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ENDOTRIVIAL MODULES: A REDUCTION TO p'-CENTRAL EXTENSIONS

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We examine how, in prime characteristic p, the group of endotrivial modules of a finite group G and the group of endotrivial modules of a quotient of G modulo a normal subgroup of order prime to p are related. There is always an inflation map, but examples show that this map is in general not surjective. We prove that the situation is controlled by a single central extension, namely, the central extension given by a p'-representation group of the quotient of G by its largest normal p'-subgroup.

1. Introduction

Endotrivial modules play an important role in the representation theory of finite groups. They have been classified in a number of special cases; see, e.g., the recent papers [Carlson et al. 2014a; Lassueur and Mazza 2015b] and the references therein. Over an algebraically closed field k of prime characteristic p, endotrivial modules for a finite group G form an abelian group $T(G)$, which is known to be finitely generated. One of the main question is to understand the structure of $T(G)$, and, in particular, of its torsion subgroup $TT(G)$.

We let $X(G)$ be the subgroup of $TT(G)$ consisting of all one-dimensional representations, that is, $X(G) \cong \mathrm{Hom}(G, k^\times)$. We also let $K(G)$ be the kernel of the restriction map $\mathrm{Res}^G_P : T(G) \to T(P)$ to a Sylow p-subgroup P of G. It is known that $X(G) \subseteq K(G) \subseteq TT(G)$ and that $K(G) = TT(G)$ in almost all cases (namely if we exclude the cases when a Sylow p-subgroup of G is cyclic, generalized quaternion, or semidihedral). Moreover, there are numerous cases, including infinite families of groups G, for which $K(G) = X(G)$. However, this is not always the case, and the structure of $K(G)$ is not understood in general.

Let $O_{p'}(G)$ denote the largest normal subgroup of G of order prime to p and set $Q := G/O_{p'}(G)$ for simplicity. There is always an inflation homomorphism

$$\mathrm{Inf}^G_Q : T(Q) \to T(G)$$

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which is easily seen to be injective. But examples show that it is in general not surjective, so we cannot expect an isomorphism between $T(G)$ and $T(Q)$. The present article analyzes how $T(G)$ and $T(Q)$ are related, by making use of a suitable central extension of Q. More precisely, associated with Q, there is a p'-representation group $\widetilde{Q}$, which is a central extension with kernel of order prime to p. This controls the behavior of projective representations of Q (in the sense of Schur). When Q is a perfect group, then $\widetilde{Q}$ is unique and is also called the universal p'-central extension of Q. When Q is not perfect, then $\widetilde{Q}$ may not be unique.

The present work is based on a key result by the first author and S. Koshitani [Koshitani and Lassueur 2016]. In the course of their investigation of endotrivial modules for a finite group with dihedral Sylow 2-subgroups, they proved a general result [op. cit., Theorem 4.4] about endotrivial modules for an arbitrary group G in the presence of a normal subgroup N of order prime to p, under mild hypotheses on G (see Hypothesis 3.1). Their result uses modules over twisted group algebras of G/N. Taking $Q = G/N$ with $N = O_{p'}(G)$, we can view such modules as modules over the ordinary group algebra of the central extension $\widetilde{Q}$. In this way, we can show that the structure of $T(G)$ is closely related to the structure of $T(\widetilde{Q})$. Our main result is as follows:

Theorem 1.1. *Let G be a finite group of p-rank at least 2 and no strongly p-embedded subgroups. Let $\widetilde{Q}$ be any p'-representation group of the group $Q := G/O_{p'}(G)$.*

(a) *There exists an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \to T(\widetilde{Q})/X(\widetilde{Q}).$$

In particular, $\Phi_{G,\widetilde{Q}}$ maps the class of $\mathrm{Inf}_Q^G(W)$ to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$, for any endotrivial kQ-module W.

(b) *The map $\Phi_{G,\widetilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \to K(\widetilde{Q})/X(\widetilde{Q}).$$

(c) *In particular, if $K(\widetilde{Q}) = X(\widetilde{Q})$, then $K(G) = X(G)$.*

We note that the construction of the map $\Phi_{G,\widetilde{Q}}$ relies on [op. cit., Theorem 4.4], which itself relies on Navarro and Robinson [Navarro and Robinson 2012], whose proof makes use of the classification of finite simple groups. This construction will be made precise in Section 4. Examples show that $\Phi_{G,\widetilde{Q}}$ is in general not surjective (see Section 7), but the theorem provides some information on $K(G)$, for all groups G such that $G/O_{p'}(G) = Q$. In particular, the question of the equality $K(G) = X(G)$ is reduced to the same question for the single group $\widetilde{Q}$.

We also conjecture that $\Phi_{G,\widetilde{Q}}$ induces an isomorphism on the torsion-free part of $T(G)$ and $T(\widetilde{Q})$ (see Section 5). Moreover, in case Q is perfect, then there is an alternative approach to $\Phi_{G,\widetilde{Q}}$ which we present in Section 6.

The two main assumptions on G in Theorem 1.1 are needed for applying the results of [Koshitani and Lassueur 2016]. However, these assumptions are not really restrictive because endotrivial modules are completely understood in the two excluded cases: they are classified if the p-rank is 1 [Mazza and Thévenaz 2007; Carlson et al. 2013], and $T(G) \cong T(H)$ if G has a strongly p-embedded subgroup H; see [Mazza and Thévenaz 2007, Lemma 2.7].

The two assumptions also allow us to prove that $T(G) \cong T(G/[G, A])$, where $A = O_{p'}(G)$, or in other words that the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with kernel A of order prime to p can always be replaced by the central extension

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \longrightarrow Q \longrightarrow 1.$$

This is explained in Section 3.

