

*Pacific
Journal of
Mathematics*

**ENDOTRIVIAL MODULES:
A REDUCTION TO p' -CENTRAL EXTENSIONS**

CAROLINE LASSUEUR AND JACQUES THÉVENAZ

ENDOTRIVIAL MODULES: A REDUCTION TO p' -CENTRAL EXTENSIONS

CAROLINE LASSUEUR AND JACQUES THÉVENAZ

We examine how, in prime characteristic p , the group of endotrivial modules of a finite group G and the group of endotrivial modules of a quotient of G modulo a normal subgroup of order prime to p are related. There is always an inflation map, but examples show that this map is in general not surjective. We prove that the situation is controlled by a single central extension, namely, the central extension given by a p' -representation group of the quotient of G by its largest normal p' -subgroup.

1. Introduction

Endotrivial modules play an important role in the representation theory of finite groups. They have been classified in a number of special cases; see, e.g., the recent papers [Carlson et al. 2014a; Lassueur and Mazza 2015b] and the references therein. Over an algebraically closed field k of prime characteristic p , endotrivial modules for a finite group G form an abelian group $T(G)$, which is known to be finitely generated. One of the main question is to understand the structure of $T(G)$, and, in particular, of its torsion subgroup $TT(G)$.

We let $X(G)$ be the subgroup of $TT(G)$ consisting of all one-dimensional representations, that is, $X(G) \cong \text{Hom}(G, k^\times)$. We also let $K(G)$ be the kernel of the restriction map $\text{Res}_p^G : T(G) \rightarrow T(P)$ to a Sylow p -subgroup P of G . It is known that $X(G) \subseteq K(G) \subseteq TT(G)$ and that $K(G) = TT(G)$ in almost all cases (namely if we exclude the cases when a Sylow p -subgroup of G is cyclic, generalized quaternion, or semidihedral). Moreover, there are numerous cases, including infinite families of groups G , for which $K(G) = X(G)$. However, this is not always the case, and the structure of $K(G)$ is not understood in general.

Let $O_{p'}(G)$ denote the largest normal subgroup of G of order prime to p and set $Q := G/O_{p'}(G)$ for simplicity. There is always an inflation homomorphism

$$\text{Inf}_Q^G : T(Q) \rightarrow T(G)$$

MSC2010: primary 20C20; secondary 20C25.

Keywords: Endotrivial modules, Schur multipliers, central extensions, perfect groups.

which is easily seen to be injective. But examples show that it is in general not surjective, so we cannot expect an isomorphism between $T(G)$ and $T(Q)$. The present article analyzes how $T(G)$ and $T(Q)$ are related, by making use of a suitable central extension of Q . More precisely, associated with Q , there is a p' -representation group \tilde{Q} , which is a central extension with kernel of order prime to p . This controls the behavior of projective representations of Q (in the sense of Schur). When Q is a perfect group, then \tilde{Q} is unique and is also called the universal p' -central extension of Q . When Q is not perfect, then \tilde{Q} may not be unique.

The present work is based on a key result by the first author and S. Koshitani [Koshitani and Lassueur 2016]. In the course of their investigation of endotrivial modules for a finite group with dihedral Sylow 2-subgroups, they proved a general result [op. cit., Theorem 4.4] about endotrivial modules for an arbitrary group G in the presence of a normal subgroup N of order prime to p , under mild hypotheses on G (see Hypothesis 3.1). Their result uses modules over twisted group algebras of G/N . Taking $Q = G/N$ with $N = O_{p'}(G)$, we can view such modules as modules over the ordinary group algebra of the central extension \tilde{Q} . In this way, we can show that the structure of $T(G)$ is closely related to the structure of $T(\tilde{Q})$. Our main result is as follows:

Theorem 1.1. *Let G be a finite group of p -rank at least 2 and no strongly p -embedded subgroups. Let \tilde{Q} be any p' -representation group of the group $Q := G/O_{p'}(G)$.*

(a) *There exists an injective group homomorphism*

$$\Phi_{G, \tilde{Q}} : T(G)/X(G) \rightarrow T(\tilde{Q})/X(\tilde{Q}).$$

In particular, $\Phi_{G, \tilde{Q}}$ maps the class of $\text{Inf}_Q^G(W)$ to the class of $\text{Inf}_{\tilde{Q}}^{\tilde{Q}}(W)$, for any endotrivial kQ -module W .

(b) *The map $\Phi_{G, \tilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G, \tilde{Q}} : K(G)/X(G) \rightarrow K(\tilde{Q})/X(\tilde{Q}).$$

(c) *In particular, if $K(\tilde{Q}) = X(\tilde{Q})$, then $K(G) = X(G)$.*

We note that the construction of the map $\Phi_{G, \tilde{Q}}$ relies on [op. cit., Theorem 4.4], which itself relies on Navarro and Robinson [Navarro and Robinson 2012], whose proof makes use of the classification of finite simple groups. This construction will be made precise in Section 4. Examples show that $\Phi_{G, \tilde{Q}}$ is in general not surjective (see Section 7), but the theorem provides some information on $K(G)$, for all groups G such that $G/O_{p'}(G) = Q$. In particular, the question of the equality $K(G) = X(G)$ is reduced to the same question for the single group \tilde{Q} .

We also conjecture that $\Phi_{G, \tilde{Q}}$ induces an isomorphism on the torsion-free part of $T(G)$ and $T(\tilde{Q})$ (see Section 5). Moreover, in case Q is perfect, then there is an alternative approach to $\Phi_{G, \tilde{Q}}$ which we present in Section 6.

The two main assumptions on G in [Theorem 1.1](#) are needed for applying the results of [\[Koshitani and Lassueur 2016\]](#). However, these assumptions are not really restrictive because endotrivial modules are completely understood in the two excluded cases: they are classified if the p -rank is 1 [\[Mazza and Thévenaz 2007; Carlson et al. 2013\]](#), and $T(G) \cong T(H)$ if G has a strongly p -embedded subgroup H ; see [\[Mazza and Thévenaz 2007, Lemma 2.7\]](#).

The two assumptions also allow us to prove that $T(G) \cong T(G/[G, A])$, where $A = O_{p'}(G)$, or in other words that the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with kernel A of order prime to p can always be replaced by the central extension

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \longrightarrow Q \longrightarrow 1.$$

This is explained in [Section 3](#).

2. Notation and preliminaries

Throughout, unless otherwise specified, we use the following notation. We let k denote an algebraically closed field of prime characteristic p . We assume that all groups are finite, and that all modules over group algebras are finitely generated, and we set $\otimes := \otimes_k$. If G is an arbitrary finite group and V is a kG -module, we denote by $\rho_V : G \rightarrow \text{GL}(V)$ the corresponding k -representation, and we denote by $\pi_V : \text{GL}(V) \rightarrow \text{PGL}(V)$ the canonical surjection. Furthermore, we denote by V^* the k -dual of V endowed with a kG -module structure via $(gf)(v) = f(g^{-1}v)$ for every $g \in G$, $f \in V^*$, $v \in V$.

