Pacific Journal of Mathematics

INFINITELY MANY POSITIVE SOLUTIONS FOR THE FRACTIONAL SCHRÖDINGER–POISSON SYSTEM

WEIMING LIU

Volume 287 No. 2 April 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.439

INFINITELY MANY POSITIVE SOLUTIONS FOR THE FRACTIONAL SCHRÖDINGER–POISSON SYSTEM

WEIMING LIU

We consider a fractional Schrödinger–Poisson system in $\mathbb{R}^3$. Under certain assumptions, we prove that the problem has infinitely many nonradial positive solutions.

1. Introduction and main result

We consider the fractional Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^s u + u + V(|x|)\Phi(x)u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = V(|x|)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1-1}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian operator for $\alpha = s, t \in (0, 1)$, $V(r)$ $(r = |x|)$ is a positive bounded function, and

$$1 < p < 2^*(s) - 1 = \frac{3+2s}{3-2s}.$$

We assume that $V(r)$ satisfies the following condition:

(V) There are constants $a > 0$, $\dfrac{3+2s}{2(3+2s+1)} < m < \dfrac{3+2s}{2}$ and $\theta > 0$ such that

$$V(r) = \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right) \quad \text{as } r \to +\infty.$$

MSC2010: 35J10, 35B99, 35J60.
Keywords: fractional Schrödinger–Poisson system, infinitely many solutions, nonradial solutions.

In (1-1), the first equation is a nonlinear fractional Schrödinger equation in which the potential Φ satisfies a nonlinear fractional Poisson equation. The study of elliptic equations involving fractional powers of the Laplacian appears to be important in many areas, including physics, biological modeling, mathematical finance and the study of standing wave solutions of certain nonlinear fractional Schrödinger equations.

Giammetta [2014] studied the evolution equation associated with the one-dimensional system

$$\begin{cases} -\Delta u + \lambda\Phi(x)u = g(u), & x \in \mathbb{R}, \\ (-\Delta)^t \Phi = \lambda u^2, & x \in \mathbb{R}. \end{cases} \tag{1-2}$$

Zhang, do Ó and Squassina [Zhang et al. 2016] established the existence of a radial ground state solution to the following fractional Schrödinger–Poisson system with a general subcritical or critical nonlinearity:

$$\begin{cases} (-\Delta)^s u + \lambda\Phi(x)u = g(u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1-3}$$

Under the assumption that the nonlinearity does not satisfy the Ambrosetti–Rabinowitz condition, Zhang [2015] used the fountain theorem to obtain the existence of infinitely many large energy solutions to the system

$$\begin{cases} (-\Delta)^s u + V(x)u + \Phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1-4}$$

When $s = t = 1$, the system reduces to the classical Schrödinger–Poisson system. In recent years, many publications have appeared on that system. Zhang [2014] studied the existence and behavior of bound states of the system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \lambda\Phi(x)u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\Phi = u^2, \ \lim_{|x|\to\infty} \Phi(x) = 0, & x \in \mathbb{R}^3, \end{cases} \tag{1-5}$$

for $\lambda > 0$ and small $\varepsilon > 0$. For $f(u) = |u|^{p-1}u$, $p \in (1,5)$, there are some results in the literature. In the case of $\varepsilon = 1$, $V(x) \equiv 1$, the existence of radially symmetric positive solutions of system (1-5) was obtained by D'Aprile and Mugnai [2004]. Azzollini and Pomponio [2008] established the existence of ground state solutions for $p \in (2,5)$. Ruiz [2006] proved that (1-5) does not admit any nontrivial solution for $1 < p \leq 2$ and possesses a positive radial solution for $2 < p < 5$. When $\lambda \equiv 1$, Ianni and Vaira [2008] considered the existence of positive bound state solutions that concentrate on the local minimum of the potential V. Furthermore, Ianni and Vaira [Ianni and Vaira 2009; Ianni 2009] investigated the radially symmetric solutions that concentrate on the spheres. Ruiz and Vaira [2011] constructed the multibump solutions whose bumps concentrate around the local minimum of the potential V. The proofs explored in [Ruiz and Vaira 2011] are based on a singular perturbation, essentially a Lyapunov–Schmidt reduction method. By using the method of invariant sets of descending flow, Liu, Wang and Zhang [Liu et al. 2016] showed that this system has infinitely many sign-changing solutions. For more related results, one can refer to [Alves and Souto 2014; Chen and Wang 2014; He and Zou 2012; Ianni and Vaira 2015; Kim and Seok 2012; Zhao et al. 2013].

In this paper, inspired by [Long et al. 2016] and [Li et al. 2010], we consider the infinitely many nonradial positive solutions of the fractional Schrödinger–Poisson system (1-1). In [Long et al. 2016], Long, Peng and Yang were concerned with the existence of infinitely many nonradial positive solutions and sign-changing solutions for the equation

$$(-\Delta)^s u + u = K(|x|)u^p, \quad u > 0,\ u \in H^s(\mathbb{R}^N).$$

In [Li et al. 2010], Li, Peng and Yan obtained infinitely many nonradial positive solutions for (1-1) with $s = t = 1$.

Compared with the operator $-\Delta$, which is local, the operator $(-\Delta)^s$ with $0 < s < 1$ on $\mathbb{R}^3$ is nonlocal. Unlike the local case $s = 1$, the leading order of the associated reduced functional in a variational reduction procedure is of polynomial instead of exponential order, due to the nonlocal effect. So we need to establish some new necessary estimates for the Lyapunov–Schmidt reduction. Also, because of the appearance of the Poisson potential Φ, problem (1-1) is more complicated than the problem in [Long et al. 2016] and [Li et al. 2010].

To the best of our knowledge, there are no results on the existence of infinitely many nonradical positive solutions to the nonlinear fractional Schrödinger–Poisson system (1-1). In this paper, we will present some results in this direction.

Now, we are able to state our main theorem.

Theorem 1.1. *If $V(r)$ satisfies (V) and $2t + 4s \geq 3$, then the problem* (1-1) *has infinitely many nonradial positive solutions.*

To prove Theorem 1.1, we will construct solutions with a large number of bumps near infinity. Since $V(r) \to 0$ as $r \to +\infty$, the solution of (1-1) can be approximated by using the solution U of the problem

$$\begin{cases} (-\Delta)^s u + u = u^p, & u > 0 \text{ in } \mathbb{R}^3, \\ u(0) = \max_{x \in \mathbb{R}^3} u(x). \end{cases} \tag{1-6}$$

It is well known that the unique solution U of (1-6) satisfies $U(x) = U(|x|)$ and $U' < 0$ (see [Frank and Lenzmann 2013; Frank et al. 2016]).

Let

$$Q_j = \left(r \cos \frac{2(j-1)\pi}{k},\ r \sin \frac{2(j-1)\pi}{k},\ 0 \right) := (Q_j', 0), \quad j = 1, 2, \ldots, k, \tag{1-7}$$

where $r \in \left[r_1 k^{\frac{3+2s}{3+2s-2m}}, r_2 k^{\frac{3+2s}{3+2s-2m}}\right]$ for some $r_2 > r_1 > 0$. Define

$$E^s = \Big\{ u : u \in H^s(\mathbb{R}^3),\ u \text{ is even in } x_h,\ h = 2, 3,\\ u(r\cos\theta, r\sin\theta, x_3) = u\Big(r\cos\Big(\theta + \frac{2\pi j}{k}\Big), r\sin\Big(\theta + \frac{2\pi j}{k}\Big), x_3\Big)\Big\}.$$

Let

$$U_r(x) = \sum_{j=1}^{k} U_{Q_j}(x), \tag{1-8}$$

where $U_{Q_j}(\cdot) = U(\cdot - Q_j)$, and Q_j is defined in (1-7).

We will prove Theorem 1.1 by proving the following result.

Theorem 1.2. *Suppose $V(r)$ satisfies (V) and $2t + 4s \geq 3$. Then there is an integer $k_0 > 0$ such that for any integer $k \geq k_0$, (1-1) has a positive solution u_k of the form*

$$u_k = U_{r_k}(x) + w_k,$$

where $w_k \in E^s$, $r_k \in \left[r_1 k^{\frac{3+2s}{3+2s-2m}}, r_2 k^{\frac{3+2s}{3+2s-2m}}\right]$ for some $r_2 > r_1 > 0$ and as $k \to +\infty$, $\|w_k\|_s \to 0$.

Remark 1.3. It follows from Theorems 1.1 and 1.2 that (1-1) has solutions with a large number of bumps near infinity. Hence the energy of these solutions can be very large.