2. Notation and preliminaries

Throughout, unless otherwise specified, we use the following notation. We let k denote an algebraically closed field of prime characteristic p. We assume that all groups are finite, and that all modules over group algebras are finitely generated, and we set $\otimes := \otimes_k$. If G is an arbitrary finite group and V is a kG-module, we denote by $\rho_V : G \to \mathrm{GL}(V)$ the corresponding k-representation, and we denote by $\pi_V : \mathrm{GL}(V) \to \mathrm{PGL}(V)$ the canonical surjection. Furthermore, we denote by V^* the k-dual of V endowed with a kG-module structure via $(gf)(v) = f(g^{-1}v)$ for every $g \in G,\ f \in V^*,\ v \in V$.

Assuming moreover that $p \mid |G|$, we recall that a kG-module V is called *endotrivial* if there is an isomorphism of kG-modules $\mathrm{End}_k(V) \cong k \oplus (\mathrm{proj})$, where k denotes the trivial kG-module and (proj) some projective kG-module, which might be zero. Any endotrivial kG-module V splits as a direct sum $V = V_0 \oplus (\mathrm{proj})$ where V_0, the projective-free part of V, is indecomposable and endotrivial. The relation

$$U \sim V \iff U_0 \cong V_0$$

is an equivalence relation on the class of endotrivial kG-modules, and $T(G)$ denotes the resulting set of equivalence classes (which we denote by square brackets). Then $T(G)$, endowed with the law $[U] + [V] := [U \otimes V]$, is an abelian group called the *group of endotrivial modules of* G. The zero element is the class $[k]$ of the trivial module and $-[V] = [V^*]$, the class of the dual module V^*. By a result of Puig, the group $T(G)$ is known to be a finitely generated abelian group; see, e.g., [Carlson et al. 2006, Corollary 2.5].

We let $X(G)$ denote the group of one-dimensional kG-modules endowed with the tensor product $\otimes$, and recall that $X(G) \cong \mathrm{Hom}(G, k^\times) \cong (G/[G, G])_{p'}$. Identifying a one-dimensional module with its class in $T(G)$, we consider $X(G)$ as a subgroup of $T(G)$.

Furthermore, if P is a Sylow p-subgroup of G, we set

$$K(G) = \mathrm{Ker}\big(\mathrm{Res}^G_P : T(G) \to T(P)\big).$$

In other words, the class of an indecomposable endotrivial kG-module V belongs to $K(G)$ if and only if $V{\downarrow}^G_P \cong k \oplus (\mathrm{proj})$, that is, in other words, V is a trivial source module. We have $X(G) \subseteq K(G)$ because any one-dimensional kP-module is trivial. Moreover, $K(G) \subseteq TT(G)$ (see [Carlson et al. 2011, Lemma 2.3]), and $K(G) = TT(G)$ unless P is cyclic, generalized quaternion, or semidihedral, by the main result of [Carlson and Thévenaz 2005].

By a central extension (E, π) of Q, we mean a group extension

$$1 \longrightarrow Z \longrightarrow E \overset{\pi}{\longrightarrow} Q \longrightarrow 1$$

with $Z = \mathrm{Ker}\,\pi$ central in E. Recall that (E, π) is said to have the *projective lifting property (relative to k)* if, for every finite-dimensional k-vector space V, every group homomorphism $\theta : Q \to \mathrm{PGL}(V)$ can be completed to a commutative diagram of group homomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & E & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda|_Z} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \theta} & & \\ 1 & \longrightarrow & k^\times \cdot \mathrm{Id}_V & \longrightarrow & \mathrm{GL}(V) & \overset{\pi_V}{\longrightarrow} & \mathrm{PGL}(V) & \longrightarrow & 1 \end{array}$$

In general, the homomorphism λ is not uniquely defined. However, by the commutativity of the diagram, the following holds:

Lemma 2.1. *In the above situation, if $\lambda, \lambda' : E \to \mathrm{GL}(V)$ are two liftings of θ to E, then there exists a degree one representation $\mu : E \to \mathrm{GL}(k)$ such that $\lambda' = \lambda \otimes \mu$.*

By results of Schur (slightly generalized for dealing with the case of characteristic p), given a finite group Q, there always exists a central extension (E, π) of Q, with kernel $M_k(Q) := \mathrm{H}^2(Q, k^\times)$, which has the projective lifting property. A *p'-representation group of Q* (or a *representation group of Q relative to k*) is a central extension $(\widetilde{Q}, \pi)$ of Q of minimal order with the projective lifting property. In this case $M_k(Q) \cong \mathrm{Ker}\,\pi \leq [\widetilde{Q}, \widetilde{Q}]$. We recall that $M_k(Q) \cong \mathrm{H}^2(Q, \mathbb{C}^\times)_{p'}$, the p'-part of the Schur multiplier of Q, and that in general a group Q with $X(Q) \neq 1$ may have several nonisomorphic p'-representation groups. Furthermore, fixing a p'-representation group $(\widetilde{Q}, \pi)$ of Q, the abelian group $M_k(Q)$ becomes isomorphic to its $k^\times$-dual via the transgression homomorphism

$$\mathrm{tr} : \mathrm{Hom}(M_k(Q), k^\times) \to \mathrm{H}^2(Q, k^\times)$$

defined by $\mathrm{tr}(\varphi) = [\varphi \circ \alpha]$, where the cocycle $\alpha \in Z^2(Q, M_k(Q))$ is in the cohomology class corresponding to the central extension $1 \to M_k(Q) \to \widetilde{Q} \xrightarrow{\pi} Q \to 1$. For further details and proofs we refer the reader to [Nagao and Tsushima 1989, Chapter 3, §5; Curtis and Reiner 1981, §11E].

