Pacific Journal of Mathematics

Volume 287 No. 2 April 2017

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.257

MAXIMAL OPERATORS FOR THE p-LAPLACIAN FAMILY

PABLO BLANC, JUAN P. PINASCO AND JULIO D. ROSSI

We prove existence and uniqueness of viscosity solutions for the problem

$$\max\{-\Delta_{p_1} u(x),\ -\Delta_{p_2} u(x)\} = f(x)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with $u = g$ on $\partial\Omega$. Here $-\Delta_p u = (N+p)^{-1}|Du|^{2-p}\operatorname{div}(|Du|^{p-2}Du)$ is the 1-homogeneous p-Laplacian and we assume that $2 \le p_1, p_2 \le \infty$. This equation appears naturally when one considers a tug-of-war game in which one of the players (the one who seeks to maximize the payoff) can choose at every step which are the parameters of the game that regulate the probability of playing a usual tug-of-war game (without noise) or playing at random. Moreover, the operator $\max\{-\Delta_{p_1} u(x),\ -\Delta_{p_2} u(x)\}$ provides a natural analogue with respect to p-Laplacians to the Pucci maximal operator for uniformly elliptic operators.

We provide two different proofs of existence and uniqueness for this problem. The first one is based in pure PDE methods (in the framework of viscosity solutions) while the second one is more connected to probability and uses game theory.

1. Introduction

In this paper our goal is to show existence and uniqueness of viscosity solutions to the Dirichlet problem for the maximal operator associated with the family of p-Laplacian operators, $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $2 \le p \le \infty$.

When one considers the family of uniformly elliptic second-order operators of the form $-\operatorname{tr}(AD^2u)$ and looks for maximal operators, one finds the so-called Pucci maximal operator, $P^+_{\lambda,\Lambda}(D^2u) = \max_{A\in\mathcal{A}} -\operatorname{tr}(AD^2u)$, where $\mathcal{A}$ is the set of uniformly elliptic matrices with ellipticity constant between λ and Λ. This maximal operator plays a crucial role in the regularity theory for uniformly elliptic second-order operators and has the following properties; see [Caffarelli and Cabré 1995]:

(1) (monotonicity) If $\lambda_1 \le \lambda_2 \le \Lambda_2 \le \Lambda_1$, then $P^+_{\lambda_2,\Lambda_2}(D^2u) \le P^+_{\lambda_1,\Lambda_1}(D^2u)$.

(2) (positive homogeneity) If $\alpha \ge 0$, then $P^+_{\lambda,\Lambda}(\alpha D^2u) = \alpha P^+_{\lambda,\Lambda}(D^2u)$.

MSC2010: 35J70, 49N70, 91A15, 91A24.

Keywords: Dirichlet boundary conditions, dynamic programming principle, p-Laplacian, tug-of-war games.

(3) (subsolutions) If u verifies $P^{+}_{\lambda,\Lambda}(D^2u) \le 0$ in the viscosity sense, then $-\operatorname{tr}(AD^2u) \le 0$ for every matrix A with ellipticity constants λ and Λ (that is, a subsolution to the maximal operator is a subsolution for every elliptic operator in the class). Therefore, from the comparison principle we get that a solution to $P^{+}_{\lambda,\Lambda}(D^2u) \le 0$ provides a lower bound for every solution of any elliptic operator in the class with the same boundary values.

If we try to reproduce these properties for the family of p-Laplacians, we are led to consider the operator $\max_{p_1 \le p \le p_2} -\Delta_p u(x)$. As we will show in this paper, this operator has similar properties to the ones that hold for the Pucci maximal operator, but with respect to the p-Laplacian family.

Hence, it is natural to consider the Dirichlet problem for the partial differential equation

$$\max_{p_1 \le p \le p_2} -\Delta_p u(x) = f(x) \tag{1-1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ for $2 \le p_1, p_2 \le \infty$. Here we have normalized the p-Laplacian and considered the operator

$$\Delta_p u = \frac{\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)}{(N+p)|\nabla u|^{p-2}},$$

which is called the 1-homogeneous p-Laplacian. We will assume that $f \equiv 0$ or that f is strictly positive or negative in Ω. We will consider solutions u (along the whole paper we consider solutions in the viscosity sense, see [Crandall et al. 1992]) to this problem with $f \equiv 0$, as p_1-p_2-harmonic functions.

Note that, formally, the 1-homogeneous p-Laplacian can be written as

$$\Delta_p u = \frac{p-2}{N+p}\Delta_\infty u + \frac{1}{N+p}\Delta u,$$

where Δu is the usual Laplacian and $\Delta_\infty u$ is the normalized ∞-Laplacian, that is,

$$\Delta u = \sum_{i=1}^{N} u_{x_i x_i} \quad \text{and} \quad \Delta_\infty u = \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{N} u_{x_i} u_{x_i x_j} u_{x_j}.$$

Therefore, we can think about the 1-homogeneous p-Laplacian as a convex combination of the Laplacian divided by $N+2$ and the ∞-Laplacian, in fact,

$$\Delta_p u = \frac{p-2}{N+p}\Delta_\infty u + \frac{N+2}{N+p}\frac{\Delta u}{N+2} = \alpha \Delta_\infty u + \theta \Delta u$$

with $\alpha = (p-2)/(N+p)$ and $\theta = 1/(N+p)$ (we reserve β for a different constant) for $2 \le p < \infty$, and $\alpha = 1$ and $\theta = 0$ for $p = \infty$.

Since we are dealing with convex combinations, equation (1-1) becomes

$$\max_{p_1 \le p \le p_2} -\Delta_p u(x) = \max\{-\Delta_{p_1} u(x),\ -\Delta_{p_2} u(x)\} = f(x), \tag{1-2}$$

with $2 \le p_1, p_2 \le \infty$.

Our main result concerning viscosity solutions to (1-2) reads as follows:

Theorem 1.1. *Let Ω be a bounded domain such that the exterior ball condition holds when $p_1 \le N$ or $p_2 \le N$. Assume that $\inf_\Omega f > 0$, $\sup_\Omega f < 0$ or $f \equiv 0$. Then, given g a continuous function defined on $\partial\Omega$, there exists a unique viscosity solution $u \in C(\overline{\Omega})$ of (1-2) with $u = g$ in $\partial\Omega$.*

Moreover, a comparison principle holds: if $u, v \in C(\overline{\Omega})$ are such that

$$\max\bigl\{-\Delta_{p_1} u, -\Delta_{p_2} u\bigr\} \le f \quad \textit{and} \quad \max\bigl\{-\Delta_{p_1} v, -\Delta_{p_2} v\bigr\} \ge f$$

are in Ω and $v \ge u$ on $\partial\Omega$, then $v \ge u$ in Ω.

In addition, we have a Hopf's lemma: let u be a supersolution to (1-2) and $x_0 \in \partial\Omega$ be such that $u(x_0) > u(x)$ for all $x \in \Omega$, then we have

$$\limsup_{t \to 0^+} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0,$$

where ν is exterior normal to $\partial\Omega$.

Remark 1.2. An analogous result holds for the equation $\min_{p_1 \le p \le p_2} -\Delta_p u(x) = f$.

Remark 1.3. For the homogeneous case, $f \equiv 0$, we have that viscosity sub- and supersolutions to the 1-homogeneous p-Laplacian,

$$-\frac{p-2}{N+p}\Delta_\infty u - \frac{1}{N+p}\Delta u = 0,$$

coincide with viscosity sub and supersolutions to the usual ($(p-1)$-homogeneous) p-Laplacian $-\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) = 0$; see [Manfredi et al. 2012b].

Therefore, for $f \equiv 0$ we are providing existence and uniqueness of viscosity solutions to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) = 0$, with $\Delta_p u$ being the usual p-Laplacian that comes from calculus of variations.

Remark 1.4. This maximal operator for the p-Laplacian family has the following properties that are analogous to the ones described above for Pucci's operator:

(1) (monotonicity) If $p_{1,1} \le p_{2,1} \le p_{2,2} \le p_{1,2}$ then

$$\max_{p_{2,1} \le p \le p_{2,2}} -\Delta_p u \le \max_{p_{1,1} \le p \le p_{1,2}} -\Delta_p u.$$

(2) (positive homogeneity) If $\alpha \ge 0$, then

$$\max_{p_1 \le p \le p_2} -\Delta_p(\alpha u) = \alpha \max_{p_1 \le p \le p_2} -\Delta_p u.$$

(3) (subsolutions) A viscosity solution u to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) \le 0$, is a viscosity solution to $-\Delta_p u(x) \le 0$ for every $p_1 \le p \le p_2$. Hence, from the comparison principle, we get that a solution to $\max_{p_1 \le p \le p_2} -\Delta_p u(x) \le 0$ provides a lower bound for every solution of any elliptic operator in the class with the same boundary values.

We have two different approaches for this problem. The first one is based on PDE tools in the framework of viscosity solutions. The second one is related to probability theory (game theory) using the game that we describe below.

Let us introduce a game that we call *unbalanced tug-of-war game with noise*. It is a two-player (Players I and II) zero-sum stochastic game. The game is played in a bounded open set $\Omega \subset \mathbb{R}^N$. Fix an $\varepsilon > 0$. At the initial time, the players place a token at a point $x_0 \in \Omega$ and Player I chooses a coin between two possible ones. They toss the chosen coin which is biased with probabilities α_i and β_i, $\alpha_i + \beta_i = 1$ and $0 \le \alpha_i, \beta_i \le 1$, $i = 1, 2$. Now, they play the tug-of-war with noise game described in [Manfredi et al. 2012b] with probabilities α_i, β_i. If they get heads (probability α_i), they toss a fair coin (with equal probability of heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$. Once the game position leaves Ω, let's say at the τ-th step, the game ends. The payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ (note that we only use the values of g in a strip of width ε around $\partial\Omega$). At the end Player II pays to Player I the amount given by the formula

$$g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n).$$

Note that the positions of the game depend on the strategies adopted by Players I and II. From this procedure we get two extreme functions, $u_{\mathrm{I}}(x_0)$ (the value of the game for Player I) and $u_{\mathrm{II}}(x_0)$ (the value of the game for Player II), that are in a sense the best expected outcomes that each player may expect choosing a strategy when the game starts at x_0. When $u_{\mathrm{I}}(x_0)$ and $u_{\mathrm{II}}(x_0)$ coincide at every $x_0 \in \Omega$ this function $u_\varepsilon := u_{\mathrm{I}} = u_{\mathrm{II}}$ is called the *value of the game*.

Theorem 1.5. *Assume that f is a Lipschitz function with* $\sup_\Omega f < 0$ *or* $\inf_\Omega f > 0$ *or $f \equiv 0$. The unbalanced tug-of-war game with noise with $\{\alpha_1, \alpha_2\} \neq \{0, 1\}$ when $f \equiv 0$ has a value and that value satisfies the dynamic programming principle, given by*

$$u_\varepsilon(x) = \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{y \in B_\varepsilon(x)} u_\varepsilon(y) + \inf_{y \in B_\varepsilon(x)} u_\varepsilon(y) \Big\} + \beta_i \fint_{B_\varepsilon(x)} u_\varepsilon(y)\, dy \right)$$

for $x \in \Omega$, with $u_\varepsilon(x) = g(x)$ for $x \notin \Omega$.

Moreover, if g is Lipschitz and Ω satisfies the exterior ball condition, then there exists a uniformly continuous function u such that

$$u_\varepsilon \to u \quad \text{uniformly in} \quad \overline{\Omega}.$$

This limit u is a viscosity solution to

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = \overline{f} & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\overline{f} = 2f$ and p_1, p_2 are given by

$$\alpha_i = \frac{p_i - 2}{p_i + N}, \qquad \beta_i = \frac{2+N}{p_i+N}, \qquad i = 1, 2.$$

Remark 1.6. When f is strictly positive or negative, we have that the game ends almost surely (a.s.). The same is true (regardless of the strategies adopted by the players) when they play with some noise at every turn, that is, when the two β_i are positive. This fact simplifies the arguments used in the proofs.

When one of the α_i is 1 (and therefore the corresponding β_i is 0) the argument is more delicate; see Section 4.

Remark 1.7. The proof of Theorem 1.5 follows from the results in Sections 4 and 5. In Section 4 we establish that the game has a value and that the value is the unique function that satisfies the dynamic programming principle (DPP). In Section 5 we prove the convergence part of the theorem. In Proposition 4.4 we establish the existence of a function satisfying the DPP. In Theorem 4.6 we prove that the function satisfying the DPP is unique and coincides with the game value, in the case $\beta_1, \beta_2 > 0$, $\sup f < 0$ or $\inf f > 0$. The same result is obtained in the remaining cases in Theorems 4.8 and 4.9. Here is where we had to assume that $\{\alpha_1, \alpha_2\} \neq \{0, 1\}$. Finally, the convergence is established in Corollaries 5.8 and 5.9.

Remark 1.8. Note that in the limit problem one only considers the values of g on $\partial\Omega$ while in the game one needs g to be defined in a bigger set. Given a Lipschitz function defined on $\partial\Omega$ we can just extend it to this larger set without affecting the Lipschitz constant. For simplicity but making an abuse of notation we also call such an extension g.

Remark 1.9. We also prove uniqueness of solutions to the DPP; see Section 4. That is, there exists a unique function verifying

$$v(x) = \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{y \in B_\varepsilon(x)} v(y) + \inf_{y \in B_\varepsilon(x)} v(y) \Big\} + \beta_i \fint_{B_\varepsilon(x)} v(y)\, dy \right),$$

for $x \in \Omega$, with $v(x) = g(x)$ for $x \notin \Omega$.

Remark 1.10. When Player II (the player who wants to minimize the expected outcome) has the choice of the probabilities α and β we end up with a solution to

$$\begin{cases} \min\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Let us make some brief comments on related work. First, let us recall that Pucci operators are crucial in regularity theory for uniformly elliptic operators, due to their natural comparison with a nondivergence linear operator with measurable coefficients. We refer to [Busca et al. 2005; Caffarelli and Cabré 1995; Felmer et al. 2006; Quaas and Sirakov 2006].

On the other hand, concerning probabilistic ideas for PDEs, the fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown the profound and powerful connection between the classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common origin in mean value properties. This approach turns out to be useful in the nonlinear theory as well, since p-harmonic functions verify an asymptotic mean value property; see, for example, [Manfredi et al. 2010; Hartenstine and Rudd 2013; Kawohl et al. 2012; Llorente 2014; 2015]. Concerning tug-of-war games and PDEs the story begins with [Peres et al. 2009; Peres and Sheffield 2008] and was extended in [Atar and Budhiraja 2010; Bjorland et al. 2012a; 2012b; Nyström and Parviainen 2014], etc. For the p-Laplacian the equivalence between viscosity and weak solutions was proved in [Julin and Juutinen 2012; Juutinen et al. 2001]. This probability approach was used to obtain regularity properties of solutions; we refer to [Armstrong and Smart 2010; Luiro and Parviainen 2015; Luiro et al. 2013; Ruosteenoja 2016].

We finish the introduction with a comment on the main technical novelties contained in this manuscript. To obtain existence and uniqueness for our maximal PDE we first use ideas and techniques from viscosity solutions theory. This part follows the usual steps (the first one shows a comparison principle and then applies Perron's method, including the construction of barriers near the boundary), but here some extra care is needed to deal with points at which the gradient of a test function vanishes. Concerning the game theoretical approach we want to emphasize that when $p_2 = \infty$ we don't know a priori that the game terminates almost surely and this fact introduces some extra difficulties. The argument that shows that there is a unique solution to the dynamic programming principle in this case is delicate; see Theorem 4.8. The proof of convergence of the values of the game as the size of the steps goes to zero is also different from previous results in the literature since here one has to take care of the strategy of the player who chooses the parameters of the game. In particular, the proof that when any of the two players pull in a fixed direction the expectation of the exit time is bounded above $C\varepsilon^2$ is more involved; see Lemma 5.2.

The paper is organized as follows: In Section 2 we prove the comparison principle and then existence and uniqueness for our problem using Perron's method; in Section 3 we introduce a precise description of the game; in Section 4 we show that the game has a value and that this value is the solution to the dynamic programming principle; and finally, in Section 5 we collect some properties of the value function of the game and show that these values converge to the unique viscosity solution of our problem.

2. Existence and uniqueness

First, let us state the definition of a viscosity solution. We have to handle some technical difficulties as the 1-homogeneous ∞-Laplacian is not well-defined when the gradient vanishes. Observing that

$$\Delta u = \operatorname{tr}(D^2 u) \quad \text{and} \quad \Delta_\infty u = \frac{\nabla u}{|\nabla u|} D^2 u \frac{\nabla u}{|\nabla u|},$$

we can write (1-2) as $F(\nabla u, D^2 u) = f$, where

$$F(v, X) = \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} - \theta_i \operatorname{tr}(X) \right\}.$$

Note that F is degenerate elliptic, that is,

$$F(v, X) \le F(v, Y) \text{ for } v \in \mathbb{R}^N \setminus \{0\} \text{ and } X, Y \in S^N \text{ provided } X \ge Y,$$

as is generally requested to work in the context of viscosity solutions.

This function $F : \mathbb{R}^N \times S^N \mapsto \mathbb{R}$ is not well-defined at $v = 0$ (here S^N denotes the set of real symmetric $N \times N$ matrices). Therefore, we need to consider the lower semicontinuous F_* and upper semicontinuous F^* envelopes of F. These functions coincide with F for $v \ne 0$, and for $v = 0$ are given by

$$F^*(0, X) = \max_{i \in \{1,2\}} \{-\alpha_i \lambda_{\min}(X) - \theta_i \operatorname{tr}(X)\},$$
$$F_*(0, X) = \max_{i \in \{1,2\}} \{-\alpha_i \lambda_{\max}(X) - \theta_i \operatorname{tr}(X)\},$$

where $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ are the minimum and maximum eigenvalues of X, respectively.

Now we are ready to give the definition for a viscosity solution to our equation.

Definition 2.1. For $2 \le p_1, p_2 \le \infty$ consider the equation

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f$$

in Ω. Then we have the following definitions:

(1) A lower semicontinuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below, we have

$$F^*\big(\nabla\phi(x), D^2\phi(x)\big) \geq f(x).$$

(2) An upper semicontinuous function u is a subsolution if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above, we have

$$F_*\big(\nabla\psi(x), D^2\psi(x)\big) \leq f(x).$$

(3) Finally, u is a viscosity solution if it is both a sub- and supersolution.

In the case $f \equiv 0$, the comparison holds for our equation as a consequence of the main result of [Koike and Kosugi 2015]. See also [Barles and Busca 2001]. Note that the comparison principle obtained in the former is slightly more general than the one obtained in the latter. We need this more general result here as our F is not necessarily continuous when the gradient vanishes. In [Koike and Kosugi 2015] a different notion of viscosity solution is considered. We remark that when a function is a viscosity sub- or supersolution in the sense of Definition 2.1 it is also that in the sense considered in [Koike and Kosugi 2015]. Therefore we can use the comparison result established there once we check their hypotheses.

Proposition 2.2. *Let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be, respectively, a viscosity subsolution and a viscosity supersolution of (1-2) with $f \equiv 0$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. We just apply the main result in [Koike and Kosugi 2015], referring to notations and details therein. To this end we need to check some conditions. First, let us show that F is elliptic. In fact, we have

$$\begin{aligned}
F(v, X - \mu v \otimes v) &= \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|}(X - \mu v \otimes v)\frac{v}{|v|} - \theta_i \operatorname{tr}(X - \mu v \otimes v) \right\} \\
&= \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} + \alpha_i \mu |v|^2 - \theta_i \operatorname{tr}(X) + \theta_i \mu |v|^2 \right\} \\
&= \max_{i \in \{1,2\}} \left\{ -\alpha_i \frac{v}{|v|} X \frac{v}{|v|} - \theta_i \operatorname{tr}(X) + \theta_i \right\} + \mu |v|^2 \\
&= F(v, X) + \mu |v|^2.
\end{aligned}$$

Moreover, F is invariant by rescaling in v and 1-homogeneous in X.

So, using the notation from [Koike and Kosugi 2015], we can take $\sigma_0(v) = |v|^2$, $\sigma_1(t) = t$ and $\rho \equiv 0$ that satisfy the conditions imposed in that paper, to obtain the comparison result. □

Now we deal with the case where f is assumed to be nontrivial and does not change sign. In fact, we assume that $\inf f > 0$ or $\sup f < 0$. We follow similar ideas to the ones in [Lu and Wang 2008].

Lemma 2.3. *If we have $u, v \in C(\overline{\Omega})$ such that*

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f \quad \textit{and} \quad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge g,$$

where $g > f$ and $v \ge u$ in $\partial\Omega$, then we have $v \ge u$ in Ω.

Proof. By adding a constant if necessary we can assume that $u, v > 0$. Arguing by contradiction we assume that

$$\max_{\overline{\Omega}}(u - v) > 0 \ge \max_{\partial\Omega}(u - v).$$

Now we double the variables and consider

$$\sup_{x,y \in \Omega} \{u(x) - v(y) - (j/2)|x - y|^2\}.$$

For large j the supremum is attained at interior points x_j, y_j such that $x_j \to \hat{x}$, $y_j \to \hat{x}$, where $\hat{x}$ is an interior point (that $\hat{x}$ cannot be on the boundary can be obtained as in [Lindqvist and Lukkari 2010]).

Now, we observe that there exists a constant C such that $j|x_j - y_j| \le C$. The theorem of sums (see Theorem 3.2 from [Crandall et al. 1992]) implies that there are symmetric matrices $\mathbb{X}_j$, $\mathbb{Y}_j$, with $\mathbb{X}_j \le \mathbb{Y}_j$ such that $(j|x_j - y_j|, \mathbb{X}_j) \in \overline{J^{2,+}}(u)(x_j)$ and $(j|x_j - y_j|, \mathbb{Y}_j) \in \overline{J^{2,-}}(v)(y_j)$, where $\overline{J^{2,+}}(u)(x_j)$ and $\overline{J^{2,-}}(v)(y_j)$ are the closures of the super- and subjets of u and v respectively. Using the equations, assuming that $x_j \ne y_j$, we have

$$\max_{i \in \{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{X}_j) \right\} \le f(y_j)$$

and

$$\max_{i \in \{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{Y}_j) \right\} \ge g(y_j).$$

Now we observe that, since $\mathbb{X}_j \le \mathbb{Y}_j$ we get

$$-\operatorname{tr}(\mathbb{X}_j) \ge -\operatorname{tr}(\mathbb{Y}_j)$$

and

$$-\left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle \ge -\left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle.$$

Hence

$$\begin{aligned} f(y_j) &\ge \max_{i \in \{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{X}_j) \right\} \\ &\ge \max_{i \in \{1,2\}} \left\{ -\alpha_i \left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle - \theta_i \operatorname{tr}(\mathbb{Y}_j) \right\} \ge g(x_j). \end{aligned}$$

This gives a contradiction passing to the limit as $j \to \infty$.

When $x_j = y_j$ we obtain

$$\max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\max}(\mathbb{Y}_j) - \theta_i \operatorname{tr}(\mathbb{Y}_j)\} \le f(y_j)$$

and

$$\max_{i\in\{1,2\}} \{-\alpha_i \lambda_{\min}(\mathbb{X}_j) - \theta_i \operatorname{tr}(\mathbb{X}_j)\} \ge g(x_j),$$

which also lead to a contradiction since $\lambda_{\max}(\mathbb{Y}_j) \ge \lambda_{\max}(\mathbb{X}_j) \ge \lambda_{\min}(\mathbb{X}_j)$.

Hence we have obtained that $u \le v$, as we wanted to prove. □

Lemma 2.4. *If $u, v \in C(\overline{\Omega})$ are such that*

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f, \quad \textit{and} \quad \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge f$$

in Ω with $\inf_\Omega f > 0$ and $v \ge u$ on $\partial\Omega$, then we have $v \ge u$ in Ω.

Proof. By adding a constant if necessary we can assume that $u, v > 0$. Let's consider $v_\delta = (1+\delta)v$, then

$$\max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} \le f < (1+\delta) f \le \max\{-\Delta_{p_1} v_\delta, -\Delta_{p_2} v_\delta\}$$

and $v_\delta \ge v \ge u$ in $\partial\Omega$. Then by the preceding lemma we conclude that $v_\delta \ge u$ in Ω for all $\delta > 0$. Making $\delta \to 0$, we get $v \ge u$ in Ω as we wanted to show. □

Remark 2.5. The above lemma is also true when $\sup_\Omega f < 0$. So, we have comparisons for the cases $\inf_\Omega f > 0$, $\sup_\Omega f < 0$ and $f \equiv 0$. From this comparison result we get uniqueness of solutions.

Now we deal with the existence of solutions. In the proof of this result we are only using that the exterior ball condition holds for Ω when $p_1 \le N$ or $p_2 \le N$.

Theorem 2.6. *Assume that $\inf f > 0$, $\sup f < 0$ or $f \equiv 0$. Then, given g a continuous function defined on $\partial\Omega$, there exists $u \in C(\overline{\Omega})$ which is a viscosity solution of (1-2) such that $u = g$ in $\partial\Omega$.*

Proof. We consider the set

$$\mathcal{A} = \{v \in C(\overline{\Omega}) : \max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} \ge f \text{ in } \Omega \text{ and } v \ge g \text{ on } \partial\Omega\},$$

where the inequality for the equation inside Ω is verified in the viscosity sense and the inequality on $\partial\Omega$ in the pointwise sense. Since $\Delta|x|^2 = 2n$ and $\Delta_\infty|x|^2 = 2$, we have that $\max\{-\Delta_{p_1} v, -\Delta_{p_2} v\} > 0$ for $v(x) = -|x|^2$. Hence we can choose K_1 such that the operator applied to $-K_1|x|^2$ is greater than $\sup f$ and then we can choose K_2 such that $K_2 - K_1|x|^2 \ge g(x)$ in $\partial\Omega$. We conclude that the function $K_2 - K_1|x|^2$ is in $\mathcal{A}$ for suitable K_1, K_2. Therefore the set $\mathcal{A}$ is not empty.

We define

$$u(x) = \inf_{v\in\mathcal{A}} v(x), \quad x \in \overline{\Omega}.$$

This infimum is finite since, as the comparison holds, we have $u(x) \geq -L_2 + L_1|x|^2$ for all $u \in \mathcal{A}$ for large L_1, L_2. The function u, being the infimum of supersolutions, is a supersolution. We already know that u is upper semicontinuous, as it is the infimum of continuous functions. Let us see that it is indeed a solution. Suppose not, then there exists $\phi \in C^2$ such that ϕ touches u at $x_0 \in \Omega$ strictly from above, but

$$\max\{-\Delta_{p_1}\phi(x_0), -\Delta_{p_2}\phi(x_0)\} > f(x_0).$$

Let us write

$$\phi(x) = \phi(x_0) + \nabla\phi(x_0)\cdot(x - x_0) + \tfrac{1}{2}\big\langle D^2\phi(x_0)(x - x_0), x - x_0\big\rangle + o\big(|x - x_0|^2\big).$$

We define $\widehat{\phi}(x) = \phi(x) - \delta$ for a small positive number δ. Then $\widehat{\phi} < u$ in a small neighborhood of x_0, contained in the set $\big\{x : \max\{-\Delta_{p_1}\phi(x), -\Delta_{p_2}\phi(x)\} > f(x)\big\}$, but $\widehat{\phi} \geq u$ outside this neighborhood, if we take δ small enough.

Now we can consider $v = \min\{\widehat{\phi}, u\}$. Since u is a viscosity supersolution in Ω and $\widehat{\phi}$ also is a viscosity supersolution in the small neighborhood of x_0, it follows that v is a viscosity supersolution. Moreover, on $\partial\Omega$, $v = u \geq g$. This implies $v \in \mathcal{A}$, but $v = \widehat{\phi} < u$ near x_0, which is a contradiction with the definition of u as the infimum in $\mathcal{A}$.

Finally, we want to prove that $u = g$ on $\partial\Omega$ and that boundary values are attained with continuity. To this end, we have to construct barriers for our operator. It is enough to prove that for every $x_0 \in \partial\Omega$ and $\varepsilon > 0$ there exists a supersolution such that $v \geq g$ on $\partial\Omega$ and $v(x_0) \leq g(x_0) + \varepsilon$, and that there exists a subsolution such that $v \leq g$ on $\partial\Omega$ and $v(x_0) \geq g(x_0) - \varepsilon$. We prove now the existence of the supersolution, and the subsolution can be obtained in a similar way.

Let us consider ϕ a radial function, $\phi(x) = \psi(r)$ with $\psi'(r) > 0$. Then

$$\Delta_\infty\phi = \psi'' \quad \text{and} \quad \Delta\phi = \psi'' + \frac{N-1}{r}\psi'$$

and we get

$$\begin{aligned}
\max_{i\in\{1,2\}}\big\{-\Delta_{p_i}\phi\big\} &= \max_{i\in\{1,2\}}\big\{-\alpha_i\,\Delta_\infty\phi - \theta_i\,\Delta\phi\big\}\\
&= \max_{i\in\{1,2\}}\Big\{-\alpha_i\,\psi'' - \theta_i\Big(\psi'' + \frac{N-1}{r}\psi'\Big)\Big\}\\
&= \max_{i\in\{1,2\}}\Big\{-\frac{p_i-2}{N+p_i}\psi'' - \frac{1}{N+p_i}\Big(\psi'' + \frac{N-1}{r}\psi'\Big)\Big\}\\
&= \max_{i\in\{1,2\}}\Big\{-\frac{p_i-1}{N+p_i}\psi'' - \frac{1}{N+p_i}\frac{N-1}{r}\psi'\Big\}.
\end{aligned}$$

We want this last expression to be greater than a positive constant.

To have a function of the form $\psi(r) = r^\gamma$ with $\gamma > 0$ that fulfills this, we need

$$\max_{i\in\{1,2\}}\Big\{-\frac{p_i-1}{N+p_i}\gamma(\gamma-1) - \frac{N-1}{N+p_i}\,\gamma\Big\}r^{\gamma-2} \geq c > 0.$$

Hence we have to choose γ according to

$$0 < \gamma < 1 - \frac{N-1}{p_i - 1}.$$

We have that such γ exists if $N < p_1$ or $N < p_2$. We will require that $\min\{p_1,p_2\} > N$, that is, $N < p_1, p_2$.

In this case we can consider $v(x) = K\phi(x-x_0) + g(x_0) + \varepsilon$ with K big enough. If $Kc > \sup f$, then v is a supersolution. We have that $v(x_0) = g(x_0) + \varepsilon$, it remains to prove that $v \geq g$ on $\partial\Omega$. Since g is continuous at x_0, there exists $\delta > 0$ such that $|g(x) - g(x_0)| < \varepsilon$ for every $x \in B_\delta(x_0)$. Then we have that $v \geq g$ on $\partial\Omega \cap B_\delta(x_0)$. Finally we can pick K such that $K\delta^\gamma + g(x_0) + \varepsilon > \sup g$, and we obtain $v \geq g$ on $\partial\Omega \cap B_\delta(x_0)^c$.

When $N \geq p_1$ or $N \geq p_2$, we can find (with similar computations) a barrier of the form $\psi(r) = -r^\gamma$ with $\gamma < 0$. Note that this function is not well-defined at 0. In this case, we have a barrier if the exterior ball condition holds. Given $x_0 \in \partial\Omega$, there exist $\lambda > 0$ and $y_0 \in \Omega^c$ such that $|x_0 - y_0| = \lambda$ and $B_\lambda(y_0) \subset \Omega^c$. We can consider $v(x) = K(\phi(x - y_0) - \phi(x_0 - y_0)) + g(x_0) + \varepsilon$ and pick K in a similar way to above. □

Now, we prove a version of the Hopf lemma for our equation. Note that since we deal with viscosity solutions, the normal derivative may not exist in a classical sense.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain with the interior ball condition and u a subsolution to (1-2) with $f \equiv 0$. Given $x_0 \in \partial\Omega$ such that $u(x_0) > u(x)$ for all $x \in \Omega$, we have*

$$\limsup_{t\to 0^+} \frac{u(x_0 - t\nu) - u(x_0)}{t} < 0.$$

where ν is exterior normal to $\partial\Omega$.

Proof. As the interior ball condition holds, we can assume there exists a ball centered at 0, contained in Ω that has x_0 in its boundary; that is, we have $B_r(0) \subset \Omega$ and $x_0 \in \partial B_r(0)$. Let us consider $\phi(x) = 1/\big(|x|^{N-2}\big) - 1/\big(r^{N-2}\big)$ if $N > 2$ and $\phi(x) = -\ln|x| + \ln(r)$ for $N = 2$. It is easy to check that

$$\Delta\phi = 0, \quad \Delta_\infty\phi \geq 0, \quad \text{in } B_r(0) \setminus \{0\}.$$

So we have

$$\begin{aligned} \max\{-\Delta_{p_1}\phi, -\Delta_{p_2}\phi\} &\leq 0 && \text{in } B_r(0)\setminus\{0\}, \\ \phi &\equiv 0 && \text{on } \partial B_r(0). \end{aligned}$$

As $u(x_0) > u(x)$ for all $x \in \Omega$, in particular on $\partial B_{r/2}(0)$, then there exists $\varepsilon > 0$ such that $u(x_0) - \varepsilon\phi \geq u$ on $\partial B_{r/2}(0)$. Therefore, by the comparison principle, we get $u(x_0) - \varepsilon\phi \geq u$ in $B_r(0) \setminus B_{r/2}(0)$ and the result follows. □

3. Unbalanced tug-of-war games with noise

In this section we introduce the game that we call *unbalanced tug-of-war game with noise*. First, let us describe the game without entering in mathematical details. It is a two-player zero-sum stochastic game. The game is played over a bounded open set $\Omega \subset \mathbb{R}^N$. An $\varepsilon > 0$ is given. Players I and II play as follows. At an initial time, they place a token at a point $x_0 \in \Omega$ and Player I chooses a coin between two possible ones (each of the two coins have different probabilities of getting a head). We think of this as choosing $i \in \{1, 2\}$. Now they play the *tug-of-war with noise* introduced in [Manfredi et al. 2012b] starting with the chosen coin. They toss the chosen coin, which is biased with probabilities α_i and β_i, where $\alpha_i + \beta_i = 1$ and $0 \le \alpha_i, \beta_i \le 1$. If they get heads (probability α_i), they toss a fair coin (with the same probability for heads and tails) and the winner of the toss moves the game position to any $x_1 \in B_\varepsilon(x_0)$ of his choice. On the other hand, if they get tails (probability β_i) the game state moves according to the uniform probability density to a random point $x_1 \in B_\varepsilon(x_0)$. Note that Player I chooses the probability of playing the usual tug-of-war game or moving at random with the choice of the first coin between two possibilities. Then they continue playing from x_1. At each turn Player I may change the choice of coin.

This procedure yields a sequence of game states $x_0, x_1, \ldots$. Once the game position leaves Ω, let's say at the τ-th step, the game ends. At that time the token will be on the compact boundary strip around Ω of width ε that we denote

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega \, : \, \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}.$$

The payoff is given by a *running payoff function* $f : \Omega \to \mathbb{R}$ and a *final payoff function* $g : \Gamma_\varepsilon \to \mathbb{R}$. At the end, Player II pays Player I the amount given by $g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$, that is, Player I will have earned

$$g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)$$

while Player II will have earned

$$-g(x_\tau) - \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n).$$

We can think of this as Player II paying Player I $\varepsilon^2 f(x_i)$ when the token leaves x_i, and $g(x_\tau)$ when the game ends.

A strategy S_{I} for Player I is a pair of collections of measurable mappings

$$S_{\mathrm{I}} = \big(\{\gamma^k\}_{k=0}^\infty, \{S_{\mathrm{I}}^k\}_{k=0}^\infty\big),$$

such that, given a partial history $(x_0, x_1, \ldots, x_k)$, Player I chooses coin 1 with probability

$$\gamma^k(x_0, x_1, \ldots, x_k) = \gamma \in [0, 1]$$

and the next game position is

$$S_{\mathrm{I}}^k(x_0, x_1, \ldots, x_k) = x_{k+1} \in B_\varepsilon(x_k)$$

if Player I wins the toss. Similarly, Player II plays according to a strategy

$$S_{\mathrm{II}} = \{S_{\mathrm{II}}^k\}_{k=0}^\infty.$$

Then, the next game position $x_{k+1} \in B_\varepsilon(x_k)$, given a partial history $(x_0, x_1, \ldots, x_k)$, is distributed according to the probability

$$\pi_{S_{\mathrm{I}}, S_{\mathrm{II}}}(x_0, x_1, \ldots, x_k, A) = \\ \frac{\beta |A \cap B_\varepsilon(x_k)|}{|B_\varepsilon(x_k)|} + \frac{\alpha}{2} \delta_{S_{\mathrm{I}}^k(x_0, x_1, \ldots, x_k)}(A) + \frac{\alpha}{2} \delta_{S_{\mathrm{II}}^k(x_0, x_1, \ldots, x_k)}(A),$$

where $\gamma = \gamma^k(x_0, x_1 \ldots, x_k)$, $\alpha = \alpha_1 \gamma + \alpha_2(1-\gamma)$, $\beta = \beta_1 \gamma + \beta_2(1-\gamma)$ and A is any measurable set (note that α and β depend on S_{I} and $(x_0, x_1, \ldots, x_k)$; we do not make this explicit to avoid overloading the notation). From now on, we shall omit k and simply denote the strategies by γ, S_{I} and S_{II}.

Let $\Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon \subset \mathbb{R}^n$ be equipped with the natural topology, and the σ-algebra $\mathcal{B}$ of the Lebesgue measurable sets. The space of all game sequences

$$H^\infty = \{x_0\} \times \Omega_\varepsilon \times \Omega_\varepsilon \times \cdots,$$

is a product space endowed with the product topology.

Let $\{\mathcal{F}_k\}_{k=0}^\infty$ denote the filtration of σ-algebras, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ which are defined as follows: $\mathcal{F}_k$ is the product σ-algebra generated by cylinder sets of the form $\{x_0\} \times A_1 \times \cdots \times A_k \times \Omega_\varepsilon \times \Omega_\varepsilon \cdots$ with $A_i \in \mathcal{B}$. For

$$\omega = (x_0, \omega_1, \ldots) \in H^\infty,$$

we define the coordinate processes

$$X_k(\omega) = \omega_k, \quad X_k : H^\infty \to \mathbb{R}^n, \quad k = 0, 1, \ldots$$

so that X_k is an $\mathcal{F}_k$-measurable random variable. Moreover, $\mathcal{F}_\infty = \sigma\left(\bigcup \mathcal{F}_k\right)$ is the smallest σ-algebra so that all X_k are $\mathcal{F}_\infty$-measurable. To denote the time when the game state reaches Γ_ε, we define a random variable

$$\tau(\omega) = \inf\{k : X_k(\omega) \in \Gamma_\varepsilon, k = 0, 1, \ldots\},$$

which is a stopping time relative to the filtration $\{\mathcal{F}_k\}_{k=0}^\infty$.

A starting point x_0 and the strategies S_{I} and S_{II} define (by Kolmogorov's extension theorem) a unique probability measure $\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}$ in H^∞ relative to the σ-algebra $\mathcal{F}^\infty$. We denote by $\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}$ the corresponding expectation.

Then, if S_{I} and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for Player I as

$$V_{x_0,\mathrm{I}}(S_{\mathrm{I}},S_{\mathrm{II}})=\begin{cases}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\big[g(X_\tau)+\varepsilon^2\sum_{n=1}^{\tau-1}f(x_n)\big] & \text{if the game ends a.s.,}\\ -\infty & \text{otherwise,}\end{cases}$$

and then the expected payoff for Player II as

$$V_{x_0,\mathrm{II}}(S_{\mathrm{I}},S_{\mathrm{II}})=\begin{cases}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\big[g(X_\tau)+\varepsilon^2\sum_{n=1}^{\tau-1}f(x_n)\big] & \text{if the game ends a.s.,}\\ +\infty & \text{otherwise.}\end{cases}$$

Note that we penalize both players when the game doesn't end almost surely.

The *value of the game for Player I* is given by

$$u_{\mathrm{I}}(x_0)=\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}V_{x_0,\mathrm{I}}(S_{\mathrm{I}},S_{\mathrm{II}}),$$

while the *value of the game for Player II* is given by

$$u_{\mathrm{II}}(x_0)=\inf_{S_{\mathrm{II}}}\sup_{S_{\mathrm{I}}}V_{x_0,\mathrm{II}}(S_{\mathrm{I}},S_{\mathrm{II}}).$$

When $u_{\mathrm{I}}=u_{\mathrm{II}}$ we say the game has a value $u:=u_{\mathrm{I}}=u_{\mathrm{II}}$. The values $u_{\mathrm{I}}(x_0)$ and $u_{\mathrm{II}}(x_0)$ are in a sense the best outcomes each player can expect when the game starts at x_0. For the measurability of the value functions we refer to [Maitra and Sudderth 1993; 1996].

Comment. It seems natural to consider a more general protocol to determine α in a prescribed closed set. It is clear that there are only two possible scenarios: At each turn, Player I wants to maximize the value of α and Player II wants to minimize it, or the converse. An expected value for α is obtained in each case assuming each player plays optimally. Depending on the value of α in each case, we are considering a game equivalent to the one that we described previously or another one where Player II gets the choice of the first coin, for certain values of α_i.

4. The game value function and the dynamic programming principle

In this section, we prove that the game has a value, that is, $u_{\mathrm{I}}=u_{\mathrm{II}}$ and that this value function satisfies the dynamic programming principle (DPP) given by

$$u(x)=\begin{cases}\varepsilon^2 f(x)+\max\limits_{i\in\{1,2\}}\left(\dfrac{\alpha_i}{2}\{\sup\limits_{B_\varepsilon(x)}u+\inf\limits_{B_\varepsilon(x)}u\}+\beta_i\fint_{B_\varepsilon(x)}u(y)\,dy\right), & x\in\Omega,\\ g(x), & x\in\Gamma_\varepsilon.\end{cases}$$

Let's see intuitively why this holds. At each step we have that Player I chooses $i \in \{1,2\}$ and then we have three possibilities:

- With probability $\alpha_i/2$, Player I moves the token, trying to maximize the expected outcome.
- With probability $\alpha_i/2$, Player II moves the token, trying to minimize the expected outcome.
- With probability β_i, the token moves at random.

Since Player I chooses i trying to maximize the expected outcome we obtain a $\max_{i\in\{1,2\}}$ in the DPP. Finally, the expected payoff at x is given by $\varepsilon^2 f(x)$ plus the expected payoff for the rest of the game.

Similar results are proved in [Antunović et al. 2012; Liu and Schikorra 2013; Luiro et al. 2013; Manfredi et al. 2012a; Peres et al. 2009; Ruosteenoja 2016]. Note that when $\alpha_1 = \alpha_2$ (and hence $\beta_1 = \beta_2$) Player I has no choice to make and we recover known results for tug-of-war games (with or without noise); see [Peres et al. 2009; Manfredi et al. 2012b]. We follow [Ruosteenoja 2016] where the idea is to prove the existence of a function satisfying the DPP and then that this function gives the game value. For the existence of a solution to the DPP we borrow some ideas from [Antunović et al. 2012], and for the uniqueness of such a solution and the existence of the value of the game we use martingales as in [Manfredi et al. 2012a]. However we will have two different cases: One, where the noise or the strict positivity (or negativity) of f assures us that the game ends almost surely, independently of the strategies adopted by the players. And another one where we have to handle the problem of getting strategies for the players to play almost optimally and to make sure that the game ends almost surely.

In what follows, $\Omega \subset \mathbb{R}^N$ is a bounded open set and $\varepsilon > 0$, $g : \Gamma_\varepsilon \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are bounded Borel functions such that $f \equiv 0$, $\inf_\Omega f > 0$ or $\sup_\Omega f < 0$.

Definition 4.1. A function u is sub-p_1-p_2-harmonious if

$$u(x) \le \begin{cases} \varepsilon^2 f(x) + \displaystyle\max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \Big\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\,dy \right) & x \in \Omega, \\ g(x) & x \in \Gamma_\varepsilon. \end{cases}$$

Analogously, a function u is super-p_1-p_2-harmonious if

$$u(x) \ge \begin{cases} \varepsilon^2 f(x) + \displaystyle\max_{i\in\{1,2\}} \left(\frac{\alpha_i}{2} \Big\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \Big\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\,dy \right), & x \in \Omega, \\ g(x), & x \in \Gamma_\varepsilon. \end{cases}$$

Finally, u is p_1-p_2-harmonious if it is both sub- and super-p_1-p_2-harmonious (i.e., the equality holds).

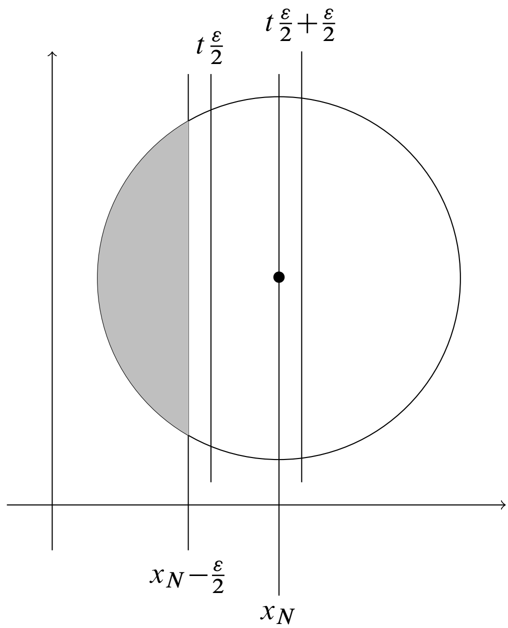

Figure 1. The partition considered in the proof of Lemma 4.2.

Here α_i and β_i are given by

$$\alpha_i = \frac{p_i - 2}{p_i + N} \quad \text{and} \quad \beta_i = \frac{N+2}{p_i + N}, \qquad i = 1, 2.$$

Our next task is to prove uniform bounds for these functions.

Lemma 4.2. *Sub-p_1-p_2-harmonious functions are uniformly bounded from above.*

Proof. We will consider the space partitioned along the x_N axis in strips of width $\varepsilon/2$. To this end we define

$$D = \frac{|\{y \in B_\varepsilon : y_N < -\varepsilon/2\}|}{|B_\varepsilon|} = \frac{|\{y \in B_1 : y_N < -1/2\}|}{|B_1|} \quad \text{and} \quad C = 1 - D.$$

The constant D gives the fraction of the ball $B_\varepsilon(x)$ covered by the shadowed section in Figure 1, $\{y \in B_\varepsilon : y_N < x_N - \varepsilon/2\}$, and C the fraction occupied by its complement.

Given $x \in \Omega$, let us consider $t \in \mathbb{R}$ such that $x_N < t\varepsilon/2 + \varepsilon/2$. We get

$$\Big\{y \in B_\varepsilon(x) : y_N < x_N - \frac{\varepsilon}{2}\Big\} \subset \Big\{z \in \mathbb{R}^N : z_N < t\frac{\varepsilon}{2}\Big\}.$$

Now, given u a sub-p_1-p_2-subharmonious function, we have that

$$u(x) \le \varepsilon^2 f(x) + \max_{i \in \{1,2\}} \bigg(\frac{\alpha_i}{2} \Big\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \Big\} + \beta_i \fint_{B_\varepsilon(x)} u(y)\, dy \bigg).$$

Now we can bound the terms in the right-hand side considering the partition given above, see Figure 1. We have

$$\begin{aligned}&\sup_{B_\varepsilon(x)} u \le \sup_{\Omega_\varepsilon} u,\\ &\inf_{B_\varepsilon(x)} u \le \sup_{\{y\in B_\varepsilon(x): y_N < x_N-\varepsilon/2\}} u \le \sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u,\end{aligned}$$

and

$$\begin{aligned}\fint_{B_\varepsilon(x)} u(y)\,dy &\le \left|\left\{y\in B_\varepsilon(x) : y_N \ge x_N - \frac{\varepsilon}{2}\right\}\right| \sup_{\{y\in B_\varepsilon(x):y_N\ge x_N-\varepsilon/2\}} u\\ &\qquad + \left|\left\{y\in B_\varepsilon(x) : y_N < x_N - \frac{\varepsilon}{2}\right\}\right| \sup_{\{y\in B_\varepsilon(x):y_N< x_N-\varepsilon/2\}} u\\ &\le C\sup_{\Omega_\varepsilon} u + D\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}u(x) &\le \varepsilon^2\sup_\Omega f + \max_{i\in\{1,2\}}\Big(\frac{\alpha_i}{2}\{\sup_{\Omega_\varepsilon} u + \sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u\}\\ &\qquad + \beta_i\big\{C\sup_{\Omega_\varepsilon} u + D\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u\big\}\Big)\\ &= \varepsilon^2\sup_\Omega f + \max_{i\in\{1,2\}}\Big(\Big\{\frac{\alpha_i}{2}+\beta_i C\Big\}\sup_{\Omega_\varepsilon} u\\ &\qquad + \Big\{\frac{\alpha_i}{2}+\beta_i D\Big\}\sup_{\Omega_\varepsilon\cap\{z_n<t\varepsilon/2\}} u\Big)\\ &= \varepsilon^2\sup_\Omega f + \max_{i\in\{1,2\}}\Big\{\frac{\alpha_i}{2}+\beta_i C\Big\}\sup_{\Omega_\varepsilon} u\\ &\qquad + \min_{i\in\{1,2\}}\Big\{\frac{\alpha_i}{2}+\beta_i D\Big\}\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u\\ &= \varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u + (1-K)\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u,\end{aligned}$$

where $K=\max_{i\in\{1,2\}}\{\alpha_i/2+\beta_i C\}$. We conclude that

$$\sup_{\Omega_\varepsilon\cap\{z_N<(t+1)\varepsilon/2\}} u_k \le \varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u_k + (1-K)\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u_k.$$

Then, inductively, we get

$$\begin{aligned}\sup_{\Omega_\varepsilon\cap\{z_N<(t+n)\varepsilon/2\}} u &\le \big(\varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u\big)\\ &\qquad \times\sum_{i=0}^{n-1}(1-K)^i + (1-K)^n\sup_{\Omega_\varepsilon\cap\{z_N<t\varepsilon/2\}} u.\end{aligned}$$

We assume without loss of generality that $\Omega\subset\{x\in\mathbb{R}^N : 0<x_N<R\}$ for some $R>0$. Now, we apply the formula for $t=0$ and n such that $n\varepsilon/2>R$, and get

$$\begin{aligned}\sup_{\Omega_\varepsilon} u &\le \big(\varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u\big)\sum_{i=0}^{n-1}(1-K)^i + (1-K)^n\sup_{\Gamma_\varepsilon} g\\ &= \big(\varepsilon^2\sup_\Omega f + K\sup_{\Omega_\varepsilon} u\big)\frac{1-(1-K)^n}{1-(1-K)} + (1-K)^n\sup_{\Gamma_\varepsilon} g\\ &= \frac{1-(1-K)^n}{K}\varepsilon^2\sup_\Omega f + \big(1-(1-K)^n\big)\sup_{\Omega_\varepsilon} u + (1-K)^n\sup_{\Gamma_\varepsilon} g.\end{aligned}$$

Hence, we obtain

$$(1-K)^n \sup_{\Omega_\varepsilon} u \le \frac{1-(1-K)^n}{K}\varepsilon^2 \sup_\Omega f + (1-K)^n \sup_{\Gamma_\varepsilon} g,$$

that gives the desired upper bound,

$$\sup_{\Omega_\varepsilon} u \le \frac{1-(1-K)^n}{K(1-K)^n}\varepsilon^2 \sup_\Omega f + \sup_{\Gamma_\varepsilon} g. \qquad \square$$

Analogously, there holds that super-p_1-p_2-harmonious functions are uniformly bounded from below.

Now with these results we can show that there exists a p_1-p_2-harmonious function as in [Liu and Schikorra 2015] applying Perron's Method. Remark that when f and g are bounded we can easily obtain the existence of sub-p_1-p_2-harmonious and super-p_1-p_2-harmonious functions.

We prefer a constructive argument (since we will use this construction again in what follows). Let $u_k : \Omega_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $u_k = g$ on Γ_ε for all $k \in \mathbb{N}$, then u_0 is sub-p_1-p_2-harmonious and

$$u_{k+1}(x) = \varepsilon^2 f(x) + \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_k + \inf_{B_\varepsilon(x)} u_k\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_k(y)\,dy\right),$$

for $x \in \Omega$.

Now, our main task is to show that this sequence converges uniformly. To this end, let us prove an auxiliary lemma where we borrow some ideas from [Antunović et al. 2012].

Lemma 4.3. *Let $x \in \Omega$, $n \in \mathbb{N}$ and fix λ_i for $i = 1, \dots, 4$, such that*

$$u_{n+1}(x) - u_n(x) \ge \lambda_1, \qquad \|u_n - u_{n-1}\|_\infty \le \lambda_2, \qquad \fint_{B_\varepsilon(x)} u_n - u_{n-1} \le \lambda_3,$$

$\lambda_3 < \lambda_1$, and $\lambda_4 > 0$. Then, for $\alpha := \max\{\alpha_1, \alpha_2\} > 0$, there exists $y \in B_\varepsilon(x)$ such that

$$\inf_{B_\varepsilon(x)} u_n \ge u_{n-1}(y) + \frac{2\lambda_1}{\alpha} - \lambda_2 - \frac{2(1-\alpha)\lambda_3}{\alpha} - \lambda_4.$$

Proof. Given $u_{n+1}(x) - u_n(x) \ge \lambda_1$, by the recursive definition, we have

$$\begin{aligned}\lambda_1 \le \varepsilon^2 f(x) &+ \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_n(y)\,dy\right) \\ &-\varepsilon^2 f(x) - \max_{i\in\{1,2\}}\left(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_{n-1} + \inf_{B_\varepsilon(x)} u_{n-1}\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_{n-1}(y)\,dy\right).\end{aligned}$$

Since $\max\{a,b\}-\max\{c,d\}\le\max\{a-c,b-d\}$, we get

$$\lambda_1 \le \max_{i\in\{1,2\}}\Bigg(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n - \sup_{B_\varepsilon(x)} u_{n-1} - \inf_{B_\varepsilon(x)} u_{n-1}\Big\} + \beta_i \fint_{B_\varepsilon(x)} u_n(y)-u_{n-1}(y)\,dy\Bigg).$$

Using that $\fint_{B_\varepsilon(x)} u_n - u_{n-1} \le \lambda_3$ we get

$$\max_{i\in\{1,2\}}\Big(\frac{\alpha_i}{2}\Big\{\sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n - \sup_{B_\varepsilon(x)} u_{n-1} - \inf_{B_\varepsilon(x)} u_{n-1}\Big\} + \beta_i\lambda_3\Big) \ge \lambda_1.$$

Now $\lambda_3<\lambda_1$ implies

$$\frac{\alpha}{2}\Big\{\sup_{B_\varepsilon(x)} u_n + \inf_{B_\varepsilon(x)} u_n - \sup_{B_\varepsilon(x)} u_{n-1} - \inf_{B_\varepsilon(x)} u_{n-1}\Big\} + (1-\alpha)\lambda_3 \ge \lambda_1.$$

We bound the difference between the suprema using $\|u_n-u_{n-1}\|_\infty\le\lambda_2$ and we obtain

$$\frac{\alpha}{2}\Big\{\inf_{B_\varepsilon(x)} u_n - \inf_{B_\varepsilon(x)} u_{n-1}\Big\} + \frac{\alpha\lambda_2}{2} + (1-\alpha)\lambda_3 \ge \lambda_1,$$

that is,

$$\inf_{B_\varepsilon(x)} u_n \ge \inf_{B_\varepsilon(x)} u_{n-1} + \frac{2\lambda_1}{\alpha} - \lambda_2 - \frac{2(1-\alpha)\lambda_3}{\alpha}.$$

Finally we can choose $y\in B_\varepsilon(x)$ such that

$$u_{n-1}(y) \le \inf_{B_\varepsilon(x)} u_{n-1} + \lambda_4,$$

which gives the desired inequality. □

Now we are ready to prove the uniform convergence and, therefore, the existence of a p_1-p_2-harmonious function.

Proposition 4.4. *The sequence u_k converges uniformly and the limit is a solution to the DPP.*

Proof. Since u_0 is sub-p_1-p_2-harmonious we have $u_1\ge u_0$. In addition, if $u_k\ge u_{k-1}$, by the recursive definition, we have $u_{k+1}\ge u_k$. Then, by induction, we obtain that the sequence of functions is an increasing sequence. Replacing $u_k\le u_{k+1}$ in the recursive definition we can see that u_k is a sub-p_1-p_2-harmonious function for all k. This gives us a uniform bound for u_k (independent of k). Hence, the u_k converge pointwise to a bounded Borel function u.

In the case $\alpha_1=\alpha_2=0$ we can pass to the limit on the recursion because of Fatou's lemma. Hence we assume $\alpha:=\max\{\alpha_1,\alpha_2\}>0$.

Now we show that the convergence is uniform. Suppose not. Observe that if $\|u_{n+1}-u_n\|_\infty\to 0$ we can extract a uniformly Cauchy subsequence, thus this subsequence converges uniformly to a limit u. This implies that the u_k converge uniformly to u, because of the monotonicity. By the recursive definition we have $\|u_{n+1}-u_n\|_\infty \geq \|u_n-u_{n-1}\|_\infty \geq 0$. Then, as we are assuming the convergence is not uniform, we have

$$\|u_{n+1}-u_n\|_\infty \to M \quad \text{and} \quad \|u_{n+1}-u_n\|_\infty \geq M$$

for some $M > 0$.

Let us observe that by Fatou's lemma it follows that

$$\lim_{n\to\infty} \int_\Omega u(y)-u_n(y)\,dy = 0,$$

so we can bound $⨍_{B_\varepsilon(x)} u_{n+1}-u_n$ uniformly on x.

Given $\delta > 0$, let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\|u_{n+1}-u_n\|_\infty \leq M+\delta \quad \text{and} \quad ⨍_{B_\varepsilon(x)} u_{n+1}-u_n < \delta,$$

for all $x \in \Omega$. We fix $k \geq 0$. Let $x_0 \in \Omega$ such that

$$u_{n_0+k}(x_0)-u_{n_0+k-1}(x_0) \geq M-\delta.$$

Now we apply Lemma 4.3 for $\lambda_1 = M-\delta,\ \lambda_2 = M+\delta,\ \lambda_3 = \delta$ and $\lambda_4 = \delta$ and we get

$$\begin{aligned} u_{n_0+k-1}(x_0), u_{n_0+k-1}(x_1) &\geq \inf_{B_\varepsilon(x_0)} u_{n_0+k-1} \\ &\geq u_{n_0+k-2}(x_1)+\frac{2(M-\delta)}{\alpha}-(M+\delta)-\frac{2(1-\alpha)}{\alpha}-\delta \\ &= u_{n_0+k-2}(x_1)+M\Big(\frac{2}{\alpha}-1\Big)-\delta\frac{4}{\alpha} \\ &\geq u_{n_0+k-2}(x_1)+M-\delta\frac{4}{\alpha}, \end{aligned}$$

for some $x_1 \in B_\varepsilon(x_0)$. Let us define $\xi = 4/\alpha$. If we repeat the argument for x_1, but now with $\lambda_1 = M-\delta\xi$, we obtain

$$u_{n_0+k-2}(x_1), u_{n_0+k-2}(x_2) \geq u_{n_0+k-3}(x_2)+M-\delta\big(\xi^2+\xi\big).$$

Inductively, we obtain a sequence $x_l,\ 1 \leq l \leq k-1$ such that

$$u_{n_0+k-l}(x_{l-1}), u_{n_0+k-l}(x_l) \geq u_{n_0+k-l-1}(x_l)+M-\delta\sum\nolimits_{t=1}^{l}\xi^t.$$

In Lemma 4.3 we require $\lambda_3 < \lambda_1$, so we need $k(\delta)$ to satisfy

$$M-\delta\sum_{t=1}^{l}\xi^t > \delta,$$

that is,

$$M > \delta \sum_{t=0}^{l} \xi^t$$

for $1 \le l \le k-1$. As the right-hand side term grows with l, it is enough to check it for $l = k-1$. Since

$$\sum_{t=1}^{l} \xi^t = \xi \frac{\xi^l - 1}{\xi - 1} \le \xi^{l+1} - 1 \le \xi^{l+1},$$

we obtain

$$u_{n_0+k-l}(x_{l-1}) \ge u_{n_0+k-l-1}(x_l) + M - \delta \xi^{l+1}.$$

Adding these inequalities for $1 \le l \le k-1$, and $u_{n_0+k}(x_0) - u_{n_0+k-1}(x_0) \ge M - \delta$ we get

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + kM - \delta \textstyle\sum_{l=0}^{k-1} \xi^{l+1}.$$

From the last inequality and the condition for $k(\delta)$, since

$$\sum_{l=0}^{k-1} \xi^{l+1} = \sum_{l=1}^{k} \xi^l \le \xi^{k+1},$$

we have

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + kM - \delta \xi^{k+1}$$

for all k such that $M > \delta \xi^{k+1}$. For $k + 1 = \lfloor \log(M/\delta)/\log \xi \rfloor$ this gives

$$u_{n_0+k}(x_0) \ge u_{n_0}(x_{k-1}) + \left(\frac{\log(M/\delta)}{\log \xi} - 3 \right) M,$$

which is a contradiction since

$$\lim_{\delta \to 0^+} \frac{\log(M/\delta)}{\log \xi} = \infty$$

and the sequence u_n is bounded. We have that $u_n \to u$ uniformly, therefore the result follows by passing to the limit in the recursive definition of u_n. In fact, that the uniform limit of the sequence u_n is a solution to the DPP is immediate since from the uniform convergence we can pass to the limit as $n \to \infty$ in all the terms of the DPP formula. □

Now we want to prove that this solution to the DPP, u, is unique and that it gives the value of the game. To this end we have to take special care of the fact that the game ends (or not) almost surely. First, we deal with the case $\beta_1, \beta_2 > 0$, $\sup_\Omega f < 0$ or $\inf_\Omega f > 0$. We apply a martingale argument to handle these cases. In other cases we also use the construction of the sequence u_k.

Lemma 4.5. *Assume that* $\beta_1, \beta_2 > 0$, $\sup f < 0$ *or* $\inf f > 0$. *Then, if the function* v *is a* p_1-p_2-*harmonious function for* g_v *and* f_v *such that* $g_v \le g_{u_{\mathrm{I}}}$ *and* $f_v \le f_{u_{\mathrm{I}}}$, *then* $v \le u_{\mathrm{I}}$.

Proof. By choosing a strategy according to the points where the maximal values of v are attained, we show that Player I can obtain a certain process which is a submartingale. The optional stopping theorem then implies that the expectation of the process under this strategy is bounded by v. Moreover, this process provides a lower bound for u_{I}.

Player II follows any strategy and Player I follows a strategy S_{I}^0 such that at $x_{k-1} \in \Omega$ he chooses γ to be 1 if

$$\begin{aligned}\frac{\alpha_1}{2}\Big\{\sup_{y\in B_\varepsilon(x)} u(y) + \inf_{y\in B_\varepsilon(x)} u(y)\Big\} + \beta_1 \fint_{B_\varepsilon(x)} u(y)\,dy \\ > \frac{\alpha_2}{2}\Big\{\sup_{y\in B_\varepsilon(x)} u(y) + \inf_{y\in B_\varepsilon(x)} u(y)\Big\} + \beta_2 \fint_{B_\varepsilon(x)} u(y)\,dy\end{aligned}$$

and 0 otherwise, and he chooses to step to a point that almost maximizes v, that is, to a point $x_k \in B_\varepsilon(x_{k-1})$ such that

$$v(x_k) \ge \sup\nolimits_{B_\varepsilon(x_{k-1})} v - \eta 2^{-k}$$

for some fixed $\eta > 0$. We start from the point x_0. It follows that

$$\begin{aligned}&\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}\Big[v(x_k) + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \,:\, x_0,\dots,x_{k-1}\Big] \\ &\qquad \ge \max_{i\in\{1,2\}} \Big(\frac{\alpha_i}{2}\Big\{\inf_{B_\varepsilon(x_{k-1})} v - \eta 2^{-k} + \sup_{B_\varepsilon(x_{k-1})} v\Big\} + \beta_i \fint_{B_\varepsilon(x_{k-1})} v\,dy\Big) \\ &\qquad\qquad + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \\ &\qquad \ge v(x_{k-1}) - \varepsilon^2 f(x_{k-1}) - \eta 2^{-k} + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k} \\ &\qquad = v(x_{k-1}) + \varepsilon^2 \sum_{n=0}^{k-2} f(x_n) - \eta 2^{-k+1},\end{aligned}$$

where we have estimated the strategy of Player II by inf and used the fact that v is p_1-p_2-harmonious. Thus

$$M_k = v(x_k) + \varepsilon^2 \sum_{n=0}^{k-1} f(x_n) - \eta 2^{-k}$$

is a submartingale.

Now we observe the following: if $\beta_1, \beta_2 > 0$ then the game ends almost surely and we can continue (see below). If $\sup f < 0$ the fact that M_k is a submartingale implies that the game ends in a finite number of moves (that can be estimated). In the case $\inf f > 0$ if the game does not end in a finite number of moves then we have to play until the accumulated payoff (recall that f gives the running payoff) is greater than v and then choose a strategy that ends the game almost surely (for example pointing to some prescribed point x_0 outside Ω).

Since $g_v \le g_{u_{\mathrm{I}}}$ and $f_v \le f_{u_{\mathrm{I}}}$, we deduce

$$\begin{aligned} u_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[g_{u_{\mathrm{I}}^{\varepsilon}}(x_\tau) + \varepsilon^2 \sum\nolimits_{n=0}^{\tau-1} f(x_n)\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}\Big[g_v(x_\tau) + \varepsilon^2 \sum\nolimits_{n=0}^{\tau-1} f(x_n) - \eta 2^{-\tau}\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \liminf_{k\to\infty} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}\Big[v(x_{\tau\wedge k}) + \varepsilon^2 \sum\nolimits_{n=0}^{(\tau-1)\wedge k} f(x_n) - \eta 2^{-(\tau\wedge k)}\Big] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}_{S^0_{\mathrm{I}},S_{\mathrm{II}}}[M_0] = v(x_0) - \eta, \end{aligned}$$

where $(\tau-1)\wedge k = \min(\tau-1, k)$, and we used Fatou's lemma as well as the optional stopping theorem for M_k. Since η is arbitrary, this proves the claim. $\square$

A symmetric result can be proved for u_{II}, hence we obtain the following result:

Theorem 4.6. *Assume that* $\beta_1, \beta_2 > 0$, $\sup f < 0$ *or* $\inf f > 0$. *Then there exists a unique* p_1-p_2*-harmonious function. Even more the game has a value, that is* $u_{\mathrm{I}} = u_{\mathrm{II}}$, *which coincides with the unique* p_1-p_2*-harmonious function.*

Proof. Let u be a p_1-p_2-harmonious function, which exits, as we know from Proposition 4.4. From the definition of the game values we know that $u_{\mathrm{I}} \le u_{\mathrm{II}}$. Then by Lemma 4.5 we have that

$$u_{\mathrm{I}} \le u_{\mathrm{II}} \le u \le u_{\mathrm{I}}.$$

Thus $u_{\mathrm{I}} = u_{\mathrm{II}} = u$. Since we can repeat the argument for any p_1-p_2-harmonious function, uniqueness follows. $\square$

Remark 4.7. Note that if we have a sub-p_1-p_2-harmonious function u, then v given by $v = u - C$ in Ω and $v = u$ in Γ_ε is sub-p_1-p_2-harmonious for every constant $C > 0$. In this way we can obtain a sub-p_1-p_2-harmonious function smaller than any super-p_1-p_2-harmonious function, and then if we start the above construction with this function we get the smallest p_1-p_2-harmonious function. That is, there exists a minimal p_1-p_2-harmonious function. We can use the analogous construction to get the largest p_1-p_2-harmonious function (the maximal p_1-p_2-harmonious function).

We now tackle the remaining case in which $f \equiv 0$ and one of the β_i is 0 (that is the same as saying that one of the α_i is equal to 1).

Theorem 4.8. *There exists a unique p_1-p_2-harmonious function when $\alpha_1 = 1$, $\alpha_2 > 0$ and $f \equiv 0$.*

Proof. Suppose not, then we have u and v such that

$$v(x) = \max\left\{\frac{1}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big), \frac{\alpha}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big) + \beta \fint_{B_\varepsilon(x)} v\right\}$$

$$u(x) = \max\left\{\frac{1}{2}\Big(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\Big), \frac{\alpha}{2}\Big(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u\Big) + \beta \fint_{B_\varepsilon(x)} u\right\}$$

in Ω and

$$u = v = g$$

on Γ_ε with

$$\|u - v\|_\infty = M > 0.$$

As we observed in Remark 4.7 we can assume $u \geq v$ (just take v as the minimal solution to the DPP). Now we want to build a point where the difference between u and v is almost attained and v has a large variation in the ball of radius ε around this point (all this has to be carefully quantified). First, we apply a compactness argument. We know that

$$\overline{\Omega}_{\varepsilon/4} \subset \bigcup_{x\in\Omega} B_{\varepsilon/2}(x).$$

As $\overline{\Omega}_{\varepsilon/4}$ is compact, there exists y_i such that

$$\overline{\Omega}_{\varepsilon/4} \subset \bigcup_{i=1}^{k} B_{\varepsilon/2}(y_i).$$

Let $A = \big\{i \in \{1, \dots, k\} : u \text{ or } v \text{ are not constant on } B_{\varepsilon/2}(y_i)\big\}$ and let $\lambda > 0$ such that, for every $i \in A$,

$$\sup_{B_\varepsilon(y_i)} u - \inf_{B_\varepsilon(y_i)} u > \left(4 + \frac{4\beta}{\alpha}\right)\lambda \quad \text{or} \quad \sup_{B_\varepsilon(y_i)} v - \inf_{B_\varepsilon(y_i)} v > 2\lambda.$$

We fix this λ. Now, for every $\delta > 0$ such that $\lambda > \delta$ and $M > \delta$, let $z \in \Omega$ such that $M - \delta < u(z) - v(z)$. Let

$$O = \{x \in \Omega : u(x) = u(z) \text{ and } v(x) = v(z)\} \subset \Omega.$$

Take $\bar{z} \in \partial O \subset \overline{\Omega}$. Letting i_0 be such that $\bar{z} \in B_{\varepsilon/2}(y_{i_0})$, we have

$$B_{\varepsilon/2}(y_{i_0}) \cap O \neq \varnothing \quad \text{and} \quad B_{\varepsilon/2}(y_{i_0}) \cap O^c \neq \varnothing,$$

hence $i_0 \in A$. Let $x_0 \in B_{\varepsilon/2}(y_{i_0}) \cap O$. In this way we have obtained x_0 such that $u(x_0) - v(x_0) > M - \delta$ and one of the following holds:

$$\sup_{B_\varepsilon(x_0)} u - \inf_{B_\varepsilon(x_0)} u > \left(4 + \frac{4\beta}{\alpha}\right)\lambda, \tag{1}$$

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda. \tag{2}$$

Let us show that in fact the second statement must hold. Suppose not, then the first holds and we have

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v \leq 2\lambda.$$

Given that

$$v(x_0) \geq \tfrac{1}{2}\big(\sup_{B_\varepsilon(x_0)} v + \inf_{B_\varepsilon(x_0)} v\big),$$

we get

$$v(x_0) + \lambda \geq \sup_{B_\varepsilon(x_0)} v.$$

Hence

$$v(x_0) + \lambda + M \geq \sup_{B_\varepsilon(x_0)} v + M \geq \sup_{B_\varepsilon(x_0)} u.$$

Further, since

$$u(x_0) - v(x_0) > M - \delta > M - \lambda,$$

we get

$$u(x_0) + 2\lambda > \sup_{B_\varepsilon(x_0)} u,$$

and

$$\sup_{B_\varepsilon(x_0)} u > \inf_{B_\varepsilon(x_0)} u + \left(4 + \frac{4\beta}{\alpha}\right)\lambda.$$

Hence

$$u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda > \inf_{B_\varepsilon(x_0)} u.$$

If we bound the integral by the value of the supremum we can control all the terms in the DPP in terms of $u(x_0)$. We have

$$\begin{aligned}
u(x_0) &= \max\left\{\frac{1}{2}\big(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u\big), \frac{\alpha}{2}\big(\sup_{B_\varepsilon(x_0)} u + \inf_{B_\varepsilon(x_0)} u\big) + \beta \fint_{B_\varepsilon(x_0)} u\right\} \\
&< \max\left\{\frac{1}{2}\left(u(x_0) + 2\lambda + u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda\right),\right. \\
&\qquad\qquad \left.\frac{\alpha}{2}\left(u(x_0) + 2\lambda + u(x_0) - \left(2 + \frac{4\beta}{\alpha}\right)\lambda\right) + \beta\big(u(x_0) + 2\lambda\big)\right\} \\
&< \max\left\{u(x_0) - \frac{4\beta}{\alpha}\lambda, u(x_0)\right\} = u(x_0),
\end{aligned}$$

which is a contradiction. Hence, the second condition must hold, that is, we have

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda.$$

Applying the DPP we get

$$v(x_0) \geq \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x_0)} v + \inf_{B_\varepsilon(x_0)} v\Big)$$

together with the fact that

$$\sup_{B_\varepsilon(x_0)} v - \inf_{B_\varepsilon(x_0)} v > 2\lambda,$$

and then we conclude that

$$v(x_0) > \inf_{B_\varepsilon(x_0)} v + \lambda.$$

We have proved that there exists x_0 such that

$$v(x_0) > \inf_{B_\varepsilon(x_0)} v + \lambda \quad \text{and} \quad u(x_0) - v(x_0) > M - \delta.$$

Now we are going to build a sequence of points where the difference between u and v is almost maximal and where the value of v decreases by at least λ in every step. Applying the DPP to $M - \delta < u(x_0) - v(x_0)$ and bounding the difference of the suprema by M we get:

$$M - \frac{2}{\alpha}\delta + \inf_{B_\varepsilon(x_0)} v < \inf_{B_\varepsilon(x_0)} u.$$

Let x_1 be such that $v(x_0) > v(x_1) + \lambda$ and $\inf_{B_\varepsilon(x_0)} v + \delta > v(x_1)$. We get

$$M - \Big(1 + \frac{2}{\alpha}\Big)\delta + v(x_1) < u(x_1).$$

To repeat this construction we need the following two results:

- In the last inequality, if δ is small enough $u(x_1) \neq v(x_1)$, hence $x_1 \in \Omega$.
- We know that $2v(x_1) \geq \inf_{B_\varepsilon(x_1)} v + \sup_{B_\varepsilon(x_1)} v > v(x_0) + \inf_{B_\varepsilon(x_1)} v$. Hence, since $v(x_0) > v(x_1) + \lambda$, we get $v(x_1) > \inf_{B_\varepsilon(x_1)} v + \lambda$.

Then we get

$$v(x_{n-1}) > v(x_n) + \lambda$$

and

$$M - \bigg(\sum_{k=0}^{n} \Big(\frac{2}{\alpha}\Big)^k\bigg)\delta + v(x_n) < u(x_n).$$

We can repeat this argument as long as

$$M - \Big(\sum_{k=0}^{n} \Big(\frac{2}{\alpha}\Big)^k \Big) \delta > 0,$$

which is a contradiction with the fact that v is bounded. □

Now we want to show that this unique function that satisfies the DPP is the game value. The key point of the proof is to construct a strategy based on the approximating sequence that we used to construct the solution.

Theorem 4.9. *Given $f \equiv 0$, the game has a value, that is $u_{\mathrm{I}} = u_{\mathrm{II}}$, which coincides with the unique p_1-p_2-harmonious function.*

Proof. Let u be the unique p_1-p_2-harmonious function (the uniqueness is given by Theorems 4.6 and 4.8). We will show that $u \le u_{\mathrm{I}}$. The analogous result can be proved for u_{II}, completing the proof.

Let us consider a sub-p_1-p_2-harmonious function u_0 which is smaller than $\inf_\Omega g$ at every point in Ω. Starting with this u_0 we build the corresponding u_k as in Proposition 4.4. We have that $u_k \to u$ as $k \to \infty$.

Now, given $\delta > 0$, let $n > 0$ be such that $u_n(x_0) > u(x_0) - \delta/2$. We build a strategy S_{I}^0 for Player I: in the first n moves, given x_{k-1} he will choose to move to a point that almost maximizes u_{n-k}, that is, he chooses $x_k \in B_\varepsilon(x_{k-1})$ such that

$$u_{n-k}(x_k) > \sup_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n}$$

and chooses γ in order to maximize

$$\frac{\alpha_i}{2}\Big\{ \inf_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n} + \sup_{B_\varepsilon(x_{k-1})} u_{n-k} \Big\} + \beta_i \fint_{B_\varepsilon(x_{k-1})} u_{n-k}\, dy.$$

After the first n moves he will follow a strategy that ends the game almost surely (for example pointing in a fix direction).

We have

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^0, S_{\mathrm{II}}}\Big[u_{n-k}(x_k) + \frac{k\delta}{2n} \,:\, x_0, \dots, x_{k-1} \Big] \\
&\quad \ge \max_{i\in\{1,2\}} \Big(\frac{\alpha_i}{2}\Big\{ \inf_{B_\varepsilon(x_{k-1})} u_{n-k} - \frac{\delta}{2n} + \sup_{B_\varepsilon(x_{k-1})} u_{n-k} \Big\} + \beta_i \fint_{B_\varepsilon(x_{k-1})} u_{n-k}\, dy \Big) + \frac{k\delta}{2n} \\
&\quad \ge u_{n-k+1}(x_{k-1}) + \frac{(k-1)\delta}{2n},
\end{aligned}$$

where we have estimated the strategy of Player II by inf and used the construction for the u_k. Thus

$$M_k = \begin{cases} u_{n-k}(x_k) + \dfrac{k\delta}{2n} - \dfrac{\delta}{2} & \text{for } 0 \le k \le n, \\ \inf_\Omega g & \text{for } k > n, \end{cases}$$

is a submartingale.

Now we have

$$\begin{aligned} u_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[g(x_\tau)] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}}[g(x_\tau)] \\ &\ge \inf_{S_{\mathrm{II}}} \liminf_{k\to\infty} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}}[M_k] \\ &\ge \inf_{S_{\mathrm{II}}} \mathbb{E}_{S^0_{\mathrm{I}}, S_{\mathrm{II}}}[M_0] = u_n(x_0) - \frac{\delta}{2} > u(x_0) - \delta, \end{aligned}$$

where $\tau \wedge k = \min(\tau, k)$, and we used the optional stopping theorem for M_k. Since δ is arbitrary, this proves the claim. $\square$

As an immediate corollary of our results in this section we obtain a comparison result for solutions to the DPP.

Corollary 4.10. *If v and u are p_1-p_2-harmonious functions for g_v, f_v and g_u, f_u, respectively such that $g_v \ge g_u$ and $f_v \ge f_u$, then $v \ge u$.*

5. Properties of harmonious functions and convergence

First, we show some properties of p_1-p_2-harmonious functions that we need to prove convergence as $\varepsilon \to 0$. We want to apply the following Arzelà–Ascoli-type lemma. For its proof, see [Manfredi et al. 2012b, Lemma 4.2].