Assuming moreover that $p \nmid |G|$, we recall that a kG -module V is called *endotrivial* if there is an isomorphism of kG -modules $\text{End}_k(V) \cong k \oplus (\text{proj})$, where k denotes the trivial kG -module and (proj) some projective kG -module, which might be zero. Any endotrivial kG -module V splits as a direct sum $V = V_0 \oplus (\text{proj})$ where V_0 , the projective-free part of V , is indecomposable and endotrivial. The relation

$$U \sim V \iff U_0 \cong V_0$$

is an equivalence relation on the class of endotrivial kG -modules, and $T(G)$ denotes the resulting set of equivalence classes (which we denote by square brackets). Then $T(G)$, endowed with the law $[U] + [V] := [U \otimes V]$, is an abelian group called the *group of endotrivial modules of G* . The zero element is the class $[k]$ of the trivial module and $-[V] = [V^*]$, the class of the dual module V^* . By a result of Puig, the group $T(G)$ is known to be a finitely generated abelian group; see, e.g., [\[Carlson et al. 2006, Corollary 2.5\]](#).

We let $X(G)$ denote the group of one-dimensional kG -modules endowed with the tensor product \otimes , and recall that $X(G) \cong \text{Hom}(G, k^\times) \cong (G/[G, G])_{p'}$. Identifying a one-dimensional module with its class in $T(G)$, we consider $X(G)$ as a subgroup of $T(G)$.

Furthermore, if P is a Sylow p -subgroup of G , we set

$$K(G) = \text{Ker}(\text{Res}_P^G : T(G) \rightarrow T(P)).$$

In other words, the class of an indecomposable endotrivial kG -module V belongs to $K(G)$ if and only if $V \downarrow_P^G \cong k \oplus (\text{proj})$, that is, in other words, V is a trivial source module. We have $X(G) \subseteq K(G)$ because any one-dimensional kP -module is trivial. Moreover, $K(G) \subseteq TT(G)$ (see [Carlson et al. 2011, Lemma 2.3]), and $K(G) = TT(G)$ unless P is cyclic, generalized quaternion, or semidihedral, by the main result of [Carlson and Thévenaz 2005].

By a central extension (E, π) of Q , we mean a group extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \text{Ker } \pi$ central in E . Recall that (E, π) is said to have the *projective lifting property (relative to k)* if, for every finite-dimensional k -vector space V , every group homomorphism $\theta : Q \rightarrow \text{PGL}(V)$ can be completed to a commutative diagram of group homomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \lambda|_Z \downarrow & & \lambda \downarrow & & \theta \downarrow & & \\ 1 & \longrightarrow & k^\times \cdot \text{Id}_V & \longrightarrow & \text{GL}(V) & \xrightarrow{\pi_V} & \text{PGL}(V) & \longrightarrow & 1 \end{array}$$

In general, the homomorphism λ is not uniquely defined. However, by the commutativity of the diagram, the following holds:

Lemma 2.1. *In the above situation, if $\lambda, \lambda' : E \rightarrow \text{GL}(V)$ are two liftings of θ to E , then there exists a degree one representation $\mu : E \rightarrow \text{GL}(k)$ such that $\lambda' = \lambda \otimes \mu$.*

By results of Schur (slightly generalized for dealing with the case of characteristic p), given a finite group Q , there always exists a central extension (E, π) of Q , with kernel $M_k(Q) := \text{H}^2(Q, k^\times)$, which has the projective lifting property. A p' -representation group of Q (or a representation group of Q relative to k) is a central extension (\tilde{Q}, π) of Q of minimal order with the projective lifting property. In this case $M_k(Q) \cong \text{Ker } \pi \leq [\tilde{Q}, \tilde{Q}]$. We recall that $M_k(Q) \cong \text{H}^2(Q, \mathbb{C}^\times)_{p'}$, the p' -part of the Schur multiplier of Q , and that in general a group Q with $X(Q) \neq 1$ may have several nonisomorphic p' -representation groups. Furthermore, fixing a p' -representation group (\tilde{Q}, π) of Q , the abelian group $M_k(Q)$ becomes isomorphic

to its k^\times -dual via the transgression homomorphism

$$\text{tr} : \text{Hom}(M_k(Q), k^\times) \rightarrow H^2(Q, k^\times)$$

defined by $\text{tr}(\varphi) = [\varphi \circ \alpha]$, where the cocycle $\alpha \in Z^2(Q, M_k(Q))$ is in the cohomology class corresponding to the central extension $1 \rightarrow M_k(Q) \rightarrow \tilde{Q} \xrightarrow{\pi} Q \rightarrow 1$. For further details and proofs we refer the reader to [Nagao and Tsushima 1989, Chapter 3, §5; Curtis and Reiner 1981, §11E].

If V, W are two finite-dimensional k -vector spaces, then the tensor product of linear maps induces a tensor product $-\otimes - : \text{PGL}(V) \times \text{PGL}(W) \rightarrow \text{PGL}(V \otimes W)$ via $\pi_V(\alpha) \otimes \pi_W(\beta) := \pi_{V \otimes W}(\alpha \otimes \beta)$ for any $\alpha \in \text{GL}(V)$ and any $\beta \in \text{GL}(W)$. Therefore, if $\mu : Q \rightarrow \text{PGL}(V)$ and $\nu : Q \rightarrow \text{PGL}(W)$ are group homomorphisms, we may define a group homomorphism $\mu \otimes \nu : Q \rightarrow \text{PGL}(V \otimes W)$ via $(\mu \otimes \nu)(q) := \mu(q) \otimes \nu(q)$ for every $q \in Q$. We shall use the following well-known results throughout:

Lemma 2.2. *Let $1 \rightarrow A \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ be an arbitrary group extension.*

- (a) *Whenever V is a kG -module such that $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$, the group homomorphism $\rho_V : G \rightarrow \text{GL}(V)$ induces a uniquely defined group homomorphism $\theta_V : Q \rightarrow \text{PGL}(V)$ such that the following diagram commutes:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \rho_V|_A \downarrow & & \rho_V \downarrow & & \theta_V \downarrow & & \\ 1 & \longrightarrow & k^\times \cdot \text{Id}_V & \longrightarrow & \text{GL}(V) & \xrightarrow{\pi_V} & \text{PGL}(V) & \longrightarrow & 1 \end{array}$$

- (b) *If V, W are kG -modules such that $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$ and $\rho_W(A) \subseteq k^\times \cdot \text{Id}_W$, then $\rho_{V \otimes W}(A) \subseteq k^\times \cdot \text{Id}_{V \otimes W}$ and we have $\theta_{V \otimes W} = \theta_V \otimes \theta_W$.*

Proof. (a) Choose a set-theoretic section $s : Q \rightarrow G$ for π and define $\theta_V := \pi_V \circ \rho_V \circ s$. Since $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$, the map θ_V is a group homomorphism making the diagram commute. Clearly θ_V is uniquely defined since π is an epimorphism.