This paper is organized as follows. In Section 2, we give some preliminaries. Then we carry out Lyapunov–Schmidt reduction in Section 3. Finally, we prove our main result in Section 4. Some technical estimates are left to the Appendix.

2. Some preliminaries

In this section, we outline the variational framework for problem (1-1) and give some preliminary lemmas. Firstly, we recall some properties of the fractional Sobolev space and some results which are important in our proof of the main theorem.

The nonlocal operator $(-\Delta)^s$ in $\mathbb{R}^3$ is defined on the Schwartz class through the Fourier transform

$$\widehat{(-\Delta)^s} f(\xi) = |\xi|^{2s} \hat{f}(\xi),$$

or via the Riesz potential. Here $\widehat{\ }$ is the Fourier transform. When f has sufficient regularity, the fractional Laplacian of a function $f : \mathbb{R}^3 \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^s f(x) = C_{3,s}\,\text{P.V.}\int_{\mathbb{R}^3} \frac{f(x)-f(y)}{|x-y|^{3+2s}}\,dy = C_{3,s}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus B_\varepsilon(x)} \frac{f(x)-f(y)}{|x-y|^{3+2s}}\,dy, \tag{2-1}$$

where $C_{3,s} = \pi^{-(2s+3/2)}\Gamma\big(\frac{3}{2}+s\big)/\Gamma(-s)$. This integral makes sense directly when $s < \frac{1}{2}$ and $f \in C^{0,\gamma}(\mathbb{R}^3)$ with $\gamma > 2s$, or if $f \in C^{1,\gamma}(\mathbb{R}^3)$ with $1+\gamma > 2s$.

When $s \in (0, 1)$, the space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3\times\mathbb{R}^3)\right\} = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1+|\xi|^{2s})|\hat{u}(\xi)|^2\,d\xi < \infty\right\}$$

and the norm is

$$\|u\|_s := \|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} |u|^2\,dx\right)^{\frac{1}{2}},$$

which is induced by the inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^3)} = \langle u, v\rangle_s + \langle u, v\rangle_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} u(x)v(x)\,dx.$$

Here the term

$$[u]_{H^s(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{\frac{1}{2}}$$

is the so-called Gagliardo (semi-)norm of u. The following identity yields the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space $H^s(\mathbb{R}^3)$:

$$[u]_{H^s(\mathbb{R}^3)} = C\left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} = C\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^3)}$$

for a suitable positive constant C depending only on s.

The homogeneous Sobolev space $D^{t,2}(\mathbb{R}^3)$ is defined by

$$D^{t,2}(\mathbb{R}^3) = \left\{u \in L^{2^*(t)}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2\,d\xi < \infty\right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{t,2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2t} |\widehat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = \|(-\Delta)^{\frac{t}{2}} u\|_{L^2(\mathbb{R}^3)}$$

and the inner product

$$(u, v)_{D^{t,2}} = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} u (-\Delta)^{\frac{t}{2}} v \, dx, \quad u, v \in D^{t,2}(\mathbb{R}^3).$$

We have the following Sobolev embedding results.

Lemma 2.1 [Di Nezza et al. 2012]. *$H^s(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in \left[2, \frac{6}{3-2s}\right]$, and locally compact whenever $q \in \left[2, \frac{6}{3-2s}\right)$.*

Lemma 2.2 [Di Nezza et al. 2012]. *For any $t \in (0, 1)$, $D^{t,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*(t)}(\mathbb{R}^3)$; i.e., there exists $S_t > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2^*(t)} \, dx \right)^{2/2^*(t)} \leq S_t \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u|^2 \, dx, \quad u \in D^{t,2}(\mathbb{R}^3).$$

Now, we recall some known results for the limit equation (1-6). In a celebrated paper, Frank and Lenzmann [2013] proved the uniqueness of the ground state solution $U(x) = U(|x|) \geq 0$ for $N = 1,\ 0 < s < 1,\ 1 < p < (N + 2s)/(N - 2s)$. Very recently, Frank, Lenzmann and Silvestre [Frank et al. 2016] obtained the nondegeneracy of ground state solutions for (1-6) in arbitrary dimension $N \geq 1$ and any admissible exponent $1 < p < (N + 2s)/(N - 2s)$.

For convenience, we summarize the properties of the ground state U of (1-6), which can be found in [Frank and Lenzmann 2013; Frank et al. 2016].

Lemma 2.3. *Let $s \in (0, 1)$ and $1 < p < (3+2s)/(3-2s)$. Then the following hold:*

(1) *Uniqueness: The ground state solution $U \in H^s(\mathbb{R}^3)$ for* (1-6) *is unique up to translations.*

(2) *Symmetry, regularity and decay: $U(x)$ is radial, positive and strictly decreasing in $|x|$. Moreover, the function U belongs to $H^{2s+1}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ and satisfies*

$$\frac{C_1}{1 + |x|^{3+2s}} \leq U(x) \leq \frac{C_2}{1 + |x|^{3+2s}}, \quad x \in \mathbb{R}^3,$$

with some constants $C_2 \geq C_1 > 0$.

(3) *Nondegeneracy: The linearized operator $L_0 = (-\Delta)^s + 1 - p|U|^{p-1}$ is nondegenerate, i.e., its kernel is given by*

$$\ker L_0 = \operatorname{span}\{\partial_{x_1} U, \partial_{x_2} U, \partial_{x_3} U\}.$$

By [Frank et al. 2016, Lemma C.2], $\partial_{x_j} U$ has the following decay estimate for $j = 1, 2, 3$:

$$|\partial_{x_j} U| \le \frac{C}{1 + |x|^{3+2s}}.$$

By Lemma 2.1, if $2t + 4s \ge 3$, $H^s(\mathbb{R}^3) \hookrightarrow L^{12/(3+2t)}(\mathbb{R}^3)$. Then, for $u \in H^s(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} u^2 v \le \|u\|^2_{12/(3+2t)} \|v\|_{2^*(t)} \le C\|u\|_s^2 \|v\|_{D^{t,2}}.$$

Hence there exists a unique Φ_u^t such that $(-\Delta)^t \Phi_u^t = V(x)u^2$ and the t-Riesz potential satisfies

$$\Phi_u^t(x) = C(t) \int_{\mathbb{R}^3} \frac{V(y)u^2(y)}{|x-y|^{3-2t}}\, dy,$$

where

$$C(t) = \frac{\Gamma\big(\frac{3}{2} - 2t\big)}{\pi^{\frac{3}{2}} 2^{2t} \Gamma(t)}.$$

Substituting Φ_u^t in (1-1), we are lead to the equation

$$(-\Delta)^s u + u + V(|x|)\Phi_u^t(x)u = |u|^{p-1}u. \tag{2-2}$$

Let us summarize some properties of $\Phi_u^t(x)$ which will be useful throughout the paper.

Lemma 2.4 [Zhang et al. 2016]. *If $t, s \in (0, 1)$ and $2t + 4s \ge 3$, then for any $u \in H^s(\mathbb{R}^3)$, we have*

(1) $u \mapsto \Phi_u^t : H^s(\mathbb{R}^3) \mapsto D^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets*;

(2) $\Phi_u^t(x) \ge 0$, $x \in \mathbb{R}^3$, *and* $\int_{\mathbb{R}^3} \Phi_u^t u^2\, dx \le C\|u\|_s^4$ *for some* $C > 0$.

3. Finite-dimensional reduction

In this section, we prove Theorem 1.1 by proving Theorem 1.2.

We assume

$$\Lambda_k := \left[\left(\frac{(3+2s)B_4}{2mB_5} - \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}},\ \left(\frac{(3+2s)B_4}{2mB_5} + \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}\right], \tag{3-1}$$

where $\alpha > 0$ is a small constant, and where B_4 and B_5 are defined in Lemma A.5.

Let $r \in \Lambda_k$. We define

$$\mathfrak{E} = \left\{ u : u \in E^s, \sum_{j=1}^{k} \int_{\mathbb{R}^3} \frac{\partial U_{Q_j}}{\partial r} U_{Q_j}^{p-1} u = 0 \right\}.$$

Define

$$I(u) = \frac{1}{2}\langle u, u\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} u^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_u^t u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \quad \forall u \in \mathfrak{E}.$$

It is easy to check that

$$\langle u_1, u_2\rangle_s + \int_{\mathbb{R}^3} u_1 u_2 - p\int_{\mathbb{R}^3} U_r^{p-1} u_1 u_2 + \int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t u_1 u_2 \\ + 2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3} \frac{V(|y|)}{|x-y|^{3-2t}} U_r u_1 \, dy\right) U_r u_2, \quad u_1 u_2 \in \mathfrak{E},$$

is a bounded bilinear functional in $\mathfrak{E}$. Hence, by the Lax–Milgram theorem there is a bounded linear operator $\mathcal{L}$ from $\mathfrak{E}$ to $\mathfrak{E}$ such that

$$\langle \mathcal{L}u_1, u_2\rangle = \langle u_1, u_2\rangle_s + \int_{\mathbb{R}^3} u_1 u_2 - p\int_{\mathbb{R}^3} U_r^{p-1} u_1 u_2 + \int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t u_1 u_2 \\ + 2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3} \frac{V(|y|)}{|x-y|^{3-2t}} U_r u_1 \, dy\right) U_r u_2, \quad u_1 u_2 \in \mathfrak{E}.$$

The following result implies that $\mathcal{L}$ is invertible in $\mathfrak{E}$.