Lemma 5.1. *Let $\{u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that*

(1) *there exists $C > 0$ such that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*

(2) *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_0$,*

$$|u_\varepsilon(x) - u_\varepsilon(y)| < \eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \quad \text{uniformly in} \quad \overline{\Omega},$$

as $\varepsilon \to 0$.

So our task now is to show that the family u_ε satisfies the hypotheses of the previous lemma. To this end we need some bounds on the expected exit time in the case of a player choose a certain strategy.

Let us start showing that u_ε are uniformly bounded. In Lemma 4.2 we obtained a bound for the value of the game for a fixed ε; here we need a bound independent of ε. To this end, let us define what we understand by pulling in one direction: we fix a direction, that is, a unitary vector v and at each turn of the game the player strategy is given as $S(x_{k-1}) = x_{k-1} + (\varepsilon - \varepsilon^3/2^k)v$.

Lemma 5.2. *In a game where a player pulls in a fixed direction the expectation of the exit time is bounded above by*

$$\mathbb{E}[\tau] \le C\varepsilon^{-2}$$

for some $C > 0$ independent of ε.

Proof. First, let us assume without loss of generality that

$$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_n < R\}$$

and that the direction that the player is pulling to is $-e_n$. Then

$$M_k = (x_k)_n + \frac{\varepsilon^3}{2^k}$$

is a supermartingale. Indeed, if the random move occurs, then we know that the expectation of $(x_{k+1})_n$ is equal to $(x_k)_n$. If the tug-of-war game is played we know that with probability one half, $(x_{k+1})_n = (x_k)_n - \varepsilon + \varepsilon^3/2^k$ and if the other player moves $(x_{k+1})_n \le (x_k)_n + \varepsilon$, so the expectation is less than or equal to $(x_k)_n + \varepsilon^3/2^{k+1}$.

Let us consider the expectation for $(M_{k+1} - M_k)^2$. If the random walk occurs, then the expectation is $\varepsilon^2/(n+2) + o(\varepsilon^2)$. Indeed,

$$\fint_{B_\varepsilon} x_n^2 = \frac{1}{n}\fint_{B_\varepsilon} |x|^2 = \frac{1}{\varepsilon^n n|B_1|}\int_0^\varepsilon r^2|\partial B_r|\,dr = \frac{|\partial B_1|}{\varepsilon^n n|B_1|}\int_0^\varepsilon r^{n+1}\,dr = \frac{\varepsilon^2}{n+2}.$$

If the tug-of-war occurs we know that with probability one half $(x_{k+1})_n = (x_k)_n - \varepsilon + \varepsilon^3/2^k$, so the expectation is greater than or equal to $\varepsilon^2/3$.

Let us consider the expectation for $M_k^2 - M_{k+1}^2$. We have

$$\mathbb{E}[M_k^2 - M_{k+1}^2] = \mathbb{E}[(M_{k+1} - M_k)^2] + 2\mathbb{E}[(M_k - M_{k+1})M_{k+1}].$$

As $(x_k)_n$ is positive, we have $2\mathbb{E}[(M_k - M_{k+1})M_{k+1}] \ge 0$. Then

$$\mathbb{E}[M_k^2 - M_{k+1}^2] \ge \varepsilon^2/(n+2),$$

so $M_k^2 + k\varepsilon^2/(n+2)$ is a supermartingale. According to the optional stopping theorem for supermartingales,

$$\mathbb{E}\left[M_{\tau\wedge k}^2 + \frac{(\tau\wedge k)\varepsilon^2}{n+2}\right] \le M_0^2.$$

We have

$$\mathbb{E}[(\tau\wedge k)]\frac{\varepsilon^2}{n+2} \le M_0^2 - E[M_{\tau\wedge k}^2] \le M_0^2.$$

Taking the limit in k, we get a bound for the expected exit time,

$$\mathbb{E}[\tau] \le (n+2)M_0^2\varepsilon^{-2},$$

so the statement holds for $C = (n+2)R^2$. □

Lemma 5.3. *An f-p_1-p_2-harmonious function u_ε with boundary values g satisfies*

(5-1) $$\inf_{y\in\Gamma_\varepsilon} g(y) + C\inf_{y\in\Omega} f(y) \le u_\varepsilon(x) \le \sup_{y\in\Gamma_\varepsilon} g(y) + C\sup_{y\in\Omega} f(y).$$

Proof. We use the connection to games. Let one of the players choose a strategy of pulling in a fixed direction. Then

$$\mathbb{E}[\tau] \le C\varepsilon^{-2},$$

and this gives the upper bound

$$\begin{aligned}\mathbb{E}\left[g(X_\tau) + \varepsilon^2\sum_{n=0}^{\tau-1} f(X_n)\right] &\le \sup_{y\in\Gamma_\varepsilon} g(y) + E[\tau]\varepsilon^2 \sup_{y\in\Omega} f(y)\\ &\le \sup_{y\in\Gamma_\varepsilon} g(y) + C\sup_{y\in\Omega} f(y).\end{aligned}$$

The lower bound follows analogously. □

Let us show now that the u_ε are asymptotically uniformly continuous. First we need a lemma that bounds the expectation for the exit time when one player is pulling towards a fixed point.

Let us consider an annular domain $B_R(y)\setminus \overline{B}_\delta(y)$ and a game played inside. In each round the token starts at a certain point x, an ε-step tug-of-war is played inside $B_R(y)$ or the token moves at random with uniform probability in $B_R(y)\cap B_\varepsilon(x)$. If an ε-step tug-of-war is played, there is a probability of one half for either player to move the token to a point of his choosing in $B_R(y)\cap B_\varepsilon(x)$. We can think there is a third player choosing whether the ε-step tug-of-war or the random move occurs. The game ends when the position reaches $\overline{B}_\delta(y)$, that is, when $x_{\tau^*}\in \overline{B}_\delta(y)$.

Lemma 5.4. *Assume that one of the players pulls towards y in the game described above. Then, no mater how many times the tug-of-war is played or the random move is done, the exit time verifies*

$$\mathbb{E}^{x_0}(\tau^*) \leq \big(C(R/\delta)\operatorname{dist}(\partial B_\delta(y), x_0) + o(1)\big)/\varepsilon^2, \tag{5-2}$$

for $x_0 \in B_R(y) \setminus \overline{B}_\delta(y)$. Here τ^* is the exit time in the previously described game and $o(1) \to 0$ as $\varepsilon \to 0$ can be taken as depending only on δ and R.

Proof. Let us denote

$$h_\varepsilon(x) = \mathbb{E}^x(\tau).$$

By symmetry, we know that h_ε is radial and it is easy to see that it is increasing in $r = |x - y|$. If we assume that the other player wants to maximize the expectation for the exit time and that the random move or tug-of-war is chosen in the same way, we have that the function h_ε satisfies a dynamic programming principle,

$$h_\varepsilon(x) = \max\left\{\frac{1}{2}\Big(\max_{B_\varepsilon(x)\cap B_R(y)} h_\varepsilon + \min_{B_\varepsilon(x)\cap B_R(y)} h_\varepsilon\Big), \fint_{B_\varepsilon(x)\cap B_R(y)} h_\varepsilon\, dz\right\} + 1,$$

by the above assumptions and that the number of steps always increases by 1 when making a step. Further, we denote $v_\varepsilon(x) = \varepsilon^2 h_\varepsilon(x)$ and obtain

$$v_\varepsilon(x) = \max\left\{\frac{1}{2}\Big(\sup_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon + \inf_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon\Big), \fint_{B_\varepsilon(x)\cap B_R(y)} v_\varepsilon\, dz\right\} + \varepsilon^2.$$

This induces us to look for a function v such that

$$v(x) \geq \fint_{B_\varepsilon(x)} v\, dz + \varepsilon^2 \quad \text{and} \quad v(x) \geq \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big) + \varepsilon^2. \tag{5-3}$$

Note that for small ε this is a sort of discrete version of the following inequalities:

$$\begin{cases} \Delta v(x) \leq -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y), \\ \Delta_\infty v(x) \leq -2, & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y). \end{cases} \tag{5-4}$$

This leads us to consider the problem

$$\begin{cases} \Delta v(x) = -2(n+2), & x \in B_{R+\varepsilon}(y) \setminus \overline{B}_\delta(y), \\ v(x) = 0, & x \in \partial B_\delta(y), \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial B_{R+\varepsilon}(y), \end{cases} \tag{5-5}$$

where $\partial v/\partial \nu$ refers to the normal derivative. The solution to this problem is radially symmetric and strictly increasing in $r = |x - y|$. It takes the form

$$v(r) = \begin{cases} -ar^2 - br^{2-N} + c & \text{if } N > 2, \text{ and} \\ -ar^2 - b\log(r) + c & \text{if } N = 2. \end{cases}$$

If we extend this v to $B_\delta(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$, it satisfies $\Delta v(x)=-2(N+2)$ in $B_{R+\varepsilon}(y)\setminus\overline{B}_{\delta-\varepsilon}(y)$. We know that

$$\Delta_\infty v = v_{rr} \le v_{rr} + \frac{N-1}{r} v_r = \Delta v.$$

Thus, v satisfies the inequalities (5-4). Then, the classical calculation shows that v satisfies (5-3) for each $B_\varepsilon(x) \subset B_{R+\varepsilon}(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$.

In addition, as v is increasing in r, it holds for each $x \in B_R(y) \setminus \overline{B}_\delta(y)$ that

$$\fint_{B_\varepsilon(x)\cap B_R(y)} v\,dz \le \fint_{B_\varepsilon(x)} v\,dz \le v(x)-\varepsilon^2$$

and

$$\tfrac{1}{2}\Big(\sup_{B_\varepsilon(x)\cap B_R(y)} v + \inf_{B_\varepsilon(x)\cap B_R(y)} v\Big) \le \tfrac{1}{2}\Big(\sup_{B_\varepsilon(x)} v + \inf_{B_\varepsilon(x)} v\Big) \le v(x)-\varepsilon^2.$$

It follows that

$$\begin{aligned}
&\mathbb{E}[v(x_k)+k\varepsilon^2 \,:\, x_0,\dots,x_{k-1}]\\
&\qquad \le \max\left\{\tfrac{1}{2}\Big(\sup_{B_\varepsilon(x_{k-1})\cap B_R(y)} v + \inf_{B_\varepsilon(x_{k-1})\cap B_R(y)} v\Big), \fint_{B_\varepsilon(x_{k-1})\cap B_R(y)} v\,dz\right\}\\
&\qquad \le v(x_{k-1}) + (k-1)\varepsilon^2,
\end{aligned}$$

if $x_{k-1} \in B_R(y) \setminus \overline{B}_\delta(y)$. Thus $v(x_k)+k\varepsilon^2$ is a supermartingale, and the optional stopping theorem yields

$$\mathbb{E}^{x_0}[v(x_{\tau^*\wedge k}) + (\tau^*\wedge k)\varepsilon^2] \le v(x_0). \tag{5-6}$$

Because $x_{\tau^*} \in \overline{B}_\delta(y) \setminus \overline{B}_{\delta-\varepsilon}(y)$, we have

$$0 \le -\mathbb{E}^{x_0}[v(x_{\tau^*})] \le o(1).$$

Furthermore, the estimate

$$0 \le v(x_0) \le C(R/\delta)\,\mathrm{dist}\big(\partial B_\delta(y), x_0\big)$$

holds for the solutions of (5-5). Thus, by passing to the limit as $k\to\infty$, we obtain

$$\varepsilon^2\mathbb{E}^{x_0}[\tau^*] \le v(x_0) - \mathbb{E}[u(x_{\tau^*})] \le C(R/\delta)\big(\mathrm{dist}(\partial B_\delta(y), x_0) + o(1)\big).$$

This completes the proof. □

Next we derive a uniform bound and estimate for the asymptotic continuity of the family of p_1-p_2-harmonious functions.

We assume here that Ω satisfies an exterior sphere condition: for each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$.

Lemma 5.5. *Let g be Lipschitz continuous in Γ_ε and f Lipschitz continuous in Ω such that $f \equiv 0$, $\inf f > 0$ or $\sup f < 0$. The p_1-p_2-harmonious function u_ε with data g and f satisfies*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le \operatorname{Lip}(g)\big(|x-y| + \delta\big) + C(R/\delta)\big(|x-y| + o(1)\big)(1 + \|f\|_\infty) + \widetilde{C}\operatorname{Lip}(f)|x-y|, \tag{5-7}$$

for every small enough $\delta > 0$ and for every two points $x, y \in \Omega \cup \Gamma_\varepsilon$. Here $o(1)$ can be taken depending only on δ and R.

Proof. The case $x, y \in \Gamma_\varepsilon$ is clear. Thus, we can concentrate on the cases $x \in \Omega$ and $y \in \Gamma_\varepsilon$ as well as $x, y \in \Omega$.

We use the connection to games. Suppose first that $x \in \Omega$ and $y \in \Gamma_\varepsilon$. By the exterior sphere condition, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. Now Player I chooses a strategy of pulling towards z, denoted by S_{I}^z. Then

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for a sufficiently large constant C, independent of ε. Indeed,

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\big[|x_k - z| : x_0, \dots, x_{k-1}\big] \\
&\quad \le \max_{i \in \{1,2\}} \left(\frac{\alpha_i}{2} \{ |x_{k-1} - z| + \varepsilon - \varepsilon^3 + |x_{k-1} - z| - \varepsilon \} + \beta_i \fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \right) \\
&\quad \le |x_{k-1} - z| + C\varepsilon^2.
\end{aligned}$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x_{k-1})} |x - z|\, dx \le |x_{k-1} - z| + C\varepsilon^2.$$

By the optional stopping theorem, this implies that

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\big[|x_\tau - z|\big] \le |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]. \tag{5-8}$$

Next we can estimate $\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau]$ by the stopping time of Lemma 5.4. Let $R > 0$ be such that $\Omega \subset B_R(z)$. Thus, by (5-2),

$$\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau] \le \varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}[\tau^*] \le C(R/\delta)\big(\operatorname{dist}(\partial B_\delta(z), x_0) + o(1)\big).$$

Since $y \in \partial B_\delta(z)$,

$$\operatorname{dist}\big(\partial B_\delta(z), x_0\big) \le |y - x_0|,$$

and thus, (5-8) implies

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^z, S_{\mathrm{II}}}\big[|x_\tau - z|\big] \le C(R/\delta)\big(|x_0 - y| + o(1)\big).$$

We get

$$g(z) - C(R/\delta)\big(|x-y| + o(1)\big) \le \mathbb{E}^{x_0}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}[\,g(x_\tau)\,].$$

Thus, we obtain

$$\begin{aligned}
\sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\Big[\, g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)\Big] \qquad&\\
\ge \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}\Big[\, g(x_\tau) + \varepsilon^2 \sum_{n=0}^{\tau-1} f(x_n)\Big]&\\
\ge g(z) - C(R/\delta)\big(|x_0-y| + o(1)\big) - \varepsilon^2 \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}[\tau]\|f\|_\infty&\\
\ge g(y) - \mathrm{Lip}(g)\delta - C(R/\delta)\big(|x_0-y| + o(1)\big)(1+\|f\|_\infty).&
\end{aligned}$$

The upper bound can be obtained by choosing for Player II a strategy where he points to z, and thus, (5-7) follows.

Finally, let $x, y \in \Omega$ and fix the strategies S_{I}, S_{II} for the game starting at x. We define a virtual game starting at y: we use the same coin tosses and random steps as the usual game starting at x. Furthermore, the players adopt their strategies S^v_{I}, S^v_{II} from the game starting at x, that is, when the game position is y_{k-1} a player chooses the step that would be taken at x_{k-1} in the game starting at x. We proceed in this way until for the first time $x_k \in \Gamma_\varepsilon$ or $y_k \in \Gamma_\varepsilon$. At that point we have

$$|x_k - y_k| = |x - y|,$$

and we may apply the previous steps that work for $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$ or for $x_k, y_k \in \Gamma_\varepsilon$.

If we are in the case $f \equiv 0$ we are done. In the case $\inf_{y\in\Omega}|f(y)| > 0$, as we know that the u_ε are uniformly bounded according to Lemma 5.3, we have that the expected exit time is bounded by

$$\widetilde{C} = \frac{\max_{y\in\Gamma_\varepsilon}|g(y)| + C\max_{y\in\Omega}|f(y)|}{\inf_{y\in\Omega}|f(y)|}.$$

So the expected difference in the running payoff in the game starting at x and the virtual one is bounded by $\widetilde{C}\,\mathrm{Lip}(f)|x-y|$, because $|x_i - y_i| = |x-y|$ for all $0 \le i \le k$. □

Corollary 5.6. *Let $\{u_\varepsilon\}$ be a family of p_1-p_2-harmonious functions. Then there exists a uniformly continuous u and a subsequence still denoted by $\{u_\varepsilon\}$ such that*

$$u_\varepsilon \to u \quad \textit{uniformly in} \quad \overline{\Omega}.$$

Proof. Using Lemmas 5.3 and 5.5 we get that the family u_ε satisfies the hypothesis of compactness in Lemma 5.1. □

Theorem 5.7. *The function u obtained as a limit in Corollary 5.6 is a viscosity solution to (1-2) when we consider the game with $f/2$ as the running payoff function.*

Proof. First, we observe that $u = g$ on $\partial\Omega$ since $u_\varepsilon = g$ on $\partial\Omega$ for all $\varepsilon > 0$. Hence, we can focus our attention on showing that u is p_1-p_2-harmonic inside Ω in the viscosity sense. To this end, we recall from [Manfredi et al. 2010] an estimate that involves the regular Laplacian ($p = 2$) and an approximation for the infinity Laplacian ($p = \infty$). Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Note that since ϕ is continuous we have

$$\min_{y\in\overline{B}_\varepsilon(x)} \phi(y) = \inf_{y\in B_\varepsilon(x)} \phi(y)$$

for all $x \in \Omega$. Let x_1^ε be the point at which ϕ attains its minimum in $\overline{B}_\varepsilon(x)$,

$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B}_\varepsilon(x)} \phi(y).$$

It follows from the Taylor expansions in [Manfredi et al. 2010] that

$$\begin{aligned}\text{(5-9)}\quad &\frac{\alpha}{2}\Big(\max_{y\in\overline{B}_\varepsilon(x)} \phi(y) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y)\Big) + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy - \phi(x) \\ &\geq \frac{\varepsilon^2}{2(n+p)}\left\{(p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right\} + o(\varepsilon^2).\end{aligned}$$

Suppose that ϕ touches u at x strictly from below. We want to prove that $F^*(\nabla\phi(x), D^2\phi(x)) \geq f(x)$. By the uniform convergence, there exists a sequence $\{x_\varepsilon\}$ converging to x such that $u_\varepsilon - \phi$ has an approximate minimum at x_ε, that is, for $\eta_\varepsilon > 0$, there exists x_ε such that

$$u_\varepsilon(x) - \phi(x) \geq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon.$$

Moreover, considering $\widetilde{\phi} = \phi - u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$, we can assume that $\phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon)$. Thus, by recalling the fact that u_ε is p_1-p_2-harmonious, we obtain

$$\eta_\varepsilon \geq \varepsilon^2 \frac{f(x_\varepsilon)}{2} - \phi(x_\varepsilon) + \max_{i\in\{1,2\}}\left\{\frac{\alpha_i}{2}\Big(\max_{\overline{B}_\varepsilon(x_\varepsilon)} \phi + \min_{\overline{B}_\varepsilon(x_\varepsilon)} \phi\Big) + \beta_i \fint_{B_\varepsilon(x_\varepsilon)} \phi(y)\,dy\right\},$$

and thus, by (5-9), and choosing $\eta_\varepsilon = o(\varepsilon^2)$, we have

$$\begin{aligned}0 \geq \frac{\varepsilon^2}{2} \max_{i\in\{1,2\}}&\left\{\alpha_i\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle + \theta_i\,\Delta\phi(x_\varepsilon)\right\} \\ &+ \varepsilon^2 \frac{f(x_\varepsilon)}{2} + o(\varepsilon^2).\end{aligned}$$

Next we need to observe that

$$\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle$$

converges to $\Delta_\infty \phi(x)$ when $\nabla\phi(x) \neq 0$ and is always bounded in the limit by $\lambda_{\min}(D^2\phi(x))$ and $\lambda_{\max}(D^2\phi(x))$. Dividing by ε^2 and letting $\varepsilon \to 0$, we get

$$F^*(\nabla\phi(x), D^2\phi(x)) \geq f(x).$$

Therefore u is a viscosity supersolution.

To prove that u is a viscosity subsolution, we use a reverse inequality to (5-9) by considering the maximum point of the test function and choose a smooth test function that touches u from above. $\square$

Now, we just observe that this probabilistic approach provides an alternative existence proof of viscosity solutions to our PDE problem.

Corollary 5.8. *Any limit function obtained as in Corollary 5.6 is a viscosity solution to the problem*

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \textit{on } \Omega, \\ u = g & \textit{on } \partial\Omega. \end{cases}$$

In particular, the problem has a solution.

We proved that the problem has an unique solution using PDE methods, therefore we conclude that we have convergence as $\varepsilon \to 0$ of u_ε (not only along subsequences).

Corollary 5.9. *It holds that*

$$u_\varepsilon \to u \quad \textit{uniformly in} \quad \overline{\Omega},$$

being u the unique solution to the problem

$$\begin{cases} \max\{-\Delta_{p_1} u, -\Delta_{p_2} u\} = f & \textit{on } \Omega, \\ u = g & \textit{on } \partial\Omega. \end{cases}$$

References

[Antunović et al. 2012] T. Antunović, Y. Peres, S. Sheffield, and S. Somersille, "Tug-of-war and infinity Laplace equation with vanishing Neumann boundary condition", *Comm. Partial Differential Equations* **37**:10 (2012), 1839–1869. MR Zbl

[Armstrong and Smart 2010] S. N. Armstrong and C. K. Smart, "An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions", *Calc. Var. Partial Differential Equations* **37**:3-4 (2010), 381–384. MR Zbl

[Atar and Budhiraja 2010] R. Atar and A. Budhiraja, "A stochastic differential game for the inhomogeneous ∞-Laplace equation", *Ann. Probab.* **38**:2 (2010), 498–531. MR Zbl

[Barles and Busca 2001] G. Barles and J. Busca, "Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term", *Comm. Partial Differential Equations* **26**:11-12 (2001), 2323–2337. MR Zbl

[Bjorland et al. 2012a] C. Bjorland, L. Caffarelli, and A. Figalli, "Non-local gradient dependent operators", *Adv. Math.* **230**:4-6 (2012), 1859–1894. MR Zbl

[Bjorland et al. 2012b] C. Bjorland, L. Caffarelli, and A. Figalli, "Nonlocal tug-of-war and the infinity fractional Laplacian", *Comm. Pure Appl. Math.* **65**:3 (2012), 337–380. MR Zbl

[Busca et al. 2005] J. Busca, M. J. Esteban, and A. Quaas, "Nonlinear eigenvalues and bifurcation problems for Pucci's operators", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22**:2 (2005), 187–206. MR Zbl

[Caffarelli and Cabré 1995] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications **43**, American Mathematical Society, Providence, RI, 1995. MR Zbl

[Crandall et al. 1992] M. G. Crandall, H. Ishii, and P.-L. Lions, "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.* (*N.S.*) **27**:1 (1992), 1–67. MR Zbl

[Felmer et al. 2006] P. L. Felmer, A. Quaas, and M. Tang, "On uniqueness for nonlinear elliptic equation involving the Pucci's extremal operator", *J. Differential Equations* **226**:1 (2006), 80–98. MR Zbl

[Hartenstine and Rudd 2013] D. Hartenstine and M. Rudd, "Statistical functional equations and p-harmonious functions", *Adv. Nonlinear Stud.* **13**:1 (2013), 191–207. MR Zbl

[Julin and Juutinen 2012] V. Julin and P. Juutinen, "A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation", *Comm. Partial Differential Equations* **37**:5 (2012), 934–946. MR Zbl

[Juutinen et al. 2001] P. Juutinen, P. Lindqvist, and J. J. Manfredi, "On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation", *SIAM J. Math. Anal.* **33**:3 (2001), 699–717. MR Zbl

[Kawohl et al. 2012] B. Kawohl, J. Manfredi, and M. Parviainen, "Solutions of nonlinear PDEs in the sense of averages", *J. Math. Pures Appl.* (9) **97**:2 (2012), 173–188. MR Zbl

[Koike and Kosugi 2015] S. Koike and T. Kosugi, "Remarks on the comparison principle for quasilinear PDE with no zeroth order terms", *Commun. Pure Appl. Anal.* **14**:1 (2015), 133–142. MR Zbl

[Lindqvist and Lukkari 2010] P. Lindqvist and T. Lukkari, "A curious equation involving the ∞-Laplacian", *Adv. Calc. Var.* **3**:4 (2010), 409–421. MR Zbl

[Liu and Schikorra 2013] Q. Liu and A. Schikorra, "A game-tree approach to discrete infinity Laplacian with running costs", preprint, 2013. arXiv

[Liu and Schikorra 2015] Q. Liu and A. Schikorra, "General existence of solutions to dynamic programming equations", *Commun. Pure Appl. Anal.* **14**:1 (2015), 167–184. MR Zbl

[Llorente 2014] J. G. Llorente, "A note on unique continuation for solutions of the ∞-mean value property", *Ann. Acad. Sci. Fenn. Math.* **39**:1 (2014), 473–483. MR Zbl

[Llorente 2015] J. G. Llorente, "Mean value properties and unique continuation", *Commun. Pure Appl. Anal.* **14**:1 (2015), 185–199. MR Zbl

[Lu and Wang 2008] G. Lu and P. Wang, "Inhomogeneous infinity Laplace equation", *Adv. Math.* **217**:4 (2008), 1838–1868. MR Zbl

[Luiro and Parviainen 2015] H. Luiro and M. Parviainen, "Regularity for nonlinear stochastic games", preprint, 2015. arXiv

[Luiro et al. 2013] H. Luiro, M. Parviainen, and E. Saksman, "Harnack's inequality for p-harmonic functions via stochastic games", *Comm. Partial Differential Equations* **38**:11 (2013), 1985–2003. MR Zbl

[Maitra and Sudderth 1993] A. Maitra and W. Sudderth, "Borel stochastic games with lim sup payoff", *Ann. Probab.* **21**:2 (1993), 861–885. MR Zbl

[Maitra and Sudderth 1996] A. P. Maitra and W. D. Sudderth, *Discrete gambling and stochastic games*, Applications of Mathematics **32**, Springer, New York, 1996. MR Zbl

[Manfredi et al. 2010] J. J. Manfredi, M. Parviainen, and J. D. Rossi, "An asymptotic mean value characterization for p-harmonic functions", *Proc. Amer. Math. Soc.* **138**:3 (2010), 881–889. MR Zbl

[Manfredi et al. 2012a] J. J. Manfredi, M. Parviainen, and J. D. Rossi, "Dynamic programming principle for tug-of-war games with noise", *ESAIM Control Optim. Calc. Var.* **18**:1 (2012), 81–90. MR Zbl

[Manfredi et al. 2012b] J. J. Manfredi, M. Parviainen, and J. D. Rossi, "On the definition and properties of p-harmonious functions", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:2 (2012), 215–241. MR Zbl

[Nyström and Parviainen 2014] K. Nyström and M. Parviainen, "Tug-of-war, market manipulation, and option pricing", *Math. Finance* (online publication December 2014).

[Peres and Sheffield 2008] Y. Peres and S. Sheffield, "Tug-of-war with noise: a game-theoretic view of the p-Laplacian", *Duke Math. J.* **145**:1 (2008), 91–120. MR Zbl

[Peres et al. 2009] Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson, "Tug-of-war and the infinity Laplacian", *J. Amer. Math. Soc.* **22**:1 (2009), 167–210. MR Zbl

[Quaas and Sirakov 2006] A. Quaas and B. Sirakov, "Existence results for nonproper elliptic equations involving the Pucci operator", *Comm. Partial Differential Equations* **31**:7-9 (2006), 987–1003. MR Zbl

[Ruosteenoja 2016] E. Ruosteenoja, "Local regularity results for value functions of tug-of-war with noise and running payoff", *Adv. Calc. Var.* **9**:1 (2016), 1–17. MR Zbl

Received June 5, 2015. Revised June 23, 2016.

PABLO BLANC
pblanc@dm.uba.ar

JUAN P. PINASCO
jpinasco@dm.uba.ar

JULIO D. ROSSI
jrossi@dm.uba.ar

(*all authors*)

DEPARTAMENTO DE MATEMÁTICA FCEYN
UNIVERSIDAD DE BUENOS AIRES
CIUDAD UNIVERSITARIA, PABELLÒN 1 (1428)
BUENOS AIRES
ARGENTINA

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.297

VAN EST ISOMORPHISM FOR HOMOGENEOUS COCHAINS

ALEJANDRO CABRERA AND THIAGO DRUMMOND

VB-groupoids define a special class of Lie groupoids which carry a compatible linear structure. We show that their differentiable cohomology admits a refinement by considering the complex of cochains which are k-homogeneous on the linear fiber. Our main result is a van Est theorem for such cochains. We also work out two applications to the general theory of representations of Lie groupoids and algebroids. The case $k = 1$ yields a van Est map for representations up to homotopy on 2-term graded vector bundles and, moreover, to a new proof of a rigidity conjecture posed by Crainic and Moerdijk. Arbitrary k-homogeneous cochains on suitable VB-groupoids lead to a novel van Est theorem for differential forms on Lie groupoids with values in a representation.

1. Introduction

The van Est theorem [1953a; 1953b; 1955a; 1955b] is a classical result relating the differentiable cohomology associated to a Lie group with the underlying Lie algebra cohomology. More precisely, given a Lie group G with Lie algebra $\mathfrak{g}$, the van Est map is a map

$$\mathrm{VE} : C^p(G) = \{f \in C^\infty(G^p) : f(g_1, \dots, g_p) = 0 \text{ if } g_i = e\} \to \mathrm{CE}(\mathfrak{g}) = \Lambda^p \mathfrak{g}^*$$

MSC2010: 22A22, 53D17.
Keywords: Lie groupoids, van Est theorem.

taking (normalized) differentiable p-cochains on G to Lie algebra p-cochains. It is defined (up to sign) by

$$\mathrm{VE}(f)(u_1,\dots,u_p)=\sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)R_{u_{\sigma(1)}}\dots R_{u_{\sigma(p)}}f, \tag{1-1}$$

where $R_u : C^p(G)\to C^{p-1}(G)$ is the operator which differentiates $f(\,\cdot\,,g_2,\dots,g_p)$ at the unit e with respect to the right-invariant vector field corresponding to u. The map VE can be seen as a model for the pullback of functions along the projection of the universal G-bundle $EG\to BG$. The van Est theorem then states that if G is (topologically) p_0-connected, the map induced by VE in cohomology is an isomorphism for $p\le p_0$ and injective for $p=p_0+1$. In the setting of Lie groupoids, the van Est theorem was first studied by A. Weinstein and P. Xu [1991] for $p_0=1$ and later generalized for arbitrary degrees by M. Crainic [2003] (see also the more recent work of D. Li-Bland and E. Meinrenken [2015]).

In this paper, we provide a refinement of this theorem for a particular class of Lie groupoids endowed with a compatible linear structure, called VB-groupoids [Pradines 1988] (see also [Bursztyn et al. 2016; Gracia-Saz and Mehta 2010; 2011]). In this case, the linear structure allows us to refine the van Est theorem by looking at *homogeneous cochains*, and we are able to derive several interesting applications from this general result.

To illustrate our approach, we examine here a simple situation involving a Lie group G and a linear representation $\rho_G : G\to \mathrm{Aut}(V)$ on a (finite-dimensional) real vector space V. The associated complex of differentiable cochains for G with values in V is $C^p(G,V)=\{f : G^p\to V : f(g_1,\dots,g_p)=0 \text{ if } g_i=e\}$ with a differential $\delta : C^p(G,V)\to C^{p+1}(G,V)$ which encodes ρ_G (see Example 2.5 below for an explicit formula). Infinitesimally, associated to the induced Lie algebra representation $\rho_{\mathfrak{g}} : \mathfrak{g}\to \mathrm{End}(V)$, we have the Chevalley–Eilenberg complex of Lie algebra cochains with values in V, namely $\mathrm{CE}^p(\mathfrak{g},V)=\Lambda^p\mathfrak{g}^*\otimes V$. In this setting, there exists a natural analogue of the van Est map

$$\Psi_\rho : C^p(G,V)\to \mathrm{CE}^p(\mathfrak{g},V). \tag{1-2}$$

How can one prove a van Est theorem for Ψ_ρ? There are two approaches: the first one is to re-prove the statement from scratch mimicking the proof of the standard case. The second one is to *deduce* the desired result from the known van Est theorem for *Lie groupoids* by relating the map (1-2) to the van Est map VE for the action groupoid $\mathcal{V}=V^*\rtimes G$. It is the second approach that we pursue in this paper.

To relate VE and Ψ_ρ, notice that both $\mathcal{V}$ and its space of p-composable arrows $B_p\mathcal{V}$ define vector bundles $\mathcal{V}\to G$ and $B_p\mathcal{V}\to G^p$, respectively. (Actually, $B_p\mathcal{V}$ is isomorphic to $V^*\times G^p$.) One can then show that the differentiable cochains

$f \in C^\infty(B_p\mathcal{V})$ which are fiberwise k-homogeneous define a subcomplex

$$C^p_{k\text{-hom}}(\mathcal{V}) \subset C^p(\mathcal{V}).$$

Analogously, the Lie algebroid $\mathfrak{v} = V^* \rtimes \mathfrak{g}$ of $\mathcal{V}$ also defines a trivial vector bundle $\mathfrak{v} \to \mathfrak{g}$ and the fiberwise k-homogeneous cochains define a subcomplex

$$\mathrm{CE}^p_{k\text{-hom}}(\mathfrak{v}) \subset \Lambda^p \mathfrak{v}^*.$$

The key point is that VE preserves k-homogeneous cochains, thus restricting to a map

$$\mathrm{VE}_{k\text{-hom}} : C^p_{k\text{-hom}}(\mathcal{V}) \to \mathrm{CE}^p_{k\text{-hom}}(v)$$

which, by a simple homological algebra argument (see page 307), is an isomorphism (resp. injective) in cohomology whenever VE is. Finally, to obtain the van Est theorem for V-valued cochains one has to verify that

$$H^p(C^\bullet_{1\text{-hom}}(\mathcal{V})) \simeq H^p(C^\bullet(G, V)),$$
$$H^p(\mathrm{CE}^\bullet_{1\text{-hom}}(\mathfrak{v})) \simeq H^p(\Lambda^\bullet \mathfrak{g}^* \otimes V),$$
$$\mathrm{VE}_{1\text{-hom}} \simeq \Psi_\rho.$$

In this paper, we follow the same reasoning but with $\mathcal{V}$ replaced by an arbitrary VB-groupoid. The main arguments follow directly as above but nontrivial computational effort needs to go into the last ingredient of the argument, namely, into relating the complexes of homogeneous (groupoid and algebroid) cochains to certain complexes already introduced in the literature from different perspectives. In particular, we obtain explicit formulas for the underlying van Est maps.

We work out two applications: in the first, we deduce a van Est theorem for representations up to homotopy in 2-term graded vector bundles [Arias Abad and Crainic 2012; 2013; Gracia-Saz and Mehta 2010; 2011] by looking at 1-homogeneous cochains and generalizing the case of ρ above, recovering results from [Arias Abad and Schätz 2011]. Moreover, we prove a cohomological vanishing result for these 1-homogeneous cochains which, in the case of the *adjoint representation*, leads to a realization of the original idea proposed in [Crainic and Moerdijk 2008] for showing a rigidity result for certain proper groupoids. (This last result was also proven in [Arias Abad and Schätz 2011] using different methods.) The second application provides a new van Est theorem for differential forms on Lie groupoids with coefficients in a representation, generalizing [Arias Abad and Crainic 2011] on the Bott–Shulmann complex and [Crainic et al. 2015b] on Spencer operators. It is interesting to notice that, in this second application, another idea is incorporated (which has its roots in *supergeometry* and was used in a Lie-theoretic context by Mehta [2009]): forms in $\Lambda^k V^*$ are k-homogeneous functions on V^k. For this application, we need the refinement of the van Est theorem in its full extent (i.e., for

k-homogeneous cochains, where k is arbitrary). Even in the particular case of differential forms with trivial coefficients, our proof of the corresponding van Est theorem is new and can be seen as illustration of the usefulness of homogeneous cochains.

Outline of the paper.

- In Section 2 we set up some notation, introduce homogeneous cochains on VB-groupoids and VB-algebroids and provide our main result: the corresponding refinement of the van Est theorem.
- In Section 3, we specialize to 1-homogeneous cochains and deduce a van Est result for representations up to homotopy. Along the way, we mention how this argument can be used to provide an alternative proof of the rigidity conjecture as originally proposed in [Crainic and Moerdijk 2008].
- In Section 4, by means of k-homogeneous cochains in suitable VB-groupoids and VB-algebroids, we prove a van Est theorem for differential forms with coefficients in a representation.

To keep the main text as simple as possible, we postpone to the Appendix some of the more technical or computational parts of the arguments in Section 4. Most of the explicit formulas contained in the Appendix follow from extensions of known lift properties of vector fields to Lie groupoids (see [Mackenzie and Xu 1994; 1998]). We would like to mention that part of this paper grew out of the project of understanding the Lie theory of multiplicative tensors on Lie groupoids [Bursztyn and Drummond $\geq$ 2017].

2. Homogeneous cochains and the van Est map for VB-groupoids

In this section, we present a refinement of groupoid and algebroid cohomology theory for VB-groupoids and VB-algebroids by considering k-homogeneous cochains. We also show that an analogue of the van Est theorem holds for such homogeneous cochains.

Homogeneous functions on vector bundles. Given any vector bundle $\pi : V \to B$, fiberwise multiplication by scalars $h : \mathbb{R} \times V \to V$ defines an action of the multiplicative monoid $\mathbb{R}$ which we shall call the *homogeneous structure of* $V \to B$. Following [Grabowski and Rotkiewicz 2009], we recall that the homogeneous structure completely characterizes the underlying vector bundle structure and that, in particular, a smooth map between the total spaces defines a vector bundle morphism if and only if it commutes with the underlying $\mathbb{R}$-actions. (See [Bursztyn et al. 2016] for applications of these ideas in a Lie-theoretic context.)

For each $k \in \mathbb{N}$, we consider

$$C^{\infty}_{k\text{-hom}}(V) := \{f \in C^{\infty}(V) : h^*_{\lambda} f = \lambda^k f \;\; \forall \lambda \in \mathbb{R}\},$$

the set of fiberwise k-homogeneous functions. Note that

$$C^\infty_{0\text{-hom}}(V) = \{f \in C^\infty(V) : \exists f_0 \in C^\infty(B) \text{ such that } f = f_0 \circ \pi\} \cong C^\infty(B).$$

Multiplication of functions gives a map $C^\infty_{k\text{-hom}}(V) \times C^\infty_{k'\text{-hom}}(V) \to C^\infty_{k+k'\text{-hom}}(V)$ and, in particular, each $C^\infty_{k\text{-hom}}(V)$ is a $C^\infty(B)$-module. In fact, $C^\infty_{k\text{-hom}}(V) \cong \Gamma(B, S^k V^*)$ for the symmetric algebra bundle $S^k V^* \to B$. The isomorphism $\Gamma(B, V^*) \cong C^\infty_{1\text{-hom}}(V)$ takes a section $\mu \in \Gamma(B, V^*)$ to the fiberwise-linear function $\ell_\mu \in C^\infty_{1\text{-hom}}(V)$ given by

$$\ell_\mu(v) = \langle \mu(b), v\rangle, \quad v \in V_b,\ b \in B.$$

The k-th derivative along the fiber defines a projection $P_{k\text{-hom}} : C^\infty(V) \to C^\infty_{k\text{-hom}}(V)$,

$$P_{k\text{-hom}}(f) = \frac{1}{k!}\frac{d^k}{d\lambda^k}(h^*_\lambda f)|_{\lambda=0}. \tag{2-1}$$

If $(x, \xi_1, \dots, \xi_n)$ are trivializing coordinates on V, then

$$P_{k\text{-hom}}(f)(x,\xi) = \sum_{k_1+\dots+k_n=k} \frac{1}{k_1!\cdots k_n!}\frac{\partial^k f}{\partial \xi_1^{k_1}\cdots\partial\xi_n^{k_n}}(x,0)\,\xi_1^{k_1}\cdots\xi_n^{k_n}.$$

Homogeneous groupoid cochains. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with source and target maps $\mathsf{s}, \mathsf{t} : \mathcal{G} \to M$, unit $\mathbf{1} : M \to \mathcal{G}$, inversion $\iota : \mathcal{G} \to \mathcal{G}$ and multiplication $\mathsf{m} : \mathcal{G}\,{}_{\mathsf{s}}\!\times_{\mathsf{t}} \mathcal{G} \to \mathcal{G}$. We denote by $B_p\mathcal{G}$ the manifold of composable p-tuples ($B_0\mathcal{G} = M$). The nerve of $\mathcal{G}$ is the simplicial manifold whose space of p-simplices is $B_p\mathcal{G}$ with the simplicial structure given by the face maps $\partial_i : B_p\mathcal{G} \to B_{p-1}\mathcal{G},\ i = 0, \dots, p$, defined by

$$\partial_i(g_1,\dots,g_p) = \begin{cases} (g_2,\dots,g_p) & \text{if } i = 0,\\ (g_1,\dots,g_{i-1}, g_i g_{i+1}, g_{i+2},\dots,g_p) & \text{if } 1 \le i \le p-1,\\ (g_1,\dots,g_{p-1}) & \text{if } i = p,\end{cases}$$

and the degeneracy maps $s_i : B_{p-1}\mathcal{G} \to B_p\mathcal{G},\ i = 0, \dots, p-1$, defined by

$$s_i(g_1,\dots,g_{p-1}) = (g_1,\dots,g_i, 1_{\mathsf{t}(g_{i+1})}, g_{i+1},\dots,g_{p-1}).$$

For $p = 1$, $\partial_0 = \mathsf{s}$, $\partial_1 = \mathsf{t}$ and $s_0 = \mathbf{1}$.

The nerve defines a functor $B_\bullet$ from the category of Lie groupoids to the category of simplicial manifolds. For a groupoid morphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$, the morphism $B\phi : B\mathcal{G}_1 \to B\mathcal{G}_2$ is defined by $B_p\phi(g_1,\dots,g_p) = (\phi(g_1),\dots,\phi(g_p))$.

The space of (*normalized*) *p-cochains* $C^p(\mathcal{G})$ on $\mathcal{G}$ consists of smooth functions $f : B_p\mathcal{G} \to \mathbb{R}$ such that $s_i^* f = 0$ for $i = 0, \dots, p-1$. These define a cochain complex with differential $\delta : C^{p-1}(\mathcal{G}) \to C^p(\mathcal{G})$ defined by

$$\delta = \sum_{i=0}^{p} (-1)^i \partial_i^*. \tag{2-2}$$

The *differentiable cohomology* of $\mathcal{G}$ is the cohomology of the complex $(C^\bullet(\mathcal{G}), \delta)$ and we denote it by $H^\bullet(\mathcal{G})$. For $f_1 \in C^p(\mathcal{G})$, $f_2 \in C^{p'}(\mathcal{G})$, the cup product $f_1 \star f_2 \in C^{p+p'}(\mathcal{G})$ is defined by

$$(f_1 \star f_2)(g_1, \dots, g_{p+p'}) = f_1(g_1, \dots, g_p) f_2(g_{p+1}, \dots, g_{p+p'}). \tag{2-3}$$

It defines an algebra structure on $C^\bullet(\mathcal{G})$ which passes to cohomology due to the Leibniz formula

$$\delta(f_1 \star f_2) = \delta(f_1) \star f_2 + (-1)^p f_1 \star \delta(f_2).$$

In the following, we investigate how the differentiable cohomology of a *VB-groupoid* interacts with its underlying homogeneous structure.

Definition 2.1. A VB-groupoid is given by a commutative square

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r] & M, \end{tikzcd} \tag{2-4}$$

where the left and right sides are Lie groupoids and the top and bottom sides are vector bundles satisfying the following compatibility condition:

$$\begin{tikzcd} \mathcal{V} \arrow[r, "h_\lambda^{\mathcal{G}}"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V} \arrow[d, shift left] \arrow[d, shift right] \\ E \arrow[r, "h_\lambda"'] & E \end{tikzcd} \tag{2-5}$$

defines a Lie groupoid morphism for each $\lambda \in \mathbb{R}$, where $h_\lambda^{\mathcal{G}} : \mathcal{V} \to \mathcal{V}$ and $h_\lambda : E \to E$ are the homogeneous structures corresponding to $\mathcal{V} \to \mathcal{G}$ and $E \to M$, respectively. We denote the structure maps of $\mathcal{V} \rightrightarrows E$ by $\mathsf{s}_\mathcal{V}, \mathsf{t}_\mathcal{V}, \mathbf{1}_\mathcal{V}, \iota_\mathcal{V}, m_\mathcal{V}$.

Instead of looking at the homogeneous structure, VB-groupoids can be alternatively defined by focusing on the fiberwise defined sum (see [Gracia-Saz and Mehta 2011]). Our choice of definition comes from [Bursztyn et al. 2016], where the two definitions are shown to be equivalent (see Theorem 3.2.3 therein).

VB-groupoids have found several applications in recent years ([Bursztyn and Cabrera 2012; Bursztyn et al. 2016; Bursztyn and Drummond ≥ 2017; Gracia-Saz and Mehta 2011; Mackenzie and Xu 1994; 1998] to mention just a few). Natural examples of VB-groupoids are given by the tangent $T\mathcal{G} \rightrightarrows TM$ and cotangent

$T^*\mathcal{G} \rightrightarrows \mathrm{Lie}(\mathcal{G})^*$ groupoids, which provide intrinsic versions of the adjoint and coadjoint representations (up to homotopy; see Section 3 below) of a Lie groupoid $\mathcal{G}$. Ordinary representations also provide examples of VB-groupoids, as we shall see in detail in Example 2.5 below.

From now on, we focus on introducing *homogeneous cochains* on a VB-groupoid and to study their properties with respect to the van Est map, while having in mind the applications to be developed in Sections 3 and 4. The first result states that $B_\bullet$ restricts to a functor from VB-groupoids to simplicial vector bundles.

Lemma 2.2. *Let $\mathcal{V} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G} \rightrightarrows M$. The space of p-composable arrows $B_p\mathcal{V}$ is a vector bundle over $B_p\mathcal{G}$. Moreover, the face and degeneracy maps are all vector bundle maps.*

Proof. Consider $\mathcal{V}^p = \mathcal{V} \times \cdots \times \mathcal{V}$ as a vector bundle over $\mathcal{G}^p$. We shall present $B_p\mathcal{V}$ as a subbundle of $\mathcal{V}^p$ restricted to $B_p\mathcal{G} \subset \mathcal{G}^p$. It follows from the commutativity of (2-4) that $B_p\mathcal{V}$ projects onto $B_p\mathcal{G}$. As $B_p\mathcal{V}$ is a smooth submanifold of $\mathcal{V}^p$, it remains to check that it is invariant by the homogeneous structure of the vector bundle $\mathcal{V}^p \to \mathcal{G}^p$ (see [Grabowski and Rotkiewicz 2009]). This is a straightforward consequence of the fact that (2-5) is a groupoid morphism. The statement regarding the face and degeneracy maps follows now from the fact that the multiplication $m_\mathcal{V} : B_2\mathcal{V} \to \mathcal{V}$ is a vector bundle map (see also [Bursztyn et al. 2016]). □

Note that the homogeneous structure $h_\lambda^{B_p\mathcal{G}} : B_p\mathcal{V} \to B_p\mathcal{V}$ of the vector bundle $B_p\mathcal{V} \to B_p\mathcal{G}$ satisfies

$$B_p h_\lambda^{\mathcal{G}} = h_\lambda^{B_p\mathcal{G}}.$$

It is now a straightforward consequence of Lemma 2.2 that homogeneous cochains define a subcomplex of the differentiable cohomology of $\mathcal{V}$.

Proposition 2.3. *Let $\mathcal{V} \rightrightarrows E$ be a VB-groupoid. If*

$$P^{\mathcal{G},p}_{k\text{-hom}} : C^\infty(B_p\mathcal{V}) \to C^\infty_{k\text{-hom}}(B_p\mathcal{V})$$

is the projection (2-1) induced by $h_\lambda^{B_p\mathcal{G}}$, then

$$P^{\mathcal{G},p+1}_{k\text{-hom}} \circ \delta = \delta \circ P^{\mathcal{G},p}_{k\text{-hom}}.$$

In particular,

$$\delta(C^\infty_{k\text{-hom}}(B_p\mathcal{V})) \subset C^\infty_{k\text{-hom}}(B_{p+1}\mathcal{V}).$$

Thus, for a VB-groupoid $\mathcal{V} \rightrightarrows E$, we define natural subcomplexes of $(C^\bullet(\mathcal{V}), \delta)$ by considering the set of fiberwise k-homogeneous functions:

$$C^\bullet_{k\text{-hom}}(\mathcal{V}) := C^\infty_{k\text{-hom}}(\mathcal{V}^{(\bullet)}) \quad \text{and} \quad H^\bullet_{k\text{-hom}}(\mathcal{V}) = H(C^\bullet_{k\text{-hom}}(\mathcal{V})).$$

Remark 2.4. For $k=0$, we have $C^\bullet_{0\text{-hom}}(\mathcal{V}) \simeq C^\bullet(\mathcal{G})$ and the cup product (2-3) on $C^\bullet(\mathcal{V})$ induces a right $C^\bullet(\mathcal{G})$-module (resp. $H^\bullet(\mathcal{G})$-module) structure on $C^\bullet_{k\text{-hom}}(\mathcal{V})$ (resp. $H^\bullet_{k\text{-hom}}(\mathcal{V})$).

Example 2.5. Let $C \to M$ be a (left) representation of the Lie groupoid $\mathcal{G} \rightrightarrows M$. The vector bundle $\mathcal{V} = \mathsf{t}^*C^* \to \mathcal{G}$ carries a VB-groupoid structure $\mathsf{t}^*C^* \rightrightarrows C^*$ defined by

$$\mathsf{s}_{\mathcal{V}}(g,\xi) = \Delta_g^*(\xi), \quad \mathsf{t}_{\mathcal{V}}(g,\xi) = \xi,$$

$$\iota_{\mathcal{V}}(g,\xi) = (g^{-1}, \Delta_g^*(\xi)), \quad \mathbf{1}_{\mathcal{V}}(\xi) = (\mathbf{1}_{\pi(\xi)}, \xi), \quad \mathsf{m}_{\mathcal{V}}((g,\xi_1),(h,\xi_2)) = (gh, \xi_1),$$

where $\Delta_g : C_{\mathsf{s}(g)} \to C_{\mathsf{t}(g)}$ is the action of $g \in \mathcal{G}$. Note that $\mathsf{t}^*C^* = C^* \rtimes \mathcal{G}$, the action groupoid for the adjoint action of $\mathcal{G}$ on C^*. As vector bundles over $B_p\mathcal{G}$, one has that $B_p(\mathsf{t}^*C^*) = \mathsf{t}_p^*C^*$, where $\mathsf{t}_p : B_p\mathcal{G} \to M$ is given by $\mathsf{t}_p(g_1,\dots,g_p) = \mathsf{t}(g_1)$ and the isomorphism is given by $((g_1,\xi_1),\dots,(g_p,\xi_p)) \mapsto ((g_1,\dots,g_p),\xi_1)$. In particular,

$$C^p_{1\text{-hom}}(\mathcal{V}) \cong \Gamma(B_p\mathcal{G}, \mathsf{t}_p^*C).$$

The right $C^\bullet(\mathcal{G})$-module structure on $C^\bullet_{1\text{-hom}}(\mathcal{V})$ corresponds to a right module structure on $\Gamma(B_\bullet\mathcal{G}, \mathsf{t}_\bullet^*C)$ given by

$$(\phi \star f)(g_1,\dots,g_{p+p'}) = \phi(g_1,\dots,g_p)\, f(g_{p+1},\dots,g_{p+p'}), \qquad f \in C^{p'}(\mathcal{G}),\ \phi \in \Gamma(B_p\mathcal{G}, \mathsf{t}_p^*C). \tag{2-6}$$

Further, the differential on $C^\bullet_{1\text{-hom}}(\mathcal{V})$ corresponds to the differential on $\Gamma(B_\bullet\mathcal{G}, \mathsf{t}_\bullet^*C)$ given by

$$(\delta\phi)(g_1,\dots,g_{p+1}) = \Delta_{g_1}(\phi(g_2,\dots,g_p)) + \sum_{i=1}^{p-1}(-1)^i\phi(g_1,\dots,g_ig_{i+1},\dots,g_p) + (-1)^p\phi(g_1,\dots,g_{p-1}).$$

Hence, as $H^\bullet(\mathcal{G})$-modules, $H^\bullet_{1\text{-hom}}(\mathcal{V}) \cong H^\bullet(\mathcal{G}, C)$, the cohomology of $\mathcal{G}$ with coefficients on the representation C (see [Crainic 2003]). More generally, $H^\bullet_{k\text{-hom}}(\mathcal{V}) \cong H^\bullet(\mathcal{G}, S^kC)$.

Homogeneous algebroid cochains. Given a VB-groupoid $\mathcal{V} \rightrightarrows E$, its Lie algebroid $\mathfrak{v} \to E$ inherits the structure of a VB-algebroid; see [Bursztyn et al. 2016]. As for VB-groupoids, we take our working definition from that paper.

Definition 2.6. A VB-algebroid is given by a commutative square

$$\begin{tikzcd} \mathfrak{v} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ E \arrow[r] & M, \end{tikzcd} \tag{2-7}$$

where the left and right sides are Lie algebroids and the top and bottom sides are vector bundles satisfying the following compatibility condition:

(2-8)
$$\begin{tikzcd} \mathfrak{v} \arrow[r, "h^{\mathfrak{g}}_{\lambda}"] \arrow[d] & \mathfrak{v} \arrow[d] \\ E \arrow[r, "h_{\lambda}"'] & E \end{tikzcd}$$

defines a Lie algebroid morphism for each $\lambda \in \mathbb{R}$, where $h^{\mathfrak{g}}_{\lambda}$ and h_{λ} are the homogeneous structures of the vector bundles $\mathfrak{v} \to \mathfrak{g}$ and $E \to M$, respectively.

Parallel to VB-groupoids, VB-algebroids together with Lie theory for VB-objects have found several applications in recent years (again, we list just a few of the available references: [Bursztyn and Cabrera 2012; Bursztyn et al. 2016; Bursztyn and Drummond $\geq$ 2017; Gracia-Saz and Mehta 2010; Mackenzie and Xu 1994; 1998]). The tangent $TA \to TM$ and the cotangent lift $T^*A \to A^*$ define examples of VB-algebroids corresponding to $T\mathcal{G}$ and $T^*\mathcal{G}$ when $A = \mathrm{Lie}(\mathcal{G})$, providing intrinsic versions of the adjoint and coadjoint representations (up to homotopy; see Section 3 below) of a Lie algebroid A. Ordinary representations of A also provide examples of VB-groupoids, as explained in Example 2.9 below. We now investigate the infinitesimal version of the notion of *homogeneous cochains*.

For any Lie algebroid $A \to M$, let $\mathrm{CE}^p(A) := \Gamma(M, \Lambda^p A^*)$ and $d : \mathrm{CE}^p(A) \to \mathrm{CE}^{p+1}(A)$ be the (Chevalley–Eilenberg) differential. The Lie algebroid cohomology $H^\bullet(A)$ is the cohomology of the complex $(\mathrm{CE}^\bullet(A), d)$. The wedge product on $\Gamma(M, \Lambda^\bullet A^*)$ induces a graded commutative algebra structure on $H^\bullet(A)$.

When considering a VB-algebroid $A = \mathfrak{v}$, the dual $\mathfrak{v}^*$ is always taken with respect to the Lie algebroid side $\mathfrak{v} \to E$, so that $\mathrm{CE}^p(\mathfrak{v}) = \Gamma(E, \Lambda^p \mathfrak{v}^*)$. The space of fiberwise (with respect to $\mathfrak{v} \to \mathfrak{g}$) k-homogeneous p-forms on $\mathfrak{v} \to E$ is

$$\Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*) := \{\alpha \in \Gamma(E, \Lambda^q \mathfrak{v}^*) : h^{\mathfrak{g}\,*}_{\lambda}\alpha = \lambda^k \alpha \ \ \forall\, \lambda \in \mathbb{R}\}. \tag{2-9}$$

The wedge product induces a map

$$\cdot \wedge \cdot : \Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*) \times \Gamma_{k'\text{-hom}}(E, \Lambda^{p'} \mathfrak{v}^*) \to \Gamma_{k+k'\text{-hom}}(E, \Lambda^{p+p'} \mathfrak{v}^*).$$

Similarly to equation (2-1), there exists a projection $P^{\mathfrak{g},p}_{k\text{-hom}} : \Gamma(E, \Lambda^p \mathfrak{v}^*) \to \Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*)$ defined by

$$P^{\mathfrak{g},p}_{k\text{-hom}}\alpha = \frac{1}{k!}\frac{d^k}{d\lambda^k}(h^{\mathfrak{g}\,*}_{\lambda}\alpha)|_{\lambda=0}. \tag{2-10}$$

Proposition 2.7. *Let $\mathfrak{v} \to E$ be a VB-algebroid. For each $k \in \mathbb{N}_0$ and every $p \geq 0$,*

$$P^{\mathfrak{g},p+1}_{k\text{-hom}} \circ d = d \circ P^{\mathfrak{g},p}_{k\text{-hom}}.$$

In particular,

$$d(\Gamma_{k\text{-hom}}(E, \Lambda^p \mathfrak{v}^*)) \subset \Gamma_{k\text{-hom}}(E, \Lambda^{p+1}\mathfrak{v}^*).$$

Proof. Since the Chevalley–Eilenberg differential d is a local operator, we can assume $\mathfrak{v} \to E$ is trivial. By looking at $h_\lambda^{\mathfrak{v}*}\alpha$ as a smooth 1-parameter family of forms, one can see that d commutes with $d/d\lambda$. The statement then follows from the fact that $h_\lambda^{\mathfrak{v}}$ is a Lie algebroid morphism and, hence, $h_\lambda^{\mathfrak{v}*}$ commutes with d. $\square$

Thus, for each $k \in \mathbb{N}_0$, the k-homogeneous forms define a subcomplex $\mathrm{CE}^\bullet_{k\text{-hom}}(\mathfrak{v})$ of $(\mathrm{CE}^\bullet(\mathfrak{v}), d)$. The notation we use is

$$\mathrm{CE}^p_{k\text{-hom}}(\mathfrak{v}) := \Gamma_{k\text{-hom}}(E, \Lambda^p\mathfrak{v}^*) \quad \text{and} \quad H^\bullet_{k\text{-hom}}(\mathfrak{v}) = H(\mathrm{CE}^\bullet_{k\text{-hom}}(\mathfrak{v})).$$

Remark 2.8. For $k = 0$, we have $\Gamma_{0\text{-hom}}(E, \Lambda^p\mathfrak{v}^*) \cong \Gamma(M, \Lambda^p\mathfrak{g}^*)$ and the wedge product turns $\Gamma_{k\text{-hom}}(E, \Lambda^\bullet\mathfrak{v}^*)$ (resp. $H^\bullet_{k\text{-hom}}(\mathfrak{v})$) into a right $\Gamma(M, \Lambda^\bullet\mathfrak{g}^*)$-module (resp. $H^\bullet(\mathfrak{g})$-module).

Example 2.9. Let $C \to M$ be a representation of the Lie algebroid $\mathfrak{g} \to M$ defined by a flat $\mathfrak{g}$-connection $\nabla : \Gamma(\mathfrak{g}) \times \Gamma(C) \to \Gamma(C)$. Consider the vector bundle $\mathfrak{v} = C^* \times_M \mathfrak{g} \to C^*$. Given $u \in \Gamma(\mathfrak{g})$, let $\chi_u : C^* \to \mathfrak{v}$ be the section given by

$$\chi_u(\xi) = (\xi, u(m)) \quad \text{for } \xi \in C^*_m. \tag{2-11}$$

The sections χ_u with u varying on $\Gamma(\mathfrak{g})$ generate $\Gamma(C^*, \mathfrak{v})$ as a $C^\infty(C^*)$-module. One can now show that the action algebroid structure $C^* \rtimes \mathfrak{g} \to C^*$, determined by

$$[\chi_{u_1}, \chi_{u_2}] = \chi_{[u_1,u_2]}, \quad u_1, u_2 \in \Gamma(\mathfrak{g}),$$

$$\rho_{\mathfrak{v}}(\chi_{u_1})(\ell_\mu) = \ell_{\nabla_{u_1}\mu}, \;\; \rho_{\mathfrak{v}}(\chi_{u_1})(f \circ \pi) = (\mathcal{L}_{\rho(u_1)} f) \circ \pi, \quad f \in C^\infty(M),\; \mu \in \Gamma(C),$$

endows $\mathfrak{v} \to C^*$ with a VB-algebroid structure, where $\pi : C^* \to M$ is the projection. The chain complex $\mathrm{CE}^\bullet_{1\text{-hom}}(\mathfrak{v})$ is naturally isomorphic to $\Gamma(\Lambda^\bullet\mathfrak{g}^* \otimes C)$ with the Koszul differential

$$\begin{aligned} d_\nabla\gamma(u_1, \dots, u_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{u_i}\gamma(u_1, \dots, \widehat{u_i}, \dots, u_{p+1}) \\ &\qquad + \sum_{1 \le i < j \le p+1} (-1)^{i+j}\gamma([u_i, u_j], u_1, \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, u_{p+1}), \end{aligned}$$

where $\gamma \in \Gamma(\Lambda^p\mathfrak{g}^* \otimes C)$. More precisely, the evaluation map $\mathrm{ev} : \mathrm{CE}^p_{1\text{-hom}}(\mathfrak{v}) \to \Gamma(\Lambda^p\mathfrak{g}^* \otimes C)$, given by

$$\langle \mathrm{ev}(\alpha)(u_1, \dots, u_p), \xi\rangle = \alpha(\chi_{u_1}(\xi), \dots, \chi_{u_p}(\xi)) \quad \text{for } u_1, \dots, u_p \in \Gamma(\mathfrak{g}),\; \xi \in C^*,$$

defines a chain isomorphism. The induced right $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$-module structure on $\Gamma(\Lambda^\bullet\mathfrak{g}^* \otimes C)$ is wedge multiplication on the right in the $\Lambda^\bullet\mathfrak{g}^*$ factor. In particular, as $H(\mathfrak{g})$-modules, $H^\bullet_{1\text{-hom}}(\mathfrak{v}) \cong H^\bullet(\mathfrak{g}, C)$, the cohomology of $\mathfrak{g}$ with values in the representation C. As for groupoids, $H^\bullet_{k\text{-hom}}(\mathfrak{v}) \cong H^\bullet(\mathfrak{g}, S^k C)$.

Van Est theorem for homogeneous cochains. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g}$. For every section $u \in \Gamma(\mathfrak{g})$, consider the corresponding right invariant vector field $\vec{u} \in \mathfrak{X}(\mathcal{G})$. In the following, we denote by $B_p u$ the vector field on the space of p-composable arrows $B_p\mathcal{G}$ given by

$$B_p u(g_1, \dots, g_p) = (\vec{u}(g_1), 0_{g_2}, \dots, 0_{g_p}). \tag{2-12}$$

Let us now recall the definition of the van Est map. First, using the degeneracy map $s_0 : B_{p-1}\mathcal{G} \to B_p\mathcal{G}$, we define $R_u : C^p(\mathcal{G}) \to C^{p-1}(\mathcal{G})$ by

$$R_u = s_0^* \circ \mathcal{L}_{B_p u}.$$

The van Est map $\mathrm{VE} : C^p(\mathcal{G}) \to \mathrm{CE}^p(\mathfrak{g})$ is defined (up to p-dependent sign) as follows [Crainic 2003]: for a p-cochain $f \in C^p(\mathcal{G})$,

$$\mathrm{VE}(f)(u_1, \dots, u_p) = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\, R_{u_{\sigma(1)}} \dots R_{u_{\sigma(p)}}(f). \tag{2-13}$$

In [Crainic 2003] it is shown that it induces a map in cohomology which preserves the corresponding product structures. We also need the following naturality result about VE.

Lemma 2.10. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Lie groupoids with Lie algebroids $\mathfrak{h}_1, \mathfrak{h}_2$, respectively. If $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a Lie groupoid morphism with the corresponding Lie algebroid morphism* $\mathrm{Lie}(\phi) : \mathfrak{h}_1 \to \mathfrak{h}_2$, *then*

$$\mathrm{VE}(B_p\phi^* f) = \mathrm{Lie}(\phi)^* \mathrm{VE}(f) \quad \forall f \in C^p(\mathcal{H}_2).$$

Proof. For any $\chi \in \Gamma(\mathfrak{h}_1)$ we can write

$$\mathrm{Lie}(\phi)(\chi) = \sum_i \gamma_i (\tilde{\chi}_i \circ \phi_0) \in \Gamma(\phi_0^* \mathfrak{h}_2),$$

where $\phi_0 = B_0\phi : M_1 \to M_2$ denotes the map between objects induced by ϕ, $\gamma_i \in C^\infty(M_1)$ and $\tilde{\chi}_i \in \Gamma(\mathfrak{h}_2)$. A direct computation shows that

$$R_\chi((B_p\phi)^* f) = \sum_i (\mathrm{t}^*_{p-1}\gamma_i)(B_{p-1}\phi)^*(R_{\tilde{\chi}_i} f) \quad \forall f \in C^p(\mathcal{H}_2).$$

If we apply the above formula p times, we notice that most of the terms in $R_{\chi_1} \dots R_{\chi_p}(B_p\phi)^* f$ will vanish since VE is defined on normalized cochains (namely, $s_i^* f = 0$). The only remaining terms are

$$\sum_{i_1, \dots, i_p} \gamma_{i_1} \dots \gamma_{i_p} \phi_0^*(R_{\tilde{\chi}_1} \dots R_{\tilde{\chi}_p} f),$$

and we thus get the statement of the lemma. □

The main result about the van Est map in the present context is as follows.

Theorem 2.11 [Crainic 2003]. *Let $\mathcal{G}$ be a Lie groupoid and let $\mathfrak{g}$ be its Lie algebroid. The van Est map (2-13) induces an algebra homomorphism*

$$\mathrm{VE} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g}).$$

Moreover, if $\mathcal{G}$ has p_0-connected source fibers, then VE *is an isomorphism in degrees $p \leq p_0$, and it is injective for $p = p_0 + 1$.*

To get our refinement of Theorem 2.11 for homogeneous cochains on VB-groupoids and algebroids, we first state a simple homological algebra fact.

Homological lemma. *Let $(C_i^\bullet, \delta_i)$ be differential complexes, $i = 1, 2$, endowed with projections $P_i : C_i^\bullet \to C_i^\bullet$ (i.e., $P_i \circ \delta_i = \delta_i \circ P_i$ and $P_i^2 = P_i$). If $F : C_1^\bullet \to C_2^\bullet$ is a morphism satisfying $F \circ P_1 = P_2 \circ F$, then for each p such that $F : H^p(C_1) \to H^p(C_2)$ is injective (resp. surjective) its restriction $F_r : H^p(S_1) \to H^p(S_2)$ is also injective (resp. surjective), where $S_i^\bullet = P_i(C_i^\bullet)$.*

We are thus left with studying the behavior of the projections onto homogeneous cochains under the van Est map. To that end, let $\mathcal{V} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G} \rightrightarrows M$ and let $\mathfrak{v} \to E$ be its Lie algebroid.

Proposition 2.12. *For each $k \in \mathbb{N}_0$ and every $p \geq 0$,*

$$\mathrm{VE} \circ P^{\mathcal{G},p}_{k\text{-hom}} = P^{\mathfrak{g},p}_{k\text{-hom}} \circ \mathrm{VE}.$$

In particular, $\mathrm{VE}(C^\infty_{k\text{-hom}}(B_p\mathcal{V})) \subset \Gamma_{k\text{-hom}}(\Lambda^p \mathfrak{v}^*)$.

Proof. Let $h^{\mathcal{G}}_\lambda : \mathcal{V} \to \mathcal{V}$ and $h^{\mathfrak{g}}_\lambda : \mathfrak{v} \to \mathfrak{v}$ be the homogeneous structures of the vector bundles $\mathcal{V} \to \mathcal{G}$ and $\mathfrak{v} \to \mathfrak{g}$, respectively. By Lemma 2.10, the fact that $h^{\mathcal{G}}_\lambda$ is a groupoid homomorphism with $\mathrm{Lie}(h^{\mathcal{G}}_\lambda) = h^{\mathfrak{g}}_\lambda$ implies that

$$\mathrm{VE} \circ h^{\mathcal{G}\,*}_\lambda = h^{\mathfrak{g}\,*}_\lambda \circ \mathrm{VE} \quad \forall\, \lambda.$$

Hence, by applying $\frac{d}{d\lambda}\big|_{\lambda=0}$ on both sides, one obtains the commutation relation between VE and the projections $P^{\cdot,p}_{\mathrm{hom},k}$. The result now follows directly. $\square$

The restriction of the van Est map to the subcomplex of k-homogeneous cochains shall be denoted by

$$\mathrm{VE}_{k\text{-hom}} := \mathrm{VE}|_{C^p_{k\text{-hom}}(\mathcal{V})} : C^p_{k\text{-hom}}(\mathcal{V}) \to \mathrm{CE}^p_{k\text{-hom}}(\mathfrak{v}).$$

Example 2.13 (0-homogeneous cochains). For $k = 0$, using the isomorphisms $C^p_{0\text{-hom}}(\mathcal{V}) \cong C^p(\mathcal{G})$ and $\mathrm{CE}^p_{0\text{-hom}}(\mathfrak{v}) \cong \mathrm{CE}^p(\mathfrak{g})$, one can check that $\mathrm{VE}_{0\text{-hom}} \cong \mathrm{VE}_{\mathcal{G}} : C^p(\mathcal{G}) \to \mathrm{CE}^p(\mathfrak{g})$. To see this, take $f \in C^p_{0\text{-hom}}(\mathcal{V})$ and $\chi_1, \ldots, \chi_p \in \mathfrak{v}$, and notice that $\mathrm{VE}_{0\text{-hom}}(f)(\chi_1, \ldots, \chi_p)$ only depends on the projections $u_i \in \mathfrak{g}$ of χ_i,

$i = 1, \ldots, p$. Hence, to compute $\mathrm{VE}_{0\text{-hom}}$, it suffices to take $\chi_1, \ldots, \chi_p$ linear sections[1] of $\mathfrak{v}$ covering $u_1, \ldots, u_p \in \Gamma(\mathfrak{g})$. In this case,

$$\mathrm{VE}_{0\text{-hom}}(f)(\chi_1, \ldots, \chi_p) = \pi_E^* \mathrm{VE}_{\mathcal{G}}(f_0)(u_1, \ldots, u_p),$$

where $f = \pi_{B_p\mathcal{G}}^* f_0$, $f_0 \in C^\infty(B_p\mathcal{G})$, and where $\pi_E : E \to M$, $\pi_{B_p\mathcal{V}} : B_p\mathcal{V} \to B_p\mathcal{G}$ are the vector bundle projections.

We are now ready to state and prove our main theorem.

Theorem 2.14. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g}$. For a VB-groupoid $\mathcal{V} \rightrightarrows E$ over $\mathcal{G}$ with underlying VB-algebroid $\mathfrak{v} \to E$, the van Est map on k-homogeneous cochains induces a module homomorphism*

$$\mathrm{VE}_{k\text{-hom}} : H^\bullet_{k\text{-hom}}(\mathcal{V}) \to H^\bullet_{k\text{-hom}}(\mathfrak{v})$$

covering the algebra homomorphism $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. Moreover, if $\mathcal{G}$ has p_0-connected source fibers, then $\mathrm{VE}_{k\text{-hom}}$ is an isomorphism for all $p \le p_0$ and it is injective for $p = p_0 + 1$.

Proof. The $H^\bullet(\mathcal{G})$-module structure on $H^\bullet_{k\text{-hom}}(\mathfrak{v})$ comes from the cup product of $C^\bullet_{k\text{-hom}}(\mathcal{V})$ and $C^\bullet_{0\text{-hom}}(\mathcal{V}) \cong C^\bullet(\mathcal{G})$. So, the first statement follows from the fact that $\mathrm{VE}_{k\text{-hom}}$ is the restriction of the van Est map of $\mathcal{V}$ to homogeneous cochains and that $\mathrm{VE}_{0\text{-hom}} \cong \mathrm{VE}_{\mathcal{G}}$.

Let us now assume that $\mathcal{G}$ has p_0-connected source fibers. First note that this implies that $\mathcal{V} \rightrightarrows E$ is also source p_0-connected. Indeed, a source fiber of $\mathcal{V} \rightrightarrows E$ is an affine bundle over the corresponding source fiber of $\mathcal{G} \rightrightarrows M$. So, the van Est theorem (Theorem 2.11) implies that $\mathrm{VE} : H^p(\mathcal{V}) \to H^p(\mathfrak{v})$ is an isomorphism for $p \le p_0$ and injective for $p = p_0 + 1$. The result now follows from Proposition 2.12 by applying the homological lemma to $F = \mathrm{VE}$, $(C_1^\bullet, \delta_1) = (C^\infty(B_\bullet\mathcal{V}), \delta)$ and $(C_2^\bullet, \delta_2) = (\Gamma(E, \Lambda^\bullet \mathfrak{v}^*), d)$ with projections $P_1 = P^{\mathcal{G},\bullet}_{k\text{-hom}} : C^\infty(B_\bullet\mathcal{V}) \to C^\infty_{k\text{-hom}}(B_\bullet\mathcal{V})$ and $P_2 = P^{\mathfrak{g},\bullet}_{k\text{-hom}} : \Gamma(E, \Lambda^\bullet \mathfrak{v}^*) \to \Gamma_{k\text{-hom}}(E, \Lambda^\bullet \mathfrak{v}^*)$. □

3. 1-homogeneous cochains and representations up to homotopy

In [Gracia-Saz and Mehta 2010; 2011], it was shown that VB-groupoids and VB-algebroids provide an intrinsic version of the notion of (2-term) representation up to homotopy, generalizing the example given in the introduction, as well as Examples 2.5 and 2.9 above. In this section, we show how Theorem 2.14, when applied to 1-homogeneous cochains, recovers a van Est result for the underlying 2-term representations up to homotopy [Arias Abad and Schätz 2011]. We also outline a new proof, realizing an original proposal [Crainic and Moerdijk 2008] of a rigidity conjecture involving the deformation cohomology underlying proper groupoids.

[1] A linear section χ of $\mathfrak{v}$ is a section $\chi : E \to \mathfrak{v}$ which is a vector bundle homomorphism covering a section $u : M \to \mathfrak{g}$ (see [Gracia-Saz and Mehta 2010]).

VB-groupoid and VB-algebroid cohomology. Following [Gracia-Saz and Mehta 2011], given a VB-groupoid $\pi : \mathcal{V} \to \mathcal{G}$ we define $C^p_{\mathrm{VB}}(\mathcal{V})$ to be the space of 1-homogeneous cochains $\phi \in C^\infty_{1\text{-hom}}(B_p\mathcal{V})$ satisfying the two additional conditions

(1) $\phi(0_g, \xi_1, \dots, \xi_{p-1}) = 0$,

(2) $\phi(0_g \cdot \xi_1, \dots, \xi_p) = \phi(\xi_1, \dots, \xi_p)$

for all $(\xi_1, \dots, \xi_p) \in B_p\mathcal{V}$ and $g \in \mathcal{G}$ such that $(0_g, \xi_1) \in B_2\mathcal{V}$. As observed in [Gracia-Saz and Mehta 2011], condition (1) above implies that $\phi(\xi_1, \xi_2, \dots, \xi_p)$ only depends on ξ_1 and on the projections $g_i = \pi(\xi_i) \in \mathcal{G}$, $i = 1, \dots, p$, while condition (2) is a left-invariance property.

It is shown in that paper that $C^\bullet_{\mathrm{VB}}(\mathcal{V})$ defines a subcomplex of $C^\bullet_{1\text{-hom}}(\mathcal{V})$. Moreover, the cup product with $C^\bullet_{0\text{-hom}}\mathcal{V} \cong C^\bullet(\mathcal{G})$ defines a right $C^\bullet(\mathcal{G})$-submodule structure on $C^\bullet_{\mathrm{VB}}(\mathcal{V})$. The next lemma relates the cohomology of the two complexes.