If V, W are two finite-dimensional k-vector spaces, then the tensor product of linear maps induces a tensor product $-\otimes- : \mathrm{PGL}(V) \times \mathrm{PGL}(W) \to \mathrm{PGL}(V \otimes W)$ via $\pi_V(\alpha) \otimes \pi_W(\beta) := \pi_{V\otimes W}(\alpha \otimes \beta)$ for any $\alpha \in \mathrm{GL}(V)$ and any $\beta \in \mathrm{GL}(W)$. Therefore, if $\mu : Q \to \mathrm{PGL}(V)$ and $\nu : Q \to \mathrm{PGL}(W)$ are group homomorphisms, we may define a group homomorphism $\mu \otimes \nu : Q \to \mathrm{PGL}(V \otimes W)$ via $(\mu \otimes \nu)(q) := \mu(q) \otimes \nu(q)$ for every $q \in Q$. We shall use the following well-known results throughout:

Lemma 2.2. *Let* $1 \to A \to G \xrightarrow{\pi} Q \to 1$ *be an arbitrary group extension.*

(a) *Whenever* V *is a* kG*-module such that* $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, *the group homomorphism* $\rho_V : G \to \mathrm{GL}(V)$ *induces a uniquely defined group homomorphism* $\theta_V : Q \to \mathrm{PGL}(V)$ *such that the following diagram commutes*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_V|_A} & & \downarrow{\scriptstyle \rho_V} & & \downarrow{\scriptstyle \theta_V} & & \\ 1 & \longrightarrow & k^\times \cdot \mathrm{Id}_V & \longrightarrow & \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) & \longrightarrow & 1 \end{array}$$

(b) *If* V, W *are* kG*-modules such that* $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$ *and* $\rho_W(A) \subseteq k^\times \cdot \mathrm{Id}_W$, *then* $\rho_{V\otimes W}(A) \subseteq k^\times \cdot \mathrm{Id}_{V\otimes W}$ *and we have* $\theta_{V\otimes W} = \theta_V \otimes \theta_W$.

Proof. (a) Choose a set-theoretic section $s : Q \to G$ for π and define $\theta_V := \pi_V \circ \rho_V \circ s$. Since $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, the map θ_V is a group homomorphism making the diagram commute. Clearly θ_V is uniquely defined since π is an epimorphism.

(b) This is a straightforward computation. □

3. Endotrivial modules and central extensions

We now fix G to be a finite group of order divisible by p, we set $A := O_{p'}(G)$ and $Q := G/A$, and we denote by $\pi_G : G \to Q$ the quotient map. Moreover, we let $(\widetilde{Q}, \pi_{\widetilde{Q}})$ be a fixed p'-representation group of Q.

Since A is a p'-subgroup of G, inflation induces an injective group homomorphism

$$\mathrm{Inf}_Q^G : T(Q) \to T(G), \qquad [V] \to [\mathrm{Inf}_Q^G(V)].$$

This is because the inflation of a projective module remains projective when the kernel A is a p'-group. We emphasize that endotrivial kG-modules cannot be recovered from endotrivial kQ-modules, as in general the inflation map Inf_Q^G is not an isomorphism; see Section 7.

Hypothesis 3.1. Assume G is a finite group fulfilling the following two conditions:

(1) the p-rank of G is greater than or equal to 2; and

(2) G has no strongly p-embedded subgroups.

The next result restates one of the main results of [Koshitani and Lassueur 2016], but in different terms. Our statement will allow us later to avoid working with modules over twisted group algebras, but simply consider the corresponding projective representations instead.

Theorem 3.2 [Koshitani and Lassueur 2016]. *Suppose G satisfies Hypothesis 3.1.*

(a) *If V is an indecomposable endotrivial kG-module, then $V{\downarrow}_A^G \cong Y \oplus \cdots \oplus Y$, where Y is a one-dimensional kA-module.*

(b) *If V is an indecomposable endotrivial kG-module, then $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$.*

Proof. (a) Since G satisfies Hypothesis 3.1, any composition factor Y of $V{\downarrow}_A^G$ is G-invariant, by [op. cit., Lemma 4.3]. Therefore $V{\downarrow}_A^G \cong Y \oplus \cdots \oplus Y$ and [op. cit., Theorem 4.4] proves that $\dim Y = 1$.

(b) This is a restatement of (a). □

Corollary 3.3. *Suppose that G satisfies Hypothesis 3.1. The inflation map*

$$\mathrm{Inf}_{G/[G,A]}^G : T(G/[G,A]) \to T(G)$$

is a group isomorphism.

Proof. Since $[G, A]$ is a normal p'-subgroup of G, the inflation map $\mathrm{Inf}_{G/[G,A]}^G$ is a well-defined injective group homomorphism. In order to prove that it is surjective, it suffices to prove that $[G, A]$ acts trivially on any indecomposable endotrivial kG-module V. But by Theorem 3.2 we have

$$\rho_V([G,A]) \subseteq [\rho_V(G), \rho_V(A)] \subseteq [\rho_V(G), k^\times \cdot \mathrm{Id}_V] = \{\mathrm{Id}_V\}.$$

Hence $[G, A]$ acts trivially on V. □

Corollary 3.3 is a reduction to the case of central extensions. Explicitly, for the study of endotrivial modules, we may always replace the given extension

$$1 \to A \to G \to Q \to 1,$$

and consider instead the central extension

$$1 \to A/[G,A] \to G/[G,A] \to Q \to 1.$$

We shall in fact not use this reduction for the proof of our main result, but rather apply directly Theorem 3.2.