Lemma 3.1. *There exists a positive constant C, independent of k, such that for any $r \in \Lambda_k$,*

$$\|\mathcal{L}u\|_s \geq C\|u\|_s, \quad u \in \mathfrak{E}.$$

Proof. We prove the lemma by contradiction. Suppose that there exist $k \to +\infty$, $r_k \in \Lambda_k$ and $u_k \in \mathfrak{E}$ with

$$\|\mathcal{L}u_k\|_s = o(1)\|u_k\|_s.$$

Then we have

$$\langle \mathcal{L}u_k, \varphi\rangle = o(1)\|u_k\|_s\|\varphi\|_s \quad \forall \varphi \in \mathfrak{E}. \tag{3-2}$$

We may assume that $\|u_k\|_s^2 = k$.

Denote

$$\Omega_j = \left\{ x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{x'}{|x'|}, \frac{Q_j'}{|Q_j'|} \right\rangle \geq \cos\frac{\pi}{k} \right\}, \quad j = 1, 2, \dots, k.$$

By symmetry, we have

$$\int_{\Omega_1}\int_{\mathbb{R}^3}\frac{(u_k(x)-u_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}\,dx\,dy + \int_{\Omega_1} u_k\varphi - p\int_{\Omega_1} U_{r_k}^{p-1}u_k\varphi + \int_{\Omega_1} V(|x|)\Phi^t_{U_{r_k}}u_k\varphi + 2\int_{\Omega_1} V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}U_{r_k}u_k\,dy\Big)U_{r_k}\varphi = \frac{1}{k}\langle \mathcal{L}u_k,\varphi\rangle = o(1)\frac{1}{\sqrt{k}}\|\varphi\|_s \quad \forall\varphi\in\mathfrak{E}. \tag{3-3}$$

Particularly, choosing $\varphi = u_k$ we get

$$\int_{\Omega_1}\int_{\mathbb{R}^3}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\Omega_1}|u_k|^2 - p\int_{\Omega_1} U_{r_k}^{p-1}|u_k|^2 + \int_{\Omega_1} V(|x|)\Phi^t_{U_{r_k}}|u_k|^2 + 2\int_{\Omega_1} V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}U_{r_k}u_k\,dy\Big)U_{r_k}u_k = o(1) \tag{3-4}$$

and

$$\int_{\Omega_1}\int_{\mathbb{R}^3}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\Omega_1}|u_k|^2 = 1. \tag{3-5}$$

Let $\tilde{u}_k(x) = u_k(x-Q_1)$. It is easy to check that for any $R>0$, we can choose k large enough such that $B_R(Q_1)\subset\Omega_1$. Consequently, (3-5) yields that

$$\int_{B_R(0)}\int_{\mathbb{R}^3}\frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{B_R(0)}|\tilde{u}_k|^2 \le 1.$$

Thus we may assume the existence of $u\in H^s(\mathbb{R}^3)$ such that as $k\to+\infty$,

$$\tilde{u}_k \rightharpoonup u \quad \text{weakly in } H^s(\mathbb{R}^3)$$

and

$$\tilde{u}_k \to u \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^3).$$

Noting that $\tilde{u}_k$ is even in x_h, $h=2,3$, we have that u is even in x_h, $h=2,3$. On the other hand, from

$$\int_{\mathbb{R}^3}\frac{\partial U_{Q_1}}{\partial r}U_{Q_1}^{p-1}u_k = 0,$$

we obtain

$$\int_{\mathbb{R}^3}\frac{\partial U}{\partial Q_1}U^{p-1}\tilde{u}_k = 0.$$

So u satisfies

$$\int_{\mathbb{R}^3} \frac{\partial U}{\partial Q_1} U^{p-1} u = 0. \tag{3-6}$$

Now we prove that u satisfies

$$(-\Delta)^s u + u - pU^{p-1} u = 0 \quad \text{in } \mathbb{R}^3.$$

Define

$$\widetilde{\mathfrak{E}} = \left\{ \varphi : \varphi \in H^s(\mathbb{R}^3),\ \int_{\mathbb{R}^3} \frac{\partial U}{\partial Q_1} U^{p-1} \varphi = 0 \right\}.$$

For any $R > 0$, let φ belong to $C_0^\infty(B_R(0)) \cap \widetilde{\mathfrak{E}}$ and be even in x_h, $h = 2, 3$. Then

$$\varphi_1(x) := \varphi(x - Q_1) \in C_0^\infty(B_R(0)).$$

We may identify $\varphi_1(x)$ as an element in $\mathfrak{E}$ by redefining the values outside Ω_1 using symmetry. Using (3-4) and Lemma A.1, we deduce that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} u\varphi - p \int_{\mathbb{R}^3} U^{p-1} u\varphi = 0. \tag{3-7}$$

Furthermore, since u is even in x_h, $h = 2, 3$, (3-7) is true for any function $\varphi \in C_0^\infty(\mathbb{R}^3)$ which is odd in x_h, $h = 2, 3$. Therefore, (3-7) holds for any $\varphi \in C_0^\infty(B_R(0)) \cap \widetilde{\mathfrak{E}}$. By the density of $C_0^\infty(\mathbb{R}^3)$ in $H^s(\mathbb{R}^3)$, we see

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} u\varphi - p \int_{\mathbb{R}^3} U^{p-1} u\varphi = 0 \quad \forall \varphi \in \widetilde{\mathfrak{E}}. \tag{3-8}$$

But (3-8) is true for $\varphi = \partial U/\partial Q_1$. Thus (3-8) holds for any $\varphi \in H^s(\mathbb{R}^3)$, and hence $u = c(\partial U/\partial Q_1)$ because u is even in x_h, $h = 2, 3$. By (3-6), we find $u = 0$. Consequently,

$$\int_{B_R(Q_1)} u_k^2 = o(1) \quad \forall R > 0.$$

Moreover, Lemma A.1 implies that for any $1 < \eta \le 3 + 2s$, there is a positive constant C such that

$$U_{Q_k}(x) \le \frac{C}{(1 + |x - Q_1|)^{3+2s-\eta}}, \quad x \in \Omega_1. \tag{3-9}$$

Thus, by (3-9) and (V), we have

$$\begin{aligned}
o(1) &= \int_{\Omega_1}\int_{\mathbb{R}^3} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\Omega_1}|u_k|^2 - p\int_{\Omega_1} U_{r_k}^{p-1}|u_k|^2 \\
&\quad + \int_{\Omega_1} V(|x|)\Phi^t_{U_{r_k}}|u_k|^2 + 2\int_{\Omega_1} V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}U_{r_k}u_k\,dy\Big)U_{r_k}u_k \\
&\ge \int_{\Omega_1}\int_{\mathbb{R}^3} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\Omega_1}|u_k|^2 \\
&\quad + C\Big(\int_{B_{\frac{R}{2}}(Q_1)} + \int_{\Omega_1\setminus B_{\frac{R}{2}}(Q_1)} \frac{1}{(1+|x-Q_1|)^{3+2s-\eta}}u_n^2\Big) + o(1) \\
&\ge \frac{1}{2} + o(1) + O_R(1),
\end{aligned}$$

which is impossible for large R. □

Proposition 3.2. *There is an integer $k_0 > 0$ such that for each $k \ge k_0$, there exists a C^1 map with respect to r from Λ_k to E^s: $\varphi = \varphi(r)$, satisfying $\varphi \in E^s$, and*

$$\Big\langle \frac{\partial J(\varphi)}{\partial\varphi}, v\Big\rangle = 0 \quad \forall v \in E^s.$$

Moreover, there is a small $\tau > 0$ such that

$$\|\varphi\|_s \le \frac{C}{r^{2m}}k^{\frac{1}{2}} + Ck^{\frac{1}{2}}\Big(\frac{k}{r}\Big)^{\frac{3+2s}{2}+\tau}. \tag{3-10}$$