Lemma 3.1. *The inclusion $\iota : C^\bullet_{\mathrm{VB}}(\mathcal{V}) \hookrightarrow C^\bullet_{1\text{-hom}}(\mathcal{V})$ induces an isomorphism of right $H^\bullet(\mathcal{G})$-modules in cohomology.*

Proof. It is enough to show that for every $\phi \in C^p_{1\text{-hom}}(\mathcal{V})$ with $\delta\phi \in C^{p+1}_{\mathrm{VB}}(\mathcal{V})$ there exists a $\psi \in C^{p-1}_{1\text{-hom}}(\mathcal{V})$ so that $\phi + \delta\psi \in C^p_{\mathrm{VB}}(\mathcal{V})$. To that end, first notice that if an arbitrary ϕ is such that both ϕ and $\delta\phi$ satisfy condition (1), then ϕ satisfies condition (2). This follows directly from evaluating

$$0 = (\delta\phi)(0_g, \xi_1, \dots, \xi_p).$$

We are thus left with showing that for each $\phi \in C^\infty_{1\text{-hom}}(B_p\mathcal{V})$ such that $\delta\phi$ satisfies (1) there exists a $\psi \in C^\infty_{1\text{-hom}}(B_{p-1}\mathcal{V})$ such that $\phi + \delta\psi$ satisfies (1). This, in turn, follows by applying recursively the following claim: if $\delta\phi$ satisfies (1) and

$$\phi(\xi_0, \dots, \xi_{p-1}) = 0 \tag{3-1}$$

for all $(\xi_0, \dots, \xi_{p-1}) \in B_p\mathcal{V}$ such that $\xi_i = 0_{g_i}$, $i = 0, \dots, l \le p-1$, then there exists a $\psi \in C^\infty_{1\text{-hom}}(B_{p-1}\mathcal{V})$ such that $\phi + \delta\psi$ satisfies (3-1) for all $(\xi_0, \dots, \xi_{p-1}) \in B_p\mathcal{V}$ such that $\xi_i = 0_{g_i}$, $i = 0, \dots, l-1$. Notice that for $l = p-1$, (3-1) follows from ϕ being homogeneous of degree 1. To prove this claim for $l < p-1$, one chooses any $\psi \in C^\infty_{1\text{-hom}}(B_{p-1}\mathcal{V})$ such that

$$\psi(\xi_1, \dots, \xi_{p-1}) = -\phi(0_{\pi(\xi_{p-1})^{-1} \cdots \pi(\xi_1)^{-1}}, \xi_1, \dots, \xi_{p-1})$$

for all $(\xi_1, \dots, \xi_{p-1}) \in B_{p-1}\mathcal{V}$ such that $\mathrm{t}_{\mathcal{V}}(\xi_1) = 0_{\mathrm{t}(\pi(\xi_1))}$. This is always possible since the subset of such elements in $B_{p-1}\mathcal{V}$ is a smooth embedded submanifold since the target map is a submersion. What needs to be shown now is

$$(\phi + \delta\psi)(\xi_0, \dots, \xi_{p-1}) = 0 \quad \forall\, (\xi_0, \dots, \xi_{p-1}) \in B_p\mathcal{V},\ \xi_i = 0_{g_i},\ i = 0, \dots, l-1.$$

Finally, this last identity follows by evaluating

$$0 = (\delta\phi)(0_{\pi(\xi_{p-1})^{-1}\cdots\pi(\xi_0)^{-1}}, \xi_0, \ldots, \xi_{p-1})$$

and using the recursion hypothesis. □

For a VB-algebroid $\mathfrak{v} \to A$, the *VB-algebroid cochain complex* is defined exactly as the complex of 1-homogeneous cochains

$$\mathrm{CE}^k_{\mathrm{VB}}(\mathfrak{v}) := \mathrm{CE}^k_{1\text{-hom}}(\mathfrak{v}).$$

The restriction of the van Est map to 1-homogeneous cochains as on page 307 provides a map $\mathrm{VE}_{1\text{-hom}} : C^\bullet_{1\text{-hom}}(\mathcal{V}) \to \mathrm{CE}^\bullet_{\mathrm{VB}}(\mathfrak{v})$. Its restriction to the subcomplex $C^\bullet_{\mathrm{VB}}(\mathcal{V}) \subset C^\bullet_{1\text{-hom}}(\mathcal{V})$ will be denoted by

$$\mathrm{VE}_{\mathrm{VB}} : C^\bullet_{\mathrm{VB}}(\mathcal{V}) \to \mathrm{CE}^\bullet_{\mathrm{VB}}(\mathfrak{v}).$$

Corollary 3.2. *With the notations above, the van Est map*

$$\mathrm{VE}_{\mathrm{VB}} : H^\bullet(C_{\mathrm{VB}}(\mathcal{V})) \to H^\bullet(\mathrm{CE}_{\mathrm{VB}}(\mathfrak{v}))$$

is a right-module homomorphism over $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. *Moreover, if $\mathcal{G}$ is source p_0-connected, then* $\mathrm{VE}_{\mathrm{VB}}$ *is an isomorphism in degree p for all $p \leq p_0$ and it is injective for $p = p_0 + 1$.*

Cohomological vanishing for proper groupoids. The VB-groupoid cohomology can be shown to be trivial in several cases as shown by the following proposition.

Proposition 3.3. *When $\mathcal{G}$ is a proper groupoid or, more generally, admits a Haar system $d\mu$ together with a cutoff function $c \in C^\infty(M)$* (see, e.g., [Arias Abad and Crainic 2013] and the proof below), *then*

$$H^p(C^\bullet_{\mathrm{VB}}(\mathcal{V})) = 0, \qquad p \geq 2.$$

Proof. The idea is to define a map $C^p_{\mathrm{VB}}(\mathcal{V}) \ni \phi \mapsto \kappa(\phi) \in C^{p-1}_{\mathrm{VB}}(\mathcal{V})$ for $p \geq 2$ by the formula

$$\kappa(\phi)(\xi_1, \ldots, \xi_{p-1}) = \int_{\mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))} \phi\big(\xi_1, \ldots, \xi_{p-1}, \sigma(h, \mathsf{s}_{\mathcal{V}}(\xi_{p-1}))\big)\, c(s(h))\, d\mu(h),$$

where $g_i = \pi(\xi_i) \in \mathcal{G}$, $i = 1, \ldots, p-1$, as before and $\sigma : \mathsf{t}^*E \to \mathcal{V}$ is any linear splitting of the epimorphism $\mathsf{t}_{\mathcal{V}} : \mathcal{V} \to \mathsf{t}^*E$. Notice that the right-hand side in the formula above is independent of the choice of σ since ϕ only depends on $(g_1, \ldots, g_{p-1}, h)$ and ξ_1. The key point is that, for $\delta\phi = 0$, $\phi \in C^p_{\mathrm{VB}}(\mathcal{V})$, $p \geq 2$, we have $\delta\kappa(\phi) = (-1)^p\phi$, hence leading to the above cohomological vanishing. This statement can be checked by direct computation: let us write $\xi_{p+1}(h) = \sigma(h, \mathsf{s}_{\mathcal{V}}(\xi_p))$ for $h \in \mathsf{t}^{-1}(\mathsf{s}(g_p))$ and $\eta_p(k) = \sigma(k, \mathsf{s}_{\mathcal{V}}(\xi_{p-1}))$ for $k \in \mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))$. Then

$$\begin{aligned}
&\delta\kappa(\phi)(\xi_1, \dots, \xi_p)\\
&= \int_{\mathsf{t}^{-1}(\mathsf{s}(g_p))} \Big[\phi(\xi_2, \dots, \xi_p, \xi_{p+1}(h)) \\
&\qquad\qquad + \sum_{i=1}^{p-1} (-1)^i \phi(\xi_1, \dots, \xi_i\xi_{i+1}, \dots, \xi_p, \xi_{p+1}(h)) \Big] c(s(h))\, d\mu(h)\\
&\qquad + (-1)^p \int_{\mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))} \phi(\xi_1, \dots, \xi_{p-1}, \eta_p(k))\, c(s(k))\, d\mu(k)\\
&= (-1)^p \int_{\mathsf{t}^{-1}(\mathsf{s}(g_p))} \big[-\phi(\xi_1, \dots, \xi_{p-1}, \xi_p\xi_{p+1}(h)) + \phi(\xi_1, \dots, \xi_p) \big] c(s(h))\, d\mu(h)\\
&\qquad + (-1)^p \int_{\mathsf{t}^{-1}(\mathsf{s}(g_{p-1}))} \phi(\xi_1, \dots, \xi_{p-1}, \eta_p(k))\, c(s(k))\, d\mu(k)\\
&= (-1)^p \phi(\xi_1, \dots, \xi_p).
\end{aligned}$$

Above, the first equality follows from the definitions of δ and κ, the second equality follows by applying $\delta\phi = 0$ inside the square brackets and, finally, the third equality follows by the normalization condition $\int_{\mathsf{t}^{-1}(x)} c(s(h))\, d\mu(h) = 1$ and by the left invariance of the measure $\int_{\mathsf{t}^{-1}(s(g))} f(gh)\, d\mu(h) = \int_{\mathsf{t}^{-1}(\mathsf{t}(g))} f(k)\, d\mu(k)$ together with the independence of $\phi(\xi_1, \dots, \xi_p)$ on the ξ_j for $j > 1$, as was mentioned before. $\square$

Let us now mention an application of the above general vanishing result, following [Crainic and Moerdijk 2008]. Given a Lie algebroid $\mathfrak{g} \to M$, there exists a complex $C^\bullet_{\mathrm{def}}(\mathfrak{g})$ controlling the deformations of $\mathfrak{g}$ and which is related to VB-cohomology as follows. Consider the induced linear Poisson structure on $\mathfrak{g}^*$, $\pi \in \Gamma(\Lambda^2 T\mathfrak{g}^*)$. The cotangent Lie algebroid $T^*\mathfrak{g} \to \mathfrak{g}^*$ has the property that its Chevalley–Eilenberg complex $(\mathrm{CE}(T^*\mathfrak{g}), d)$ is isomorphic to the Poisson complex $(\mathfrak{X}(\mathfrak{g}^*), [\pi, \cdot\,])$; see [Mackenzie and Xu 1994]. Under this isomorphism, the subcomplex $\mathrm{CE}^\bullet_{\mathrm{VB}}(T^*\mathfrak{g}) \subset \mathrm{CE}^\bullet(T^*\mathfrak{g})$ corresponds to the so-called *linear Poisson complex* $\mathfrak{X}_{\mathrm{lin}}(\mathfrak{g}^*)$ of $\mathfrak{g}^*$. On the other hand, Proposition 7 in [Crainic and Moerdijk 2008] shows that $\mathfrak{X}^\bullet_{\mathrm{lin}}(\mathfrak{g}^*) \cong C^\bullet_{\mathrm{def}}(\mathfrak{g})$, so that

$$\mathrm{CE}^\bullet_{\mathrm{VB}}(T^*\mathfrak{g}) \cong \mathfrak{X}^\bullet_{\mathrm{lin}}(\mathfrak{g}^*) \cong C^\bullet_{\mathrm{def}}(\mathfrak{g}).$$

On the groupoid side, for a Lie groupoid $\mathcal{G} \rightrightarrows M$, the complex $C_{\mathrm{VB}}(T^*\mathcal{G})$ was shown in [Crainic et al. 2015a] to be isomorphic to the complex $C_{\mathrm{def}}(\mathcal{G})$ controlling deformations of the Lie groupoid structure.

In this context, Corollary 3.2 recovers a result from [Crainic et al. 2015a]: the map

$$\mathrm{VE}_{\mathrm{def}} : H^\bullet_{\mathrm{def}}(\mathcal{G}) \to H^\bullet_{\mathrm{def}}(\mathfrak{g})$$

defines a (graded) module homomorphism covering $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$ which induces isomorphisms in degrees $p \le p_0$ and a monomorphism in degree $p = p_0 + 1$ when $\mathcal{G}$ is source p_0-connected.

By combining this result with our general vanishing criteria (Proposition 3.3 above), we further obtain an independent proof of the (cohomological) rigidity conjecture of [Crainic and Moerdijk 2008]: if $\mathcal{G}$ is proper and source 2-connected, then $H^2_{\mathrm{def}}(\mathfrak{g}) = 0$. Note that the map $\mathrm{VE}_{\mathrm{def}}$ is the "lin version" of the van Est map which was assumed to exist by Crainic and Moerdijk [2008] as a step towards proving their conjecture.

Remark 3.4. The conjecture was originally proved in [Arias Abad and Schätz 2011] using a van Est result for representations up to homotopy. In particular, they used a vanishing result for cohomologies with coefficients in representations up to homotopy established in [Arias Abad and Crainic 2013]. Our vanishing result should be considered as a geometric counterpart to theirs in the 2-term case (see below).

Splittings and representations up to homotopy. VB-groupoids and VB-algebroids can be (noncanonically) *split* into the base Lie groupoid and Lie algebroid data and representation-like information on the fibers (recall Examples 2.5 and 2.9). It turns out that the correct notion encoding this split data is that of (2-term) *representations up to homotopy* [Arias Abad and Crainic 2012; 2013; Gracia-Saz and Mehta 2010; 2011], which we now recall.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\mathfrak{g} \to M$ and $\mathcal{E} = C[1] \oplus E$ a graded vector bundle over M with C in degree -1 and E in degree 0. The associated space of $\mathcal{E}$-valued (normalized) p-cochains is defined as

$$C(\mathcal{G}, \mathcal{E})^p$$
$$:= \{\mu := (\mu_E, \mu_C) \in \Gamma(B_p\mathcal{G}; \mathsf{t}_p^* E) \oplus \Gamma(B_{p+1}\mathcal{G}; \mathsf{t}_{p+1}^* C) \mid s_i^*\mu_E = 0,\, s_i^*\mu_C = 0\},$$

where $s_i : B_\bullet\mathcal{G} \to B_{\bullet+1}\mathcal{G}$ is the i-th degeneracy map. There is a (right) $C^\bullet(\mathcal{G})$-module structure on $C(\mathcal{G}, \mathcal{E})^\bullet$ defined by $\mu \star f = (\mu_E \star f, \mu_C \star f)$, where each component is given by formula (2-6). A representation up to homotopy of $\mathcal{G}$ on $\mathcal{E}$ is an $\mathbb{R}$-linear map $\mathcal{D}_{\mathcal{G}} : C(\mathcal{G}, \mathcal{E})^\bullet \to C(\mathcal{G}, \mathcal{E})^{\bullet+1}$ satisfying $\mathcal{D}_{\mathcal{G}}^2 = 0$ and

$$\mathcal{D}_{\mathcal{G}}(\mu \star f) = \mathcal{D}_{\mathcal{G}}(\mu) \star f + (-1)^p \mu \star (\delta f), \quad \mu \in C(\mathcal{G}, \mathcal{E})^p,\ f \in C^{p'}(\mathcal{G}).$$

The resulting cohomology is denoted by $H(\mathcal{G}, \mathcal{E})$. Note that $\star$ defines a right $H(\mathcal{G})$-module structure on $H(\mathcal{G}, \mathcal{E})$.

A representation up to homotopy on $\mathcal{E}$ can be alternatively given by quasiactions Δ^E and Δ^C of $\mathcal{G}$ on E and C, respectively, a bundle map $\partial : C \to E$ and a smooth correspondence which, for each $(g_1, g_2) \in B_2\mathcal{G}$, gives a linear map $\Omega_{(g_1,g_2)} : E|_{\mathsf{s}(g_2)} \to C|_{\mathsf{t}(g_1)}$ satisfying certain structural equations (see [Arias Abad and Crainic 2013; Gracia-Saz and Mehta 2011]). Moreover, in analogy with the case of an ordinary representation (cf. Example 2.5), a representation up to homotopy of $\mathcal{G}$ on $\mathcal{E}$ endows $\mathcal{V} = \mathsf{s}^*E^* \oplus_{\mathcal{G}} \mathsf{t}^*C^* \rightrightarrows C^*$ with a VB-groupoid structure [Gracia-Saz and Mehta 2011]. The structure maps are given by

$$\begin{aligned} &\mathsf{s}_{\mathcal{V}}(\xi, g, \eta) = (\Delta_g^C)^*\xi - \partial^*\eta,\ \mathsf{t}_{\mathcal{V}}(\xi, g, \eta) = \xi, \quad \xi \in C^*|_{\mathsf{t}(g)},\ \eta \in E^*|_{\mathsf{s}(g)}, \\ &(\xi_1, g_1, \eta_1)\cdot(\xi_2, g_2, \eta_2) = \big(\xi_1,\ g_1g_2,\ \Omega^*_{(g_1,g_2)}\xi_1 + (\Delta^E_{g_2})^*\eta_1 + \eta_2\big) \end{aligned} \tag{3-2}$$

for compatible arrows and $\mathbf{1}_{\mathcal{V}}(\xi) = (\xi, 1_m, 0)$ for $\xi \in C^*|_m$. Finally, in [Gracia-Saz and Mehta 2011] the authors show that every VB-groupoid can be presented (noncanonically) in this form, thus establishing a correspondence between VB-groupoids and 2-term representations up to homotopy of G.

The above correspondence between VB-groupoid structures and representations up to homotopy can be understood from the following relation between the cochain complex associated to $\mathcal{E}$ and that of 1-homogeneous cochains on $\mathcal{V}$. Consider the map $\Psi : C(\mathcal{G}, \mathcal{E})^p \to C^\infty_{1\text{-hom}}(B_{p+1}\mathcal{V})$ defined by

$$\begin{aligned} \Psi(\mu)((\xi_1, g_1, \eta_1), \ldots, (\xi_{p+1}, g_{p+1}, \eta_{p+1})) \\ = \langle \eta_1, \mu_E(g_2, \ldots, g_{p+1})\rangle + \langle \xi_1, \mu_C(g_1, \ldots, g_{p+1})\rangle. \end{aligned} \tag{3-3}$$

In [Gracia-Saz and Mehta 2011] (see Theorem 5.6), it is proven that $\Psi : C(\mathcal{G}, \mathcal{E})^\bullet \to C^{\bullet+1}_{1\text{-hom}}(\mathcal{V})$ is a monomorphism of graded $C(\mathcal{G})$-modules satisfying

$$\Psi \circ (-\mathcal{D}_{\mathcal{G}}) = \delta \circ \Psi$$

whose image coincides with the VB-groupoid cochain complex $C^\bullet_{\mathrm{VB}}(\mathcal{V}) \subset C^\bullet_{1\text{-hom}}(\mathcal{V})$ (shifted by one, hence the minus sign in the equation above). We then obtain the next lemma as a direct consequence of Lemma 3.1.

Lemma 3.5. *The map $\Psi : H^\bullet(\mathcal{G}, \mathcal{E}) \to H^{\bullet+1}_{1\text{-hom}}(\mathcal{V})$ induced in cohomology is an isomorphism of right $H^\bullet(\mathcal{G})$-modules.*

Infinitesimal counterpart. Let $\mathfrak{g}$ be a Lie algebroid and $\mathcal{E}$ be as before, and consider

$$\Omega(\mathfrak{g}, \mathcal{E})^p = \Gamma(\Lambda^p\mathfrak{g}^* \otimes E) \oplus \Gamma(\Lambda^{p+1}\mathfrak{g}^* \otimes C).$$

The space $\Omega(\mathfrak{g}, \mathcal{E})$ is a right $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$-module with multiplication defined by wedge product on the right on the $\Lambda^\bullet\mathfrak{g}^*$ factor. A representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$ is an $\mathbb{R}$-linear map $\mathcal{D}_{\mathfrak{g}} : \Omega(\mathfrak{g}, \mathcal{E})^\bullet \to \Omega(\mathfrak{g}, \mathcal{E})^{\bullet+1}$ satisfying $\mathcal{D}_{\mathfrak{g}}^2 = 0$ and

$$\mathcal{D}_{\mathfrak{g}}(\omega \wedge \beta) = \mathcal{D}_{\mathfrak{g}}(\omega) \wedge \beta + (-1)^p \omega \wedge d\beta, \quad \omega \in \Omega(\mathfrak{g}, \mathcal{E})^p,\ \beta \in \Gamma(\Lambda\mathfrak{g}^*).$$

We denote the cohomology of $(\Omega(\mathfrak{g}, \mathcal{E}), \mathcal{D}_{\mathfrak{g}})$ by $H(\mathfrak{g}, \mathcal{E})$.

As in the VB-groupoid case, VB-algebroid structures on $\mathfrak{v} = C^* \times_M \mathfrak{g} \times_M E^* \to C^*$ are in one-to-one correspondence with representations up to homotopy of $\mathfrak{g}$ on

$\mathcal{E} = C[1] \oplus E$ (see [Gracia-Saz and Mehta 2010]). We recall here how this correspondence can be seen from the cohomological perspective. The space of sections $\Gamma(C^*, \mathfrak{v})$ is generated, as a $C^\infty(C^*)$-module, by sections:

$$\chi_u(\xi) = (\xi, u(m), 0), \quad \Upsilon_\eta(\xi) = (\xi, 0, \eta(m))$$

for $\xi \in C^*|_m$, $u \in \Gamma(\mathfrak{g})$, $\eta \in \Gamma(E^*)$. Define a map

$$\mathrm{ev} : \mathrm{CE}^{p+1}_{1\text{-hom}}(\mathfrak{v}) \to \Omega(\mathfrak{g}, \mathcal{E})^p, \quad \mathrm{ev}(\alpha) = (\hat{\alpha}_E, \hat{\alpha}_C), \tag{3-4}$$

where $\hat{\alpha}_E \in \Gamma(\Lambda^p \mathfrak{g}^* \otimes E)$ and $\hat{\alpha}_C \in \Gamma(\Lambda^{p+1} \mathfrak{g}^* \otimes C)$, by

$$\langle \hat{\alpha}_E(u_1, \ldots, u_p), \eta \rangle = \alpha(\Upsilon_\eta, \chi_{u_1}, \ldots, \chi_{u_p}) \in C^\infty_{0\text{-hom}}(C^*) \cong C^\infty(M),$$
$$\hat{\alpha}_C(u_1, \ldots, u_{p+1}) = \alpha(\chi_{u_1}, \ldots, \chi_{u_{p+1}}) \in C^\infty_{1\text{-hom}}(C^*) \cong \Gamma(C)$$

for $u_1, \ldots, u_{p+1} \in \Gamma(\mathfrak{g})$, $\eta \in \Gamma(E^*)$.

Lemma 3.6. *Under the identification* $\Gamma(\Lambda^\bullet \mathfrak{g}^*) \cong \mathrm{CE}_{hom,0}(\mathfrak{v})$, *the map* ev *is a* (*right*) $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$*-module isomorphism.*

Proof. Let $\{\xi^k\}_{k=1}^{\mathrm{rank}(C^*)}$, $\{\gamma^j\}_{j=1}^{\mathrm{rank}(\mathfrak{g}^*)}$ and $\{e_i\}_{i=1}^{\mathrm{rank}(E)}$ be local frames for C^*, $\mathfrak{g}^*$ and E respectively. We identify e_i (resp. γ^j) with the corresponding section of $\mathfrak{v}^*$: $C^*|_m \ni \xi \mapsto (\xi, 0, e_i(m))$ (resp. $\xi \mapsto (\xi, \gamma^j(m), 0)$). Locally, any element $\alpha \in \mathrm{CE}^{p+1}_{1\text{-hom}}(\mathfrak{v})$ is written as

$$\alpha(m, \xi) = a_k A^k_{j_1 \ldots j_{p+1}}(m) \gamma^{j_1} \wedge \cdots \wedge \gamma^{j_{p+1}} + B^i_{j_1 \ldots j_p}(m) e_i \wedge \gamma^{j_1} \wedge \cdots \wedge \gamma^{j_p},$$

where $\xi = a_k \xi^k(m)$. From the definition, one sees that

$$A^k_{j_1 \ldots j_{p+1}}(m) = \langle \hat{\alpha}_C(u_{j_1}, \ldots, u_{j_{p+1}}), \xi^k(m) \rangle,$$
$$B^i_{j_1 \ldots j_p}(m) = \langle \hat{\alpha}_E(u_{j_1, \ldots, u_{j_p}}), \eta^i(m) \rangle,$$

where $\{u_j\}$, $\{\eta^i\}$ are local frames for $\mathfrak{g}$ and E^* dual to $\{\gamma^j\}$, $\{e_i\}$, respectively. It is now straightforward to prove the statement. □

Hence, the operator $\mathcal{D}_\mathfrak{g}$ defined by $\mathcal{D}_\mathfrak{g} \circ \mathrm{ev} = \mathrm{ev} \circ (-d)$, where d is the Chevalley–Eilenberg differential of $\mathfrak{v}$, defines a representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$. (Note that ev shifts degree by minus one, hence the sign in the definition of $\mathcal{D}_\mathfrak{g}$.) It is shown in [Gracia-Saz and Mehta 2010] that, moreover, every VB-algebroid can be split as $\mathfrak{v} \simeq C^{\infty *} \times_M \mathfrak{g} \times_M E^* \to C^*$, thus establishing a correspondence between VB-algebroids and 2-term representations up to homotopy of $\mathfrak{g}$.

Given a representation up to homotopy $\mathcal{D}_\mathcal{G} : C(\mathcal{G}, \mathcal{E}) \to C(\mathcal{G}, \mathcal{E})$ of $\mathcal{G}$ on $\mathcal{E}$, the VB-groupoid $\mathcal{V} \rightrightarrows C^*$ defined by (3-2), seen as a Lie groupoid over C^*, has a Lie algebroid whose underlying bundle is precisely $\mathfrak{v} = C^* \times_M \mathfrak{g} \times_M E^* \to C^*$. In this case, the above construction of $\mathcal{D}_\mathfrak{g}$ can understood as the differentiation of the representation $\mathcal{D}_\mathcal{G}$, namely, $\mathcal{D}_\mathfrak{g} = \mathrm{Lie}(\mathcal{D}_\mathcal{G})$. (See also [Arias Abad and Schätz 2011].)

Remark 3.7. A representation up to homotopy of $\mathfrak{g}$ on $\mathcal{E}$ can be alternatively described by a map $\partial : C \to E$, $\mathfrak{g}$-connections ∇^E and ∇^C on E and C, respectively, and a curvature term $R \in \Gamma(\Lambda^2\mathfrak{g}^* \otimes \mathrm{Hom}(E, C))$ satisfying some compatibility equations (see [Arias Abad and Crainic 2012; Gracia-Saz and Mehta 2010]). We refer to [Brahic et al. 2014] for the formulas of the operators $(\partial, \nabla^E, \nabla^C, R)$ corresponding to $\mathrm{Lie}(\mathcal{D}_{\mathcal{G}})$ in terms of the data defining $\mathcal{D}_{\mathcal{G}}$.

Van Est theorem for representations up to homotopy. Define $\mathrm{VE_{rep}} : C(\mathcal{G}, \mathcal{E})^p \to \Omega(\mathfrak{g}, \mathcal{E})^p$ by $\mathrm{VE_{rep}} := \mathrm{ev} \circ \mathrm{VE_{1\text{-}hom}} \circ \Psi$. Diagrammatically,

$$
\begin{tikzcd}
C(G, \mathcal{E})^k \arrow[r, "\Psi"] \arrow[d, "\mathrm{VE_{rep}}"'] & C^\infty_{\text{1-hom}}(\mathcal{V}^{(k+1)}) \arrow[d, "\mathrm{VE_{1\text{-}hom}}"] \\
\Omega(\mathfrak{g}, \mathcal{E})^k & \mathrm{CE}^{k+1}_{\text{1-hom}}(\mathfrak{v}) \arrow[l, "\mathrm{ev}"']
\end{tikzcd}
\tag{3-5}
$$

It is clear from the previous discussion that $\mathrm{VE_{rep}}$ induces a map in cohomology.

Theorem 3.8. *The van Est map* $\mathrm{VE_{rep}} : H^\bullet(\mathcal{G}, \mathcal{E}) \to H^\bullet(\mathfrak{g}, \mathcal{E})$ *is a right module homomorphism over* $\mathrm{VE}_{\mathcal{G}} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. *Moreover, if $\mathcal{G}$ is source p_0-connected, then the induced map in cohomology* $\mathrm{VE_{rep}} : H^p(\mathcal{G}, \mathcal{E}) \to H^p(\mathfrak{g}, \mathcal{E})$ *is an isomorphism for $-1 \le p \le p_0 - 1$ and it is injective for $p = p_0$.*

Proof. This is a straightforward consequence of Theorem 2.14 and Lemmas 3.5 and 3.6. Notice the shift in grading for which one has isomorphisms. This arises because one has to apply Theorem 2.14 to $C^\infty_{\text{1-hom}}(B_{k+1}\mathcal{V}) \to \mathrm{CE}^{k+1}_{\text{1-hom}}(\mathfrak{v})$ in order to analyze $C(\mathcal{G}, \mathcal{E})^k \to \Omega^k(\mathfrak{g}, \mathcal{E})$. □

The fact that the above cohomology groups are isomorphic was also proven in [Arias Abad and Schätz 2011] using different techniques (in the more general setting of representations on arbitrarily graded vector bundles). Notice that, from our perspective, it just arises as a refinement of the usual van Est map for $\mathcal{V}$ for 1-homogeneous cochains.

Remark 3.9 (formulas for $\mathrm{VE_{rep}}$). For $u \in \Gamma(\mathfrak{g})$, define the map $R_u : C^p(\mathcal{G}, \mathcal{E}) \to C^{p-1}(\mathcal{G}, \mathcal{E})$ by

$$(R_u\mu_C)(g_1, \dots, g_p) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Delta^C_{\phi^u_\epsilon(\mathtt{t}(g_1))^{-1}} \mu_C(\phi^u_\epsilon(\mathtt{t}(g_1)), g_1, \dots, g_p),$$

where $\phi^u_\epsilon : M \to \mathcal{G}$ is the flow of the right-invariant vector field $\vec{u}$ and the definition $R_u\mu_E$ is analogous. Note that our conventions are different from those in [Arias Abad and Schätz 2011]. One can now check the identities

$$R_{\chi_u}\Psi(\mu) = \Psi(R_u\mu_C, 0), \qquad R_{\Upsilon_\eta}\Psi(\mu) = q^*\langle \mu_E, \eta\rangle,$$
$$R_{\chi_v}R_{\Upsilon_\eta}\Psi(\mu) = q^*\langle R_v\mu_E, \eta\rangle, \quad R_{\Upsilon_\eta}R_{\chi_v}\Psi(\mu) = 0,$$

where $q : B_\bullet\mathcal{V} \to B_\bullet\mathcal{G}$ is the projection map. Using these identities, it is now straightforward to check that

$$\mathrm{VE}_{\mathrm{rep}}(\mu) = (\hat{\mu}_E, \hat{\mu}_C) \in \Gamma(\Lambda^p\mathfrak{g}^* \otimes E) \oplus \Gamma(\Lambda^{p+1}\mathfrak{g}^* \otimes C)$$

is given by

$$\hat{\mu}_E(u_1, \ldots, u_p) = (-1)^p \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) R_{u_{\sigma(1)}} \ldots R_{u_{\sigma(p)}} \mu_0,$$

$$\hat{\mu}_C(u_1, \ldots, u_{p+1}) = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}(\sigma) R_{u_{\sigma(1)}} \ldots R_{u_{\sigma(p+1)}} \mu_C.$$

4. Differential forms with values in a representation

In this section, we study differential forms on a Lie groupoid $\mathcal{G}$ with values in a representation $C \to M$. These objects were introduced in [Crainic et al. 2015b] together with their infinitesimal counterparts, the Spencer operators. We here provide a van Est theorem for them as an application of our main result. The key idea is to reinterpret forms as homogeneous functions.

Van Est theorem for differential forms with coefficients. We start this section by formally defining the ingredients entering the van Est theorem for forms with coefficients (Theorem 4.4 below) without any reference to the VB-groupoids and algebroids. Later, we show how VB-groupoids and VB-algebroids provide a useful framework for interpreting many of the definitions and for giving a proof of Theorem 4.4.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $C \to M$ be a representation of $\mathcal{G}$ and consider the map $\mathtt{t}_p : B_p\mathcal{G} \to M$, $\mathtt{t}_p(g_1, \ldots, g_p) = \mathtt{t}(g_1)$. When no confusion arises, we omit the reference to p and simply denote $\mathtt{t}_p$ by $\mathtt{t}$. The space of q-differential forms on the nerve of $\mathcal{G}$ with coefficients in C is $\Omega^q(B_\bullet\mathcal{G}, \mathtt{t}^*C)$. It carries a differential $\delta : \Omega^q(B_{p-1}\mathcal{G}, \mathtt{t}^*C) \to \Omega^q(B_p\mathcal{G}, \mathtt{t}^*C)$ defined by

$$\delta\omega|_{(g_1,\ldots,g_p)} = \Delta_{g_1} \circ \partial_0^*\omega + \sum_{i=1}^{p}(-1)^i \partial_i^*\omega \quad \text{for } p \geq 2,$$

$$\delta\omega|_g = \Delta_g \circ \mathtt{s}^*\omega - \mathtt{t}^*\omega \quad \text{for } p = 1.$$

It is straightforward to check that $\delta^2 = 0$.

Note that, for $\omega \in \Omega^q(\mathcal{G}, \mathtt{t}^*C)$,

$$\delta\omega|_{(g_1,g_2)} = \Delta_{g_1} \circ \mathrm{pr}_2^*\omega - m^*\omega + \mathrm{pr}_1^*\omega,$$

where $\mathrm{pr}_i(g_1, g_2) = g_i$ for $i = 1, 2$. In this case, a form $\omega \in \Omega^q(\mathcal{G}, \mathtt{t}^*C)$ which satisfies $\delta\omega = 0$ is called *multiplicative* (see [Crainic et al. 2015b]). Note that

$\Omega^q(B_\bullet\mathcal{G}, \mathrm{t}^*C)$ is a right dg-module for $C^\bullet(\mathcal{G})$ with the module structure defined as usual by

$$(\omega\star f)|_{(g_1,\dots,g_{p+p'})}=\omega|_{(g_1,\dots,g_p)} f(g_{p+1},\dots,g_{p+p'}),\ \ \omega\in\Omega^q(B_p\mathcal{G},\mathrm{t}^*C),\ f\in C^{p'}(\mathcal{G}).$$

Remark 4.1. In the case of trivial coefficients (i.e., when C is the trivial line bundle), the de Rham differential turns $\Omega^q(B_p\mathcal{G}, t^*C) = \Omega^q(B_p\mathcal{G})$ into a double complex known as the Bott–Shulman double complex associated to $\mathcal{G}$ (see [Arias Abad and Crainic 2011]). In the remainder of this paper, we focus on the cohomology of δ alone and leave the investigation of compatible double complex structures (corresponding to "multiplicative linear flat connections") for future work.

Let $\mathfrak{g} \to M$ be the Lie algebroid of $\mathcal{G}$. Similarly to [Arias Abad and Crainic 2011], we define the Weil complex $W^{p,q}(\mathfrak{g}, C)$ to be the space of sequences $c = (c_0, c_1, \dots)$, where each

$$c_k : \underbrace{\Gamma(\mathfrak{g}) \times \cdots \times \Gamma(\mathfrak{g})}_{p-k \text{ times}} \to \Omega^{q-k}(M, S^k\mathfrak{g}^* \otimes C)$$

is an $\mathbb{R}$-linear skew-symmetric map whose failure at being $C^\infty(M)$-linear is controlled by

$$\begin{aligned}(4\text{-}1)\quad & c_k(fu_1,\dots,u_{p-k}|\cdot) \\ &= fc_k(u_1,\dots,u_{p-k}|\cdot)+df\wedge c_{k+1}(u_2,\dots,u_{p-k}|u_1,\cdot)\quad \forall f\in C^\infty(M).\end{aligned}$$

For each q, the complex $W^{\bullet,q}(\mathfrak{g}, C)$ carries a differential $\mathrm{d}_W : W^{p,q}(\mathfrak{g}, C) \to W^{p+1,q}(\mathfrak{g}, C)$, which we now define. First, note that $\Omega^i(M, S^j\mathfrak{g}^* \otimes C)$ is a module for the Lie algebra $\Gamma(\mathfrak{g})$. Indeed, for $\alpha \in \Omega^i(M)$ and $P \in \Gamma(S^j\mathfrak{g}^* \otimes C)$,

$$u \cdot (\alpha \otimes P) = (\mathcal{L}_{\rho(u)}\alpha) \otimes P + \alpha \otimes (u \cdot P), \quad u \in \Gamma(\mathfrak{g}),$$

defines an action of $\Gamma(\mathfrak{g})$ on $\Omega^i(M, S^j\mathfrak{g}^* \otimes C)$, where

$$(u \cdot P)(v_1,\dots,v_k) = \nabla_u P(v_1,\dots,v_k) - \sum_{i=1}^k P(v_1,\dots,[u, v_i],\dots,v_k),$$

and $\nabla : \Gamma(\mathfrak{g}) \times \Gamma(C) \to \Gamma(C)$ is the $\mathfrak{g}$-connection giving the representation C. Now, d_W is defined by

$$\begin{aligned}(4\text{-}2)\quad & \mathrm{d}_W(c)_k(u_1,\dots,u_{p-k+1}|v_1,\dots,v_k) \\ &= (-1)^k\Big(d_{\mathrm{CE}}(c_k)(u_1,\dots,u_{p-k+1}|v_1,\dots,v_k) \\ &\qquad - \sum_{j=1}^k i_{\rho(v_j)}c_{k-1}(u_1,\dots,u_{p-k+1}|v_1,\dots,\widehat{v_j},\dots,v_k)\Big),\end{aligned}$$

where d_{CE} is the Chevalley–Eilenberg differential on $C^\bullet(\Gamma(\mathfrak{g}), \Omega^{q-k}(M, S^k\mathfrak{g}^*\otimes C))$. There is a right $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$-module structure on $W^{\bullet,q}(\mathfrak{g}, C)$. It is defined, for $\beta \in \Gamma(\Lambda^{p'}\mathfrak{g}^*)$ and $c \in W^{p,q}(\mathfrak{g}, C)$, by

$$(c\wedge\beta)_k(u_1, \dots, u_{p+p'-k}|\cdot)$$
$$= \sum_{\sigma\in S(p-k,p')} \mathrm{sgn}(\sigma)\, c_k(u_{\sigma(1)}, \dots, u_{\sigma(p-k)})\, \beta(u_{\sigma(p-k+1)}, \dots, u_{\sigma(p+p'-k)}),$$

where $S(p-k, p')$ is the space of $(p-k, p')$-unshuffles.

Proposition 4.2. *$W^{\bullet,q}(\mathfrak{g}, C)$ is a right dg-module for $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$.*

This result will follow from an evaluation isomorphism similar to (3-4) (see Proposition 4.12 below) between $W^{\bullet,q}(\mathfrak{g}, C)$ and another right dg-module for $\Gamma(\Lambda^\bullet\mathfrak{g}^*)$. It is important to remark that all the signs appearing in the above formula for d_W, as well as in formula (4-4) below, are natural consequences of a simple ordering convention in the definition of this evaluation isomorphism.

Remark 4.3. For $p = 0$ we have $W^{0,q}(\mathfrak{g}, C) = \Omega^q(M, C)$. In this case, for $c \in W^{0,q}(\mathfrak{g}, C)$ we have $\mathrm{d}_W(c) = (\mathrm{d}_W(c)_0, \mathrm{d}_W(c)_1)$, where $\mathrm{d}_W(c)_0 : \Gamma(\mathfrak{g}) \to \Omega^q(M,C)$ and $\mathrm{d}_W(c)_1 \in \Omega^{q-1}(M, \mathfrak{g}^*\otimes C)$ are given by

$$\mathrm{d}_W(c)_0(u) = u\cdot c \quad\text{and}\quad \mathrm{d}_W(c)_1(v) = i_{\rho(v)}c.$$

For $W^{1,q}(\mathfrak{g}, C)$, its elements are $c = (c_0, c_1)$, where $c_0 : \Gamma(g) \to \Omega^q(M, C)$ and $c_1 \in \Omega^{q-1}(M, \mathfrak{g}^*\otimes C) \cong \mathrm{Hom}(\mathfrak{g}, \Lambda^{q-1}T^*M\otimes C)$. In this case,

$$\begin{aligned}
\mathrm{d}_W(c)_0(u_1, u_2) &= u_1\cdot c_0(u_2) - u_2\cdot c_0(u_1) - c_0([u_1, u_2]),\\
\mathrm{d}_W(c)_1(u|v) &= i_{\rho(v)}c_0(u) - u\cdot c_1(v) + c_1([u, v]),\\
\mathrm{d}_W(c)_2(v_1, v_2) &= -i_{\rho(v_1)}c_1(v_2) - i_{\rho(v_2)}c_1(v_2).
\end{aligned}$$

Note that, in the case $p = 1$, the equation $\mathrm{d}(c) = 0$ is equivalent to (c_0, c_1) being a C-valued Spencer operator on $\mathfrak{g}$ [Crainic et al. 2015b] and, thus, in particular, to (c_0, c_1) being an infinitesimally multiplicative form [Arias Abad and Crainic 2011] when $C = \mathbb{R}$, with the trivial representation.

Van Est map. Given $u \in \Gamma(\mathfrak{g})$, let $\phi^u_\epsilon : \mathcal{G} \to \mathcal{G}$ be the flow of the right-invariant vector field $\vec{u}$. The flow of the corresponding vector field $B_pu \in \mathfrak{X}(B_p\mathcal{G})$ is given by

$$\psi^u_\epsilon(g_1, \dots, g_p) = (\phi^u_\epsilon(g_1), g_2, \dots, g_p).$$

Define operators $R_u : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to \Omega^q(B_{p-1}\mathcal{G}, \mathrm{t}^*C)$ and $J_u : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to \Omega^{q-1}(B_{p-1}\mathcal{G}, \mathrm{t}^*C)$ by

$$\begin{aligned}
R_u\omega|_{(g_1,\dots,g_{p-1})} &= s_0^*\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Delta_{\phi^u_\epsilon(\mathrm{t}(g_1)))^{-1}} \circ \psi^{u\,*}_\epsilon\omega\right),\\
J_u\omega &= s_0^* i_{B_pu}\omega.
\end{aligned} \tag{4-3}$$

The van Est map $\mathrm{VE}_\Omega : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to W^{p,q}(\mathfrak{g}, C)$, defined by $\mathrm{VE}_\Omega(\omega) = (c_0(\omega), c_1(\omega), \dots)$, has each $c_k(\omega)$ given by

$$c_k(\omega)(u_1, \dots, u_{p-k}|v_1, \dots, v_k) = (-1)^{k(k-1)/2} \sum_{\sigma \in S(p)} \mathrm{sgn}(\sigma)(-1)^{\epsilon(\sigma,k)} D_{\sigma(1)} \dots D_{\sigma(p)}\omega, \tag{4-4}$$

where

$$D_j = \begin{cases} J_{v_j} & \text{if } j \in \{1, \dots, k\}, \\ R_{u_{j-k}} & \text{if } j \in \{k+1, \dots, p\}, \end{cases} \tag{4-5}$$

and

$$\epsilon(\sigma, k) = \#\{(i, j) \in \{1, \dots, k\} \times \{1, \dots, k\} \mid i < j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

Theorem 4.4. VE_Ω *induces a map on cohomology* $\mathrm{VE}_\Omega : H^\bullet(\Omega^q(B_\bullet\mathcal{G}, \mathrm{t}^*C)) \to H^\bullet(W^{\bullet,q}(\mathfrak{g}, C))$ *which is a right module homomorphism over* $\mathrm{VE}_\mathcal{G} : H^\bullet(\mathcal{G}) \to H^\bullet(\mathfrak{g})$. *Moreover, if* $\mathcal{G}$ *is source* p_0*-connected, then*

$$VE_\Omega : H^p(\Omega^q(B_\bullet\mathcal{G}, \mathrm{t}^*C)) \to H^p(W^{\bullet,q}(\mathfrak{g}, C))$$

is an isomorphism for $p \leq p_0$ *and it is injective for* $p = p_0 + 1$, *for each fixed* q.

In the remainder of the paper, we prove Theorem 4.4 by showing how it can be framed as a van Est result for a class of VB-groupoids. Notice that the above theorem recovers Theorem 5.1 of [Arias Abad and Crainic 2011] (up to some sign conventions) when $C = M \times \mathbb{R}$ with the trivial representation. It is interesting that, even in this particular case, our proof is independent of the one given in that paper.

Forms as functions. The key idea in the proof of Theorem 4.4 is that differential forms can be seen as homogeneous functions on an appropriate space. In this subsection, we elaborate on this classical viewpoint.

Let $V_1, \dots, V_{q+1}$ be vector bundles over B and consider the fiber product $\prod_{j=1}^{q+1} V_j = V_1 \times_B \cdots \times_B V_{q+1}$ with the natural vector bundle structure over B (the Whitney sum $V_1 \oplus \cdots \oplus V_{q+1} \to B$).

Simple functions. For $i = 1, \dots, q+1$, let $0_i : \prod_{j \neq i} V_j \to \prod_j V_j$ be the inclusion which puts a zero in the i-th coordinate. Then a function $f \in C^\infty\big(\prod_j V_j\big)$ is said to be *simple* if

$$0_i^* f = 0 \quad \forall\, i = 1, \dots, q+1.$$

For a subset $I \subset \{1, \dots, q+1\}$, denote by $|I|$ its cardinality and by $0_I : \prod_{j \notin I} V_j \to \prod_j V_j$ the inclusion which puts a zero in the entries indicated by the elements of I.

Define $P_{(l)} : C^\infty\big(\prod_j V_j\big) \to C^\infty\big(\prod_j V_j\big)$, $l = -1, 0, 1, \dots, q$, by

$$P_{(-1)}(f) = f,$$
$$P_{(l)}(f) = P_{(l-1)}(f) - \sum_{|I|=q+1-l} 0_I^* P_{(l-1)}(f) \quad \text{for } l = 0, \dots, q. \tag{4-6}$$

Each $P_{(l)}$, $l = 0, \dots, q$, is a projection onto the space of functions of $\prod_j V_j$ which vanishes whenever $q+1-l$ entries are zero. In particular, $P_{\mathrm{spl}} := P_{(q)}$ is a projection onto the space of simple functions.

Multilinearity and skew-symmetry. The map

$$\Gamma(B, V_1^* \otimes \cdots \otimes V_{q+1}^*) \to C^\infty\big(\textstyle\prod_{j=1}^{q+1} V_j\big),$$
$$\mu_1 \otimes \cdots \otimes \mu_{q+1} \mapsto (\ell_{\mu_1} \circ \mathrm{pr}_1) \cdots (\ell_{\mu_{q+1}} \circ \mathrm{pr}_{q+1}),$$

is a monomorphism of $C^\infty(B)$-modules, where $\mathrm{pr}_i : \prod_{j=1}^{q+1} V_j \mapsto V_i$ is the projection onto the i-th summand. It follows from Taylor's theorem that its image is the space of simple $(q+1)$-homogeneous functions.

We are mainly interested in the case $V_1 = \cdots = V_q = V$ and $V_{q+1} = W^*$ and we denote the q-fold fiber product $V \times_B \cdots \times_B V$ by $\bigtimes_B^q V$. A function $f \in C^\infty\big(\bigtimes_B^q V \times_B W^*\big)$ is said to be *skew-symmetric* if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(q)}, \xi) = \mathrm{sgn}(\sigma) f(v_1, \dots, v_q, \xi) \quad \forall\, v_i \in V,\ \xi \in W^*,\ \sigma \in S_q.$$

The map $P_{\mathrm{sk}} : C^\infty\big(\bigtimes_B^q V \times_B W^*\big) \to C^\infty\big(\bigtimes_B^q V \times_B W^*\big)$, defined by

$$P_{\mathrm{sk}}(f) = \frac{1}{q!} \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma) f \circ \sigma, \tag{4-7}$$

is a projection onto the space of skew-symmetric functions, where S_q is the symmetric group and $\sigma : \bigtimes_B^q V \times_B W^* \to \bigtimes_B^q V \times_B W^*$ is the permutation of the first q entries belonging to V according to σ. Let us define

$$\mathfrak{F} : \Gamma(B, \Lambda^q V^* \otimes W) \to C^\infty\big(\bigtimes_B^q V \times W^*\big),$$
$$\omega = (\mu_1 \wedge \cdots \wedge \mu_q) \otimes \xi \mapsto q!\, P_{\mathrm{sk}}\big((\ell_{\mu_1} \circ \mathrm{pr}_1) \cdots (\ell_{\mu_q} \circ \mathrm{pr}_q)(\ell_\xi \circ \mathrm{pr}_{q+1})\big). \tag{4-8}$$

It is straightforward to check that $\mathfrak{F}$ is a monomorphism of $C^\infty(B)$-modules whose image is the space of simple, skew-symmetric $(q+1)$-homogeneous functions. We denote the image of $\mathfrak{F}$ by $C^\infty_{\mathrm{ext}}\big(\bigtimes_B^q V \times W^*\big)$. The projections P_{sk}, P_{spl} and $P_{q+1\text{-hom}}$ commute with each other, and so

$$P_{\mathrm{ext}} := P_{\mathrm{sk}} \circ P_{\mathrm{spl}} \circ P_{q+1\text{-hom}} : C^\infty\big(\bigtimes_B^q V \times_B W^*\big) \to C^\infty_{\mathrm{ext}}\big(\bigtimes_B^q V \times_B W^*\big) \tag{4-9}$$

is a projection onto $C^\infty_{\mathrm{ext}}\big(\bigtimes_B^q V \times_B W^*\big)$.

Example 4.5. For $V = B \times \mathbb{R}^n$, let $\{\theta_1, \dots, \theta_m\}$ be a local frame for W and $\{e^1, \dots, e^n\}$ be a global frame for V^*. A point $p \in \bigtimes_B^q V \times_B W^*$, $q \leq n$, has coordinates

$$p = (x, \underline{y_1}, \dots, \underline{y_q}, \xi_1, \dots, \xi_m), \quad x \in B, \ \underline{y_j} = (y_{1,j}, \dots, y_{n,j}) \in \mathbb{R}^n, \ \xi_l \in \mathbb{R}.$$

For a function $f \in C^\infty\big(\bigtimes_B^q V \times_B W^*\big)$, we have $P_{\mathrm{ext}} f = \dfrac{1}{q!}\mathfrak{F}(\omega_f)$, where $\omega_f \in \Gamma(B, \Lambda^q V^* \otimes W)$ is given by

$$\omega_f(p) = \sum_{1 \leq k_1 < \cdots < k_q \leq n} \sum_{i=1}^m \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma) \frac{\partial^{q+1} f}{\partial y_{k_{\sigma(1)},1} \cdots \partial y_{k_{\sigma(q)},q} \, \partial \xi_i}(x, 0)\, e^{k_1} \wedge \cdots \wedge e^{k_q} \otimes \theta_i(x).$$

The VB-groupoid behind the curtains. We define here the VB-groupoid whose differentiable cochain complex contains the complex of differential forms with coefficients. Later on, we show how the Weil complex is embedded in the Chevalley–Eilenberg complex of its Lie algebroid.

Differential forms with coefficients. Let $T\mathcal{G} \rightrightarrows TM$ be the tangent groupoid, obtained by taking the derivative of all the structure maps defining $\mathcal{G}$. Let us introduce the VB-groupoid $\mathbb{G}_q \rightrightarrows \mathbb{M}_q$ defined by

$$\begin{array}{ccc} \mathbb{G}_q = \underbrace{T\mathcal{G} \times_{\mathcal{G}} \cdots \times_{\mathcal{G}} T\mathcal{G}}_{q \text{ times}} \times_{\mathcal{G}} \mathsf{t}^*C^* & \longrightarrow & \mathcal{G} \\ \downdownarrows & & \downdownarrows \\ \mathbb{M}_q = \underbrace{TM \times_M \cdots \times_M TM}_{q \text{ times}} \times_M C^* & \longrightarrow & M \end{array} \tag{4-10}$$

where the structure maps are defined[2] componentwise and $\mathsf{t}^*C^* \rightrightarrows C^*$ is the action groupoid corresponding to the right action of $\mathcal{G}$ (see Example 2.5) on C^* obtained by taking adjoints. We frequently omit the subscript q when no confusion arises. The q-fold fiber products on (4-10) are also denoted as $\bigtimes_{\mathcal{G}}^q T\mathcal{G} \times_{\mathcal{G}} \mathsf{t}^*C^*$ and $\bigtimes_M^q TM \times_M C^*$.

Lemma 4.6. *The space of p-composable arrows $B_p\mathbb{G}$ is isomorphic as a vector bundle over $B_p\mathcal{G}$ to the q-fold fiber product $TB_p\mathcal{G} \times_{B_p\mathcal{G}} \cdots \times_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{B_p\mathcal{G}} \mathsf{t}^*C^*$. More concisely,*

$$B_p\mathbb{G} = B_p\big(\bigtimes_{\mathcal{G}}^q T\mathcal{G} \times_{\mathcal{G}} \mathsf{t}^*C^*\big) \cong \big(\bigtimes_{B_p\mathcal{G}}^q TB_p\mathcal{G}\big) \times_{B_p\mathcal{G}} \mathsf{t}^*C^*. \tag{4-11}$$

[2]There is a more general fact playing a role here: Whitney sums of VB-groupoids yield VB-groupoids (see [Bursztyn and Cabrera 2012]).

The proof consists in simply defining the isomorphism

$$B_p\mathbb{G} \ni (\underline{U}^{(1)}, \dots, \underline{U}^{(p)}) \mapsto \big((U_1^{(1)}, \dots, U_1^{(p)}), \dots, (U_q^{(1)}, \dots, U_q^{(p)}), (g_1, \dots, g_p, \xi_1)\big),$$

where each $\underline{U}^{(i)} = (U_1^{(i)}, \dots, U_q^{(i)}, (g_i, \xi_i)) \in \mathbb{G}$.

One important consequence of the isomorphism (4-11) is that the space of differential forms $\Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$ can be identified with a subspace of $C^\infty(B_p\mathbb{G})$, which we denote by $C^\infty_{\mathrm{ext}}(B_p\mathbb{G})$. It is the image of the map (4-8):

$$\mathfrak{F} : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to C^\infty\big(\big(\textstyle\times^q_{B_p\mathcal{G}} TB_p\mathcal{G}\big) \times_{B_p\mathcal{G}} \mathrm{t}^*C^*\big) \cong C^\infty(B_p\mathbb{G}). \tag{4-12}$$

In order to characterize $C^\infty_{\mathrm{ext}}(B_p\mathbb{G})$ more explicitly, note that, given a permutation $\sigma \in S_q$, the permutation map

$$\sigma_{\mathcal{G}} : \textstyle\times^q_{\mathcal{G}} T\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^*C \to \times^q_{\mathcal{G}} T\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^*C$$

is a groupoid morphism and, under the isomorphism (4-11),

$$B_p\sigma_{\mathcal{G}} \cong \sigma_{B_p\mathcal{G}} \tag{4-13}$$

for the corresponding permutation map

$$\sigma_{B_p\mathcal{G}} : \textstyle\times^q_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{B_p\mathcal{G}} \mathrm{t}^*C \to \times^q_{B_p\mathcal{G}} TB_p\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^*C.$$

Similarly, the zero maps $0_i^{\mathcal{G}} : \mathbb{G}_{q-1} \to \mathbb{G}_q$ $(i = 1, \dots, q)$ and $0_{q+1}^{\mathcal{G}} : \times^q_{\mathcal{G}} T\mathcal{G} \to \mathbb{G}_q$ are groupoid morphisms and

$$B_p 0_i^{\mathcal{G}} \cong 0_i^{B_p\mathcal{G}} \quad \forall\, i = 1, \dots, q+1. \tag{4-14}$$

Hence,

$$C^\infty_{\mathrm{ext}}(B_p\mathbb{G}) = \big\{ f \in C^\infty_{(q+1)\text{-hom}}(B_p\mathbb{G}) \mid (B_p\sigma_{\mathcal{G}})^* f = \mathrm{sgn}(\sigma) f,\ (B_p 0_i^{\mathcal{G}})^* f = 0 \quad \forall\, \sigma \in S_q,\ i = 1, \dots, q+1 \big\}.$$

Note that the projection (4-9) gives here, under the isomorphism (4-11), a projection $P_{\mathrm{ext},\mathcal{G}} : C^\infty(B_p\mathbb{G}) \to C^\infty_{\mathrm{ext}}(B_p\mathbb{G})$.

Proposition 4.7. *The projection $P_{\mathrm{ext},\mathcal{G}}$ satisfies*

$$P_{\mathrm{ext},\mathcal{G}} \circ \delta = \delta \circ P_{\mathrm{ext},\mathcal{G}}.$$

In particular, $C^\infty_{\mathrm{ext}}(B_\bullet\mathbb{G})$ is a subcomplex.

Proof. The result follows directly from (4-13), (4-14) and the fact that

$$(B_{p+1}\phi)^* \delta f = \delta (B_p\phi)^* f$$

for an arbitrary groupoid morphism $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ and $f \in C^p(\mathcal{H}_2)$. □

In the following, we denote by $C^\bullet_{\mathrm{ext}}(\mathbb{G})$ and $H^\bullet_{\mathrm{ext}}(\mathbb{G})$ the complex $(C^\infty_{\mathrm{ext}}(B_\bullet\mathbb{G}), \delta)$ and its cohomology, respectively.

Proposition 4.8. *The map $\mathfrak{F} : \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \to C^p_{\mathrm{ext}}(\mathbb{G})$ is a dg-module isomorphism.*

Proof. Let $\partial_i : B_{p+1}\mathcal{G} \to B_p\mathcal{G}$ and $\not\partial_i : B_{p+1}\mathbb{G} \to B_p\mathbb{G}$, $i = 0, \dots, p+1$, be the face maps and let $s_j : B_{p-1}\mathcal{G} \to B_p\mathcal{G}$ and $\mathbb{s}_j : B_{p-1}\mathbb{G} \to B_p\mathbb{G}$, $j = 0, \dots, p-1$, be the degeneracy maps for $\mathcal{G}$ and $\mathbb{G}$, respectively. The result follows from the fact that

$$\not\partial_0^*\mathfrak{F}_\omega = \mathfrak{F}_{g_1\cdot\partial_0^*\omega}, \quad \not\partial_i^*\mathfrak{F}_\omega = \mathfrak{F}_{\partial_i^*\omega}, \quad \mathbb{s}_j^*\mathfrak{F}_\omega = \mathfrak{F}_{s_j^*\omega} \qquad \forall\, \omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C),$$

when restricted to the fiber over $(g_1, \dots, g_p) \in B_p\mathcal{G}$. □

Remark 4.9. The framework presented here can be used to define multiplicativity for differential forms on a Lie groupoid with values in a 2-term representation up to homotopy. This was done in [Egea 2016] by simply changing t^*C^* to $\mathcal{V} = \mathrm{s}^*E^* \oplus \mathrm{t}^*C^*$ with the VB-groupoid structure defined by (3-2).

Weil complex. The Lie algebroid $\mathbb{A}_q \to \mathbb{M}$ of the Lie groupoid (4-10) $\mathbb{G}_q \rightrightarrows \mathbb{M}_q$ is the q-fold fiber-product[3] $\bigtimes^q_{\mathfrak{g}} T\mathfrak{g} \times_{\mathfrak{g}} \pi^*C^* \to \bigtimes^q_M TM \times_M C^*$, where $\pi : \mathfrak{g} \to M$ denotes the projection map of the Lie algebroid of $\mathcal{G}$.

Definition 4.10. Let $\alpha \in \Gamma(\mathbb{M}, \Lambda^\bullet\mathbb{A}^*)$. We say that α is skew-symmetric with respect to $\mathbb{A} \to \mathfrak{g}$ if

$$\sigma_{\mathfrak{g}}^*\alpha = \mathrm{sgn}(\sigma)\alpha \quad \forall\, \sigma \in S_q, \tag{4-15}$$

where $\sigma_{\mathfrak{g}} : \mathbb{A}_q \to \mathbb{A}_q$ permutes the q-coordinates on $\bigtimes^q_{\mathfrak{g}} T\mathfrak{g}$ according to σ. Similarly, α is multilinear with respect to $\mathbb{A} \to \mathfrak{g}$ if

$$h_\lambda^{\mathfrak{g}\,*}\alpha = \lambda^{q+1}\alpha, \tag{4-16}$$

$$(0_i^{\mathfrak{g}})^*\alpha = 0 \quad \forall\, i = 1, \dots, q+1, \tag{4-17}$$

where $\mathfrak{h}^{\mathfrak{g}}_\lambda : \mathbb{A}_q \to \mathbb{A}_q$ is the homogeneous structure of the vector bundle $\mathbb{A}_q \to \mathfrak{g}$, and $0^{\mathfrak{g}}_i : \mathbb{A}_{q-1} \to \mathbb{A}_q$ and $0^{\mathfrak{g}}_{q+1} : \bigtimes^q_{\mathfrak{g}} T\mathfrak{g} \to \mathbb{A}_q$, $i = 1, \dots, q$, are the zero maps.

Let $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}_q^*)$ be the subspace of $\Gamma(\mathbb{M}, \Lambda^p\mathbb{A}_q^*)$ of skew-symmetric multilinear forms with respect to $\mathbb{A} \to \mathfrak{g}$. In particular, $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}_q^*)$ is a subset of $\Gamma_{(q+1)\text{-hom}}(\mathbb{M}, \Lambda^p\mathbb{A}_q)$. In the following, we frequently omit the reference to q on the Lie algebroid $\mathbb{A}_q$. There exists a projection $P_{\mathrm{ext},\mathfrak{g}} : \Gamma(\mathbb{M}, \Lambda^p\mathbb{A}) \to \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*)$ obtained exactly as (4-9) composing the projection $P^{\mathfrak{g},p}_{(q+1)\text{-hom}}$ (2-10) with the ones constructed from the zero maps $0^{\mathfrak{g}}_i$ and permutations $\sigma^{\mathfrak{g}}$ exactly as in (4-6) and (4-7), respectively.

[3] As with VB-groupoids, Whitney sums of VB-algebroids yield VB-algebroids. Moreover, Whitney sums are preserved by the Lie functor (see [Bursztyn and Cabrera 2012]).

Proposition 4.11. *The projection* $P_{\mathrm{ext},\mathfrak{g}}$ *satisfies*

$$P_{\mathrm{ext},\mathfrak{g}} \circ d = d \circ P_{\mathrm{ext},\mathfrak{g}}.$$

In particular, $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*)$ *is a subcomplex of* $\mathrm{CE}^\bullet(\mathbb{A})$.

Proof. The result follows from the fact that the maps $h^{\mathfrak{g}}_\lambda$, $0^{\mathfrak{g}}_i$, $\sigma_{\mathfrak{g}}$ are all Lie algebroid morphisms. In fact,

$$h^{\mathfrak{g}}_\lambda = \mathrm{Lie}(\mathfrak{h}^{\mathcal{G}}_\lambda), \quad 0^{\mathfrak{g}}_i = \mathrm{Lie}(0^{\mathcal{G}}_i), \quad \sigma_{\mathfrak{g}} = \mathrm{Lie}(\sigma_{\mathcal{G}}) \tag{4-18}$$

for the corresponding maps $\mathfrak{h}^{\mathcal{G}}_\lambda$, $0^{\mathcal{G}}_i$, $\sigma_{\mathcal{G}}$ on the Lie groupoid $\mathbb{G}$. □

In the following, we shall denote by $\mathrm{CE}^\bullet_{\mathrm{ext}}(\mathbb{A})$ and by $H^\bullet_{\mathrm{ext}}(\mathbb{A})$ the complex $(\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*), d)$ and its cohomology, respectively. Note that $\mathrm{CE}^\bullet_{\mathrm{ext}}(\mathbb{A})$ is a right dg-module for $\Gamma(\Lambda^\bullet \mathfrak{g}) \cong \Gamma_{0\text{-hom}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*)$ by considering the wedge product.

Proposition 4.12. *There exists a right* $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$*-module isomorphism* $\mathrm{ev} : \mathrm{CE}^\bullet_{\mathrm{ext}}(\mathbb{A}) \to W^{\bullet,q}(\mathfrak{g}, C)$ *satisfying*

$$\mathrm{ev} \circ d = \mathrm{d}_W \circ \mathrm{ev}.$$

We refer to the Appendix (see Proposition A.3) for a proof. It is important to note that Proposition 4.12 implies that $W^{\bullet,q}(\mathfrak{g}, C)$ is a right dg-module for $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$ as stated in Proposition 4.2. It is also worth noting that ev is a map defined similarly to (3-4) (i.e., it evaluates an element $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ on a set of generators of $\Gamma(\mathbb{M}, \mathbb{A})$ to give the sequence $(c_0, c_1, \dots) \in W^{p,q}(\mathfrak{g}, C)$).

Remark 4.13. An alternative characterization of $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ can be given by seeing vector bundles as Lie groupoids (with multiplication given by addition on the fibers). Set $\mathbb{A}^{(p)} = \times^p_{\mathbb{M}} \mathbb{A}$ and $\mathfrak{g}^{(p)} = \times^p_M \mathfrak{g}$. One has $\mathbb{A}^{(p)} = B_p \mathbb{A}$ and $\mathfrak{g}^{(p)} = B_p \mathfrak{g}$. In particular, the isomorphism (4-11) implies that

$$\mathbb{A}^{(p)} \cong \times^q_{\mathfrak{g}^{(p)}} T\mathfrak{g}^{(p)} \times_{\mathfrak{g}^{(p)}} \pi^* C^* \tag{4-19}$$

as vector bundles over $\mathfrak{g}^{(p)}$, where $\pi : \mathfrak{g}^{(p)} \to M$ is defined (following the previous convention for $t : \mathcal{G}^{(p)} \to M$) as $\pi(u_1, \dots, u_p) = \pi(u_1)$. Hence, $\Omega^q(\mathfrak{g}^{(p)}, \pi^* C)$, the space of differential forms on $\mathfrak{g}^{(p)}$ with values on C, can be embedded as a subspace of $C^\infty(\mathbb{A}^{(p)})$ via (4-8). Similarly, $\Gamma(\mathbb{M}, \Lambda^p \mathbb{A})$ can also be embedded as a subspace of $C^\infty(\mathbb{A}^{(p)})$. One can now check that

$$\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*) = \Gamma(\mathbb{M}, \Lambda^p \mathbb{A}) \cap \Omega^q(\mathfrak{g}^{(p)}, \pi^* C).$$

In the case where $C = \mathbb{R}$, with the trivial representation, Li-Bland and Meinrenken [2015] gave a similar characterization of the Weil algebra as a subspace of differential forms on $\mathfrak{g}$. In this context, the case $p = 1$ was already studied by Bursztyn, Cabrera and Ortiz [Bursztyn and Cabrera 2012; Bursztyn et al. 2009].

Proof of the van Est theorem for differential forms with coefficients. Let $\mathrm{VE} : (C^\bullet(\mathbb{G}), \delta_{\mathbb{G}}) \to (\Gamma(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*), d)$ be the van Est map (2-13) for the groupoid $\mathbb{G} \rightrightarrows \mathbb{M}$.

Proposition 4.14. *We have*

$$\mathrm{VE} \circ P^{\mathcal{G},p}_{\mathrm{ext}} = P^{\mathfrak{g},p}_{\mathrm{ext}} \circ \mathrm{VE}.$$

In particular, $\mathrm{VE}(C^\infty_{\mathrm{ext}}(B_p\mathbb{G})) \subset \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$.

Proof. From Proposition 2.12, one already has that VE satisfies $\mathrm{VE} \circ P^{\mathcal{G},p}_{q+1\text{-hom}} = P^{\mathfrak{g},p}_{q+1\text{-hom}} \circ \mathrm{VE}$. It remains to show that VE commutes with the projections associated to the skew-symmetry and the simplicity properties. But this follows from Lemma 2.10 together with the relations (4-13), (4-14) and (4-18). □

Let $\mathrm{VE}_{\mathrm{ext}} : C^\infty_{\mathrm{ext}}(B_p\mathbb{G}) \to \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ be the restriction of the van Est map.

Lemma 4.15. *The following diagram commutes*:

$$\begin{tikzcd} \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C) \arrow[r, "\mathfrak{F}"] \arrow[d, "\mathrm{VE}_\Omega"'] & C^\infty_{\mathrm{ext}}(B_p\mathbb{G}) \arrow[d, "\mathrm{VE}_{\mathrm{ext}}"] \\ W^{p,q}(\mathfrak{g}, C) & \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*) \arrow[l, "\mathrm{ev}"'] \end{tikzcd} \tag{4-20}$$

The proof of Lemma 4.15 consists of a direct but technical verification that we postpone until the Appendix (see page 328). Finally, we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. As ev and $\mathfrak{F}$ are dg-module isomorphisms, it remains to show that $\mathrm{VE}_{\mathrm{ext}}$ induces isomorphisms on the cohomology $H^p(C^\infty_{\mathrm{ext}}(B_\bullet\mathbb{G})) \to H^p(\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^\bullet \mathbb{A}^*))$ for $p \le p_0$ and a monomorphism for $p = p_0 + 1$. Since the ordinary van Est map $\mathrm{VE}_{\mathbb{G}}$ for $\mathbb{G}$ satisfies the above, the theorem then follows from the homological lemma by means of the underlying projections exactly as in the proof of Theorem 2.14. □

Remark 4.16. The space $\Omega^\bullet(B_\bullet\mathcal{G}, \mathrm{t}^*C)$ is a bigraded right module for the bigraded algebra $\Omega^\bullet(B_\bullet\mathcal{G})$ with the cup product [Dupont 1978]. The multiplication is given by

$$\omega \cup \eta = (-1)^{qp'} \mathrm{pr}^*\omega \wedge \mathrm{pr}'^*\eta, \quad \omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C),\ \eta \in \Omega^{q'}(B_{p'}\mathcal{G}),$$

where $\mathrm{pr} : B_{p+p'}\mathcal{G} \to B_p\mathcal{G}$ (resp. $\mathrm{pr}' : B_{p+p'}\mathcal{G} \to B_{p'}\mathcal{G}$) is the projection onto the first p arrows (resp. last p' arrows). It is interesting to note that such module structure can also be described within the VB-groupoid context. Indeed, by considering the projections $\widetilde{\mathrm{pr}} : \mathbb{G}^{q+q'} \to \mathbb{G}^q$ and $\widetilde{\mathrm{pr}}' : \mathbb{G}^{q+q'} \to \times^{q'}_{\mathcal{G}} T\mathcal{G}$, one can check that $\mathfrak{F}_{\omega\cup\eta} \in C^\infty(B_{p+p'}\mathbb{G}^{q+q'})$ can be obtained from $(B_p\widetilde{\mathrm{pr}})^*\mathfrak{F}_\omega \in C^\infty(B_p\mathbb{G}^{q+q'})$ and $(B_{p'}\widetilde{\mathrm{pr}}')^*\mathfrak{F}_\eta \in C^\infty(B_{p'}\mathbb{G}^{q+q'})$ by skew-symmetrizing their cup product

$$(B_p\widetilde{\mathrm{pr}})^*\mathfrak{F}_\omega \star (B_{p'}\widetilde{\mathrm{pr}}')^*\mathfrak{F}_\eta \in C^\infty(B_{p+p'}\mathbb{G}^{q+q'}).$$

Similarly, one can define a bigraded module structure on $W^{\bullet,\bullet}(\mathfrak{g}, C)$ for the Weil algebra $W^{\bullet,\bullet}(\mathfrak{g})$ [Arias Abad and Crainic 2011] using the wedge product for their models as subcomplexes of the Chevalley–Eilenberg complexes. These bigraded module structures should be useful for studying "multiplicative linear flat" connections on C.

Appendix: Formulas for the evaluation map

We turn to the proof of Lemma 4.15 relating the formula for VE_Ω with the standard van Est map for $\mathbb{G}$ and $\mathbb{A}$. In the process, we also give a detailed description (see (A-8) below) of the map $\mathrm{ev} : \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}_q^*) \to W^{p,q}(\mathfrak{g}, C)$, making use of special sections of $\mathbb{A}_q$.

Special sections. Let $TB \to B$ be the tangent bundle of B. Given a vector field $X \in \mathfrak{X}(B)$, let $X^T, X^{\mathrm{v}} \in \mathfrak{X}(TB)$ be its *tangent* and *vertical lift* respectively.[4] Define vector fields $X^{T,q}$ and $X^{\mathrm{v},q}_{(j)}$, $j = 1, \dots, q$, on the manifold $\bigtimes_B^q TB$ as follows:

$$X^{T,q}(v_1, \dots, v_q) = (X^T(v_1), \dots, X^T(v_q)), \tag{A-1}$$

$$X^{\mathrm{v},q}_{(j)}(v_1, \dots, v_q) = (0_{v_1}, \dots, X^{\mathrm{v}}(v_j), \dots, 0_{v_q}). \tag{A-2}$$

Let now $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $\pi : \mathfrak{g} \to M$. For a representation $C \to M$ of $\mathcal{G}$, consider the Lie groupoid (4-10), $\bigtimes_{\mathcal{G}}^q T\mathcal{G} \times_{\mathcal{G}} \times \mathrm{t}^* C^* \rightrightarrows \bigtimes_M^q TM \times_M C^*$, with corresponding Lie algebroid $\bigtimes_{\mathfrak{g}}^q T\mathfrak{g} \times_{\mathfrak{g}} \pi^* C^*$. For a section $u : M \to \mathfrak{g}$, let $Tu : TM \to T\mathfrak{g}$ be its derivative and $\chi_u : C^* \to \pi^* C^* = C^* \times_M \mathfrak{g}$ the section defined by (2-11). The expressions

$$\mathbb{T}u(x_1, \dots, x_q, \xi) = (Tu(x_1), \dots, Tu(x_q), \chi_u(\xi)),$$

$$\mathbb{Z}_i u(x_1, \dots, x_q, \xi) = \left(T0(x_1), \dots, T0(x_i) + \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\epsilon u(m)), \dots, T0(x_q), 0_\xi \right),$$

for $i = 1, \dots, q$, $x_1, \dots, x_q \in T_m M$, $\xi \in C^*_m$ and $m \in M$, define sections of the Lie algebroid $\mathbb{A} = \bigtimes_{\mathfrak{g}}^q T\mathfrak{g} \times_{\mathfrak{g}} \pi^* C^* \to \mathbb{M} = \bigtimes_M^q TM \times_M C^*$. It is known that $\mathbb{T}u$ and $\mathbb{Z}_i u$, $i = 1, \dots, q$, generate $\Gamma(\mathbb{M}, \mathbb{A})$ as a $C^\infty(\mathbb{M})$ module.[5]

Lemma A.1. *As vector fields on $B_p(\bigtimes_{\mathcal{G}}^q T\mathcal{G} \times_{\mathcal{G}} \mathrm{t}^* C^*) \cong \bigtimes_{B_p\mathcal{G}}^q TB_p\mathcal{G} \times_{B_p\mathcal{G}} \mathrm{t}^* C^*$, the following identities hold*:

$$B_p(\mathbb{T}u) = ((B_p u)^{T,q}, X_u), \tag{A-3}$$

$$B_p(\mathbb{Z}_i v) = ((B_p v)^{\mathrm{v},q}_{(i)}, 0), \tag{A-4}$$

[4] The flow at time ϵ of X^T (resp. X^{v}) is the derivative of the flow at time ϵ of X (resp. translation by ϵX).

[5] This follows from a general result regarding *core* and *linear sections* of double vector bundles (see [Mackenzie 2011]).

where $X_u \in \mathfrak{X}(\mathrm{t}^*C^*)$ is the vector field whose time-ϵ flow is given by

$$(g_1, \ldots, g_p, \xi) \mapsto (\psi_u^\epsilon(g_1, \ldots, g_p), \Delta^*_{\phi_u^\epsilon(\mathrm{t}(g_1))^{-1}}(\xi)).$$

In particular, for $\omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$,

$$\mathbb{s}_0^*\mathcal{L}_{B_p(\mathbb{T}u)}\mathfrak{F}_\omega = \mathfrak{F}_{R_u\omega}, \tag{A-5}$$

$$\mathbb{s}_0^*\mathcal{L}_{B_p(\mathbb{Z}_i u)}\mathfrak{F}_\omega = (-1)^{i-1}\mathfrak{F}_{J_u\omega} \circ \mathrm{pr}_{(i)}^{p-1,q}, \tag{A-6}$$

where R_u and J_u were defined in (4-3),

$$\mathrm{pr}_{(i)}^{\cdot,q} : \times^q_{B.\mathcal{G}} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathrm{t}^*C^* \to \times^{q-1}_{B.\mathcal{G}} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathrm{t}^*C^*$$

is the projection which forgets the i-th component and $\mathbb{s}_0$ is the first degeneracy map for $\mathbb{G}$.

Proof. For $u \in \Gamma(\mathfrak{g})$, consider the sections $\mathbb{Z}u, Tu$ of $T\mathfrak{g} \to TM$, where

$$\mathbb{Z}u(x) = T0(x) + \frac{d}{d\epsilon}\bigg|_{\epsilon=0}(\epsilon u(m)), \quad x \in T_mM.$$

One has that $\overrightarrow{Tu} = \vec{u}^T$ and $\overrightarrow{\mathbb{Z}u} = \vec{u}^{\mathrm{v}}$ as vector fields on $T\mathcal{G} \rightrightarrows TM$ (see [Mackenzie and Xu 1994]). Also, the flow of the right invariant vector field $\overrightarrow{\chi_u} \in \mathfrak{X}(\mathrm{t}^*C^*)$ is given by

$$(g, \xi) \mapsto (\phi_u^\epsilon(g), \phi_u^\epsilon(\mathrm{t}(g))^{-1} \cdot \xi).$$

The identities (A-3) and (A-4) now follow from analyzing the flows together with the rearrangement isomorphism (4-11). Hence, for $\omega \in \Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$,

$$\begin{aligned}
&(\mathcal{L}_{B_p(\mathbb{T}u)}\mathfrak{F}_\omega)|_{(\bar{U}_1,\ldots,\bar{U}_q,(g_1,\ldots,g_p,\xi))} \\
&\quad = \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\mathfrak{F}_\omega\big(T\psi_u^\epsilon(\bar{U}_1), \ldots, T\psi_u^\epsilon(\bar{U}_q), (\psi_u^\epsilon(g_1, \ldots, g_p), \phi_u^\epsilon(\mathrm{t}(g_1))^{-1} \cdot \xi)\big) \\
&\quad = \big\langle \xi, \phi(\mathrm{t}(g_1))^{-1} \cdot (\psi_u^\epsilon)^*\omega(\bar{U}_1, \ldots, \bar{U}_q)\big\rangle.
\end{aligned}$$

Now, (A-5) follows from the commutation relations on Proposition 4.8. The identity (A-6) follows similarly. ☐

The evaluation map. We now describe the chain isomorphism $\mathrm{ev}: \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*) \to W^{p,q}(\mathfrak{g}, C)$. First, for $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*) \subset \Gamma(\mathbb{M}, \Lambda^p\mathbb{A}^*)$, define

$$\tilde{c}_k(\alpha) : \big(\times^{p-k}\Gamma(\mathfrak{g})\big) \times \big(\times^k \Gamma(\mathfrak{g})\big) \to C^\infty\big(\times^q_M TM \times_M C^*\big)$$

as

$$\tilde{c}_k(\alpha)(u_1, \ldots, u_{p-k}|v_1, \ldots, v_k) = \alpha(\mathbb{Z}_1v_1, \ldots, \mathbb{Z}_kv_k, \mathbb{T}u_1, \ldots, \mathbb{T}u_{p-k}).$$

Lemma A.2. *There exists a map* $c_k(\alpha) : \times^p \Gamma(\mathfrak{g}) \to \Omega^{q-k}(M, C)$ *such that*

$$\tilde{c}_k(\alpha) = \mathfrak{F}_{c_k(\alpha)} \circ \mathrm{pr}_{[1,k]} \quad \forall \alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}_q^*), \tag{A-7}$$

where $\mathrm{pr}_{[1,k]} : \times_M^q TM \times_M C^* \to \times_M^{q-k} TM \times_M C^*$ *is the projection which forgets the first k entries.*

Proof. The multilinearity with respect to both vector bundle structures, $\mathbb{A} \to \mathbb{M}$ and $\mathbb{A} \to \mathfrak{g}$, implies that $\tilde{c}_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k) = F_\alpha \circ \mathrm{pr}_{[1,k]}$, where $F_\alpha \in C^\infty\big(\times^{q-k} TM \times_M C^*\big)$ is given by

$$\begin{aligned} &F_\alpha(y_1, \dots, y_{q-k}, \xi) \\ &\qquad = \tilde{c}_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k)(\underbrace{0_m, \dots, 0_m}_{k \text{ times}}, y_1, \dots, y_{q-k}, \xi). \end{aligned}$$

We now have to check that $F_\alpha \in C^\infty_{\mathrm{ext}}\big(\times^{q-k} TM \times_M C^*\big)$, i.e., F is $(q-k+1)$-homogeneous, simple and skew-symmetric. The homogeneity of F_α follows from the homogeneity of α together with the linearity of the sections $\mathbb{T}u$ and the properties of the section $\mathbb{Z}_j v$:

$$\begin{aligned} \mathbb{Z}_j(v)|_{(0_m, \dots, 0_m, \lambda y_1, \dots, \lambda y_{q-k}, \lambda\xi)} &= h^{\mathfrak{g}}_\lambda\Big(\mathbb{Z}_j\Big(\frac{1}{\lambda} v\Big)\Big|_{(0_m, \dots, 0_m, y_1, \dots, y_{q-k}, \xi)}\Big), \\ \mathbb{Z}_j(\lambda v) &= \lambda \cdot \mathbb{Z}_j(v), \end{aligned}$$

where $\lambda > 0$ and $\cdot$ stands for the multiplication for $\mathbb{A} \to \mathbb{M}$. The simplicity of F_α follows from the identity

$$(F_\alpha \circ 0_i) \circ \mathrm{pr}_{[1,k]} = ((0^{\mathfrak{g}}_{k+i})^* \alpha)(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) = 0$$

for $i = 1, \dots, q-k+1$. Finally, let $\sigma \in S_{q-k} \subset S_q$, seen as the subgroup acting as the identity on $\{1, \dots, k\}$. One can check that

$$\begin{aligned} (F_\alpha \circ \sigma_M) \circ \mathrm{pr}_{[1,k]} &= (\sigma_{\mathfrak{g}}^* \alpha)(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) \\ &= \mathrm{sgn}(\sigma) \alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) \\ &= \mathrm{sgn}(\sigma) F_\alpha \circ \mathrm{pr}_{[1,k]}. \end{aligned}$$

This shows that $F_\alpha \in C^\infty_{\mathrm{ext}}\big(\times^{q-k} TM \times_M C^*\big)$ and, therefore, there exists $c_k(\alpha) : \times^p \Gamma(\mathfrak{g}) \to \Omega^{q-k}(M, C)$ such that $F_\alpha = \mathfrak{F}_{c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k)}$. □

Our aim is to prove that

$$\mathrm{ev}(\alpha) = (c_0(\alpha), c_1(\alpha), \dots) \tag{A-8}$$

defines a map from $\Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ into $W^{p,q}(\mathfrak{g}, C)$. First note that the sequence $(c_0(\alpha), c_1(\alpha), \dots)$ completely determines $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$. Indeed, as $\Gamma(\mathbb{M}, \mathbb{A})$ is generated as a $C^\infty(\mathbb{M})$-module by sections of the type $\mathbb{T}u$, $\mathbb{Z}_i v$, any element of

$\Gamma(\mathbb{M}, \Lambda^p \mathbb{A}^*)$ is determined by its values on these sections. Now, one can check that, for $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$,

$$i_{\mathbb{Z}_j v} i_{\mathbb{Z}_j w} \alpha = 0 \quad \text{for } j = 1, \dots, q, \tag{A-9}$$

and, for a permutation $\sigma \in S_q$,

$$\begin{aligned}\alpha(\mathbb{Z}_{\sigma(1)} v_1, \dots, \mathbb{Z}_{\sigma(k)} v_k, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}) \\ = \operatorname{sgn}(\sigma) \sigma_M^* (\alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k})).\end{aligned} \tag{A-10}$$

Hence, to recover α from its values on the sections $\mathbb{T} u$, $\mathbb{Z}_i v$, it suffices to know the values of α encoded on the sequence $(c_0(\alpha), c_1(\alpha), \dots)$. The next result gives the desired proof of Proposition 4.12.