- (b) This is a straightforward computation. □

3. Endotrivial modules and central extensions

We now fix G to be a finite group of order divisible by p , we set $A := O_{p'}(G)$ and $Q := G/A$, and we denote by $\pi_G : G \rightarrow Q$ the quotient map. Moreover, we let $(\tilde{Q}, \pi_{\tilde{Q}})$ be a fixed p' -representation group of Q .

Since A is a p' -subgroup of G , inflation induces an injective group homomorphism

$$\text{Inf}_Q^G : T(Q) \rightarrow T(G), \quad [V] \rightarrow [\text{Inf}_Q^G(V)].$$

This is because the inflation of a projective module remains projective when the kernel A is a p' -group. We emphasize that endotrivial kG -modules cannot be

recovered from endotrivial kQ -modules, as in general the inflation map Inf_Q^G is not an isomorphism; see [Section 7](#).

Hypothesis 3.1. Assume G is a finite group fulfilling the following two conditions:

- (1) the p -rank of G is greater than or equal to 2; and
- (2) G has no strongly p -embedded subgroups.

The next result restates one of the main results of [\[Koshitani and Lassueur 2016\]](#), but in different terms. Our statement will allow us later to avoid working with modules over twisted group algebras, but simply consider the corresponding projective representations instead.

Theorem 3.2 [\[Koshitani and Lassueur 2016\]](#). *Suppose G satisfies [Hypothesis 3.1](#).*

- (a) *If V is an indecomposable endotrivial kG -module, then $V \downarrow_A^G \cong Y \oplus \cdots \oplus Y$, where Y is a one-dimensional kA -module.*
- (b) *If V is an indecomposable endotrivial kG -module, then $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$.*

Proof. (a) Since G satisfies [Hypothesis 3.1](#), any composition factor Y of $V \downarrow_A^G$ is G -invariant, by [\[op. cit., Lemma 4.3\]](#). Therefore $V \downarrow_A^G \cong Y \oplus \cdots \oplus Y$ and [\[op. cit., Theorem 4.4\]](#) proves that $\dim Y = 1$.

(b) This is a restatement of (a). □

Corollary 3.3. *Suppose that G satisfies [Hypothesis 3.1](#). The inflation map*

$$\text{Inf}_{G/[G,A]}^G : T(G/[G,A]) \rightarrow T(G)$$

is a group isomorphism.

Proof. Since $[G,A]$ is a normal p' -subgroup of G , the inflation map $\text{Inf}_{G/[G,A]}^G$ is a well-defined injective group homomorphism. In order to prove that it is surjective, it suffices to prove that $[G,A]$ acts trivially on any indecomposable endotrivial kG -module V . But by [Theorem 3.2](#) we have

$$\rho_V([G,A]) \subseteq [\rho_V(G), \rho_V(A)] \subseteq [\rho_V(G), k^\times \cdot \text{Id}_V] = \{\text{Id}_V\}.$$

Hence $[G,A]$ acts trivially on V . □

[Corollary 3.3](#) is a reduction to the case of central extensions. Explicitly, for the study of endotrivial modules, we may always replace the given extension

$$1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1,$$

and consider instead the central extension

$$1 \rightarrow A/[G,A] \rightarrow G/[G,A] \rightarrow Q \rightarrow 1.$$

We shall in fact not use this reduction for the proof of our main result, but rather apply directly [Theorem 3.2](#).

Lemma 3.4. *Let $(\tilde{Q}, \pi_{\tilde{Q}})$ be a p' -representation group of Q . Then $X(\tilde{Q}) = \text{Inf}_{\tilde{Q}}^Q(X(Q))$, hence $X(\tilde{Q}) \cong X(Q)$.*

Proof. We apply the fact, mentioned in Section 2, that $\text{Ker } \pi_{\tilde{Q}} \subseteq [\tilde{Q}, \tilde{Q}]$. This implies that any one-dimensional representation of \tilde{Q} has $\text{Ker } \pi_{\tilde{Q}}$ in its kernel, hence is inflated from $\tilde{Q}/\text{Ker } \pi_{\tilde{Q}} \cong Q$.

Another way of seeing the same thing is to associate to the central extension

$$1 \longrightarrow M_k(Q) \longrightarrow \tilde{Q} \xrightarrow{\pi_{\tilde{Q}}} Q \longrightarrow 1$$

the Hochschild–Serre five-term exact sequence

$$1 \longrightarrow \text{Hom}(Q, k^\times) \xrightarrow{\text{Inf}} \text{Hom}(\tilde{Q}, k^\times) \xrightarrow{\text{Res}} \text{Hom}(M_k(Q), k^\times) \xrightarrow{\text{tr}} \text{H}^2(Q, k^\times) \xrightarrow{\text{Inf}} \text{H}^2(\tilde{Q}, k^\times).$$

Since the transgression map tr is an isomorphism, the first map Inf must be an isomorphism as well. \square

4. Proof of Theorem 1.1

Keep the notation of the previous section. Moreover, given an endotrivial kG -module V such that $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$, we let

$$\theta_V : Q \rightarrow \text{PGL}(V)$$

denote the induced homomorphism constructed in Lemma 2.2(a). The projective lifting property for the central extension $(\tilde{Q}, \pi_{\tilde{Q}})$ allows us to fix a representation

$$\rho_{V_{\tilde{Q}}} : \tilde{Q} \rightarrow \text{GL}(V)$$

lifting θ_V to \tilde{Q} . We denote by $V_{\tilde{Q}}$ the corresponding $k\tilde{Q}$ -module.

Lemma 4.1. *Let V be an endotrivial kG -module such that $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$. Then $V_{\tilde{Q}}$ is an endotrivial $k\tilde{Q}$ -module.*

Proof. We have to work with two group extensions

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi_G} Q \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow M \longrightarrow \tilde{Q} \xrightarrow{\pi_{\tilde{Q}}} Q \longrightarrow 1,$$

where $M := M_k(Q)$. Both A and M have order prime to p .