Lemma 3.4. *Let $(\widetilde{Q}, \pi_{\widetilde{Q}})$ be a p'-representation group of Q. Then $X(\widetilde{Q}) = \mathrm{Inf}_Q^{\widetilde{Q}}(X(Q))$, hence $X(\widetilde{Q}) \cong X(Q)$.*

Proof. We apply the fact, mentioned in Section 2, that $\mathrm{Ker}\,\pi_{\widetilde{Q}} \subseteq [\widetilde{Q}, \widetilde{Q}]$. This implies that any one-dimensional representation of $\widetilde{Q}$ has $\mathrm{Ker}\,\pi_{\widetilde{Q}}$ in its kernel, hence is inflated from $\widetilde{Q}/\,\mathrm{Ker}\,\pi_{\widetilde{Q}} \cong Q$.

Another way of seeing the same thing is to associate to the central extension

$$1 \longrightarrow M_k(Q) \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

the Hochschild–Serre five-term exact sequence

$$1 \longrightarrow \mathrm{Hom}(Q, k^\times) \xrightarrow{\mathrm{Inf}} \mathrm{Hom}(\widetilde{Q}, k^\times) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(M_k(Q), k^\times) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(Q, k^\times) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(\widetilde{Q}, k^\times).$$

Since the transgression map tr is an isomorphism, the first map Inf must be an isomorphism as well. □

4. Proof of Theorem 1.1

Keep the notation of the previous section. Moreover, given an endotrivial kG-module V such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, we let

$$\theta_V : Q \to \mathrm{PGL}(V)$$

denote the induced homomorphism constructed in Lemma 2.2(a). The projective lifting property for the central extension $(\widetilde{Q}, \pi_{\widetilde{Q}})$ allows us to fix a representation

$$\rho_{V_{\widetilde{Q}}} : \widetilde{Q} \to \mathrm{GL}(V)$$

lifting θ_V to $\widetilde{Q}$. We denote by $V_{\widetilde{Q}}$ the corresponding $k\widetilde{Q}$-module.

Lemma 4.1. *Let V be an endotrivial kG-module such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$. Then $V_{\widetilde{Q}}$ is an endotrivial $k\widetilde{Q}$-module.*

Proof. We have to work with two group extensions

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi_G} Q \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow M \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1,$$

where $M := M_k(Q)$. Both A and M have order prime to p.

Let $P \in \mathrm{Syl}_p(G)$, set $\bar{P} := AP/A \in \mathrm{Syl}_p(Q)$, and let $\iota_P : P \to AP$ be the inclusion map, so that

$$\phi := \pi_G \circ \iota_P : P \to \bar{P}$$

is an isomorphism. Next choose $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{Q})$ such that $M\widetilde{P}/M = \bar{P} \in \mathrm{Syl}_p(Q)$. Let $\iota_{\widetilde{P}} : \widetilde{P} \to M\widetilde{P}$ be the inclusion map, so that $\psi := \pi_{\widetilde{Q}} \circ \iota_{\widetilde{P}} : \widetilde{P} \to \bar{P}$ is an isomorphism. Consider now the two commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & \overline{P} \\ {\scriptstyle\rho_V}\downarrow & & \downarrow{\scriptstyle\theta_V} \\ \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \widetilde{P} & \xrightarrow{\psi} & \overline{P} \\ {\scriptstyle\rho_{\widetilde{Q}}}\downarrow & & \downarrow{\scriptstyle\theta_V} \\ \mathrm{GL}(V) & \xrightarrow{\pi_V} & \mathrm{PGL}(V) \end{array}$$

where we write $\rho_{\widetilde{Q}} := \rho_{V_{\widetilde{Q}}}$ for simplicity. Since ϕ and ψ are isomorphisms, for any $u \in \overline{P}$,

$$\pi_V \rho_V \phi^{-1}(u) = \theta_V(u) = \pi_V \rho_{\widetilde{Q}} \psi^{-1}(u).$$

We claim that if two elements $u_1, u_2 \in \mathrm{GL}(V)$ have p-power order and satisfy $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = u_2$. Postponing the proof of the claim, we deduce that

$$\rho_V \phi^{-1}(u) = \rho_{\widetilde{Q}} \psi^{-1}(u),$$

because they have p-power order. This means that the representations $(\rho_V)|_P$ and $(\rho_{\widetilde{Q}})|_{\widetilde{P}}$, transported via isomorphisms to representations of $\overline{P}$, are equal. Now, a module is endotrivial if and only if its restriction to a Sylow p-subgroup is; see [Carlson et al. 2006, Proposition 2.6]. Moreover, this property is preserved when transported via group isomorphisms. Since V is endotrivial, so is $V{\downarrow_P}$, hence so is $V_{\widetilde{Q}}{\downarrow_{\widetilde{P}}}$, and it follows that $V_{\widetilde{Q}}$ is endotrivial.

We are left with the proof of the claim. If $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = \alpha u_2$ where $\alpha \in k^\times$. For some large enough power p^n, we have $u_1^{p^n} = u_2^{p^n} = 1$. Therefore we obtain

$$1 = u_1^{p^n} = (\alpha u_2)^{p^n} = \alpha^{p^n} u_2^{p^n} = \alpha^{p^n}.$$

But there are no nontrivial p-th roots of unity in $k^\times$, so $\alpha = 1$, hence $u_1 = u_2$. □

Proposition 4.2. *Assume G satisfies Hypothesis 3.1. Then there is an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \to T(\widetilde{Q})/X(\widetilde{Q})$$

defined by $\Phi_{G,\widetilde{Q}}([V]+X(G)) := [V_{\widetilde{Q}}]+X(\widetilde{Q})$ for any indecomposable endotrivial kG-module V. Moreover, for any endotrivial kQ-module W, the homomorphism $\Phi_{G,\widetilde{Q}}$ maps the class of $\mathrm{Inf}_Q^G(W)$ to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$.