Proposition A.3. *Given* $\alpha \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*)$, *one has that*

(1) $c_k(\alpha)$ *is skew-symmetric on the* u *entries*;

(2) $c_k(\alpha)$ *is symmetric on the* v *entries*;

(3) *given* $f \in C^\infty(M)$,

$$c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, f v_k) = f c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k),$$

$$\begin{aligned}c_k(\alpha)(f u_1, \dots, u_{p-k} | v_1, \dots, v_k) = f c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k) \\ + df \wedge c_{k+1}(\alpha)(u_2, \dots, u_{p-k} | v_1, \dots, v_k, u_1).\end{aligned}$$

In particular, each c_k *can be viewed as an* $\mathbb{R}$*-linear skew-symmetric map* $c_k : \times^{p-k} \Gamma(\mathfrak{g}) \to \Omega^{q-k}(M, S^k \mathfrak{g}^* \otimes C)$. *Moreover, the map* $\mathrm{ev} : \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p \mathbb{A}^*) \to W^{p,q}(\mathfrak{g}, C)$ *defined by* (A-8) *is a right* $\Gamma(\Lambda^\bullet \mathfrak{g}^*)$*-module isomorphism satisfying*

$$\mathrm{ev} \circ \mathrm{d}_{\mathrm{ext}} = \mathrm{d}_W \circ \mathrm{ev}.$$

Proof.

(1) This follows directly from the skew-symmetry of α with respect to $\mathbb{A} \to \mathbb{M}$.

(2) Let $\sigma \in S_k \subset S_q$, seen as the subgroup acting as the identity on $\{k+1, \dots, q\}$. From (A-10) and the skew-symmetry of α with respect to $\mathbb{A} \to \mathbb{M}$,

$$\begin{aligned}&\alpha(\mathbb{Z}_1 v_{\sigma(1)}, \dots, \mathbb{Z}_k v_{\sigma(k)}, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}) \\ &\quad = \operatorname{sgn}(\sigma) \alpha(\mathbb{Z}_{\sigma(1)} v_{\sigma(1)}, \dots, \mathbb{Z}_{\sigma(k)} v_{\sigma(k)}, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}) \\ &\quad = (\operatorname{sgn}(\sigma))^2 \alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}).\end{aligned}$$

In the second equality we have used the fact that

$$\alpha(\mathbb{T} u_1, \dots, \mathbb{T} u_{p-k}, \mathbb{Z}_{\sigma(1)} v_{\sigma(1)}, \dots, \mathbb{Z}_{\sigma(k)} v_{\sigma(k)}) \in C^\infty\big(\times^q TM \times_M C^*\big)$$

does not depend on the first k coordinates.

(3) One can check that

$$\mathbb{Z}_i(fv) = (f \circ \pi) \cdot \mathbb{Z}_i v,$$
$$\mathbb{T}(fu) = (f \circ \pi) \cdot \mathbb{T}u + \sum_{j=1}^{q} (\ell_{df} \circ \mathrm{pr}_j) \cdot \mathbb{Z}_j u,$$

where all the sums and scalar multiplications are with respect to $\pi : \mathbb{A} \to \mathbb{M}$, $\mathrm{pr}_j : \times_M^q TM \times_M C^* \to TM$ is the projection onto the j-th factor and $\ell_{df} \in C^\infty(TM)$ is the linear function corresponding to $df \in \Omega^1(M)$. To simplify notation, we identify $\Omega^{q-k}(M, C)$ with its image on $C^\infty\big(\times^{q-k} TM \times_M C^*\big)$ under $\mathfrak{F}$ in the following. The first equation of (3) is now straightforward to check. As for the second, it follows from (A-9) and (A-10) that

$$\begin{aligned}
&c_k(\alpha)(fu_1, \dots, u_{p-k} | v_1, \dots, v_k) \circ \mathrm{pr}_{[1,k]} \\
&= (f \circ \pi)\alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k}) \\
&\quad + \sum_{j=k+1}^{q} (\ell_{df} \circ \mathrm{pr}_j)\alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{Z}_j u_1, \mathbb{T}u_2, \dots, \mathbb{T}u_{p-k}) \\
&= (f \circ \pi)c_k(\alpha)(u_1, \dots, u_{p-k} | v_1, \dots, v_k) \circ \mathrm{pr}_{[1,k]} \\
&\quad + \underbrace{\sum_{j=k+1}^{q} (-1)^{j-k-1}(\ell_{df} \circ \mathrm{pr}_j)\alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{Z}_{k+1} u_1, \mathbb{T}u_2, \dots, \mathbb{T}u_{p-k}) \circ \sigma_M^j}_{(*)},
\end{aligned}$$

where $\sigma^j \in S_q$ is the cycle $(j\ \ j{-}1 \cdots k{+}2\ \ k{+}1)$, for $k+1 \le j \le q$, which has sign equal to $(-1)^{j-k-1}$. It is now straightforward to check that $(*)$ equals $df \wedge c_{k+1}(u_2, \dots, u_{p-k} | v_1, \dots, v_k, u_1) \circ \mathrm{pr}_{[1,k]}$.

It remains to prove that ev is a dg-module isomorphism. Let us first prove that ev commutes with the multiplication. Let $\beta \in \Gamma(\Lambda^{p'} \mathfrak{g}^*) \cong \Gamma_{0\text{-hom}}(\mathbb{M}, \Lambda^{p'} \mathbb{A}^*)$ and consider $\mathrm{ev}(\alpha \wedge \beta) = (c_0(\alpha \wedge \beta), c_1(\alpha \wedge \beta), \dots)$. By definition,

$$\begin{aligned}
&c_k(\alpha \wedge \beta)(u_1, \dots, u_{p+p'-k} | v_1, \dots, v_k) \circ \mathrm{pr}_{[1,k]} \\
&\qquad = (\alpha \wedge \beta)(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p+p'-k}) \\
&\qquad = \sum_{\sigma \in S(p-k, p')} \mathrm{sgn}(\sigma)\alpha(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_{\sigma(1)}, \dots, \mathbb{T}u_{\sigma(p-k)}) \\
&\qquad\qquad\qquad \times \beta(\mathbb{T}u_{\sigma(p-k+1)}, \dots, \mathbb{T}u_{\sigma(p+p'-k)}),
\end{aligned}$$

where $S(p-k, p')$ is the space of $(p-k, p')$-unshuffles and the last equality follows from the fact that the contraction of β with any section of type $\mathbb{Z}_. v_.$ is zero. The result now follows easily.

Finally, to prove that ev intertwines the differential, consider

$$\mathrm{ev}(d\alpha) = (c_0(d\alpha), \dots, c_k(d\alpha), \dots),$$

where

$$
\begin{aligned}
&c_k(d\alpha)(u_1,\dots,u_{p+1-k}|v_1,\dots,v_k)\circ \mathrm{pr}_{[1,k]}\\
&\quad = d\alpha(\mathbb{Z}_1 v_1,\dots,\mathbb{Z}_k v_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p+1-k})\\
&\quad = \underbrace{\sum_{j=1}^{k}(-1)^{j+1}\mathcal{L}_{\rho(\mathbb{Z}_j v_j)}\alpha(\mathbb{Z}_1 v_1,\dots,\widehat{\mathbb{Z}_j v_j},\dots)}_{(A)}\\
&\qquad\quad + \underbrace{\sum_{i=1}^{p-k+1}(-1)^{i+k+1}\mathcal{L}_{\rho(\mathbb{T}u_i)}\alpha(\mathbb{Z}_1 v_1,\dots,\widehat{\mathbb{T}u_i},\dots)}_{(B)}\\
&\qquad\qquad + \underbrace{\sum_{1\le i<j\le p+1-k}(-1)^{i+j}\alpha([\mathbb{T}u_i,\mathbb{T}u_j],\dots,\widehat{Tu_i},\dots,\widehat{Tu_j},\dots)}_{(C)}\\
&\qquad\qquad\quad + \underbrace{\sum_{i=1}^{p+1-k}\sum_{j=1}^{k}(-1)^{j+(k+i)}\alpha([\mathbb{Z}_j v_j,\mathbb{T}u_i],\dots,\widehat{\mathbb{Z}_j v_j},\dots,\widehat{\mathbb{T}u_i},\dots)}_{(D)}.
\end{aligned}
$$

Notice that there are no terms containing $[\mathbb{Z}_{j_1}v_{j_1},\mathbb{Z}_{j_2}v_{j_2}]$ since these brackets are all zero. To study the remaining terms, we use some properties of the tangent Lie algebroid $T\mathfrak{g}\to TM$ (see [Mackenzie and Xu 1994]) and the action algebroid $C^*\times_M\mathfrak{g}\to C^*$.

(A): From (A-10),

$$
\begin{aligned}
&\alpha(\mathbb{Z}_1 v_1,\dots,\widehat{\mathbb{Z}_j v_j},\dots,\mathbb{Z}_k v_k,\mathbb{T}u_1,\dots,\mathbb{T}u_{p+k-1})\\
&\qquad\qquad = (-1)^{k-j}\sigma_M^*(\mathfrak{F}_{c_{k-1}(\alpha)(u_1,\dots,u_{p+1-k}|v_1,\dots,v_{j-1},v_{j+1},\dots,v_k)}\circ\mathrm{pr}_{[1,k-1]}),
\end{aligned}
$$

where $\sigma=(j\ k)(j\ k{-}1)\cdots(j\ j{+}1)\in S_q$. Now,

$$
\begin{aligned}
\mathrm{pr}_{[1,k-1]}\circ\sigma_M(x_1,\dots,x_q,\xi) &= (x_j,x_{k+1},\dots,x_q,\xi),\\
\rho(\mathbb{Z}_j v_j) &= (\rho(v_j)^{\mathrm{v},q}_{(j)},0)
\end{aligned}
$$

and

$$
\mathcal{L}_{(X^{\mathrm{v},q}_{(1)},0)}\mathfrak{F}_\omega=\mathfrak{F}_{i_X\omega}\circ\mathrm{pr}_{(1)}\quad \forall\, X\in\mathfrak{X}(M),\ \omega\in\Omega^q(M,C),
$$

where $\mathrm{pr}_{(1)}:\times^q TM\times_M C^*\to\times^{q-1}TM\times_M C^*$ is the projection which forgets the first component. These facts imply that

$$
(A)=(-1)^{k+1}\sum_{j=1}^{k}\big(i_{\rho(v_j)}c_{k-1}(\alpha)(u_1,\dots,u_{p+1-k}|v_1,\dots,v_{j-1},v_{j+1},\dots,v_k)\big)\circ\mathrm{pr}_{[1,k]}.
$$

(B): The fact that $\rho(\mathbb{T}u_i) = (\rho(u_i)^T, \rho(\chi_u))$, where $(\,\cdot\,)^T$ stands for tangent lift, implies that, for $\omega \in \Omega^q(M, C)$, $\mathcal{L}_{\rho(\mathbb{T}u_i)}\mathfrak{F}_\omega = \mathfrak{F}_{u_i\cdot\omega}$, where $u\cdot(\beta\otimes\mu) = \mathcal{L}_{\rho(u)}\beta\otimes\mu + \beta\otimes\nabla_u\mu$, $\beta \in \Omega^q(M)$, $\mu \in \Gamma(C)$ and ∇ is the $\mathfrak{g}$-connection on C defining the representation of $\mathfrak{g}$ on C. Hence,

$$(B) = (-1)^k \sum_{i=1}^{p-k+1} (-1)^{i+1}\big(u_i \cdot c_k(u_1, \dots, \widehat{u_i}, \dots, u_{p+1-k}|v_1, \dots, v_k)\big) \circ \mathrm{pr}_{[1,k]}.$$

(C) and (D): From the identities $[\mathbb{T}u_i, \mathbb{T}u_j] = \mathbb{T}[u_i, u_j]$, $[\mathbb{T}u_i, \mathbb{Z}_j v_j] = \mathbb{Z}_j[u_i, v_j]$, it is straightforward to check that

$$\begin{aligned}(C) = (-1)^k \sum_{1\le i_1<i_j\le p-k+1} (-1)^{i_1+i_2}\big(c_k(\alpha)([u_{i_1}, u_{i_2}], u_1, \dots, \widehat{u_{i_1}}, \dots, \widehat{u_{i_2}},&\\ \dots, u_{p-k+1}|v_1, \dots, v_k)\big) \circ \mathrm{pr}_{[1,k]},&\end{aligned}$$

$$\begin{aligned}(D) = (-1)^k \sum_{i=1}^{p-k+1}\sum_{j=1}^{k}(-1)^i\big(c_k(\alpha)(u_1, \dots, \widehat{u_i},&\\ \dots, u_{p+1-k}|v_1, \dots, [u_i, v_j], \dots, v_k)\big) \circ \mathrm{pr}_{[1,k]}.&\end{aligned}$$

Hence,

$$\begin{aligned}(A) + (B) + (C) + (D) = \mathrm{d}_W(c(\alpha))_k(u_1, \dots, u_{p-k+1}|v_1, \dots, v_k) \circ \mathrm{pr}_{[1,k]}&\\ \Longrightarrow\quad c(d\alpha)_k = \mathrm{d}_W(c(\alpha))_k,&\end{aligned}$$

as we wanted. □

Proof of Lemma 4.15.

Lemma 4.15 rephrased. *Let ω be an element of $\Omega^q(B_p\mathcal{G}, \mathrm{t}^*C)$ and consider* $\mathrm{VE}_\Omega(\omega) = (c_0(\omega), c_1(\omega), \dots)$ *as defined in (4-4). Also, let* $\alpha = \mathrm{VE}_{\mathrm{ext}}(\mathfrak{F}_\omega) \in \Gamma_{\mathrm{ext}}(\mathbb{M}, \Lambda^p\mathbb{A}^*)$ *and consider* $\mathrm{ev}(\alpha) = (c_0(\alpha), c_1(\alpha), \dots)$ *defined by (A-7). Then*

$$c_k(\omega) = c_k(\alpha) \quad \forall\, k \ge 0.$$

Proof. From (2-13),

$$\begin{aligned}\mathfrak{F}_{c_k(\alpha)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)} \circ \mathrm{pr}_{[1,k]} &= \mathrm{VE}_{\mathrm{ext}}(\mathfrak{F}_\omega)(\mathbb{Z}_1 v_1, \dots, \mathbb{Z}_k v_k, \mathbb{T}u_1, \dots, \mathbb{T}u_{p-k})\\ &= \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) R_{\chi_{\sigma(1)}} \dots R_{\chi_{\sigma(p)}}\mathfrak{F}_\omega,\end{aligned}$$

where $\chi_i = \mathbb{Z}_i v_i$ (resp. $\mathbb{T}u_{i-k}$) if $i \in \{1, \dots, k\}$ (resp. if $i \in \{k+1, \dots, p\}$). The main ingredients of the proof are the identities from Lemma A.1:

$$\begin{aligned}R_{\mathbb{T}u_i}\mathfrak{F}_\omega &= \mathbb{s}_0^*\mathcal{L}_{B_p\mathbb{T}u_i}\mathfrak{F}_\omega = \mathfrak{F}_{R_{u_i}\omega},\\ R_{\mathbb{Z}_i v_i}\mathfrak{F}_\omega &= (-1)^{i-1}\mathfrak{F}_{J_{v_i}\omega} \circ \mathrm{pr}_{(i)}^{p-1,q},\end{aligned}$$

where $\mathrm{pr}^{\cdot,q}_{(i)} : \times^{q}_{B.\mathcal{G}} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathrm{t}^*C^* \to \times^{q-1}_{B.\mathcal{G}} TB.\mathcal{G} \times_{B.\mathcal{G}} \mathrm{t}^*C^*$ is the projection which forgets the i-th component. In the rest of the proof, the difficulty lies in the combinatorics needed to count the number of -1's appearing due to the presence of the sections $\mathbb{Z}_i v_i$.

Let $0 \le r \le p$, $1 \le s \le q$ and $1 \le i, j \le s$. For $\eta \in \Omega^{s-1}(B_r\mathcal{G}, \mathrm{t}^*C)$, one can check that

$$R_{\mathbb{T}u}(\mathfrak{F}_\eta \circ \mathrm{pr}^{r,s}_{(i)}) = \mathfrak{F}_{R_u\eta} \circ \mathrm{pr}^{r-1,s}_{(i)}$$

and

$$R_{\mathbb{Z}_j v}(\mathfrak{F}_\eta \circ \mathrm{pr}^{r,s}_{(i)}) = \begin{cases} (-1)^{j-1}\mathfrak{F}_{J_v\eta} \circ \mathrm{pr}^{r-1,s-1}_{(j)} \circ \mathrm{pr}^{r-1,s}_{(i)} & \text{if } i > j, \\ -(-1)^{j-1}\mathfrak{F}_{J_v\eta} \circ \mathrm{pr}^{r-1,s-1}_{(i)} \circ \mathrm{pr}^{r-1,s}_{(j)} & \text{if } i < j, \\ 0 & \text{if } i = j. \end{cases}$$

Let us now fix a permutation $\sigma \in S_q$. For $1 \le l \le k$, let $j_l = \sigma^{-1}(l)$ for $l \ge 1$ and set $j_0 = 0$. Denote by τ the permutation of $\{0, 1, \dots, k\}$ such that $j_{\tau(0)} < \cdots < j_{\tau(k)}$. One can now prove by induction that, for $j_{\tau(l)} \le r < j_{\tau(l+1)}$,

$$R_{\chi_{\sigma(r+1)}} \dots R_{\chi_{\sigma(p)}}\mathfrak{F}_\omega = \delta(k,l)\big(\mathfrak{F}_{D_{\sigma(r+1)}\dots D_{\sigma(p)}\omega} \circ \mathrm{pr}^{r,q-k+l}_{(i_1)} \circ \cdots \circ \mathrm{pr}^{r,q}_{(i_{k-l})}\big), \tag{A-11}$$

where the D_j are the operators (4-5), $\{i_1 < \cdots < i_{k-l}\} = \{\tau(l+1), \dots, \tau(k)\}$ and

$$\delta(k,l) = (-1)^{k-l}(-1)^{\tau(l+1)+\cdots+\tau(k)}(-1)^{N(\tau,l)}$$

with

$$N(\tau,l) = \#\big\{(i,j) \in \{l+1,\dots,k\} \times \{l+1,\dots,k\} \mid i < j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\big\}.$$

Note that, for $l = 0$,

$$\delta(k,0) = (-1)^k(-1)^{1+\cdots+k}(-1)^{\epsilon(\sigma,k)} = (-1)^{k(k-1)/2}(-1)^{\epsilon(\sigma,k)}.$$

In particular, when $r = 0$, we have

$$\begin{aligned}
&\mathfrak{F}_{c_k(\alpha)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)} \circ \mathrm{pr}_{[1,k]} \\
&\quad = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) R_{\chi_{\sigma(1)}} \dots R_{\chi_{\sigma(p)}}\mathfrak{F}_\omega \\
&\quad = (-1)^{k(k-1)/2} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)(-1)^{\epsilon(\sigma,k)}\mathfrak{F}_{D_{\sigma(1)}\dots D_{\sigma(p)}\omega} \circ \mathrm{pr}^{0,q-k}_{(1)} \circ \cdots \circ \mathrm{pr}^{0,q}_{(k)} \\
&\quad = \mathfrak{F}_{c_k(\omega)(u_1,\dots,u_{p-k}|v_1,\dots,v_k)} \circ \mathrm{pr}_{[1,k]}.
\end{aligned}$$

□

Acknowledgments

We would like to thank H. Bursztyn and O. Brahic for useful discussions. Cabrera would also like to thank R. Mehta for his insightful ideas in the early stages of this work. The authors are also especially grateful to Matias del Hoyo for pointing out an incompleteness in the proof of Lemma 3.1 and for helping to complete it.

References

[Arias Abad and Crainic 2011] C. Arias Abad and M. Crainic, "The Weil algebra and the Van Est isomorphism", *Ann. Inst. Fourier* (*Grenoble*) **61**:3 (2011), 927–970. MR Zbl

[Arias Abad and Crainic 2012] C. Arias Abad and M. Crainic, "Representations up to homotopy of Lie algebroids", *J. Reine Angew. Math.* **663** (2012), 91–126. MR Zbl

[Arias Abad and Crainic 2013] C. Arias Abad and M. Crainic, "Representations up to homotopy and Bott's spectral sequence for Lie groupoids", *Adv. Math.* **248** (2013), 416–452. MR Zbl

[Arias Abad and Schätz 2011] C. Arias Abad and F. Schätz, "Deformations of Lie brackets and representations up to homotopy", *Indag. Math.* (*N.S.*) **22**:1-2 (2011), 27–54. MR Zbl

[Brahic et al. 2014] O. Brahic, A. Cabrera, and C. Ortiz, "Obstructions to the integrability of VB-algebroids", preprint, 2014. arXiv

[Bursztyn and Cabrera 2012] H. Bursztyn and A. Cabrera, "Multiplicative forms at the infinitesimal level", *Math. Ann.* **353**:3 (2012), 663–705. MR Zbl

[Bursztyn and Drummond ≥ 2017] H. Bursztyn and T. Drummond, "Lie theory of multiplicative tensors", work in progress.

[Bursztyn et al. 2009] H. Bursztyn, A. Cabrera, and C. Ortiz, "Linear and multiplicative 2-forms", *Lett. Math. Phys.* **90**:1-3 (2009), 59–83. MR Zbl

[Bursztyn et al. 2016] H. Bursztyn, A. Cabrera, and M. del Hoyo, "Vector bundles over Lie groupoids and algebroids", *Adv. Math.* **290** (2016), 163–207. MR Zbl

[Crainic 2003] M. Crainic, "Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes", *Comment. Math. Helv.* **78**:4 (2003), 681–721. MR Zbl

[Crainic and Moerdijk 2008] M. Crainic and I. Moerdijk, "Deformations of Lie brackets: cohomological aspects", *J. Eur. Math. Soc.* (*JEMS*) **10**:4 (2008), 1037–1059. MR Zbl

[Crainic et al. 2015a] M. Crainic, J. N. Mestre, and I. Struchiner, "Deformations of Lie groupoids", preprint, 2015. arXiv

[Crainic et al. 2015b] M. Crainic, M. A. Salazar, and I. Struchiner, "Multiplicative forms and Spencer operators", *Math. Z.* **279**:3-4 (2015), 939–979. MR Zbl

[Dupont 1978] J. L. Dupont, *Curvature and characteristic classes*, Lecture Notes in Mathematics **640**, Springer, 1978. MR Zbl

[Egea 2016] L. Egea, *VB-groupoid cocycles and their applications to multiplicative structures*, Ph.D. thesis, Instituto Nacional de Mathemática Pura e Aplicada, 2016.

[van Est 1953a] W. T. van Est, "Group cohomology and Lie algebra cohomology in Lie groups, I", *Nederl. Akad. Wetensch. Proc. Ser. A.* **15** (1953), 484–492. MR Zbl

[van Est 1953b] W. T. van Est, "Group cohomology and Lie algebra cohomology in Lie groups, II", *Nederl. Akad. Wetensch. Proc. Ser. A.* **15** (1953), 493–504. MR Zbl

[van Est 1955a] W. T. van Est, "On the algebraic cohomology concepts in Lie groups, I", *Nederl. Akad. Wetensch. Proc. Ser. A.* **17** (1955), 225–233. MR Zbl

[van Est 1955b] W. T. van Est, "On the algebraic cohomology concepts in Lie groups, II", *Nederl. Akad. Wetensch. Proc. Ser. A.* **17** (1955), 286–294. MR Zbl

[Grabowski and Rotkiewicz 2009] J. Grabowski and M. Rotkiewicz, "Higher vector bundles and multi-graded symplectic manifolds", *J. Geom. Phys.* **59**:9 (2009), 1285–1305. MR Zbl

[Gracia-Saz and Mehta 2010] A. Gracia-Saz and R. A. Mehta, "Lie algebroid structures on double vector bundles and representation theory of Lie algebroids", *Adv. Math.* **223**:4 (2010), 1236–1275. MR Zbl

[Gracia-Saz and Mehta 2011] A. Gracia-Saz and R. A. Mehta, "VB-groupoids and representation theory of Lie groupoids", preprint, 2011. arXiv

[Li-Bland and Meinrenken 2015] D. Li-Bland and E. Meinrenken, "On the van Est homomorphism for Lie groupoids", *Enseign. Math.* **61**:1-2 (2015), 93–137. MR Zbl

[Mackenzie 2011] K. C. H. Mackenzie, "Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids", *J. Reine Angew. Math.* **658** (2011), 193–245. MR Zbl

[Mackenzie and Xu 1994] K. C. H. Mackenzie and P. Xu, "Lie bialgebroids and Poisson groupoids", *Duke Math. J.* **73**:2 (1994), 415–452. MR Zbl

[Mackenzie and Xu 1998] K. C. H. Mackenzie and P. Xu, "Classical lifting processes and multiplicative vector fields", *Quart. J. Math. Oxford Ser.* (2) **49**:193 (1998), 59–85. MR Zbl

[Mehta 2009] R. A. Mehta, "Q-groupoids and their cohomology", *Pacific J. Math.* **242**:2 (2009), 311–332. MR Zbl

[Pradines 1988] J. Pradines, "Remarque sur le groupoïde cotangent de Weinstein–Dazord", *C. R. Acad. Sci. Paris Sér. I Math.* **306**:13 (1988), 557–560. MR Zbl

[Weinstein and Xu 1991] A. Weinstein and P. Xu, "Extensions of symplectic groupoids and quantization", *J. Reine Angew. Math.* **417** (1991), 159–189. MR Zbl

Received May 14, 2016. Revised June 28, 2016.

ALEJANDRO CABRERA
DEPARTAMENTO DE MATEMATICA APLICADA, INSTITUTO DE MATEMATICA
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CAIXA POSTAL 68530
21941-909 RIO DE JANEIRO-RJ
BRAZIL
acabrera@labma.ufrj.br

THIAGO DRUMMOND
DEPARTAMENTO DE MATEMATICA, INSTITUTO DE MATEMATICA
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CAIXA POSTAL 68530
21941-909 RIO DE JANEIRO-RJ
BRAZIL
drummond@im.ufrj.br

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.337

THE RICCI–BOURGUIGNON FLOW

GIOVANNI CATINO, LAURA CREMASCHI, ZINDINE DJADLI,
CARLO MANTEGAZZA AND LORENZO MAZZIERI

We present some results on a family of geometric flows introduced by J. P. Bourguignon in 1981 that generalize the Ricci flow. For suitable values of the scalar parameter involved in these flows, we prove short time existence and provide curvature estimates. We also state some results on the associated solitons.

1. Introduction

In this paper we consider an n-dimensional, compact, smooth, Riemannian manifold M (without boundary) whose metric $g = g(t)$ is evolving according to the flow equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\rho \mathrm{R} g = -2(\mathrm{Ric} - \rho \mathrm{R} g) \tag{1-1}$$

where Ric is the Ricci tensor of the manifold, R its scalar curvature and ρ is a real constant. This family of geometric flows contains, as a special case, the Ricci flow, setting $\rho = 0$. Moreover, by a suitable rescaling in time, when ρ is nonpositive, they can be seen as an interpolation between the Ricci flow and the Yamabe flow (see [Brendle 2005; Schwetlick and Struwe 2003; Ye 1994], for instance), obtained as a limit when $\rho \to -\infty$.

It should be noticed that for special values of the constant ρ the tensor $\mathrm{Ric} - \rho \mathrm{R} g$ appearing on the right-hand side of the evolution equation is of special interest. In particular,

- $\rho = 1/2$, the Einstein tensor $\mathrm{Ric} - \frac{\mathrm{R}}{2} g$,
- $\rho = 1/n$, the traceless Ricci tensor $\mathrm{Ric} - \frac{\mathrm{R}}{n} g$,
- $\rho = 1/2(n-1)$, the Schouten tensor $\mathrm{Ric} - \frac{\mathrm{R}}{2(n-1)} g$,
- $\rho = 0$, the Ricci tensor Ric.

MSC2010: 53C21.
Keywords: Ricci flow, Ricci-Bourguignon flow, short time existence.

In dimension two, the first three tensors are zero, hence the flow is static, and in higher dimension the values of ρ are strictly ordered as above, in descending order.

Apart from these special values of ρ, for which we will call the associated flows by the name of the corresponding tensor, in general we will refer to the evolution equation defined by the PDE system (1-1) as the Ricci–Bourguignon flow (or shortly RB flow).

The study of these flows was proposed by Jean-Pierre Bourguignon [1981, Question 3.24], building on some unpublished work of Lichnerowicz in the sixties and a paper of Aubin [1970]. In 2003, Fischer [2004] studied a conformal version of this problem where the scalar curvature is constrained along the flow. In 2011, Lu, Qing and Zheng [Lu et al. 2014] also proved some results on the conformal Ricci–Bourguignon flow. Some results concerning solitons of the Ricci–Bourguignon flow (called *gradient ρ-Einstein solitons*) can be found in [Catino and Mazzieri 2016; Catino et al. 2015b].

We will see in the next section that when ρ is larger than $1/2(n-1)$ the principal symbol of the operator in the right hand side of the second order quasilinear parabolic PDE (1-1) has negative eigenvalues, not allowing even a short time existence result for the flow for general initial data (manifold M and initial metric g_0). On the contrary, the main task of Section 2 will be to show that for any $\rho < 1/2(n-1)$, every initial compact Riemannian manifold (M, g_0) has a unique smooth solution $g(t)$ solving the flow equation (1-1), with $g(0) = g_0$, at least in a positive time interval.

However, the problem of knowing whether the "critical" *Schouten flow*

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\mathrm{Ric} + \dfrac{\mathrm{R}}{n-1} g, \\ g(0) = g_0, \end{cases} \tag{1-2}$$

when $\rho = 1/2(n-1)$, admits or not a short time solution for general initial manifolds and metrics remains open, when $n \geq 3$.

We will see that if $\rho \leq 1/2(n-1)$, the principal symbol of the elliptic operator is nonnegative definite and it actually contains some zero eigenvalues due to the diffeomorphism invariance of the geometric flow. When $\rho < 1/2(n-1)$, these zero eigenvalues are the only ones, while all the others are actually positive, hence, they can be dealt with (as is customary by now) by means of the so-called DeTurck's trick [1983; 2003]. In the case of the Schouten flow $\rho = 1/2(n-1)$ instead, the principal symbol contains an extra zero eigenvalue besides the ones due to the diffeomorphism invariance, preventing this argument from being sufficient to conclude and to give a general short time existence result.

We mention that the presence of this extra zero eigenvalue should be expected, as the Cotton tensor, which is obtained from the Schouten tensor A by

$$\mathrm{C}_{ijk} = \nabla_k \mathrm{A}_{ij} - \nabla_j \mathrm{A}_{ik} = \nabla_k \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{ik} - \frac{1}{2(n-1)}(\nabla_k \mathrm{R} g_{ij} - \nabla_j \mathrm{R} g_{ik}),$$

satisfies the following invariance under the conformal change of metric $\widetilde{g} = e^{2u} g$,

$$e^{3u}\widetilde{\mathrm{C}}_{ijk} = \mathrm{C}_{ijk} + (n-2)\mathrm{W}_{ijkl}\nabla^l u;$$

see [Catino et al. 2016, equation 3.35]. Recently, Delay [2014], following the work of Fischer and Marsden, gave some evidence on the fact that DeTurck's trick should fail for the Schouten tensor.

In Section 3, we will compute the evolution equations for the curvature.

In Section 4, by means of the maximum principle, we derive, from the evolution of the curvature, some conditions on the curvature which are preserved by the RB flow. In particular, we show that the Hamilton–Ivey estimate in dimension three holds.

In Section 5, we establish some *a priori* estimates on the Riemann tensor and prove that, if a compact solution of the flow exists up to a finite maximal time T, then the Riemann tensor is unbounded when approaching T.

Finally, in the last section we discuss the structure and the classification of the solitons of the RB flow.

1A. ***Notation and preliminaries.*** The Riemann curvature operator of a Riemannian manifold (M, g) of dimension n is defined as in [Gallot et al. 1990] by

$$\mathrm{Riem}(X, Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z,$$

and we will denote by $d\mu_g$ the canonical volume measure associated to the metric g.

In a local coordinate system, the components of the $(3, 1)$-Riemann curvature tensor are given by $\mathrm{R}^l_{ijk}(\partial/\partial x^l) = \mathrm{Riem}(\partial/\partial x^i, \partial/\partial x^j)\partial/\partial x^k$, and we denote by $\mathrm{R}_{ijkl} = g_{lm}\mathrm{R}^m_{ijk}$ its $(4, 0)$-version.

With this choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v, w, v, w) = \mathrm{R}_{ijkl}v^i w^j v^k w^l > 0$.

The Ricci tensor is obtained as the contraction $\mathrm{R}_{ik} = g^{jl}\mathrm{R}_{ijkl}$, and $\mathrm{R} = g^{ik}\mathrm{R}_{ik}$ will denote the scalar curvature.

The so-called Weyl tensor is then defined by the decomposition formula (see [Gallot et al. 1990, Chapter 3, Section K]) of the Riemann tensor in dimension $n \geq 3$,

$$\mathrm{W}_{ijkl} = \mathrm{R}_{ijkl} + \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) - \frac{1}{n-2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il}). \tag{1-3}$$

The tensor W satisfies all the symmetries of the curvature tensor and all its traces with the metric are zero, as can be easily seen from the above formula.

In dimension three, W is identically zero for every Riemannian manifold (M, g), and it becomes relevant when $n \geq 4$ since it vanishes if and only if (M, g) is locally conformally flat. This latter condition means that around every point $p \in M$ there is a conformal deformation $\widetilde{g}_{ij} = e^f g_{ij}$ of the original metric g, such that the new metric is flat, namely, the Riemann tensor associated to $\widetilde{g}$ is zero in U_p (here $f : U_p \to \mathbb{R}$ is a smooth function defined in a open neighborhood U_p of p).

2. Short time existence

Theorem 2.1. *Let $\rho < 1/2(n-1)$. Then, the evolution equation (1-1) has a unique solution for a positive time interval on any smooth, n-dimensional, compact Riemannian manifold M (without boundary) for any initial metric g_0.*

Proof. We first compute the linearized operator DL_{g_0} of the operator $L = -2(\mathrm{Ric} - \rho \mathrm{R} g)$ at a metric g_0. The Ricci tensor and the scalar curvature have the following linearizations (see [Besse 1987, Theorem 1.174] or [Topping 2006]), where we use the metric g_0 to lower and raise indices and to take traces:

$$\begin{aligned} D\mathrm{Ric}_{g_0}(h)_{ik} &= \tfrac{1}{2}(-\Delta h_{ik} - \nabla_i \nabla_k \operatorname{tr}(h) + \nabla_i \nabla^t h_{tk} + \nabla_k \nabla^t h_{it}) + \mathrm{LOT}, \\ D\mathrm{R}_{g_0}(h) &= -\Delta(\operatorname{tr} h) + \nabla^s \nabla^t h_{st} + \mathrm{LOT}. \end{aligned}$$

Here LOT stands for *lower order terms*.

Then, the linearization of L at g_0 is given by

$$\begin{aligned} DL_{g_0}(h)_{ik} &= -2(D\mathrm{Ric}_{g_0}(h)_{ik} - \rho D\mathrm{R}_{g_0}(h)(g_0)_{ik}) + 2\rho \mathrm{R}_{g_0} h_{ik} \\ &= \Delta h_{ik} + \nabla_i \nabla_k \operatorname{tr}(h) - \nabla_i \nabla^t h_{tk} - \nabla_k \nabla^t h_{it} \\ &\qquad\qquad - 2\rho(\Delta(\operatorname{tr} h) - \nabla^s \nabla^t h_{st})(g_0)_{ik} + \mathrm{LOT}, \end{aligned}$$

for every bilinear form $h \in \Gamma(S^2 M)$. Now, we obtain the principal symbol of the linearized operator in the direction of an arbitrary cotangent vector ξ by replacing each covariant derivative ∇_α appearing in the higher order terms with the corresponding component ξ_α:

$$\begin{aligned} \sigma_\xi(DL_{g_0})(h)_{ik} = \xi^t \xi_t h_{ik} &+ \xi_i \xi_k \operatorname{tr}_{g_0}(h) - \xi_i \xi^t h_{kt} - \xi_k \xi^t h_{it} \\ &- 2\rho \xi^t \xi_t \operatorname{tr}_{g_0}(h)(g_0)_{ik} + 2\rho \xi^t \xi^s h_{ts}(g_0)_{ik}. \end{aligned}$$

As usual, since the symbol is homogeneous we can assume that $|\xi|_{g_0} = 1$ and we perform all the computations in an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of $T_p M$ such that $\xi = g_0(e_1, \cdot)$, that is, $\xi_i = 0$ for $i \neq 1$.

Hence we obtain

$$\sigma_\xi(DL_{g_0})(h)_{ik} = h_{ik} + \delta_{i1}\delta_{k1} \operatorname{tr}_{g_0}(h) - \delta_{i1} h_{k1} - \delta_{k1} h_{i1} - 2\rho \operatorname{tr}_{g_0}(h)\delta_{ik} + 2\rho h_{11} \delta_{ik},$$

which can be represented in the coordinate system

$$(h_{11}, h_{22}, \dots, h_{nn}, h_{12}, \dots, h_{1n}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

for any $h \in \Gamma(S^2M)$, by the following matrix

$$\sigma_\xi(DL_{g_0}) = \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1-2\rho & \cdots & 1-2\rho \\ \vdots & & A[n-1] & \\ 0 & & & \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{(n-1)(n-2)/2} \end{array}\right),$$

where $A[n-1]$ is the $(n-1) \times (n-1)$ matrix given by

$$A[n-1] = \begin{pmatrix} 1-2\rho & -2\rho & \cdots & -2\rho \\ -2\rho & 1-2\rho & \cdots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \cdots & 1-2\rho \end{pmatrix}.$$

We can see that there are at least n null eigenvalues, as would be expected by the diffeomorphism invariance of the operator L, and $(n-1)(n-2)/2$ eigenvalues equal to 1. The remaining $n-1$ eigenvalues can be computed by the following lemma which is easily proved by induction on the dimension of A.

Lemma 2.2. *Let $A[m]$ be the $m \times m$ matrix*

$$A[m] = \begin{pmatrix} 1-2\rho & -2\rho & \cdots & -2\rho \\ -2\rho & 1-2\rho & \cdots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \cdots & 1-2\rho \end{pmatrix}. \tag{2-1}$$

Then we have

$$\det(A[m] - \lambda \mathrm{Id}_m) = (1-\lambda)^{(m-1)}(1-2m\rho-\lambda).$$

Using this lemma, we conclude that the eigenvalues of the principal symbol of DL_{g_0} are 0 with multiplicity n, 1 with multiplicity $\frac{1}{2}(n+1)(n-2)$ and $1-2(n-1)\rho$ with multiplicity 1.

Now we apply the so-called *DeTurck's trick* [1983; 2003] to show that the RB flow is equivalent to a Cauchy problem for a strictly parabolic operator, modulo the action of the diffeomorphism group of M. Let $V : \Gamma(S^2M) \to \Gamma(TM)$ be

"DeTurck's" vector field defined by

$$
\begin{aligned}
V^j(g) &= -g_0^{jk} g^{pq} \nabla_p\big(\tfrac{1}{2}\operatorname{tr}_g(g_0) g_{qk} - (g_0)_{qk}\big) \\
&= -\tfrac{1}{2} g_0^{jk} g^{pq} (\nabla_k (g_0)_{pq} - \nabla_p (g_0)_{qk} - \nabla_q (g_0)_{pk}),
\end{aligned}
\tag{2-2}
$$

where g_0 is a fixed Riemannian metric on M and g_0^{jk} are the components of the inverse matrix of g_0.

DeTurck's trick (see [DeTurck 1983, 2003] for details) states that in order to show the smooth existence part of the theorem, we only need to check that the operator $D(L-\mathscr{L}_V)_{g_0}$ is strongly elliptic, where $\mathscr{L}_V$ is the Lie derivative operator in the direction of V.

The principal symbol of this latter operator, with the same notation used above, is well known and is given by

$$\sigma_\xi(D\mathscr{L}_V)_{g_0}(h)_{ik} = \delta_{i1}\delta_{k1}\operatorname{tr}_{g_0}(h) - \delta_{i1}h_{k1} - \delta_{k1}h_{i1}.$$

Then we can easily see that the linearized DeTurck–Ricci–Bourguignon operator has principal symbol in the direction ξ, with respect to an orthonormal basis $\{\xi^\flat, e_2, \dots, e_n\}$, given by

$$
\sigma_\xi((D(L-\mathscr{L}_V)_{g_0}) = \left(\begin{array}{c|c|c}
\begin{matrix} 1 & -2\rho & \cdots & -2\rho \\ \vdots & & A[n-1] & \\ 0 & & & \end{matrix} & 0 & 0 \\
\hline
0 & \mathrm{Id}_{(n-1)} & 0 \\
\hline
0 & 0 & \mathrm{Id}_{(n-1)(n-2)/2}
\end{array}\right),
$$

expressed in the coordinate system

$$(h_{11}, h_{22}, \dots, h_{nn}, h_{12}, h_{13}, \dots, h_{1n}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

for any $h \in \Gamma(S^2M)$.

Using Lemma 2.2 again, this matrix has $\frac{1}{2}n(n+1)-1$ eigenvalues equal to 1 and 1 eigenvalue equal to $1-2(n-1)\rho$. Therefore, by DeTurck's trick, a sufficient condition for the existence of a solution is that $\rho < 1/(2(n-1))$.

The uniqueness part of the theorem is proven in the same way as for the Ricci flow (see [Hamilton 1995]). The RB flow is equivalent, via the one parameter group of diffeomorphisms generated by DeTurck's vector field, to the DeTurck–RB flow which is strictly parabolic. On the other hand, the one parameter group of diffeomorphisms satisfies the harmonic map flow introduced by Eells and Sampson

[1964], which is also parabolic. These two facts imply the uniqueness of the solution for the RB flow (see [Chow and Knopf 2004, Chapter 3, Section 4] for more details). □

3. Evolution of the curvature

3A. ***The evolution of curvature.*** As the metric tensor evolves by

$$\frac{\partial}{\partial t} g_{ij} = -2(\mathrm{R}_{ij} - \rho \mathrm{R} g_{ij}),$$

it is easy to see, differentiating the identity $g_{ij} g^{jl} = \delta_i^l$, that

$$\frac{\partial}{\partial t} g^{jl} = 2(\mathrm{Ric}^{jl} - \rho \mathrm{R} g^{jl}). \tag{3-1}$$

It follows that the canonical volume measure μ satisfies

$$\frac{d\mu}{dt} = \frac{\partial}{\partial t}\sqrt{\det g_{ij}}\mathscr{L}^n = \frac{\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial}{\partial t} g_{ij}}{2}\mathscr{L}^n = (n\rho - 1)\mathrm{R}\sqrt{\det g_{ij}}\mathscr{L}^n = (n\rho-1)\mathrm{R}\mu.$$

Computing in a normal coordinate system, the evolution equation for the Christoffel symbols is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} =& \frac{1}{2} g^{il}\Big\{\frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial t} g_{kl}\Big) + \frac{\partial}{\partial x_k}\Big(\frac{\partial}{\partial t} g_{jl}\Big) - \frac{\partial}{\partial x_l}\Big(\frac{\partial}{\partial t} g_{jk}\Big)\Big\} \\
&+ \frac{1}{2}\frac{\partial}{\partial t} g^{il}\Big\{\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk}\Big\} \\
=& \frac{1}{2} g^{il}\Big\{\nabla_j\Big(\frac{\partial}{\partial t} g_{kl}\Big) + \nabla_k\Big(\frac{\partial}{\partial t} g_{jl}\Big) - \nabla_l\Big(\frac{\partial}{\partial t} g_{jk}\Big)\Big\} \\
=& - g^{il}\{\nabla_j(\mathrm{R}_{kl} - \rho\mathrm{R} g_{kl}) + \nabla_k(\mathrm{R}_{jl} - \rho\mathrm{R} g_{jl}) - \nabla_l(\mathrm{R}_{jk} - \rho\mathrm{R} g_{jk})\} \\
=& - \nabla_j\mathrm{R}^i_k - \nabla_k\mathrm{R}^i_j - \nabla^i\mathrm{R}_{jk} + \rho(\nabla_j\mathrm{R}\delta^i_k + \nabla_k\mathrm{R}\delta^i_j + \nabla^i\mathrm{R} g_{jk}).
\end{aligned}$$

Proposition 3.1. *Along the RB flow on a n-dimensional Riemannian manifold (M, g), the curvature tensor, the Ricci tensor and the scalar curvature satisfy the following evolution equations*:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R}_{ijkl} =& \Delta\mathrm{R}_{ijkl} + 2(\mathrm{B}_{ijkl} - \mathrm{B}_{ijlk} - \mathrm{B}_{iljk} + \mathrm{B}_{ikjl}) \\
&- g^{pq}(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}) \\
&- \rho(\nabla_i\nabla_k\mathrm{R} g_{jl} - \nabla_i\nabla_l\mathrm{R} g_{jk} - \nabla_j\nabla_k\mathrm{R} g_{il} + \nabla_j\nabla_l\mathrm{R} g_{ik}) + 2\rho\mathrm{R}\mathrm{R}_{ijkl},
\end{aligned} \tag{3-2}$$

where the tensor B *is defined as* $\mathrm{B}_{ijkl} = g^{pq} g^{rs} \mathrm{R}_{ipjr} \mathrm{R}_{kqls}$,

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{R}_{ik} &= \Delta \mathrm{R}_{ik} + 2g^{pq}g^{rs}\mathrm{R}_{pirk}\mathrm{R}_{qs} - 2g^{pq}\mathrm{R}_{pi}\mathrm{R}_{qk} \\ &\quad - (n-2)\rho\nabla_i\nabla_k\mathrm{R} - \rho\Delta\mathrm{R}g_{ik},\end{aligned} \tag{3-3}$$

$$\frac{\partial}{\partial t}\mathrm{R} = (1-2(n-1)\rho)\Delta\mathrm{R} + 2|\mathrm{Ric}|^2 - 2\rho\mathrm{R}^2. \tag{3-4}$$

Proof. The following computation is analogous to the one for the Ricci flow performed by Hamilton [1982].

By the first variation formula for the (4, 0)-Riemann tensor (see [Besse 1987, Theorem 1.174] or [Topping 2006]), we have in general

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Riem}(X,Y,W,Z) &= \tfrac{1}{2}(h(\mathrm{Riem}(X,Y)W,Z) - h(\mathrm{Riem}(X,Y)Z,W)) \\ &\quad -\tfrac{1}{2}(-\nabla^2_{Y,W}h(X,Z) - \nabla^2_{X,Z}h(Y,W) + \nabla^2_{X,W}h(Y,Z) + \nabla^2_{Y,Z}h(X,W)),\end{aligned}$$

where $X, Y, W, Z \in \Gamma(TM)$ are vector fields and $h = (\partial/\partial t)g$. Along the RB flow $h = -2(\mathrm{Ric} - \rho\mathrm{R}g)$, and therefore

$$\begin{aligned}&\frac{\partial}{\partial t}\mathrm{Riem}(X, Y, W, Z) \\ &= -\mathrm{Ric}(\mathrm{Riem}(X, Y)W, Z) + \mathrm{Ric}(\mathrm{Riem}(X, Y)Z, W) \\ &\qquad -\nabla^2_{Y,W}\mathrm{Ric}(X, Z) - \nabla^2_{X,Z}\mathrm{Ric}(Y, W) + \nabla^2_{X,W}\mathrm{Ric}(Y, Z) + \nabla^2_{Y,Z}\mathrm{Ric}(X, W) \\ &\qquad -\rho(-\nabla^2_{Y,W}\mathrm{R}g(X, Z) - \nabla^2_{X,Z}\mathrm{R}g(Y, W) + \nabla^2_{X,W}\mathrm{R}g(Y, Z) + \nabla^2_{Y,Z}\mathrm{R}g(X, W)) \\ &\qquad + 2\rho\mathrm{R}\mathrm{Riem}(X, Y, W, Z).\end{aligned}$$

Using the second Bianchi identity and the commutation formula for second covariant derivatives, we get the following equation for the Laplacian of the Riemann tensor:

$$\begin{aligned}&\Delta\mathrm{Riem}(X, Y, W, Z) \\ &= -\nabla^2_{Y,W}\mathrm{Ric}(X, Z) - \nabla^2_{X,Z}\mathrm{Ric}(Y, W) + \nabla^2_{X,W}\mathrm{Ric}(Y, Z) + \nabla^2_{Y,Z}\mathrm{Ric}(X, W) \\ &\qquad - \mathrm{Ric}(\mathrm{Riem}(W, Z)Y, X) + \mathrm{Ric}(\mathrm{Riem}(W, Z)X, Y) - 2(\mathrm{B}(X, Y, W, Z) \\ &\qquad -\mathrm{B}(X, Y, Z, W) + \mathrm{B}(X, W, Y, Z) - \mathrm{B}(X, Z, Y, W)).\end{aligned}$$

Plugging it into the evolution equation, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\mathrm{Riem}(X,Y, W, Z) \\ &= \Delta\mathrm{Riem}(X,Y, W, Z) - \rho(\nabla^2\mathrm{R} \owedge g)(X,Y, W, Z) \\ &\qquad + 2(\mathrm{B}(X,Y, W, Z) - \mathrm{B}(X,Y, Z, W) + \mathrm{B}(X, W,Y, Z) - \mathrm{B}(X, Z,Y, W)) \\ &\qquad - \mathrm{Ric}(\mathrm{Riem}(X,Y)W, Z) + \mathrm{Ric}(\mathrm{Riem}(X,Y)Z, W) - \mathrm{Ric}(\mathrm{Riem}(W, Z)X,Y) \\ &\qquad + \mathrm{Ric}(\mathrm{Riem}(W, Z)Y, X) + 2\rho\mathrm{R}\mathrm{Riem}(X,Y, W, Z),\end{aligned}$$

which is formula (3-2) once written in coordinates. Here the symbol $\owedge$ denotes the Kulkarni–Nomizu product of two symmetric bilinear forms p and q, defined by

$$
\begin{aligned}
&(p \owedge q)(X, Y, Z, T) \\
&\qquad = p(X, Z)q(Y, T) + p(Y, T)q(X, Z) - p(X, T)q(Y, Z) - p(Y, Z)q(X, T),
\end{aligned}
$$

for every tangent vector fields $X, Y, Z, T \in \Gamma(TM)$.

Taking into account the evolution equation for the inverse of the metric (3-1), contracting equation (3-2) and using again the second Bianchi identity, formula (3-3) follows (see [Hamilton 1982] for details). Contracting again one gets the evolution equation (3-4) for the scalar curvature. □

3B. ***Uhlenbeck's trick and the evolution of the curvature operator.*** In this subsection we want to study the evolution equation of the curvature operator, as was done for the Ricci flow by Hamilton [1986].

First of all, we simplify the expression of the reaction term in equation (3-2) by means of the so-called Uhlenbeck's trick [Hamilton 1986]. Briefly, we will relate the curvature tensor of the evolving metric to an evolving tensor of an abstract bundle with the same symmetries of the curvature (see Proposition 3.4) and a nicer evolution equation; afterwards we will find a suitable orthonormal moving frame of $(TM, g(t))$ and write the evolution equation of the coordinates of the Riemann tensor with respect to this frame. The result will be a system of *scalar* evolution equations and no more a tensorial equation (see [Chow and Knopf 2004] for more details on this method in the case of Ricci flow).

Let $(M, g(t))_{t\in[0,T)}$ be the solution of the RB flow with initial data g_0 and consider on the tangent bundle TM the family of endomorphisms $\{\varphi(t)\}_{t\in[0,T)}$ defined by the evolution equation

$$
\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \mathrm{Ric}^{\#}_{g(t)} \circ \varphi(t) - \rho \mathrm{R}_{g(t)}\varphi(t), \\ \varphi(0) = \mathrm{Id}_{TM}, \end{cases} \tag{3-5}
$$

where $\mathrm{Ric}^{\#}_{g(t)}$ is the endomorphism of the tangent bundle canonically associated to the Ricci tensor by raising an index.

For every point p of the manifold M, the evolution equation (3-5) represents a system of linear ODEs on the fiber T_pM; therefore a unique solution exists as long as the RB flow exists. Moreover, if we let $(h(t))_{t\in[0,T)}$ be the family of bundle metrics defined by $h(t) = \varphi(t)^*(g(t))$, where $\varphi(t)$ satisfies system (3-5), then $h(t) = g_0$ for every $t \in [0, T)$. As

$$
\text{for all } t \in [0, T), \qquad \varphi(t) : (TM, g_0) \to (TM, g(t))
$$

is a bundle isometry, the pullback via $\varphi(t)$ of the Levi-Civita connection associated to $g(t)$ is a connection on TM compatible with the metric g_0. In the following, we will denote by (V, h) the vector bundle (TM, g_0) in order to stress the fact that we are not considering the Levi-Civita connection associated to g_0, but the family of time-dependent connections $D(t)$ defined via the bundle isometries $\varphi(t)$.

The following lemma states some basic properties of these pullback connections:

Lemma 3.2 (see [Chow and Knopf 2004, Chapter 6, Section 2]). *Let $D(t) : \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$ be the pullback connection defined by*

$$D(t)_X \zeta = \varphi(t)^*(\nabla_X^{g(t)}(\varphi(t)(\zeta))),$$

for all $t \in [0, T)$, for all $X \in \Gamma(TM)$, for all $\zeta \in \Gamma(V)$, where $\nabla^{g(t)}$ is the Levi-Civita connection of $g(t)$.

Let again $D(t)$ be the canonical extension to the tensor powers of V and T be a covariant tensor on M. Then, for every $t \in [0, T)$ and $X \in \Gamma(TM)$ we have

$$D(t)_X(\varphi(t)^*(T)) = \varphi(t)^*(\nabla_X^{g(t)} T).$$

In particular, $D(t)_X h = \varphi^*(\nabla_X^{g(t)} g(t)) = 0$, so every connection of the family $D(t)$ is compatible with the bundle metric h on V.

Let $D^2 : \Gamma(TM) \times \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$ be the second covariant derivative defined by

$$D^2_{X,Y}(\zeta) = D_X(D_Y \zeta) - D_{\nabla_X^{g(t)} Y} \zeta, \qquad \text{for all } X, Y \in \Gamma(TM), \text{ for all } \zeta \in \Gamma(V),$$

and the rough Laplacian defined by $\Delta_D = \mathrm{tr}_g(D^2)$. Then, for every covariant tensor T on M, we have

$$D^2_{X,Y}(\varphi^*(T)) = \varphi^*(\nabla^2_{X,Y} T) \qquad \text{for all } X, Y \in \Gamma(TM), \tag{3-6}$$

$$\Delta_D(\varphi^*(T)) = \varphi^*(\Delta_g T). \tag{3-7}$$

Remark 3.3. Let $\mathcal{R} \in \mathrm{End}(\Lambda^2 M)$ be the *Riemann curvature operator* defined by

$$\langle \mathcal{R}(X \wedge Y), W \wedge Z \rangle = \mathrm{Riem}(X, Y, W, Z), \tag{3-8}$$

where $\langle \cdot, \cdot \rangle$ is the linear extension of g to the exterior powers of TM.

In the following, we use a convention on the Lie algebra structure of $\Lambda^2 M$ which is different from the original one chosen by Hamilton [1986]. More precisely, with his convention, the eigenvalues of the curvature operator are twice the sectional curvatures, whereas with our convention the curvature operator has the sectional curvatures as eigenvalues. In particular, every formula differs from the corresponding one in the usual theory of the Ricci flow by a factor of 2 (see also [Chow and Knopf 2004, Chapter 6, Section 3] for the details). We recall that $\mathcal{R}$ can be considered as an element of $\Gamma(S^2(\Lambda^2 M))$, and the following equations hold:

$$\mathrm{R} = 2 \sum_{i<k} \mathcal{R}^{(ik)}_{(ik)}, \quad (\mathcal{R}^2)_{ijkl} = \mathrm{B}_{ijkl} - \mathrm{B}_{ijlk}, \quad (\mathcal{R}\#\mathcal{R})_{ijkl} = \mathrm{B}_{ikjl} - \mathrm{B}_{iljk},$$

where B is defined as in Proposition 3.1. For more details on the structure of the curvature operator we refer the reader again to [Chow and Knopf 2004, Chapter 6, Section 3].

We now consider the pullback of the Riemann curvature tensor and the curvature operator.

Proposition 3.4. *Let* Piem *be the pullback of the Riemann curvature tensor via the family of bundle isometries* $\{\varphi(t)\}_{t\in[0,T)}$. *The following statements hold true*:

(1) Piem *has the same symmetry properties as* Riem, *i.e.*, *it can be seen as an element of* $\Gamma(S^2(\Lambda^2 V))$ *and it satisfies the first Bianchi identity*;

(2) *For every* $p \in M$ *and* $t \in [0,T)$ *the* ***algebraic curvature operator*** $\mathcal{P}(p,t) \in \mathrm{End}(\Lambda^2 V_p)$ (*see Remark 3.7*), *defined by* $\varphi\circ\mathcal{P} = \mathcal{R}\circ\varphi$ *has the same eigenvalues as* $\mathcal{R}(p,t)$. *In particular*, $\mathcal{P}$ *is positive* (*nonnegative*) *definite if and only if* $\mathcal{R}$ *is positive* (*nonnegative*) *definite*;

(3) $\mathrm{Pic}(t) = \mathrm{tr}_h(\mathrm{Piem}(t)) = \varphi(t)^*(\mathrm{Ric}_{g(t)})$;

(4) $\mathrm{P} = \mathrm{tr}_h(\mathrm{Pic}(t)) = \mathrm{R}_{g(t)}$;

(5) $\mathrm{B}(\mathrm{Piem}) = \varphi^*(\mathrm{B}(\mathrm{Riem}))$, *where* B *is defined the same way as in Proposition 3.1 for a generic element of* $S^2(\Lambda^2 V^*)$.

Finally, we can compute the evolution of Piem and $\mathcal{P}$.

Proposition 3.5. *The tensors* Piem *and* $\mathcal{P}$ *satisfy respectively the following evolution equations*

$$\begin{aligned}(3\text{-}9)\quad \frac{\partial}{\partial t}(\mathrm{Piem})_{abcd} &= \Delta_D(\mathrm{Piem})_{abcd} - \rho(\varphi^*(\nabla^2\mathrm{R}) \owedge h)_{abcd} \\ &\quad + 2(\mathrm{B}(\mathrm{Piem})_{abcd} - \mathrm{B}(\mathrm{Piem})_{abdc} + \mathrm{B}(\mathrm{Piem})_{acbd} - \mathrm{B}(\mathrm{Piem})_{adbc}) \\ &\quad - 2\rho \mathrm{P}\mathrm{Piem}_{abcd},\end{aligned}$$

$$(3\text{-}10)\qquad \frac{\partial}{\partial t}\mathcal{P} = \Delta_D\mathcal{P} - 2\rho\varphi^*(\nabla^2\,\mathrm{tr}_h(\mathcal{P})) \owedge h + 2\mathcal{P}^2 + 2\mathcal{P}^{\#} - 4\rho\,\mathrm{tr}_h(\mathcal{P})\mathcal{P},$$

where $\mathrm{tr}_h(\mathcal{P}(t)) = \mathrm{tr}_{g(t)}(\mathcal{R}(t)) = \frac{1}{2}\mathrm{R}(t)$.

Remark 3.6. On the right-hand side of (3-9) the term $\varphi^*(\nabla^2\mathrm{R})$ appears (i.e., the pullback of the Hessian of the scalar curvature, seen as a symmetric 2-form on the tangent bundle) and it cannot be expressed in terms of the connection $D(t)$.

Proof. Let $\zeta_1, \dots, \zeta_4$ be sections of V; then combining the evolution equations of the Riemann tensor (3-2) and of the bundle isometry φ (3-5), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\mathrm{Piem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\\
&= \varphi^*\Big(\frac{\partial}{\partial t}\mathrm{Riem}\Big)(\zeta_1, \zeta_2, \zeta_3, \zeta_4) + \mathrm{Riem}\Big(\frac{\partial\varphi}{\partial t}(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big)\\
&\quad + \mathrm{Riem}\Big(\varphi(\zeta_1), \frac{\partial\varphi}{\partial t}(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big) + \mathrm{Riem}\Big(\varphi(\zeta_1), \varphi(\zeta_2), \frac{\partial\varphi}{\partial t}(\zeta_3), \varphi(\zeta_4)\Big)\\
&\quad + \mathrm{Riem}\Big(\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \frac{\partial\varphi}{\partial t}(\zeta_4)\Big)\\
&= \varphi^*(\Delta_g \mathrm{Riem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \rho\varphi^*(\nabla^2\mathrm{R} \owedge g)(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\\
&\quad + 2\varphi^*(\mathrm{B}(\mathrm{Riem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \mathrm{B}(\mathrm{Riem})(\zeta_1, \zeta_2, \zeta_4, \zeta_3) - \mathrm{B}(\mathrm{Riem})(\zeta_1, \zeta_4, \zeta_2, \zeta_3)\\
&\quad + \mathrm{B}(\mathrm{Riem})(\zeta_1, \zeta_3, \zeta_2, \zeta_4)) + 2\rho\mathrm{R}\varphi^*(\mathrm{Riem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\\
&\quad - \mathrm{Riem}(\mathrm{Ric}^\# \circ \varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)) - \mathrm{Riem}(\varphi(\zeta_1), \mathrm{Ric}^\# \circ \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4))\\
&\quad - \mathrm{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), \mathrm{Ric}^\# \circ \varphi(\zeta_3), \varphi(\zeta_4)) - \mathrm{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \mathrm{Ric}^\# \circ \varphi(\zeta_4))\\
&\quad + \mathrm{Riem}((\mathrm{Ric}^\# \circ \varphi - \rho\mathrm{R}\varphi)(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4))\\
&\quad + \mathrm{Riem}(\varphi(\zeta_1), (\mathrm{Ric}^\# \circ \varphi - \rho\mathrm{R}\varphi)(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4))\\
&\quad + \mathrm{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), (\mathrm{Ric}^\# \circ \varphi - \rho\mathrm{R}\varphi)(\zeta_3), \varphi(\zeta_4))\\
&\quad + \mathrm{Riem}(\varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), (\mathrm{Ric}^\# \circ \varphi - \rho\mathrm{R}\varphi)(\zeta_4))\\
&= \Delta_D(\mathrm{Piem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \rho(\varphi^*(\nabla^2\mathrm{R}) \owedge h)(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\\
&\quad + 2\varphi^*(\mathrm{B}(\mathrm{Piem})(\zeta_1, \zeta_2, \zeta_3, \zeta_4) - \mathrm{B}(\mathrm{Piem})(\zeta_1, \zeta_2, \zeta_4, \zeta_3) - \mathrm{B}(\mathrm{Piem})(\zeta_1, \zeta_4, \zeta_2, \zeta_3)\\
&\quad + \mathrm{B}(\mathrm{Piem})(\zeta_1, \zeta_3, \zeta_2, \zeta_4)) - 2\rho\mathrm{P}\mathrm{Piem}(\zeta_1, \zeta_2, \zeta_3, \zeta_4),
\end{aligned}
$$

where we used several identities stated above. For $\zeta_1, \dots, \zeta_4$ belonging to a local frame we get the desired equation (3-9).

Combining the evolution equation for Piem with the formulas in Remark 3.3, we find the evolution equation of $\mathcal{P}$. $\square$

Remark 3.7. It must be noticed that, even though for every $p \in M$ and $t \in [0, T)$, the tensor $\mathcal{P}(p, t)$ belongs to the set of algebraic curvature operators $\mathcal{C}_b(V_p)$, in general it does not coincide with the curvature operator of the pullback connection $D(t)$. In the present literature the pullback tensor is always denoted by Riem and this abuse of notation is somehow misleading, suggesting wrongly that $\mathrm{Piem}(t) = \varphi(t)^*(\mathrm{Riem}_{g(t)})$ is equal to $\mathrm{Riem}_{\varphi(t)^*(g(t))} = \mathrm{Riem}_h$, but this is no longer true for general isomorphisms of the tangent bundle (however it is true for $\varphi \in \mathrm{Diff}(M)$).

By Uhlenbeck's trick, the evolution equation (3-10) for $\mathcal{P}$ allows a simpler use of the maximum principle for tensors as the reaction term is nicer and the metric on $S^2(\Lambda^2 V)$ is independent of time. Moreover, the subsets of $S^2(\Lambda^2 V)$ preserved by such PDE correspond to curvature conditions preserved under the RB flow.

4. Preserved curvature conditions

In this section we will use the maximum principle in various formulations in order to find curvature conditions which are preserved by the RB flow.

4A. ***The scalar curvature.*** We begin by considering the evolution equation for the scalar curvature (3-4), which behaves as under the Ricci flow.

Proposition 4.1. *Let $(M, g(t))_{t\in[0,T)}$ be a compact maximal solution of the RB flow (1-1). If $\rho \le 1/(2(n-1))$, the minimum of the scalar curvature is nondecreasing along the flow. In particular if $\mathrm{R}_{g(0)} \ge \alpha$, for some $\alpha \in \mathbb{R}$, then $\mathrm{R}_{g(t)} \ge \alpha$ for every $t \in [0, T)$. Moreover if $\alpha > 0$ then $T \le n/(2(1-n\rho)\alpha)$.*

Proof. As $\rho \le 1/(2(n-1)) \le 1/n$, for any $n > 1$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{R} &= (1-2(n-1)\rho)\Delta\mathrm{R} + 2|\mathrm{Ric}|^2 - 2\rho\mathrm{R}^2\\ &\ge (1-2(n-1)\rho)\Delta\mathrm{R} + 2\mathrm{R}^2/n - 2\rho\mathrm{R}^2\\ &\ge (1-2(n-1)\rho)\Delta\mathrm{R},\end{aligned}$$

hence, by the maximum principle, the minimum of the scalar curvature is nondecreasing along the RB flow on a compact manifold. In particular, for every $\alpha \in \mathbb{R}$, the condition $\mathrm{R} \ge \alpha$ is preserved.

Finally, integrating the inequality

$$\frac{\partial}{\partial t}\mathrm{R}_{\min} \ge 2\Big(\frac{1}{n} - \rho\Big)\mathrm{R}_{\min}^2,$$

that holds almost everywhere for $t \in [0, T)$ (by Hamilton's trick (see [Hamilton 1997], [Mantegazza 2011, Lemma 2.1.3])), we obtain

$$\mathrm{R}_{\min}(t) \ge \frac{n\alpha}{n-2(1-n\rho)\alpha t}, \tag{4-1}$$

which, for $\alpha > 0$, gives the estimate on the maximal time of existence. $\square$

Remark 4.2. In the special case of the Schouten flow (when $\rho = 1/2(n-1)$), we actually have

$$\frac{\partial}{\partial t}\mathrm{R} \ge \frac{n-2}{n(n-1)}\mathrm{R}^2,$$

at every point of the manifold, which implies that the scalar curvature is pointwise nondecreasing and diverges in finite time.

Remark 4.3. By means of the strong maximum principle, it follows that if the initial manifold has nonnegative scalar curvature, then either the manifold is Einstein ($\mathrm{Ric}=0$) or the scalar curvature becomes positive for every positive time under any RB flow with $\rho \leq 1/(2(n-1))$.

Proposition 4.4. *Let $(M, g(t))_{t\in(-\infty,0]}$ be a compact, n-dimensional, ancient solution of the RB flow (1-1). If $\rho \leq 1/(2(n-1))$ then, either $\mathrm{R}>0$ or $\mathrm{Ric}\equiv 0$ on $M\times(-\infty, 0]$.*

Proof. As $g(t)$ is an ancient solution, for every $t_0 < t_1 \leq 0$, we can define $\widetilde{g}(s) = g(s+t_0)$, which is a solution of the RB flow for $s\in[0, t_1-t_0]$. Then we have $\widetilde{\mathrm{R}}_{\min}(0) = \mathrm{R}_{\min}(t_0)$, hence, from formula (4-1)

$$\widetilde{\mathrm{R}}_{\min}(s) \geq \frac{n}{n\widetilde{\mathrm{R}}^{-1}_{\min}(0) - 2(1-n\rho)s},$$

for every $s\in(0, t_1-t_0]$. In particular, we have

$$\mathrm{R}_{\min}(t_1) = \widetilde{\mathrm{R}}_{\min}(t_1-t_0) \geq \frac{n}{n\mathrm{R}^{-1}_{\min}(t_0) - 2(1-n\rho)(t_1-t_0)}.$$

If $\mathrm{R}_{\min}(t_0)\geq 0$, by Proposition 4.1, it follows that $\mathrm{R}_{\min}(t_1)\geq 0$, so we can assume that $\mathrm{R}_{\min}(t_0)<0$, hence

$$\mathrm{R}_{\min}(t_1) \geq \frac{n}{n\mathrm{R}^{-1}_{\min}(t_0) - 2(1-n\rho)(t_1-t_0)} > -\frac{n}{2(1-n\rho)(t_1-t_0)},$$

for every $t_1 < t_0$, and sending t_0 to $-\infty$, we still conclude that $\mathrm{R}_{\min}(t_1)\geq 0$. Since this holds for every $t_1\leq 0$ the previous remark implies the result. □

4B. ***Maximum principle for uniformly elliptic operators.*** Let M be a smooth compact manifold, $g(t)$, $t\in[0,T)$, a family of Riemannian metrics on M and $(E, h(t))$ $t\in[0,T)$, be a real vector bundle on M, endowed with a (possibly time-dependent) bundle metric. Let $D(t):\Gamma(TM)\times\Gamma(E)\to\Gamma(E)$ be a family of linear connections on E, compatible at each time with the bundle metric $h(t)$. We have already seen in Section 3B how to define the second covariant derivative, using also the Levi-Civita connections $\nabla_{g(t)}$ associated to the Riemannian metrics on M.

Definition 4.5. We consider a second order linear operator $\mathcal{L}$ on $\Gamma(E)$ that lacks a 0-th order term, and hence can be written in a local frame field $\{e_i\}_{i=1,\dots,n}$ of TM

$$\mathcal{L} = a^{ij} D^2_{e_i e_j} - b^i D_{e_i} \tag{4-2}$$

where $a = a^{ij} e_i\otimes e_j \in \Gamma(S^2(TM))$ is a symmetric $(0,2)$-tensor and $b = b^i e_i$ is a smooth vector field. We say $\mathcal{L}$ is *uniformly elliptic* if a is uniformly positive definite.

Remark 4.6. In the previous definition, both the coefficients and the connections are in general time-dependent and we say that $\mathcal{L}$ is uniformly elliptic if it is so for every $t \in [0, T)$ uniformly in time.

Weinberger [1975] proved the maximum principle for systems of solutions of a time-dependent heat equation in Euclidean space; Hamilton [1986] treated the general case of a vector bundle over an evolving Riemannian manifold. Here we present a slight generalization of Hamilton's theorem for parabolic equations with uniformly elliptic operator (Savas-Halilaj and Smoczyk [2014, Theorem 2.2] proved a "static" version). As before, $(M, g(t))$ is a smooth compact manifold equipped with a family of Riemannian metrics; we consider a real vector bundle E over M, equipped with a fixed bundle metric h and a family of time-dependent connections $D(t)$ compatible at every time with h.

Definition 4.7. Let $S \subset E$ be a subbundle and denote $S_p = S \cap E_p$ for every $p \in M$. We say that S is *invariant under parallel translation* with respect to D, if for every curve $\gamma : [0, l] \to M$ and vector $v \in S_{\gamma(0)}$, the unique parallel (with respect to D) section $v(s) \in E_{\gamma(s)}$ along $\gamma(s)$ with $v(0) = v$ is contained in S.

Theorem 4.8 (vectorial maximum principle). *Let $u : [0, T) \to \Gamma(E)$ be a smooth solution of the following parabolic equation*

$$\frac{\partial}{\partial t} u = \mathcal{L}u + F(u, t), \tag{4-3}$$

where $\mathcal{L}$ is a uniformly elliptic operator as defined in (4-2) and $F : E \times [0, T) \to E$ is a continuous map, locally Lipschitz in the E factor, which is also fiber-preserving, i.e., $F(v, t) \in E_p$ for every $p \in M$, $v \in E_p$, $t \in [0, T)$.

Let $K \subset E$ be a closed subbundle (for the metric h), invariant under parallel translation with respect to $D(t)$, for every $t \in [0, T)$, and convex in the fibers, i.e., $K_p = K \cap E_p$ is convex for every $p \in M$.

Suppose that K is preserved by the ODE associated to (4-3), i.e., for every $p \in M$ and $U_0 \in K_p$, the solution $U(t)$ of

$$\begin{cases} \frac{dU}{dt} &= F_p(U(t), t), \\ U(0) &= U_0. \end{cases} \tag{4-4}$$

remains in K_p. Then, if u is contained in K at time 0, u remains in K, i.e., $u(p, t) \in K_p$ for every $p \in M$, $t \in [0, T)$.

Proof. (Sketch) We can follow exactly the detailed proof written in [Chow et al. 2008, Chapter 10, Section 3], provided that we generalize their Lemma 10.34 to the analogue one for uniformly elliptic operator (see again [Savas-Halilaj and Smoczyk 2014, Lemma 2.2]): if $K \subset E$ satisfies all the hypotheses of Theorem 4.8 and

$u \in \Gamma(E)$ is a smooth section of E, then

$$u(p) \in K_p \quad \text{for all } p \in M \quad \Longrightarrow \quad \mathcal{L}(u)_p \in C_{u(p)} K_p \quad \text{for all } p \in M,$$

where $C_{u(p)} K_p$ is the tangent cone of the convex set K_p at $u(p)$. □

There is a further generalization of this maximum principle which allows the subset K to be time-dependent.

Theorem 4.9 (vectorial maximum principle, time-dependent set). *Let $u : [0, T) \to \Gamma(E)$ be a smooth solution of the parabolic equation (4-3), with the notations of the previous theorem. For every $t \in [0, T)$, let $K(t) \subset E$ be a closed subbundle (for the metric h), invariant under parallel translation with respect to $D(t)$, convex in the fibers and such that the spacetime track*

$$\mathcal{T} = \{(v, t) \in E \times \mathbb{R} : v \in K(t), t \in [0, T)\}$$

is closed in $E \times [0, T)$. Suppose that, for every $t_0 \in [0, T)$, $K(t_0)$ is preserved by the ODE associated, i.e., for any $p \in M$, any solution $U(t)$ of the ODE that starts in $K(t_0)_p$ remains in $K(t)_p$, as long as it exists. Then, if $u(0)$ is contained in $K(0)$, $u(p, t) \in K(t)_p$ for ever $p \in M$, $t \in [0, T)$.

The proof of this theorem, when K depends continuously on time and F does not depend on time is due to Bohm and Wilking [2007, Theorem 1.1]. In the general case the proof can be found in [Chow et al. 2008, Chapter 10, Section 5], with the usual adaptation to the uniformly elliptic case.

As remarked before, the evolution equation (3-2) of the Riemann tensor has some mixed products of type Riem $*$ Ric which makes it difficult to understand the behavior of the reaction term. On the other hand, if we perform Uhlenbeck's trick, the evolution equation (3-9) becomes a little nicer and can be used to understand how the RB flow affects the geometry.

More precisely, we use the evolution equation (3-10) for the algebraic curvature operator $\mathcal{P} \in \Gamma(S^2(\Lambda^2 V^*))$ to prove that the cone of nonnegative curvature operators is preserved by the RB flow.

Proposition 4.10. *Let $(M, g(t))_{t\in[0,T)}$ be a compact solution of the RB flow (1-1) with $\rho < 1/(2(n-1))$ and such that the initial data g_0 has nonnegative curvature operator. Then $\mathcal{R}_{g(t)} \geq 0$ for every $t \in [0, T)$.*

Proof. We recall the evolution equation (3-10) for $\mathcal{P} = \varphi^{-1} \circ \mathcal{R} \circ \varphi$,

$$\frac{\partial}{\partial t}\mathcal{P} = \Delta_D \mathcal{P} - 2\rho\varphi^*(\nabla^2 \operatorname{tr}_h(\mathcal{P})) \owedge h + 2\mathcal{P}^2 + 2\mathcal{P}^\# - 4\rho \operatorname{tr}_h(\mathcal{P})\mathcal{P},$$

where $\operatorname{tr}_h(\mathcal{P}(t)) = 1/2\mathrm{R}(t)$ is half of the scalar curvature of the metric $g(t)$. By Proposition 3.4, it suffices to show that the nonnegativity of $\mathcal{P}$ is preserved by equation (3-10). We want to apply the vectorial maximum principle Theorem 4.8, and therefore we must show that

$$\mathcal{L}(Q) = \Delta_D Q - 2\rho\varphi^*(\nabla^2 \operatorname{tr}_h(Q)) \owedge h$$

is a uniformly elliptic operator on the bundle $(\Gamma(S^2(\Lambda^2 V^*)), h, D(t))$.