Let $P \in \text{Syl}_p(G)$, set $\bar{P} := AP/A \in \text{Syl}_p(Q)$, and let $\iota_P : P \rightarrow AP$ be the inclusion map, so that

$$\phi := \pi_G \circ \iota_P : P \rightarrow \bar{P}$$

is an isomorphism. Next choose $\tilde{P} \in \text{Syl}_p(\tilde{Q})$ such that $M\tilde{P}/M = \bar{P} \in \text{Syl}_p(Q)$. Let $\iota_{\tilde{P}} : \tilde{P} \rightarrow M\tilde{P}$ be the inclusion map, so that $\psi := \pi_{\tilde{Q}} \circ \iota_{\tilde{P}} : \tilde{P} \rightarrow \bar{P}$ is an

isomorphism. Consider now the two commutative diagrams:

$$\begin{array}{ccc}
 P & \xrightarrow{\phi} & \bar{P} \\
 \rho_V \downarrow & & \theta_V \downarrow \\
 \text{GL}(V) & \xrightarrow{\pi_V} & \text{PGL}(V)
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \tilde{P} & \xrightarrow{\psi} & \bar{P} \\
 \rho_{\tilde{Q}} \downarrow & & \theta_V \downarrow \\
 \text{GL}(V) & \xrightarrow{\pi_V} & \text{PGL}(V)
 \end{array}$$

where we write $\rho_{\tilde{Q}} := \rho_{V_{\tilde{Q}}}$ for simplicity. Since ϕ and ψ are isomorphisms, for any $u \in \bar{P}$,

$$\pi_V \rho_V \phi^{-1}(u) = \theta_V(u) = \pi_V \rho_{\tilde{Q}} \psi^{-1}(u).$$

We claim that if two elements $u_1, u_2 \in \text{GL}(V)$ have p -power order and satisfy $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = u_2$. Postponing the proof of the claim, we deduce that

$$\rho_V \phi^{-1}(u) = \rho_{\tilde{Q}} \psi^{-1}(u),$$

because they have p -power order. This means that the representations $(\rho_V)|_P$ and $(\rho_{\tilde{Q}})|_{\tilde{P}}$, transported via isomorphisms to representations of \bar{P} , are equal. Now, a module is endotrivial if and only if its restriction to a Sylow p -subgroup is; see [Carlson et al. 2006, Proposition 2.6]. Moreover, this property is preserved when transported via group isomorphisms. Since V is endotrivial, so is $V \downarrow_P$, hence so is $V_{\tilde{Q}} \downarrow_{\tilde{P}}$, and it follows that $V_{\tilde{Q}}$ is endotrivial.

We are left with the proof of the claim. If $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = \alpha u_2$ where $\alpha \in k^\times$. For some large enough power p^n , we have $u_1^{p^n} = u_2^{p^n} = 1$. Therefore we obtain

$$1 = u_1^{p^n} = (\alpha u_2)^{p^n} = \alpha^{p^n} u_2^{p^n} = \alpha^{p^n}.$$

But there are no nontrivial p -th roots of unity in k^\times , so $\alpha = 1$, hence $u_1 = u_2$. \square

Proposition 4.2. *Assume G satisfies Hypothesis 3.1. Then there is an injective group homomorphism*

$$\Phi_{G, \tilde{Q}} : T(G)/X(G) \rightarrow T(\tilde{Q})/X(\tilde{Q})$$

defined by $\Phi_{G, \tilde{Q}}([V] + X(G)) := [V_{\tilde{Q}}] + X(\tilde{Q})$ for any indecomposable endotrivial kG -module V . Moreover, for any endotrivial kQ -module W , the homomorphism $\Phi_{G, \tilde{Q}}$ maps the class of $\text{Inf}_Q^G(W)$ to the class of $\text{Inf}_{\tilde{Q}}^{\tilde{Q}}(W)$.

Proof. First, Lemma 4.1 allows us to define a map $\phi : T(G) \rightarrow T(\tilde{Q})/X(\tilde{Q})$ by setting $\phi([V]) := [V_{\tilde{Q}}] + X(\tilde{Q})$ for any $[V] \in T(G)$ such that $\rho_V(A) \subseteq k^\times \cdot \text{Id}_V$. The definition of $\phi([V])$ does not depend on the choice of $V_{\tilde{Q}}$, for if $\rho_{V'_{\tilde{Q}}}$ is a second lifting of θ_V to \tilde{Q} , then by Lemma 2.1 there exists $X \in X(\tilde{Q})$ such that $V'_{\tilde{Q}} \cong V_{\tilde{Q}} \otimes X$, hence $\phi([V_{\tilde{Q}}]) = \phi([V'_{\tilde{Q}}])$.

Next, let V, W be two indecomposable endotrivial kG -modules. Theorem 3.2 implies that $\rho_{V \otimes W}(A) = (\rho_V \otimes \rho_W)(A) \subseteq k^\times \cdot \text{Id}_{V \otimes W}$. Thus, by Lemma 2.2(b),

$\theta_V \otimes_W = \theta_V \otimes \theta_W$, and it is easy to verify that $\rho_{V_{\tilde{Q}}} \otimes \rho_{W_{\tilde{Q}}}$ lifts $\theta_V \otimes \theta_W$ to \tilde{Q} . Therefore, by [Lemma 2.1](#), there exists $X \in X(\tilde{Q})$ such that $(V \otimes W)_{\tilde{Q}} \cong V_{\tilde{Q}} \otimes W_{\tilde{Q}} \otimes X$. This shows that ϕ is a group homomorphism.

It is clear that $\text{Ker } \phi = X(G)$, since by construction $\dim V_{\tilde{Q}} = \dim V$ for any indecomposable endotrivial kG -module V . As a result, ϕ induces the required homomorphism $\Phi_{G, \tilde{Q}}$.

Finally, if W is any endotrivial kQ -module, then the $k\tilde{Q}$ -module constructed from $V = \text{Inf}_Q^G(W)$ is easily seen to be the inflated module $V_{\tilde{Q}} = \text{Inf}_{\tilde{Q}}^{\tilde{Q}}(W)$, because the map $\theta_V : Q \rightarrow \text{PGL}(V)$ comes from a group homomorphism $Q \rightarrow \text{GL}(V)$. This shows that the class of $\text{Inf}_Q^G(W)$ is mapped to the class of $\text{Inf}_{\tilde{Q}}^{\tilde{Q}}(W)$ under the map $\Phi_{G, \tilde{Q}}$, proving the additional statement. \square

Corollary 4.3. *Assume G satisfies [Hypothesis 3.1](#). If \tilde{Q}_1 and \tilde{Q}_2 are two nonisomorphic p' -representation groups of Q , then*

$$\Phi_{\tilde{Q}_1, \tilde{Q}_2} : T(\tilde{Q}_1)/X(\tilde{Q}_1) \rightarrow T(\tilde{Q}_2)/X(\tilde{Q}_2)$$

is an isomorphism.