Proof. Write

$$J(\varphi) = I(U_r+\varphi), \quad \varphi \in E^s.$$

By direct computation, we have

$$\begin{aligned}
J(\varphi) &= I(U_r+\varphi) \\
&= \frac{1}{2}\langle U_r+\varphi, U_r+\varphi\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}(U_r+\varphi)^2 \\
&\qquad + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r+\varphi}(U_r+\varphi)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} \\
&= \frac{1}{2}\langle U_r, U_r\rangle_s + \langle U_r,\varphi\rangle_s + \frac{1}{2}\langle\varphi,\varphi\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}U_r^2 + \frac{1}{2}\int_{\mathbb{R}^3}\varphi^2 + \int_{\mathbb{R}^3}U_r\varphi \\
&\qquad + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r+\varphi}(U_r+\varphi)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} U_r^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r} U_r^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |U_r|^{p+1} \\
&\quad + \int_{\mathbb{R}^3}\Big(\sum_{j=1}^k U_{Q_j}^p - U_r^p\Big)\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r} U_r\varphi \\
&\quad + \frac{1}{2}\langle\varphi,\varphi\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}\varphi^2 - \frac{p}{2}\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r}\varphi^2 \\
&\quad + \int_{\mathbb{R}^3} V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}} U_r\varphi\, dy\Big) U_r\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi} U_r\varphi \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi}\varphi^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} + \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r|^{p+1} \\
&\quad + \int_{\mathbb{R}^3}|U_r|^p\varphi + \frac{p}{2}\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi^2.
\end{aligned}
$$

Hence,

$$J(\varphi) = J(0) + f(\varphi) + \tfrac{1}{2}\langle\mathcal{L}\varphi,\varphi\rangle + R(\varphi),$$

where

$$f(\varphi) = \int_{\mathbb{R}^3}\Big(\sum_{j=1}^k U_{Q_j}^p - U_r^p\Big)\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r} U_r\varphi. \tag{3-11}$$

We notice that $\mathcal{L}$ is the bounded linear map from E^s to E^s in Lemma 2.1, and

$$
\begin{aligned}
R(\varphi) &= \int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi} U_r\varphi + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi}\varphi^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} \\
&\qquad + \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r|^{p+1} + \int_{\mathbb{R}^3}|U_r|^p\varphi + \frac{p}{2}\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi^2.
\end{aligned}
$$

It is not difficult to verify that $f(\varphi)$ is a bounded linear functional in E^s, so there exists an $f_k \in E^s$ such that

$$f(\varphi) = \langle f_k, \varphi\rangle.$$

Thus, to find a critical point for $J(\varphi)$, we only need to solve

$$f_k + \mathcal{L}\varphi + R'(\varphi) = 0. \tag{3-12}$$

From Lemma 3.1 we know $\mathcal{L}$ is invertible. Therefore, (3-12) can be rewritten as

$$\varphi = \mathcal{A}(\varphi) =: -\mathcal{L}^{-1} f_k - \mathcal{L}^{-1} R'(\varphi).$$

Set

$$\mathcal{N} = \Big\{\varphi : \varphi \in E^s,\ \|\varphi\|_s \le \frac{1}{r^{2m-\tau}} k^{\frac{1}{2}} + k^{\frac{1}{2}}\Big(\frac{k}{r}\Big)^{\frac{3+2s+\tau}{2}}\Big\},$$

where $\tau > 0$ is small.

When $1 < p \le 2$, we can verify that

$$\|R'(\varphi)\|_s \le C\|\varphi\|_s^p.$$

Hence Lemma 3.3 below implies

$$\begin{aligned} \|\mathcal{A}(\varphi)\|_s & \\ &\le C\|f_k\|_s + C\|\varphi\|_s^p \\ &\le \frac{C}{r^{2m}} k^{\frac{1}{2}} + Ck^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s}{2}+\tau} + C\left(\frac{1}{r^{2m-\tau}} k^{\frac{1}{2}} + k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}\right)^p \\ &\le \frac{1}{r^{2m-\tau}} k^{\frac{1}{2}} + k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}. \end{aligned} \tag{3-13}$$

Thus, $\mathcal{A}$ maps $\mathcal{N}$ into $\mathcal{N}$ when $1 < p \le 2$.

Meanwhile, when $1 < p \le 2$, we see

$$\|R''(\varphi)\|_s \le C\|\varphi\|_s^{p-1}.$$

Thus,

$$\begin{aligned} \|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\|_s &= \|\mathcal{L}^{-1}R'(\varphi_1) - \mathcal{L}^{-1}R'(\varphi_2)\|_s \\ &\le C\|R'(\varphi_1) - R'(\varphi_2)\|_s \\ &\le C\|R''(\varepsilon\varphi_1 + (1-\varepsilon)\varphi_2)\|_s\|\varphi_1 - \varphi_2\|_s \\ &\le C\big(\|\varphi_1\|_s^{p-1} + \|\varphi_2\|_s^{p-1}\big)\|\varphi_1 - \varphi_2\|_s \le \tfrac{1}{2}\|\varphi_1 - \varphi_2\|_s, \end{aligned}$$

where $\varepsilon \in (0, 1)$.

Thus, we have proved that when $1 < p \le 2$, $\mathcal{A}$ is a contraction map.

When $p > 2$, by Remark A.2, the Hölder inequality, the Sobolev inequality, and Lemmas 2.2 and 2.4, we get

$$\begin{aligned} &|\langle R'(\varphi), \xi\rangle| \\ &\quad= \bigg|2\int_{\mathbb{R}^3} V(|x|)\bigg(\int_{\mathbb{R}^3} \frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\bigg)U_r\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t U_r\xi \\ &\qquad\quad + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t\varphi\xi - \int_{\mathbb{R}^3}|U_r+\varphi|^p\xi + \int_{\mathbb{R}^3}|U_r|^p\xi + p\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi\xi\bigg| \\ &\quad\le \bigg|2\int_{\mathbb{R}^3} V(|x|)\bigg(\int_{\mathbb{R}^3} \frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\bigg)U_r\varphi \\ &\qquad\qquad\qquad + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t U_r\xi + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t\varphi\xi\bigg| \\ &\qquad + \bigg|\int_{\mathbb{R}^3}|U_r+\varphi|^p\xi - \int_{\mathbb{R}^3}|U_r|^p\xi - p\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi\xi\bigg| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{r^m}\int_{\mathbb{R}^3}|\Phi_\varphi^t|^{\frac12}|\Phi_\xi^t|^{\frac12}U_r|\varphi| \\
&\quad + \frac{C}{r^m}\left(\int_{\mathbb{R}^3}|\Phi_\varphi^t|^{\frac{6}{3-2t}}\right)^{\frac{3-2t}{6}}\left(\int_{\mathbb{R}^3}|\xi|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}}\left(\int_{\mathbb{R}^3}|U_r|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}} \\
&\quad + \frac{C}{r^m}\left(\int_{\mathbb{R}^3}|\Phi_\varphi^t|^{\frac{6}{3-2t}}\right)^{\frac{3-2t}{6}}\left(\int_{\mathbb{R}^3}|\xi|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}}\left(\int_{\mathbb{R}^3}|\varphi|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}} \\
&\quad + C\int_{\mathbb{R}^3}|U_r|^{p-2}|\varphi|^2|\xi| \\
&\le \frac{C}{r^m}\|\varphi\|_s^2\|\xi\|_s + \frac{C}{r^m}k^{\frac{3+2t}{12}}\|\Phi_\varphi^t\|_{D^{t,2}}\|\xi\|_s + \frac{C}{r^m}\|\Phi_\varphi^t\|_{D^{t,2}}\|\xi\|_s\|\varphi\|_s \\
&\quad + C\left(\int_{\mathbb{R}^3}\big(|U_r|^{p-2}|\varphi|^2\big)^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\|\xi\|_s \\
&\le \frac{C}{r^m}\|\varphi\|_s^2\|\xi\|_s + \frac{C}{r^m}k^{\frac{3+2t}{12}}\|\varphi\|_s^2\|\xi\|_s + \frac{C}{r^m}\|\varphi\|_s^3\|\xi\|_s \\
&\quad + C\left(\int_{\mathbb{R}^3}|\varphi|^{\frac{2p+2}{p}}\right)^{\frac{p}{p+1}}\|\xi\|_s.
\end{aligned}
$$