As $\mathcal{L}$ is a linear second order operator, we compute as usual its principal symbol in the arbitrary direction ξ. In order to simplify the computations, we choose opportune frames at every point $p \in M$ and time $t \in [0, T)$. Then let $\{e_i\}_{i=1,\dots,n}$ be an orthonormal basis of (V_p, h_p) such that $\xi = h_p(e_1, \cdot)$. According to Uhlenbeck's trick (Section 3B) and the convention on algebraic curvature operators (Section 3B) we have that $\{f_i = \varphi(t)_p(e_i)\}_{i_1,\dots,n}$ is an orthonormal basis of T_pM with respect to $g(t)_p$, the components of $\varphi(t)_p$ with these choices are $\varphi_i^a = \delta_i^a$, and $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis of $\Lambda^2 V_p$. Hence, the principal symbol of the operator $\mathcal{L}$ written in these frames is

$$\begin{aligned}\sigma_\xi(\mathcal{L}Q)_{(ij)(kl)} &= \xi^p\xi_p Q_{(ij)(kl)} - 2\rho\delta_i^a\delta_j^b\delta_k^c\delta_l^d \operatorname{tr}_h(Q)(\xi\otimes\xi \owedge h)_{(ab)(cd)}\\ &= |\xi|^2 Q_{(ij)(kl)} - 2\rho \operatorname{tr}_h(Q)(\xi\otimes\xi \owedge h)_{(ij)(kl)}\\ &= Q_{(ij)(kl)} - 2\rho\Big(\sum_{p<q} Q_{(pq)(pq)}\Big)\delta_i^1\delta_k^1\delta_{jl},\end{aligned}$$

where we used that $|\xi| = 1$, $i < j$ and $k < l$ in the last step. Now it is easy to see that the matrix representing the symbol has the following form:

$$\sigma_\xi(\mathcal{L}) = \left(\begin{array}{c|c|c} A[n-1] & \begin{matrix} -2\rho & \cdots & 2\rho \\ \vdots & \ddots & \vdots \\ -2\rho & \cdots & -2\rho \end{matrix} & 0 \\ \hline 0 & \mathrm{Id}_{(n-1)(n-2)/2} & 0 \\ \hline 0 & 0 & \mathrm{Id}_{N(N-1)/2} \end{array}\right),$$

where we have ordered the components as follows: first the $n-1$ ones of the form $(1j)(1j)$ with $j > 1$, then the $(n-1)(n-2)/2$ ones of the form $(ij)(ij)$ with $1 < i < j$, and last the $N(N-1)/2$ "nondiagonal" ones, with $N = n(n-1)/2$ and A is the matrix defined in (2-1).

By Lemma 2.2 the eigenvalues of the symbol are 1 with multiplicity $\frac{1}{2}N(N+1)-1$ and $1-2(n-1)\rho$ with multiplicity 1, since $\rho < 1/2(n-1)$ the operator $\mathcal{L}$ is uniformly elliptic.

We next consider the reaction term $F(Q) = 2(Q^2 + Q^\# - 2\rho \operatorname{tr}_h(Q)Q)$. Clearly F is continuous, locally Lipschitz and fiber-preserving. Let $\Omega \subset \Gamma(S^2(\Lambda^2 V^*))$ be the set of nonnegative algebraic curvature operators, where we have identified $S^2(\Lambda^2 V^*) \simeq \operatorname{End}_{SA}(\Lambda^2 V)$ via the metric h. We observe that $\Omega = \{Q : \lambda_N(Q_p) \geq 0\}$, where $N = n(n-1)/2$ and λ_N is the least eigenvalue of Q_p. Hence Ω is clearly closed, by [Chow et al. 2008, Lemma 10.11] it is invariant under parallel translation with respect to every connection $D(t)$ and it is convex, provided that the function $Q \mapsto \lambda_N(Q_p)$ is concave. We can rewrite

$$\lambda_N(Q_p) = \inf_{\{v \in \Lambda^2 V_p : |v|_h = 1\}} h(Q_p(v), v),$$

so it is easy to conclude, by the bilinearity of the metric h and the concavity of inf, that the function defining Ω is actually concave and so its superlevels are convex. In order to finish the proof we have to show that the ODE $dQ/dt = F(Q)$ preserves Ω. Now, by standard facts in convex analysis, we only need to prove that $F_p(Q_p) \in T_{Q_p}\Omega_p$ for every $p \in M$ such that $Q_p \in \partial\Omega_p$, where $\partial\Omega_p$ is the set of $Q_p \in \Omega_p$ where there is $v \in \Lambda^2 V_p$ such that $Q_p(v, v) = 0$ and the tangent cone is

$$T_{Q_p}\Omega_p = \{S_p \in S^2(\Lambda^2 V_p^*) : S_p(v, v) \geq 0 \text{ for every } v \in \Lambda^2 V_p \text{ such that } Q_p(v, v) = 0\}$$

Let $v \in \Lambda^2 V_p$ and $\{\theta_\alpha\}$ be respectively a null eigenvector of Q_p and an orthonormal basis of $\Lambda^2 V_p$ that diagonalizes Q_p. Clearly $v = v^\alpha \theta_\alpha$ and $(Q_p)_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$. with $\lambda_\alpha \geq 0$. Then $(Q_p^2)_{\alpha\beta} = \lambda_\alpha^2 \delta_{\alpha\beta}$ and $(Q_p^\#)_{\alpha\beta} = \frac{1}{2}(c_\alpha^{\gamma\nu})^2 \lambda_\gamma \lambda_\nu \delta_{\alpha\beta}$ and

$$F_p(Q_p)(v, v) = \lambda_\alpha^2 (v^\alpha)^2 + \tfrac{1}{2}(c_\alpha^{\gamma\nu})^2 \lambda_\gamma \lambda_\nu (v^\alpha)^2 \geq 0. \qquad \square$$

4C. ***The evolution of the Weyl tensor.*** By means of the evolution equations found for the curvatures, we are also able to write the equation satisfied by the Weyl tensor along the RB flow (1-1). In [Catino and Mantegazza 2011] the authors compute the evolution equation of the Weyl tensor during the Ricci flow (see [Catino et al. 2015a] for a significant application of this formula) and we use most of their computations.

Proposition 4.11. *During the RB flow of an n-dimensional Riemannian manifold (M, g) the Weyl tensor satisfies the following evolution equation*:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{W}_{ijkl} &= \Delta \mathrm{W}_{ijkl} + 2(\mathrm{B(W)}_{ijkl} - \mathrm{B(W)}_{ijlk} - \mathrm{B(W)}_{iljk} + \mathrm{B(W)}_{ikjl}) \\
&\qquad + 2\rho \mathrm{R}\mathrm{W}_{ijkl} - g^{pq}(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql}) \\
&\qquad\quad + \frac{2}{(n-2)^2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{1}{(n-2)}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} \\
&\qquad\qquad - \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{Ric} \owedge g)_{ijkl} + \frac{\mathrm{R}^2 - |\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g \owedge g)_{ijkl},
\end{aligned}$$

where $\mathrm{B}(W)_{ijkl} = g^{pq} g^{rs} \mathrm{W}_{ipjr} \mathrm{W}_{kqls}$.

Proof. By recalling the decomposition formula for the Weyl tensor (1-3) we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{W} &= \frac{\partial}{\partial t}\mathrm{Riem} + \frac{1}{2(n-1)(n-2)}\Big(\frac{\partial}{\partial t}\mathrm{R}g \owedge g + 2\frac{\partial}{\partial t}g \owedge g\Big) \\ &\qquad\qquad - \frac{1}{n-2}\Big(\frac{\partial}{\partial t}\mathrm{Ric} \owedge g + \mathrm{Ric} \owedge \frac{\partial}{\partial t}g\Big) \\ &= \mathcal{L}_{II} + \mathcal{L}_0,\end{aligned}$$

where $\mathcal{L}_{II}$ is the second order term in the curvatures and $\mathcal{L}$ the 0-th one. We deal first with the higher order term; plugging in the evolution equations of Riem, Ric and R (Proposition 3.1) we get

$$\begin{aligned}\mathcal{L}_{II} &= \Delta\mathrm{Riem} - \rho(\nabla^2\mathrm{R} \owedge g) + \frac{1-2(n-1)\rho}{2(n-1)(n-2)}\Delta\mathrm{R}g \owedge g \\ &\qquad - \frac{1}{n-2}(\Delta\mathrm{Ric} \owedge g - (n-2)\rho\nabla^2\mathrm{R} \owedge g - \rho\Delta\mathrm{R}g \owedge g) \\ &= \Delta\mathrm{Riem} + \frac{1-2(n-1)\rho + 2(n-1)\rho}{2(n-1)(n-2)}\Delta\mathrm{R}g \owedge g - \frac{1}{n-2}\Delta\mathrm{Ric} \owedge g \\ &= \Delta\mathrm{W}.\end{aligned}$$

Then we consider the lower order terms

$$\begin{aligned}(\mathcal{L}_0)_{ijkl} &= 2(B(\mathrm{Riem})_{ijkl} - B(\mathrm{Riem})_{ijlk} - B(\mathrm{Riem})_{iljk} + B(\mathrm{Riem})_{ikjl}) \\ &\quad - g^{pq}(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}) \\ &\quad + 2\rho\mathrm{R}\Big(\mathrm{W} - \frac{1}{2(n-1)(n-2)}\mathrm{R}g \owedge g + \frac{1}{n-2}\mathrm{Ric} \owedge g\Big)_{ijkl} \\ &\quad + \frac{1}{2(n-1)(n-2)}(2|\mathrm{Ric}|^2 g \owedge g - 2\rho\mathrm{R}^2 g \owedge g - 4\mathrm{R}\mathrm{Ric} \owedge g + 4\rho\mathrm{R}^2 g \owedge g)_{ijkl} \\ &\quad - \frac{1}{n-2}[2(\mathrm{Riem} * \mathrm{Ric}) \owedge g - 2\mathrm{Ric}^2 \owedge g - 2\mathrm{Ric} \owedge \mathrm{Ric} + 2\rho\mathrm{R}\mathrm{Ric} \owedge g]_{ijkl} \\ &= 2(B(\mathrm{Riem})_{ijkl} - B(\mathrm{Riem})_{ijlk} - B(\mathrm{Riem})_{iljk} + B(\mathrm{Riem})_{ikjl}) \\ &\quad - g^{pq}(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}) + 2\rho\mathrm{R}\mathrm{W}_{ijkl} \\ &\quad - \frac{2}{n-2}[(\mathrm{Riem} * \mathrm{Ric}) \owedge g - \mathrm{Ric}^2 \owedge g - \mathrm{Ric} \owedge \mathrm{Ric}]_{ijkl} \\ &\quad - \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{Ric} \owedge g)_{ijkl} + \frac{|\mathrm{Ric}|^2}{(n-1)(n-2)}(g \owedge g)_{ijkl},\end{aligned}$$

where $(\mathrm{Riem} * \mathrm{Ric})_{ab} = \mathrm{R}_{apbq}\mathrm{R}_{st}g^{ps}g^{qt}$ and $(\mathrm{Ric}^2)_{ab} = \mathrm{R}_{ap}\mathrm{R}_{bq}g^{pq}$.

Now we deal separately with every term containing the full curvature Riem, using its decomposition formula, expanding the Kulkarni–Nomizu products and then contracting again. We have that

$$[(g \owedge g) * \mathrm{Ric}]_{ab} = 2[\mathrm{R}g - \mathrm{Ric}]_{ab},\ [(\mathrm{Ric} \owedge g) * \mathrm{Ric}]_{ab} = [-2\mathrm{Ric}^2 + \mathrm{R}\mathrm{Ric} + |\mathrm{Ric}|^2 g]_{ab}.$$

Hence

$$(4\text{-}5)\quad (\mathrm{Riem} * \mathrm{Ric}) \owedge g = (\mathrm{W} * \mathrm{Ric}) \owedge g - \frac{2}{n-2}\mathrm{Ric}^2 \owedge g + \frac{n\mathrm{R}}{(n-1)(n-2)}\mathrm{Ric} \owedge g + \frac{(n-1)|\mathrm{Ric}|^2 - \mathrm{R}^2}{(n-1)(n-2)} g \owedge g.$$

Then

$$\begin{aligned}\mathrm{R}_{qi}\mathrm{R}_{pjkl}g^{pq} &= \mathrm{R}_{qi}\Big(\mathrm{W}_{pjkl} - \frac{\mathrm{R}}{(n-1)(n-2)}(g_{pk}g_{jl} - g_{pl}g_{jk})\Big)g^{pq} \\ &\qquad + \frac{1}{n-2}\mathrm{R}_{qi}(\mathrm{R}_{pk}g_{jl} + \mathrm{R}_{jl}g_{pk} - \mathrm{R}_{pl}g_{jk} - \mathrm{R}_{jk}g_{pl})g^{pq} \\ &= \mathrm{R}_{qi}\mathrm{W}_{pjkl}g^{pq} - \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk}) \\ &\qquad + \frac{1}{n-2}(\mathrm{R}^2_{ik}g_{jl} - \mathrm{R}^2_{il}g_{jk} + \mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{il}\mathrm{R}_{jk}).\end{aligned}$$

Interchanging the index and using the symmetry properties we get

$$\begin{aligned}(4\text{-}6)\quad & g^{pq}(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}) \\ &= g^{pq}(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql}) \\ &\quad + \frac{2}{n-2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{2}{n-2}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} - \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{Ric} \owedge g)_{ijkl}.\end{aligned}$$

Finally the "B"-terms:

$$\begin{aligned}&B(\mathrm{Riem})_{abcd} \\ &= \Big(\mathrm{W} - \frac{\mathrm{R}}{2(n-1)(n-2)} g \owedge g + \frac{1}{n-2}\mathrm{Ric} \owedge g\Big)_{apbq} \\ &\qquad \Big(\mathrm{W} - \frac{\mathrm{R}}{2(n-1)(n-2)} g \owedge g + \frac{1}{n-2}\mathrm{Ric} \owedge g\Big)_{csdt} g^{ps}g^{qt}\end{aligned}$$

$$(\mathrm{W}_{apbq}(g \owedge g)_{csdt} + (g \owedge g)_{apbq}\mathrm{W}_{csdt})g^{ps}g^{qt} = -2\mathrm{W}_{adbc} - 2\mathrm{W}_{cbda}$$

$$\begin{aligned}&(\mathrm{W}_{apbq}(\mathrm{Ric} \owedge g)_{csdt} + (\mathrm{Ric} \owedge g)_{apbq}\mathrm{W}_{csdt})g^{ps}g^{qt} \\ &= (\mathrm{W} * \mathrm{Ric})_{ab}g_{cd} + (\mathrm{W} * \mathrm{Ric})_{cd}g_{ab} \\ &\qquad -(\mathrm{W}_{cbdp}\mathrm{R}_{aq} + \mathrm{W}_{cpda}\mathrm{R}_{bq} + \mathrm{W}_{adbp}\mathrm{R}_{cq} + \mathrm{W}_{apbd}\mathrm{R}_{dq})g^{pq}\end{aligned}$$

$$\begin{aligned}&(g \owedge g)_{apbd}(g \owedge g)_{csdt}g^{ps}g^{qt} \\ &= 4((n-2)g_{ab}g_{cd} + g_{ac}g_{bd})((\mathrm{Ric} \owedge g)_{apbq}(g \owedge g)_{csdt} + (\mathrm{Ric} \owedge g)_{csdt}(g \owedge g)_{apbq})g^{ps}g^{qt} \\ &= 2((n-4)\mathrm{R}_{ab}g_{cd} + (n-4)\mathrm{R}_{cd}g_{ab} + 2\mathrm{R}_{ac}g_{bd} + 2\mathrm{R}_{bd}g_{ac}) \\ &\qquad (\mathrm{Ric} \owedge g)_{abpq}(\mathrm{Ric} \owedge g)_{csdt}g^{ps}g^{qt} \\ &= -2\mathrm{R}^2_{ab}g_{cd} - 2\mathrm{R}^2_{cd}g_{ab} + \mathrm{R}^2_{ac}g_{bd} + \mathrm{R}^2_{bd}g_{ac} \\ &\qquad + (n-4)\mathrm{R}_{ab}\mathrm{R}_{cd} + 2\mathrm{R}_{ac}\mathrm{R}_{bd} + \mathrm{R}(\mathrm{R}_{ab}g_{cd} + \mathrm{R}_{cd}g_{ab}) + |\mathrm{Ric}|^2 g_{ab}g_{cd}\end{aligned}$$

Now, adding the same type quantities for the different index permutations and using the symmetry properties of W we obtain

$$
\begin{aligned}
(4\text{-}7)\quad & B(\mathrm{Riem})_{ijkl} - B(\mathrm{Riem})_{ijlk} - B(\mathrm{Riem})_{iljk} + B(\mathrm{Riem})_{ikjl} \\
&= B(\mathrm{W})_{ijkl} - B(\mathrm{W})_{ijlk} - B(\mathrm{W})_{iljk} + B(\mathrm{W})_{ikjl} \\
&+ \frac{1}{n-2}((\mathrm{W} * \mathrm{Ric}) \owedge g)_{ijkl} - \frac{1}{(n-2)^2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{1}{2(n-2)}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} \\
&+ \frac{\mathrm{R}}{(n-1)(n-2)^2}(\mathrm{Ric} \owedge g)_{ijkl} + \Big(\frac{|\mathrm{Ric}|^2}{2(n-2)^2} - \frac{\mathrm{R}^2}{2(n-1)(n-2)^2}\Big)(g \owedge g)_{ijkl}.
\end{aligned}
$$

We are ready to complete the computation of the 0-th order term in the evolution equation, using the previous formulas (4-5), (4-6), (4-7):

$$
\begin{aligned}
(\mathcal{L}_0)_{ijkl} = 2(B(\mathrm{W})_{ijkl} - B(\mathrm{W})_{ijlk} - B(\mathrm{W})_{iljk} + B(\mathrm{W})_{ikjl}) + 2\rho \mathrm{R}\mathrm{W}_{ijkl} \\
- g^{pq}(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql}) \\
+ \frac{2}{(n-2)^2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{1}{(n-2)}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} \\
- \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{Ric} \owedge g)_{ijkl} + \frac{\mathrm{R}^2 - |\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g \owedge g)_{ijkl} \quad \square
\end{aligned}
$$

4D. ***Conditions preserved in dimension three.*** In general dimension, it is very hard to find other curvature conditions preserved by the flow, and this is due principally to the complex structure of the reaction terms; for example in the evolution equation satisfied by the Ricci tensor (3-3), the reaction terms involve the full curvature tensor. Therefore it is easier to restrict our attention to the three dimensional case, in which the Weyl part of the Riemann tensor vanishes and all the geometric information is encoded in the Ricci tensor.

In the special case of dimension three, we can also use the evolution equation (3-10) of the pullback of the curvature operator to obtain more refined conditions preserved, because we can rewrite the ODE associated to the evolution of $\mathcal{P}$ as a system of ODEs in the eigenvalues of $\mathcal{P}$ that, by Proposition 3.4, are nothing but the sectional curvatures of $\mathcal{R}$. This point of view was introduced for the Ricci flow by Hamilton [1997] and can be easily generalized to the RB flow as follows:

Lemma 4.12. *If $n = 3$, then $\mathcal{P}_p$ has 3 eigenvalues, λ, μ, ν, and the ODE fiberwise associated to equation (3-10) can be written as the following system*:

$$
(4\text{-}8)\qquad
\begin{cases}
\frac{d\lambda}{dt} &= 2\lambda^2 + 2\mu\nu - 4\rho\lambda(\lambda+\mu+\nu),\\
\frac{d\mu}{dt} &= 2\mu^2 + 2\lambda\nu - 4\rho\mu(\lambda+\mu+\nu),\\
\frac{d\nu}{dt} &= 2\nu^2 + 2\lambda\mu - 4\rho\nu(\lambda+\mu+\nu).
\end{cases}
$$

In particular, if we assume $\lambda(0) \geq \mu(0) \geq \nu(0)$, then $\lambda(t) \geq \mu(t) \geq \nu(t)$ as long as the solution of the system exists.

Proof. We can pointwise identify V_p with an orthonormal frame of $\mathbb{R}^3$ with the standard basis. Then $\Lambda^2 V_p \simeq \mathfrak{so}(3)$ with the standard structure constants and if an algebraic operator Q_p is diagonal, both Q_p^2 and $Q_p^\#$ are diagonal with respect to the same basis (for the detailed computation of this fact, see [Chow and Knopf 2004, Chapter 6.4]). Hence the ODE $(d/dt)Q_p = F_p(Q_p)$, associated fiberwise to equation (3-10), preserves the eigenvalues of Q_p, that is, if $Q_p(0)$ is diagonal with respect to an orthonormal basis, $Q_p(t)$ stays diagonal with respect to the same basis and the ODE can be rewritten as the system (4-8) in the eigenvalues.

To prove the last statement, we observe that

$$\frac{d}{dt}(\lambda-\mu) = 2(\lambda-\mu)((1-2\rho)(\lambda+\mu)-(1+2\rho)\nu),$$
$$\frac{d}{dt}(\mu-\nu) = 2(\mu-\nu)((1-2\rho)(\mu+\nu)-(1+2\rho)\lambda). \qquad \square$$

Remark 4.13. We already proved that the differential operator in the evolution equation of $\mathcal{P}$ is uniformly elliptic if $\rho < 1/2(n-1)$, that is, $\rho < \frac{1}{4}$ in dimension three. Therefore any geometric condition expressed in terms of the eigenvalues is preserved along the RB flow if the cone identified by the condition is closed, convex and preserved by the system (4-8).

By using this method, we can prove:

Proposition 4.14. *Let $(M, g(t))_{t\in[0,T)}$ be a compact, three dimensional solution of the RB flow (1-1). If $\rho < \frac{1}{4}$, then*

(i) *nonnegative Ricci curvature is preserved along the flow*;

(ii) *nonnegative sectional curvature is preserved along the flow*;

(iii) *the pinching inequality* $\mathrm{Ric} \geq \varepsilon \mathrm{R} g$ *is preserved along the flow for any* $\varepsilon \leq \frac{1}{3}$.

Proof.

(i) If $\mathrm{Ric}(g(0)) \geq 0$, then $\mathrm{Ric}_{g(t)} \geq 0$. The eigenvalues of Ric are the pairwise sums of the sectional curvatures. Hence the condition is identified by the cone

$$K_p = \{Q_p : (\mu+\nu)(Q_p) \geq 0\}.$$

The closedness is obvious; in order to see that K_p is convex, we observe that the greatest eigenvalue can be characterized by $\lambda(Q_p) = \max\{Q_p(v, v) : v \in V_p |v|_h = 1\}$. Hence K_p is convex. Then the function $Q_p \mapsto \mu(Q_p)+\nu(Q_p) = \mathrm{tr}(Q_p)-\lambda(Q_p)$ is concave and this implies that its superlevels are convex. By system (4-8) we obtain

$$\frac{d}{dt}(\mu+\nu) = 2\mu^2+2\nu^2+2\lambda(\mu+\nu)-4\rho(\mu+\nu)\,\mathrm{tr}(Q_p).$$

There is the stationary solution corresponding to $\mu(0) = 0 = \nu(0)$. Otherwise, whenever $\mu(t_0)+\nu(t_0) = 0$ with $\mu(t_0) \neq 0$ and $\nu(t_0) \neq 0$, $(d/dt)(\mu+\nu)(t_0) = 2(\mu^2+\nu^2)(t_0) > 0$, then K is preserved.

(ii) If $\mathrm{Sec}(g(0)) \geq 0$, then $\mathrm{Sec}_{g(t)} \geq 0$. This condition is the nonnegativity of $\mathcal{P}$, already proved in general dimension in Proposition 4.10, identified by the cone $K_p = \{Q_p : \nu(Q_p) \geq 0\}$, which is convex as a superlevel of a concave function. We suppose that $\nu(t_0) = 0$. Then

$$\frac{d}{dt}\nu(t_0) = 2\lambda(t_0)\mu(t_0) \geq 0,$$

since the order between the eigenvalues is preserved and therefore $\lambda(t_0) \geq \mu(t_0) \geq 0$.

(iii) For every $\varepsilon \in \left(0, \frac{1}{3}\right]$, if $\mathrm{Ric}(g(0)) - \varepsilon \mathrm{R}(g(0))g(0) \geq 0$, then $\mathrm{Ric}_{g(t)} - \varepsilon \mathrm{R}_{g(t)} g(t) \geq 0$. Translating in terms of eigenvalues of $\mathcal{P}$, the condition means $\mu(Q_p) + \nu(Q_p) - 2\varepsilon \operatorname{tr}(Q_p) \geq 0$; that is, $\lambda(Q_p) \leq (1-2\varepsilon)/(2\varepsilon)(\mu(Q_p) + \nu(Q_p))$. Then the right cone is

$$K_p = \{Q_p : \lambda(Q_p) - C(\varepsilon)(\mu(Q_p) + \nu(Q_p)) \leq 0\},$$

where $C(\varepsilon) = (1-2\varepsilon)/(2\varepsilon) \in \left[\frac{1}{2}, +\infty\right)$. The defining function is the sum of two convex functions, hence its sublevels are convex. Now, for $C = \frac{1}{2}$, that corresponds to $\varepsilon = \frac{1}{3}$, and we have $\lambda(0) = \mu(0) = \nu(0)$ at each point of M; that is, the initial metric $g(0)$ has constant sectional curvature and this condition is preserved along the flow.

For $C > \frac{1}{2}$, we suppose $\lambda(t_0) = C(\mu(t_0) + \nu(t_0))$, then

$$\begin{aligned}
&\frac{d}{dt}(\lambda - C(\mu+\nu))(t_0) \\
&= 2[\lambda^2 + \mu\nu - C(\mu^2 + \nu^2 + \lambda(\mu+\nu)) - 2\rho \operatorname{tr}(Q_p)(\lambda - C(\mu+\nu))](t_0) \\
&= 2[C^2(\mu(t_0)+\nu(t_0))^2 + \mu(t_0)\nu(t_0) - C(\mu(t_0)^2 + \nu(t_0)^2) - C^2(\mu(t_0)+\nu(t_0))^2] \\
&\leq (1-2C)(\mu(t_0)^2 + \nu(t_0)^2) \leq 0,
\end{aligned}$$

which completes the proof. □

4E. *Hamilton–Ivey estimate.* A remarkable property of the three dimensional Ricci flow is the pinching estimate, independently proved by Hamilton [1995] and Ivey [1993], which says that positive sectional curvature dominates negative sectional curvature during the Ricci flow, that is, if the initial metric g_0 has a negative sectional curvature somewhere, the Ricci flow starting at g_0 evolves the scalar curvature towards the positive semiaxis in future times, which means that there will be a greater (in absolute value) positive sectional curvature.

We have generalized the pinching estimate and some consequences for positive values of the parameter ρ. In the same notation used before, let $\lambda \geq \mu \geq \nu$ be the ordered eigenvalues of the curvature operator.

Theorem 4.15 (Hamilton–Ivey estimate). *Let $(M, g(t))$ be a solution of the RB on a compact three-manifold such that the initial metric satisfies the normalizing assumption* $\min_{p\in M} \nu_p(0) \geq -1$. *If $\rho \in \left[0, \frac{1}{6}\right)$, then at any point (p,t) where $\nu_p(t) < 0$, the scalar curvature satisfies*

$$\mathrm{R} \geq |\nu|(\log|\nu| + \log(1+2(1-6\rho)t) - 3). \tag{4-9}$$

Proof. We would like to apply the maximum principle for time-dependent sets in Theorem 4.9. Hence we need to express condition (4-9) in terms of a family of closed, convex, invariant subsets of $S^2(\Lambda^2 V^*)$, where $(V, h(t), D(t))$ is the usual bundle isomorphism of the tangent bundle defined via Uhlenbeck's trick (Section 3B). Moreover, by [Chow et al. 2008, Lemma 10.11], we already know that, for any $t \in [0, T)$, the set

$$K_p(t) = \left\{ \begin{array}{l} Q_p : \mathrm{tr}(Q_p) \geq -\frac{3}{1+2(1-6\rho)t} \quad \text{and if} \quad \nu(Q_p) \leq -\frac{1}{1+2(1-6\rho)t}, \\ \text{then } \mathrm{tr}(Q_p) \geq |\nu(Q_p)|(\log|\nu(Q_p)| + \log(1+2(1-6\rho)t) - 3) \end{array} \right\}$$

defines a closed invariant subset of $S^2(\Lambda^2 V^*)$. Since, for $\rho \in \left[0, \frac{1}{6}\right)$, $K(t)$ depends continuously on time, the spacetime track of $K(t)$ is closed in $S^2(\Lambda^2 V^*)$.

Now we show that $K_p(t)$ is convex for every $p \in M$ and $t \in [0, T)$. Following [Chow and Knopf 2004, Lemma 9.5], we consider the map

$$\Phi : S^2(\Lambda^2 V_p^*) \to \mathbb{R}^2, \qquad \Phi(Q_p) = (|\nu(Q_p)|, \mathrm{tr}(Q_p))$$

Clearly, we have that $Q_p \in K_p(t)$ if and only if $\Phi(Q_p) \in A(t)$, where

$$A(t) = \left\{ \begin{array}{l} (x, y) \in \mathbb{R}^2 : y \geq -\frac{3}{1+2(1-6\rho)t}; \quad y \geq -3x; \\ \text{if } x \geq \frac{1}{1+2(1-6\rho)t}, \text{ then } y \geq x(\log x + \log(1+2(1-6\rho)t) - 3) \end{array} \right\}$$

is a convex subset of $\mathbb{R}^2$. Then in order to show that $K_p(t)$ is convex it is sufficient to show that the segment between any two algebraic operators in $K_p(t)$ is sent by the map Φ into $A(t)$. Therefore let $Q_p, Q'_p \in K_p(t)$, $s \in [0, 1]$ and $Q_p(s) = sQ_p + (1-s)Q'_p$. About the first defining condition for $A(t)$, the trace is a linear functional, hence it is obviously fulfilled by $Q_p(s)$, while the second condition is satisfied by any algebraic operator.

The third condition is a bit tricky. If $\nu(Q_p), \nu(Q'_p) > -1/(1+(1-6\rho)t)$, then the condition is empty for every point of the segment because ν is a concave function. By continuity we can assume that $\nu(Q_p(s)) \leq -1/(1+(1-6\rho)t)$ without loss of generality for every $s \in [0, 1]$, and hence $x(Q_p(s)) = -\nu(Q_p(s))$ is a convex function and $x(Q_p(s)) \leq sx(Q_p) + (1-s)x(Q'_p)$. On the other hand the second condition implies that $x(Q_p(s)) \geq -y(Q_p(s))/3 = -\frac{1}{3}(sy(Q_p) + (1-s)y(Q'_p))$. Then $\Phi(Q_p(s))$ belongs to the trapezium of vertices

$$\Phi(Q_p), \left(-\tfrac{1}{3}y(Q_p), y(Q_p)\right), \Phi(Q'_p), \left(-\tfrac{1}{3}y(Q'_p), y(Q'_p)\right),$$

which is contained in $A(t)$, as its vertices are, and $A(t)$ is convex.

Now we prove that $K(t)$ is preserved by the system (4-8). By taking the sum of the three equations in the system (see also Remark 4.13) we get

$$\frac{d}{dt}\operatorname{tr}(Q_p) \geq \frac{4}{3}(1-3\rho)\operatorname{tr}(Q_p)^2.$$

By hypothesis, $\nu(Q_p)(0) \geq -1$, hence $\operatorname{tr}(Q_p)(0) \geq -3$ for every $p \in M$ and by integrating the previous inequality,

$$\operatorname{tr}(Q_p)(t) \geq -\frac{3}{1+4(1-3\rho)t} \geq -\frac{3}{1+2(1-6\rho)t},$$

which holds for any $\rho \in \left[0, \frac{1}{6}\right)$.

In order to prove that the second inequality is preserved, too, we consider, for every $p \in M$ such that $\nu(Q_p)(0) < 0$, the function

$$f(t) = \frac{\operatorname{tr}(Q_p)}{-\nu(Q_p)} - \log(-\nu(Q_p)) - \log(1+2(1-6\rho)t), \tag{4-10}$$

and we compute its derivative along the flow:

$$\begin{aligned}\frac{d}{dt}f =& \frac{1}{\nu^2}[(-2\nu)(\lambda^2+\mu^2+\nu^2+\lambda\mu+\lambda\nu+\mu\nu-2\rho(\lambda+\mu+\nu)^2)\\ &\qquad +2(\lambda+\mu+\nu)(\nu^2+\lambda\mu-2\rho\nu(\lambda+\mu+\nu))]\\ &\qquad\qquad -\frac{2}{\nu}(\nu^2+\lambda\mu-2\rho\nu(\lambda+\mu+\nu)) - \frac{2(1-6\rho)}{1+2(1-6\rho)t}\\ =&\frac{2}{\nu^2}[-\nu(\lambda^2+\mu^2+\lambda\mu)+\lambda\mu(\lambda+\mu)-\nu^3+2\rho\nu^2(\lambda+\mu+\nu)] - \frac{2(1-6\rho)}{1+2(1-6\rho)t}.\end{aligned}$$

As in the case of the Ricci flow, it is easy to see that the quantity $-\nu(\lambda^2+\mu^2+\lambda\mu)+\lambda\mu(\lambda+\mu)$ is always nonnegative if $\nu < 0$. In fact, if $\mu > 0$ it is obvious, whereas if $\mu \leq 0$ one has

$$-\nu(\lambda^2+\mu^2+\lambda\mu)+\lambda\mu(\lambda+\mu) = (\mu-\nu)(\lambda^2+\mu^2+\lambda\mu)-\mu^3 \geq 0.$$

Hence we get

$$\frac{d}{dt}f(t) \geq -2\nu + 4\rho(\lambda+\mu+\nu) - \frac{2(1-6\rho)}{1+2(1-6\rho)t}. \tag{4-11}$$

If $\rho \geq 0$, since $\lambda+\mu+\nu \geq 3\nu$, we obtain

$$\frac{d}{dt}f \geq -2(1-6\rho)\left(\nu + \frac{1}{1+2(1-6\rho)t}\right) \geq 0$$

whenever $\nu \leq -1/(1+2(1-6\rho)t)$ and $\rho \leq \frac{1}{6}$.

Hence, if (λ, μ, ν) is a solution of system (4-8) in $[0, T)$ with $(\lambda(0), \mu(0), \nu(0)) \in K_p(0)$, we suppose that there is $t_1 > 0$ such that $\nu(t_1) < -1/(1+2(1-6\rho)t_1)$. Then either $\nu(t) < -1/(1+2(1-6\rho)t)$ for any $t \in [0, t_1]$, or there exists $t_0 < t_1$ such that $\nu(t_0) = -1/(1+2(1-6\rho)t_0)$ and $\nu(t) < -1/(1+2(1-6\rho)t)$ for any $t \in (t_0, t_1]$. In the first case, by hypothesis we obtain $f(0) \geq -3$ and $(d/dt) f(t) \geq 0$ for any $t \in [0, t_1]$, therefore $f(t_1) \geq -3$; in the second case $f(t_0) = (\lambda + \mu + \nu)(t_0)/-\nu(t_0) \geq -3$ and $(d/dt) f(t) \geq 0$ for any $t \in [t_0, t_1]$, therefore again $f(t_1) \geq -3$, which is equivalent to the second inequality. □

Remark 4.16. The extra term $4\rho(\lambda + \mu + \nu)$ on the key equation (4-11) requires strong assumptions on the parameter ρ since we have no information on the sign of the trace. However, combining equation (4-11) with Proposition 4.4, we can enlarge the range of ρ to $\left[0, \frac{1}{4}\right)$, simply by dropping the extra term, nonnegative for ancient solutions and therefore conclude that an ancient solution to the RB flow on a compact three-manifold with bounded scalar curvature has nonnegative sectional curvature for any value of $\rho \in \left[0, \frac{1}{4}\right)$ (see [Chow and Knopf 2004, Corollary 9.8]).

Proposition 4.17. *Let $(M, g(t))_{t \in (-\infty, 0]}$ be a compact, three dimensional, ancient solution of the RB flow (1-1) with uniformly bounded scalar curvature. If $\rho \in \left[0, \frac{1}{4}\right)$ then the sectional curvature is nonnegative.*

5. Curvature estimates

5A. *Technical lemmas.* Before proving the curvature estimates for the RB flow, we need some technical results, the first being the following proposition:

Proposition 5.1. *Let $k \in \mathbb{N}$, $p \in [1, +\infty]$ and $q \in [1, +\infty)$. There exists a constant $C(n, k, p, q)$ such that for all $0 \leq j \leq k$ and all tensors T*

$$\|\nabla^j T\|_{r_j} \leq C \|T\|_p^{1-j/k} \|\nabla^k T\|_q^{j/k},$$

where $1/r_j = (1 - j/k)/p + j/k/q$.

To prove this proposition, we need several lemmas.

Lemma 5.2. *Let $p \in [1, +\infty]$, $q \in [1, +\infty)$ and $r \in [2, +\infty)$ such that $2/r = 1/p + 1/q$. There exists a constant $C(n, r)$ such that for all tensors T,*

$$\|\nabla T\|_r^2 \leq C \|T\|_p \|\nabla^2 T\|_q.$$

Proof.

$$\begin{aligned}\|\nabla T\|_r^r &= \int_M \langle \nabla T, |\nabla T|^{r-2}\nabla T\rangle \, d\mu_g \\ &= -\int_M \langle T, \nabla\left(|\nabla T|^{r-2}\nabla T\right)\rangle \, d\mu_g \\ &= -\int_M \langle T, (r-2)\nabla^2 T|\nabla T|^{r-3}\nabla T\rangle \, d\mu_g - \int_M \langle T, |\nabla T|^{r-2}\nabla^2 T\rangle \, d\mu_g \\ &\le C\int_M |T||\nabla^2 T||\nabla T|^{r-2} \, d\mu_g \\ &\le C\|T\|_p \|\nabla^2 T\|_q \|\nabla T\|_r^{r-2},\end{aligned}$$

using Hölder's inequality with $(r-2)/r + 1/p + 1/q = 1$. This ends the proof of this lemma. □

Lemma 5.3 [Hamilton 1982, Corollary 12.5]. *Let $k \in \mathbb{N}$. If $f : \{0, \dots, k\} \to \mathbb{R}$ satisfies for all $0 < j < k$*

$$f(j) \le C f(j-1)^{\frac{1}{2}} f(j+1)^{\frac{1}{2}},$$

where C is a positive constant, then for all $0 \le j \le k$,

$$f(j) \le C^{j(k-j)} f(0)^{1-j/k} f(k)^{j/k}.$$

Proof of Proposition 5.1. We apply Lemma 5.3 with $f(j) = \|\nabla^j T\|_{r_j}$. Since $2/r_j = 1/r_{j-1} + 1/r_{j+1}$, Lemma 5.2 shows that there exists $C(n, k, p, q)$ such that

$$f(j) \le C f(j-1)^{\frac{1}{2}} f(j+1)^{\frac{1}{2}},$$

and then Lemma 5.3 gives Proposition 5.1, since $r_0 = p$ a $r_k = q$. □

Lemma 5.4. *For all tensors of the form $S * T$, there exists C depending on the dimension and the coefficients in the expression such that*

$$|S * T| \le C|S||T|.$$

Proof. By the Cauchy–Schwarz inequality, $(\mathrm{tr}_g T)^2 = (g^{\alpha\beta} T_{\alpha\beta})^2 \le n T_{\alpha\beta} T^{\alpha\beta} = n|T|^2$. Then

$$|S * T| \le C(n)|S \otimes T \otimes g^{\otimes j} \otimes (g^{-1})^{\otimes k} \le C(n) n^{\frac{j+k}{2}} |S||T|. \qquad \square$$

Let $k \in \mathbb{N}$, and set, for a tensor T, $F_g(T) = \sum_{j+l=k;\, j,l \ge 0} \nabla^j T * \nabla^l T * \nabla^k T$.

Lemma 5.5. *Let $k \in \mathbb{N}$. Let $p \in [2, +\infty]$ and $q \in [2, +\infty)$ such that $1/p + 2/q = 1$. There exists $C(n, k, p, q, F)$ such that for all tensors T,*

$$\int_M |F_g(T)| \, d\mu_g \le C\|T\|_p \|\nabla^k T\|_q^2.$$

Proof. Let us consider one term in $F_g(T)$ that can be written $\nabla^j T * \nabla^l T * \nabla^k T$, $j, l \geq 0$. We set

$$\frac{1}{r_j} = \frac{1 - \frac{j}{k}}{p} + \frac{\frac{j}{k}}{q} \quad \text{and} \quad \frac{1}{r_l} = \frac{1 - \frac{l}{k}}{p} + \frac{\frac{l}{k}}{q}.$$

Since $1/r_j + 1/r_j + 1/q = 1$, by Lemma 5.4 and Hölder's inequality we have

$$\begin{aligned}\int_M |\nabla^j T * \nabla^l T * \nabla^k T| \, d\mu_g &\leq C' \int_M |\nabla^j T||\nabla^l T||\nabla^k T| \, d\mu_g \\ &\leq C' \|\nabla^j T\|_{r_j} \|\nabla^l T\|_{r_l} \|\nabla^k T\|_q,\end{aligned}$$

Then, by applying Proposition 5.1 to the first two factors, we get

$$\int_M |\nabla^j T * \nabla^l T * \nabla^k T| \, d\mu_g \leq C \|T\|_p \|\nabla^k T\|_q^2.$$

The result follows since $F_g(T)$ is a linear combination of such terms. □

5B. *Curvature estimates.*

Theorem 5.6. *Assume $\rho < 1/(2(n-1))$. If $g(t)$ is a compact solution of the RB flow for $t \in [0, T)$ such that*

$$\sup_{(x,t) \in M \times [0,T)} |\mathrm{Riem}(x,t)| \leq K,$$

then for all $k \in \mathbb{N}$ there exists a constant $C(n, \rho, k, K, T)$ such that for all $t \in (0, T]$

$$\|\nabla^k \mathrm{Riem}_{g(t)}\|_2^2 \leq \frac{C}{t^k} \sup_{t \in [0,T)} \|\mathrm{Riem}_{g(t)}\|_2^2.$$

Proof. A direct computation gives

$$\begin{aligned}\frac{\partial}{\partial t}|\mathrm{Riem}|^2 &= \Delta(|\mathrm{Riem}|^2) - 2|\nabla \mathrm{Riem}|^2 - 8\rho \mathrm{R}_{ij} \nabla^i \nabla^j \mathrm{R} + \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \\ \frac{\partial}{\partial t}\mathrm{R}^2 &= (1 - 2(n-1)\rho)\Delta(\mathrm{R}^2) - 2(1 - 2(n-1)\rho)|\nabla \mathrm{R}|^2 + 4\mathrm{R}|\mathrm{Ric}|^2 - 4\rho \mathrm{R}^3.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\int_M |\mathrm{Riem}|^2 \, d\mu_g &= -2\int_M |\nabla \mathrm{Riem}|^2 \, d\mu_g - 8\rho \int_M \mathrm{R}_{ij} \nabla^i \nabla^j \mathrm{R} \, d\mu_g \\ &\quad + \int_M \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \, d\mu_g \\ \frac{d}{dt}\int_M \mathrm{R}^2 d\mu_g &= -2(1 - 2(n-1)\rho)\int_M |\nabla \mathrm{R}|^2 d\mu_g + \int_M \mathrm{Riem} * \mathrm{Riem} * \mathrm{Riem} \, d\mu_g.\end{aligned}$$

Now we want to compute $\int_M \mathrm{R}_{ij}\nabla^i\nabla^j\mathrm{R}\,d\mu_g$. Using the Bianchi identity we have

$$\int_M \mathrm{R}_{ij}\nabla^i\nabla^j\mathrm{R}\,d\mu_g = -\frac{1}{2}\int_M |\nabla\mathrm{R}|^2\,d\mu_g\,.$$

We conclude that

$$\begin{aligned}\frac{d}{dt}\int_M |\mathrm{Riem}|^2\,d\mu_g &= -2\int_M |\nabla\mathrm{Riem}|^2\,d\mu_g + 4\rho\int_M |\nabla\mathrm{R}|^2\,d\mu_g\\ &\quad + \int_M \mathrm{Riem}*\mathrm{Riem}*\mathrm{Riem}\,d\mu_g\end{aligned}$$

and

$$\frac{d}{dt}\int_M \mathrm{R}^2\,d\mu_g = -2(1-2(n-1)\rho)\int_M |\nabla\mathrm{R}|^2\,d\mu_g + \int_M \mathrm{Riem}*\mathrm{Riem}*\mathrm{Riem}\,d\mu_g.$$

As we did before, a straightforward computation gives:

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla^k\mathrm{Riem}|^2\,d\mu_g &= -2\int_M |\nabla^{k+1}\mathrm{Riem}|^2\,d\mu_g + 4\rho\int_M |\nabla^{k+1}\mathrm{R}|^2\,d\mu_g\\ &\quad + \sum_{j+l=k;\,j,l\geq 0}\int_M \nabla^j\mathrm{Riem}*\nabla^l\mathrm{Riem}*\nabla^k\mathrm{Riem}\,d\mu_g\\ \frac{d}{dt}\int_M |\nabla^k\mathrm{R}|^2\,d\mu_g &= -2(1-2(n-1)\rho)\int_M |\nabla^{k+1}\mathrm{R}|^2\,d\mu_g\\ &\quad + \sum_{j+l=k;\,j,l\geq 0}\int_M \nabla^j\mathrm{Riem}*\nabla^l\mathrm{Riem}*\nabla^k\mathrm{Riem}\,d\mu_g.\end{aligned}$$

Consider

$$\mathcal{A}_k := \int_M |\nabla^k\mathrm{Riem}|^2\,d\mu_g + \frac{4|\rho|}{(1-2(n-1)\rho)}\int_M |\nabla^k\mathrm{R}|^2\,d\mu_g,$$

and set $f_k(t) := \sum_{j=0}^{k}(\beta^j t^j/j!)\mathcal{A}_j$, where $\beta := \min(1,\, 1-2(n-1)\rho)$. We have

$$f_k'(t) = \sum_{j=0}^{k-1}\frac{\beta^j t^j}{j!}(\mathcal{A}_j' + \beta\mathcal{A}_{j+1}) + \frac{\beta^k t^k}{k!}\mathcal{A}_k'. \tag{5-1}$$

We have by a direct computation, for any j:

$$\begin{aligned}\mathcal{A}_j' + \beta\mathcal{A}_{j+1} &= (-2+\beta)\|\nabla^{j+1}\mathrm{Riem}\|_2^2 + \Big(4\rho - 8|\rho| + \frac{4\beta|\rho|}{1-2(n-1)\rho}\Big)\|\nabla^{j+1}\mathrm{R}\|_2^2\\ &\quad + \sum_{i+l=j,i,l\geq 0}\int_M \nabla^i\mathrm{Riem}*\nabla^l\mathrm{Riem}*\nabla^j\mathrm{Riem}\,d\mu_g.\end{aligned}$$

We need to estimate

$$\sum_{i+l=j,i,l\geq 0}\int_M \nabla^i \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^j \mathrm{Riem}\, d\mu_g.$$

For this we use Lemma 5.5 with $p=+\infty$ and $q=2$:

$$\sum_{i+l=j,i,l\geq 0}\int_M \nabla^i \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^j \mathrm{Riem}\, d\mu_g \leq C\|\mathrm{Riem}\|_\infty \|\nabla^j \mathrm{Riem}\|_2^2.$$

Using Proposition 5.1, with $k=j+1$ we get

$$\sum_{i+l=k}\int_M \nabla^i \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^j \mathrm{Riem}\, d\mu_g \leq C\|\mathrm{Riem}\|_\infty (\|\mathrm{Riem}\|_2^2)^{\frac{1}{j+1}} (\|\nabla^{j+1}\mathrm{Riem}\|_2^2)^{\frac{j}{j+1}},$$

where $i,l\geq 0$. Now we apply Young's inequality $ab \leq a^p/p + b^q/q$, where

$$a = C\|\mathrm{Riem}\|_\infty (\|\mathrm{Riem}\|_2^2)^{\frac{1}{j+1}}, \quad b = (\|\nabla^{j+1}\mathrm{Riem}\|_2^2)^{\frac{j}{j+1}}$$

and $p=j+1$, $q=(j+1)/j$. We use the hypothesis on the boundedness of $\|\mathrm{Riem}\|_\infty$ and we obtain

$$\sum_{i+l=j}\int_M \nabla^i \mathrm{Riem} * \nabla^l \mathrm{Riem} * \nabla^j \mathrm{Riem}\, d\mu_g \leq C'(n,\rho,j,K)\|\mathrm{Riem}\|_2^2 + \|\nabla^{j+1}\mathrm{Riem}\|_2^2,$$

where $i,l\geq 0$. Putting this last inequality in the previous computation, we obtain

$$\begin{aligned}\mathcal{A}_j' + \beta\mathcal{A}_{j+1} &\leq (-1+\beta)\|\nabla^{j+1}\mathrm{Riem}\|_2^2 + \Big(4\rho - 8|\rho| + \frac{4\beta|\rho|}{1-2(n-1)\rho}\Big)\|\nabla^{j+1}\mathrm{R}\|_2^2 \\ &\qquad\qquad + C'(n,\rho,j,K)\|\mathrm{Riem}\|_2^2 \\ &\leq C'(n,\rho,j,K)\|\mathrm{Riem}\|_2^2,\end{aligned}$$

where we use the facts that $-1+\beta\leq 0$ and $4\rho - 8|\rho| + 4|\rho|\beta/(1-2(n-1)\rho)\leq 0$. The same estimates holds for the last term in equation (5-1), since

$$\mathcal{A}_k' \leq \mathcal{A}_k' + \beta\mathcal{A}_{k+1} \leq C'(n,\rho,k,K)\|\mathrm{Riem}\|_2^2$$

Therefore

$$\begin{aligned}f_k'(t) &\leq \sum_{j=0}^k \frac{\beta^j t^j}{j!} C'(n,\rho,j,K)\|\mathrm{Riem}\|_2^2 \\ &\leq \overline{C}(n,\rho,k,K)\|\mathrm{Riem}\|_2^2 (e^{\beta t}-1) \leq \widetilde{C}(n,\rho,k,K,T)\|\mathrm{Riem}\|_2^2.\end{aligned}$$

Since $f_k(0) = \mathcal{A}_0 \le C(\rho, n)\|\mathrm{Riem}\|_2^2$, by integrating the previous inequality we finally get

$$\|\nabla^k \mathrm{Riem}\|_2^2 \le \mathcal{A}_k \le \frac{k!}{\beta^k t^k} f_k(t) \le \frac{\widehat{C}}{t^k}\big[f_k(0) + \widetilde{C} t \|\mathrm{Riem}\|_2^2\big]$$
$$\le \frac{\widehat{C}[C(\rho, n) + \widetilde{C} t]}{t^k} \|\mathrm{Riem}\|_2^2 \le \frac{C}{t^k} \|\mathrm{Riem}\|_2^2,$$

which concludes the proof of the theorem. □

5C. ***Long time existence.*** In this section we will prove the following result.

Theorem 5.7. *Assume $\rho < 1/(2(n-1))$. If $g(t)$ is a compact solution of the RB flow on a maximal time interval $[0, T)$, $T < +\infty$, then*

$$\limsup_{t \to T} \max_M |\mathrm{Riem}(\,\cdot\,, t)| = +\infty.$$

Proof. This proof follows exactly the one given by Hamilton for the Ricci flow (see [Hamilton 1982, Section 14]). First of all we observe that, if the Riemann tensor is uniformly bounded as $t \to T$ and $T < +\infty$, then also its L^2-norm is uniformly bounded, because from the previous computations, for $\mathcal{A}_0 = \|\mathrm{Riem}\|_2^2 + 4|\rho|/(1 - 2(n-1)\rho)\|\mathrm{R}\|_2^2$, so we have $\mathcal{A}_0' \le C\mathcal{A}_0$.

Then, by Theorem 5.6, we get, for any $j \in \mathbb{N}$

$$\|\nabla^j \mathrm{Riem}\|_2^2 \le C_j.$$

Now, by using the interpolation inequalities in Proposition 5.1 with $p = \infty$, $q = 2$, we immediately get the estimates

$$\|\nabla^j \mathrm{Riem}\|_{\frac{2k}{j}} \le C_{j,k},$$

for all $j \in \mathbb{N}$ and $k \ge j$. Therefore, by interpolation the same result holds for a generic exponent r, with a constant that depends on j and r.

Now, let $E_j := |\nabla^j \mathrm{Riem}|^2$. Then, for all $r < +\infty$ we have

$$\int_M (|E_j|^r + |\nabla E_j|^r)\, d\mu_g \le C'_{j,r}.$$

Thus, by the Sobolev inequality, if $r > j$, one has

$$\max_M |E_j|^r \le C_t \int_M (|E_j|^r + |\nabla E_j|^r)\, d\mu_g.$$

Notice that the constant C_t depends on the metric $g(t)$, but it does not depend on the derivatives of $g(t)$. Moreover, from [Hamilton 1982, Lemma 14.2], it follows that the metrics are all equivalent. Hence, the constant C_t is uniformly bounded as $t \to T$

and, from the previous estimates, it follows that, if $|\mathrm{Riem}| \le C$ on $M \times [0, T)]$, for every $j \in \mathbb{N}$ one has

$$\max_M |\nabla^j \mathrm{Riem}| \le C_j,$$

where the constant C_j depends only on the initial value of the metric and the constant C.

Arguing now as in [Hamilton 1982, Section 14], it follows that the metrics $g(t)$ converge to some limit metric $g(T)$ in the C^∞ topology (with all their time/space ordinary partial derivatives, once written in local coordinates), hence, we can restart the flow with this initial metric $g(T)$, obtaining a smooth flow in some larger time interval $[0, T+\delta)$, in contradiction with the fact that T was the maximal time of smooth existence. This completes the proof of Theorem 5.7. □

Acknowledgements

We want to thank Mauro Carfora for several suggestions and comments. The authors are partially supported by the GNAMPA project "Equazioni di evoluzione geometriche e strutture di tipo Einstein". Laura Cremaschi is partially supported by the PRIN 2010–2011 project "Calcolo delle Variazioni". Zindine Djadli is partially supported by the grant GTO: ANR-12-BS01-0004.

References

[Aubin 1970] T. Aubin, "Métriques riemanniennes et courbure", *J. Differential Geometry* **4** (1970), 383–424. MR Zbl

[Besse 1987] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik (3) **10**, Springer, Berlin, 1987. MR Zbl

[Böhm and Wilking 2007] C. Böhm and B. Wilking, "Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature", *Geom. Funct. Anal.* **17**:3 (2007), 665–681. MR Zbl

[Bourguignon 1981] J.-P. Bourguignon, "Ricci curvature and Einstein metrics", pp. 42–63 in *Global differential geometry and global analysis* (Berlin, 1979), edited by D. Ferus et al., Lecture Notes in Math. **838**, Springer, Berlin, 1981. MR Zbl

[Brendle 2005] S. Brendle, "Convergence of the Yamabe flow for arbitrary initial energy", *J. Differential Geom.* **69**:2 (2005), 217–278. MR Zbl

[Catino and Mantegazza 2011] G. Catino and C. Mantegazza, "The evolution of the Weyl tensor under the Ricci flow", *Ann. Inst. Fourier* (*Grenoble*) **61**:4 (2011), 1407–1435. MR Zbl

[Catino and Mazzieri 2016] G. Catino and L. Mazzieri, "Gradient Einstein solitons", *Nonlinear Anal.* **132** (2016), 66–94. MR Zbl

[Catino et al. 2015a] G. Catino, C. Mantegazza, and L. Mazzieri, "Locally conformally flat ancient Ricci flows", *Analysis & PDE* **8**:2 (2015), 365–371.

[Catino et al. 2015b] G. Catino, L. Mazzieri, and S. Mongodi, "Rigidity of gradient Einstein shrinkers", *Commun. Contemp. Math.* **17**:6 (2015), art. id. 1550046, 18 pp. MR

[Catino et al. 2016] G. Catino, P. Mastrolia, D. D. Monticelli, and M. Rigoli, "Conformal Ricci solitons and related integrability conditions", *Adv. Geom.* **16**:3 (2016), 301–328. MR

[Chow and Knopf 2004] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs **110**, American Mathematical Society, Providence, RI, 2004. MR Zbl

[Chow et al. 2008] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, II: Analytic aspects*, Mathematical Surveys and Monographs **144**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Delay 2014] E. Delay, "Inversion d'opérateurs de courbures au voisinage de la métrique euclidienne", preprint, 2014. arXiv

[DeTurck 1983] D. M. DeTurck, "Deforming metrics in the direction of their Ricci tensors", *J. Differential Geom.* **18**:1 (1983), 157–162. MR Zbl

[DeTurck 2003] D. M. DeTurck, "Deforming metrics in the direction of their Ricci tensors (improved version)", pp. 163–165 in *Collected papers on Ricci flow*, edited by H. D. Cao et al., Series in Geometry and Topology **37**, Int. Press, Somerville, MA, 2003.

[Eells and Sampson 1964] J. Eells, Jr. and J. H. Sampson, "Harmonic mappings of Riemannian manifolds", *Amer. J. Math.* **86** (1964), 109–160. MR Zbl

[Fischer 2004] A. E. Fischer, "An introduction to conformal Ricci flow", *Classical Quantum Gravity* **21**:3 (2004), S171–S218. MR Zbl

[Gallot et al. 1990] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 2nd ed., Springer, Berlin, 1990. MR Zbl

[Hamilton 1982] R. S. Hamilton, "Three-manifolds with positive Ricci curvature", *J. Differential Geom.* **17**:2 (1982), 255–306. MR Zbl

[Hamilton 1986] R. S. Hamilton, "Four-manifolds with positive curvature operator", *J. Differential Geom.* **24**:2 (1986), 153–179. MR Zbl

[Hamilton 1995] R. S. Hamilton, "The formation of singularities in the Ricci flow", pp. 7–136 in *Surveys in differential geometry* (Cambridge, 1993), vol. 2, edited by S.-T. Yau, Int. Press, Cambridge, 1995. MR Zbl

[Hamilton 1997] R. S. Hamilton, "Four-manifolds with positive isotropic curvature", *Comm. Anal. Geom.* **5**:1 (1997), 1–92. MR Zbl

[Ivey 1993] T. Ivey, "Ricci solitons on compact three-manifolds", *Differential Geom. Appl.* **3**:4 (1993), 301–307. MR Zbl

[Lu et al. 2014] P. Lu, J. Qing, and Y. Zheng, "A note on conformal Ricci flow", *Pacific J. Math.* **268**:2 (2014), 413–434. MR Zbl

[Mantegazza 2011] C. Mantegazza, *Lecture notes on mean curvature flow*, Progress in Mathematics **290**, Birkhäuser, Basel, 2011. MR Zbl

[Savas-Halilaj and Smoczyk 2014] A. Savas-Halilaj and K. Smoczyk, "Bernstein theorems for length and area decreasing minimal maps", *Calc. Var. Partial Differential Equations* **50**:3–4 (2014), 549–577. MR Zbl

[Schwetlick and Struwe 2003] H. Schwetlick and M. Struwe, "Convergence of the Yamabe flow for "large" energies", *J. Reine Angew. Math.* **562** (2003), 59–100. MR Zbl

[Topping 2006] P. Topping, *Lectures on the Ricci flow*, London Mathematical Society Lecture Note Series **325**, Cambridge Univ. Press, 2006. MR Zbl

[Weinberger 1975] H. F. Weinberger, "Invariant sets for weakly coupled parabolic and elliptic systems", *Rend. Mat.* (6) **8** (1975), 295–310. MR Zbl

[Ye 1994] R. Ye, "Global existence and convergence of Yamabe flow", *J. Differential Geom.* **39**:1 (1994), 35–50. MR Zbl

Received March 2, 2016. Revised September 16, 2016.

GIOVANNI CATINO
DIPARTIMENTO DI MATEMATICA
POLITECNICO DI MILANO
VIA BONARDI 9
I-20133 MILANO
ITALY
giovanni.catino@polimi.it

LAURA CREMASCHI
SCUOLA NORMALE SUPERIORE DI PISA
PIAZZA DEI CAVALIERI 7
I-56126 PISA
ITALY
laura.cremaschi@sns.it

ZINDINE DJADLI
INSTITUT FOURIER
UNIVERSITÉ GRENOBLE
100 RUE DES MATHS
38402 ST MARTIN D'HERES
FRANCE
and
LABORATORIO FIBONACCI
PIAZZA DEI CAVALIERI 7
I-56126 PISA
ITALY
Zindine.Djadli@ujf-grenoble.fr

CARLO MANTEGAZZA
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI
UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II
VIA CINTIA
MONTE S. ANGELO
I-80126 NAPOLI
ITALY
carlo.mantegazza@sns.it

LORENZO MAZZIERI
DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DEGLI STUDI DI TRENTO
VIA SOMMARIVE 14
I-38123 POVO (TN)
ITALY
lorenzo.mazzieri@unitn.it

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.371

THE NORMAL FORM THEOREM AROUND POISSON TRANSVERSALS

PEDRO FREJLICH AND IOAN MĂRCUȚ

Dedicated to Alan Weinstein on the occasion of his 70*th birthday*

We prove a normal form theorem for Poisson structures around Poisson transversals (also called cosymplectic submanifolds), which simultaneously generalizes Weinstein's symplectic neighborhood theorem from symplectic geometry and Weinstein's splitting theorem. Our approach turns out to be essentially canonical, and as a byproduct, we obtain an equivariant version of the latter theorem.

1. Introduction

This paper is devoted to the study of semilocal properties of *Poisson transversals*. These are submanifolds X of a Poisson manifold (M, π) that meet each symplectic leaf of π *transversally* and *symplectically*. A Poisson transversal X carries a canonical Poisson structure, whose leaves are the intersections of leaves of π with X, and are endowed with the pullback symplectic structure.

Even though this class of submanifolds has very rarely been dealt with in full generality — much to our dismay and surprise — Poisson transversals permeate the whole theory of Poisson manifolds, often playing a quite fundamental role. This lack of specific attention is especially intriguing since they are a special case of several distinguished classes of submanifolds which have aroused interest lately: Poisson transversals are Lie–Dirac submanifolds [Xu 2003], Poisson–Dirac submanifolds

MSC2010: primary 53D17; secondary 53D05.
Keywords: differential geometry, symplectic geometry, Poisson manifolds, Poisson groupoids and algebroids.

[Crainic and Fernandes 2004], and also pre-Poisson submanifolds [Cattaneo and Zambon 2009] (see also [Zambon 2011] for a survey on submanifolds in Poisson geometry).

No wonder, then, that Poisson transversals have shown up already in the infancy of Poisson geometry, in the foundational paper of Weinstein [1983]. Namely, if L is a symplectic leaf and $x \in L$, then a submanifold X that intersects L transversally at x and has complementary dimension is a Poisson transversal, and its induced Poisson structure governs much of the geometry transverse to L. In fact, a small enough tubular neighborhood of L in M will have the property that all its fibers are Poisson transversals. Such fibrations are nowadays called *Poisson fibrations*, and were studied by Vorobjev [2001] — mostly in connection with the local structure around symplectic leaves — and also by Fernandes and Brahic [2008]. That Poisson fibrations are related to Haefliger's formalism of geometric structures described by groupoid-valued cocycles (see [Haefliger 1958] and also [Gromov 1986]) — of which the "automatic transversality" of Lemma 7 is also reminiscent — should not escape notice. In fact, in physics literature, Poisson fibrations have long been known in the guise of *second class constraints*, and motivated the introduction by P. Dirac [1950] of what we know today as the induced Dirac bracket, which in our language is the induced Poisson structure on the fibers.

The role played by Poisson transversals in Poisson geometry is similar to that played by symplectic submanifolds in symplectic geometry and by transverse submanifolds in foliation theory (see the examples in the next section). The key observation is that the transverse geometry around a Poisson transversal X is of nonsingular and *contravariant* nature: it behaves more like a 2-form than as a bivector in the directions conormal to X. This allows us to make particularly effective use of the tools of "contravariant geometry". In the core of our arguments lies the fact that the contravariant exponential map $\exp_{\mathcal{X}}$ associated to a Poisson spray $\mathcal{X}$ gives rise to a tubular neighborhood adapted to $X \subset (M, \pi)$, in complete analogy with the classical construction of a tubular neighborhood of a submanifold X in a Riemannian manifold (M, g), thus effectively reducing many problems to the symplectic case.

The main result of this paper is a local normal form theorem around Poisson transversals, which simultaneously generalizes Weinstein's splitting theorem [1983] and Weinstein's symplectic neighborhood theorem [1971]. At a Poisson transversal X of (M, π), the restriction of the Poisson bivector $\pi|_X \in \Gamma(\wedge^2 TM|_X)$ determines

- a Poisson structure on X, denoted π_X,
- a nondegenerate, fiberwise 2-form on the conormal bundle $p : N^*X \to X$, denoted

$$w_X \in \Gamma(\wedge^2 NX).$$

Let $\widetilde{\sigma}$ be a closed 2-form on N^*X that extends $\sigma := -w_X$, i.e., which restricts on $T(N^*X)|_X = TX \oplus N^*X$ to the trivial extension of σ by zero.

To such an extension we associate a Poisson structure $\pi(\widetilde{\sigma})$ on an open set $U(\widetilde{\sigma}) \subset N^*X$ around X. The symplectic leaves of $\pi(\widetilde{\sigma})$ are in one-to-one correspondence with the leaves of π_X; namely if (L, ω_L) is a leaf of π_X, the corresponding leaf of $\pi(\widetilde{\sigma})$ is an open set $\widetilde{L} \subset p^{-1}(L)$ around L endowed with the 2-form $\omega_{\widetilde{L}} := p^*(\omega_L) + \widetilde{\sigma}|_{\widetilde{L}}$. The Poisson manifold $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$ is the local model of π around X. We will provide a more conceptual description of the local model using Dirac geometry.

Theorem 1 (normal form theorem). *Let (M, π) be a Poisson manifold and $X \subset M$ be an embedded Poisson transversal. An open neighborhood of X in (M, π) is Poisson diffeomorphic to an open neighborhood of X in the local model $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$.*

Under stronger assumptions (which always hold around points in X) we can provide an even more explicit description of the normal form. Assuming *symplectic* triviality of the conormal bundle to X, the theorem implies a generalized version of the Weinstein splitting theorem, expressing the Poisson as a product, i.e., in the form (1) below. This coincides with Weinstein's setting when we look at (small) Poisson transversals of complementary dimension to a symplectic leaf.

The proof of Theorem 1 relies on the symplectic realization constructed in [Crainic and Mărcuț 2011] with the aid of global Poisson geometry, and on elementary Dirac-geometric techniques; the former is the crucial ingredient that allows us to have a good grasp of directions conormal to the Poisson transversal, and the latter furnishes the appropriate language to deal with objects which have mixed covariant-contravariant behavior. As an illustration of the strength and canonicity of our methods, we present as an application the proof of an equivariant version of Weinstein's splitting theorem. Other applications of the normal form theorem, which reveal the Poisson-topological aspects of Poisson transversals, will be treated elsewhere.

Theorem 2. *Let (M, π) be a Poisson manifold and let G be a compact Lie group acting by Poisson diffeomorphisms on M. If $x \in M$ is a fixed point of G, then there are coordinates $(p_1, \dots, p_n, q_1, \dots, q_n, y_1, \dots, y_m) \in \mathbb{R}^{2n+m}$ centered at x such that*

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{j,k=1}^{m} \varpi_{j,k}(y) \frac{\partial}{\partial y_j} \wedge \frac{\partial}{\partial y_k}, \tag{1}$$

and in these coordinates G acts linearly and keeps the subspaces $\mathbb{R}^{2n} \times \{0\}$ and $\{0\} \times \mathbb{R}^m$ invariant.

This answers in the negative a question posed by Miranda and Zung [2006] about the necessity of the "tameness" condition they assume in their proof of this result. We wish to thank Miranda for bringing this problem to our attention.

We should probably also say a few words about terminology. Poisson transversals are also referred to as *cosymplectic submanifolds* in the literature, and this is motivated by the fact that the conormal directions to such a submanifold are symplectic, i.e., the Poisson tensor is nondegenerate on the conormal bundle to the submanifold. Even though this nomenclature is perfectly reasonable, there are several reasons why we decided not to use this name. Foremost among these:

(1) There is already a widely used notion of a cosymplectic manifold, defined as a manifold of dimension $2n+1$, endowed with a closed 1-form θ and a closed 2-form ω such that $\theta\wedge\omega^n$ is a volume form.

(2) The general point of view of transverse geometric structures is of great insight into Poisson transversals when we rephrase the problem in terms of Dirac structures and contravariant geometry. Moreover, the proximity between the dual pairs used in the proof of the normal form theorem, and the gadget of Morita equivalence, which is known to govern the transverse geometry to the symplectic leaves, is too obvious to ignore.

2. Some basic properties of Poisson transversals

Let (M,π) be a Poisson manifold. A *Poisson transversal* in M is an embedded submanifold $X\subset M$ that meets each symplectic leaf of π *transversally* and *symplectically*. We translate both these conditions algebraically. Let $x\in X$ and let (L,ω) be the symplectic leaf through x. Transversality translates to

$$T_xX+T_xL=T_xM.$$

Taking annihilators in this equation, we obtain that $N_x^*X\cap\ker(\pi_x^\sharp)=\{0\}$, or equivalently, that the restriction of $\pi^\sharp$ to N_x^*X is injective:

$$0\to N_x^*X\xrightarrow{\pi_x^\sharp}T_xM. \tag{2}$$

For the second condition, note that the kernel of $\omega_x|_{T_xX\cap T_xL}$ is $T_xX\cap\pi_x^\sharp(N_x^*X)$. So the condition that $T_xX\cap T_xL$ be a symplectic subspace is equivalent to

$$T_xX\cap\pi_x^\sharp(N_x^*X)=\{0\}. \tag{3}$$

Since T_xX and N_x^*X have complementary dimensions, (2) and (3) imply the following decomposition, which is equivalent to X being a Poisson transversal:

$$TX\oplus\pi^\sharp(N^*X)=TM|_X. \tag{4}$$

The decomposition of the tangent bundle (4) canonically gives an embedded normal bundle, denoted

$$NX := \pi^\sharp(N^*X) \subset TM|_X,$$

and a corresponding decomposition for the cotangent bundle

$$N^*X \oplus N^\circ X = T^*M|_X.$$

For $\xi \in N^*_x X$ and $\eta \in N^\circ_x X$, we have that $\pi^\sharp(\xi) \in N_x X$, hence $\pi(\xi, \eta) = 0$. This implies that $\pi|_X$ has no mixed component in the decomposition

$$\wedge^2 TM|_X = \wedge^2 TX \oplus (TX \otimes NX) \oplus \wedge^2 NX.$$

Therefore $\pi|_X$ splits as

$$\pi|_X = \pi_X + w_X, \quad \pi_X \in \Gamma(\wedge^2 TX), \quad w_X \in \Gamma(\wedge^2 NX).$$

It is well known that these two tensors satisfy the following properties, but for completeness we include a proof.

Lemma 3. *The bivector π_X is Poisson and w_X, regarded as a 2-form on N^*X, is fiberwise nondegenerate.*

Proof. To prove that π_X is Poisson, we will use Dirac-geometric techniques (for other approaches, see [Crainic and Fernandes 2004; Xu 2003]; for the basics of Dirac geometry, see [Bursztyn and Radko 2003]). It suffices to show that the pullback via the inclusion $i : X \to M$ of the Dirac structure $L_\pi := \{\pi^\sharp(\xi) + \xi : \xi \in T^*M\}$ equals the almost Dirac structure $L_{\pi_X} := \{\pi_X^\sharp(\xi) + \xi : \xi \in T^*X\}$, since this makes L_{π_X} automatically involutive, and hence π_X Poisson. But to show this it suffices to prove the following inclusion:

$$\begin{aligned} L_{\pi_X} &= \{\pi_X^\sharp(\xi) + \xi : \xi \in T^*X\} = \{\pi_X^\sharp(i^*\eta) + i^*\eta : \eta \in N^\circ X\} \\ &= \{\pi^\sharp(\eta) + i^*\eta : \eta \in N^\circ X\} \subset i^* L_\pi, \end{aligned}$$

where we used that $w_X^\sharp(\eta) = 0$, for $\eta \in N^\circ X$.

The map $w_X^\sharp : N^*X \to NX$ is just the restriction of π, which, by the decomposition (4), is a linear isomorphism. □

We recall three natural instances of Poisson transversals, which appear throughout Poisson geometry:

Example 4. If π is nondegenerate then X is a Poisson transversal if and only if X is a symplectic submanifold of (M, π).

Example 5. If L is the symplectic leaf of (M, π) through a point $x \in M$, a submanifold X that intersects L transversally at x and is of complementary dimension is a Poisson transversal around x.

Example 6. If (M, π) is a regular Poisson manifold with underlying foliation $\mathcal{F}$ of codimension q, then every submanifold X of dimension q that is transverse to $\mathcal{F}$ is a Poisson transversal.

A very useful — and somewhat surprising — fact about Poisson transversals is that they behave well with respect to Poisson maps:

Lemma 7. *Let $\varphi : (M_0, \pi_0) \to (M_1, \pi_1)$ be a Poisson map and $X_1 \subset M_1$ be a Poisson transversal. Then*:

(1) *φ is transverse to X_1.*

(2) *$X_0 := \varphi^{-1}(X_1)$ is also a Poisson transversal.*

(3) *φ restricts to a Poisson map $\varphi|_{X_0} : (X_0, \pi_{X_0}) \to (X_1, \pi_{X_1})$.*

(4) *The differential of φ along X_0 restricts to a fiberwise linear isomorphism between embedded normal bundles $\varphi_*|_{NX_0} : NX_0 \to NX_1$.*

(5) *The map $F : N^*X_0 \to N^*X_1,\ F(\xi) = (\varphi^*)^{-1}(\xi),\ \xi \in N^*X_0$ is a fiberwise linear symplectomorphism between the symplectic vector bundles*

$$F : (N^*X_0, w_{X_0}) \to (N^*X_1, w_{X_1}).$$

Corollary 8. *Let (M, π) be a Poisson manifold, $X \subset M$ be a Poisson transversal and $W \subset M$ be a Poisson submanifold. Then W and X intersect transversally, and $X \cap W$ is*

- *a Poisson transversal in $(W, \pi|_W)$, and*
- *a Poisson submanifold of (X, π_X).*

Proof of Lemma 7. Consider $x \in X_0$ and let $y := \varphi(x) \in X_1$. Since φ is a Poisson map we have:

$$\pi_1^\sharp(\eta) = \varphi_*\big(\pi_0^\sharp(\varphi^*\eta)\big), \quad \text{for all } \eta \in T_y^*M_1,$$

therefore $\pi_1^\sharp(T_y^*M_1) \subset \varphi_*(T_xM_0)$. But X_1 being a Poisson transversal now implies that φ is transverse to X_1:

$$T_yM_1 = T_yX_1 + \pi_1^\sharp(T_y^*M_1) = T_yX_1 + \varphi_*(T_xM_0).$$

In particular, X_0 is a submanifold of M_0. To show that X_0 is a Poisson transversal, we will prove that the decomposition $TX_0 \oplus \pi_0^\sharp(N^*X_0) = TM_0|_{X_0}$ holds. Note first that

$$T_xX_0 = (\varphi_*)^{-1}(T_yX_1) \quad \text{and} \quad N_x^*X_0 = \varphi^*(N_y^*X_1).$$

Let $v \in T_xM_0$, and decompose $\varphi_*v = u + \pi_1^\sharp(\eta)$, with $u \in T_yX_1$ and $\eta \in N_y^*X_1$. Then $\varphi^*\eta \in N_x^*X_0$ and $w := v - \pi_0^\sharp(\varphi^*\eta)$ projects to u, hence $w \in T_xX_0$. This shows that $v = w + \pi_0^\sharp(\varphi^*\eta) \in T_xX_0 + \pi_0^\sharp(N_x^*X_0)$, hence

$$T_xM_0 = T_xX_0 + \pi_0^\sharp(N_x^*X_0).$$

Counting dimensions, we conclude that this is a direct sum decomposition, and therefore X_0 is a Poisson transversal.

Note, moreover, that φ_* preserves the embedded normal bundles

$$\varphi_*(N_xX_0) = \varphi_*\big(\pi_0^\sharp(N_x^*X_0)\big) = \varphi_*\big(\pi_0^\sharp(\varphi^*(N_y^*X_1))\big) = \pi_1^\sharp(N_y^*X_1) = N_yX_1,$$

and because they have the same rank, $\varphi_*|_{NX_0}$ is a fiberwise isomorphism. Since we also have $\varphi_*(T_xX_0) \subset T_yX_1$, the Poisson condition $\varphi_*(\pi_{0,x}) = \pi_{1,y}$ implies that $\varphi_*(\pi_{X_0,x}) = \pi_{X_1,y}$ and $\varphi_*(w_{X_0,x}) = w_{X_1,y}$. This implies (3) and (4). □

3. The local model

The local model around a Poisson transversal depends on an extra choice:

Definition 9. Let (E, σ) be a symplectic vector bundle over X. A *closed extension* of σ is a closed 2-form $\widetilde{\sigma}$ defined on a neighborhood of X in E, such that its restriction to $TE|_X = TX \oplus E$ equals the trivial extension of σ to $TE|_X$. We denote the space of all closed extensions by $\Upsilon(E, \sigma)$.

Closed extensions always exist, and can be constructed employing the standard de Rham homotopy operator (see, e.g., the extension theorem in [Weinstein 1977]).

In the warm-up for the construction below of the local model, let us revisit the three instances which are generalized by our main result.

Example 10 (Weinstein's symplectic neighborhood theorem [1971]). Let (M, ω) be a symplectic manifold, and $(X, \omega_X) \subset M$ be a symplectic submanifold. The symplectic orthogonal of TX, denoted by $E := TX^\omega$, is a symplectic vector bundle with bilinear form $\sigma := \omega|_E$. The local model around X is given by the closed 2-form $\widetilde{\sigma} + p^*(\omega_X)$ on E, where $p : E \to X$ is the projection and $\widetilde{\sigma} \in \Upsilon(E, \sigma)$. Weinstein's symplectic neighborhood theorem says that a neighborhood of X in (M, ω) is symplectomorphic to a neighborhood of X in $(E, \widetilde{\sigma} + p^*(\omega_X))$.

Example 11 (Weinstein's splitting theorem [1983]). Let (M, π) be a Poisson manifold and let $x \in M$. Let also (L, ω) be the symplectic leaf through $x \in M$, and (X, π_X) a Poisson transversal at x, of complementary dimension. The local model around x is given by the product of Poisson manifolds

$$(T_xL, \omega_x^{-1}) \times (X, \pi_X).$$

Weinstein's splitting theorem (or Darboux–Weinstein theorem) asserts that (M, π) is Poisson diffeomorphic around x to an open set around $(0, x)$ in the local model.