Proof. Let V be an indecomposable $k\tilde{Q}_1$ -module. By construction

$$\Phi_{\tilde{Q}_1, \tilde{Q}_2}([V] + X(\tilde{Q}_1)) = [W] + X(\tilde{Q}_2),$$

where $W := V_{\tilde{Q}_2}$ is a $k\tilde{Q}_2$ -module such that ρ_W lifts $\theta_V : Q \rightarrow \text{PGL}(V)$ to \tilde{Q}_2 . But then ρ_V lifts $\theta_W = \theta_V$ to \tilde{Q}_1 , so that by construction

$$\Phi_{\tilde{Q}_2, \tilde{Q}_1}([W] + X(\tilde{Q}_2)) = [V] + X(\tilde{Q}_1).$$

In other words, $\Phi_{\tilde{Q}_1, \tilde{Q}_2} \circ \Phi_{\tilde{Q}_2, \tilde{Q}_1} = \text{Id}$. Similarly $\Phi_{\tilde{Q}_2, \tilde{Q}_1} \circ \Phi_{\tilde{Q}_1, \tilde{Q}_2} = \text{Id}$. \square

Corollary 4.4. *Assume G satisfies [Hypothesis 3.1](#). The map $\Phi_{G, \tilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G, \tilde{Q}} : K(G)/X(G) \rightarrow K(\tilde{Q})/X(\tilde{Q}).$$

In particular, if $K(\tilde{Q}) \cong X(\tilde{Q})$, then $K(G) \cong X(G)$.

Proof. Let $P \in \text{Syl}_p(G)$ and let V be an indecomposable endotrivial kG -module. As in the proof of [Lemma 4.1](#), the two modules $V \downarrow_P^G$ and $V_{\tilde{Q}} \downarrow_{\tilde{P}}^{\tilde{Q}}$ are isomorphic, provided we view them as modules over the group \bar{P} via the isomorphisms $P \cong \bar{P}$ and $\tilde{P} \cong \bar{P}$. It follows that V has a trivial source if and only if $V_{\tilde{Q}}$ has. Therefore $\Phi_{G, \tilde{Q}}$ restricts to an injective group homomorphism

$$\Phi_{G, \tilde{Q}} : K(G)/X(G) \rightarrow K(\tilde{Q})/X(\tilde{Q}).$$

The special case follows. \square

[Proposition 4.2](#) together with [Corollary 4.4](#) prove [Theorem 1.1](#).

5. Conjecture on the torsion-free part

We keep the notation of the previous sections. Let $TF(G) = T(G)/TT(G)$, the torsion-free part of the group of endotrivial modules. Since $X(G) \subseteq TT(G)$, the map

$$\Phi_{G, \tilde{Q}} : T(G)/X(G) \rightarrow T(\tilde{Q})/X(\tilde{Q})$$

induces an injective group homomorphism

$$\Psi_{G, \tilde{Q}} : TF(G) \rightarrow TF(\tilde{Q}).$$

We know that $\Phi_{G, \tilde{Q}}$ is in general not surjective, but we conjecture that $\Psi_{G, \tilde{Q}}$ is surjective.

Conjecture 5.1. (a) The map $\text{Inf}_Q^G : TF(Q) \rightarrow TF(G)$ is an isomorphism.

(b) The map $\Psi_{G, \tilde{Q}} : TF(G) \rightarrow TF(\tilde{Q})$ is an isomorphism.

Note that (b) follows from (a), by applying (a) to both $\text{Inf}_Q^G : TF(Q) \rightarrow TF(G)$ and $\text{Inf}_{\tilde{Q}}^{\tilde{Q}} : TF(Q) \rightarrow TF(\tilde{Q})$ and composing, because the map $\Psi_{G, \tilde{Q}} : TF(G) \rightarrow TF(\tilde{Q})$ is the identity on modules inflated from Q .

Part (a) of [Conjecture 5.1](#) is in fact a consequence of any of the two conjectures made in [\[Carlson et al. 2014b\]](#). First, Conjecture 10.1 in that reference asserts that, if a group homomorphism $\phi : G \rightarrow G'$ induces an isomorphism between the corresponding p -fusion systems, then ϕ should induce an isomorphism $TF(G') \xrightarrow{\sim} TF(G)$. In the special case where ϕ is the quotient map $\phi : G \rightarrow Q = G/O_{p'}(G)$, it is well-known that the fusion systems are isomorphic, so we would obtain the isomorphism $TF(Q) \xrightarrow{\sim} TF(G)$ of [Conjecture 5.1](#) above. This special case is explicitly mentioned at the end of Section 10 in [\[op. cit.\]](#).

Conjecture 9.2 in [\[op. cit.\]](#) asserts that the group $TF(G)$ should be generated by endotrivial modules lying in the principal block. Since $O_{p'}(G)$ acts trivially on any module lying in the principal block of G , such a module is inflated from Q , so the inflation map $\text{Inf}_Q^G : TF(Q) \rightarrow TF(G)$ in [Conjecture 5.1](#) above should be an isomorphism.

[Example 7.3](#) below illustrates a method allowing one to prove that the maps in [Conjecture 5.1](#) are isomorphisms in specific cases.

6. The perfect case

When the group $Q = G/O_{p'}(G)$ is perfect, there is an alternative approach to the construction of the injective group homomorphism of [Theorem 1.1\(a\)](#) using universal central extensions.

Recall that a *universal p' -central extension* of an arbitrary finite group Q is by definition a central extension

$$1 \longrightarrow M_{p'} \longrightarrow \tilde{Q} \xrightarrow{\pi_{\tilde{Q}}} Q \longrightarrow 1$$

with $M_{p'} = \text{Ker } \pi_{\tilde{Q}}$ of order prime to p and satisfying the following universal property: For any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \text{Ker } \pi$ of order prime to p , there exists a unique group homomorphism $\phi : \tilde{Q} \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & M_{p'} & \longrightarrow & \tilde{Q} & \xrightarrow{\pi_{\tilde{Q}}} & Q \longrightarrow 1 \\ & & \phi|_{M_{p'}} \downarrow & & \phi \downarrow & & \text{Id} \downarrow \\ 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q \longrightarrow 1 \end{array}$$

A standard argument shows that if a universal p' -central extension $(\tilde{Q}, \pi_{\tilde{Q}})$ exists, then it is unique up to isomorphism.

Lemma 6.1. *If $(\tilde{Q}, \pi_{\tilde{Q}})$ is a universal p' -central extension of a finite group R , then $(\tilde{Q}, \pi_{\tilde{Q}})$ is p' -representation group of Q .*

Proof. Let $(\check{Q}, \pi_{\check{Q}})$ be an arbitrary p' -representation group of Q . Let V be a finite-dimensional k -vector space and $\theta : Q \rightarrow \text{PGL}(V)$ a group homomorphism. Because $(\check{Q}, \pi_{\check{Q}})$ has the projective lifting property and $(\tilde{Q}, \pi_{\tilde{Q}})$ is universal, there exist group homomorphisms $\tilde{\theta} : \check{Q} \rightarrow \text{GL}(V)$ and $\phi : \tilde{Q} \rightarrow \check{Q}$ such that $\tilde{\theta} \circ \phi$ lifts θ . Therefore $(\tilde{Q}, \pi_{\tilde{Q}})$ has the projective lifting property as well.