Hence, we deduce that

$$\|R'(\varphi)\|_s \le C\big(\|\varphi\|_s^2 + \|\varphi\|_s^3\big).$$

For the estimate of $\|R''(\varphi)\|_s$, we have

$$
\begin{aligned}
|R''(\varphi)(\xi,\eta)| = \Big|\, & 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\eta\xi\,dy\right)U_r\varphi \\
& + 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\right)U_r\eta \\
& + 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\eta\,dy\right)U_r\xi \\
& + 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\right)\varphi\eta \\
& + \int_{\mathbb{R}^3}V(|x|)\Phi_\varphi^t\xi\eta - p\int_{\mathbb{R}^3}(U_r+\varphi)^{p-1}\xi\eta + p\int_{\mathbb{R}^3}U_r^{p-1}\xi\eta\,\Big| \\
\le{}& C\big(\|\varphi\|_s + \|\varphi\|_s^2\big)\|\xi\|_s\|\eta\|_s,
\end{aligned}
$$

which implies

$$\|R''(\varphi)\|_s \le C\big(\|\varphi\|_s + \|\varphi\|_s^2\big).$$

Thus, we can conclude that

$$
\begin{aligned}
\text{(3-14)}\quad \|\mathcal{A}(\varphi)\|_s &\le C\|f_k\|_s + C\|\varphi\|_s^2 \\
&\le \frac{C}{r^{2m}}k^{\frac{1}{2}} + Ck^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s}{2}+\tau} + C\left(\frac{1}{r^{2m-\tau}}k^{\frac{1}{2}} + k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}\right)^2 \\
&\le \frac{1}{r^{2m-\tau}}k^{\frac{1}{2}} + k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\|_s &= \|\mathcal{L}^{-1}R'(\varphi_1) - \mathcal{L}^{-1}R'(\varphi_2)\|_s \\
&\le C\|R'(\varphi_1) - R'(\varphi_2)\|_s \\
&\le C\|R''(\varepsilon\varphi_1 + (1-\varepsilon)\varphi_2)\|_s\|\varphi_1 - \varphi_2\|_s \\
&\le \tfrac{1}{2}\|\varphi_1 - \varphi_2\|_s,
\end{aligned}
$$

where $\varepsilon \in (0, 1)$. Hence, $\mathcal{A}$ is also a contraction map from $\mathcal{N}$ to $\mathcal{N}$.

Now applying the contraction mapping theorem, we can find a unique φ such that (3-12) holds. Moreover, it follows from (3-13) and (3-14) that (3-10) holds. □

Lemma 3.3. *There exist constants $C > 0$ and $\tau > 0$ small enough such that*

$$
\|f_k\|_s \le \frac{C}{r^{2m}}k^{\frac{1}{2}} + Ck^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s}{2}+\tau}.
$$

Proof. We recall

$$
\text{(3-15)}\qquad f(\varphi) = \int_{\mathbb{R}^3}\left(\sum_{j=1}^{k} U_{Q_j}^p - U_r^p\right)\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t U_r\varphi.
$$

Using $U_{Q_j} \le U_{Q_1}$, $x \in \Omega_1$, $\dfrac{3+2s}{3+2s+1} < 2m < 3+2s$ and Lemma A.1, we obtain

(3-16)

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left|U_r^p - \sum_{j=1}^{k} U_{Q_j}^p\right||\varphi| \\
&= k\int_{\Omega_1}\left|U_r^p - \sum_{j=1}^{k} U_{Q_j}^p\right||\varphi| \\
&\le Ck\int_{\Omega_1} U_{Q_1}^{p-1}\sum_{j=2}^{k} U_{Q_j}|\varphi| \\
&\le Ck\left(\int_{\Omega_1}\left(U_{Q_1}^{p-1}\sum_{j=2}^{k} U_{Q_j}\right)^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\left(\int_{\Omega_1}|\varphi|^{p+1}\right)^{\frac{1}{p+1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le Ck\Bigl(\int_{\Omega_1}\Bigl(\frac{1}{(1+|x-Q_1|)^{(3+2s)(p-1)+\frac{3+2s}{2}-\sigma}}\Bigl(\frac{k}{r}\Bigr)^{\frac{3+2s}{2}+\sigma}\Bigr)^{\frac{p+1}{p}}\Bigr)^{\frac{p}{p+1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\Bigl(\int_{\Omega_1}|\varphi|^{p+1}\Bigr)^{\frac{1}{p+1}} \\
&\le Ck^{\frac{p}{p+1}}\Bigl(\frac{k}{r}\Bigr)^{\frac{3+2s}{2}+\sigma}\Bigl(\int_{\Omega_1}\Bigl(\frac{1}{(1+|x-Q_1|)^{(3+2s)(p-1)+\frac{3+2s}{2}-\sigma}}\Bigr)^{\frac{p+1}{p}}\Bigr)^{\frac{p}{p+1}}\|\varphi\|_s \\
&\le Ck^{\frac{1}{2}}\Bigl(\frac{k}{r}\Bigr)^{\frac{3+2s}{2}+\tau}\|\varphi\|_s,
\end{aligned}
$$

where $\tau > 0$ is a small constant and $\sigma \in \bigl(0, \frac{3+2s}{2}\bigr)$.

On the other hand, by Lemma A.4 and Remark A.2, we have

$$
\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r}U_r\varphi \le \frac{C}{r^{2m}}\Bigl(\int_{\mathbb{R}^3}U_r^2\Bigr)^{\frac{1}{2}}\Bigl(\int_{\mathbb{R}^3}\varphi^2\Bigr)^{\frac{1}{2}} \le \frac{C}{r^{2m}}k^{\frac{1}{2}}\|\varphi\|_s. \tag{3-17}
$$

Inserting (3-16) and (3-17) into (3-15), we can complete the proof. □

4. Proof of the main result

Proof of Theorem 1.2. Let $\varphi(r)$ be the map obtained in Proposition 3.2. Define

$$
\mathcal{F}(r) = I(U_r + \varphi(r)) \quad \forall r \in \Lambda_k.
$$

It is well known that if r is a critical point of $\mathcal{F}(r)$, then $U_r + \varphi(r)$ is a solution of (1-1) (see [Cao and Tang 2006]). As a consequence, in order to complete the proof of the proposition, we only need to prove that $\mathcal{F}(r)$ has a critical point in Λ_k.

Hence, by Proposition 3.2 and Lemma A.5, we have

$$
\begin{aligned}
\mathcal{F}(r) &= I(U_r) + f(\varphi) + \tfrac{1}{2}\langle \mathcal{L}\varphi, \varphi\rangle + R(\varphi) \\
&= I(U_r) + O\bigl(\|f_k\|_s\|\varphi\|_s + \|\varphi\|_s^2\bigr) \\
&= kB_3 - kB_4\Bigl(\frac{k}{r}\Bigr)^{3+2s} + k\frac{B_5}{r^{2m}} + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k}K(Q_j - Q_1)\int_{\mathbb{R}^3}U^2 \\
&\qquad\qquad + kO\Bigl(\frac{1}{r^{2m+\tau}}\Bigr) + O\Bigl(\frac{1}{r^{2m}}k^{\frac{1}{2}} + k^{\frac{1}{2}}\Bigl(\frac{k}{r}\Bigr)^{\frac{3+2s}{2}+\tau}\Bigr)^2 \\
&= kB_3 - kB_4\Bigl(\frac{k}{r}\Bigr)^{3+2s} + k\frac{B_5}{r^{2m}} \\
&\qquad\qquad + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k}K(Q_j - Q_1)\int_{\mathbb{R}^3}U^2 + kO\Bigl(\frac{1}{r^{2m+\tau}}\Bigr),
\end{aligned}
$$

where B_3, B_4 and B_5 are defined in Lemma A.5.

We consider its maximum with respect to r:

$$\max\{\mathcal{F}(r) : r \in \Lambda_k\}. \tag{4-1}$$

Assume that (4-1) is achieved by some r_k in Λ_k. We will prove that r_k is an interior point of Λ_k.

