mat.* **41**:2 (2003), 381–400. MR Zbl

[Warszawski 2015] T. Warszawski, "(Weak) m-extremals and m-geodesics", *Complex Var. Elliptic Equ.* **60**:8 (2015), 1077–1105. MR Zbl

[Wikström 2001] F. Wikström, "Non-linearity of the pluricomplex Green function", *Proc. Amer. Math. Soc.* **129**:4 (2001), 1051–1056. MR Zbl

[Wikström 2003] F. Wikström, "Computing the pluricomplex Green function with two poles", *Experiment. Math.* **12**:3 (2003), 375–384. MR Zbl

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