Proof. First, Lemma 4.1 allows us to define a map $\phi : T(G) \to T(\widetilde{Q})/X(\widetilde{Q})$ by setting $\phi([V]) := [V_{\widetilde{Q}}] + X(\widetilde{Q})$ for any $[V] \in T(G)$ such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$. The definition of $\phi([V])$ does not depend on the choice of $V_{\widetilde{Q}}$, for if $\rho_{V'_{\widetilde{Q}}}$ is a second lifting of θ_V to $\widetilde{Q}$, then by Lemma 2.1 there exists $X \in X(\widetilde{Q})$ such that $V'_{\widetilde{Q}} \cong V_{\widetilde{Q}} \otimes X$, hence $\phi([V_{\widetilde{Q}}]) = \phi([V'_{\widetilde{Q}}])$.

Next, let V, W be two indecomposable endotrivial kG-modules. Theorem 3.2 implies that $\rho_{V\otimes W}(A) = (\rho_V \otimes \rho_W)(A) \subseteq k^\times \cdot \mathrm{Id}_{V\otimes W}$. Thus, by Lemma 2.2(b),

$\theta_{V\otimes W} = \theta_V \otimes \theta_W$, and it is easy to verify that $\rho_{V_{\widetilde{Q}}} \otimes \rho_{W_{\widetilde{Q}}}$ lifts $\theta_V \otimes \theta_W$ to $\widetilde{Q}$. Therefore, by Lemma 2.1, there exists $X \in X(\widetilde{Q})$ such that $(V \otimes W)_{\widetilde{Q}} \cong V_{\widetilde{Q}} \otimes W_{\widetilde{Q}} \otimes X$. This shows that ϕ is a group homomorphism.

It is clear that $\operatorname{Ker}\phi = X(G)$, since by construction $\dim V_{\widetilde{Q}} = \dim V$ for any indecomposable endotrivial kG-module V. As a result, ϕ induces the required homomorphism $\Phi_{G,\widetilde{Q}}$.

Finally, if W is any endotrivial kQ-module, then the $k\widetilde{Q}$-module constructed from $V = \operatorname{Inf}_Q^G(W)$ is easily seen to be the inflated module $V_{\widetilde{Q}} = \operatorname{Inf}_Q^{\widetilde{Q}}(W)$, because the map $\theta_V : Q \to \mathrm{PGL}(V)$ comes from a group homomorphism $Q \to \mathrm{GL}(V)$. This shows that the class of $\operatorname{Inf}_Q^G(W)$ is mapped to the class of $\operatorname{Inf}_Q^{\widetilde{Q}}(W)$ under the map $\Phi_{G,\widetilde{Q}}$, proving the additional statement. □

Corollary 4.3. *Assume G satisfies Hypothesis 3.1. If $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are two nonisomorphic p'-representation groups of Q, then*

$$\Phi_{\widetilde{Q}_1,\widetilde{Q}_2} : T(\widetilde{Q}_1)/X(\widetilde{Q}_1) \to T(\widetilde{Q}_2)/X(\widetilde{Q}_2)$$

is an isomorphism.

Proof. Let V be an indecomposable $k\widetilde{Q}_1$-module. By construction

$$\Phi_{\widetilde{Q}_1,\widetilde{Q}_2}([V] + X(\widetilde{Q}_1)) = [W] + X(\widetilde{Q}_2),$$

where $W := V_{\widetilde{Q}_2}$ is a $k\widetilde{Q}_2$-module such that ρ_W lifts $\theta_V : Q \to \mathrm{PGL}(V)$ to $\widetilde{Q}_2$. But then ρ_V lifts $\theta_W = \theta_V$ to $\widetilde{Q}_1$, so that by construction

$$\Phi_{\widetilde{Q}_2,\widetilde{Q}_1}([W] + X(\widetilde{Q}_2)) = [V] + X(\widetilde{Q}_1).$$

In other words, $\Phi_{\widetilde{Q}_1,\widetilde{Q}_2} \circ \Phi_{\widetilde{Q}_2,\widetilde{Q}_1} = \mathrm{Id}$. Similarly $\Phi_{\widetilde{Q}_2,\widetilde{Q}_1} \circ \Phi_{\widetilde{Q}_1,\widetilde{Q}_2} = \mathrm{Id}$. □

Corollary 4.4. *Assume G satisfies Hypothesis 3.1. The map $\Phi_{G,\widetilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \to K(\widetilde{Q})/X(\widetilde{Q}).$$

In particular, if $K(\widetilde{Q}) \cong X(\widetilde{Q})$, then $K(G) \cong X(G)$.

Proof. Let $P \in \mathrm{Syl}_p(G)$ and let V be an indecomposable endotrivial kG-module. As in the proof of Lemma 4.1, the two modules $V{\downarrow}_P^G$ and $V_{\widetilde{Q}}{\downarrow}_{\widetilde{P}}^{\widetilde{Q}}$ are isomorphic, provided we view them as modules over the group $\bar{P}$ via the isomorphisms $P \cong \bar{P}$ and $\widetilde{P} \cong \bar{P}$. It follows that V has a trivial source if and only if $V_{\widetilde{Q}}$ has. Therefore $\Phi_{G,\widetilde{Q}}$ restricts to an injective group homomorphism

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \to K(\widetilde{Q})/X(\widetilde{Q}).$$

The special case follows. □

Proposition 4.2 together with Corollary 4.4 prove Theorem 1.1.

5. Conjecture on the torsion-free part

We keep the notation of the previous sections. Let $TF(G) = T(G)/TT(G)$, the torsion-free part of the group of endotrivial modules. Since $X(G) \subseteq TT(G)$, the map

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \to T(\widetilde{Q})/X(\widetilde{Q})$$

induces an injective group homomorphism

$$\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q}).$$

We know that $\Phi_{G,\widetilde{Q}}$ is in general not surjective, but we conjecture that $\Psi_{G,\widetilde{Q}}$ is surjective.