Consider the following smooth function in Λ_k:

$$g(r) := -B_4\left(\frac{k}{r}\right)^{3+2s} + \frac{B_5}{r^{2m}}.$$

Then

$$g'(r) = (3+2s)B_4\frac{k^{3+2s}}{r^{4+2s}} - \frac{2mB_5}{r^{2m+1}}.$$

It is easy to check that $g(r)$ has a maximum point $\tilde{r}_k$, satisfying

$$g'(\tilde{r}_k) = 0.$$

Thus

$$\tilde{r}_k = \left(\frac{(3+2s)B_4}{2mB_5}\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}.$$

By direct computation, we observe that

(4-2)

$$\begin{aligned}
\mathcal{F}(r_k) \geq \mathcal{F}(\tilde{r}_k) &\geq kB_3 - kB_4\left(\frac{k}{\tilde{r}_k}\right)^{3+2s} + k\frac{B_5}{\tilde{r}_k^{2m}} + kO\left(\frac{1}{\tilde{r}_k^{2m+\tau}}\right)\\
&= kB_3 + k\frac{B_5}{\tilde{r}_k^{2m}}\left(1-\frac{2m}{3+2s}\right) + kO\left(\frac{1}{\tilde{r}_k^{2m+\tau}}\right)\\
&= kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}}B_4^{-\frac{2m}{3+2s-2m}}\left(\frac{3+2s}{2m}-1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}}\\
&\qquad + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big).
\end{aligned}$$

On the other hand, if we suppose that

$$r_k = \left(\frac{(3+2s)B_4}{2mB_5} - \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}},$$

then

(4-3)

$$\begin{aligned}
\mathcal{F}(r_k) &= kB_3 + kB_5\left(1-\frac{2m}{3+2s}\right)\left(\frac{(3+2s)B_4}{2mB_5}-\alpha\right)^{\frac{-2m}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}}\\
&\qquad + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big)
\end{aligned}$$

$$= kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}} B_4^{-\frac{2m}{3+2s-2m}} \left(\frac{3+2s}{2m} - 1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} \times \left(1 - \frac{2m\alpha B_5}{(3+2s)B_4}\right)^{-\frac{2m}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}}$$
$$+ kO\big(r^{-\frac{2m(3-2t)}{3+2s}-2m}\big) + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big)$$
$$< kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}} B_4^{-\frac{2m}{3+2s-2m}} \left(\frac{3+2s}{2m} - 1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}}$$
$$+ kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big).$$

This is a contradiction to (4-2).

Similarly

$$\mathcal{F}\left(\left(\frac{(3+2s)B_4}{2mB_5} + \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}\right)$$
$$< kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}} B_4^{-\frac{2m}{3+2s-2m}} \left(\frac{3+2s}{2m} - 1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}} + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big).$$

Hence we can check that (4-1) is achieved by some r_k which is in the interior of Λ_k. As a result, r_k is a critical point of $\mathcal{F}(r)$. Therefore

$$U_{r_k} + \varphi(r_k)$$

is a solution of (1-1). □

Appendix: Some technical estimates

In this section, we give some estimates of the energy expansion for the approximate solutions. Firstly, we recall

$$Q_j = \left(r\cos\frac{2(j-1)\pi}{k},\ r\sin\frac{2(j-1)\pi}{k},\ 0\right), \quad j = 1, \dots, k,$$

$$\Omega_j = \left\{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{x'}{|x'|}, \frac{Q_j'}{|Q_j'|}\right\rangle \geq \cos\frac{\pi}{k}\right\}, \quad j = 1, 2, \dots, k,$$

and

$$I(u) = \frac{1}{2}\langle u, u\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} u^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_u^t u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1},$$

where Φ_u^t is the solution of $(-\Delta)^t \Phi_u^t = V(|x|)u^2$.

Recall that U is the unique solution of

$$\begin{cases}(-\Delta)^s u + u = u^p, & u > 0 \text{ in } \mathbb{R}^3,\\ u(0) = \max_{x\in\mathbb{R}^3} u(x).\end{cases}$$

Let K be the solution of

$$\begin{cases}(-\Delta)^t v = U^2 & \text{in } \mathbb{R}^3,\\ v \in D^{t,2}(\mathbb{R}^3).\end{cases}$$

Then K is radial, and $r^{3-2t}K(r) \to K_0 > 0$ as $r \to +\infty$.

To begin, we give the following lemmas.

Lemma A.1 [Long et al. 2016, Lemma A.2]. *For any $x \in \Omega_1$, and $\eta \in (1, 3+2s]$, there are constants $C, B > 0$ such that*

$$\sum_{i=2}^{k} U_{Q_i}(x) \le C \frac{1}{(1+|x-Q_1|)^{3+2s-\eta}} \frac{k^\eta}{r^\eta} \le C\frac{k^\eta}{r^\eta}$$

and

$$\sum_{j=2}^{k} \frac{1}{|Q_j - Q_1|^\eta} = B\left(\frac{k}{r}\right)^\eta + O\left(\frac{k}{|r|^\eta}\right).$$

Remark A.2. It follows from Lemma A.1 that U_r is bounded.

Lemma A.3 [Wei and Zhao 2013, Lemma 13.1]. *Assume that $0 < m < 3$ and $n > m$. Then*

$$\int_{\mathbb{R}^3} \frac{1}{|y-x|^{3-m}} \frac{1}{(1+|x|)^n}\,dx \le \begin{cases} C(1+|y|)^{m-n} & \text{if } n < 3,\\ C(1+|y|)^{m-3}\big[1+\log(1+|y|)\big] & \text{if } n = 3,\\ C(1+|y|)^{m-3} & \text{if } n > 3.\end{cases}$$

Now, we estimate Φ_{U_r} and $I(U_r)$.

Lemma A.4. *We have*

$$\Phi^t_{U_r}(y) = \frac{a}{r^m}\sum_{j=1}^{k} K(y-Q_j) + O\left(\sum_{j=1}^{k} \frac{1}{r^{m+\tau}} \frac{1}{(1+|y-Q_j|)^{3-2t}}\right).$$

Proof. For any $\beta > 0$, we get

$$\frac{1}{|y+Q_j|^\beta} = \frac{1}{|Q_j|^\beta}\left(1 + O\left(\frac{|y|}{|Q_j|}\right)\right), \quad y \in B_{\frac{r}{2}}(0).$$

By Lemmas A.1 and A.3, we are led to

(A-1)

$$
\begin{aligned}
&\int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} U_r^2(x)\,dx \\
&= \int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}^2(x)\,dx \\
&\qquad + O\left(\int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1} \sum_{j=2}^{k} U_{Q_j}\,dx + \int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} \left(\sum_{j=2}^{k} U_{Q_j} \right)^2 dx \right) \\
&= \int_{\Omega_1 \cap B_{\frac{r}{2}}(0)} \left(\frac{a}{|x+Q_1|^m} + O\left(\frac{1}{|x+Q_1|^{m+\theta}} \right) \right) \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx \\
&\qquad + \int_{\Omega_1 \cap B^C_{\frac{r}{2}}(0)} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}^2(x)\,dx \\
&\qquad + O\left(\left(\frac{k}{r} \right)^{3+2s} \int_{\Omega_1 \cap B_{\frac{r}{2}}(0)} \frac{V(|x+Q_1|)}{|y-x-Q_1|^{3-2t}} U(x)\,dx \right. \\
&\qquad\qquad\qquad\qquad \left. + \left(\frac{k}{r} \right)^{3+2s} \int_{\Omega_1 \cap B^C_{\frac{r}{2}}(0)} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}(x)\,dx \right) \\
&\qquad + O\left(\left(\frac{k}{r} \right)^{2\eta} \int_{\Omega_1} \frac{1}{|y-x-Q_1|^{3-2t}} \frac{1}{(1+|x|)^{2(3+2s-\eta)}}\,dx \right) \\
&= \frac{a}{r^m} \int_{\Omega_1 \cap B_{\frac{r}{2}}(0)} \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx \\
&\qquad + O\left(\frac{1}{r^{m+\tau}} \int_{\Omega_1 \cap B_{\frac{r}{2}}(0)} \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx \right) \\
&\qquad + O\left(\frac{1}{r^m} \int_{\Omega_1 \cap B^C_{\frac{r}{2}}(0)} \frac{U(x)}{|y-x-Q_1|^{3-2t}} \frac{1}{r^{3+2s}}\,dx \right) \\
&\qquad + O\left(\left(\frac{k}{r} \right)^{3+2s} \int_{\Omega_1 \cap B_{\frac{r}{2}}(0)} \frac{V(|x+Q_1|)}{|y-x-Q_1|^{3-2t}} U(x)\,dx \right. \\
&\qquad\qquad\qquad\qquad \left. + \left(\frac{k}{r} \right)^{3+2s} \int_{\Omega_1 \cap B^C_{\frac{r}{2}}(0)} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}(x)\,dx \right) \\
&\qquad + O\left(\left(\frac{k}{r} \right)^{2\eta} \int_{\Omega_1} \frac{1}{|y-x-Q_1|^{3-2t}} \frac{1}{(1+|x|)^{2(3+2s-\eta)}}\,dx \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a}{r^m}\int_{\mathbb{R}^3} \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx + O\biggl(\frac{1}{r^m}\int_{\Omega_1\cap B_{\frac{r}{2}}(0)} \frac{U(x)}{|y-x-Q_1|^{3-2t}}U(x)\,dx\biggr)\\
&\quad + O\biggl(\frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_1|)^{3-2t}}\biggr)\\
&\quad + O\biggl(\frac{1}{r^{m+\tau}}\int_{\mathbb{R}^3} \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx\biggr) + O\biggl(\frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_1|)^{3-2t}}\biggr)\\
&\quad + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{3+2s}\frac{1}{r^m}\int_{\mathbb{R}^3}\frac{1}{|y-x-Q_1|^{3-2t}}\frac{1}{(1+|x|)^{3+2s}}\,dx\biggr)\\
&\quad + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{2\eta}\frac{1}{(1+|y-Q_1|)^{3-2t}}\biggr)\\
&= \frac{a}{r^m}K(y-Q_1) + O\biggl(\frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_1|)^{3-2t}}\biggr),
\end{aligned}
$$

where $\tau > 0$ is small and we choose $\eta = \frac{1}{2}(3+2s) \in (1, 3+2s]$.