Example 12 (transversals to foliations). Let M be a manifold carrying a smooth (regular) foliation $\mathcal{F}$, and let $X \subset M$ be a submanifold transverse to $\mathcal{F}$,

$$T_x X + T_x \mathcal{F} = T_x M, \quad \text{for all } x \in X.$$

Let $\mathcal{F}_X$ be the induced foliation on X. The local model of the foliation $\mathcal{F}$ around X is $(NX, p^*\mathcal{F}_X)$, where $p : NX \to X$ is the normal bundle to X; note that the leaves of the local model are of the form $p^{-1}(L)$, for L a leaf of $\mathcal{F}_X$. To build an isomorphism between $\mathcal{F}$ and its model around X, consider a metric g on $T\mathcal{F}$ and let $\exp_g : T\mathcal{F} \supset U \to M$ denote the leafwise exponential map of g, i.e., for each leaf L, $\exp_g : (TL \cap U) \to L$ is the (Levi-Civita) exponential map of the Riemannian manifold $(L, g|_L)$. Then $T\mathcal{F}_X^{\perp} \subset T\mathcal{F}|_X$ is a complement to TX in $TM|_X$, and the composition

$$NX \xrightarrow{\sim} T\mathcal{F}_X^{\perp} \xrightarrow{\exp_g} M$$

pulls the foliation $\mathcal{F}$ to the local model.

The idea for constructing the local model around a Poisson transversal is to put the foliation in normal form in the sense of Example 12, and then perform Weinstein's construction of Example 10 along all symplectic leaves simultaneously.

Let (E, σ) be a symplectic vector bundle over a Poisson manifold (X, π_X) with projection $p : E \to X$ and consider a closed extension $\widetilde{\sigma} \in \Upsilon(E, \sigma)$. As mentioned in the introduction, the symplectic leaves of the local model are $(\widetilde{L}, \omega_{\widetilde{L}})$, for (L, ω_L) a symplectic leaf of (X, π_X), where $\widetilde{L} \subset p^{-1}(L)$ is an open set containing L and

$$\omega_{\widetilde{L}} := \widetilde{\sigma}|_{\widetilde{L}} + p^*(\omega_L).$$

To show this construction yields a smooth Poisson bivector around X, we rewrite it using the language of Dirac geometry. Let L_{π_X} be the Dirac structure corresponding to π_X. Dirac structures can be pulled back along submersions. The pullback of L_{π_X} to E, denoted by $p^*(L_{\pi_X})$, has presymplectic leaves $(p^{-1}(L), p^*(\omega_L))$, where (L, ω_L) is a symplectic leaf of π_X. Finally, the gauge transform by $\widetilde{\sigma}$, denoted by $p^*(L_{\pi_X})^{\widetilde{\sigma}}$, has the required effect: it adds to each leaf the restriction of $\widetilde{\sigma}$.

Lemma 13. *Let (E, σ) be a symplectic vector bundle over a Poisson manifold (X, π_X), and let $\widetilde{\sigma} \in \Upsilon(E, \sigma)$ be a closed extension of σ. On a neighborhood $U(\widetilde{\sigma})$ of X in E, we have that the Dirac structure*

$$L(\widetilde{\sigma}) := p^*(L_{\pi_X})^{\widetilde{\sigma}}$$

corresponds to a Poisson structure $\pi(\widetilde{\sigma})$ that decomposes along X as

$$\pi(\widetilde{\sigma})|_X = \pi_X + \sigma^{-1} \in \Gamma(\wedge^2 TX) \oplus \Gamma(\wedge^2 E).$$

Equivalently, (X, π_X) is a Poisson transversal for $\pi(\widetilde{\sigma})$, the canonical normal bundle is $E \subset TE|_X$, and the induced nondegenerate bivector is $w_X = \sigma^{-1}$.

Proof. The condition that $L(\widetilde{\sigma})$ be Poisson is open, thus it suffices to show that $L(\widetilde{\sigma})$ has the expected form along X. This can be easily checked, since

$$p^*(L_{\pi_X})|_X = \{\pi_X^\sharp(\xi) + Y + \xi : \xi \in T^*X,\ Y \in E\},$$

and therefore

$$\begin{aligned} L(\widetilde{\sigma})|_X &= \{\pi_X^\sharp(\xi) + Y + \xi + \iota_Y\sigma : \xi \in T^*X,\ Y \in E\} \\ &= \{\pi_X^\sharp(\xi) + (\sigma^{-1})^\sharp(\eta) + \xi + \eta : \xi \in T^*X,\ \eta \in E^*\} \\ &= \{(\pi_X + \sigma^{-1})^\sharp(\theta) + \theta : \theta \in T^*E|_X\}. \end{aligned}$$ □

Definition 14. The Poisson manifold $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$ from the lemma is called the *local model* associated to (E, σ) and (X, π_X).

If X is a Poisson transversal of a Poisson manifold (M, π), π_X is the induced Poisson structure on X, $E = N^*X$ is the conormal bundle to X and

$$\sigma = -w_X = -(\pi|_{N^*X}),$$

then $(U(\widetilde{\sigma}), \pi(\widetilde{\sigma}))$ is called the *local model of* π *around* X.

Remark 15. We point out that there is a choice in having the local models of π around X live in the *conormal* bundle to X, as opposed to its normal bundle NX, as is typically the case for normal form theorems. In fact, since

$$w_X : (N^*X, -w_X) \to (NX, w_X^{-1})$$

is an isomorphism of symplectic vector bundles, we can translate canonically all our constructions to NX via w_X.

That we chose N^*X instead of NX is meant to emphasize that we regard the conormal N^*X as the more appropriate notion of "contravariant normal", an opinion which is corroborated by the scheme of proof of Theorem 1, where we spread out a tubular neighborhood of X by following contravariant geodesics starting in directions conormal to X.

The construction of the local model depends on the choice of a closed extension. A Poisson version of the Moser argument, which first appeared in [Alekseev and Meinrenken 2007] (see also [Alekseev and Meinrenken 2016]) will be later employed to prove that different extensions induce isomorphic local models.

Lemma 16 (Moser lemma). *Suppose we are given a path of Poisson structures of the form* $t \mapsto \pi_t := \pi^{t\,d\alpha}$, *where* π *is a Poisson structure and* $\alpha \in \Omega^1(M)$. *Then the isotopy* $\phi_{\mathcal{V}}^{t,s}$ *generated by the time-dependent vector field* $\mathcal{V}_t := -\pi_t^\sharp(\alpha)$ *stabilizes* π_t:

$$\phi_{\mathcal{V}*}^{t,s}\pi_s = \pi_t,$$

whenever this is defined.

Proof. Recall that Poisson cohomology is computed by the complex $(\mathfrak{X}^\bullet(M), d_\pi)$, where $d_\pi : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$ is defined by $d_\pi := [\pi, \cdot]$ and $[\cdot, \cdot]$ stands for the Schouten bracket on multivector fields. Moreover, π, regarded as a map $\pi^\sharp : T^*M \to TM$, induces a chain map

$$(-1)^{\bullet+1} \wedge^\bullet \pi^\sharp : (\Omega^\bullet(M), d) \to (\mathfrak{X}^\bullet(M), d_\pi),$$

from the de Rham complex of differential forms, see, e.g., [Dufour and Zung 2005]. In particular,

$$L_{\mathcal{V}_t}\pi_t = \big[\pi_t, \pi_t^\sharp(\alpha)\big] = d_{\pi_t}\pi_t^\sharp(\alpha) = -\wedge^2 \pi_t^\sharp(d\alpha).$$

As maps, this can be written as

$$(L_{\mathcal{V}_t}\pi_t)^\sharp = \pi_t^\sharp \circ (d\alpha)^\flat \circ \pi_t^\sharp,$$

where $(d\alpha)^\flat : TM \to T^*M$ stands for $d\alpha$ regarded as a map. Also, by the very definition of gauge transformation, we have the identity $\pi^\sharp = \pi_t^\sharp \circ \big(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\big)$, whence

$$\begin{aligned}
0 = \frac{d\pi^\sharp}{dt} &= \frac{d\pi_t^\sharp}{dt} \circ \big(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\big) + \pi_t^\sharp \circ (d\alpha)^\flat \circ \pi^\sharp \\
&= \left(\frac{d\pi_t^\sharp}{dt} + \pi_t^\sharp \circ (d\alpha)^\flat \circ \pi_t^\sharp\right) \circ \big(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\big) \\
&= \left(\frac{d\pi_t}{dt} + L_{\mathcal{V}_t}\pi_t\right)^\sharp \circ \big(\mathrm{id} + t(d\alpha)^\flat \circ \pi^\sharp\big).
\end{aligned}$$

Finally, we obtain

$$\frac{d}{dt}(\phi_{\mathcal{V}}^{t,s})^*\pi_t = (\phi_{\mathcal{V}}^{t,s})^*\Big(L_{\mathcal{V}_t}\pi_t + \frac{d\pi_t}{dt}\Big) = 0,$$

showing that $(\phi_{\mathcal{V}}^{t,s})^*\pi_t = \pi_s$. □

Next, we show that different choices of closed extensions yield isomorphic local models.

Lemma 17. *If (E, σ) is a symplectic vector bundle over a Poisson manifold (X, π_X), then all corresponding local models are isomorphic around X by diffeomorphisms that fix X up to first order.*

Proof. If $\widetilde{\sigma}_1 \in \Upsilon(E, \sigma)$ is a second extension, $\widetilde{\sigma}_1 - \widetilde{\sigma}$ is a closed 2-form on E that vanishes on $TE|_X$. Since the inclusion $X \subset E$ is a homotopy equivalence, $\widetilde{\sigma}_1 - \widetilde{\sigma}$ is exact, and one can choose a primitive $\eta \in \Omega^1(E)$ that vanishes on $TE|_X$. Actually, by the relative Poincaré lemma in [Weinstein 1977], one may choose η with vanishing first derivatives along X. Denote $\pi(\widetilde{\sigma})$ and $\pi(\widetilde{\sigma} + d\eta)$ by π_0 and π_1, respectively. Then π_1 is the gauge transform by $d\eta$ of π_0, denoted $\pi_1 = \pi_0^{d\eta}$. These bivectors can be interpolated by the family of Poisson structures

$$\pi_t := \pi_0^{t d\eta}, \quad t \in [0,1].$$

Now, π_t corresponds to the smooth family of Dirac structures $L_t := p^*(L_{\pi_X})^{\widetilde{\sigma}+t d\eta}$, and the set $U \subset \mathbb{R} \times E$ of those points (t,x) where $L_{t,x}$ is Poisson is open. Since $[0,1] \times X \subset U$, there is an open neighborhood V of X in E such that $[0,1] \times V \subset U$. Thus, π_t is defined on V for all $t \in [0,1]$. By the Moser lemma (Lemma 16), we see that the flow of the time-dependent vector field

$$Y_t := -\pi_t^\sharp(\eta)$$

trivializes the family, i.e., $(\phi_Y^{t,s})^*(\pi_t) = \pi_s$ whenever it is defined. Since η and its first derivatives vanish along X, it follows that $\phi_Y^{t,s}$ fixes X and that its differential is the identity on $TE|_X$. Arguing as before, the set where $\phi_Y^{t,0}$ is defined up to $t = 1$ contains an open neighborhood $V' \subset V$ of X, so we obtain a Poisson diffeomorphism

$$\phi_Y^{1,0} : (V', \pi_0) \xrightarrow{\sim} \big(\phi_Y^{1,0}(V'), \pi_1\big). \qquad \square$$

4. The normal form theorem

The normal form theorem (Theorem 1) for a Poisson structure (M,π) around a Poisson transversal X states that π and its local model (built out of $\pi|_X$) are isomorphic around X. In the symplectic case, this follows from the Moser argument in a straightforward manner. For general Poisson manifolds, the proof is more involved. The main difficulty is to put the foliation in normal form; namely, to find a tubular neighborhood of X along the leaves of π. If the foliation is regular, such a construction can be performed by restricting a metric to the leaves and taking leafwise the Riemannian exponential (cf. Example 12). If π is not regular, it is not a priori clear if these maps glue to a smooth tubular neighborhood of X in M. We will use instead a contravariant version of this argument in which we replace the classical exponential from Riemannian geometry by its Poisson-geometric analog: the contravariant exponential. The more surprising outcome is that a contravariant exponential not only puts the foliation in normal form, but also provides a closed extension *and* the required isomorphism to the local model. A funny consequence is that a choice of Poisson spray $\mathcal{X}$ for (M,π) puts *all* of its Poisson transversals in normal form canonically and simultaneously!

We start by recalling some notions and results from contravariant geometry.

Definition 18. A *Poisson spray* $\mathcal{X} \in \mathfrak{X}^1(T^*M)$ on a Poisson manifold (M,π) is a vector field on T^*M such that

(1) $p_*\mathcal{X}(\xi) = \pi^\sharp(\xi)$, for all $\xi \in T^*M$,

(2) $m_t^* \mathcal{X} = t\mathcal{X}$, for all $t > 0$,

where $p : T^*M \to M$ is the projection and $m_t : T^*M \to T^*M$ is the multiplication by t. The flow $\phi^t_{\mathcal{X}}$ of $\mathcal{X}$ is called the *geodesic flow*.

The *contravariant exponential* of $\mathcal{X}$ is the map

$$\exp_{\mathcal{X}} : U \to M, \qquad \xi \mapsto p \circ \phi^1_{\mathcal{X}}(\xi),$$

on an open set $U \subset T^*M$ where the geodesic flow is defined up to time 1. By abuse of notation, we will write $\exp_{\mathcal{X}} : T^*M \to M$, as if it were defined on T^*M.

Poisson sprays exist on every Poisson manifold. For example, if ∇ is a connection on T^*M, then the map that associates to $\xi \in T^*M$ the horizontal lift of $\pi^\sharp(\xi)$ is a Poisson spray.

The main feature of Poisson sprays is that they produce symplectic realizations.

Theorem 19 [Crainic and Mărcuț 2011]. *Given (M, π) a Poisson manifold and $\mathcal{X}$ a Poisson spray, there exists an open neighborhood $\Sigma \subset T^*M$ of the zero section, on which the average of the canonical symplectic structure $\omega_{\mathrm{can}} \in \Omega^2(T^*M)$ under the geodesic flow*

$$\Omega_{\mathcal{X}} := \int_0^1 (\phi^t_{\mathcal{X}})^* \omega_{\mathrm{can}} \, dt, \tag{5}$$

is a symplectic structure on Σ, and the projection $p : (\Sigma, \Omega_{\mathcal{X}}) \to (M, \pi)$ is a symplectic realization (i.e., a surjective Poisson submersion).

Let $X \subset (M, \pi)$ be a Poisson transversal. As before, we denote by π_X the induced Poisson structure on X, and by $w_X := \pi|_{N^*X}$. We are ready to state the main result of this paper.

Theorem 20 (detailed version of Theorem 1). *Let (M, π) be a Poisson manifold and let $X \subset M$ be a Poisson transversal. A Poisson spray $\mathcal{X}$ induces a closed extension of $\sigma := -w_X$ in a neighborhood of X in N^*X, given by*

$$\widetilde{\sigma}_{\mathcal{X}} := -\Omega_{\mathcal{X}}|_{N^*X} \in \Upsilon(N^*X, \sigma).$$

The corresponding local model $\pi(\widetilde{\sigma}_{\mathcal{X}})$ is isomorphic to π around X. Explicitly, a Poisson diffeomorphism between open sets around X is given by the map

$$\exp_{\mathcal{X}}|_{N^*X} : (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}})) \xrightarrow{\sim} (M, \pi).$$

For the proof of Theorem 20, we need some properties of dual pairs. Recall from [Weinstein 1971]:

Definition 21. A *dual pair* consists of a symplectic manifold (Σ, Ω), two Poisson manifolds (M_0, π_0) and (M_1, π_1), and two Poisson submersions

$$(M_0, \pi_0) \xleftarrow{s} (\Sigma, \Omega) \xrightarrow{t} (M_1, \pi_1)$$

with symplectically orthogonal fibers:

$$\ker ds^{\Omega} = \ker dt.$$

The pair is called a *full dual pair*, if s and t are surjective.

Dual pairs and Poisson transversals interact pretty well, as the following shows:

Lemma 22. *Let* $(M_0, \pi_0) \xleftarrow{s} (\Sigma, \Omega) \xrightarrow{t} (M_1, \pi_1)$ *be a dual pair, and let* $X_0 \subset M_0$ *and* $X_1 \subset M_1$ *be Poisson transversals. Then* $\overline{\Sigma} := s^{-1}(X_0) \cap t^{-1}(X_1)$ *is a symplectic submanifold that fits into the dual pair*

$$(X_0, \pi_{X_0}) \xleftarrow{s} (\overline{\Sigma}, \Omega|_{\overline{\Sigma}}) \xrightarrow{t} (X_1, \pi_{X_1}).$$

Proof. First note that $\overline{\Sigma}$ is the inverse image of the Poisson transversal $X_0 \times X_1$ under the Poisson map

$$(s, t) : (\Sigma, \Omega) \to (M_0, \pi_0) \times (M_1, \pi_1).$$

By Lemma 7, (s, t) is transverse to $X_0 \times X_1$, $\overline{\Sigma}$ is a symplectic manifold and (s, t) restricts to a Poisson map

$$(s, t) : (\overline{\Sigma}, \Omega|_{\overline{\Sigma}}) \to (X_0, \pi_{X_0}) \times (X_1, \pi_{X_1}).$$

It remains to show that the maps

$$\overline{s} := s|_{\overline{\Sigma}} : \overline{\Sigma} \to X_0 \quad \text{and} \quad \overline{t} := t|_{\overline{\Sigma}} : \overline{\Sigma} \to X_1$$

are submersions with symplectically orthogonal fibers. Let $m_i := \dim(M_i)$ and $x_i := \dim(X_i)$. The fact that s and t are submersions with orthogonal fibers implies that $\dim(\Sigma) = m_0 + m_1$. By transversality of (s, t) and $X_0 \times X_1$, we have that $\operatorname{codim}(\overline{\Sigma}) = \operatorname{codim}(X_0 \times X_1)$; thus $\dim(\overline{\Sigma}) = x_0 + x_1$. Now, for a point $p \in \overline{\Sigma}$, one clearly has $\ker d_p\overline{t} \subset (\ker d_p\overline{s})^{\Omega|_{\overline{\Sigma}}}$, and since $\overline{\Sigma}$ is symplectic, it follows that

$$\dim(\ker d_p\overline{s}) + \dim(\ker d_p\overline{t}) \leq \dim(\overline{\Sigma}) = x_0 + x_1.$$

On the other hand, we have that $\dim(\ker d_p\overline{s}) \geq \dim(\overline{\Sigma}) - \dim(X_0) = x_1$, and similarly $\dim(\ker d_p\overline{t}) \geq x_0$, so we obtain $\dim(\ker d_p\overline{s}) = x_1$ and $\dim(\ker d_p\overline{t}) = x_0$. This implies that $d_p\overline{s}$ and $d_p\overline{t}$ are surjective, and that $\ker d_p\overline{s}$ and $\ker d_p\overline{t}$ are symplectically orthogonal. □

Lemma 23 shows how π_0, π_1 and Ω are related.

Lemma 23. *Let* $(M_0, \pi_0) \xleftarrow{s} (\Sigma, \Omega) \xrightarrow{t} (M_1, \pi_1)$ *be a dual pair. Then the Dirac structures* L_{π_i} *corresponding to* π_i *satisfy the following relation*:

$$s^*(L_{\pi_0})^{-\Omega} = t^*(L_{-\pi_1}).$$

Proof. An element $\chi \in s^*(L_{\pi_0})^{-\Omega}$ is of the form

$$\chi = Y + s^*\xi - \iota_Y \Omega, \quad \text{where } \xi \in T^*M_0,\ s_*Y = \pi_0^\sharp(\xi).$$

Then, since $s_*\Omega^{-1}(s^*\xi) = \pi_0^\sharp(\xi)$, we have that

$$Y - (\Omega^{-1})^\sharp(s^*\xi) \in \ker ds = (\Omega^{-1})^\sharp(t^*T^*M_1).$$

Hence there is $\eta \in T^*M_1$ such that

$$Y = (\Omega^{-1})^\sharp(s^*\xi) - (\Omega^{-1})^\sharp(t^*\eta).$$

Applying t_* and Ω (separately) to both sides, we find that

$$t_*Y = -t_*(\Omega^{-1})^\sharp(t^*\eta) = -\pi_1^\sharp(\eta) \quad \text{and} \quad s^*\xi - \iota_Y\Omega = t^*\eta,$$

and hence

$$\chi = Y + s^*\xi - \iota_Y\Omega = Y + t^*\eta \in t^*(L_{-\pi_1}).$$

This shows one inclusion; the other follows by symmetry. □

As a first step towards the proof of Theorem 20, we analyze what happens infinitesimally.

Lemma 24. *We have that $\widetilde{\sigma}_{\mathcal{X}}$ extends σ, $\widetilde{\sigma}_{\mathcal{X}} \in \Upsilon(N^*X, \sigma)$, and that $\exp_{\mathcal{X}}$ is a diffeomorphism between open sets around X.*

Proof. We identify the zero section of T^*M with M, and for $x \in M$, we identify $T_x(T^*M) = T_xM \oplus T_x^*M$. The properties of the Poisson spray imply that the geodesic flow fixes M, and that its differential along M is given (see [Crainic and Mărcuț 2011]) by

$$d_x\phi_{\mathcal{X}}^t : T_xM \oplus T_x^*M \to T_xM \oplus T_x^*M, \quad (Y,\xi) \mapsto (Y + t\pi^\sharp(\xi), \xi).$$

In particular, $\exp_{\mathcal{X}} = p \circ \phi_{\mathcal{X}}^1$ is a diffeomorphism around X, restricting to the identity along X, and the following formula for $\Omega_{\mathcal{X}}$ holds along M:

$$\Omega_{\mathcal{X}}((Y_1,\xi_1),(Y_2,\xi_2)) = \xi_2(Y_1) - \xi_1(Y_2) + \pi(\xi_1,\xi_2).$$

Taking $(Y_i,\xi_i) \in T_xX \oplus N_x^*X = T_x(N^*X)$, for $x \in X$, we obtain

$$\Omega_{\mathcal{X}}((Y_1,\xi_1),(Y_2,\xi_2)) = \pi(\xi_1,\xi_2) = w_X(\xi_1,\xi_2),$$

showing that $\widetilde{\sigma}_{\mathcal{X}} \in \Upsilon(N^*X, -w_X)$. □

Next, we observe that Theorem 19 implies the existence of self-dual pairs.

Lemma 25. *Let $\mathcal{X}$ be a Poisson spray on the Poisson manifold (M,π), and denote by $\Omega_{\mathcal{X}}$ the symplectic form from* Theorem 19. *On an open neighborhood of the zero* *section $\Sigma \subset T^*M$ we have a full dual pair*:

$$(M,\pi) \xleftarrow{p} (\Sigma, \Omega_{\mathcal{X}}) \xrightarrow{\exp_{\mathcal{X}}} (M,-\pi).$$

Proof. Let Σ be an open neighborhood of the zero section on which the geodesic flow $\phi^t_{\mathcal{X}}$ is defined for all $t \in [0,1]$, and on which $\Omega_{\mathcal{X}}$ is nondegenerate. In the proof of Theorem 19 from [Crainic and Mărcuț 2011] it is shown that the symplectic orthogonals of the fibers p are the fibers of $\exp_{\mathcal{X}}$. To show that $\exp_{\mathcal{X}}$ pushes $\Omega^{-1}_{\mathcal{X}}$ down to a bivector on M, one could invoke Libermann's theorem, and then, using the formulas from the proof of Lemma 24, one could check that along the zero section this bivector is indeed $-\pi$. We adopt a more direct approach. First, note that $-\mathcal{X}$ is a Poisson spray for $-\pi$, and that on $\Sigma_- := \phi^1_{\mathcal{X}}(\Sigma)$, the geodesic flow of $-\mathcal{X}$ is defined up to time 1. Moreover, $\Omega_{-\mathcal{X}}$ is nondegenerate on Σ_-, because

$$\begin{aligned}(\phi^1_{\mathcal{X}})^*\Omega_{-\mathcal{X}} &= \int_0^1 (\phi^1_{\mathcal{X}})^*(\phi^t_{-\mathcal{X}})^*\omega_{\mathrm{can}}\,dt = \int_0^1 (\phi^{1-t}_{\mathcal{X}})^*\omega_{\mathrm{can}}\,dt = \int_0^1 (\phi^t_{\mathcal{X}})^*\omega_{\mathrm{can}}\,dt\\ &= \Omega_{\mathcal{X}}.\end{aligned}$$

This also finishes the proof, since $\exp_{\mathcal{X}}$ is the composition of Poisson maps:

$$(\Sigma, \Omega_{\mathcal{X}}) \xrightarrow{\phi^1_{\mathcal{X}}} (\Sigma_-, \Omega_{-\mathcal{X}}) \xrightarrow{p} (M,-\pi). \qquad \square$$

We are ready to conclude the proof.

Proof of Theorem 20. We use the self-dual pair from Lemma 25, which, by abuse of notation, we write as if it were defined on the entire T^*M:

$$(M,\pi) \xleftarrow{p} (T^*M, \Omega_{\mathcal{X}}) \xrightarrow{\exp_{\mathcal{X}}} (M,-\pi).$$

Using Lemma 22, we take the preimage under $(p, \exp_{\mathcal{X}})$ of $X \times M$ to obtain a new dual pair (again, the maps are defined only around X),

$$(X,\pi_X) \xleftarrow{p} (T^*M|_X, \Omega_{\mathcal{X}}|_{T^*M|_X}) \xrightarrow{\exp_{\mathcal{X}}|_{T^*M|_X}} (M,-\pi).$$

By Lemma 23, we have the following equality of Dirac structures:

$$p^*(L_{\pi_X})^{-\Omega_{\mathcal{X}}|_{T^*M|_X}} = (\exp_{\mathcal{X}}|_{T^*M|_X})^*(L_\pi).$$

Since the left-hand side restricts along N^*X to the Dirac structure of the local model $\pi(\widetilde{\sigma}_{\mathcal{X}})$, we have

$$L_{\pi(\widetilde{\sigma}_{\mathcal{X}})} = (\exp_{\mathcal{X}}|_{N^*X})^*(L_\pi).$$

Since $\exp_{\mathcal{X}}|_{N^*X}$ is a diffeomorphism around X (Lemma 24), we see that it is a Poisson diffeomorphism around X:

$$\exp_{\mathcal{X}}|_{N^*X} : (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}})) \xrightarrow{\sim} (M,\pi). \qquad \square$$

5. Application: equivariant Weinstein splitting theorem

As an application of the normal form theorem (or rather of its proof), we obtain an equivariant version of Weinstein's splitting theorem around fixed points. A version of this result with extra assumptions was obtained in [Miranda and Zung 2006].

Theorem 26 (detailed version of Theorem 2). *Let (M, π) be a Poisson manifold and G a compact Lie group acting by Poisson diffeomorphisms on M. If $x \in M$ is a fixed point of G, then there are coordinates $(p_1, \dots, p_n, q_1, \dots, q_n, y_1, \dots, y_m) \in \mathbb{R}^{2n+m}$ centered at x such that*

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{j,k=1}^{m} \varpi_{j,k}(y) \frac{\partial}{\partial y_j} \wedge \frac{\partial}{\partial y_k}, \quad \varpi_{j,k}(0) = 0,$$

and in these coordinates G acts linearly and keeps the subspaces $\mathbb{R}^{2n} \times \{0\}$ and $\{0\} \times \mathbb{R}^m$ invariant.

In other words, (M, π) is G-equivariantly Poisson diffeomorphic around x to an open set around $(0, x)$ in the product

$$(T_x L, \omega_x^{-1}) \times (X, \pi_X), \tag{6}$$

where (L, ω) is the symplectic leaf through x, X is a G-invariant Poisson transversal of complementary dimension, and G acts diagonally on (6).

On equivariant symplectic trivializations. In the proof of Theorem 26 we will use a lemma on equivariant trivializations of symplectic vector bundles, which we present here. We start with a result about symplectic vector spaces:

Lemma 27. *Let (V, ω_0) be a symplectic vector space. There exist an open neighborhood $\mathcal{U}(\omega_0)$ of ω_0 in $\wedge^2 V^*$, invariant under the group $\mathrm{Sp}(V, \omega_0)$ of linear symplectomorphisms of ω_0, and a smooth map*

$$b : \mathcal{U}(\omega_0) \to \mathrm{Gl}(V), \quad \omega \mapsto b_\omega,$$

satisfying

$$b_\omega^*(\omega_0) = \omega, \quad b_{\omega_0} = \mathrm{id}, \quad s^{-1} \circ b_\omega \circ s = b_{s^*(\omega)},$$

for all $\omega \in \mathcal{U}(\omega_0)$ and all $s \in \mathrm{Sp}(V, \omega_0)$.

Proof. On the open set $\mathbb{O} := \mathbb{C} \setminus (-\infty, 0]$ consider the holomorphic square root,

$$\sqrt{(\cdot)} : \mathbb{O} \to \mathbb{C}, \quad \sqrt{e^{a+i\theta}} := e^{a/2+i\theta/2}, \quad a \in \mathbb{R},\ \theta \in (-\pi, \pi).$$

Denote the set of linear isomorphisms of V with eigenvalues in $\mathbb{O}$ by $\mathbb{O}(V) \subset \mathrm{Gl}(V)$. By holomorphic functional calculus [Wikipedia 2013], there is an "extension" of the square root to $\mathbb{O}(V)$, which satisfies

$$\left(\sqrt{x}\right)^2 = x,\quad \sqrt{x^{-1}} = \left(\sqrt{x}\right)^{-1},\quad \sqrt{(y\circ x\circ y^{-1})} = y\circ\sqrt{x}\circ y^{-1},\quad \sqrt{x^*} = \left(\sqrt{x}\right)^*,$$

for every $x \in \mathbb{O}(V)$ and every linear isomorphism $y : V \to W$.

Consider $\mathcal{U}(\omega_0) := \{\omega_0 \circ x \mid x \in \mathbb{O}(V)\}$, and define the map

$$b : \mathcal{U}(\omega_0) \to \mathrm{Gl}(V),\quad b_\omega := \sqrt{\omega_0^{-1}\circ\omega}.$$

Note that via the identification $\wedge^2 V^* \subset \mathrm{Hom}(V, V^*)$, the action of $\mathrm{Gl}(V)$ on $\wedge^2 V^*$ becomes $y^*(\omega) = y^* \circ \omega \circ y$. Let $\omega = \omega_0 \circ x \in \mathcal{U}(\omega_0)$, with $x \in \mathbb{O}(V)$ and $s \in \mathrm{Sp}(V, \omega_0)$. The following shows that $\mathcal{U}(\omega_0)$ is $\mathrm{Sp}(V, \omega_0)$-invariant:

$$s^*(\omega) = s^* \circ \omega_0 \circ x \circ s = (s^* \circ \omega_0 \circ s) \circ (s^{-1} \circ x \circ s) = \omega_0 \circ s^{-1} \circ x \circ s \in \mathcal{U}(\omega_0).$$

For the next condition, note first that

$$b_\omega^* = \left(\sqrt{\omega_0^{-1}\circ\omega}\right)^* = \sqrt{\omega\circ\omega_0^{-1}} = \omega_0 \circ b_\omega \circ \omega_0^{-1},$$

therefore

$$b_\omega^*(\omega_0) = b_\omega^* \circ \omega_0 \circ b_\omega = \omega_0 \circ b_\omega^2 = \omega.$$

Finally, for $s \in \mathrm{Sp}(V, \omega_0)$, we have that

$$\begin{aligned} s^{-1}\circ b_\omega \circ s &= \sqrt{s^{-1}\circ\omega_0^{-1}\circ\omega\circ s} = \sqrt{s^{-1}\circ\omega_0^{-1}\circ(s^*)^{-1}\circ s^*\circ\omega\circ s} \\ &= \sqrt{(s^*(\omega_0))^{-1}\circ s^*(\omega)} = b_{s^*(\omega)}. \end{aligned}$$ □

Remark 28. The lemma can also be proved using the Moser argument. First note that $\mathcal{U}(\omega_0)$ can be described as the set of 2-forms $\omega \in \wedge^2 V^*$ for which $\omega_t := t\omega_0 + (1-t)\omega$ is nondegenerate for all $t \in [0, 1]$. The 2-form $\omega - \omega_0$ has a canonical primitive given by $\eta := \frac{1}{2}\iota_\xi(\omega - \omega_0)$, where ξ is the Euler vector field of V. Let $X_t(\omega)$ be the time-dependent vector field defined by the equation $\iota_{X_t(\omega)}\omega_t = \eta$. The Moser argument shows that the time t flow of $X_t(\omega)$ pulls $t\omega_0 + (1-t)\omega$ to ω, and one can easily check that b_ω is the time-one flow of $X_t(\omega)$.

Lemma 29. *Let $(E, \sigma) \to X$ be a symplectic vector bundle, and let G be a compact group acting on E by symplectic vector bundle automorphisms. If $x \in X$ is a fixed point, there exist an invariant open set $U \subset X$ around x and a G-equivariant symplectic vector bundle isomorphism,*

$$(E, \sigma|_U) \xrightarrow{\sim} (E_x \times U, \sigma_x),$$

where the action of G on $E_x \times U$ is the product one.

Proof. We first construct a G-equivariant product decomposition. Let U be a G-invariant open set over which E trivializes, and fix a trivialization $E|_U \cong E_x \times U$. The action of G on $E_x \times U$ is of the form $g(e, y) = (\rho_y(g)e, gy)$. To make the action diagonal, we apply the vector bundle isomorphism,

$$\alpha : E_x \times U \xrightarrow{\sim} E_x \times U, \quad (e, y) \mapsto (A_y(e), y), \quad A_y := \int_G \rho_x(g)^{-1}\rho_y(g)d\mu(g),$$

where μ is the Haar measure on G. Note that A_y is a linear isomorphism for y near x, and that it satisfies

$$A_{gy} \circ \rho_y(g) = \rho_x(g) \circ A_y.$$

Thus, by shrinking U, we may assume that the action on $E_x \times U$ is the product action, which we simply denote by $g(e, y) = (ge, gy)$.

The symplectic structures are given by a smooth family $\{\sigma_y\}_{y \in U}$ of bilinear forms on E_x. This family is G-invariant, in the sense that it satisfies

$$\sigma_{gy} = (g^{-1})^*(\sigma_y), \quad g \in G, \ y \in U.$$

Consider the open set $\mathcal{U}(\sigma_x) \subset \wedge^2 E_x^*$ and the map $b : \mathcal{U}(\sigma_x) \to \mathrm{Gl}(E_x)$ from the previous lemma. By shrinking U, we may assume that $\sigma_y \in \mathcal{U}(\sigma_x)$, for all $y \in U$. Since $b_{\sigma_y}^*(\sigma_x) = \sigma_y$, we have a "canonical" symplectic trivialization:

$$\beta : E_x \times U \xrightarrow{\sim} E_x \times U, \quad (e, y) \mapsto (b_{\sigma_y} e, y).$$

Now $g^{-1} : E_x \to E_x$ preserves σ_x, so

$$b_{\sigma_{gy}} = b_{(g^{-1})^*\sigma_y} = g \circ b_{\sigma_y} \circ g^{-1}.$$

Equivalently, the map β is G-equivariant:

$$\beta(ge, gy) = (b_{\sigma_{gy}} ge, gy) = (g b_{\sigma_y} e, gy) = g\beta(e, y).$$

Thus, $\beta \circ \alpha$ is an isomorphism of symplectic vector bundles that trivializes the symplectic structure, and turns the G-action into the product one. □

Proof of Theorem 26. We split the proof into four steps.

<u>Step 1: a G-invariant transversal</u>. Let (L, ω) denote the leaf through x. Since x is a fixed point, it follows that G preserves L. Thus G acts by symplectomorphisms on (L, ω).

We fix $X \subset M$, a G-invariant transversal through x such that $\dim(L)+\dim(X) = \dim(M)$. The existence of such a transversal follows from Bochner's linearization theorem: the action around x is isomorphic to the linear action of G on $T_x M$; by choosing a G-invariant inner product on $T_x M$, we let X be an invariant ball around the origin in the orthogonal complement of $T_x L$.

Let $\pi|_X = \pi_X + w_X$ denote the decomposition of π along X. Then G acts by Poisson diffeomorphisms on (X, π_X), and by symplectic vector bundle automorphisms on $(N^*X, -w_X)$.

Step 2: the G-invariant spray. Let $\mathcal{X}$ be a G-invariant Poisson spray. Such a vector field can be constructed by averaging any Poisson spray; the conditions that a vector field on T^*M be a Poisson spray are affine. The flow of $\mathcal{X}$ is therefore G-equivariant. By Theorem 20, and with the notations used there, we obtain a G-equivariant Poisson diffeomorphism around X,

$$\exp_{\mathcal{X}} : (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}})) \to (M, \pi),$$

where $\widetilde{\sigma}_{\mathcal{X}} \in \Upsilon(N^*X, -w_X)$ is automatically G-invariant.

Step 3: a G-equivariant symplectic trivialization. Note first that w_X, regarded as a map $N^*X \to TM|_X$, yields a symplectic isomorphism,

$$w_{X,x} : (N_x^*X, -w_{X,x}) \xrightarrow{\sim} (T_xL, \omega_x).$$

This remark and Lemma 29 imply that around the fixed point x, by shrinking X if necessary, we can simultaneously trivialize the bundle $(N^*X, -w_X)$ symplectically and turn the action to a product action, hence, we obtain a G-equivariant symplectic vector bundle isomorphism

$$\Psi : \big(\mathrm{pr}_2 : (T_xL, \omega_x) \times X \to X\big) \xrightarrow{\sim} \big(p : (N^*X, -w_X) \to X\big),$$

where the action on $T_xL \times X$ is the product action. Therefore, $\widetilde{\omega}_{\mathcal{X}} := \Psi^*(\widetilde{\sigma}_{\mathcal{X}})$ is a closed G-invariant extension of ω_x, i.e., $\widetilde{\omega}_{\mathcal{X}} \in \Upsilon(T_xL \times X, \omega_x)$. Moreover, the map

$$\Psi : (T_xL \times X, \pi(\widetilde{\omega}_{\mathcal{X}})) \xrightarrow{\sim} (N^*X, \pi(\widetilde{\sigma}_{\mathcal{X}}))$$

is a G-equivariant Poisson diffeomorphism, where $\pi(\widetilde{\omega}_{\mathcal{X}})$ denotes the Poisson structure around X corresponding to the Dirac structure $\mathrm{pr}_2^*(L_{\pi_X})^{\widetilde{\sigma}_{\mathcal{X}}}$.

Step 4: the G-equivariant Moser argument. Note that ω_x has a second extension to $T_xL \times X$ given by $\overline{\omega}_x := \mathrm{pr}_1^*(\omega_x)$. The corresponding local model is the Poisson structure from the statement

$$(T_xL \times X, \pi(\overline{\omega}_x)) = (T_xL, \omega_x^{-1}) \times (X, \pi_X).$$

By Steps 2 and 3, we are left to find a G-equivariant diffeomorphism around X that sends $\pi(\widetilde{\omega}_{\mathcal{X}})$ to $\pi(\overline{\omega}_x)$. For this we need the equivariant version of Lemma 17, whose proof can be easily adapted to this setting: first, note that the 2-form $\overline{\omega}_x - \widetilde{\omega}_{\mathcal{X}}$ has a primitive

$$\eta \in \Omega^1(T_xL \times X)$$

such that $\eta_{(0,y)} = 0$ for all $y \in X$. Since both $\overline{\omega}_X$ and $\widetilde{\omega}_{\mathcal{X}}$ are G-invariant, by averaging, we can make η G-invariant as well. Consider the time-dependent vector field,

$$Y_t := -\pi_t^{\sharp}(\eta),$$

where $\pi_t := \pi(\widetilde{\omega}_{\mathcal{X}})^{t\,d\eta}$. The time-one flow $\phi_Y^{1,0}$ sends $\pi_0 = \pi(\widetilde{\omega}_{\mathcal{X}})$ to $\pi_1 = \pi(\overline{\omega}_X)$. Since both π_t and η are G-invariant, it follows that $\phi_Y^{1,0}$ is G-equivariant as well. This concludes the proof. $\square$

Acknowledgements

We would like to thank Marius Crainic for useful discussions. Frejlich was supported by the NWO Vrije Competitie project "Flexibility and Rigidity of Geometric Structures" no. 612.001.101 and IMPA (CAPES-FORTAL project), and Mărcuţ by the ERC Starting Grant no. 279729.

References

[Alekseev and Meinrenken 2007] A. Alekseev and E. Meinrenken, "Ginzburg–Weinstein via Gelfand–Zeitlin", *J. Differential Geom.* **76**:1 (2007), 1–34. MR Zbl

[Alekseev and Meinrenken 2016] A. Alekseev and E. Meinrenken, "Linearization of Poisson Lie group structures", *J. Symplectic Geom.* **14**:1 (2016), 227–267. MR Zbl

[Brahic and Fernandes 2008] O. Brahic and R. L. Fernandes, "Poisson fibrations and fibered symplectic groupoids", pp. 41–59 in *Poisson geometry in mathematics and physics*, edited by G. Dito et al., Contemp. Math. **450**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Bursztyn and Radko 2003] H. Bursztyn and O. Radko, "Gauge equivalence of Dirac structures and symplectic groupoids", *Ann. Inst. Fourier* (*Grenoble*) **53**:1 (2003), 309–337. MR Zbl

[Cattaneo and Zambon 2009] A. S. Cattaneo and M. Zambon, "Coisotropic embeddings in Poisson manifolds", *Trans. Amer. Math. Soc.* **361**:7 (2009), 3721–3746. MR Zbl

[Crainic and Fernandes 2004] M. Crainic and R. L. Fernandes, "Integrability of Poisson brackets", *J. Differential Geom.* **66**:1 (2004), 71–137. MR Zbl

[Crainic and Mărcuţ 2011] M. Crainic and I. Mărcuţ, "On the existence of symplectic realizations", *J. Symplectic Geom.* **9**:4 (2011), 435–444. MR Zbl

[Dirac 1950] P. A. M. Dirac, "Generalized Hamiltonian dynamics", *Canadian J. Math.* **2** (1950), 129–148. MR Zbl

[Dufour and Zung 2005] J.-P. Dufour and N. T. Zung, *Poisson structures and their normal forms*, Progress in Mathematics **242**, Birkhäuser, Basel, 2005. MR Zbl

[Gromov 1986] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik (3) **9**, Springer, Berlin, 1986. MR Zbl

[Haefliger 1958] A. Haefliger, "Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes", *Comment. Math. Helv.* **32** (1958), 248–329. MR Zbl

[Miranda and Zung 2006] E. Miranda and N. T. Zung, "A note on equivariant normal forms of Poisson structures", *Math. Res. Lett.* **13**:5-6 (2006), 1001–1012. MR Zbl

[Vorobjev 2001] Y. Vorobjev, "Coupling tensors and Poisson geometry near a single symplectic leaf", pp. 249–274 in *Lie algebroids and related topics in differential geometry* (Warsaw, 2000), edited by J. Kubarski et al., Banach Center Publ. **54**, Polish Acad. Sci. Inst. Math., Warsaw, 2001. MR Zbl

[Weinstein 1971] A. Weinstein, "Symplectic manifolds and their Lagrangian submanifolds", *Advances in Math.* **6** (1971), 329–346. MR Zbl

[Weinstein 1977] A. Weinstein, *Lectures on symplectic manifolds*, C.B.M.S. Regional Conference Series in Math. **29**, American Mathematical Society, Providence, RI, 1977. MR Zbl

[Weinstein 1983] A. Weinstein, "The local structure of Poisson manifolds", *J. Differential Geom.* **18**:3 (1983), 523–557. MR Zbl

[Wikipedia 2013] "Holomorphic functional calculus", Wikipedia entry, 2013, http://tinyurl.com/wiki-holomorphic.

[Xu 2003] P. Xu, "Dirac submanifolds and Poisson involutions", *Ann. Sci. École Norm. Sup.* (4) **36**:3 (2003), 403–430. MR Zbl

[Zambon 2011] M. Zambon, "Submanifolds in Poisson geometry: a survey", pp. 403–420 in *Complex and differential geometry*, edited by W. Ebeling et al., Springer Proc. Math. **8**, Springer, Berlin, 2011. MR Zbl

Received January 18, 2016. Revised July 11, 2016.

PEDRO FREJLICH
DEPARTAMENTO DE MATEMÁTICA
PUC RIO DE JANEIRO
RUA MARQUÊS DE SÃO VICENTE 225
GÁVEA
22451-900 RIO DE JANEIRO
BRAZIL
frejlich.math@gmail.com

IOAN MĂRCUŢ
INSTITUTE FOR MATHEMATICS, ASTROPHYSICS AND PARTICLE PHYSICS
RADBOUD UNIVERSITY NIJMEGEN
6500 GL
NIJMEGEN
THE NETHERLANDS
i.marcut@math.ru.nl

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.393

SOME CLOSURE RESULTS FOR $\mathcal{C}$-APPROXIMABLE GROUPS

DEREK F. HOLT AND SARAH REES

We investigate closure results for $\mathcal{C}$-approximable groups, for certain classes $\mathcal{C}$, of groups with invariant length functions. In particular we prove, each time for certain (but not necessarily the same) classes $\mathcal{C}$ that: (i) the direct product of two $\mathcal{C}$-approximable groups is $\mathcal{C}$-approximable; (ii) the restricted standard wreath product $G \wr H$ is $\mathcal{C}$-approximable when G is $\mathcal{C}$-approximable and H is residually finite; and (iii) a group G with normal subgroup N is $\mathcal{C}$-approximable when N is $\mathcal{C}$-approximable and G/N is amenable. Our direct product result is valid for LEF, weakly sofic and hyperlinear groups, as well as for all groups that are approximable by finite groups equipped with commutator-contractive invariant length functions (considered by A. Thom). Our wreath product result is valid for weakly sofic groups, and we prove it separately for sofic groups. This last result has recently been generalised by Hayes and Sale, who proved that the restricted standard wreath product of any two sofic groups is sofic. Our result on extensions by amenable groups is valid for weakly sofic groups, and was proved by Elek and Szabó (2006) for sofic groups N.

1. Introduction

Our interest in $\mathcal{C}$-approximable groups stems from the fact that, by making an appropriate choice of the class $\mathcal{C}$, the definition of a $\mathcal{C}$-approximable group equates to that of one of a variety of classes of groups currently of interest, including sofic groups, hyperlinear groups, weakly sofic groups, linear sofic groups, and LEF groups. Hence techniques that apply to one such class can often be applied to another. In this article we develop some general techniques to establish some closure properties for many of these classes, specifically for direct products, for wreath products with residually finite groups, and for extensions by amenable groups. We shall refer to closure results in the literature, mostly for specific classes of $\mathcal{C}$-approximable groups; in some cases our proofs have been inspired by the proofs of those. We are grateful to the anonymous referee of the paper for a careful reading and several helpful comments and corrections.

MSC2010: primary 20F65; secondary 20E22.
Keywords: C-approximable group, sofic, hyperlinear, weakly sofic, linear sofic.

Our definition of a $\mathcal{C}$-approximable group is taken from [Thom 2012, Definition 1.6] and specialises to the definitions of sofic and hyperlinear groups in [Capraro and Lupini 2015]; we shall discuss some of the alternative definitions later on in this section. Our definition requires the concept of an *invariant length function* on a group K; that is, a map $\ell : K \to [0, 1]$ such that, for all $x, y \in K$:

$$\ell(x) = 0 \Longleftrightarrow x = 1, \qquad \ell(x^{-1}) = \ell(x),$$
$$\ell(xy) \leq \ell(x) + \ell(y), \quad \ell(xyx^{-1}) = \ell(y).$$

Every group admits the trivial length function ℓ_0 defined by $\ell_0(x) = 1$ if $x \neq 1$, $\ell_0(1) = 0$, and may admit many others. The Hamming norm, which computes the proportion of points moved by a permutation of a finite set, gives an invariant length function for finite symmetric groups.

In the following definition $\mathcal{C}$ is understood to be a set of pairs, each pair consisting of a group K together with an invariant length function ℓ_K on K; so the same group may occur in $\mathcal{C}$ with more than one length function. For a group K, the statement $K \in \mathcal{C}$ means that K is the group in at least one such pair.

Definition 1.1. (1) For a group G, a map $\delta : G \to \mathbb{R}$ (for which we write δ_g rather than $\delta(g)$) is a *weight function for* G if $\delta_1 = 0$ and $\delta_g > 0$ for all $1 \neq g \in G$.

(2) Let G be a group with weight function δ, let K be a group with invariant length function ℓ_K, let $\epsilon > 0$, and let F be a finite subset of G. Then the map $\phi : G \to K$ is an $(F, \epsilon, \delta, \ell_K)$-*quasihomomorphism* if
- $\phi(1) = 1$,
- $\forall g, h \in F$, $\ell_K(\phi(gh)\phi(h)^{-1}\phi(g)^{-1}) \leq \epsilon$, and
- $\forall g \in F \setminus \{1\}$, $\ell_K(\phi(g)) \geq \delta_g$.

(3) Let $\mathcal{C}$ be a class of groups with associated invariant length functions. Then a group G is $\mathcal{C}$-*approximable* if it has a weight function δ, such that, for each $\epsilon > 0$ and for each finite subset F of G, there exists an $(F, \epsilon, \delta, \ell_K)$-quasihomomorphism $\phi : G \to K$ for some $(K, \ell_K) \in \mathcal{C}$.

Since these conditions cannot possibly be satisfied if $\delta_g > 1$ for some $g \in G$, we shall always assume that $\delta_g \leq 1$.

In particular, sofic groups are precisely those groups that are $\mathcal{C}$-approximable with respect to the class $\mathcal{C}$ of finite symmetric groups with length function defined by the Hamming norms, and with weight functions of the form $\delta_g = c$ for all $1 \neq g \in G$, for some fixed constant $c > 0$; see [Pestov and Kwiatkowska 2009, Theorem 5.2].

The (normalised) Hilbert–Schmidt norm on the set of $n \times n$ complex matrices $A = (a_{ij})$ is defined by

$$\|(a_{ij})\|_{\mathrm{HS}_n} := \sqrt{\tfrac{1}{n}\textstyle\sum_{i,j}|a_{ij}|^2} = \sqrt{\tfrac{1}{n}\mathrm{Tr}(A^*A)}.$$

The hyperlinear groups are precisely those groups that are $\mathcal{C}$-approximable with respect to the class $\mathcal{C}$ of finite-dimensional unitary groups with length function defined by $\ell(g) = \frac{1}{2}\|g - I_n\|_{\mathrm{HS}_n}$, and with the same weight functions as for sofic groups; see [Pestov and Kwiatkowska 2009, Theorem 4.2]. Furthermore, weakly sofic groups, linear sofic groups and LEF groups can all be defined as $\mathcal{C}$-approximable groups, where the classes $\mathcal{C}$ are (respectively) the class $\mathcal{F}$ of all finite groups equipped with all associated invariant length functions, the groups $\mathrm{GL}_n(\mathbb{C})$ equipped with the norm $\ell(g) = \frac{1}{n}\mathrm{rk}(I_n - g)$ [Arzhantseva and Păunescu 2017], and the finite groups equipped with the trivial length function. We refer the reader to [Arzhantseva and Gal 2013; Ciobanu et al. 2014; Elek and Szabó 2006; 2011; Păunescu 2011; Stolz 2013] for a number of closure results involving various of these classes of groups.

Following [Thom 2012] we say that an invariant length function $\ell : K \to [0, 1]$ is *commutator-contractive* if it satisfies the condition

$$\ell([x, y]) \leq 4\ell(x)\ell(y) \quad \forall x, y \in K.$$

Note that the trivial length function is commutator-contractive. Let $\mathcal{F}_C$ be the class of all finite groups, each equipped with all commutator-contractive length functions. The main result of [Thom 2012] is that Higman's group [1951] is not $\mathcal{F}_C$-approximable. This group is widely seen as a candidate for a first example of a nonsofic group.

There are many variations in the literature of the definition of a $\mathcal{C}$-approximable group, not all of which are believed to be equivalent in general to our basic definition, although the paucity of known examples of groups that are not $\mathcal{C}$-approximable makes it difficult to prove their inequivalence.

Some definitions, such as [Glebsky 2015, Definition 2] and [Stolz 2013, §2] allow invariant length functions to take values in $[0, \infty)$ rather than in $[0, 1]$. This does not affect the classes of sofic, hyperlinear, linear sofic and LEF groups, since the length functions used in these classes all have range $[0, 1]$. It is also easily seen that the class of weakly sofic groups is not changed by this variant since, if a group is weakly sofic using length functions with range $[0, \infty)$, and ℓ_K is such a length function on a finite group K, then simply by replacing $\ell_K(g)$ by the new length function $\max(\ell_K(g), 1)$, we can show that G is weakly sofic using length functions with range $[0, 1]$. So this variation in the range of permissible length functions does not appear to us to be significant.

The more substantial variants involve the condition

$$\forall g \in F,\ \ \ell_K(\phi(g)) \geq \delta_g$$

in the definition of $\mathcal{C}$-approximability. These are discussed in Section 2 of [Stolz 2013]. The group G is said to have the *discrete $\mathcal{C}$-approximation property* if the weight function for G can be chosen to be constant on all nonidentity elements. It is said to have the *strong discrete $\mathcal{C}$-approximation property* if the condition above is replaced by

$$\forall g \in F,\ \ \ell_K(\phi(g)) \geq \mathrm{diam}(K) - \epsilon,$$

where $\mathrm{diam}(K)$ is defined to be $\sup\{\ell_K(x) : x \in K\}$, and ϵ is as in Definition 1.1(3). By choosing the weight function $\delta_g = \mathrm{diam}(G)/2$ for all $g \in G \setminus \{1\}$, we see immediately that the strong discrete $\mathcal{C}$-approximation property implies the discrete $\mathcal{C}$-approximation property, which clearly implies that G is $\mathcal{C}$-approximable using our definition. But the converse implications are not clear, and may not hold in general.

The definition given for sofic groups in [Elek and Szabó 2006] enforces the strong discrete approximation property. But it is shown in [Capraro and Lupini 2015, Exercise II.1.8] that, for this class, any $\mathcal{C}$-approximable group has the strong discrete $\mathcal{C}$-approximation property.

It is proved in [Arzhantseva and Păunescu 2017, Proposition 5.13] that linearly sofic groups have the discrete $\mathcal{C}$-approximation property, but it appears to be unknown whether they have the strong discrete $\mathcal{C}$-approximation property.

Hyperlinear groups do not have the strong $\mathcal{C}$-approximation property, and we are grateful to the referee for pointing this out to us. The diameter of the unitary group $\mathcal{U}(n)$ with length function defined as above by $\ell(g) = \frac{1}{2}\|g - I_n\|_{\mathrm{HS}_n}$ is 1. By using the identity

$$\|g - h\|^2_{\mathrm{HS}_n} + \|g + h\|^2_{\mathrm{HS}_n} = 4$$

for $g, h \in \mathcal{U}(n)$ and putting $h = I_n$, we see that, if $1 - \ell(g)$ is small, then g is close to $-I_n$ with respect to the Hilbert–Schmidt metric. So if $1 - \ell(g_1)$ and $1 - \ell(g_2)$ are both small, then g_1g_2 is close to I_n and hence $\ell(g_1g_2)$ is close to 0. It follows that a hyperlinear group with the strong discrete $\mathcal{C}$-approximation property must be finite with order at most 2.

For hyperlinear groups, it is true that, for any finite $F \subseteq G$ and $\epsilon > 0$, there exists an approximately multiplicative map $\phi : G \to \mathcal{U}(n)$ for which $|\mathrm{Tr}(\phi(g))/n| < \epsilon$ for all $g \in F \setminus \{1\}$. This was first proved in [Elek and Szabó 2005] using ideas introduced in [Rădulescu 2008].

It is not difficult to show that the classes of $\mathcal{F}$-approximable (i.e., weakly sofic) and $\mathcal{F}_C$-approximable groups both have the strong discrete $\mathcal{C}$-approximation property. For a finite subset F of a group G in one of these two classes, and $\epsilon > 0$, let $c = \min\{\delta_g : g \in F\}$, and let $\phi : G \to K$ be an $(F, c\epsilon, \delta, \ell_K)$-quasihomomorphism. Then, by replacing ℓ_K by the length function $\ell'_K(x) := \min(\ell_K(x)/c, 1)$, which is commutator-contractive if ℓ_K is, we see that ϕ is an $(F, \epsilon, \delta, \ell'_K)$-quasihomomorphism for which $\ell'_K(\phi(g)) = 1$ for all $g \in F$, so G has the strong discrete $\mathcal{C}$-approximation property.

We prove our closure results for direct products, wreath products, and extensions by amenable groups in Sections 2, 3 and 4, and 5, respectively. To prove the last of these, on extensions of $\mathcal{C}$-approximable groups N by amenable groups, we need to assume that the group N has the discrete $\mathcal{C}$-approximation property. For each of our closure results, it is straightforward to show that, if the groups that are assumed to be $\mathcal{C}$-approximable have the discrete or the strong discrete $\mathcal{C}$-approximation property, then so does the group G that is proved to be $\mathcal{C}$-approximable.

Concerning free products, we note that it is proved in [Elek and Szabó 2006, Theorem 1], [Stolz 2013, Theorem 5.6] and [Popa 1995; Voiculescu 1998], respectively, that the classes of sofic, linear sofic, and hyperlinear groups are closed under free products; further it is proved in [Brown et al. 2008] that free products of hyperlinear groups amalgamated over amenable subgroups are hyperlinear. We thank the referee for bringing to our attention the results for hyperlinear groups. We are unaware of any corresponding results for weakly sofic groups, and our efforts to prove such a result have so far been unsuccessful.

2. The direct product result

In order to state and prove our closure result for direct products of $\mathcal{C}$-approximable groups, we must construct an appropriate invariant length function for the direct product of two groups in $\mathcal{C}$. Suppose that $(J, \ell_J), (K, \ell_K) \in \mathcal{C}$. Then, for $p \in \mathbb{N} \cup \{\infty\}$, we define the functions $L^p_{\ell_J,\ell_K} : J \times K \to [0, 1]$ by

$$L^p_{\ell_J,\ell_K}(x, y) := \sqrt[p]{\tfrac{1}{2}(\ell_J(x)^p + \ell_K(y)^p)}, \quad p \in \mathbb{N},$$

and $L^\infty_{\ell_J,\ell_K}(x, y) := \max(\ell_J(x), \ell_K(y))$. We write just $L^p(x, y)$ when there is no ambiguity.

Note that $L^p(x, y) \le L^\infty(x, y) \le 1$ for all $p \ge 1$.

It follows immediately from Minkowski's inequality (basically the triangle inequality for the L^p norm) that L^p satisfies the rule

$$L^p(x_1x_2, y_1y_2) \le L^p(x_1, y_1) + L^p(x_2, y_2),$$

and hence is an invariant length function on $J \times K$. As we shall see below, we can use L^p (for some choice of p) to deduce the closure of $\mathcal{C}$-approximable groups under direct products provided that $(J \times K, L^p) \in \mathcal{C}$.

Theorem 2.1. *Let $\mathcal{C}$ be a class of groups with associated invariant length functions and suppose that, for some fixed $p \in \mathbb{N} \cup \{\infty\}$, and for any groups $J, K \in \mathcal{C}$,*

$$(J, \ell_J), (K, \ell_K) \in \mathcal{C} \Rightarrow (J \times K, L^p) \in \mathcal{C}.$$

Then the direct product $G \times H$ of two $\mathcal{C}$-approximable groups G and H is also $\mathcal{C}$-approximable.

Proof. Suppose that $\mathcal{C}$, p satisfy the conditions of the theorem.

Let G and H be $\mathcal{C}$-approximable with associated weight functions δ^G and δ^H. We define the weight function $\delta^{G\times H}$ by

$$\delta^{G\times H}(g,h) := \sqrt[p]{\tfrac{1}{2}(\delta^G(g)^p + \delta^H(h)^p)}.$$

Now suppose that $\epsilon > 0$ is given, and let F be a finite subset of $G \times H$. Then we can find finite subsets $F_G \subseteq G$, $F_H \subseteq H$ such that $F \subseteq F_G \times F_H$, pairs $(J, \ell_J), (K, \ell_K) \in \mathcal{C}$, an $(F_G, \epsilon, \delta^G, \ell_J)$-quasihomomorphism $\phi_G : G \to J$, and an $(F_H, \epsilon, \delta^H, \ell_K)$-quasihomomorphism $\phi_H : H \to K$.

We define $\phi : G \times H \to M := J \times K$ by $\phi(g,h) := (\phi_G(g), \phi_H(h))$ and $\ell_M(x,y) := L^p(x,y)$.

We verify easily that, for $(g_1,h_1), (g_2,h_2) \in F$, and hence $g_1, g_2 \in F_G$ and $g_2, h_2 \in F_H$,

$$\begin{aligned}&\ell_M(\phi(g_1g_2, h_1h_2)\phi(g_2,h_2)^{-1}\phi(g_1,h_1)^{-1})\\ &\qquad = L^p(\phi_G(g_1g_2)\phi_G(g_2)^{-1}\phi_G(g_1)^{-1}, \phi_H(h_1h_2)\phi_H(h_2)^{-1}\phi_H(h_1)^{-1}) \le \epsilon,\end{aligned}$$

and the other conditions are similarly verified. □

We can apply the result to deduce closure under direct products for the classes of weakly sofic groups, LEF groups, hyperlinear groups, linear sofic groups and Thom's class [2012] of $\mathcal{F}_C$-approximable groups.

For weakly sofic groups, the condition holds for any p, and for LEF groups it holds for $p = \infty$.

When ℓ_J, ℓ_K are Hilbert–Schmidt norms in the same dimension n, the function L^2 matches the Hilbert–Schmidt norm in dimension $2n$; observing that whenever G maps by a quasihomomorphism to a linear group in dimension m it also maps to a linear group in dimension rm, for any r, via a quasihomomorphism with the same parameters (the composite of the original quasihomomorphism and a diagonal map), we see that in essence the theorem applies with $p = 2$ to prove closure under direct products for the class of hyperlinear groups. Similarly it applies when $p = 1$ to prove closure under direct products for the class of linear sofic groups.

But for Hamming norms ℓ_J, ℓ_K, the function $L^p_{\ell_J,\ell_K}$ is not a Hamming norm, and hence we cannot deduce the closure of the class of sofic groups under direct products from this result.

Of course all of these specific closure results are already known, and the corresponding result for sofic groups is proved in [Elek and Szabó 2006].

The following lemma together with Theorem 2.1 shows that the class of $\mathcal{F}_C$-approximable groups is closed under direct products.

Lemma 2.2. *Suppose that the groups J, K have commutator-contractive length functions $\ell_J : J \to [0, 1]$, $\ell_K : K \to [0, 1]$. Then L^∞, as defined above, is a commutator-contractive length function for their direct product.*

Proof. Let $(g_1, h_1), (g_2, h_2) \in G \times H$. Then

$$\begin{aligned} L^\infty([(g_1, h_1), (g_2, h_2)]) &= L^\infty([g_1, g_2], [h_1, h_2]) \\ &= \max(l_J([g_1, g_2]), l_K([h_1, h_2])) \\ &\le \max(4l_J(g_1)l_J(g_2), 4l_K(h_1)l_K(h_2)) \\ &\le 4\max(l_J(g_1), l_K(h_1)) \max(l_J(g_2), l_K(h_2)) \\ &= 4L^\infty(g_1, h_1)L^\infty(g_2, h_2). \end{aligned}$$ □

This result does not hold in general for L^p with $p \in [1, \infty)$.

3. The wreath product result

By definition, the restricted standard wreath product $W = G \wr H$ of two groups G, H is a semidirect product $H \ltimes B$. The *base group* B of W is the direct product of copies of G, one for each $h \in H$, and is viewed as the set of all functions $b : H \to G$ with finite support (that is, with $b(h)$ trivial for all but finitely many $h \in H$). Elements of B are multiplied componentwise; that is, $b_1b_2(h) = b_1(h)b_2(h)$ for $b_1, b_2 \in B$, $h \in H$. For $b \in B$, we denote by b^{-1} the function in B defined by $b^{-1}(h) = b(h)^{-1}$. The (right) action of H on B is defined by the rule $b^h(h') = b(h'h^{-1})$; we often abbreviate $(b^h)^{-1} = (b^{-1})^h$ as b^{-h}. So the elements of W have the form hb with $h \in H$, $b \in B$, and $(h_1b_1)(h_2b_2) = h_1h_2b_1^{h_2}b_2$, while $(h, b)^{-1} = (h^{-1}, b^{-h^{-1}})$.

To let us state and prove our closure result for wreath products of $\mathcal{C}$-approximable groups, we need to construct an appropriate invariant length function for the wreath product $J \wr X$ of a group $J \in \mathcal{C}$ by a finite group X.

Where B' is the base group of $J \wr X$, we define $\ell_J^X : J \wr X \to [0, 1]$ as follows. For $b' \in B'$, we put

$$\ell_L^X(b') = \max_{x \in X} \ell_J(b'(x)),$$

and then, for $x \neq 1$, put

$$\ell_J^X(xb') = 1.$$

It is straightforward to verify that ℓ_J^X is an invariant length function.

Theorem 3.1. *Let $\mathcal{C}$ be a class of groups with associated invariant length functions and suppose that, for all $(J, \ell_J) \in \mathcal{C}$ and all finite groups X, the wreath product $(J \wr X, \ell_J^X)$ is in $\mathcal{C}$. Suppose the group G is $\mathcal{C}$-approximable and the group H is residually finite. Then the restricted standard wreath product $G \wr H$ is $\mathcal{C}$-approximable.*

Proof. Suppose that G is $\mathcal{C}$-approximable with associated weight function δ, and that H is residually finite, and let $W = G \wr H$ be the restricted standard wreath product. Let B be the base group.

We define the weight function $\beta : W \to \mathbb{R}$ as follows:

$$\beta_{hb} = \begin{cases} 1 & \text{if } h \neq 1, \\ \max_{k \in H} \delta_{b(k)} & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$ be given, and let $F = \{h_i b_i : 1 \leq i \leq r\}$ be a finite subset of W. Our aim is to find $(K, \ell_K) \in \mathcal{C}$ and an $(F, \epsilon, \beta_W, \ell_K)$-quasihomomorphism $\psi : W \to K$.

Let E be a finite subset of H that contains

(i) h_i for $1 \leq i \leq r$;

(ii) all $h \in H$ with $b_j(h) \neq 1$ for some j with $1 \leq j \leq r$; and

(iii) all $h \in H$ with $b_j(hh_i^{-1}) \neq 1$ for some i, j with $1 \leq i \leq r,\ 1 \leq j \leq r$.

Choose $N \trianglelefteq H$ with H/N finite such that the images in H/N of the elements of E are all distinct and the images of $E \setminus \{1\}$ are nontrivial.

Let $D = \{b_j(h) : 1 \leq j \leq r,\ h \in H\}$. Then D is a finite subset of G so, by our definition of $\mathcal{C}$-approximability, for a given $\epsilon > 0$, there exists $(J, \ell_J) \in \mathcal{C}$, and a $(D, \epsilon, \delta, \ell_J)$-quasihomomorphism $\phi : G \to J$.

We will approximate W by $K := J \wr (H/N)$, and let ℓ_K be the length function $\ell_J^{H/N}$ defined above. Let B' be the base group of K, that is, the group of finitely supported functions from H/N to J.

We define $\psi : W \to K$ as follows. Suppose that $b \in B$, and $h, k \in H$. Note that our choice of N ensures that $E \cap kN$ is either empty or consists of a single element $k' \in kN$. We let $\psi(hb) := \bar{h}\hat{b}$, where we write $\bar{h}$ for hN and $\hat{b} : H/N \to J$ is defined by the rule

$$\hat{b}(kN) = \begin{cases} 1 & \text{when } E \cap kN = \varnothing, \\ \phi(b(k')) & \text{when } E \cap kN = \{k'\}. \end{cases}$$

We claim that ψ has the appropriate properties. Certainly $\psi(1) = 1$.

We first verify the required lower bound on $\ell_K(\psi(hb))$ for elements $hb \in F$. If $h \neq 1$ then our choice of N ensures that $\bar{h} \neq 1$, and so $\ell_K(\psi(hb)) = 1 = \beta_{hb}$.

If $h = 1$, then (where the maximum of an empty set of numbers in $[0, 1]$ is defined to be 0),

$$\begin{aligned} \ell_K(\psi(hb)) &= \ell_K(\psi(b)) = \ell_K(\hat{b}) \\ &= \max_{kN \in H/N \,:\, \{k'\} := kN \cap E \neq \varnothing} \ell_J(\phi(b(k'))) \\ &= \max_{k' \in E} \ell_J(\phi(b(k'))) \\ &\geq \max_{k' \in E} \delta_{b(k')} = \max_{k' \in H} \delta_{b(k')} = \beta_b. \end{aligned}$$

The equality of the two maxima in the final line follows from the definition of E, which ensures that $b(k)=1$ for any $k \in H \setminus E$ and hence that, for such k, $\delta_{b(k)}=0$.

It remains to show that, for $h_i b_i, h_j b_j \in F$,

$$l_K(\psi(h_i b_i h_j b_j)(\psi(h_i b_i)\psi(h_j b_j))^{-1}) \le \epsilon.$$

We have

$$\psi(h_i b_i h_j b_j) = \psi(h_i h_j b_i^{h_j} b_j) = \overline{h_i h_j}\ \widehat{b_i^{h_j} b_j},$$

and

$$\psi(h_i b_i)\psi(h_j b_j) = (\bar{h}_i \hat{b}_i)(\bar{h}_j \hat{b}_j) = \bar{h}_i \bar{h}_j \hat{b}_i^{\bar{h}_j} \hat{b}_j.$$

Since l_K is invariant under conjugation, the length we need is that of the element

$$b' := \widehat{b_i^{h_j} b_j}\, \hat{b}_j^{-1} \big(\hat{b}_i^{\bar{h}_j}\big)^{-1}$$

of B'. By definition, $\ell_K(b') = \max_{kN \in H/N} \ell_J(b'(kN))$. So choose a coset kN. We want to bound $\ell_J(b'(kN))$ for each such choice. We have

$$\begin{aligned} b'(kN) &= \widehat{b_i^{h_j} b_j}(kN)(\hat{b}_j(kN))^{-1}(\hat{b}_i^{\bar{h}_j}(kN))^{-1} \\ &= \widehat{b_i^{h_j} b_j}(kN)(\hat{b}_j(kN))^{-1}(\hat{b}_i(kh_j^{-1}N))^{-1} \\ &= \begin{cases} (\hat{b}_i(kh_j^{-1}N))^{-1} & \text{if } kN \cap E = \varnothing, \qquad (1) \\ \phi(b_i(k'h_j^{-1})b_j(k'))(\phi(b_j(k')))^{-1}(\hat{b}_i(kh_j^{-1}N))^{-1} & \text{if } kN \cap E = \{k'\}, \qquad (2) \end{cases} \end{aligned}$$

since in case (1) we have $\widehat{b_i^{h_j} b_j}(kN) = \hat{b}_j(kN) = 1$, and in case (2), we have $\widehat{b_i^{h_j} b_j}(kN) = \phi((b_i^{h_j} b_j)(k')) = \phi(b_i(k'h_j^{-1})b_j(k'))$, and $\hat{b}_j(kN) = \phi(b_j(k'))$.

When $E \cap kh_j^{-1}N = \varnothing$, we have $\hat{b}_i(kh_j^{-1}N) = 1$. In that case, by the definition of E, we also have $b_i(k'h_j^{-1}) = 1$ and so, in both case (1) and case (2), we deduce that $b'(kN) = 1$ and $\ell_J(b'(kN)) = 0$.

Otherwise $E \cap kh_j^{-1}N$ is nonempty, and its single element is equal to $k''h_j^{-1}$, for some $k'' \in kN$.

Suppose first that $b_i(k''h_j^{-1}) = 1$, and hence again we have $\hat{b}_i(kh_j^{-1}N) = 1$. If we are in case (2) then we must also have $b_i(k'h_j^{-1}) = 1$, since if $b_i(k'h_j^{-1}) \ne 1$, then condition (ii) of the definition of E gives $k'h_j^{-1} \in E$, and so $k' = k''$, contradicting $b_i(k''h_j^{-1}) = 1$. Then, just as above, we see that in both cases (1) and (2) we again get $b'(kN) = 1$ and $\ell_J(b'(kN)) = 0$.

Otherwise $b_i(k''h_j^{-1}) \ne 1$ and condition (iii) of the definition of E gives $k'' \in E$ and hence we are in case (2) with $k' = k''$. Then

$$b'(kN) = \phi(b_i(k'h_j^{-1})b_j(k'))\phi(b_j(k'))^{-1}\phi(b_i(k'h_j^{-1}))^{-1}.$$

Since ϕ was assumed to be a $(D, \epsilon, \delta, \ell_J)$-quasihomomorphism, $\ell_J(b'(kN)) \le \epsilon$ and, since this is true for all $kN \in H/N$, we get $\ell_K(b') \le \epsilon$ as required. $\square$

The conditions of the theorem clearly hold for the class $\mathcal{F}$, as well as for finite groups equipped with the trivial length function, and hence the classes of weakly sofic and LEF groups are both closed under restricted wreath products with residually finite groups. The following lemma together with Theorem 2.1 shows that the class of $\mathcal{F}_C$-approximable groups is also closed under restricted wreath products with residually finite groups.

Lemma 3.2. *Let J be a group equipped with an invariant function ℓ_J. If ℓ_J is commutator-contractive, then so is ℓ_J^X, for any finite group X.*

Proof. We consider the commutator of two elements $x_1 b_1$ and $x_2 b_2$ in J.

First suppose that x_1 and x_2 are both nontrivial. Then $\ell_J^X(x_1 b_1) = \ell_J^X(x_2 b_2) = 1$, and so the inequality holds trivially.

Now suppose that $x_1 = x_2 = 1$. Then

$$\begin{aligned}
\ell_J^X([b_1, b_2]) &= \max\nolimits_{x\in X} \ell_J([b_1, b_2](x)) \\
&= \max\nolimits_{x\in X} \ell_J([b_1(x), b_2(x)]) \\
&\le 4 \max\nolimits_{x\in X} \ell_J(b_1(x))\ell_J(b_2(x)) \\
&\le 4 \max\nolimits_{x\in X} \ell_J(b_1(x)) \max\nolimits_{y\in X} \ell_J(b_2(y)) \\
&= 4\ell_J^X(b_1)\ell_J^X(b_2).
\end{aligned}$$

Finally suppose that $x_1 = 1,\ x_2 \ne 1$ (the other case is very similar). Then

$$\begin{aligned}
\ell_J^X([b_1, x_2 b_2]) &= \ell_J^X(b_1^{-1} b_2^{-1} x_2^{-1} b_1 x_2 b_2) \\
&= \ell_J^X(b_1^{-1} b_2^{-1} b_1^{x_2} b_2) \\
&= \max\nolimits_{x\in X} \ell_J(b_1(x)^{-1} b_2(x)^{-1} b_1^{x_2}(x) b_2(x)) \\
&= \max\nolimits_{x\in X} \ell_J(b_1(x)^{-1} b_2(x)^{-1} b_1(x x_2^{-1}) b_2(x)) \\
&\le \max\nolimits_{x\in X} (\ell_J(b_1(x)^{-1}) + \ell_J(b_2(x)^{-1} b_1(x x_2^{-1}) b_2(x))) \\
&= \max\nolimits_{x\in X} (\ell_J(b_1(x)^{-1}) + \ell_J(b_1(x x_2^{-1}))) \\
&\le \max\nolimits_{x\in X} (\ell_J(b_1(x)^{-1})) + \max\nolimits_{y\in X} (\ell_J(b_1(y))) \\
&\le 2 \max\nolimits_{x\in X} (\ell_J(b_1(x)^{-1}) = 2\ell_J^X(b_1).
\end{aligned}$$ $\square$

4. The wreath product result for sofic groups

We prove now the corresponding result for sofic groups. For this, we are not free to choose our own norm function on the wreath product, but we must use the Hamming distance norm. The proof is nevertheless very similar in structure to that of Theorem 3.1. We use the definition of sofic groups given in [Elek and Szabó 2006] where, rather than having a weight function on the group G, we require that, for finite $F \subseteq G$, the proportion of moved points of elements of $F \setminus \{1\}$ in an (F, ϵ)-quasiaction of G on a finite set is at least $1-\epsilon$.

We note that this result has recently been generalised by Hayes and Sale [2016], who proved that the restricted standard wreath product of any two sofic groups is sofic.

Theorem 4.1. *The restricted standard wreath product $G \wr H$ of a sofic group G and a residually finite group H is sofic.*

Proof. Assume that G is sofic and H is residually finite, and let $W = G \wr H$ be the restricted standard wreath product. So, as in the proof of Theorem 3.1, W is the semidirect product of its base group B by H.

Let $F = \{h_i b_i : 1 \leq i \leq r\}$ be a finite subset of W. Then, for a given $\epsilon > 0$, we need to find an (F, ϵ)-quasiaction of W on some finite set Y.

We define the finite subset E of H, the normal subgroup N of H, and the finite subset D of G exactly as in the proof of Theorem 3.1. So, in particular, for any $k \in H$, $E \cap kN$ is either empty or consists of a single element $k' \in kN$. Let $m = |H/N|$.

Then, by [Elek and Szabó 2006, Lemma 2.1], for a given $\epsilon > 0$, there is a $(D, \epsilon/m)$-quasiaction $\phi : G \to \mathrm{Sym}(X)$ of G on some finite set X, and we may assume that $\phi(1) = 1$. Since we can choose both m and X to be arbitrarily large for given D and ϵ, we may assume that $|X|^{-m/2} < \epsilon$.

Let $Y = X^{H/N}$ be the set of functions $\delta : H/N \to X$. So $|Y| = |X|^m$. We define $\psi : W \to \mathrm{Sym}(Y)$ as follows. (The image of ψ is contained in the primitive wreath product of $\mathrm{Sym}(X)$ and H/N, as defined in [Dixon and Mortimer 1996, §2.6].)

For $b \in B,\ h, k \in H$, let $\delta^{\psi(hb)}(kN) := \delta(kh^{-1}N)^{\tau(b,k)}$, where

$$\tau(b, k) := \begin{cases} 1 & \text{when } E \cap kN = \varnothing, \\ \phi(b(k')) & \text{when } E \cap kN = \{k'\}. \end{cases}$$

We claim that ψ is an (F, ϵ)-quasiaction of W on Y. Observe first that $\psi(1) = 1$.

We check next that, for each $h_i b_i \in F \setminus \{1\}$, $\psi(h_i b_i)$ is $(1-\epsilon)$-different from 1. If $h_i \neq 1$ then, by assumption, $h_i \notin N$, so $kh_i^{-1}N \neq kN$ for all $kN \in H/N$. So, if $\delta \in Y$ is a fixed point of $\psi(h_i b_i)$, then the value of $\delta(kN)$ is uniquely determined by that of $\delta(kh_i^{-1}N)$ for each $kN \in H/N$, so the proportion of fixed points is at most $|X|^{m/2}/|X|^m = |X|^{-m/2}$, which we assumed to be less than ϵ.