Now, because $(\tilde{Q}, \pi_{\tilde{Q}})$ is universal, it is easy to see that $X(\tilde{Q}) = X(Q) = 1$. Therefore the Hochschild–Serre 5-term exact sequence associated to $(\tilde{Q}, \pi_{\tilde{Q}})$ is:

$$1 \longrightarrow 1 \longrightarrow 1 \longrightarrow \text{Hom}(M_{p'}, k^\times) \xrightarrow{\text{tr}} \text{H}^2(Q, k^\times) \xrightarrow{\text{Inf}} \text{H}^2(\tilde{Q}, k^\times)$$

Thus the transgression map $\text{tr} : \text{Hom}(M_{p'}, k^\times) \rightarrow \text{H}^2(Q, k^\times) = M_k(Q)$ is injective. But $M_{p'} \cong \text{Hom}(M_{p'}, k^\times)$, therefore by minimality of $(\check{Q}, \pi_{\check{Q}})$, we have $|M_{p'}| = |M_k(Q)|$ and $|\tilde{Q}| = |\check{Q}|$, proving that $(\tilde{Q}, \pi_{\tilde{Q}})$ is a p' -representation group of Q . \square

Lemma 6.2. *Any finite perfect group Q admits a universal p' -central extension.*

Proof. Since Q is a perfect group, it is well-known that Q has a representation group relative to \mathbb{C} , say $(\hat{Q}, \pi_{\hat{Q}})$, which is unique up to isomorphism and that

$$\text{Ker}(\pi_{\hat{Q}}) =: M \cong M_{\mathbb{C}}(Q) = \text{H}^2(Q, \mathbb{C}^\times),$$

the Schur multiplier of Q . Moreover, $(\hat{Q}, \pi_{\hat{Q}})$ is a universal central extension of Q , in particular perfect; see [Rotman 1995, Theorem 11.11]. Thus, for any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

where $Z = \text{Ker } \pi$, there exists a unique group homomorphism $\psi : \hat{Q} \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & M & \longrightarrow & \hat{Q} & \xrightarrow{\pi_{\hat{Q}}} & Q \longrightarrow 1 \\ & & \psi|_M \downarrow & & \psi \downarrow & & \text{Id} \downarrow \\ 1 & \longrightarrow & Z & \longrightarrow & E & \xrightarrow{\pi} & Q \longrightarrow 1 \end{array}$$

If Z has order prime to p , then the p -part M_p of M lies in the kernel of $\psi|_M$. Passing to the quotient by M_p , we define $\tilde{Q} := \hat{Q}/M_p$ and denote by $\phi : \tilde{Q} \rightarrow E$ the map induced by ψ . Thus we obtain an induced central extension

$$1 \longrightarrow M_{p'} \longrightarrow \tilde{Q} \xrightarrow{\pi_{\tilde{Q}}} Q \longrightarrow 1$$

where $M_{p'} := M/M_p$, a universal p' -central extension of Q by construction. □

Given an arbitrary group extension $1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$ with perfect quotient Q and kernel A of order prime to p , there is an induced p' -central extension:

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \xrightarrow{\pi_G} Q \longrightarrow 1$$

Moreover, by the above, Q admits a universal p' -central extension, which is in fact a p' -representation group $(\tilde{Q}, \pi_{\tilde{Q}})$ of Q . Therefore, by the universal property, there exists a unique group homomorphism $\phi_G : \tilde{Q} \rightarrow G/[G, A]$ lifting the identity on Q .

Lemma 6.3. *The homomorphism $\phi_G : \tilde{Q} \rightarrow G/[G, A]$ induces a group homomorphism*

$$\phi_G^* : T(G/[G, A]) \rightarrow T(\tilde{Q})$$

such that $\phi_G^* = \text{Inf}_{\text{Im}(\phi_G)}^{\tilde{Q}} \circ \text{Res}_{\text{Im}(\phi_G)}^{G/[G, A]}$. Moreover, both $\text{Inf}_{\text{Im}(\phi_G)}^{\tilde{Q}}$ and $\text{Res}_{\text{Im}(\phi_G)}^{G/[G, A]}$ preserve indecomposability of endotrivial modules.

Proof. The kernel of ϕ_G is contained in $\text{Ker } \pi_{\tilde{Q}} = M_{p'}$, which is a p' -group. Therefore, there is an induced inflation map $\text{Inf}_{\text{Im}(\phi_G)}^{\tilde{Q}} : T(\text{Im}(\phi_G)) \rightarrow T(\tilde{Q})$, preserving indecomposability of endotrivial modules.

Since $\text{Im}(\phi_G)$ maps onto Q via π_G , the group $G/[G, A]$ is the product of $\text{Im}(\phi_G)$ and the central p' -subgroup $A/[G, A]$. It follows that $\text{Im}(\phi_G)$ is a normal subgroup of $G/[G, A]$ of index prime to p . Therefore, the restriction to $\text{Im}(\phi_G)$ of any indecomposable endotrivial $k(G/[G, A])$ -module remains indecomposable and is endotrivial [Carlson et al. 2009, Proposition 3.1].

We define ϕ_G^* to be the composite of $\text{Inf}_{\text{Im}(\phi_G)}^{\tilde{Q}}$ and $\text{Res}_{\text{Im}(\phi_G)}^{G/[G, A]}$. □

Composing the group homomorphism

$$\phi_G^* : T(G/[G, A]) \rightarrow T(\tilde{Q})$$

with the inverse of the isomorphism

$$\text{Inf}_{G/[G, A]}^G : T(G/[G, A]) \rightarrow T(G)$$

of [Corollary 3.3](#), we obtain a group homomorphism

$$\Phi : T(G) \rightarrow T(\tilde{Q}).$$

We now show that this provides the alternative approach to the map of [Theorem 1.1](#).

Proposition 6.4. *Suppose that G satisfies [Hypothesis 3.1](#) and that Q is perfect.*

- (a) $\text{Ker } \Phi = X(G)$.
- (b) *The induced injective group homomorphism*

$$\bar{\Phi} : T(G)/X(G) \rightarrow T(\tilde{Q}) = T(\tilde{Q})/X(\tilde{Q})$$

coincides with the map $\Phi_{G, \tilde{Q}}$ of [Theorem 1.1](#).

Proof. Consider the map $\phi_G^* : T(G/[G, A]) \rightarrow T(\tilde{Q})$ of [Lemma 6.3](#). It is clear that the image of a one-dimensional module is one-dimensional, hence trivial since $X(\tilde{Q}) = 1$ by [Lemma 3.4](#). Therefore $X(G) \subseteq \text{Ker } \Phi$. It follows that Φ induces a group homomorphism $\bar{\Phi}$ as in the statement.