So

$$
\Phi^t_{U_r}(y) = \frac{a}{r^m}\sum_{j=1}^k K(y-Q_j) + O\biggl(\sum_{j=1}^k \frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_j|)^{3-2t}}\biggr). \qquad \square
$$

Lemma A.5. *We have*

$$
\begin{aligned}
&I(U_r)\\
&= kB_3 - kB_4\Bigl(\frac{k}{r}\Bigr)^{3+2s} + k\frac{B_5}{r^{2m}} + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^k K(Q_j-Q_1)\int_{\mathbb{R}^3}U^2 + kO\Bigl(\frac{1}{r^{2m+\tau}}\Bigr),
\end{aligned}
$$

where $B_3 = \bigl(\frac{1}{2}-\frac{1}{p+1}\bigr)\int_{\mathbb{R}^3} U^{p+1}$, $B_4 = \frac{1}{2}B_2$, $B_5 = \frac{a^2}{4}\int_{\mathbb{R}^3} KU^2$ *and* $\tau > 0$ *is small.*

Proof. Recall that

$$
I(U_r) = \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} U_r^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r}U_r^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r|^{p+1}.
$$

By direct computation, we obtain

$$
\begin{aligned}
\text{(A-2)}\quad \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}U_r^2 &= \frac{1}{2}\sum_{j=1}^k\langle U_{Q_j}, U_{Q_j}\rangle_s + \frac{1}{2}\sum_{i\neq j}\langle U_{Q_j}, U_{Q_j}\rangle_s\\
&\quad + \frac{1}{2}\sum_{j=1}^k\int_{\mathbb{R}^3}U_{Q_j}^2 + \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^3}U_{Q_i}U_{Q_j}\\
&= \frac{k}{2}\int_{\mathbb{R}^3}U^{p+1} + \frac{k}{2}\sum_{j=2}^k\int_{\mathbb{R}^3}U_{Q_1}^pU_{Q_j}.
\end{aligned}
$$

By the result in [Long et al. 2016], we know that

$$\sum_{j=2}^{k}\int_{\mathbb{R}^3}U_{Q_1}^{p}U_{Q_j}=\sum_{j=2}^{k}\frac{B_1}{|Q_1-Q_j|^{3+2s}}+O\biggl(\sum_{j=2}^{k}\frac{1}{|Q_1-Q_j|^{3+2s+\tau}}\biggr),\tag{A-3}$$

where B_1 is a positive constant and $\tau>0$ is small enough. We also obtain

$$\begin{aligned}\frac{1}{p+1}\int_{\Omega_1}|U_r|^{p+1}&=\frac{1}{p+1}\int_{\Omega_1}\biggl(U_{Q_1}+\sum_{j=2}^{k}U_{Q_j}\biggr)^{p+1}\\&=\frac{1}{p+1}\int_{\Omega_1}|U_{Q_1}|^{p+1}+\int_{\Omega_1}|U_{Q_1}|^{p}\sum_{j=2}^{k}U_{Q_j}\\&\quad+O\biggl(\int_{\Omega_1}|U_{Q_1}|^{p-1}\biggl(\sum_{j=2}^{k}U_{Q_j}\biggr)^2\biggr)+O\biggl(\int_{\Omega_1}\biggl(\sum_{j=2}^{k}U_{Q_j}\biggr)^{p+1}\biggr)\\&=\frac{1}{p+1}\int_{\Omega_1}|U_{Q_1}|^{p+1}+\int_{\Omega_1}|U_{Q_1}|^{p}\sum_{j=2}^{k}U_{Q_j}\\&\quad+O\biggl(\biggl(\sum_{j=2}^{k}\frac{1}{|Q_j-Q_1|^{\frac{3}{2}+2s}}\biggr)^2\biggr)\\&\quad\quad+O\biggl(\biggl(\sum_{j=2}^{k}\frac{1}{|Q_j-Q_1|^{3+2s-\frac{3+(p-1)s}{p+1}}}\biggr)^{p+1}\biggr)\\&=\frac{1}{p+1}\int_{\mathbb{R}^3}|U|^{p+1}+\int_{\Omega_1}|U_{Q_1}|^{p}\sum_{j=2}^{k}U_{Q_j}+O\biggl(\frac{k}{r}\biggr)^{3+4s}.\end{aligned}\tag{A-4}$$

Using (A-3) and Lemma A.4, we see that

(A-5)

$$\begin{aligned}&\int_{\mathbb{R}^3}V(|x|)\Phi_{U_r}^{t}U_r^2\\&\quad=k\int_{\Omega_1}V(|x|)\Phi_{U_r}^{t}U_r^2\\&\quad=k\int_{\Omega_1}V(|x|)\biggl(\frac{a}{r^m}\sum_{j=1}^{k}K(x-Q_j)\\&\qquad\qquad+O\biggl(\sum_{j=1}^{k}\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr)\biggr)\biggl(U_{Q_1}+O\biggl(\frac{k}{r}\biggr)^{3+2s}\biggr)^2\end{aligned}$$

$$
\begin{aligned}
&= k\int_{\Omega_1} V(|x|)\frac{a}{r^m}\sum_{j=1}^{k} K(x-Q_j)U_{Q_1}^2 \\
&\quad + k\int_{\Omega_1} V(|x|)O\biggl(\sum_{j=1}^{k}\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr)U_{Q_1}^2 \\
&\quad + kO\biggl(\frac{1}{r^{m+\tau}}\int_{\Omega_1} V(|x|)\frac{a}{r^m}\sum_{j=1}^{k} K(x-Q_j)\biggr) \\
&\quad + kO\biggl(\int_{\Omega_1} V(|x|)\sum_{j=1}^{k}\frac{1}{r^{2m+2\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr) \\
&= k\int_{\Omega_1} V(|x|)\frac{a}{r^m}K(x-Q_1)U_{Q_1}^2 + k\int_{\Omega_1} V(|x|)\frac{a}{r^m}\sum_{j=2}^{k} K(x-Q_j)U_{Q_1}^2 \\
&\quad + k\int_{\Omega_1} V(|x|)O\biggl(\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_1|)^{3-2t}}\biggr)U_{Q_1}^2 \\
&\quad + k\int_{\Omega_1} V(|x|)O\biggl(\sum_{j=2}^{k}\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr)U_{Q_1}^2 \\
&\quad + kO\biggl(\int_{\mathbb{R}^3} V(|x|)\sum_{j=1}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr) \\
&= \frac{ka}{r^m}\int_{\Omega_1\cap B_{\frac{r}{2}}(0)}\biggl(\frac{a}{|x+Q_1|^m} + O\biggl(\frac{1}{|x+Q_1|^{m+\theta}}\biggr)\biggr)K(x)U^2(x) \\
&\quad + kO\biggl(\int_{\Omega_1\cap B^C_{\frac{r}{2}}(0)}\frac{1}{r^{2m}}K(x-Q_j)U_{Q_1}^2\biggr) + \frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j-Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\biggl(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1-Q_j|^{3-2t}}\biggr) + kO\biggl(\frac{1}{r^{2m+\tau}}\biggr) \\
&= \frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j-Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\biggl(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1-Q_j|^{3-2t}}\biggr) + kO\biggl(\frac{1}{r^{2m+\tau}}\biggr).
\end{aligned}
$$