If, on the other hand, $h_i = 1$ and $b_i \neq 1$, then there exists $h \in E$ with $b_i(h) \neq 1$. Now an element $\delta \in Y$ is fixed by $\psi(h_i b_i) = \psi(b_i)$ if and only if $\delta(kN)$ is fixed by $\tau(b, k)$ for all $kN \in H/N$. Hence, in particular, for a fixed point δ, we have $\delta(hN) = \delta(hN)^{\tau(b_i, h)}$, and so $\delta(hN)$ is a fixed point of $\tau(b_i, h) = \phi(b_i(h))$. Since the proportion of such points in X is, by assumption, at most ϵ, the same is true for $\psi(b_i)$.

Finally we need to verify that $\psi(h_ib_i)\psi(h_jb_j)$ is ϵ-similar to $\psi(h_ih_jb_i^{h_j}b_j)$ for each i, j with $1 \le i, j \le r$; that is, that the two permutations agree on at least a proportion $1-\epsilon$ of the points.

Now

$$\delta^{\psi(h_ib_i)\psi(h_jb_j)}(kN) = (\delta^{\psi(h_ib_i)}(kh_j^{-1}N))^{\tau(b_j,k)} = \delta(kh_j^{-1}h_i^{-1}N)^{\tau(b_i,kh_j^{-1})\tau(b_j,k)},$$

and

$$\delta^{\psi(h_ih_jb_i^{h_j}b_j)}(kN) = \delta(kh_j^{-1}h_i^{-1}N)^{\tau(b_i^{h_j}b_j,k)},$$

so we need to compare $\tau(b_i, kh_j^{-1})\tau(b_j, k)$ with $\tau(b_i^{h_j}b_j, k)$.

The argument is very similar to that in the analogous part of the proof of Theorem 3.1. We are in one of two cases. Either

(1) $E \cap kN = \varnothing$, in which case $\tau(b_j, k) = \tau(b_i^{h_j}b_j, k) = 1$, or

(2) $E \cap kN = \{k'\}$, for some $k' \in K$, and so $\tau(b_j, k) = \phi(b_j(k'))$, and $\tau(b_i^{h_j}b_j, k) = \phi((b_i^{h_j}b_j)(k')) = \phi(b_i(k'h_j^{-1})b_j(k'))$.

When $E \cap kh_j^{-1}N = \varnothing$, then $b_i(k'h_j^{-1}) = 1$ and, in both case (1) and case (2), $\tau(b_i, kh_j^{-1})\tau(b_j, k) = \tau(b_i^{h_j}b_j, k)$.

Otherwise, $E \cap kh_j^{-1}N = \{k''h_j^{-1}\}$ for some $k'' \in kN$.

Suppose first that $b_i(k''h_j^{-1}) = 1$. If we are in case (2) then $b_i(k'h_j^{-1}) = 1$, since otherwise, just as in the proof of Theorem 3.1, condition (ii) of the definition of E gives $k'h_j^{-1} \in E$, and so $k' = k''$, and we have a contradiction. Hence, in both case (1) and case (2) we again have $\tau(b_i, kh_j^{-1})\tau(b_j, k) = \tau(b_i^{h_j}b_j, k)$.

Otherwise $b_i(k''h_j^{-1}) \neq 1$, and then, again just as in the proof of Theorem 3.1, condition (iii) of the definition of E gives $k'' \in E$. Hence we are in case (2) and $k' = k''$. Then

$$\tau(b_i, gh_j^{-1})\tau(b_j, g) = \phi(b_i(k'h_j^{-1}))\phi(b_j(k'))$$

and

$$\tau(b_i^{h_j}b_j, g) = \phi(b_i(k'h_j^{-1})b_j(k')).$$

Since $b_i(k'h_j^{-1}),\ b_j(k') \in D$, the fact that ϕ is a $(D, \epsilon/m)$-quasiaction implies that the proportion of the points of X on which the permutations $\phi(b_i(k'h_j^{-1})b_j(k'))$ and $\phi(b_i(k'h_j^{-1}))\phi(b_j(k'))$ have the same image is at least $1-\epsilon/m$.

It follows that the proportion of elements $\delta \in Y$ with

$$\delta^{\psi(h_ib_i)\psi(h_jb_j)}(kN) = \delta^{\psi(h_ih_jb_i^{h_j}b_j)}(kN)$$

is at least $1-\epsilon/m$. But $\delta^{\psi(h_ib_i)\psi(h_jb_j)} = \delta^{\psi(h_ih_jb_i^{h_j}b_j)}$ if and only if they take the same values on all $kN \in H/N$, and the proportion of $\delta \in Y$ for which this is true is at least $1-\epsilon$. ∎

5. Extensions by amenable groups

In Section 3 we defined the restricted standard wreath product $G \wr H$ of groups G, H. In this section, we shall need wreath products by permutation groups. For a group K and a finite set A, we define the permutation wreath product $W = K \wr \mathrm{Sym}(A)$ as $W = \mathrm{Sym}(A) \ltimes B$ where the base group is now the set of all functions $b : A \to K$. As before, we define $b_1 b_2(a) := b_1(a) b_2(a)$ for $b_1, b_2 \in B$, $a \in A$, and we define the action of $\mathrm{Sym}(A)$ on B by the rule $b^\alpha(a) = b(a^{\alpha^{-1}})$, for $\alpha \in \mathrm{Sym}(A)$, $a \in A$. Much as before, elements of the wreath product are represented as pairs (α, b) with $\alpha \in \mathrm{Sym}(A)$ and $b \in B$, multiplied according to the rule $(\alpha_1, b_1)(\alpha_2, b_2) = (\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$, and with $(\alpha, b)^{-1} = (\alpha^{-1}, b^{-\alpha^{-1}})$.

In general the length function for finite wreath products that we used in the proof of Theorem 3.1 is not suitable for the proof of Theorem 5.1 below. So we need to define a different one.

Given an invariant length function ℓ_K on K, we can define an invariant length function $\hat{\ell}_K^A$ on W by

$$\hat{\ell}_K^A(\alpha, b) = \frac{1}{|A|}\Bigg(\sum_{a \in A \,:\, a^\alpha = a} \ell_K(b(a)) + \sum_{a \in A \,:\, a^\alpha \neq a} 1 \Bigg).$$

Most of the conditions for $\hat{\ell}_K^A$ to be an invariant length function are straightforward consequences of the conditions on ℓ_K. The verification of

$$\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2) \le \hat{\ell}_K^A(\alpha_1, b_1) + \hat{\ell}_K^A(\alpha_2, b_2)$$

may require a little more thought. For this, we consider the terms corresponding to the various $a \in A$ in the three sums that make up $\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$, $\hat{\ell}_K^A(\alpha_1, b_1)$, and $\hat{\ell}_K^A(\alpha_2, b_2)$. We see that, for each $a \in A$ with $a^{\alpha_1} \neq a$ or $a^{\alpha_2} \neq a$, the term in $\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$ is at most $1/|A|$, but at least one of the two nonnegative terms in $\hat{\ell}_K^A(\alpha_1, b_1)$ and $\hat{\ell}_K^A(\alpha_2, b_2)$ is equal to $1/|A|$. On the other hand, for $a \in A$ with $a^{\alpha_1} = a$ and $a^{\alpha_2} = a$, the term corresponding to a in $\hat{\ell}_K^A(\alpha_1\alpha_2, b_1^{\alpha_2} b_2)$ is

$$\frac{1}{|A|}\ell_K(b_1^{\alpha_2}(a) b_2(a)) = \frac{1}{|A|}\ell_K(b_1(a) b_2(a)) \le \frac{1}{|A|}(\ell_K(b_1(a)) + \ell_K(b_2(a)),$$

which is the corresponding term in $\hat{\ell}_K^A(\alpha_1, b_1) + \hat{\ell}_K^A(\alpha_2, b_2)$.

Theorem 5.1. *Let $\mathcal{C}$ be a class of groups with associated invariant length functions and suppose that, for all $(K, \ell_K) \in \mathcal{C}$ and all finite sets A, the wreath product $(K \wr \mathrm{Sym}(A), \hat{\ell}_K^A)$ is in $\mathcal{C}$. Suppose that the group G has a normal subgroup N with the discrete $\mathcal{C}$-approximation property* (as defined in Section 1) *such that G/N is amenable. Then G has the discrete $\mathcal{C}$-approximation property.*

This result has already been proved for sofic groups [Elek and Szabó 2006, Theorem 1 (3)] and linear sofic groups [Stolz 2013, Theorem 5.3]. However, in order to avoid confusion we should comment that, while the above result considers extensions G of $\mathcal{C}$-approximable normal subgroups N with G/N amenable, by contrast, [Arzhantseva and Gal 2013, Theorem 7] considers extensions G of finitely generated residually finite normal subgroups N for which G/N is in a selected class $\mathcal{R}$ of groups (including groups that are residually amenable groups, LEF, LEA, sofic or surjunctive).

Proof. The proof is based on the corresponding proof in [Elek and Szabó 2006, Theorem 1 (3)] for sofic groups N.

By assumption, the normal subgroup N of G is $\mathcal{C}$-approximable using a weight function δ that takes a constant value c on all elements of $N \setminus \{1\}$. Since we can reduce the value of c without affecting the $\mathcal{C}$-approximability of N, we may assume that $c < 1$. If $N \neq \{1\}$ then we define the weight function β of G by $\beta_g = c$ for all $g \neq 1$, and if $N = \{1\}$, then we define β by $\beta_g = \frac{1}{2}$ for all $g \neq 1$.

For $g \in G$, let $\bar{g}$ be the homomorphic image of g in G/N and let $\sigma : G/N \to G$ be a section (so $\overline{\sigma(h)} = h$ for all $h \in G/N$), where $\sigma(\bar{1}) = 1$. We can lift σ to a map from G to G for which the image of $g \in G$ is $\sigma(\bar{g})$; we shall abuse notation and call that map σ as well.

To verify the $\mathcal{C}$-approximability condition on G, let F be a finite subset of G and let $\epsilon > 0$. We may assume that $\epsilon < \min\bigl(\frac{1}{2}, 1 - c\bigr)$.

The amenability of G/N ensures the existence of a finite subset $\overline{A}$ of G/N containing the identity element such that $|\overline{A}\bar{g} \setminus \overline{A}| \leq \epsilon |\overline{A}|$ for all $g \in F \cup F^{-1} \cup F^2 \cup F^{-2}$. Let $A = \sigma(\overline{A})$; note that all points of A are fixed by the map $\sigma : G \to G$. We define a map $\phi : G \to \mathrm{Sym}(A)$ as follows:

$$\text{for } g \in G,\ a \in A, \quad a^{\phi(g)} := \begin{cases} \sigma(ag) & \text{if } \overline{ag} \in \overline{A}, \\ \text{any choice with } \phi(g) \in \mathrm{Sym}(A) & \text{otherwise.} \end{cases}$$

Let $E = N \cap (A \cdot F \cdot A^{-1})$. The $\mathcal{C}$-approximability of N ensures the existence of an $(E, \epsilon, \delta, \ell_K)$-quasihomomorphism $\psi : N \to K$ with $(K, \ell_K) \in \mathcal{C}$.

Now we let $W = K \wr \mathrm{Sym}(A) = \mathrm{Sym}(A) \ltimes B$ and define $\Phi : G \to W$ by $\Phi(g) = (\phi(g), b)$ where, for $a \in A$, $b(a) := \psi(\sigma(ag^{-1})ga^{-1})$.

We show first that $\hat{\ell}_K^A(\Phi(g)) \geq \beta_g$ for $g \in F$. If $g \notin N$ then, since $\phi(g)$ moves all points $a \in A$ for which $\overline{ag} \in \overline{A}$, we have

$$\hat{\ell}_K^A(\Phi(g)) \geq 1 - \epsilon > \tfrac{1}{2} = \delta_g.$$

If $g \in N \setminus \{1\}$ then $\overline{ag^{-1}} = \bar{a}$, so $\sigma(ag^{-1}) = a$ for all $a \in A$, and $\hat{\ell}_K^A(\Phi(g))$ is the average over $a \in A$ of $\ell_K(\psi(aga^{-1}))$. But since each $aga^{-1} \in E \setminus \{1\}$, these all exceed δ_g.

Now let $g, h \in F$. We aim to show that

$$\hat{\ell}_K^A(\Phi(gh)\Phi(h)^{-1}\Phi(g)^{-1}) \leq 5\epsilon.$$

For $a \in A$, we have

$$\Phi(g) = (\phi(g), b), \qquad \text{where } b(a) = \psi(\sigma(ag^{-1})ga^{-1}),$$

$$\Phi(h) = (\phi(h), c), \qquad \text{where } c(a) = \psi(\sigma(ah^{-1})ha^{-1}),$$

$$\Phi(gh) = (\phi(gh), d), \qquad \text{where } d(a) = \psi(\sigma(ah^{-1}g^{-1})gha^{-1}),$$

$$\begin{aligned}\Phi(g)\Phi(h) = (\phi(g)\phi(h), b^{\phi(h)}c),\\ \text{where } (b^{\phi(h)}c)(a) &= b^{\phi(h)}(a)c(a) = b(a^{\phi(h)^{-1}})c(a)\\ &= \psi(\sigma(a^{\phi(h)^{-1}}g^{-1})ga^{-\phi(h)^{-1}})\psi(\sigma(ah^{-1})ha^{-1})\end{aligned}$$

(where, for $a, k \in G$, we write a^{-k} as shorthand for $(a^{-1})^k = (a^k)^{-1}$). Then

$$\begin{aligned}\Phi(gh)(\Phi(g)\Phi(h))^{-1} &= (\phi(gh), d)(\phi(g)\phi(h), b^{\phi}(h)c)^{-1}\\ &= (\phi(gh), d)((\phi(g)\phi(h))^{-1}, (b^{\phi}(h)c)^{-(\phi(g)\phi(h))^{-1}})\\ &= (\phi(gh)(\phi(g)\phi(h))^{-1}, (d(b^{\phi}(h)c)^{-1})^{(\phi(g)\phi(h))^{-1}}).\end{aligned}$$

Now, for a proportion of at least $1-2\epsilon$ of the points $a \in A$, we have both $\overline{ah^{-1}} \in \overline{A}$ and $\overline{ah^{-1}g^{-1}} \in \overline{A}$. For those points a, we have $a^{\phi(h)^{-1}} = \sigma(ah^{-1})$ and so the final expression for $(b^{\phi(h)}c)(a)$ above becomes

$$\psi(\sigma(ah^{-1}g^{-1})g\sigma(ah^{-1})^{-1}) \times \psi(\sigma(ah^{-1})ha^{-1}),$$

and we see that the image of a under the second component of $\Phi(gh)(\Phi(g)\Phi(h))^{-1}$ is equal to a conjugate of

$$\psi(xy)\psi(y)^{-1}\psi(x)^{-1},$$

where $x = \sigma(ah^{-1}g^{-1})g\sigma(ah^{-1})^{-1}$ and $y = \sigma(ah^{-1})ha^{-1}$. The elements x, y are both in the finite subset E of G, and hence, since ψ is a quasihomomorphism, $\ell_K(\psi(xy)\psi(y)^{-1}\psi(x)^{-1}) < \epsilon$, and we deduce that

$$\ell_K((d(b^{\phi(h)}c)^{-1})^{(\phi(g)\phi(h))^{-1}})(a)) < \epsilon,$$

for at least a proportion $1-2\epsilon$ of the points of A.

Our choice of A ensures also that $\phi(gh)(\phi(g)\phi(h))^{-1}(a) = a$ for at least a proportion $1-2\epsilon$ of the points a of A.

Now, for at least a proportion $1-4\epsilon$ of the points of A, the conditions of both of the last two paragraphs hold, and so we can deduce

$$\hat{\ell}_K^A(\Phi(gh)\Phi(h)^{-1}\Phi(g)^{-1}) < \epsilon(1-4\epsilon) + 4\epsilon < 5\epsilon. \qquad \square$$

In particular, by taking $\mathcal{C} = \mathcal{F}$ with each $K \in \mathcal{F}$ associated with all possible length functions, we see that the class of weakly sofic groups is closed under extension by amenable groups.

In general, ℓ_K commutator-contractive does not imply that $\hat{\ell}_K^A$ is commutator-contractive. But if, instead, we define ℓ_K^A as we did in Section 3 (that is, for $b \in B$, $\ell_K^A(b) = \max_{a \in A} \ell_K(b(a))$, and $\ell_K^A(\alpha b) = 1$ when $1 \neq \alpha \in \mathrm{Sym}(A)$) then, as we proved in Lemma 3.2, ℓ_K^A is commutator-contractive.

Our proof of Theorem 5.1 does not always work with this commutator-contractive norm, but it does work if $\phi : G/N \to A$ is a homomorphism. In particular, when $G/N \cong (\mathbb{Z}, +)$, we can choose A to be $\{x \in \mathbb{Z} : -m \leq x \leq m\}$ for some m and define ϕ to be addition modulo $2m+1$. So, by applying this repeatedly, we have:

Proposition 5.2. *The class of $\mathcal{F}_c$-approximable groups is closed under extension by polycyclic groups.*

References

[Arzhantseva and Gal 2013] G. Arzhantseva and S. Gal, "On approximation properties of semi-direct products of groups", preprint, 2013. arXiv

[Arzhantseva and Păunescu 2017] G. Arzhantseva and L. Păunescu, "Linear sofic groups and algebras", *Trans. Amer. Math. Soc.* **369**:4 (2017), 2285–2310. MR Zbl

[Brown et al. 2008] N. P. Brown, K. J. Dykema, and K. Jung, "Free entropy dimension in amalgamated free products", *Proc. Lond. Math. Soc.* (3) **97**:2 (2008), 339–367. MR Zbl

[Capraro and Lupini 2015] V. Capraro and M. Lupini, *Introduction to sofic and hyperlinear groups and Connes' embedding conjecture*, Lecture Notes in Mathematics **2136**, Springer, 2015. MR Zbl

[Ciobanu et al. 2014] L. Ciobanu, D. F. Holt, and S. Rees, "Sofic groups: graph products and graphs of groups", *Pacific J. Math.* **271**:1 (2014), 53–64. MR Zbl

[Dixon and Mortimer 1996] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics **163**, Springer, 1996. MR Zbl

[Elek and Szabó 2005] G. Elek and E. Szabó, "Hyperlinearity, essentially free actions and L^2-invariants: the sofic property", *Math. Ann.* **332**:2 (2005), 421–441. MR Zbl

[Elek and Szabó 2006] G. Elek and E. Szabó, "On sofic groups", *J. Group Theory* **9**:2 (2006), 161–171. MR Zbl

[Elek and Szabó 2011] G. Elek and E. Szabó, "Sofic representations of amenable groups", *Proc. Amer. Math. Soc.* **139**:12 (2011), 4285–4291. MR Zbl

[Glebsky 2015] L. Glebsky, "Approximation of groups, characterizations of sofic groups, and equations over groups", preprint, 2015. arXiv

[Hayes and Sale 2016] B. Hayes and A. Sale, "The wreath product of two sofic groups is sofic", preprint, 2016. arXiv

[Higman 1951] G. Higman, "A finitely generated infinite simple group", *J. London Math. Soc.* (2) **26** (1951), 61–64. MR Zbl

[Păunescu 2011] L. Păunescu, "On sofic actions and equivalence relations", *J. Funct. Anal.* **261**:9 (2011), 2461–2485. MR Zbl

[Pestov and Kwiatkowska 2009] V. G. Pestov and A. Kwiatkowska, "An introduction to hyperlinear and sofic groups", preprint, 2009. arXiv

[Popa 1995] S. Popa, "Free-independent sequences in type II_1 factors and related problems", pp. 187–202 in *Recent advances in operator algebras* (Orléans, 1992), Astérisque **232**, 1995. MR Zbl

[Rădulescu 2008] F. Rădulescu, "The von Neumann algebra of the non-residually finite Baumslag group $\langle a, b | ab^3a^{-1} = b^2 \rangle$ embeds into R^ω", pp. 173–185 in *Hot topics in operator theory*, edited by R. G. Douglas et al., Theta Ser. Adv. Math. **9**, Theta, Bucharest, 2008. MR Zbl

[Stolz 2013] A. Stolz, "Properties of linearly sofic groups", preprint, 2013. arXiv

[Thom 2012] A. Thom, "About the metric approximation of Higman's group", *J. Group Theory* **15**:2 (2012), 301–310. MR Zbl

[Voiculescu 1998] D. Voiculescu, "A strengthened asymptotic freeness result for random matrices with applications to free entropy", *Internat. Math. Res. Notices* 1 (1998), 41–63. MR Zbl

Received January 8, 2016. Revised September 25, 2016.

DEREK F. HOLT
MATHEMATICS INSTITUTE
UNIVERSITY OF WARWICK
COVENTRY
CV4 7AL
UNITED KINGDOM
d.f.holt@warwick.ac.uk

SARAH REES
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEWCASTLE
NEWCASTLE
NE1 7RU
UNITED KINGDOM
sarah.rees@ncl.ac.uk

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.411

COMAN CONJECTURE FOR THE BIDISC

ŁUKASZ KOSIŃSKI, PASCAL J. THOMAS AND WŁODZIMIERZ ZWONEK

We show the equality between the Lempert function and the Green function with two poles with equal weights in the bidisc, thus giving the positive answer to a conjecture of Coman in the simplest unknown case. Actually, we prove a slightly more general equality which in some sense is natural when studied from the point of view of the Nevanlinna–Pick problem in the bidisc.

1. Presentation of the problem and its history

Let D be a domain in $\mathbb{C}^n$ and let $\varnothing \neq P := \{p_1, \dots, p_N\} \subset D$ where $p_j \neq p_k$, $j \neq k$. Let also $\nu : P \to (0, \infty)$. Denote $\nu_j := \nu(p_j)$. Let $z \in D$.

Define $l_D(z; P; \nu) := l_D(z; (p_1, \nu_1), \dots, (p_N, \nu_N))$ as the infimum of the numbers

$$\sum_{j=1}^{N} \nu_j \log |\lambda_j|$$

such that there is an analytic disc $\psi : \mathbb{D} \to D$ with $\psi(0) = z$, $\psi(\lambda_j) = p_j$, $j = 1, \dots, N$.

Recall that $l_D(z; P; \nu) = \min\{l_D(z; A; \nu_{|A}) : \varnothing \neq A \subset P\}$ (see [Nikolov and Pflug 2006] for arbitrary D or [Wikström 2001] for D convex). The last equality will be of interest for us since in the case of taut domains (convex and bounded domains are taut) the infimum in the definition of $l_D(z; P; \nu)$ will be attained by some analytic disc defining $l_D(z; A; \nu_{|A})$ for some $\varnothing \neq A \subset P$.

The function $l_D(\,\cdot\,; P; \nu)$ is called *the Lempert function with the poles at P and with the weight function ν (or weights ν_j).*

Analoguously we define *the pluricomplex Green function $g_D(z; P; \nu)$ with the poles at P and the weight function ν* as the supremum of numbers $u(z)$ over all

The authors were supported by the grant of the Polish National Centre number UMO-2013/08/M/ST1/00986 promoting the cooperation between the groups of complex analysis in the Paul Sabatier University in Toulouse and the Jagiellonian University in Kraków.

MSC2010: primary 32U35; secondary 30E05.

Keywords: Green function, Lempert function, Carathéodory pseudodistance, Coman conjecture, m-extremal, m-complex geodesic, bidisc.

negative plurisubharmonic functions $u : D \to [-\infty, 0)$ with logarithmic poles at P, i.e., such that

$$u(\cdot) - \nu_j \log \| \cdot - p_j \|$$

is bounded above near p_j, $j = 1, \dots, N$.

It is trivial that $g_D(z; P; \nu) \leq l_D(z; P; \nu)$. D. Coman [2000] conjectured the equality $l_D(\cdot; P; \nu) = g_D(\cdot; P; \nu)$ for all convex domains D.

The conjecture has an obvious motivation in the Lempert Theorem [1981] which implies the equality in the case $N = 1$, and in the fact that the equality in the case of the unit ball and two poles with equal weights ($D = \mathbb{B}_n,\ N = 2,\ \nu_1 = \nu_2$) holds (see [Coman 2000] and also [Edigarian and Zwonek 1998]).

The conjecture turned out to be false. The first counterexample was found in [Carlehed and Wiegerinck 2003] ($D := \mathbb{D}^2,\ N = 2$ and different weights). Later a counterexample was found in the case of the bidisc ($D = \mathbb{D}^2$) with $N = 4$ and all weights equal (see [Thomas and Trao 2003]).

The simplest nontrivial case that was not clear yet was the case of the bidisc, two poles and equal weights. Recall that a partial positive answer in this case was found in [Carlehed 1999] (see also [Edigarian and Zwonek 1998]) in the case the poles were lying on $\mathbb{D} \times \{0\}$. In [Wikström 2003] numerical computations were carried out which strongly suggested that the equality in the case $D = \mathbb{D}^2,\ N = 2,\ \nu_1 = \nu_2$ should hold. The aim of this paper is to show that actually the Coman conjecture holds in the bidisc ($D = \mathbb{D}^2$), $N = 2$, two arbitrary poles and $\nu_1 = \nu_2$. In our proof we show even more: the equality of the Carathéodory function (defined below) and the Lempert function with two poles and equal weights in the bidisc. The methods we use originated with the study of the Nevanlinna–Pick problem for the bidisc.

2. Nevanlinna–Pick problem, m-complex geodesics, formulation of the solution

As already mentioned, the aim of the paper is to show a more general result than one claimed in the Coman conjecture for the bidisc, two poles and equal weights. To formulate the main result we need to introduce a new function. Since we shall be interested in equal weights we restrict ourselves from now on to the case when $\nu \equiv 1$. To make the presentation clearer we adopt the notation

$$d_D(z, \{p_1, \dots, p_N\}) := d_D(z; \{(p_1, 1), \dots, (p_N, 1)\})$$

($d = l$ or g) where the $p_j \in D$ are pairwise disjoint, $j = 1, \dots, N$.

Let us recall the definition of *the Carathéodory function with the poles at* p_j (*with weights equal to one*)

(1) $c_D(z, p_1, \dots, p_N) := \sup\{\log|F(z)| : F \in \mathcal{O}(D, \mathbb{D}), F(p_j) = 0, j = 1, \dots, N\}.$

It is simple to see that

$$c_D(\,\cdot\,, p_1, \dots, p_N) \le g_D(\,\cdot\,, p_1, \dots, p_N) \le l_D(\,\cdot\,, p_1, \dots, p_N).$$

Our main result is the following:

Theorem 1. *Let $p, q \in \mathbb{D}^2$ be two distinct points. Then*

$$c_{\mathbb{D}^2}(z; p; q) = l_{\mathbb{D}^2}(z; p; q) \quad \textit{for} \quad z \in \mathbb{D}^2.$$

Note that the function F for which the supremum in the definition of the Carathéodory function is attained always exists. On the other hand, in the case where D is a taut domain, for a point $z \in D$ and pole set P there are always a set $\varnothing \ne Q = \{q_1, \dots, q_M\} \subset P$ and a mapping $f \in \mathcal{O}(\mathbb{D}, D)$, $\lambda_j \in \mathbb{D}$ such that $f(0) = z$, $f(\lambda_j) = q_j$, $j = 1, \dots, M$ and $l_D(z; P) = l_D(z; Q) = \sum_{j=1}^{M} \log|\lambda_j|$. Consequently, in case the equality $c_D(z; p_1, \dots, p_N) = l_D(z; p_1; \dots; p_N)$ holds, there exist $f \in \mathcal{O}(\mathbb{D}, D)$, $F \in \mathcal{O}(D, \mathbb{D})$ such that $f(0) = z$, $f(\lambda_j) = q_j$, $F(q_j) = 0$, $|F(0)| = \prod_{j=1}^{M} |\lambda_j|$, $j = 1, \dots, M$, and (thus) $F \circ f$ is a finite Blaschke product of degree $M \le N$. This observation leads us to introduce and consider the notions of m-extremals and m-geodesics.

First recall that given a system of m pairwise different numbers $(\lambda_1, \dots, \lambda_m)$, $\lambda_j \in \mathbb{D}$ and a domain $D \subset \mathbb{C}^n$, a holomorphic mapping $f : \mathbb{D} \to D$ is called *a (weak) m-extremal* for $(\lambda_1, \dots, \lambda_m)$ if there is no holomorphic mapping $g : \mathbb{D} \to D$ such that $g(\mathbb{D}) \Subset D$ and $g(\lambda_j) = f(\lambda_j)$, $j = 1, \dots, m$. In case f is m-extremal with respect to any choice of m pairwise different arguments the mapping f is called *m-extremal*. A holomorphic mapping $f : \mathbb{D} \to D$ is called *an m-geodesic* if there is an $F \in \mathcal{O}(D, \mathbb{D})$ such that $F \circ f$ is a finite Blaschke product of degree at most $m - 1$. The function F will be called *the left inverse to f*. It is immediate to see that any m-geodesic is an m-extremal.

The notions of (weak) m-extremals and m-geodesics, which have clear origin in Nevanlinna–Pick problems for functions in the unit disk, have been recently introduced and studied in [Agler et al. 2013; 2015], [Kosiński and Zwonek 2016a], [Kosiński 2014] and [Warszawski 2015]. It is worth recalling that the description of m-extremals in the unit disc is classical and well known. The mapping $h \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ is m-extremal for $(\lambda_1, \dots, \lambda_m)$, $\lambda_j \in \mathbb{D}$ if and only if h is a finite Blaschke product of degree at most $m - 1$. Moreover, in such a case the interpolating function is uniquely determined (see [Pick 1915]).

The remark after Theorem 1 on the form of functions for which the extremum in the definition of the Lempert function may be attained may be formulated as follows. For any taut domain D, for any system of poles $P = \{p_1, \dots, p_N\} \subset D$ and any $z \in D \setminus P$ there are a subset $Q = \{q_1, \dots, q_M\} \subset P$ and $f \in \mathcal{O}(\mathbb{D}, D)$ such that $f(\lambda_j) = q_j$, $j = 1, \dots, M$, $f(0) = z$, and f is a weak $(M + 1)$-extremal for

$(0, \lambda_1, \dots, \lambda_M)$. Assuming additionally the equality $c_D(z; P) = l_D(z; P)$ would then imply the existence of a special $(M+1)$-geodesic, the one having some subset $Q \subset P$ in its image but such that the left inverse F maps the whole set P to 0. Consequently a necessary (but not sufficient!) condition for having the desired equality at z for the set of poles P is the existence of some $(M+1)$-geodesic passing through a subset $Q \subset P$ and mapping 0 to z.

Below we present a result on uniqueness of left inverses for m-geodesics in convex domains in $\mathbb{C}^2$ which we shall use in a (very special) case of the bidisc. The result is a simple generalization of a similar result formulated for 2-geodesics that can be found in [Kosiński and Zwonek 2016b] (however, for the clarity of the presentation we restrict ourselves to dimension two). We also present its proof here for the sake of completeness.

Lemma 2. *Let D be a convex domain in $\mathbb{C}^2$, $\lambda_j \in \mathbb{D}$, $j = 1, \dots, m$, $m \geq 2$, be pairwise different and let $f, g : \mathbb{D} \to D$ be such that $f(\lambda_j) = g(\lambda_j) =: z_j$ and $f \not\equiv g$. Assume additionally that $F, G \in \mathcal{O}(D, \mathbb{D})$ are such that $F \circ f$ and $G \circ g$ are Blaschke products of degree at most $m-1$. Then $F \equiv G$. Moreover, for any $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{D}$ such that $\mu f(\lambda) + (1-\mu)g(\lambda) \in D$ we have the equality*

$$F(\mu f(\lambda) + (1-\mu)g(\lambda)) = F(f(\lambda)).$$

Proof. For $t \in [0, 1]$ define $h_t := tf + (1-t)f \in \mathcal{O}(\mathbb{D}, D)$. Then $h_t(\lambda_j) = z_j$, $j = 1, \dots, m$, so, due to the uniqueness of the solution of the extremal problem in the disk, we get that $F \circ h_t \equiv G \circ h_t =: B$, $t \in [0, 1]$, is a finite Blaschke product of degree $\leq m-1$. Consequently, we get the equality $F \equiv G$ on the set

$$\{tf(\lambda) + (1-t)g(\lambda) = g(\lambda) + t(f(\lambda) - g(\lambda)) : t \in [0, 1], \lambda \in \mathbb{D}\}.$$

Moreover, the identity principle (applied to the map $\mu \mapsto F(\mu f(\lambda) + (1-\mu)g(\lambda))$) implies that

$$F(\mu f(\lambda) + (1-\mu)g(\lambda)) = G(\mu f(\lambda) + (1-\mu)g(\lambda)) = B(\lambda)$$

for all $(\mu, \lambda) \in V$ where V is the set (domain) of all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{D}$ such that

$$\Phi(\mu, \lambda) := \mu f(\lambda) + (1-\mu)g(\lambda) = g(\lambda) + \mu(f(\lambda) - g(\lambda)) \in D.$$

Note that $V \supset [0, 1] \times \mathbb{D}$. The equality mentioned earlier gives, in particular, $F \equiv G$ on $\Phi(V)$.

Let $\varnothing \neq U \Subset \mathbb{D}$ be a domain such that $f(\lambda) \neq g(\lambda)$, $\lambda \in \bar{U}$, and $B_{|U}$ is injective.

Let $V \supset \Omega := U_1 \times U \supset [0, 1] \times U$ be a domain. We claim that $\Phi_{|\Omega}$ is injective, which would finish the proof as in such a case $\Phi(\Omega)$ would be open and then the application of the identity principle would imply that $F \equiv G$ on D.

To see the injectivity take $(\mu_1, \lambda_1), (\mu_2, \lambda_2) \in \Omega$ such that $\Phi(\mu_1, \lambda_1) = \Phi(\mu_2, \lambda_2)$. Then $B(\lambda_1) = B(\lambda_2)$ so the injectivity of $B_{|U}$ implies that $\lambda_1 = \lambda_2$ which (since $f(\lambda_1) - g(\lambda_1) = f(\lambda_2) - g(\lambda_2) \neq 0$) gives the equality $\mu_1 = \mu_2$. □

3. Properties of extremals for the Lempert function in case the Coman conjecture holds

Let us now restrict our considerations to the case of the bidisc and two poles $p, q \in \mathbb{D}^2$, $p \neq q$. Without loss of generality we may assume that $z = (0, 0)$. Simple continuity properties of the Lempert and Carathéodory function allow us to reduce the Coman conjecture to the proof of the equality

$$c(p, q) := c_{\mathbb{D}^2}((0, 0), p, q) = l_{\mathbb{D}^2}((0, 0), p, q) =: l(p, q)$$

for (p, q) from some open, dense subset of $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$ to be defined later ($\triangle$ denotes the diagonal in the corresponding Cartesian product $X \times X$, here $X = \mathbb{D}^2$).

Below we shall present the starting point for our considerations. The proof contains the reasoning which will lead us to the structure of the proof of the equality $c(p, q) = l(p, q)$ presented later.

Lemma 3. *Let $p, q \in \mathbb{D}^2 \setminus \Delta$ be such that $|p_1| \neq |p_2|$, $|q_1| \neq |q_2|$, $p_1 \neq q_1$ and $p_2 \neq q_2$. Then the equality $c(p, q) = l(p, q)$ holds if an only if one of the following conditions is satisfied*:

(1) *up to a permutation of coordinates $|p_2| < |p_1|$, $|q_2| < |q_1|$ and $m(p_2/p_1, q_2/q_1) \leq m(p_1, q_1)$, or $p_2 = \omega p_1$, $q_2 = \omega q_1$ for some unimodular ω, where m is the Möbius distance on the disc*, *see Section 4*,

(2) *there exist α, β, c in the unit disc*, *a unimodular constant ω*, *and $t \in (0, 1)$ such that an analytic disc where m_α, m_β are* (*idempotent*) *Möbius maps*

$$\varphi(\lambda) = \lambda(m_\alpha(\lambda), \omega m_\beta(\lambda)), \quad \lambda \in \mathbb{D},$$

satisfies $\varphi(c) = p$ and $\varphi(m_\gamma(c)) = q$, *where $\gamma = t\alpha + (1 - t)\beta$.*

In order to prove Lemma 3 we need the following technical result:

Lemma 4. *Let $\alpha, \beta \in \mathbb{D}$, $\alpha \neq \beta$, $t \in [0, 1]$, $\omega, \tau \in \mathbb{T}$. Define*

$$\varphi(\lambda) := \lambda(m_\alpha(\lambda), \omega m_\beta(\lambda))$$

and let

$$G(x) := \frac{tx_1 + (1 - t)\bar{\omega}x_2 + \tau\bar{\omega}x_1x_2}{1 + \tau((1 - t)x_1 + t\bar{\omega}x_2)}, \quad x = (x_1, x_2) \in \mathbb{D}^2. \tag{2}$$

Set $G(\varphi(\lambda)) =: \lambda f(\lambda)$, $\lambda \in \mathbb{D}$. Denote $f(0) = \gamma := t\alpha + (1 - t)\beta$. Then f is an automorphism of $\mathbb{D}$ (*equal to m_γ*) *if and only if $\tau = (\overline{\alpha - \beta})/(\alpha - \beta)$.*

Proof. The proof of the above lemma reduces to showing that in the inequality $|f'(0)|/(1-|f(0)|^2) \leq 1$ the equality holds if and only if $\tau = (\overline{\alpha - \beta})/(\alpha - \beta)$ which is elementary although tedious. □

Proof of necessity in Lemma 3. Assume that we have the equality for (p, q). There are two possibilities (up to a permutation of variables p and q):

(i) there exist a holomorphic $\varphi : \mathbb{D} \to \mathbb{D}^2$, $F : \mathbb{D}^2 \to \mathbb{D}$ and $c \in (0, 1)$ such that $\varphi(0) = (0, 0)$, $\varphi(c) = p$ and $F(p) = F(q) = c$, $F(0, 0) = 0$.

Then $F(\varphi(\lambda)) = \lambda$, so $\varphi(\lambda) = (\omega\lambda, \psi(\lambda))$ where $|\omega| = 1$ (up to switching coordinates). If $\psi \notin \mathrm{Aut}(\mathbb{D})$ then Lemma 2 implies that $F(z) = \bar{\omega} z_1$ so $p_1 = q_1$ and $|p_2| \leq |p_1|$.

The second subcase is when $\psi \in \mathrm{Aut}(\mathbb{D})$ and $\psi(0) = 0$. But then $|p_1| = |p_2|$.

(ii) The function φ realizing the infimum is a weak 3-extremal with respect to $(0, c, d)$ such that $\varphi(0) = (0, 0)$, $\varphi(c) = p$, $\varphi(d) = q$. The special left inverse $F : \mathbb{D}^2 \to \mathbb{D}$ would satisfy the equalities $F(p) = F(q) = 0$ and $F(0) = cd$. Consequently $F \circ \varphi = m_c m_d$. We have two possibilities:

(a) φ is a geodesic (2-extremal). This holds if either

- $|p_2| < |p_1|$, $|q_2| < |q_1|$ and $m(p_2/p_1, p_2/p_1) \leq m(p_1, q_1)$, or
- $|p_1| < |p_2|$, $|q_1| < |q_2|$ and $m(p_1/p_2, q_1/q_2) \leq m(p_2, q_2)$, or
- $p_2 = \omega p_1$ and $q_2 = \omega q_1$ for some unimodular ω.

(b) φ is not a 2-extremal. First note that $\varphi(\lambda) = \lambda\psi(\lambda)$ where ψ is a 2-extremal (geodesic). Consequently, up to a permutation of the coordinates, $\varphi(\lambda) = \lambda(m(\lambda), h(\lambda))$, where m is some Möbius map and $h \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. In the case h is not a Möbius map the mapping φ is not uniquely determined — in the sense that for the triple $(0, c, d)$ there also exists another 3-extremal mapping $\tilde{\varphi}$ which maps this triple of numbers to the same triple of points. But existence of the left inverse already gives its uniqueness (see Lemma 2); moreover, it follows from the same lemma that $F(\lambda m(\lambda), \mu) = m_c(\lambda) m_d(\lambda)$ for any $\mu \in \mathbb{D}$, which easily implies that $F(z) = a(z_1)$, where a is some Möbius map. But the last property may hold only if $p_1 = q_1$.

Thus the generic case for φ being a 3-extremal from the definition of the Lempert function which are not 2-extremals is the one given by the formula

$$\varphi(\lambda) = \lambda(\omega' m_\alpha(\lambda), \omega m_\beta(\lambda)), \quad \lambda \in \mathbb{D}, \tag{3}$$

where $\alpha, \beta \in \mathbb{D}$ and $\omega', \omega \in \mathbb{T}$. Multiplying α, β, c, d by a unimodular constant one may assume that $\omega' = 1$.

Our aim is now to show what the necessary form of functions $F \in \mathcal{O}(\mathbb{D}^2, \mathbb{D})$ such that $F \circ \varphi$ is a Blaschke product should be. We present below the reasoning, employing some results of McCarthy and Agler. Let us also mention that G. Knese (personal communication, 2014) let us know about another approach which leads to the same form of left inverses.

We are looking for a form of a function $F : \mathbb{D}^2 \to \mathbb{D}$ such that $F \circ \varphi = m_c m_d$. Set $G := m_{cd} \circ F$. Clearly $G \circ \varphi(0) = 0$, so it suffices to consider the following situation:

$$G(\lambda m_\alpha(\lambda), \omega\lambda m_\beta(\lambda)) = \lambda m_\gamma(\lambda), \quad \lambda \in \mathbb{D}.$$

We are looking for a formula for G. Note that we consider only the case when $\alpha \neq \beta$. The cases $\gamma = \alpha$ or $\beta = \gamma$ are also excluded.

Assuming that G and γ do exist consider the following Pick problem:

$$\begin{cases} (0,0) \mapsto 0 \\ (\gamma m_\alpha(\gamma), \omega\gamma m_\beta(\gamma)) \mapsto 0, \\ (\lambda' m_\alpha(\lambda'), \omega\lambda' m_\beta(\lambda')) \mapsto \lambda' m_\gamma(\lambda'), \end{cases}$$

where λ' is any point in $\mathbb{D}$, $\lambda' \neq \lambda$. It is quite clear that this problem is strictly 2-dimensional, extremal and nondegenerate (with the notions understood as defined in [Agler and McCarthy 2002, Chapter 12], itself drawing from [Agler and McCarthy 2000] where the terminology is slightly different). Therefore, it follows from [Agler and McCarthy 2002, Theorem 12.13, p. 201–204] that the above problem has a unique solution which is given by a rational inner function of degree 2, with no terms in x_1^2 or x_2^2. It is easily seen that the solution to this problem is a left inverse we are looking for. Therefore,

$$G(x) = \frac{Ax_1 + Bx_2 + Cx_1x_2}{1 + Dx_1 + Ex_2 + Gx_1x_2}.$$

Now we proceed in a standard way: comparing multiplicities in the poles of m_α and m_β, etc. After additional calculations we get that $A + \omega B = 1$ and then

$$G(x) = \frac{tx_1 + (1-t)\bar{\omega}x_2 - \eta x_1x_2}{1 - ((1-t)x_1 + tx_2)\omega}, \tag{4}$$

where $t \in (0, 1)$ and $\eta \in \mathbb{T}$. In particular, $\gamma = t\alpha + (1-t)\beta$. It is clear that $d = m_\gamma(c)$, which finishes the proof of necessity. □

Proof of sufficiency in Lemma 3. Assume first that condition (1) is satisfied. In other words there is $\psi \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$ is such that $\psi(p_1) = p_2/p_1$, $\psi(q_1) = q_2/q_1$. Let $F(z) := m_{p_1}(z_1)m_{q_1}(z_1)$, $z \in \mathbb{D}^2$. Put

$$\varphi(\lambda) := (\lambda, \lambda\psi(\lambda)), \quad \lambda \in \mathbb{D}.$$

Observe that $\varphi(0) = (0,0)$, $\varphi(p_1) = p$, $\varphi(q_1) = q$, $F(0,0) = p_1q_1$ and $F(p) = F(q) = 0$ which give the equality

$$c(p;q) \le l(p;q) \le \log|p_1q_1| \le c(p;q).$$

Now suppose that (2) holds, i.e., the analytic disc $\lambda \mapsto \varphi(\lambda) = \lambda(m_\alpha(\lambda), \omega m_\beta(\lambda))$ satisfies $\varphi(c) = p$ and $\varphi(m_\gamma(c)) = q$. Let G be the function given by the formula (2) with $\tau = (\overline{\alpha - \beta})/(\alpha - \beta)$. It follows from Lemma 4 that $G(\varphi(\lambda)) = \lambda m_\gamma(\lambda)$ for $\lambda \in \mathbb{D}$. In particular

$$F := m_{cm_{\gamma(c)}} \circ G$$

satisfies $F \circ \varphi = \tau m_c m_{m_\gamma(c)}$ for some $\tau \in \mathbb{T}$. This gives the equality

$$c(\varphi(c), \varphi(m_\gamma(c))) = l(\varphi(c), \varphi(m_\gamma(c))). \qquad \square$$

The above result is a key one — it will turn out that the set of pairs of points $(\varphi(\lambda), \varphi(m_\gamma(\lambda)))$ (parametrized by $(\alpha, \beta, c, t, \omega)$) will build an open set, which together with the one constructed with the help of extremals for the Lempert functions being 2-geodesics will be dense in $\mathbb{D}^2 \times \mathbb{D}^2$ — that will complete the proof.

4. Proof of the equality $c(p; q) = l(p; q)$

To prove the Coman conjecture for the bidisc we consider open sets in $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$ whose union forms a dense subset of $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$ and on each part the desired equality holds. Let us denote $\sigma(p, q) := ((p_2, p_1), (q_2, q_1))$, $p, q \in \mathbb{D}^2$. Define U as the set of points $(p; q) \in \mathbb{D}^2 \times \mathbb{D}^2$ satisfying the following inequalities

$$(5) \qquad |p_2| < |p_1|, |q_2| < |q_1| \quad \text{and} \quad m(p_2/p_1, q_2/q_1) < m(p_1, q_1),$$

where m is the Möbius distance on the unit disc given by the formula $m(\lambda_1, \lambda_2) := \left|(\lambda_1 - \lambda_2)/(1 - \bar{\lambda}_1 \lambda_2)\right|$.

Denote

$$\Omega_1 := U \cup \sigma(U).$$

The equality on Ω_1 was proved in Lemma 3.

We shall consider now the set given by 3-geodesics that are not 2-geodesics and that appeared in Lemma 3.

Consider a real-analytic mapping

$$\Phi : \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{T} \times (0, 1) \to \mathbb{D}^2 \times \mathbb{D}^2$$

given by the formula (below and in the sequel $\gamma := t\alpha + (1 - t)\beta$)

$$(\alpha, \beta, c, \omega, t) \mapsto \big(\varphi_{\alpha,\beta,\omega}(c), \varphi_{\alpha,\beta,\omega}(m_\gamma(c))\big),$$

where

$$\varphi_{\alpha,\beta,\omega}(\zeta) := (\omega \zeta m_\alpha(\zeta), \zeta m_\beta(\zeta)), \quad \zeta \in \mathbb{D}.$$

Motivated by the considerations in Section 3 we define open sets.

Denote $\mathcal{A} := \{(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2 : p_1 = q_1 \quad \text{or} \quad p_2 = q_2\}$ and

$$F_1 := \{(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2 : |p_2| > |p_1| \quad \text{and} \quad |q_2| < |q_1|\}. \tag{6}$$

We also define the set F_2 as the set of points $(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2$ satisfying the following inequalities:

$$|p_2| < |p_1| \quad \text{and} \quad |q_2| < |q_1| \quad \text{and} \quad m\left(\frac{p_2}{p_1}, \frac{q_2}{q_1}\right) > m(p_1, q_1). \tag{7}$$

Let $F_3 = \sigma(F_1)$, and $F_4 = \sigma(F_2)$. Let $E_j := F_j \setminus \mathcal{A}$.

Define

$$\Omega_2 := E_1 \cup E_2 \cup E_3 \cup E_4.$$

Certainly the sets E_j are disjoint and open. Moreover, they are connected. Actually, $\mathcal{A}$ is an analytic set so it is sufficient to show the connectivity of F_j. But F_1 is the image of $\mathbb{D} \times \mathbb{D}_* \times \mathbb{D}_* \times \mathbb{D}$ under the mapping $\lambda \mapsto (\lambda_1\lambda_2, \lambda_2, \lambda_4, \lambda_3\lambda_4)$. On the other hand the set F_2 is the image, under the mapping $\lambda \mapsto (\lambda_1, \lambda_1\lambda_2, \lambda_3, \lambda_3\lambda_4)$ of the set $B := \{\lambda \in \mathbb{D}_* \times \mathbb{D} \times \mathbb{D}_* \times \mathbb{D} : m(\lambda_1, \lambda_3) < m(\lambda_2, \lambda_4)\}$. To show connectedness of the last set it suffices to show that $\tilde{B} := \{\lambda \in \mathbb{D} \times \mathbb{D} \times \mathbb{D} \times \mathbb{D} : m(\lambda_1, \lambda_3) < m(\lambda_2, \lambda_4)\}$ is connected, as B is obtained from $\tilde{B}$ by removing an analytic set. This is the case because any point $\lambda \in \tilde{B}$ may be joined by the curve $[0, 1] \ni t \mapsto (t\lambda_1, \lambda_2, t\lambda_3, \lambda_4)$ with $(0, \lambda_2, 0, \lambda_4)$. And now it is sufficient to see that the set $\{0\} \times \mathbb{D}_* \times \{0\} \times \mathbb{D}_*$ is arc-connected.

Let $G_j := \Phi^{-1}(E_j)$. To finish the proof of the assertion it suffices to show that

$$\Phi|_{G_j} : G_j \to E_j$$

is surjective. In fact, in such a case $\Phi(G_j) = E_j$ so the equality $l = c$ holds on Ω_2, which together with Ω_1 builds a dense subset of $\mathbb{D}^2 \times \mathbb{D}^2 \setminus \triangle$.

Therefore, to finish the proof of the theorem we go to the proof of the surjectivity of the mappings defined above.

Without loss of generality we may restrict to the cases $j = 1, 2$.

First note that the sets G_j are nonempty. Therefore, to finish the proof it is sufficient to show that $\Phi(G_j)$ is open and closed in E_j.

First we show that $\Phi(G_j)$ is closed. The proof may be conducted with the standard sequence procedure; however, we shall make use of considerations that were given in Section 3.

Take (p, q) in the closure of $\Phi(G_j)$ with respect to E_j. The continuity property implies that $c(p, q) = l(p, q)$. It follows immediately from Lemma 3 that (p, q) lies in $\Phi(G_j)$.

To show that the image is open it suffices to prove that Φ is locally injective.

So assume that $\Phi(\alpha, \beta, c, \omega, t) = \Phi(\tilde{\alpha}, \tilde{\beta}, \tilde{c}, \tilde{\omega}, \tilde{t})$.

Let $\varphi := \varphi_{\alpha,\beta,\omega}$, $\tilde{\varphi} := \varphi_{\tilde{\alpha},\tilde{\beta},\tilde{\omega}}$.

Let $F(x) = (\bar{\omega} t x_1 + (1-t)x_2 + \eta x_1 x_2)/(1 - ((1-t)\bar{\omega} x_1 + t x_2)\eta)$, where η is properly chosen. It simply follows from the previous discussion that F is a left inverse to both φ and $\tilde{\varphi}$. Therefore, $F = \tilde{F}$, where $\tilde{F}$ denotes the appropriate left inverse to $\tilde{\varphi}$. Thus $t = \tilde{t}$ and $\omega = \tilde{\omega}$. Moreover, $cm_\gamma(c) = \tilde{c} m_{\tilde{\gamma}}(\tilde{c}) =: l \neq 0$. Therefore, it suffices to show the local injectivity of the function

$$\Psi : (\alpha, \beta, c) \mapsto \left(cm_\alpha(c), \frac{l}{c} m_\alpha\left(\frac{l}{c}\right), cm_\beta(c), \frac{l}{c} m_\beta\left(\frac{l}{c}\right) \right)$$

defined for $(\alpha, \beta, c) \in \mathbb{D}^3$ such that $(z, w) = \Phi(\alpha, \beta, c)$ satisfies $|z_1| \neq |z_2|$, $|w_1| \neq |w_2|$, $z_1 \neq w_1$ and $z_2 \neq w_2$ (in particular, $\alpha \neq \beta$, $c \neq 0$).

Proposition 5. *Ψ is locally injective. Moreover, Ψ is two-to-one.*

Proof. Observe first that $\Psi(\alpha, \beta, c) = \Psi(-\alpha, -\beta, -c)$. Therefore, to get the assertion, it suffices to show that for fixed points $z := (z_1, z_2)$, $w := (w_1, w_2)$ such that $z_1 \neq z_2$, $w_1 \neq w_2$, $z_1 \neq w_1$ and $z_2 \neq w_2$ the equation $\Psi(\alpha, \beta, c) = (z_1, z_2, w_1, w_2)$ has at most two solutions.

From the equation we deduce that

$$\alpha = c\frac{z_2(1 - z_1/l)}{z_2 - z_1} + \frac{1}{c}\frac{z_1(z_2 - l)}{z_2 - z_1}, \quad \text{and} \quad \bar{\alpha} = c\frac{1 - z_2/l}{z_1 - z_2} + \frac{1}{c}\frac{z_1 - l}{z_1 - z_2},$$
$$\beta = c\frac{w_2(1 - w_1/l)}{w_2 - w_1} + \frac{1}{c}\frac{w_1(w_2 - l)}{w_2 - w_1}, \quad \text{and} \quad \bar{\beta} = c\frac{1 - w_2/l}{w_1 - w_2} + \frac{1}{c}\frac{w_1 - l}{w_1 - w_2}.$$

We can write the above equations in the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = M \begin{pmatrix} c \\ 1/c \end{pmatrix}, \quad \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = N \begin{pmatrix} c \\ 1/c \end{pmatrix},$$

where $M, N \in \mathbb{C}^{2\times 2}$. Set $v := \binom{c}{1/c}$. The equations imply that $Mv = \bar{N}\bar{v}$.

Notice that

$$\det M = \frac{z_2(1 - z_1/l)w_1(w_2 - l) - w_2(1 - w_1/l)z_1(z_2 - l)}{(z_2 - z_1)(w_2 - w_1)},$$
$$\det N = \frac{(1 - z_2/l)(w_1 - l) - (1 - w_2/l)(z_1 - l)}{(z_2 - z_1)(w_2 - w_1)}.$$

The hypotheses made on z and w ensure that $(1-z_2/l)(w_1-l)$ and $(1-w_2/l)(z_1-l)$ cannot vanish simultaneously, so if $\det N = 0$, we see that the equation $\det M = 0$ reduces to $z_2 w_1 - z_1 w_2 = 0$. Since $l \neq 0$, this together with $\det N = 0$ would imply $z_1 = z_2$ or $z_1 = w_1$, which is excluded. Therefore at least one of the matrices M or N is invertible. Suppose for now that M is invertible, we have $v = P\bar{v}$, with $P := M^{-1}\bar{N}$. Since $\bar{v} = \bar{P}v$, we see that $v = P\bar{P}v$.

Since $M\begin{pmatrix} l \\ 1 \end{pmatrix} = \begin{pmatrix} l \\ l \end{pmatrix}$ and $N\begin{pmatrix} l \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\bar{P}P\begin{pmatrix} \bar{l} \\ 1 \end{pmatrix} = |l|^{-2}\begin{pmatrix} \bar{l} \\ 1 \end{pmatrix}$, so that we have an eigenvalue $|l|^{-2} > 1$ of $\bar{P}P$, and $P\bar{P} \neq I$. So $\dim\ker(I - P\bar{P}) \leq 1$, which means, since v cannot be 0, that there is a nonzero vector $w \in \mathbb{C}^2$, depending only on z, w, l, such that v is collinear to w, which implies $c^2 = w_1/w_2$. So we have at most two possible values for (α, β, c).

If $\det M = 0$, then N is invertible and we reason in the same way starting from $v = N^{-1}\bar{M}v$. □

References

[Agler and McCarthy 2000] J. Agler and J. E. McCarthy, "The three point Pick problem on the bidisk", *New York J. Math.* **6** (2000), 227–236. MR Zbl

[Agler and McCarthy 2002] J. Agler and J. E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics **44**, American Mathematical Society, Providence, RI, 2002. MR Zbl

[Agler et al. 2013] J. Agler, Z. A. Lykova, and N. J. Young, "Extremal holomorphic maps and the symmetrized bidisc", *Proc. Lond. Math. Soc.* (3) **106**:4 (2013), 781–818. MR Zbl

[Agler et al. 2015] J. Agler, Z. A. Lykova, and N. J. Young, "3-extremal holomorphic maps and the symmetrized bidisc", *J. Geom. Anal.* **25**:3 (2015), 2060–2102. MR Zbl

[Carlehed 1999] M. Carlehed, "Potentials in pluripotential theory", *Ann. Fac. Sci. Toulouse Math.* (6) **8**:3 (1999), 439–469. MR Zbl

[Carlehed and Wiegerinck 2003] M. Carlehed and J. Wiegerinck, "Le cône des fonctions plurisousharmoniques négatives et une conjecture de Coman", *Ann. Polon. Math.* **80** (2003), 93–108. MR Zbl

[Coman 2000] D. Coman, "The pluricomplex Green function with two poles of the unit ball of $\mathbb{C}^n$", *Pacific J. Math.* **194**:2 (2000), 257–283. MR Zbl

[Edigarian and Zwonek 1998] A. Edigarian and W. Zwonek, "Invariance of the pluricomplex Green function under proper mappings with applications", *Complex Variables Theory Appl.* **35**:4 (1998), 367–380. MR Zbl

[Kosiński 2014] L. Kosiński, "Spectral Nevanlinna–Pick problem and weak extremals in the symmetrized bidisc", preprint, 2014. arXiv

[Kosiński and Zwonek 2016a] Ł. Kosiński and W. Zwonek, "Extremal holomorphic maps in special classes of domains", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **16**:1 (2016), 159–182. MR Zbl

[Kosiński and Zwonek 2016b] Ł. Kosiński and W. Zwonek, "Nevanlinna–Pick problem and uniqueness of left inverses in convex domains, symmetrized bidisc and tetrablock", *J. Geom. Analysis* **26**:3 (2016), 1863–1890. MR Zbl

[Lempert 1981] L. Lempert, "La métrique de Kobayashi et la représentation des domaines sur la boule", *Bull. Soc. Math. France* **109**:4 (1981), 427–474. MR Zbl

[Nikolov and Pflug 2006] N. Nikolov and P. Pflug, "The multipole Lempert function is monotone under inclusion of pole sets", *Michigan Math. J.* **54**:1 (2006), 111–116. MR Zbl

[Pick 1915] G. Pick, "Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden", *Math. Ann.* **77**:1 (1915), 7–23. MR

[Thomas and Trao 2003] P. J. Thomas and N. V. Trao, "Pluricomplex Green and Lempert functions for equally weighted poles", *Ark. Mat.* **41**:2 (2003), 381–400. MR Zbl

[Warszawski 2015] T. Warszawski, "(Weak) m-extremals and m-geodesics", *Complex Var. Elliptic Equ.* **60**:8 (2015), 1077–1105. MR Zbl

[Wikström 2001] F. Wikström, "Non-linearity of the pluricomplex Green function", *Proc. Amer. Math. Soc.* **129**:4 (2001), 1051–1056. MR Zbl

[Wikström 2003] F. Wikström, "Computing the pluricomplex Green function with two poles", *Experiment. Math.* **12**:3 (2003), 375–384. MR Zbl

Received November 16, 2014. Revised July 21, 2015.

ŁUKASZ KOSIŃSKI
INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
JAGIELLONIAN UNIVERSITY
ŁOJASIEWICZA 6
30-348 KRAKÓW
POLAND
lukasz.kosinski@im.uj.edu.pl

PASCAL J. THOMAS
UNIVERSITÉ PAUL SABATIER
IMJ, 118 ROUTE DE NARBONNE
31062 TOULOUSE CEDEX 9
FRANCE
pascal.thomas@math.univ-toulouse.fr

WŁODZIMIERZ ZWONEK
INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
JAGIELLONIAN UNIVERSITY
ŁOJASIEWICZA 6
30-348 KRAKÓW
POLAND
Wlodzimierz.Zwonek@im.uj.edu.pl

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.423

ENDOTRIVIAL MODULES: A REDUCTION TO p'-CENTRAL EXTENSIONS

CAROLINE LASSUEUR AND JACQUES THÉVENAZ

We examine how, in prime characteristic p, the group of endotrivial modules of a finite group G and the group of endotrivial modules of a quotient of G modulo a normal subgroup of order prime to p are related. There is always an inflation map, but examples show that this map is in general not surjective. We prove that the situation is controlled by a single central extension, namely, the central extension given by a p'-representation group of the quotient of G by its largest normal p'-subgroup.

1. Introduction

Endotrivial modules play an important role in the representation theory of finite groups. They have been classified in a number of special cases; see, e.g., the recent papers [Carlson et al. 2014a; Lassueur and Mazza 2015b] and the references therein. Over an algebraically closed field k of prime characteristic p, endotrivial modules for a finite group G form an abelian group $T(G)$, which is known to be finitely generated. One of the main question is to understand the structure of $T(G)$, and, in particular, of its torsion subgroup $TT(G)$.

We let $X(G)$ be the subgroup of $TT(G)$ consisting of all one-dimensional representations, that is, $X(G) \cong \mathrm{Hom}(G, k^{\times})$. We also let $K(G)$ be the kernel of the restriction map $\mathrm{Res}^G_P : T(G) \to T(P)$ to a Sylow p-subgroup P of G. It is known that $X(G) \subseteq K(G) \subseteq TT(G)$ and that $K(G) = TT(G)$ in almost all cases (namely if we exclude the cases when a Sylow p-subgroup of G is cyclic, generalized quaternion, or semidihedral). Moreover, there are numerous cases, including infinite families of groups G, for which $K(G) = X(G)$. However, this is not always the case, and the structure of $K(G)$ is not understood in general.

Let $O_{p'}(G)$ denote the largest normal subgroup of G of order prime to p and set $Q := G/O_{p'}(G)$ for simplicity. There is always an inflation homomorphism

$$\mathrm{Inf}^G_Q : T(Q) \to T(G)$$

MSC2010: primary 20C20; secondary 20C25.
Keywords: Endotrivial modules, Schur multipliers, central extensions, perfect groups.

which is easily seen to be injective. But examples show that it is in general not surjective, so we cannot expect an isomorphism between $T(G)$ and $T(Q)$. The present article analyzes how $T(G)$ and $T(Q)$ are related, by making use of a suitable central extension of Q. More precisely, associated with Q, there is a p'-representation group $\widetilde{Q}$, which is a central extension with kernel of order prime to p. This controls the behavior of projective representations of Q (in the sense of Schur). When Q is a perfect group, then $\widetilde{Q}$ is unique and is also called the universal p'-central extension of Q. When Q is not perfect, then $\widetilde{Q}$ may not be unique.

The present work is based on a key result by the first author and S. Koshitani [Koshitani and Lassueur 2016]. In the course of their investigation of endotrivial modules for a finite group with dihedral Sylow 2-subgroups, they proved a general result [op. cit., Theorem 4.4] about endotrivial modules for an arbitrary group G in the presence of a normal subgroup N of order prime to p, under mild hypotheses on G (see Hypothesis 3.1). Their result uses modules over twisted group algebras of G/N. Taking $Q = G/N$ with $N = O_{p'}(G)$, we can view such modules as modules over the ordinary group algebra of the central extension $\widetilde{Q}$. In this way, we can show that the structure of $T(G)$ is closely related to the structure of $T(\widetilde{Q})$. Our main result is as follows:

Theorem 1.1. *Let G be a finite group of p-rank at least 2 and no strongly p-embedded subgroups. Let $\widetilde{Q}$ be any p'-representation group of the group $Q := G/O_{p'}(G)$.*

(a) *There exists an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \to T(\widetilde{Q})/X(\widetilde{Q}).$$

In particular, $\Phi_{G,\widetilde{Q}}$ maps the class of $\mathrm{Inf}_Q^G(W)$ to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$, for any endotrivial kQ-module W.

(b) *The map $\Phi_{G,\widetilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G) \to K(\widetilde{Q})/X(\widetilde{Q}).$$

(c) *In particular, if $K(\widetilde{Q}) = X(\widetilde{Q})$, then $K(G) = X(G)$.*

We note that the construction of the map $\Phi_{G,\widetilde{Q}}$ relies on [op. cit., Theorem 4.4], which itself relies on Navarro and Robinson [Navarro and Robinson 2012], whose proof makes use of the classification of finite simple groups. This construction will be made precise in Section 4. Examples show that $\Phi_{G,\widetilde{Q}}$ is in general not surjective (see Section 7), but the theorem provides some information on $K(G)$, for all groups G such that $G/O_{p'}(G) = Q$. In particular, the question of the equality $K(G) = X(G)$ is reduced to the same question for the single group $\widetilde{Q}$.

We also conjecture that $\Phi_{G,\widetilde{Q}}$ induces an isomorphism on the torsion-free part of $T(G)$ and $T(\widetilde{Q})$ (see Section 5). Moreover, in case Q is perfect, then there is an alternative approach to $\Phi_{G,\widetilde{Q}}$ which we present in Section 6.

The two main assumptions on G in Theorem 1.1 are needed for applying the results of [Koshitani and Lassueur 2016]. However, these assumptions are not really restrictive because endotrivial modules are completely understood in the two excluded cases: they are classified if the p-rank is 1 [Mazza and Thévenaz 2007; Carlson et al. 2013], and $T(G) \cong T(H)$ if G has a strongly p-embedded subgroup H; see [Mazza and Thévenaz 2007, Lemma 2.7].

The two assumptions also allow us to prove that $T(G) \cong T(G/[G, A])$, where $A = O_{p'}(G)$, or in other words that the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

with kernel A of order prime to p can always be replaced by the central extension

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \longrightarrow Q \longrightarrow 1.$$

This is explained in Section 3.

2. Notation and preliminaries

Throughout, unless otherwise specified, we use the following notation. We let k denote an algebraically closed field of prime characteristic p. We assume that all groups are finite, and that all modules over group algebras are finitely generated, and we set $\otimes := \otimes_k$. If G is an arbitrary finite group and V is a kG-module, we denote by $\rho_V : G \to \mathrm{GL}(V)$ the corresponding k-representation, and we denote by $\pi_V : \mathrm{GL}(V) \to \mathrm{PGL}(V)$ the canonical surjection. Furthermore, we denote by V^* the k-dual of V endowed with a kG-module structure via $(gf)(v) = f(g^{-1}v)$ for every $g \in G$, $f \in V^*$, $v \in V$.

Assuming moreover that $p \mid |G|$, we recall that a kG-module V is called *endotrivial* if there is an isomorphism of kG-modules $\mathrm{End}_k(V) \cong k \oplus (\mathrm{proj})$, where k denotes the trivial kG-module and (proj) some projective kG-module, which might be zero. Any endotrivial kG-module V splits as a direct sum $V = V_0 \oplus (\mathrm{proj})$ where V_0, the projective-free part of V, is indecomposable and endotrivial. The relation

$$U \sim V \iff U_0 \cong V_0$$

is an equivalence relation on the class of endotrivial kG-modules, and $T(G)$ denotes the resulting set of equivalence classes (which we denote by square brackets). Then $T(G)$, endowed with the law $[U] + [V] := [U \otimes V]$, is an abelian group called the *group of endotrivial modules of* G. The zero element is the class $[k]$ of the trivial module and $-[V] = [V^*]$, the class of the dual module V^*. By a result of Puig, the group $T(G)$ is known to be a finitely generated abelian group; see, e.g., [Carlson et al. 2006, Corollary 2.5].

We let $X(G)$ denote the group of one-dimensional kG-modules endowed with the tensor product $\otimes$, and recall that $X(G) \cong \mathrm{Hom}(G, k^\times) \cong (G/[G, G])_{p'}$. Identifying a one-dimensional module with its class in $T(G)$, we consider $X(G)$ as a subgroup of $T(G)$.

Furthermore, if P is a Sylow p-subgroup of G, we set

$$K(G) = \mathrm{Ker}\big(\mathrm{Res}^G_P : T(G) \to T(P)\big).$$

In other words, the class of an indecomposable endotrivial kG-module V belongs to $K(G)$ if and only if $V{\downarrow}^G_P \cong k \oplus (\mathrm{proj})$, that is, in other words, V is a trivial source module. We have $X(G) \subseteq K(G)$ because any one-dimensional kP-module is trivial. Moreover, $K(G) \subseteq TT(G)$ (see [Carlson et al. 2011, Lemma 2.3]), and $K(G) = TT(G)$ unless P is cyclic, generalized quaternion, or semidihedral, by the main result of [Carlson and Thévenaz 2005].

By a central extension (E, π) of Q, we mean a group extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \mathrm{Ker}\,\pi$ central in E. Recall that (E, π) is said to have the *projective lifting property (relative to k)* if, for every finite-dimensional k-vector space V, every group homomorphism $\theta : Q \to \mathrm{PGL}(V)$ can be completed to a commutative diagram of group homomorphisms:

$$\begin{tikzcd}
1 \arrow[r] & Z \arrow[r] \arrow[d, "\lambda|_Z"'] & E \arrow[r, "\pi"] \arrow[d, "\lambda"'] & Q \arrow[r] \arrow[d, "\theta"'] & 1 \\
1 \arrow[r] & k^\times \cdot \mathrm{Id}_V \arrow[r] & \mathrm{GL}(V) \arrow[r, "\pi_V"] & \mathrm{PGL}(V) \arrow[r] & 1
\end{tikzcd}$$

In general, the homomorphism λ is not uniquely defined. However, by the commutativity of the diagram, the following holds:

Lemma 2.1. *In the above situation, if $\lambda, \lambda' : E \to \mathrm{GL}(V)$ are two liftings of θ to E, then there exists a degree one representation $\mu : E \to \mathrm{GL}(k)$ such that $\lambda' = \lambda \otimes \mu$.*

By results of Schur (slightly generalized for dealing with the case of characteristic p), given a finite group Q, there always exists a central extension (E, π) of Q, with kernel $M_k(Q) := \mathrm{H}^2(Q, k^\times)$, which has the projective lifting property. A *p'-representation group of Q* (or a *representation group of Q relative to k*) is a central extension $(\widetilde{Q}, \pi)$ of Q of minimal order with the projective lifting property. In this case $M_k(Q) \cong \mathrm{Ker}\,\pi \leq [\widetilde{Q}, \widetilde{Q}]$. We recall that $M_k(Q) \cong \mathrm{H}^2(Q, \mathbb{C}^\times)_{p'}$, the p'-part of the Schur multiplier of Q, and that in general a group Q with $X(Q) \neq 1$ may have several nonisomorphic p'-representation groups. Furthermore, fixing a p'-representation group $(\widetilde{Q}, \pi)$ of Q, the abelian group $M_k(Q)$ becomes isomorphic to its $k^{\times}$-dual via the transgression homomorphism

$$\mathrm{tr}: \mathrm{Hom}(M_k(Q), k^{\times}) \to \mathrm{H}^2(Q, k^{\times})$$

defined by $\mathrm{tr}(\varphi) = [\varphi \circ \alpha]$, where the cocycle $\alpha \in Z^2(Q, M_k(Q))$ is in the cohomology class corresponding to the central extension $1 \to M_k(Q) \to \widetilde{Q} \xrightarrow{\pi} Q \to 1$. For further details and proofs we refer the reader to [Nagao and Tsushima 1989, Chapter 3, §5; Curtis and Reiner 1981, §11E].

If V, W are two finite-dimensional k-vector spaces, then the tensor product of linear maps induces a tensor product $-\otimes- : \mathrm{PGL}(V) \times \mathrm{PGL}(W) \to \mathrm{PGL}(V \otimes W)$ via $\pi_V(\alpha) \otimes \pi_W(\beta) := \pi_{V\otimes W}(\alpha \otimes \beta)$ for any $\alpha \in \mathrm{GL}(V)$ and any $\beta \in \mathrm{GL}(W)$. Therefore, if $\mu : Q \to \mathrm{PGL}(V)$ and $\nu : Q \to \mathrm{PGL}(W)$ are group homomorphisms, we may define a group homomorphism $\mu \otimes \nu : Q \to \mathrm{PGL}(V \otimes W)$ via $(\mu \otimes \nu)(q) := \mu(q) \otimes \nu(q)$ for every $q \in Q$. We shall use the following well-known results throughout:

Lemma 2.2. *Let $1 \to A \to G \xrightarrow{\pi} Q \to 1$ be an arbitrary group extension.*

(a) *Whenever V is a kG-module such that $\rho_V(A) \subseteq k^{\times} \cdot \mathrm{Id}_V$, the group homomorphism $\rho_V : G \to \mathrm{GL}(V)$ induces a uniquely defined group homomorphism $\theta_V : Q \to \mathrm{PGL}(V)$ such that the following diagram commutes*:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "\rho_V|_A"'] & G \arrow[r, "\pi"] \arrow[d, "\rho_V"'] & Q \arrow[r] \arrow[d, "\theta_V"'] & 1 \\
1 \arrow[r] & k^{\times} \cdot \mathrm{Id}_V \arrow[r] & \mathrm{GL}(V) \arrow[r, "\pi_V"] & \mathrm{PGL}(V) \arrow[r] & 1
\end{tikzcd}$$

(b) *If V, W are kG-modules such that $\rho_V(A) \subseteq k^{\times} \cdot \mathrm{Id}_V$ and $\rho_W(A) \subseteq k^{\times} \cdot \mathrm{Id}_W$, then $\rho_{V\otimes W}(A) \subseteq k^{\times} \cdot \mathrm{Id}_{V\otimes W}$ and we have $\theta_{V\otimes W} = \theta_V \otimes \theta_W$.*

Proof. (a) Choose a set-theoretic section $s : Q \to G$ for π and define $\theta_V := \pi_V \circ \rho_V \circ s$. Since $\rho_V(A) \subseteq k^{\times} \cdot \mathrm{Id}_V$, the map θ_V is a group homomorphism making the diagram commute. Clearly θ_V is uniquely defined since π is an epimorphism.

(b) This is a straightforward computation. □

3. Endotrivial modules and central extensions

We now fix G to be a finite group of order divisible by p, we set $A := O_{p'}(G)$ and $Q := G/A$, and we denote by $\pi_G : G \to Q$ the quotient map. Moreover, we let $(\widetilde{Q}, \pi_{\widetilde{Q}})$ be a fixed p'-representation group of Q.

Since A is a p'-subgroup of G, inflation induces an injective group homomorphism

$$\mathrm{Inf}_Q^G : T(Q) \to T(G), \qquad [V] \to [\mathrm{Inf}_Q^G(V)].$$

This is because the inflation of a projective module remains projective when the kernel A is a p'-group. We emphasize that endotrivial kG-modules cannot be recovered from endotrivial kQ-modules, as in general the inflation map Inf_Q^G is not an isomorphism; see Section 7.

Hypothesis 3.1. Assume G is a finite group fulfilling the following two conditions:

(1) the p-rank of G is greater than or equal to 2; and

(2) G has no strongly p-embedded subgroups.

The next result restates one of the main results of [Koshitani and Lassueur 2016], but in different terms. Our statement will allow us later to avoid working with modules over twisted group algebras, but simply consider the corresponding projective representations instead.

Theorem 3.2 [Koshitani and Lassueur 2016]. *Suppose G satisfies Hypothesis 3.1.*

(a) *If V is an indecomposable endotrivial kG-module, then $V{\downarrow}_A^G \cong Y \oplus \cdots \oplus Y$, where Y is a one-dimensional kA-module.*

(b) *If V is an indecomposable endotrivial kG-module, then $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$.*

Proof. (a) Since G satisfies Hypothesis 3.1, any composition factor Y of $V{\downarrow}_A^G$ is G-invariant, by [op. cit., Lemma 4.3]. Therefore $V{\downarrow}_A^G \cong Y \oplus \cdots \oplus Y$ and [op. cit., Theorem 4.4] proves that $\dim Y = 1$.

(b) This is a restatement of (a). □

Corollary 3.3. *Suppose that G satisfies Hypothesis 3.1. The inflation map*

$$\mathrm{Inf}_{G/[G,A]}^G : T(G/[G,A]) \to T(G)$$

is a group isomorphism.

Proof. Since $[G, A]$ is a normal p'-subgroup of G, the inflation map $\mathrm{Inf}_{G/[G,A]}^G$ is a well-defined injective group homomorphism. In order to prove that it is surjective, it suffices to prove that $[G, A]$ acts trivially on any indecomposable endotrivial kG-module V. But by Theorem 3.2 we have

$$\rho_V([G, A]) \subseteq [\rho_V(G), \rho_V(A)] \subseteq [\rho_V(G), k^\times \cdot \mathrm{Id}_V] = \{\mathrm{Id}_V\}.$$

Hence $[G, A]$ acts trivially on V. □

Corollary 3.3 is a reduction to the case of central extensions. Explicitly, for the study of endotrivial modules, we may always replace the given extension

$$1 \to A \to G \to Q \to 1,$$

and consider instead the central extension

$$1 \to A/[G, A] \to G/[G, A] \to Q \to 1.$$

We shall in fact not use this reduction for the proof of our main result, but rather apply directly Theorem 3.2.

Lemma 3.4. *Let $(\widetilde{Q}, \pi_{\widetilde{Q}})$ be a p'-representation group of Q. Then $X(\widetilde{Q}) = \mathrm{Inf}_Q^{\widetilde{Q}}(X(Q))$, hence $X(\widetilde{Q}) \cong X(Q)$.*

Proof. We apply the fact, mentioned in Section 2, that $\mathrm{Ker}\,\pi_{\widetilde{Q}} \subseteq [\widetilde{Q}, \widetilde{Q}]$. This implies that any one-dimensional representation of $\widetilde{Q}$ has $\mathrm{Ker}\,\pi_{\widetilde{Q}}$ in its kernel, hence is inflated from $\widetilde{Q}/\mathrm{Ker}\,\pi_{\widetilde{Q}} \cong Q$.

Another way of seeing the same thing is to associate to the central extension

$$1 \longrightarrow M_k(Q) \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

the Hochschild–Serre five-term exact sequence

$$1 \longrightarrow \mathrm{Hom}(Q, k^\times) \xrightarrow{\mathrm{Inf}} \mathrm{Hom}(\widetilde{Q}, k^\times) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(M_k(Q), k^\times) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(Q, k^\times) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(\widetilde{Q}, k^\times).$$

Since the transgression map tr is an isomorphism, the first map Inf must be an isomorphism as well. $\square$

4. Proof of Theorem 1.1

Keep the notation of the previous section. Moreover, given an endotrivial kG-module V such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$, we let

$$\theta_V : Q \to \mathrm{PGL}(V)$$

denote the induced homomorphism constructed in Lemma 2.2(a). The projective lifting property for the central extension $(\widetilde{Q}, \pi_{\widetilde{Q}})$ allows us to fix a representation

$$\rho_{V_{\widetilde{Q}}} : \widetilde{Q} \to \mathrm{GL}(V)$$

lifting θ_V to $\widetilde{Q}$. We denote by $V_{\widetilde{Q}}$ the corresponding $k\widetilde{Q}$-module.