Conjecture 5.1. (a) The map $\mathrm{Inf}_Q^G : TF(Q) \to TF(G)$ is an isomorphism.

(b) The map $\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q})$ is an isomorphism.

Note that (b) follows from (a), by applying (a) to both $\mathrm{Inf}_Q^G : TF(Q) \to TF(G)$ and $\mathrm{Inf}_Q^{\widetilde{Q}} : TF(Q) \to TF(\widetilde{Q})$ and composing, because the map $\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q})$ is the identity on modules inflated from Q.

Part (a) of Conjecture 5.1 is in fact a consequence of any of the two conjectures made in [Carlson et al. 2014b]. First, Conjecture 10.1 in that reference asserts that, if a group homomorphism $\phi : G \to G'$ induces an isomorphism between the corresponding p-fusion systems, then ϕ should induce an isomorphism $TF(G') \xrightarrow{\sim} TF(G)$. In the special case where ϕ is the quotient map $\phi : G \to Q = G/O_{p'}(G)$, it is well-known that the fusion systems are isomorphic, so we would obtain the isomorphism $TF(Q) \xrightarrow{\sim} TF(G)$ of Conjecture 5.1 above. This special case is explicitly mentioned at the end of Section 10 in [op. cit].

Conjecture 9.2 in [op. cit.] asserts that the group $TF(G)$ should be generated by endotrivial modules lying in the principal block. Since $O_{p'}(G)$ acts trivially on any module lying in the principal block of G, such a module is inflated from Q, so the inflation map $\mathrm{Inf}_Q^G : TF(Q) \to TF(G)$ in Conjecture 5.1 above should be an isomorphism.

Example 7.3 below illustrates a method allowing one to prove that the maps in Conjecture 5.1 are isomorphisms in specific cases.

6. The perfect case

When the group $Q = G/O_{p'}(G)$ is perfect, there is an alternative approach to the construction of the injective group homomorphism of Theorem 1.1(a) using universal central extensions.

Recall that a *universal p'-central extension* of an arbitrary finite group Q is by definition a central extension

$$1 \longrightarrow M_{p'} \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

with $M_{p'} = \operatorname{Ker} \pi_{\widetilde{Q}}$ of order prime to p and satisfying the following universal property: For any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \operatorname{Ker} \pi$ of order prime to p, there exists a unique group homomorphism $\phi : \widetilde{Q} \to E$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M_{p'} & \longrightarrow & \widetilde{Q} & \xrightarrow{\pi_{\widetilde{Q}}} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi|_{M_{p'}}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q & \longrightarrow & 1 \end{array}$$

A standard argument shows that if a universal p'-central extension $(\widetilde{Q}, \pi_{\widetilde{Q}})$ exists, then it is unique up to isomorphism.

Lemma 6.1. *If $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is a universal p'-central extension of a finite group R, then $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is p'-representation group of Q.*

Proof. Let $(\check{Q}, \pi_{\check{Q}})$ be an arbitrary p'-representation group of Q. Let V be a finite-dimensional k-vector space and $\theta : Q \to \mathrm{PGL}(V)$ a group homomorphism. Because $(\check{Q}, \pi_{\check{Q}})$ has the projective lifting property and $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is universal, there exist group homomorphisms $\tilde{\theta} : \check{Q} \to \mathrm{GL}(V)$ and $\phi : \widetilde{Q} \to \check{Q}$ such that $\tilde{\theta} \circ \phi$ lifts θ. Therefore $(\widetilde{Q}, \pi_{\widetilde{Q}})$ has the projective lifting property as well.

Now, because $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is universal, it is easy to see that $X(\widetilde{Q}) = X(Q) = 1$. Therefore the Hochschild–Serre 5-term exact sequence associated to $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is:

$$1 \longrightarrow 1 \longrightarrow 1 \longrightarrow \operatorname{Hom}(M_{p'}, k^\times) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(Q, k^\times) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(\widetilde{Q}, k^\times)$$

Thus the transgression map $\mathrm{tr} : \operatorname{Hom}(M_{p'}, k^\times) \to \mathrm{H}^2(Q, k^\times) = M_k(Q)$ is injective. But $M_{p'} \cong \operatorname{Hom}(M_{p'}, k^\times)$, therefore by minimality of $(\check{Q}, \pi_{\check{Q}})$, we have $|M_{p'}| = |M_k(Q)|$ and $|\widetilde{Q}| = |\check{Q}|$, proving that $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is a p'-representation group of Q. $\square$

Lemma 6.2. *Any finite perfect group Q admits a universal p'-central extension.*

Proof. Since Q is a perfect group, it is well-known that Q has a representation group relative to $\mathbb{C}$, say $(\hat{Q}, \pi_{\hat{Q}})$, which is unique up to isomorphism and that

$$\operatorname{Ker}(\pi_{\hat{Q}}) =: M \cong M_{\mathbb{C}}(Q) = \mathrm{H}^2(Q, \mathbb{C}^\times),$$

the Schur multiplier of Q. Moreover, $(\hat{Q}, \pi_{\hat{Q}})$ is a universal central extension of Q, in particular perfect; see [Rotman 1995, Theorem 11.11]. Thus, for any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

where $Z = \operatorname{Ker} \pi$, there exists a unique group homomorphism $\psi : \hat{Q} \to E$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & \hat{Q} & \xrightarrow{\pi_{\hat{Q}}} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi|_M} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q & \longrightarrow & 1 \end{array}$$