Our assumption on G implies that, if V is an endotrivial kG -module, then $[G, A]$ acts trivially on V ([Corollary 3.3](#)). Moreover, $\rho_V : G/[G, A] \rightarrow \text{GL}(V)$ lifts $\theta_V : Q \rightarrow \text{PGL}(V)$, as in [Section 4](#). It is then clear that $\rho_V \phi_G : \tilde{Q} \rightarrow \text{GL}(V)$ also lifts $\theta_V : Q \rightarrow \text{PGL}(V)$. Therefore, the definition of $\Phi_{G, \tilde{Q}}$ (see [Proposition 4.2](#)) shows that the class of V is mapped by $\Phi_{G, \tilde{Q}}$ to the class of the module $V_{\tilde{Q}}$ corresponding to the representation $\rho_V \phi_G$. In other words, $[V_{\tilde{Q}}] = \Phi([V])$ and this shows that $\Phi_{G, \tilde{Q}}$ coincides with $\bar{\Phi}$.

Finally, since $\Phi_{G, \tilde{Q}}$ is injective and is equal to $\bar{\Phi}$, we have $\text{Ker } \bar{\Phi} = \{0\}$. Therefore we obtain $\text{Ker } \Phi = X(G)$. □

Remark 6.5. The proof we give above shows that [Proposition 6.4](#) remains valid if the assumption that Q is perfect is replaced by the assumption that Q admits a universal p' -central extension. It is proved in [[Lassueur and Thévenaz 2017](#)] that this happens if and only if $X(Q) = 1$, that is, Q is p' -perfect. Here, for simplicity, we restrict ourselves to the perfect case.

7. Examples

In this final section, we provide various examples, in particular illustrating cases where the morphism $\Phi_{G, \tilde{Q}}$ is not surjective.

Example 7.1. Suppose that Q is simple and take $G = Q$, hence $A = O_{p'}(G) = \{1\}$. Then $\Phi_{Q, \tilde{Q}}$ is just the inflation map $T(Q) \rightarrow T(\tilde{Q})$. If Q is a finite simple group listed in the table below, then it is known that its unique p' -representation group \tilde{Q} has indecomposable endotrivial modules lying in faithful p -blocks, namely not inflated from Q .

Q	p	\tilde{Q}	$T(Q)$	$T(\tilde{Q})$
\mathfrak{A}_6	3	2. \mathfrak{A}_6	$\mathbb{Z} \oplus \mathbb{Z}/4$	$\mathbb{Z} \oplus \mathbb{Z}/8$
\mathfrak{A}_6	2	3. \mathfrak{A}_6	\mathbb{Z}^2	$\mathbb{Z}^2 \oplus \mathbb{Z}/3$
M_{22}	3	4. M_{22}	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$
J_3	2	3. J_3	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}/3$
Ru	3	2. Ru	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/4$
Fi_{22}	5	6. Fi_{22}	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/2$

The results concerning the sporadic groups can be found in [Lassueur and Mazza 2015b, Table 3], and those about the alternating group \mathfrak{A}_6 in [Lassueur and Mazza 2015a, Theorems A and B] together with [Carlson et al. 2009, Theorems A and B].

Further examples are given by the exceptional covering group $2.F_4(2)$ of the exceptional group of Lie type $F_4(2)$, which possesses simple torsion endotrivial modules lying in faithful blocks in characteristics 5 and 7 [Lassueur and Malle 2015, Proposition 5.5], although the full structure of the group of endotrivial modules has not been determined in these cases.

Example 7.2. Assume $p > 2$, let $n \geq \max\{2p, p + 4\}$ be an integer and denote by $\tilde{\mathfrak{S}}_n$ and $\hat{\mathfrak{S}}_n$ the two isoclinic p' -representation groups of the symmetric group \mathfrak{S}_n . Corollary 4.3 yields

$$T(\tilde{\mathfrak{S}}_n)/X(\tilde{\mathfrak{S}}_n) \cong T(\hat{\mathfrak{S}}_n)/X(\hat{\mathfrak{S}}_n).$$

However, Lassueur and Mazza [2015a, Theorem B, parts (1) and (2)] prove a stronger result, namely

$$T(\tilde{\mathfrak{S}}_n) = \text{Inf}_{\tilde{\mathfrak{S}}_n}^{\tilde{\mathfrak{S}}_n}(T(\mathfrak{S}_n)) \quad \text{and} \quad T(\hat{\mathfrak{S}}_n) = \text{Inf}_{\hat{\mathfrak{S}}_n}^{\hat{\mathfrak{S}}_n}(T(\mathfrak{S}_n)).$$

Consequently, given any finite group G such that $G/O_{p'}(G)$ is isomorphic to one of \mathfrak{S}_n , $\tilde{\mathfrak{S}}_n$ or $\hat{\mathfrak{S}}_n$ (with $n \geq \max\{2p, p + 4\}$), by Theorem 1.1 there exist injective group homomorphisms

$$T(\mathfrak{S}_n)/X(\mathfrak{S}_n) \longrightarrow T(G)/X(G) \xrightarrow{\Phi_{G, \hat{\mathfrak{S}}_n}} T(\hat{\mathfrak{S}}_n)/X(\hat{\mathfrak{S}}_n) \xrightarrow{\sim} T(\mathfrak{S}_n)/X(\mathfrak{S}_n),$$

where the first map is induced by inflation. Hence we have $T(G)/X(G) \cong T(\mathfrak{S}_n)/X(\mathfrak{S}_n)$. Recall that the structure of $T(\mathfrak{S}_n)$ is known [Carlson et al. 2009].

Example 7.3. In this final example, we outline a method which allows us to show that the maps $\text{Inf}_Q^{\tilde{Q}}$ is an isomorphism on the torsion-free part of the groups of endotrivial modules of Q and \tilde{Q} in some concrete cases.

Specifically, we may use the fact that endotrivial modules are liftable to characteristic zero, and afford characters taking root of unity values at p -singular conjugacy classes; see [Lassueur et al. 2016, Theorem 1.3 and Corollary 2.3]. Therefore, if for every faithful p -block B of $k\tilde{Q}$ (of full defect) no elements of $\mathbb{Z} \text{Irr}_{\mathbb{C}}(B)$ take root of unity values at p -singular conjugacy classes of \tilde{Q} , then any endotrivial $k\tilde{Q}$ -module is inflated from Q , hence

$$\text{Inf}_{\tilde{Q}}^Q : TF(Q) \rightarrow TF(\tilde{Q})$$

is an isomorphism.