Above all, we deduce that

(A-6)

$$
\begin{aligned}
I(U_r)
&= \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} U_r^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t U_r^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |U_r|^{p+1} \\
&= \frac{k}{2}\int_{\mathbb{R}^3} U^{p+1} + \frac{k}{2}\sum_{j=2}^{k}\int_{\mathbb{R}^3} U_{Q_1}^p U_{Q_j} - \frac{k}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} - k\int_{\Omega_1} |U_{Q_1}|^p \sum_{j=2}^{k} U_{Q_j} \\
&\quad + kO\left(\frac{k}{r}\right)^{3+4s} + \frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\left(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1 - Q_j|^{3-2t}}\right) + kO\left(\frac{1}{r^{2m+\tau}}\right) \\
&= \frac{k}{2}\int_{\mathbb{R}^3} U^{p+1} + \frac{k}{2}\sum_{j=2}^{k}\int_{\mathbb{R}^3} U_{Q_1}^p U_{Q_j} - \frac{k}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} - k\int_{\Omega_1} |U_{Q_1}|^p \sum_{j=2}^{k} U_{Q_j} \\
&\quad + kO\left(\frac{k}{r}\right)^{3+4s} + \frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\left(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1 - Q_j|^{3-2t}}\right) + kO\left(\frac{1}{r^{2m+\tau}}\right) \\
&= \frac{k}{2}\int_{\mathbb{R}^3} U^{p+1} + \frac{k}{2}\sum_{j=2}^{k}\int_{\mathbb{R}^3} U_{Q_1}^p U_{Q_j} - \frac{k}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} \\
&\quad - k\int_{\mathbb{R}^3} |U_{Q_1}|^p \sum_{j=2}^{k} U_{Q_j} + kO\left(\frac{k}{r}\right)^{3+2s+\tau} + kO\left(\frac{k}{r}\right)^{3+4s} \\
&\quad + \frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\left(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1 - Q_j|^{3-2t}}\right) + kO\left(\frac{1}{r^{2m+\tau}}\right) \\
&= k\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} U^{p+1} - \frac{k}{2}\sum_{j=2}^{k}\frac{B_1}{|Q_1 - Q_j|^{3+2s}} \\
&\quad + kO\left(\sum_{j=2}^{k}\frac{1}{|Q_1 - Q_j|^{3+2s+\tau}}\right) + kO\left(\frac{k}{r}\right)^{3+2s+\tau} + kO\left(\frac{k}{r}\right)^{3+4s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2+\frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j-Q_1)\int_{\mathbb{R}^3} U^2\\
&+kO\biggl(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1-Q_j|^{3-2t}}\biggr)+kO\Bigl(\frac{1}{r^{2m+\tau}}\Bigr)\\
={}&k\Bigl(\frac{1}{2}-\frac{1}{p+1}\Bigr)\int_{\mathbb{R}^3} U^{p+1}-\frac{k}{2}B_2\Bigl(\frac{k}{r}\Bigr)^{3+2s}+\frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2\\
&+\frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j-Q_1)\int_{\mathbb{R}^3} U^2+kO\Bigl(\frac{1}{r^{2m+\tau}}\Bigr). \qquad\square
\end{aligned}
$$

Acknowledgements

The author thanks the referee and Professor Robert Finn for helpful discussions and suggestions. This paper was supported by the NSFC (Nos. 11601139, 11301204).

References

[Alves and Souto 2014] C. O. Alves and M. A. S. Souto, "Existence of least energy nodal solution for a Schrödinger–Poisson system in bounded domains", *Z. Angew. Math. Phys.* **65**:6 (2014), 1153–1166. MR Zbl

[Azzollini and Pomponio 2008] A. Azzollini and A. Pomponio, "Ground state solutions for the nonlinear Schrödinger–Maxwell equations", *J. Math. Anal. Appl.* **345**:1 (2008), 90–108. MR Zbl

[Cao and Tang 2006] D. Cao and Z. Tang, "Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields", *J. Differential Equations* **222**:2 (2006), 381–424. MR Zbl

[Chen and Wang 2014] S. Chen and C. Wang, "Existence of multiple nontrivial solutions for a Schrödinger–Poisson system", *J. Math. Anal. Appl.* **411**:2 (2014), 787–793. MR Zbl

[D'Aprile and Mugnai 2004] T. D'Aprile and D. Mugnai, "Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations", *Proc. Roy. Soc. Edinburgh Sect. A* **134**:5 (2004), 893–906. MR Zbl

[Di Nezza et al. 2012] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces", *Bull. Sci. Math.* **136**:5 (2012), 521–573. MR Zbl

[Frank and Lenzmann 2013] R. L. Frank and E. Lenzmann, "Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$", *Acta Math.* **210**:2 (2013), 261–318. MR Zbl

[Frank et al. 2016] R. L. Frank, E. Lenzmann, and L. Silvestre, "Uniqueness of radial solutions for the fractional Laplacian", *Comm. Pure Appl. Math.* **69**:9 (2016), 1671–1726. Zbl

[Giammetta 2014] A. R. Giammetta, "Fractional Schrödinger–Poisson–Slater system in one dimension", preprint, 2014. arXiv

[He and Zou 2012] X. He and W. Zou, "Existence and concentration of ground states for Schrödinger–Poisson equations with critical growth", *J. Math. Phys.* **53**:2 (2012), 023702-1–19. MR Zbl

[Ianni 2009] I. Ianni, "Solutions of the Schrödinger–Poisson problem concentrating on spheres, II: Existence", *Math. Models Methods Appl. Sci.* **19**:6 (2009), 877–910. MR Zbl

[Ianni and Vaira 2008] I. Ianni and G. Vaira, "On concentration of positive bound states for the Schrödinger–Poisson problem with potentials", *Adv. Nonlinear Stud.* **8**:3 (2008), 573–595. MR Zbl

[Ianni and Vaira 2009] I. Ianni and G. Vaira, "Solutions of the Schrödinger–Poisson problem concentrating on spheres, I: Necessary conditions", *Math. Models Methods Appl. Sci.* **19**:5 (2009), 707–720. MR Zbl

[Ianni and Vaira 2015] I. Ianni and G. Vaira, "Non-radial sign-changing solutions for the Schrödinger–Poisson problem in the semiclassical limit", *NoDEA Nonlinear Differential Equations Appl.* **22**:4 (2015), 741–776. MR Zbl

[Kim and Seok 2012] S. Kim and J. Seok, "On nodal solutions of the nonlinear Schrödinger–Poisson equations", *Commun. Contemp. Math.* **14**:6 (2012), art. id 1250041. MR Zbl

[Li et al. 2010] G. Li, S. Peng, and S. Yan, "Infinitely many positive solutions for the nonlinear Schrödinger–Poisson system", *Commun. Contemp. Math.* **12**:6 (2010), 1069–1092. MR Zbl

[Liu et al. 2016] Z. Liu, Z.-Q. Wang, and J. Zhang, "Infinitely many sign-changing solutions for the nonlinear Schrödinger–Poisson system", *Ann. Mat. Pura Appl.* (4) **195**:3 (2016), 775–794. MR Zbl

[Long et al. 2016] W. Long, S. Peng, and J. Yang, "Infinitely many positive and sign-changing solutions for nonlinear fractional scalar field equations", *Discrete Contin. Dyn. Syst.* **36**:2 (2016), 917–939. MR Zbl

[Ruiz 2006] D. Ruiz, "The Schrödinger–Poisson equation under the effect of a nonlinear local term", *J. Funct. Anal.* **237**:2 (2006), 655–674. MR Zbl

[Ruiz and Vaira 2011] D. Ruiz and G. Vaira, "Cluster solutions for the Schrödinger–Poisson–Slater problem around a local minimum of the potential", *Rev. Mat. Iberoam.* **27**:1 (2011), 253–271. MR Zbl

[Wei and Zhao 2013] J. Wei and C. Zhao, "Non-compactness of the prescribed Q-curvature problem in large dimensions", *Calc. Var. Partial Differential Equations* **46**:1 (2013), 123–164. MR Zbl

[Zhang 2014] J. Zhang, "The existence and concentration of positive solutions for a nonlinear Schrödinger–Poisson system with critical growth", *J. Math. Phys.* **55**:3 (2014), 031507-1–14. MR Zbl

[Zhang 2015] J. Zhang, "Existence and multiplicity results for the fractional Schrödinger–Poisson systems", preprint, 2015. arXiv

[Zhang et al. 2016] J. Zhang, J. M. do Ó, and M. Squassina, "Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity", *Adv. Nonlinear Stud.* **16**:1 (2016), 15–30. MR Zbl

[Zhao et al. 2013] L. Zhao, H. Liu, and F. Zhao, "Existence and concentration of solutions for the Schrödinger–Poisson equations with steep well potential", *J. Differential Equations* **255**:1 (2013), 1–23. MR Zbl

Received January 31, 2016. Revised June 7, 2016.

WEIMING LIU
SCHOOL OF MATHEMATICS AND STATISTICS
HUBEI NORMAL UNIVERSITY
HUANGSHI, 435002
CHINA
whu.027@163.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 287 No. 2 April 2017

0030-8730(201704)287:2;1-U