If Z has order prime to p, then the p-part M_p of M lies in the kernel of $\psi|_M$. Passing to the quotient by M_p, we define $\widetilde{Q} := \hat{Q}/M_p$ and denote by $\phi : \widetilde{Q} \to E$ the map induced by ψ. Thus we obtain an induced central extension

$$1 \longrightarrow M_{p'} \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

where $M_{p'} := M/M_p$, a universal p'-central extension of Q by construction. □

Given an arbitrary group extension $1 \to A \to G \to Q \to 1$ with perfect quotient Q and kernel A of order prime to p, there is an induced p'-central extension:

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \xrightarrow{\pi_G} Q \longrightarrow 1$$

Moreover, by the above, Q admits a universal p'-central extension, which is in fact a p'-representation group $(\widetilde{Q}, \pi_{\widetilde{Q}})$ of Q. Therefore, by the universal property, there exists a unique group homomorphism $\phi_G : \widetilde{Q} \to G/[G, A]$ lifting the identity on Q.

Lemma 6.3. *The homomorphism $\phi_G : \widetilde{Q} \to G/[G, A]$ induces a group homomorphism*

$$\phi_G^* : T(G/[G, A]) \to T(\widetilde{Q})$$

such that $\phi_G^* = \operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}} \circ \operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$. Moreover, both $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}}$ and $\operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$ preserve indecomposability of endotrivial modules.

Proof. The kernel of ϕ_G is contained in $\operatorname{Ker} \pi_{\widetilde{Q}} = M_{p'}$, which is a p'-group. Therefore, there is an induced inflation map $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}} : T(\operatorname{Im}(\phi_G)) \to T(\widetilde{Q})$, preserving indecomposability of endotrivial modules.

Since $\operatorname{Im}(\phi_G)$ maps onto Q via π_G, the group $G/[G, A]$ is the product of $\operatorname{Im}(\phi_G)$ and the central p'-subgroup $A/[G, A]$. It follows that $\operatorname{Im}(\phi_G)$ is a normal subgroup of $G/[G, A]$ of index prime to p. Therefore, the restriction to $\operatorname{Im}(\phi_G)$ of any indecomposable endotrivial $k(G/[G, A])$-module remains indecomposable and is endotrivial [Carlson et al. 2009, Proposition 3.1].

We define ϕ_G^* to be the composite of $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}}$ and $\operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$. □

Composing the group homomorphism

$$\phi_G^* : T(G/[G, A]) \to T(\widetilde{Q})$$

with the inverse of the isomorphism

$$\mathrm{Inf}_{G/[G,A]}^{G} : T(G/[G, A]) \to T(G)$$

of Corollary 3.3, we obtain a group homomorphism

$$\Phi : T(G) \to T(\widetilde{Q}).$$

We now show that this provides the alternative approach to the map of Theorem 1.1.

Proposition 6.4. *Suppose that G satisfies Hypothesis 3.1 and that Q is perfect.*

(a) $\mathrm{Ker}\,\Phi = X(G)$.

(b) *The induced injective group homomorphism*

$$\bar{\Phi} : T(G)/X(G) \to T(\widetilde{Q}) = T(\widetilde{Q})/X(\widetilde{Q})$$

coincides with the map $\Phi_{G,\widetilde{Q}}$ of Theorem 1.1.

Proof. Consider the map $\phi_G^* : T(G/[G, A]) \to T(\widetilde{Q})$ of Lemma 6.3. It is clear that the image of a one-dimensional module is one-dimensional, hence trivial since $X(\widetilde{Q}) = 1$ by Lemma 3.4. Therefore $X(G) \subseteq \mathrm{Ker}\,\Phi$. It follows that Φ induces a group homomorphism $\bar{\Phi}$ as in the statement.

Our assumption on G implies that, if V is an endotrivial kG-module, then $[G, A]$ acts trivially on V (Corollary 3.3). Moreover, $\rho_V : G/[G, A] \to \mathrm{GL}(V)$ lifts $\theta_V : Q \to \mathrm{PGL}(V)$, as in Section 4. It is then clear that $\rho_V\,\phi_G : \widetilde{Q} \to \mathrm{GL}(V)$ also lifts $\theta_V : Q \to \mathrm{PGL}(V)$. Therefore, the definition of $\Phi_{G,\widetilde{Q}}$ (see Proposition 4.2) shows that the class of V is mapped by $\Phi_{G,\widetilde{Q}}$ to the class of the module $V_{\widetilde{Q}}$ corresponding to the representation $\rho_V\,\phi_G$. In other words, $[V_{\widetilde{Q}}] = \Phi([V])$ and this shows that $\Phi_{G,\widetilde{Q}}$ coincides with $\bar{\Phi}$.

Finally, since $\Phi_{G,\widetilde{Q}}$ is injective and is equal to $\bar{\Phi}$, we have $\mathrm{Ker}\,\bar{\Phi} = \{0\}$. Therefore we obtain $\mathrm{Ker}\,\Phi = X(G)$. □

Remark 6.5. The proof we give above shows that Proposition 6.4 remains valid if the assumption that Q is perfect is replaced by the assumption that Q admits a universal p'-central extension. It is proved in [Lassueur and Thévenaz 2017] that this happens if and only if $X(Q) = 1$, that is, Q is p'-perfect. Here, for simplicity, we restrict ourselves to the perfect case.

7. Examples

In this final section, we provide various examples, in particular illustrating cases where the morphism $\Phi_{G,\widetilde{Q}}$ is not surjective.

Example 7.1. Suppose that Q is simple and take $G = Q$, hence $A = O_{p'}(G) = \{1\}$. Then $\Phi_{Q,\widetilde{Q}}$ is just the inflation map $T(Q) \to T(\widetilde{Q})$. If Q is a finite simple group listed in the table below, then it is known that its unique p'-representation group $\widetilde{Q}$ has indecomposable endotrivial modules lying in faithful p-blocks, namely not inflated from Q.