This was used [Lassueur and Mazza 2015a, Theorem B] in the case that $Q = \mathfrak{S}_n$, $n \geq \max\{2p, p + 4\}$ (as mentioned in Example 7.2 above), as well as for a large number of sporadic simple groups Q [Lassueur and Mazza 2015b, Lemmas 4.3 and 6.2]. More precisely, in characteristic $p = 2$ for $Q = M_{12}, M_{22}, J_2, HS, McL, Ru, Suz, ON, Fi_{22}, Co_1, Fi'_{24}$, or B ; in characteristic $p = 3$ for $Q = M_{12}, J_2, HS, Suz, Fi_{22}, Co_1$, or B ; in characteristic $p = 5$ for $Q = J_2, HS, Ru, Suz, Co_1, Fi'_{24}$, or B ; and in characteristic $p = 7$ for $Q = Co_1, Fi'_{24}$, or B .

Acknowledgements

The authors are indebted to Shigeo Koshitani for several useful discussions and for providing references.

References

- [Carlson and Thévenaz 2005] J. F. Carlson and J. Thévenaz, “The classification of torsion endo-trivial modules”, *Ann. of Math.* (2) **162**:2 (2005), 823–883. [MR](#) [Zbl](#)
- [Carlson et al. 2006] J. F. Carlson, N. Mazza, and D. K. Nakano, “Endotrivial modules for finite groups of Lie type”, *J. Reine Angew. Math.* **595** (2006), 93–119. [MR](#) [Zbl](#)
- [Carlson et al. 2009] J. F. Carlson, N. Mazza, and D. K. Nakano, “Endotrivial modules for the symmetric and alternating groups”, *Proc. Edinb. Math. Soc.* (2) **52**:1 (2009), 45–66. [MR](#) [Zbl](#)
- [Carlson et al. 2011] J. F. Carlson, N. Mazza, and J. Thévenaz, “Endotrivial modules for p -solvable groups”, *Trans. Amer. Math. Soc.* **363**:9 (2011), 4979–4996. [MR](#) [Zbl](#)
- [Carlson et al. 2013] J. F. Carlson, N. Mazza, and J. Thévenaz, “Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup”, *J. Eur. Math. Soc. (JEMS)* **15**:1 (2013), 157–177. [MR](#) [Zbl](#)
- [Carlson et al. 2014a] J. F. Carlson, N. Mazza, and D. K. Nakano, “Endotrivial modules for the general linear group in a nondefining characteristic”, *Math. Z.* **278**:3-4 (2014), 901–925. [MR](#) [Zbl](#)
- [Carlson et al. 2014b] J. F. Carlson, N. Mazza, and J. Thévenaz, “Torsion-free endotrivial modules”, *J. Algebra* **398** (2014), 413–433. [MR](#) [Zbl](#)
- [Curtis and Reiner 1981] C. W. Curtis and I. Reiner, *Methods of representation theory, I: With applications to finite groups and orders*, Wiley, New York, 1981. [MR](#) [Zbl](#)

- [Koshitani and Lassueur 2016] S. Koshitani and C. Lassueur, “Endo-trivial modules for finite groups with dihedral Sylow 2-subgroup”, *J. Group Theory* **19**:4 (2016), 635–660. [MR](#) [Zbl](#)
- [Lassueur and Malle 2015] C. Lassueur and G. Malle, “Simple endotrivial modules for linear, unitary and exceptional groups”, *Math. Z.* **280**:3-4 (2015), 1047–1074. [MR](#) [Zbl](#)
- [Lassueur and Mazza 2015a] C. Lassueur and N. Mazza, “Endotrivial modules for the Schur covers of the symmetric and alternating groups”, *Algebr. Represent. Theory* **18**:5 (2015), 1321–1335. [MR](#) [Zbl](#)
- [Lassueur and Mazza 2015b] C. Lassueur and N. Mazza, “Endotrivial modules for the sporadic simple groups and their covers”, *J. Pure Appl. Algebra* **219**:9 (2015), 4203–4228. [MR](#) [Zbl](#)
- [Lassueur and Thévenaz 2017] C. Lassueur and J. Thévenaz, “Universal p' -central extensions”, *Expo. Math.* (2017).
- [Lassueur et al. 2016] C. Lassueur, G. Malle, and E. Schulte, “Simple endotrivial modules for quasi-simple groups”, *J. Reine Angew. Math.* **712** (2016), 141–174. [MR](#) [Zbl](#)
- [Mazza and Thévenaz 2007] N. Mazza and J. Thévenaz, “Endotrivial modules in the cyclic case”, *Arch. Math. (Basel)* **89**:6 (2007), 497–503. [MR](#) [Zbl](#)
- [Nagao and Tsushima 1989] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press, Boston, 1989. [MR](#) [Zbl](#)
- [Navarro and Robinson 2012] G. Navarro and G. R. Robinson, “On endo-trivial modules for p -solvable groups”, *Math. Z.* **270**:3-4 (2012), 983–987. [MR](#) [Zbl](#)
- [Rotman 1995] J. J. Rotman, *An introduction to the theory of groups*, 4th ed., Graduate Texts in Mathematics **148**, Springer, 1995. [MR](#) [Zbl](#)

Received January 4, 2016. Revised May 26, 2016.

CAROLINE LASSUEUR
FB MATHEMATIK
TECHNISCHE UNIVERSITÄT KAISERSLAUTERN
POSTFACH 3049
D-67653 KAISERSLAUTERN
GERMANY
lassueur@mathematik.uni-kl.de

JACQUES THÉVENAZ
SECTION DE MATHÉMATIQUES
EPFL
STATION 8
CH-1015 LAUSANNE
SWITZERLAND
jacques.thevenaz@epfl.ch

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jlhu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.


The subscription price for 2017 is US \$450/year for the electronic version, and \$625/year for print and electronic.

Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 287 No. 2 April 2017

Maximal operators for the p -Laplacian family	257
PABLO BLANC, JUAN P. PINASCO and JULIO D. ROSSI	
Van Est isomorphism for homogeneous cochains	297
ALEJANDRO CABRERA and THIAGO DRUMMOND	
The Ricci–Bourguignon flow	337
GIOVANNI CATINO, LAURA CREMASCHI, ZINDINE DJADLI, CARLO MANTEGAZZA and LORENZO MAZZIERI	
The normal form theorem around Poisson transversals	371
PEDRO FREJLICH and IOAN MĂRCUȚ	
Some closure results for \mathcal{C} -approximable groups	393
DEREK F. HOLT and SARAH REES	
Coman conjecture for the bidisc	411
ŁUKASZ KOSIŃSKI, PASCAL J. THOMAS and WŁODZIMIERZ ZWONEK	
Endotrivial modules: a reduction to p' -central extensions	423
CAROLINE LASSUEUR and JACQUES THÉVENAZ	
Infinitely many positive solutions for the fractional Schrödinger–Poisson system	439
WEIMING LIU	
A Gaussian upper bound of the conjugate heat equation along Ricci-harmonic flow	465
XIAN-GAO LIU and KUI WANG	
Approximation to an extremal number, its square and its cube	485
JOHANNES SCHLEISCHITZ	