Q	p	$\widetilde{Q}$	$T(Q)$	$T(\widetilde{Q})$
$\mathfrak{A}_6$	3	$2\,.\,\mathfrak{A}_6$	$\mathbb{Z}\oplus\mathbb{Z}/4$	$\mathbb{Z}\oplus\mathbb{Z}/8$
$\mathfrak{A}_6$	2	$3\,.\,\mathfrak{A}_6$	$\mathbb{Z}^2$	$\mathbb{Z}^2\oplus\mathbb{Z}/3$
M_{22}	3	$4\,.\,M_{22}$	$\mathbb{Z}\oplus(\mathbb{Z}/2)^2$	$\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/4$
J_3	2	$3\,.\,J_3$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/3$
Ru	3	$2\,.\,Ru$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/4$
Fi_{22}	5	$6\,.\,Fi_{22}$	$\mathbb{Z}\oplus(\mathbb{Z}/2)^2$	$\mathbb{Z}\oplus\mathbb{Z}/6\oplus\mathbb{Z}/2$

The results concerning the sporadic groups can be found in [Lassueur and Mazza 2015b, Table 3], and those about the alternating group $\mathfrak{A}_6$ in [Lassueur and Mazza 2015a, Theorems A and B] together with [Carlson et al. 2009, Theorems A and B].

Further examples are given by the exceptional covering group $2.F_4(2)$ of the exceptional group of Lie type $F_4(2)$, which possesses simple torsion endotrivial modules lying in faithful blocks in characteristics 5 and 7 [Lassueur and Malle 2015, Proposition 5.5], although the full structure of the group of endotrivial modules has not been determined in these cases.

Example 7.2. Assume $p > 2$, let $n \geq \max\{2p,\, p+4\}$ be an integer and denote by $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ the two isoclinic p'-representation groups of the symmetric group $\mathfrak{S}_n$. Corollary 4.3 yields

$$T(\widetilde{\mathfrak{S}}_n)/X(\widetilde{\mathfrak{S}}_n) \cong T(\widehat{\mathfrak{S}}_n)/X(\widehat{\mathfrak{S}}_n).$$

However, Lassueur and Mazza [2015a, Theorem B, parts (1) and (2)] prove a stronger result, namely

$$T(\widetilde{\mathfrak{S}}_n) = \mathrm{Inf}_{\mathfrak{S}_n}^{\widetilde{\mathfrak{S}}_n}(T(\mathfrak{S}_n)) \quad \text{and} \quad T(\widehat{\mathfrak{S}}_n) = \mathrm{Inf}_{\mathfrak{S}_n}^{\widehat{\mathfrak{S}}_n}(T(\mathfrak{S}_n)).$$

Consequently, given any finite group G such that $G/O_{p'}(G)$ is isomorphic to one of $\mathfrak{S}_n$, $\widetilde{\mathfrak{S}}_n$ or $\widehat{\mathfrak{S}}_n$ (with $n \geq \max\{2p,\, p+4\}$), by Theorem 1.1 there exist injective group homomorphisms

$$T(\mathfrak{S}_n)/X(\mathfrak{S}_n) \longrightarrow T(G)/X(G) \xrightarrow{\Phi_{G,\widehat{\mathfrak{S}}_n}} T(\widehat{\mathfrak{S}}_n)/X(\widehat{\mathfrak{S}}_n) \xrightarrow{\sim} T(\mathfrak{S}_n)/X(\mathfrak{S}_n),$$

where the first map is induced by inflation. Hence we have $T(G)/X(G) \cong T(\mathfrak{S}_n)/X(\mathfrak{S}_n)$. Recall that the structure of $T(\mathfrak{S}_n)$ is known [Carlson et al. 2009].

Example 7.3. In this final example, we outline a method which allows us to show that the maps $\mathrm{Inf}_Q^{\widetilde{Q}}$ is an isomorphism on the torsion-free part of the groups of endotrivial modules of Q and $\widetilde{Q}$ in some concrete cases.

Specifically, we may use the fact that endotrivial modules are liftable to characteristic zero, and afford characters taking root of unity values at p-singular conjugacy classes; see [Lassueur et al. 2016, Theorem 1.3 and Corollary 2.3]. Therefore, if for every faithful p-block B of $k\widetilde{Q}$ (of full defect) no elements of $\mathbb{Z}\,\mathrm{Irr}_{\mathbb{C}}(B)$ take root of unity values at p-singular conjugacy classes of $\widetilde{Q}$, then any endotrivial $k\widetilde{Q}$-module is inflated from Q, hence

$$\mathrm{Inf}_Q^{\widetilde{Q}} : TF(Q) \to TF(\widetilde{Q})$$

is an isomorphism.

This was used [Lassueur and Mazza 2015a, Theorem B] in the case that $Q = \mathfrak{S}_n$, $n \geq \max\{2p, p+4\}$ (as mentioned in Example 7.2 above), as well as for a large number of sporadic simple groups Q [Lassueur and Mazza 2015b, Lemmas 4.3 and 6.2]. More precisely, in characteristic $p = 2$ for $Q = M_{12}$, M_{22}, J_2, HS, McL, Ru, Suz, ON, Fi_{22}, Co_1, Fi'_{24}, or B; in characteristic $p = 3$ for $Q = M_{12}$, J_2, HS, Suz, Fi_{22}, Co_1, or B; in characteristic $p = 5$ for $Q = J_2$, HS, Ru, Suz, Co_1, Fi'_{24}, or B; and in characteristic $p = 7$ for $Q = Co_1$, Fi'_{24}, or B.

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