Lemma 4.1. *Let V be an endotrivial kG-module such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$. Then $V_{\widetilde{Q}}$ is an endotrivial $k\widetilde{Q}$-module.*

Proof. We have to work with two group extensions

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi_G} Q \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow M \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1,$$

where $M := M_k(Q)$. Both A and M have order prime to p.

Let $P \in \mathrm{Syl}_p(G)$, set $\bar{P} := AP/A \in \mathrm{Syl}_p(Q)$, and let $\iota_P : P \to AP$ be the inclusion map, so that

$$\phi := \pi_G \circ \iota_P : P \to \bar{P}$$

is an isomorphism. Next choose $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{Q})$ such that $M\widetilde{P}/M = \bar{P} \in \mathrm{Syl}_p(Q)$. Let $\iota_{\widetilde{P}} : \widetilde{P} \to M\widetilde{P}$ be the inclusion map, so that $\psi := \pi_{\widetilde{Q}} \circ \iota_{\widetilde{P}} : \widetilde{P} \to \bar{P}$ is an isomorphism. Consider now the two commutative diagrams:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\rho_V"'] & \bar{P} \arrow[d, "\theta_V"] \\ \mathrm{GL}(V) \arrow[r, "\pi_V"] & \mathrm{PGL}(V) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{P} \arrow[r, "\psi"] \arrow[d, "\rho_{\widetilde{Q}}"'] & \bar{P} \arrow[d, "\theta_V"] \\ \mathrm{GL}(V) \arrow[r, "\pi_V"] & \mathrm{PGL}(V) \end{tikzcd}$$

where we write $\rho_{\widetilde{Q}} := \rho_{V_{\widetilde{Q}}}$ for simplicity. Since ϕ and ψ are isomorphisms, for any $u \in \bar{P}$,

$$\pi_V \rho_V \phi^{-1}(u) = \theta_V(u) = \pi_V \rho_{\widetilde{Q}} \psi^{-1}(u).$$

We claim that if two elements $u_1, u_2 \in \mathrm{GL}(V)$ have p-power order and satisfy $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = u_2$. Postponing the proof of the claim, we deduce that

$$\rho_V \phi^{-1}(u) = \rho_{\widetilde{Q}} \psi^{-1}(u),$$

because they have p-power order. This means that the representations $(\rho_V)|_P$ and $(\rho_{\widetilde{Q}})|_{\widetilde{P}}$, transported via isomorphisms to representations of $\bar{P}$, are equal. Now, a module is endotrivial if and only if its restriction to a Sylow p-subgroup is; see [Carlson et al. 2006, Proposition 2.6]. Moreover, this property is preserved when transported via group isomorphisms. Since V is endotrivial, so is $V{\downarrow_P}$, hence so is $V_{\widetilde{Q}}{\downarrow_{\widetilde{P}}}$, and it follows that $V_{\widetilde{Q}}$ is endotrivial.

We are left with the proof of the claim. If $\pi_V(u_1) = \pi_V(u_2)$, then $u_1 = \alpha u_2$ where $\alpha \in k^\times$. For some large enough power p^n, we have $u_1^{p^n} = u_2^{p^n} = 1$. Therefore we obtain

$$1 = u_1^{p^n} = (\alpha u_2)^{p^n} = \alpha^{p^n} u_2^{p^n} = \alpha^{p^n}.$$

But there are no nontrivial p-th roots of unity in $k^\times$, so $\alpha = 1$, hence $u_1 = u_2$. □

Proposition 4.2. *Assume G satisfies Hypothesis 3.1. Then there is an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \to T(\widetilde{Q})/X(\widetilde{Q})$$

defined by $\Phi_{G,\widetilde{Q}}([V] + X(G)) := [V_{\widetilde{Q}}] + X(\widetilde{Q})$ for any indecomposable endotrivial kG-module V. Moreover, for any endotrivial kQ-module W, the homomorphism $\Phi_{G,\widetilde{Q}}$ maps the class of $\mathrm{Inf}_Q^G(W)$ to the class of $\mathrm{Inf}_Q^{\widetilde{Q}}(W)$.

Proof. First, Lemma 4.1 allows us to define a map $\phi : T(G) \to T(\widetilde{Q})/X(\widetilde{Q})$ by setting $\phi([V]) := [V_{\widetilde{Q}}] + X(\widetilde{Q})$ for any $[V] \in T(G)$ such that $\rho_V(A) \subseteq k^\times \cdot \mathrm{Id}_V$. The definition of $\phi([V])$ does not depend on the choice of $V_{\widetilde{Q}}$, for if $\rho_{V'_{\widetilde{Q}}}$ is a second lifting of θ_V to $\widetilde{Q}$, then by Lemma 2.1 there exists $X \in X(\widetilde{Q})$ such that $V'_{\widetilde{Q}} \cong V_{\widetilde{Q}} \otimes X$, hence $\phi([V_{\widetilde{Q}}]) = \phi([V'_{\widetilde{Q}}])$.

Next, let V, W be two indecomposable endotrivial kG-modules. Theorem 3.2 implies that $\rho_{V\otimes W}(A) = (\rho_V \otimes \rho_W)(A) \subseteq k^\times \cdot \mathrm{Id}_{V\otimes W}$. Thus, by Lemma 2.2(b),

$\theta_{V\otimes W}=\theta_V\otimes\theta_W$, and it is easy to verify that $\rho_{V_{\widetilde{Q}}}\otimes\rho_{W_{\widetilde{Q}}}$ lifts $\theta_V\otimes\theta_W$ to $\widetilde{Q}$. Therefore, by Lemma 2.1, there exists $X\in X(\widetilde{Q})$ such that $(V\otimes W)_{\widetilde{Q}}\cong V_{\widetilde{Q}}\otimes W_{\widetilde{Q}}\otimes X$. This shows that ϕ is a group homomorphism.

It is clear that $\operatorname{Ker}\phi = X(G)$, since by construction $\dim V_{\widetilde{Q}} = \dim V$ for any indecomposable endotrivial kG-module V. As a result, ϕ induces the required homomorphism $\Phi_{G,\widetilde{Q}}$.

Finally, if W is any endotrivial kQ-module, then the $k\widetilde{Q}$-module constructed from $V=\operatorname{Inf}_Q^G(W)$ is easily seen to be the inflated module $V_{\widetilde{Q}}=\operatorname{Inf}_Q^{\widetilde{Q}}(W)$, because the map $\theta_V : Q\to \mathrm{PGL}(V)$ comes from a group homomorphism $Q\to \mathrm{GL}(V)$. This shows that the class of $\operatorname{Inf}_Q^G(W)$ is mapped to the class of $\operatorname{Inf}_Q^{\widetilde{Q}}(W)$ under the map $\Phi_{G,\widetilde{Q}}$, proving the additional statement. □

Corollary 4.3. *Assume G satisfies Hypothesis 3.1. If $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are two nonisomorphic p'-representation groups of Q, then*

$$\Phi_{\widetilde{Q}_1,\widetilde{Q}_2} : T(\widetilde{Q}_1)/X(\widetilde{Q}_1)\to T(\widetilde{Q}_2)/X(\widetilde{Q}_2)$$

is an isomorphism.

Proof. Let V be an indecomposable $k\widetilde{Q}_1$-module. By construction

$$\Phi_{\widetilde{Q}_1,\widetilde{Q}_2}([V]+X(\widetilde{Q}_1)) = [W]+X(\widetilde{Q}_2),$$

where $W := V_{\widetilde{Q}_2}$ is a $k\widetilde{Q}_2$-module such that ρ_W lifts $\theta_V : Q\to \mathrm{PGL}(V)$ to $\widetilde{Q}_2$. But then ρ_V lifts $\theta_W=\theta_V$ to $\widetilde{Q}_1$, so that by construction

$$\Phi_{\widetilde{Q}_2,\widetilde{Q}_1}([W]+X(\widetilde{Q}_2)) = [V]+X(\widetilde{Q}_1).$$

In other words, $\Phi_{\widetilde{Q}_1,\widetilde{Q}_2}\circ\Phi_{\widetilde{Q}_2,\widetilde{Q}_1}=\mathrm{Id}$. Similarly $\Phi_{\widetilde{Q}_2,\widetilde{Q}_1}\circ\Phi_{\widetilde{Q}_1,\widetilde{Q}_2}=\mathrm{Id}$. □

Corollary 4.4. *Assume G satisfies Hypothesis 3.1. The map $\Phi_{G,\widetilde{Q}}$ induces by restriction an injective group homomorphism*

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G)\to K(\widetilde{Q})/X(\widetilde{Q}).$$

In particular, if $K(\widetilde{Q})\cong X(\widetilde{Q})$, then $K(G)\cong X(G)$.

Proof. Let $P\in \mathrm{Syl}_p(G)$ and let V be an indecomposable endotrivial kG-module. As in the proof of Lemma 4.1, the two modules $V{\downarrow}_P^G$ and $V_{\widetilde{Q}}{\downarrow}_{\widetilde{P}}^{\widetilde{Q}}$ are isomorphic, provided we view them as modules over the group $\bar{P}$ via the isomorphisms $P\cong\bar{P}$ and $\widetilde{P}\cong\bar{P}$. It follows that V has a trivial source if and only if $V_{\widetilde{Q}}$ has. Therefore $\Phi_{G,\widetilde{Q}}$ restricts to an injective group homomorphism

$$\Phi_{G,\widetilde{Q}} : K(G)/X(G)\to K(\widetilde{Q})/X(\widetilde{Q}).$$

The special case follows. □

Proposition 4.2 together with Corollary 4.4 prove Theorem 1.1.

5. Conjecture on the torsion-free part

We keep the notation of the previous sections. Let $TF(G) = T(G)/TT(G)$, the torsion-free part of the group of endotrivial modules. Since $X(G) \subseteq TT(G)$, the map

$$\Phi_{G,\widetilde{Q}} : T(G)/X(G) \to T(\widetilde{Q})/X(\widetilde{Q})$$

induces an injective group homomorphism

$$\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q}).$$

We know that $\Phi_{G,\widetilde{Q}}$ is in general not surjective, but we conjecture that $\Psi_{G,\widetilde{Q}}$ is surjective.

Conjecture 5.1. (a) The map $\mathrm{Inf}_Q^G : TF(Q) \to TF(G)$ is an isomorphism.

(b) The map $\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q})$ is an isomorphism.

Note that (b) follows from (a), by applying (a) to both $\mathrm{Inf}_Q^G : TF(Q) \to TF(G)$ and $\mathrm{Inf}_Q^{\widetilde{Q}} : TF(Q) \to TF(\widetilde{Q})$ and composing, because the map $\Psi_{G,\widetilde{Q}} : TF(G) \to TF(\widetilde{Q})$ is the identity on modules inflated from Q.

Part (a) of Conjecture 5.1 is in fact a consequence of any of the two conjectures made in [Carlson et al. 2014b]. First, Conjecture 10.1 in that reference asserts that, if a group homomorphism $\phi : G \to G'$ induces an isomorphism between the corresponding p-fusion systems, then ϕ should induce an isomorphism $TF(G') \xrightarrow{\sim} TF(G)$. In the special case where ϕ is the quotient map $\phi : G \to Q = G/O_{p'}(G)$, it is well-known that the fusion systems are isomorphic, so we would obtain the isomorphism $TF(Q) \xrightarrow{\sim} TF(G)$ of Conjecture 5.1 above. This special case is explicitly mentioned at the end of Section 10 in [op. cit].

Conjecture 9.2 in [op. cit.] asserts that the group $TF(G)$ should be generated by endotrivial modules lying in the principal block. Since $O_{p'}(G)$ acts trivially on any module lying in the principal block of G, such a module is inflated from Q, so the inflation map $\mathrm{Inf}_Q^G : TF(Q) \to TF(G)$ in Conjecture 5.1 above should be an isomorphism.

Example 7.3 below illustrates a method allowing one to prove that the maps in Conjecture 5.1 are isomorphisms in specific cases.

6. The perfect case

When the group $Q = G/O_{p'}(G)$ is perfect, there is an alternative approach to the construction of the injective group homomorphism of Theorem 1.1(a) using universal central extensions.

Recall that a *universal p'-central extension* of an arbitrary finite group Q is by definition a central extension

$$1 \longrightarrow M_{p'} \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

with $M_{p'} = \operatorname{Ker} \pi_{\widetilde{Q}}$ of order prime to p and satisfying the following universal property: For any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

with $Z = \operatorname{Ker} \pi$ of order prime to p, there exists a unique group homomorphism $\phi : \widetilde{Q} \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & M_{p'} \arrow[r] \arrow[d, "\phi|_{M_{p'}}"'] & \widetilde{Q} \arrow[r, "\pi_{\widetilde{Q}}"] \arrow[d, "\phi"'] & Q \arrow[r] \arrow[d, "\mathrm{Id}"'] & 1 \\
1 \arrow[r] & Z \arrow[r] & E \arrow[r, "\pi"] & Q \arrow[r] & 1
\end{tikzcd}$$

A standard argument shows that if a universal p'-central extension $(\widetilde{Q}, \pi_{\widetilde{Q}})$ exists, then it is unique up to isomorphism.

Lemma 6.1. *If $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is a universal p'-central extension of a finite group R, then $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is p'-representation group of Q.*

Proof. Let $(\check{Q}, \pi_{\check{Q}})$ be an arbitrary p'-representation group of Q. Let V be a finite-dimensional k-vector space and $\theta : Q \to \mathrm{PGL}(V)$ a group homomorphism. Because $(\check{Q}, \pi_{\check{Q}})$ has the projective lifting property and $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is universal, there exist group homomorphisms $\tilde{\theta} : \check{Q} \to \mathrm{GL}(V)$ and $\phi : \widetilde{Q} \to \check{Q}$ such that $\tilde{\theta} \circ \phi$ lifts θ. Therefore $(\widetilde{Q}, \pi_{\widetilde{Q}})$ has the projective lifting property as well.

Now, because $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is universal, it is easy to see that $X(\widetilde{Q}) = X(Q) = 1$. Therefore the Hochschild–Serre 5-term exact sequence associated to $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is:

$$1 \longrightarrow 1 \longrightarrow 1 \longrightarrow \mathrm{Hom}(M_{p'}, k^{\times}) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(Q, k^{\times}) \xrightarrow{\mathrm{Inf}} \mathrm{H}^2(\widetilde{Q}, k^{\times})$$

Thus the transgression map $\mathrm{tr} : \mathrm{Hom}(M_{p'}, k^{\times}) \to \mathrm{H}^2(Q, k^{\times}) = M_k(Q)$ is injective. But $M_{p'} \cong \mathrm{Hom}(M_{p'}, k^{\times})$, therefore by minimality of $(\check{Q}, \pi_{\check{Q}})$, we have $|M_{p'}| = |M_k(Q)|$ and $|\widetilde{Q}| = |\check{Q}|$, proving that $(\widetilde{Q}, \pi_{\widetilde{Q}})$ is a p'-representation group of Q. $\square$

Lemma 6.2. *Any finite perfect group Q admits a universal p'-central extension.*

Proof. Since Q is a perfect group, it is well-known that Q has a representation group relative to $\mathbb{C}$, say $(\hat{Q}, \pi_{\hat{Q}})$, which is unique up to isomorphism and that

$$\mathrm{Ker}(\pi_{\hat{Q}}) =: M \cong M_{\mathbb{C}}(Q) = \mathrm{H}^2(Q, \mathbb{C}^{\times}),$$

the Schur multiplier of Q. Moreover, $(\hat{Q}, \pi_{\hat{Q}})$ is a universal central extension of Q, in particular perfect; see [Rotman 1995, Theorem 11.11]. Thus, for any central extension

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

where $Z = \operatorname{Ker} \pi$, there exists a unique group homomorphism $\psi : \hat{Q} \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] \arrow[d, "\psi|_M"'] & \hat{Q} \arrow[r, "\pi_{\hat{Q}}"] \arrow[d, "\psi"'] & Q \arrow[r] \arrow[d, "\mathrm{Id}"'] & 1 \\
1 \arrow[r] & Z \arrow[r] & E \arrow[r, "\pi"] & Q \arrow[r] & 1
\end{tikzcd}$$

If Z has order prime to p, then the p-part M_p of M lies in the kernel of $\psi|_M$. Passing to the quotient by M_p, we define $\widetilde{Q} := \hat{Q}/M_p$ and denote by $\phi : \widetilde{Q} \to E$ the map induced by ψ. Thus we obtain an induced central extension

$$1 \longrightarrow M_{p'} \longrightarrow \widetilde{Q} \xrightarrow{\pi_{\widetilde{Q}}} Q \longrightarrow 1$$

where $M_{p'} := M/M_p$, a universal p'-central extension of Q by construction. □

Given an arbitrary group extension $1 \to A \to G \to Q \to 1$ with perfect quotient Q and kernel A of order prime to p, there is an induced p'-central extension:

$$1 \longrightarrow A/[G, A] \longrightarrow G/[G, A] \xrightarrow{\pi_G} Q \longrightarrow 1$$

Moreover, by the above, Q admits a universal p'-central extension, which is in fact a p'-representation group $(\widetilde{Q}, \pi_{\widetilde{Q}})$ of Q. Therefore, by the universal property, there exists a unique group homomorphism $\phi_G : \widetilde{Q} \to G/[G, A]$ lifting the identity on Q.

Lemma 6.3. *The homomorphism $\phi_G : \widetilde{Q} \to G/[G, A]$ induces a group homomorphism*

$$\phi_G^* : T(G/[G, A]) \to T(\widetilde{Q})$$

such that $\phi_G^* = \operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}} \circ \operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$. Moreover, both $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}}$ and $\operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$ preserve indecomposability of endotrivial modules.

Proof. The kernel of ϕ_G is contained in $\operatorname{Ker} \pi_{\widetilde{Q}} = M_{p'}$, which is a p'-group. Therefore, there is an induced inflation map $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}} : T(\operatorname{Im}(\phi_G)) \to T(\widetilde{Q})$, preserving indecomposability of endotrivial modules.

Since $\operatorname{Im}(\phi_G)$ maps onto Q via π_G, the group $G/[G, A]$ is the product of $\operatorname{Im}(\phi_G)$ and the central p'-subgroup $A/[G, A]$. It follows that $\operatorname{Im}(\phi_G)$ is a normal subgroup of $G/[G, A]$ of index prime to p. Therefore, the restriction to $\operatorname{Im}(\phi_G)$ of any indecomposable endotrivial $k(G/[G, A])$-module remains indecomposable and is endotrivial [Carlson et al. 2009, Proposition 3.1].

We define ϕ_G^* to be the composite of $\operatorname{Inf}_{\operatorname{Im}(\phi_G)}^{\widetilde{Q}}$ and $\operatorname{Res}_{\operatorname{Im}(\phi_G)}^{G/[G,A]}$. □

Composing the group homomorphism

$$\phi_G^* : T(G/[G, A]) \to T(\widetilde{Q})$$

with the inverse of the isomorphism

$$\mathrm{Inf}_{G/[G,A]}^{G} : T(G/[G, A]) \to T(G)$$

of Corollary 3.3, we obtain a group homomorphism

$$\Phi : T(G) \to T(\widetilde{Q}).$$

We now show that this provides the alternative approach to the map of Theorem 1.1.

Proposition 6.4. *Suppose that G satisfies Hypothesis 3.1 and that Q is perfect.*

(a) $\mathrm{Ker}\,\Phi = X(G)$.

(b) *The induced injective group homomorphism*

$$\bar{\Phi} : T(G)/X(G) \to T(\widetilde{Q}) = T(\widetilde{Q})/X(\widetilde{Q})$$

coincides with the map $\Phi_{G,\widetilde{Q}}$ of Theorem 1.1.

Proof. Consider the map $\phi_G^* : T(G/[G, A]) \to T(\widetilde{Q})$ of Lemma 6.3. It is clear that the image of a one-dimensional module is one-dimensional, hence trivial since $X(\widetilde{Q}) = 1$ by Lemma 3.4. Therefore $X(G) \subseteq \mathrm{Ker}\,\Phi$. It follows that Φ induces a group homomorphism $\bar{\Phi}$ as in the statement.

Our assumption on G implies that, if V is an endotrivial kG-module, then $[G, A]$ acts trivially on V (Corollary 3.3). Moreover, $\rho_V : G/[G, A] \to \mathrm{GL}(V)$ lifts $\theta_V : Q \to \mathrm{PGL}(V)$, as in Section 4. It is then clear that $\rho_V\,\phi_G : \widetilde{Q} \to \mathrm{GL}(V)$ also lifts $\theta_V : Q \to \mathrm{PGL}(V)$. Therefore, the definition of $\Phi_{G,\widetilde{Q}}$ (see Proposition 4.2) shows that the class of V is mapped by $\Phi_{G,\widetilde{Q}}$ to the class of the module $V_{\widetilde{Q}}$ corresponding to the representation $\rho_V\,\phi_G$. In other words, $[V_{\widetilde{Q}}] = \Phi([V])$ and this shows that $\Phi_{G,\widetilde{Q}}$ coincides with $\bar{\Phi}$.

Finally, since $\Phi_{G,\widetilde{Q}}$ is injective and is equal to $\bar{\Phi}$, we have $\mathrm{Ker}\,\bar{\Phi} = \{0\}$. Therefore we obtain $\mathrm{Ker}\,\Phi = X(G)$. □

Remark 6.5. The proof we give above shows that Proposition 6.4 remains valid if the assumption that Q is perfect is replaced by the assumption that Q admits a universal p'-central extension. It is proved in [Lassueur and Thévenaz 2017] that this happens if and only if $X(Q) = 1$, that is, Q is p'-perfect. Here, for simplicity, we restrict ourselves to the perfect case.

7. Examples

In this final section, we provide various examples, in particular illustrating cases where the morphism $\Phi_{G,\widetilde{Q}}$ is not surjective.

Example 7.1. Suppose that Q is simple and take $G = Q$, hence $A = O_{p'}(G) = \{1\}$. Then $\Phi_{Q,\widetilde{Q}}$ is just the inflation map $T(Q) \to T(\widetilde{Q})$. If Q is a finite simple group listed in the table below, then it is known that its unique p'-representation group $\widetilde{Q}$ has indecomposable endotrivial modules lying in faithful p-blocks, namely not inflated from Q.

Q	p	$\widetilde{Q}$	$T(Q)$	$T(\widetilde{Q})$
$\mathfrak{A}_6$	3	$2\,.\,\mathfrak{A}_6$	$\mathbb{Z} \oplus \mathbb{Z}/4$	$\mathbb{Z} \oplus \mathbb{Z}/8$
$\mathfrak{A}_6$	2	$3\,.\,\mathfrak{A}_6$	$\mathbb{Z}^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}/3$
M_{22}	3	$4\,.\,M_{22}$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$
J_3	2	$3\,.\,J_3$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/3$
Ru	3	$2\,.\,Ru$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/4$
Fi_{22}	5	$6\,.\,Fi_{22}$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z} \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/2$

The results concerning the sporadic groups can be found in [Lassueur and Mazza 2015b, Table 3], and those about the alternating group $\mathfrak{A}_6$ in [Lassueur and Mazza 2015a, Theorems A and B] together with [Carlson et al. 2009, Theorems A and B].

Further examples are given by the exceptional covering group $2.F_4(2)$ of the exceptional group of Lie type $F_4(2)$, which possesses simple torsion endotrivial modules lying in faithful blocks in characteristics 5 and 7 [Lassueur and Malle 2015, Proposition 5.5], although the full structure of the group of endotrivial modules has not been determined in these cases.

Example 7.2. Assume $p > 2$, let $n \geq \max\{2p, p+4\}$ be an integer and denote by $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ the two isoclinic p'-representation groups of the symmetric group $\mathfrak{S}_n$. Corollary 4.3 yields

$$T(\widetilde{\mathfrak{S}}_n)/X(\widetilde{\mathfrak{S}}_n) \cong T(\widehat{\mathfrak{S}}_n)/X(\widehat{\mathfrak{S}}_n).$$

However, Lassueur and Mazza [2015a, Theorem B, parts (1) and (2)] prove a stronger result, namely

$$T(\widetilde{\mathfrak{S}}_n) = \mathrm{Inf}_{\mathfrak{S}_n}^{\widetilde{\mathfrak{S}}_n}(T(\mathfrak{S}_n)) \quad \text{and} \quad T(\widehat{\mathfrak{S}}_n) = \mathrm{Inf}_{\mathfrak{S}_n}^{\widehat{\mathfrak{S}}_n}(T(\mathfrak{S}_n)).$$

Consequently, given any finite group G such that $G/O_{p'}(G)$ is isomorphic to one of $\mathfrak{S}_n$, $\widetilde{\mathfrak{S}}_n$ or $\widehat{\mathfrak{S}}_n$ (with $n \geq \max\{2p, p+4\}$), by Theorem 1.1 there exist injective group homomorphisms

$$T(\mathfrak{S}_n)/X(\mathfrak{S}_n) \longrightarrow T(G)/X(G) \xrightarrow{\Phi_{G,\widehat{\mathfrak{S}}_n}} T(\widehat{\mathfrak{S}}_n)/X(\widehat{\mathfrak{S}}_n) \xrightarrow{\sim} T(\mathfrak{S}_n)/X(\mathfrak{S}_n),$$

where the first map is induced by inflation. Hence we have $T(G)/X(G) \cong T(\mathfrak{S}_n)/X(\mathfrak{S}_n)$. Recall that the structure of $T(\mathfrak{S}_n)$ is known [Carlson et al. 2009].

Example 7.3. In this final example, we outline a method which allows us to show that the maps $\mathrm{Inf}_Q^{\widetilde{Q}}$ is an isomorphism on the torsion-free part of the groups of endotrivial modules of Q and $\widetilde{Q}$ in some concrete cases.

Specifically, we may use the fact that endotrivial modules are liftable to characteristic zero, and afford characters taking root of unity values at p-singular conjugacy classes; see [Lassueur et al. 2016, Theorem 1.3 and Corollary 2.3]. Therefore, if for every faithful p-block B of $k\widetilde{Q}$ (of full defect) no elements of $\mathbb{Z}\,\mathrm{Irr}_{\mathbb{C}}(B)$ take root of unity values at p-singular conjugacy classes of $\widetilde{Q}$, then any endotrivial $k\widetilde{Q}$-module is inflated from Q, hence

$$\mathrm{Inf}_Q^{\widetilde{Q}} : TF(Q) \to TF(\widetilde{Q})$$

is an isomorphism.

This was used [Lassueur and Mazza 2015a, Theorem B] in the case that $Q = \mathfrak{S}_n$, $n \geq \max\{2p, p+4\}$ (as mentioned in Example 7.2 above), as well as for a large number of sporadic simple groups Q [Lassueur and Mazza 2015b, Lemmas 4.3 and 6.2]. More precisely, in characteristic $p = 2$ for $Q = M_{12}$, M_{22}, J_2, HS, McL, Ru, Suz, ON, Fi_{22}, Co_1, Fi_{24}', or B; in characteristic $p = 3$ for $Q = M_{12}$, J_2, HS, Suz, Fi_{22}, Co_1, or B; in characteristic $p = 5$ for $Q = J_2$, HS, Ru, Suz, Co_1, Fi_{24}', or B; and in characteristic $p = 7$ for $Q = Co_1$, Fi_{24}', or B.

Acknowledgements

The authors are indebted to Shigeo Koshitani for several useful discussions and for providing references.

References

[Carlson and Thévenaz 2005] J. F. Carlson and J. Thévenaz, "The classification of torsion endo-trivial modules", *Ann. of Math.* (2) **162**:2 (2005), 823–883. MR Zbl

[Carlson et al. 2006] J. F. Carlson, N. Mazza, and D. K. Nakano, "Endotrivial modules for finite groups of Lie type", *J. Reine Angew. Math.* **595** (2006), 93–119. MR Zbl

[Carlson et al. 2009] J. F. Carlson, N. Mazza, and D. K. Nakano, "Endotrivial modules for the symmetric and alternating groups", *Proc. Edinb. Math. Soc.* (2) **52**:1 (2009), 45–66. MR Zbl

[Carlson et al. 2011] J. F. Carlson, N. Mazza, and J. Thévenaz, "Endotrivial modules for p-solvable groups", *Trans. Amer. Math. Soc.* **363**:9 (2011), 4979–4996. MR Zbl

[Carlson et al. 2013] J. F. Carlson, N. Mazza, and J. Thévenaz, "Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup", *J. Eur. Math. Soc.* (*JEMS*) **15**:1 (2013), 157–177. MR Zbl

[Carlson et al. 2014a] J. F. Carlson, N. Mazza, and D. K. Nakano, "Endotrivial modules for the general linear group in a nondefining characteristic", *Math. Z.* **278**:3-4 (2014), 901–925. MR Zbl

[Carlson et al. 2014b] J. F. Carlson, N. Mazza, and J. Thévenaz, "Torsion-free endotrivial modules", *J. Algebra* **398** (2014), 413–433. MR Zbl

[Curtis and Reiner 1981] C. W. Curtis and I. Reiner, *Methods of representation theory, I: With applications to finite groups and orders*, Wiley, New York, 1981. MR Zbl

[Koshitani and Lassueur 2016] S. Koshitani and C. Lassueur, "Endo-trivial modules for finite groups with dihedral Sylow 2-subgroup", *J. Group Theory* **19**:4 (2016), 635–660. MR Zbl

[Lassueur and Malle 2015] C. Lassueur and G. Malle, "Simple endotrivial modules for linear, unitary and exceptional groups", *Math. Z.* **280**:3-4 (2015), 1047–1074. MR Zbl

[Lassueur and Mazza 2015a] C. Lassueur and N. Mazza, "Endotrivial modules for the Schur covers of the symmetric and alternating groups", *Algebr. Represent. Theory* **18**:5 (2015), 1321–1335. MR Zbl

[Lassueur and Mazza 2015b] C. Lassueur and N. Mazza, "Endotrivial modules for the sporadic simple groups and their covers", *J. Pure Appl. Algebra* **219**:9 (2015), 4203–4228. MR Zbl

[Lassueur and Thévenaz 2017] C. Lassueur and J. Thévenaz, "Universal p'-central extensions", *Expo. Math.* (2017).

[Lassueur et al. 2016] C. Lassueur, G. Malle, and E. Schulte, "Simple endotrivial modules for quasi-simple groups", *J. Reine Angew. Math.* **712** (2016), 141–174. MR Zbl

[Mazza and Thévenaz 2007] N. Mazza and J. Thévenaz, "Endotrivial modules in the cyclic case", *Arch. Math.* (*Basel*) **89**:6 (2007), 497–503. MR Zbl

[Nagao and Tsushima 1989] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press, Boston, 1989. MR Zbl

[Navarro and Robinson 2012] G. Navarro and G. R. Robinson, "On endo-trivial modules for p-solvable groups", *Math. Z.* **270**:3-4 (2012), 983–987. MR Zbl

[Rotman 1995] J. J. Rotman, *An introduction to the theory of groups*, 4th ed., Graduate Texts in Mathematics **148**, Springer, 1995. MR Zbl

Received January 4, 2016. Revised May 26, 2016.

CAROLINE LASSUEUR
FB MATHEMATIK
TECHNISCHE UNIVERSITAT KAISERSLAUTERN
POSTFACH 3049
D-67653 KAISERSLAUTERN
GERMANY
lassueur@mathematik.uni-kl.de

JACQUES THÉVENAZ
SECTION DE MATHÉMATIQUES
EPFL
STATION 8
CH-1015 LAUSANNE
SWITZERLAND
jacques.thevenaz@epfl.ch

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.439

INFINITELY MANY POSITIVE SOLUTIONS FOR THE FRACTIONAL SCHRÖDINGER–POISSON SYSTEM

WEIMING LIU

We consider a fractional Schrödinger–Poisson system in $\mathbb{R}^3$. Under certain assumptions, we prove that the problem has infinitely many nonradial positive solutions.

1. Introduction and main result

We consider the fractional Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^s u + u + V(|x|)\Phi(x)u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = V(|x|)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1-1}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian operator for $\alpha = s, t \in (0, 1)$, $V(r)$ $(r = |x|)$ is a positive bounded function, and

$$1 < p < 2^*(s) - 1 = \frac{3+2s}{3-2s}.$$

We assume that $V(r)$ satisfies the following condition:

(V) There are constants $a > 0$, $\dfrac{3+2s}{2(3+2s+1)} < m < \dfrac{3+2s}{2}$ and $\theta > 0$ such that

$$V(r) = \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right) \quad \text{as } r \to +\infty.$$

MSC2010: 35J10, 35B99, 35J60.
Keywords: fractional Schrödinger–Poisson system, infinitely many solutions, nonradial solutions.

In (1-1), the first equation is a nonlinear fractional Schrödinger equation in which the potential Φ satisfies a nonlinear fractional Poisson equation. The study of elliptic equations involving fractional powers of the Laplacian appears to be important in many areas, including physics, biological modeling, mathematical finance and the study of standing wave solutions of certain nonlinear fractional Schrödinger equations.

Giammetta [2014] studied the evolution equation associated with the one-dimensional system

$$\begin{cases} -\Delta u + \lambda\Phi(x)u = g(u), & x \in \mathbb{R}, \\ (-\Delta)^t \Phi = \lambda u^2, & x \in \mathbb{R}. \end{cases} \tag{1-2}$$

Zhang, do Ó and Squassina [Zhang et al. 2016] established the existence of a radial ground state solution to the following fractional Schrödinger–Poisson system with a general subcritical or critical nonlinearity:

$$\begin{cases} (-\Delta)^s u + \lambda\Phi(x)u = g(u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1-3}$$

Under the assumption that the nonlinearity does not satisfy the Ambrosetti–Rabinowitz condition, Zhang [2015] used the fountain theorem to obtain the existence of infinitely many large energy solutions to the system

$$\begin{cases} (-\Delta)^s u + V(x)u + \Phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1-4}$$

When $s = t = 1$, the system reduces to the classical Schrödinger–Poisson system. In recent years, many publications have appeared on that system. Zhang [2014] studied the existence and behavior of bound states of the system

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \lambda\Phi(x)u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\Phi = u^2,\ \lim_{|x|\to\infty}\Phi(x) = 0, & x \in \mathbb{R}^3, \end{cases} \tag{1-5}$$

for $\lambda > 0$ and small $\varepsilon > 0$. For $f(u) = |u|^{p-1}u$, $p \in (1,5)$, there are some results in the literature. In the case of $\varepsilon = 1$, $V(x) \equiv 1$, the existence of radially symmetric positive solutions of system (1-5) was obtained by D'Aprile and Mugnai [2004]. Azzollini and Pomponio [2008] established the existence of ground state solutions for $p \in (2,5)$. Ruiz [2006] proved that (1-5) does not admit any nontrivial solution for $1 < p \le 2$ and possesses a positive radial solution for $2 < p < 5$. When $\lambda \equiv 1$, Ianni and Vaira [2008] considered the existence of positive bound state solutions that concentrate on the local minimum of the potential V. Furthermore, Ianni and Vaira [Ianni and Vaira 2009; Ianni 2009] investigated the radially symmetric solutions that concentrate on the spheres. Ruiz and Vaira [2011] constructed the multibump solutions whose bumps concentrate around the local minimum of the potential V. The proofs explored in [Ruiz and Vaira 2011] are based on a singular perturbation, essentially a Lyapunov–Schmidt reduction method. By using the method of invariant sets of descending flow, Liu, Wang and Zhang [Liu et al. 2016] showed that this system has infinitely many sign-changing solutions. For more related results, one can refer to [Alves and Souto 2014; Chen and Wang 2014; He and Zou 2012; Ianni and Vaira 2015; Kim and Seok 2012; Zhao et al. 2013].

In this paper, inspired by [Long et al. 2016] and [Li et al. 2010], we consider the infinitely many nonradial positive solutions of the fractional Schrödinger–Poisson system (1-1). In [Long et al. 2016], Long, Peng and Yang were concerned with the existence of infinitely many nonradial positive solutions and sign-changing solutions for the equation

$$(-\Delta)^s u + u = K(|x|)u^p, \quad u > 0,\ u \in H^s(\mathbb{R}^N).$$

In [Li et al. 2010], Li, Peng and Yan obtained infinitely many nonradial positive solutions for (1-1) with $s = t = 1$.

Compared with the operator $-\Delta$, which is local, the operator $(-\Delta)^s$ with $0 < s < 1$ on $\mathbb{R}^3$ is nonlocal. Unlike the local case $s = 1$, the leading order of the associated reduced functional in a variational reduction procedure is of polynomial instead of exponential order, due to the nonlocal effect. So we need to establish some new necessary estimates for the Lyapunov–Schmidt reduction. Also, because of the appearance of the Poisson potential Φ, problem (1-1) is more complicated than the problem in [Long et al. 2016] and [Li et al. 2010].

To the best of our knowledge, there are no results on the existence of infinitely many nonradical positive solutions to the nonlinear fractional Schrödinger–Poisson system (1-1). In this paper, we will present some results in this direction.

Now, we are able to state our main theorem.

Theorem 1.1. *If $V(r)$ satisfies (V) and $2t + 4s \geq 3$, then the problem (1-1) has infinitely many nonradial positive solutions.*

To prove Theorem 1.1, we will construct solutions with a large number of bumps near infinity. Since $V(r) \to 0$ as $r \to +\infty$, the solution of (1-1) can be approximated by using the solution U of the problem

$$\begin{cases} (-\Delta)^s u + u = u^p, & u > 0 \text{ in } \mathbb{R}^3, \\ u(0) = \max_{x \in \mathbb{R}^3} u(x). \end{cases} \tag{1-6}$$

It is well known that the unique solution U of (1-6) satisfies $U(x) = U(|x|)$ and $U' < 0$ (see [Frank and Lenzmann 2013; Frank et al. 2016]).

Let

$$Q_j = \left(r\cos\frac{2(j-1)\pi}{k},\, r\sin\frac{2(j-1)\pi}{k},\, 0 \right) := (Q_j', 0), \quad j = 1, 2, \ldots, k, \tag{1-7}$$

where $r \in \big[r_1 k^{\frac{3+2s}{3+2s-2m}}, r_2 k^{\frac{3+2s}{3+2s-2m}}\big]$ for some $r_2 > r_1 > 0$. Define

$$E^s = \bigg\{u : u \in H^s(\mathbb{R}^3),\ u \text{ is even in } x_h,\ h = 2, 3,\\ u(r\cos\theta, r\sin\theta, x_3) = u\bigg(r\cos\Big(\theta + \frac{2\pi j}{k}\Big), r\sin\Big(\theta + \frac{2\pi j}{k}\Big), x_3\bigg)\bigg\}.$$

Let

$$U_r(x) = \sum_{j=1}^{k} U_{Q_j}(x), \tag{1-8}$$

where $U_{Q_j}(\cdot) = U(\cdot - Q_j)$, and Q_j is defined in (1-7).

We will prove Theorem 1.1 by proving the following result.

Theorem 1.2. *Suppose $V(r)$ satisfies (V) and $2t + 4s \geq 3$. Then there is an integer $k_0 > 0$ such that for any integer $k \geq k_0$, (1-1) has a positive solution u_k of the form*

$$u_k = U_{r_k}(x) + w_k,$$

where $w_k \in E^s$, $r_k \in \big[r_1 k^{\frac{3+2s}{3+2s-2m}}, r_2 k^{\frac{3+2s}{3+2s-2m}}\big]$ for some $r_2 > r_1 > 0$ and as $k \to +\infty$, $\|w_k\|_s \to 0$.

Remark 1.3. It follows from Theorems 1.1 and 1.2 that (1-1) has solutions with a large number of bumps near infinity. Hence the energy of these solutions can be very large.

This paper is organized as follows. In Section 2, we give some preliminaries. Then we carry out Lyapunov–Schmidt reduction in Section 3. Finally, we prove our main result in Section 4. Some technical estimates are left to the Appendix.

2. Some preliminaries

In this section, we outline the variational framework for problem (1-1) and give some preliminary lemmas. Firstly, we recall some properties of the fractional Sobolev space and some results which are important in our proof of the main theorem.

The nonlocal operator $(-\Delta)^s$ in $\mathbb{R}^3$ is defined on the Schwartz class through the Fourier transform

$$\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s} \hat{f}(\xi),$$

or via the Riesz potential. Here $\widehat{\ }$ is the Fourier transform. When f has sufficient regularity, the fractional Laplacian of a function $f : \mathbb{R}^3 \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^s f(x) = C_{3,s}\,\mathrm{P.V.}\int_{\mathbb{R}^3} \frac{f(x)-f(y)}{|x-y|^{3+2s}}\,dy = C_{3,s}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus B_\varepsilon(x)} \frac{f(x)-f(y)}{|x-y|^{3+2s}}\,dy, \tag{2-1}$$

where $C_{3,s} = \pi^{-(2s+3/2)}\Gamma\big(\frac{3}{2}+s\big)/\Gamma(-s)$. This integral makes sense directly when $s < \frac{1}{2}$ and $f \in C^{0,\gamma}(\mathbb{R}^3)$ with $\gamma > 2s$, or if $f \in C^{1,\gamma}(\mathbb{R}^3)$ with $1+\gamma > 2s$.

When $s \in (0,1)$, the space $H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3)$ is defined by

$$\begin{aligned} H^s(\mathbb{R}^3) &= \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3\times\mathbb{R}^3) \right\} \\ &= \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1+|\xi|^{2s})|\hat{u}(\xi)|^2\,d\xi < \infty \right\} \end{aligned}$$

and the norm is

$$\|u\|_s := \|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} |u|^2\,dx \right)^{\frac{1}{2}},$$

which is induced by the inner product

$$\begin{aligned} \langle u,v\rangle_{H^s(\mathbb{R}^3)} &= \langle u,v\rangle_s + \langle u,v\rangle_{L^2(\mathbb{R}^3)} \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbb{R}^3} u(x)v(x)\,dx. \end{aligned}$$

Here the term

$$[u]_{H^s(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy \right)^{\frac{1}{2}}$$

is the so-called Gagliardo (semi-)norm of u. The following identity yields the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space $H^s(\mathbb{R}^3)$:

$$[u]_{H^s(\mathbb{R}^3)} = C\left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2\,d\xi \right)^{\frac{1}{2}} = C\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^3)}$$

for a suitable positive constant C depending only on s.

The homogeneous Sobolev space $D^{t,2}(\mathbb{R}^3)$ is defined by

$$D^{t,2}(\mathbb{R}^3) = \left\{ u \in L^{2^*(t)}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2\,d\xi < \infty \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{t,2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2t} |\widehat{u}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} = \|(-\Delta)^{\frac{t}{2}} u\|_{L^2(\mathbb{R}^3)}$$

and the inner product

$$(u, v)_{D^{t,2}} = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} u (-\Delta)^{\frac{t}{2}} v \, dx, \quad u, v \in D^{t,2}(\mathbb{R}^3).$$

We have the following Sobolev embedding results.

Lemma 2.1 [Di Nezza et al. 2012]. *$H^s(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in \left[2, \frac{6}{3-2s}\right]$, and locally compact whenever $q \in \left[2, \frac{6}{3-2s}\right)$.*

Lemma 2.2 [Di Nezza et al. 2012]. *For any $t \in (0, 1)$, $D^{t,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*(t)}(\mathbb{R}^3)$; i.e., there exists $S_t > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2^*(t)} \, dx\right)^{2/2^*(t)} \le S_t \int_{\mathbb{R}^3} |(-\Delta)^{\frac{t}{2}} u|^2 \, dx, \quad u \in D^{t,2}(\mathbb{R}^3).$$

Now, we recall some known results for the limit equation (1-6). In a celebrated paper, Frank and Lenzmann [2013] proved the uniqueness of the ground state solution $U(x) = U(|x|) \ge 0$ for $N = 1,\ 0 < s < 1,\ 1 < p < (N + 2s)/(N - 2s)$. Very recently, Frank, Lenzmann and Silvestre [Frank et al. 2016] obtained the nondegeneracy of ground state solutions for (1-6) in arbitrary dimension $N \ge 1$ and any admissible exponent $1 < p < (N + 2s)/(N - 2s)$.

For convenience, we summarize the properties of the ground state U of (1-6), which can be found in [Frank and Lenzmann 2013; Frank et al. 2016].

Lemma 2.3. *Let $s \in (0, 1)$ and $1 < p < (3 + 2s)/(3 - 2s)$. Then the following hold*:

(1) *Uniqueness*: *The ground state solution $U \in H^s(\mathbb{R}^3)$ for (1-6) is unique up to translations.*

(2) *Symmetry*, *regularity and decay*: *$U(x)$ is radial, positive and strictly decreasing in $|x|$. Moreover, the function U belongs to $H^{2s+1}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ and satisfies*

$$\frac{C_1}{1 + |x|^{3+2s}} \le U(x) \le \frac{C_2}{1 + |x|^{3+2s}}, \quad x \in \mathbb{R}^3,$$

with some constants $C_2 \ge C_1 > 0$.

(3) *Nondegeneracy*: *The linearized operator $L_0 = (-\Delta)^s + 1 - p|U|^{p-1}$ is nondegenerate, i.e., its kernel is given by*

$$\ker L_0 = \operatorname{span}\{\partial_{x_1} U, \partial_{x_2} U, \partial_{x_3} U\}.$$

By [Frank et al. 2016, Lemma C.2], $\partial_{x_j} U$ has the following decay estimate for $j = 1, 2, 3$:

$$|\partial_{x_j} U| \le \frac{C}{1+|x|^{3+2s}}.$$

By Lemma 2.1, if $2t+4s \ge 3$, $H^s(\mathbb{R}^3) \hookrightarrow L^{12/(3+2t)}(\mathbb{R}^3)$. Then, for $u \in H^s(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} u^2 v \le \|u\|^2_{12/(3+2t)} \|v\|_{2^*(t)} \le C\|u\|_s^2 \|v\|_{D^{t,2}}.$$

Hence there exists a unique Φ_u^t such that $(-\Delta)^t \Phi_u^t = V(x)u^2$ and the t-Riesz potential satisfies

$$\Phi_u^t(x) = C(t) \int_{\mathbb{R}^3} \frac{V(y)u^2(y)}{|x-y|^{3-2t}}\, dy,$$

where

$$C(t) = \frac{\Gamma\left(\frac{3}{2}-2t\right)}{\pi^{\frac{3}{2}} 2^{2t} \Gamma(t)}.$$

Substituting Φ_u^t in (1-1), we are lead to the equation

$$(-\Delta)^s u + u + V(|x|)\Phi_u^t(x) u = |u|^{p-1}u. \tag{2-2}$$

Let us summarize some properties of $\Phi_u^t(x)$ which will be useful throughout the paper.

Lemma 2.4 [Zhang et al. 2016]. *If $t, s \in (0,1)$ and $2t + 4s \ge 3$, then for any $u \in H^s(\mathbb{R}^3)$, we have*

1. $u \mapsto \Phi_u^t : H^s(\mathbb{R}^3) \mapsto D^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets*;
2. $\Phi_u^t(x) \ge 0$, $x \in \mathbb{R}^3$, *and* $\int_{\mathbb{R}^3} \Phi_u^t u^2\, dx \le C\|u\|_s^4$ *for some* $C > 0$.

3. Finite-dimensional reduction

In this section, we prove Theorem 1.1 by proving Theorem 1.2.

We assume

$$\Lambda_k := \left[\left(\frac{(3+2s)B_4}{2mB_5} - \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}, \left(\frac{(3+2s)B_4}{2mB_5} + \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}\right], \tag{3-1}$$

where $\alpha > 0$ is a small constant, and where B_4 and B_5 are defined in Lemma A.5.

Let $r \in \Lambda_k$. We define

$$\mathfrak{E} = \left\{ u : u \in E^s, \sum_{j=1}^{k} \int_{\mathbb{R}^3} \frac{\partial U_{Q_j}}{\partial r} U_{Q_j}^{p-1} u = 0 \right\}.$$

Define

$$I(u) = \frac{1}{2}\langle u, u\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} u^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_u^t u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} \quad \forall u \in \mathfrak{E}.$$

It is easy to check that

$$\langle u_1, u_2\rangle_s + \int_{\mathbb{R}^3} u_1 u_2 - p\int_{\mathbb{R}^3} U_r^{p-1} u_1 u_2 + \int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t u_1 u_2 \\ + 2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3} \frac{V(|y|)}{|x-y|^{3-2t}} U_r u_1\, dy\right) U_r u_2, \quad u_1 u_2 \in \mathfrak{E},$$

is a bounded bilinear functional in $\mathfrak{E}$. Hence, by the Lax–Milgram theorem there is a bounded linear operator $\mathcal{L}$ from $\mathfrak{E}$ to $\mathfrak{E}$ such that

$$\langle \mathcal{L}u_1, u_2\rangle = \langle u_1, u_2\rangle_s + \int_{\mathbb{R}^3} u_1 u_2 - p\int_{\mathbb{R}^3} U_r^{p-1} u_1 u_2 + \int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t u_1 u_2 \\ + 2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3} \frac{V(|y|)}{|x-y|^{3-2t}} U_r u_1\, dy\right) U_r u_2, \quad u_1 u_2 \in \mathfrak{E}.$$

The following result implies that $\mathcal{L}$ is invertible in $\mathfrak{E}$.

Lemma 3.1. *There exists a positive constant C, independent of k, such that for any $r \in \Lambda_k$,*

$$\|\mathcal{L}u\|_s \geq C\|u\|_s, \quad u \in \mathfrak{E}.$$

Proof. We prove the lemma by contradiction. Suppose that there exist $k \to +\infty$, $r_k \in \Lambda_k$ and $u_k \in \mathfrak{E}$ with

$$\|\mathcal{L}u_k\|_s = o(1)\|u_k\|_s.$$

Then we have

$$\langle \mathcal{L}u_k, \varphi\rangle = o(1)\|u_k\|_s\|\varphi\|_s \quad \forall \varphi \in \mathfrak{E}. \tag{3-2}$$

We may assume that $\|u_k\|_s^2 = k$.

Denote

$$\Omega_j = \left\{ x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{x'}{|x'|}, \frac{Q_j'}{|Q_j'|} \right\rangle \geq \cos\frac{\pi}{k} \right\}, \quad j = 1, 2, \ldots, k.$$

By symmetry, we have

$$\begin{aligned}(3\text{-}3)\quad &\int_{\Omega_1}\int_{\mathbb{R}^3}\frac{(u_k(x)-u_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}\,dx\,dy\\ &\quad+\int_{\Omega_1}u_k\varphi-p\int_{\Omega_1}U_{r_k}^{p-1}u_k\varphi+\int_{\Omega_1}V(|x|)\Phi^t_{U_{r_k}}u_k\varphi\\ &\qquad+2\int_{\Omega_1}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}U_{r_k}u_k\,dy\right)U_{r_k}\varphi\\ &\qquad\qquad=\frac{1}{k}\langle\mathcal{L}u_k,\varphi\rangle=o(1)\frac{1}{\sqrt{k}}\|\varphi\|_s\quad\forall\varphi\in\mathfrak{E}.\end{aligned}$$

Particularly, choosing $\varphi=u_k$ we get

$$\begin{aligned}(3\text{-}4)\quad &\int_{\Omega_1}\int_{\mathbb{R}^3}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\int_{\Omega_1}|u_k|^2-p\int_{\Omega_1}U_{r_k}^{p-1}|u_k|^2\\ &+\int_{\Omega_1}V(|x|)\Phi^t_{U_{r_k}}|u_k|^2+2\int_{\Omega_1}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}U_{r_k}u_k\,dy\right)U_{r_k}u_k\\ &=o(1)\end{aligned}$$

and

$$(3\text{-}5)\qquad \int_{\Omega_1}\int_{\mathbb{R}^3}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\int_{\Omega_1}|u_k|^2=1.$$

Let $\tilde{u}_k(x)=u_k(x-Q_1)$. It is easy to check that for any $R>0$, we can choose k large enough such that $B_R(Q_1)\subset\Omega_1$. Consequently, (3-5) yields that

$$\int_{B_R(0)}\int_{\mathbb{R}^3}\frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\int_{B_R(0)}|\tilde{u}_k|^2\le 1.$$

Thus we may assume the existence of $u\in H^s(\mathbb{R}^3)$ such that as $k\to+\infty$,

$$\tilde{u}_k\rightharpoonup u\quad\text{weakly in } H^s(\mathbb{R}^3)$$

and

$$\tilde{u}_k\to u\quad\text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^3).$$

Noting that $\tilde{u}_k$ is even in x_h, $h=2,3$, we have that u is even in x_h, $h=2,3$. On the other hand, from

$$\int_{\mathbb{R}^3}\frac{\partial U_{Q_1}}{\partial r}U_{Q_1}^{p-1}u_k=0,$$

we obtain

$$\int_{\mathbb{R}^3}\frac{\partial U}{\partial Q_1}U^{p-1}\tilde{u}_k=0.$$

So u satisfies

$$\int_{\mathbb{R}^3} \frac{\partial U}{\partial Q_1} U^{p-1} u = 0. \tag{3-6}$$

Now we prove that u satisfies

$$(-\Delta)^s u + u - pU^{p-1} u = 0 \quad \text{in } \mathbb{R}^3.$$

Define

$$\widetilde{\mathfrak{E}} = \left\{ \varphi : \varphi \in H^s(\mathbb{R}^3),\ \int_{\mathbb{R}^3} \frac{\partial U}{\partial Q_1} U^{p-1} \varphi = 0 \right\}.$$

For any $R > 0$, let φ belong to $C_0^\infty(B_R(0)) \cap \widetilde{\mathfrak{E}}$ and be even in x_h, $h = 2, 3$. Then

$$\varphi_1(x) := \varphi(x - Q_1) \in C_0^\infty(B_R(0)).$$

We may identify $\varphi_1(x)$ as an element in $\mathfrak{E}$ by redefining the values outside Ω_1 using symmetry. Using (3-4) and Lemma A.1, we deduce that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{3+2s}}\, dx\, dy + \int_{\mathbb{R}^3} u\varphi - p \int_{\mathbb{R}^3} U^{p-1} u\varphi = 0. \tag{3-7}$$

Furthermore, since u is even in x_h, $h = 2, 3$, (3-7) is true for any function $\varphi \in C_0^\infty(\mathbb{R}^3)$ which is odd in x_h, $h = 2, 3$. Therefore, (3-7) holds for any $\varphi \in C_0^\infty(B_R(0)) \cap \widetilde{\mathfrak{E}}$. By the density of $C_0^\infty(\mathbb{R}^3)$ in $H^s(\mathbb{R}^3)$, we see

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{3+2s}}\, dx\, dy + \int_{\mathbb{R}^3} u\varphi - p \int_{\mathbb{R}^3} U^{p-1} u\varphi = 0 \quad \forall \varphi \in \widetilde{\mathfrak{E}}. \tag{3-8}$$

But (3-8) is true for $\varphi = \partial U / \partial Q_1$. Thus (3-8) holds for any $\varphi \in H^s(\mathbb{R}^3)$, and hence $u = c(\partial U / \partial Q_1)$ because u is even in x_h, $h = 2, 3$. By (3-6), we find $u = 0$. Consequently,

$$\int_{B_R(Q_1)} u_k^2 = o(1) \quad \forall R > 0.$$

Moreover, Lemma A.1 implies that for any $1 < \eta \le 3 + 2s$, there is a positive constant C such that

$$U_{Q_k}(x) \le \frac{C}{(1 + |x - Q_1|)^{3+2s-\eta}}, \quad x \in \Omega_1. \tag{3-9}$$

Thus, by (3-9) and (V), we have

$$\begin{aligned} o(1) &= \int_{\Omega_1}\int_{\mathbb{R}^3}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\int_{\Omega_1}|u_k|^2-p\int_{\Omega_1}U_{r_k}^{p-1}|u_k|^2 \\ &\quad+\int_{\Omega_1}V(|x|)\Phi^t_{U_{r_k}}|u_k|^2+2\int_{\Omega_1}V(|x|)\biggl(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}U_{r_k}u_k\,dy\biggr)U_{r_k}u_k \\ &\ge \int_{\Omega_1}\int_{\mathbb{R}^3}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\int_{\Omega_1}|u_k|^2 \\ &\quad+C\biggl(\int_{B_{\frac{R}{2}}(Q_1)}+\int_{\Omega_1\setminus B_{\frac{R}{2}}(Q_1)}\frac{1}{(1+|x-Q_1|)^{3+2s-\eta}}u_n^2\biggr)+o(1) \\ &\ge \frac{1}{2}+o(1)+O_R(1), \end{aligned}$$

which is impossible for large R. □

Proposition 3.2. *There is an integer $k_0>0$ such that for each $k\ge k_0$, there exists a C^1 map with respect to r from Λ_k to E^s: $\varphi=\varphi(r)$, satisfying $\varphi\in E^s$, and*

$$\biggl\langle\frac{\partial J(\varphi)}{\partial\varphi},v\biggr\rangle=0\quad\forall v\in E^s.$$

Moreover, there is a small $\tau>0$ such that

$$\|\varphi\|_s\le\frac{C}{r^{2m}}k^{\frac{1}{2}}+Ck^{\frac{1}{2}}\Bigl(\frac{k}{r}\Bigr)^{\frac{3+2s}{2}+\tau}. \tag{3-10}$$

Proof. Write

$$J(\varphi)=I(U_r+\varphi),\quad \varphi\in E^s.$$

By direct computation, we have

$$\begin{aligned} J(\varphi)&=I(U_r+\varphi) \\ &=\frac{1}{2}\langle U_r+\varphi,U_r+\varphi\rangle_s+\frac{1}{2}\int_{\mathbb{R}^3}(U_r+\varphi)^2 \\ &\qquad+\frac{1}{4}\int_{\mathbb{R}^3}V(|x|)\Phi^t_{U_r+\varphi}(U_r+\varphi)^2-\frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} \\ &=\frac{1}{2}\langle U_r,U_r\rangle_s+\langle U_r,\varphi\rangle_s+\frac{1}{2}\langle\varphi,\varphi\rangle_s+\frac{1}{2}\int_{\mathbb{R}^3}U_r^2+\frac{1}{2}\int_{\mathbb{R}^3}\varphi^2+\int_{\mathbb{R}^3}U_r\varphi \\ &\qquad+\frac{1}{4}\int_{\mathbb{R}^3}V(|x|)\Phi^t_{U_r+\varphi}(U_r+\varphi)^2-\frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} U_r^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r} U_r^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |U_r|^{p+1} \\
&\quad + \int_{\mathbb{R}^3}\Big(\sum_{j=1}^k U^p_{Q_j} - U^p_r\Big)\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r} U_r\varphi \\
&\quad + \frac{1}{2}\langle \varphi,\varphi\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}\varphi^2 - \frac{p}{2}\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r}\varphi^2 \\
&\quad + \int_{\mathbb{R}^3} V(|x|)\Big(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}} U_r\varphi\,dy\Big) U_r\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi} U_r\varphi \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi}\varphi^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} + \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r|^{p+1} \\
&\quad + \int_{\mathbb{R}^3}|U_r|^p\varphi + \frac{p}{2}\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi^2.
\end{aligned}
$$

Hence,

$$J(\varphi) = J(0) + f(\varphi) + \tfrac{1}{2}\langle \mathcal{L}\varphi,\varphi\rangle + R(\varphi),$$

where

$$f(\varphi) = \int_{\mathbb{R}^3}\Big(\sum_{j=1}^k U^p_{Q_j} - U^p_r\Big)\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r} U_r\varphi. \tag{3-11}$$

We notice that $\mathcal{L}$ is the bounded linear map from E^s to E^s in Lemma 2.1, and

$$
\begin{aligned}
R(\varphi) = \int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi} U_r\varphi + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi^t_{\varphi}\varphi^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r+\varphi|^{p+1} \\
+ \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r|^{p+1} + \int_{\mathbb{R}^3}|U_r|^p\varphi + \frac{p}{2}\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi^2.
\end{aligned}
$$

It is not difficult to verify that $f(\varphi)$ is a bounded linear functional in E^s, so there exists an $f_k \in E^s$ such that

$$f(\varphi) = \langle f_k, \varphi\rangle.$$

Thus, to find a critical point for $J(\varphi)$, we only need to solve

$$f_k + \mathcal{L}\varphi + R'(\varphi) = 0. \tag{3-12}$$

From Lemma 3.1 we know $\mathcal{L}$ is invertible. Therefore, (3-12) can be rewritten as

$$\varphi = \mathcal{A}(\varphi) =: -\mathcal{L}^{-1} f_k - \mathcal{L}^{-1} R'(\varphi).$$

Set

$$\mathcal{N} = \Big\{\varphi : \varphi \in E^s,\ \|\varphi\|_s \le \frac{1}{r^{2m-\tau}} k^{\frac{1}{2}} + k^{\frac{1}{2}}\Big(\frac{k}{r}\Big)^{\frac{3+2s+\tau}{2}}\Big\},$$

where $\tau > 0$ is small.

When $1 < p \le 2$, we can verify that

$$\|R'(\varphi)\|_s \le C\|\varphi\|_s^p.$$

Hence Lemma 3.3 below implies

$$\begin{aligned}
\|\mathcal{A}(\varphi)\|_s
&\le C\|f_k\|_s + C\|\varphi\|_s^p\\
&\le \frac{C}{r^{2m}}k^{\frac12} + Ck^{\frac12}\left(\frac{k}{r}\right)^{\frac{3+2s}{2}+\tau} + C\left(\frac{1}{r^{2m-\tau}}k^{\frac12} + k^{\frac12}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}\right)^p\\
&\le \frac{1}{r^{2m-\tau}}k^{\frac12} + k^{\frac12}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}.
\end{aligned} \tag{3-13}$$

Thus, $\mathcal{A}$ maps $\mathcal{N}$ into $\mathcal{N}$ when $1 < p \le 2$.

Meanwhile, when $1 < p \le 2$, we see

$$\|R''(\varphi)\|_s \le C\|\varphi\|_s^{p-1}.$$

Thus,

$$\begin{aligned}
\|\mathcal{A}(\varphi_1) - \mathcal{A}(\varphi_2)\|_s &= \|\mathcal{L}^{-1}R'(\varphi_1) - \mathcal{L}^{-1}R'(\varphi_2)\|_s\\
&\le C\|R'(\varphi_1) - R'(\varphi_2)\|_s\\
&\le C\|R''(\varepsilon\varphi_1 + (1-\varepsilon)\varphi_2)\|_s\|\varphi_1 - \varphi_2\|_s\\
&\le C\big(\|\varphi_1\|_s^{p-1} + \|\varphi_2\|_s^{p-1}\big)\|\varphi_1 - \varphi_2\|_s \le \tfrac12\|\varphi_1 - \varphi_2\|_s,
\end{aligned}$$

where $\varepsilon \in (0, 1)$.

Thus, we have proved that when $1 < p \le 2$, $\mathcal{A}$ is a contraction map.

When $p > 2$, by Remark A.2, the Hölder inequality, the Sobolev inequality, and Lemmas 2.2 and 2.4, we get

$$\begin{aligned}
&|\langle R'(\varphi), \xi\rangle|\\
&\quad= \left|2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\right)U_r\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t U_r\xi\right.\\
&\qquad\left. + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t\varphi\xi - \int_{\mathbb{R}^3}|U_r+\varphi|^p\xi + \int_{\mathbb{R}^3}|U_r|^p\xi + p\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi\xi\right|\\
&\quad\le \left|2\int_{\mathbb{R}^3} V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\right)U_r\varphi\right.\\
&\qquad\qquad\qquad\left. + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t U_r\xi + \int_{\mathbb{R}^3} V(|x|)\Phi_\varphi^t\varphi\xi\right|\\
&\qquad + \left|\int_{\mathbb{R}^3}|U_r+\varphi|^p\xi - \int_{\mathbb{R}^3}|U_r|^p\xi - p\int_{\mathbb{R}^3}|U_r|^{p-1}\varphi\xi\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{r^m}\int_{\mathbb{R}^3}|\Phi_\varphi^t|^{\frac12}|\Phi_\xi^t|^{\frac12}U_r|\varphi| \\
&\quad + \frac{C}{r^m}\left(\int_{\mathbb{R}^3}|\Phi_\varphi^t|^{\frac{6}{3-2t}}\right)^{\frac{3-2t}{6}}\left(\int_{\mathbb{R}^3}|\xi|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}}\left(\int_{\mathbb{R}^3}|U_r|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}} \\
&\quad + \frac{C}{r^m}\left(\int_{\mathbb{R}^3}|\Phi_\varphi^t|^{\frac{6}{3-2t}}\right)^{\frac{3-2t}{6}}\left(\int_{\mathbb{R}^3}|\xi|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}}\left(\int_{\mathbb{R}^3}|\varphi|^{\frac{12}{3+2t}}\right)^{\frac{3+2t}{12}} \\
&\quad + C\int_{\mathbb{R}^3}|U_r|^{p-2}|\varphi|^2|\xi| \\
&\le \frac{C}{r^m}\|\varphi\|_s^2\|\xi\|_s + \frac{C}{r^m}k^{\frac{3+2t}{12}}\|\Phi_\varphi^t\|_{D^{t,2}}\|\xi\|_s + \frac{C}{r^m}\|\Phi_\varphi^t\|_{D^{t,2}}\|\xi\|_s\|\varphi\|_s \\
&\quad + C\left(\int_{\mathbb{R}^3}\big(|U_r|^{p-2}|\varphi|^2\big)^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\|\xi\|_s \\
&\le \frac{C}{r^m}\|\varphi\|_s^2\|\xi\|_s + \frac{C}{r^m}k^{\frac{3+2t}{12}}\|\varphi\|_s^2\|\xi\|_s + \frac{C}{r^m}\|\varphi\|_s^3\|\xi\|_s \\
&\quad + C\left(\int_{\mathbb{R}^3}|\varphi|^{\frac{2p+2}{p}}\right)^{\frac{p}{p+1}}\|\xi\|_s.
\end{aligned}
$$

Hence, we deduce that

$$\|R'(\varphi)\|_s \le C\big(\|\varphi\|_s^2 + \|\varphi\|_s^3\big).$$

For the estimate of $\|R''(\varphi)\|_s$, we have

$$
\begin{aligned}
|R''(\varphi)(\xi,\eta)| = \bigg| & 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\eta\xi\,dy\right)U_r\varphi \\
&+ 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\right)U_r\eta \\
&+ 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\eta\,dy\right)U_r\xi \\
&+ 2\int_{\mathbb{R}^3}V(|x|)\left(\int_{\mathbb{R}^3}\frac{V(|y|)}{|x-y|^{3-2t}}\varphi\xi\,dy\right)\varphi\eta \\
&+ \int_{\mathbb{R}^3}V(|x|)\Phi_\varphi^t\xi\eta - p\int_{\mathbb{R}^3}(U_r+\varphi)^{p-1}\xi\eta + p\int_{\mathbb{R}^3}U_r^{p-1}\xi\eta\bigg| \\
\le C\big(&\|\varphi\|_s + \|\varphi\|_s^2\big)\|\xi\|_s\|\eta\|_s,
\end{aligned}
$$

which implies

$$\|R''(\varphi)\|_s \le C\big(\|\varphi\|_s + \|\varphi\|_s^2\big).$$

Thus, we can conclude that

$$\begin{aligned}\|\mathcal{A}(\varphi)\|_s &\le C\|f_k\|_s + C\|\varphi\|_s^2 \\ &\le \frac{C}{r^{2m}}k^{\frac{1}{2}} + Ck^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s}{2}+\tau} + C\left(\frac{1}{r^{2m-\tau}}k^{\frac{1}{2}} + k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}\right)^2 \\ &\le \frac{1}{r^{2m-\tau}}k^{\frac{1}{2}} + k^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s+\tau}{2}}\end{aligned} \tag{3-14}$$

and

$$\begin{aligned}\|\mathcal{A}(\varphi_1)-\mathcal{A}(\varphi_2)\|_s &= \|\mathcal{L}^{-1}R'(\varphi_1)-\mathcal{L}^{-1}R'(\varphi_2)\|_s \\ &\le C\|R'(\varphi_1)-R'(\varphi_2)\|_s \\ &\le C\|R''(\varepsilon\varphi_1+(1-\varepsilon)\varphi_2)\|_s\|\varphi_1-\varphi_2\|_s \\ &\le \tfrac{1}{2}\|\varphi_1-\varphi_2\|_s,\end{aligned}$$

where $\varepsilon \in (0,1)$. Hence, $\mathcal{A}$ is also a contraction map from $\mathcal{N}$ to $\mathcal{N}$.

Now applying the contraction mapping theorem, we can find a unique φ such that (3-12) holds. Moreover, it follows from (3-13) and (3-14) that (3-10) holds. □

Lemma 3.3. *There exist constants $C > 0$ and $\tau > 0$ small enough such that*

$$\|f_k\|_s \le \frac{C}{r^{2m}}k^{\frac{1}{2}} + Ck^{\frac{1}{2}}\left(\frac{k}{r}\right)^{\frac{3+2s}{2}+\tau}.$$

Proof. We recall

$$f(\varphi) = \int_{\mathbb{R}^3}\left(\sum_{j=1}^{k} U_{Q_j}^p - U_r^p\right)\varphi + \int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t U_r\varphi. \tag{3-15}$$

Using $U_{Q_j} \le U_{Q_1}$, $x \in \Omega_1$, $\frac{3+2s}{3+2s+1} < 2m < 3+2s$ and Lemma A.1, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\left|U_r^p - \sum_{j=1}^{k} U_{Q_j}^p\right||\varphi| \\ &= k\int_{\Omega_1}\left|U_r^p - \sum_{j=1}^{k} U_{Q_j}^p\right||\varphi| \\ &\le Ck\int_{\Omega_1} U_{Q_1}^{p-1}\sum_{j=2}^{k} U_{Q_j}|\varphi| \\ &\le Ck\left(\int_{\Omega_1}\left(U_{Q_1}^{p-1}\sum_{j=2}^{k} U_{Q_j}\right)^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}}\left(\int_{\Omega_1}|\varphi|^{p+1}\right)^{\frac{1}{p+1}}\end{aligned} \tag{3-16}$$

$$\le Ck\Big(\int_{\Omega_1}\Big(\frac{1}{(1+|x-Q_1|)^{(3+2s)(p-1)+\frac{3+2s}{2}-\sigma}}\Big(\frac{k}{r}\Big)^{\frac{3+2s}{2}+\sigma}\Big)^{\frac{p+1}{p}}\Big)^{\frac{p}{p+1}} \times\Big(\int_{\Omega_1}|\varphi|^{p+1}\Big)^{\frac{1}{p+1}}$$

$$\le Ck^{\frac{p}{p+1}}\Big(\frac{k}{r}\Big)^{\frac{3+2s}{2}+\sigma}\Big(\int_{\Omega_1}\Big(\frac{1}{(1+|x-Q_1|)^{(3+2s)(p-1)+\frac{3+2s}{2}-\sigma}}\Big)^{\frac{p+1}{p}}\Big)^{\frac{p}{p+1}}\|\varphi\|_s$$

$$\le Ck^{\frac{1}{2}}\Big(\frac{k}{r}\Big)^{\frac{3+2s}{2}+\tau}\|\varphi\|_s,$$

where $\tau > 0$ is a small constant and $\sigma \in \big(0, \frac{3+2s}{2}\big)$.

On the other hand, by Lemma A.4 and Remark A.2, we have

$$\text{(3-17)}\qquad \int_{\mathbb{R}^3} V(|x|)\Phi^t_{U_r}U_r\varphi \le \frac{C}{r^{2m}}\Big(\int_{\mathbb{R}^3}U_r^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}\varphi^2\Big)^{\frac{1}{2}} \le \frac{C}{r^{2m}}k^{\frac{1}{2}}\|\varphi\|_s.$$

Inserting (3-16) and (3-17) into (3-15), we can complete the proof. □

4. Proof of the main result

Proof of Theorem 1.2. Let $\varphi(r)$ be the map obtained in Proposition 3.2. Define

$$\mathcal{F}(r) = I(U_r + \varphi(r)) \quad \forall r \in \Lambda_k.$$

It is well known that if r is a critical point of $\mathcal{F}(r)$, then $U_r + \varphi(r)$ is a solution of (1-1) (see [Cao and Tang 2006]). As a consequence, in order to complete the proof of the proposition, we only need to prove that $\mathcal{F}(r)$ has a critical point in Λ_k.

Hence, by Proposition 3.2 and Lemma A.5, we have

$$\begin{aligned}
\mathcal{F}(r) &= I(U_r) + f(\varphi) + \tfrac{1}{2}\langle\mathcal{L}\varphi,\varphi\rangle + R(\varphi)\\
&= I(U_r) + O\big(\|f_k\|_s\|\varphi\|_s + \|\varphi\|_s^2\big)\\
&= kB_3 - kB_4\Big(\frac{k}{r}\Big)^{3+2s} + k\frac{B_5}{r^{2m}} + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k}K(Q_j-Q_1)\int_{\mathbb{R}^3}U^2\\
&\qquad + kO\Big(\frac{1}{r^{2m+\tau}}\Big) + O\Big(\frac{1}{r^{2m}}k^{\frac{1}{2}} + k^{\frac{1}{2}}\Big(\frac{k}{r}\Big)^{\frac{3+2s}{2}+\tau}\Big)^2\\
&= kB_3 - kB_4\Big(\frac{k}{r}\Big)^{3+2s} + k\frac{B_5}{r^{2m}}\\
&\qquad + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k}K(Q_j-Q_1)\int_{\mathbb{R}^3}U^2 + kO\Big(\frac{1}{r^{2m+\tau}}\Big),
\end{aligned}$$

where B_3, B_4 and B_5 are defined in Lemma A.5.

We consider its maximum with respect to r:

$$\max\{\mathcal{F}(r) : r \in \Lambda_k\}. \tag{4-1}$$

Assume that (4-1) is achieved by some r_k in Λ_k. We will prove that r_k is an interior point of Λ_k.

Consider the following smooth function in Λ_k:

$$g(r) := -B_4\left(\frac{k}{r}\right)^{3+2s} + \frac{B_5}{r^{2m}}.$$

Then

$$g'(r) = (3+2s)B_4\frac{k^{3+2s}}{r^{4+2s}} - \frac{2mB_5}{r^{2m+1}}.$$

It is easy to check that $g(r)$ has a maximum point $\tilde{r}_k$, satisfying

$$g'(\tilde{r}_k) = 0.$$

Thus

$$\tilde{r}_k = \left(\frac{(3+2s)B_4}{2mB_5}\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}.$$

By direct computation, we observe that

(4-2)

$$\begin{aligned}\mathcal{F}(r_k) \geq \mathcal{F}(\tilde{r}_k) &\geq kB_3 - kB_4\left(\frac{k}{\tilde{r}_k}\right)^{3+2s} + k\frac{B_5}{\tilde{r}_k^{2m}} + kO\left(\frac{1}{\tilde{r}_k^{2m+\tau}}\right)\\ &= kB_3 + k\frac{B_5}{\tilde{r}_k^{2m}}\left(1-\frac{2m}{3+2s}\right) + kO\left(\frac{1}{\tilde{r}_k^{2m+\tau}}\right)\\ &= kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}}B_4^{-\frac{2m}{3+2s-2m}}\left(\frac{3+2s}{2m}-1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}}\\ &\qquad + kO\left(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\right).\end{aligned}$$

On the other hand, if we suppose that

$$r_k = \left(\frac{(3+2s)B_4}{2mB_5} - \alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}},$$

then

(4-3)

$$\begin{aligned}\mathcal{F}(r_k) &= kB_3 + kB_5\left(1-\frac{2m}{3+2s}\right)\left(\frac{(3+2s)B_4}{2mB_5} - \alpha\right)^{\frac{-2m}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}}\\ &\qquad + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 + kO\left(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\right)\end{aligned}$$

$$\begin{aligned}
&= kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}} B_4^{-\frac{2m}{3+2s-2m}} \left(\frac{3+2s}{2m}-1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} \\
&\qquad\qquad \times \left(1-\frac{2m\alpha B_5}{(3+2s)B_4}\right)^{-\frac{2m}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}} \\
&\quad + kO\big(r^{-\frac{2m(3-2t)}{3+2s}-2m}\big) + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big) \\
&< kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}} B_4^{-\frac{2m}{3+2s-2m}} \left(\frac{3+2s}{2m}-1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}} \\
&\quad + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big).
\end{aligned}$$

This is a contradiction to (4-2).

Similarly

$$\begin{aligned}
&\mathcal{F}\left(\left(\frac{(3+2s)B_4}{2mB_5}+\alpha\right)^{\frac{1}{3+2s-2m}} k^{\frac{3+2s}{3+2s-2m}}\right) \\
&\quad < kB_3 + kB_5^{\frac{3+2s}{3+2s-2m}} B_4^{-\frac{2m}{3+2s-2m}} \left(\frac{3+2s}{2m}-1\right)\left(\frac{3+2s}{2m}\right)^{-\frac{3+2s}{3+2s-2m}} k^{\frac{-2m(3+2s)}{3+2s-2m}} \\
&\qquad\qquad + kO\big(k^{-\frac{2m(3+2s)}{3+2s-2m}-\tau}\big).
\end{aligned}$$

Hence we can check that (4-1) is achieved by some r_k which is in the interior of Λ_k. As a result, r_k is a critical point of $\mathcal{F}(r)$. Therefore

$$U_{r_k} + \varphi(r_k)$$

is a solution of (1-1). □

Appendix: Some technical estimates

In this section, we give some estimates of the energy expansion for the approximate solutions. Firstly, we recall

$$Q_j = \left(r\cos\frac{2(j-1)\pi}{k},\ r\sin\frac{2(j-1)\pi}{k},\ 0\right), \quad j = 1, \ldots, k,$$

$$\Omega_j = \left\{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{x'}{|x'|}, \frac{Q_j'}{|Q_j'|}\right\rangle \geq \cos\frac{\pi}{k}\right\}, \quad j = 1, 2, \ldots, k,$$

and

$$I(u) = \frac{1}{2}\langle u, u\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} u^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_u^t u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1},$$

where Φ_u^t is the solution of $(-\Delta)^t \Phi_u^t = V(|x|)u^2$.

Recall that U is the unique solution of

$$\begin{cases}(-\Delta)^s u + u = u^p, & u > 0 \text{ in } \mathbb{R}^3,\\ u(0) = \max_{x\in\mathbb{R}^3} u(x).\end{cases}$$

Let K be the solution of

$$\begin{cases}(-\Delta)^t v = U^2 & \text{in } \mathbb{R}^3,\\ v \in D^{t,2}(\mathbb{R}^3).\end{cases}$$

Then K is radial, and $r^{3-2t}K(r) \to K_0 > 0$ as $r \to +\infty$.

To begin, we give the following lemmas.

Lemma A.1 [Long et al. 2016, Lemma A.2]. *For any $x \in \Omega_1$, and $\eta \in (1, 3+2s]$, there are constants $C, B > 0$ such that*

$$\sum_{i=2}^{k} U_{Q_i}(x) \le C \frac{1}{(1+|x-Q_1|)^{3+2s-\eta}} \frac{k^\eta}{r^\eta} \le C \frac{k^\eta}{r^\eta}$$

and

$$\sum_{j=2}^{k} \frac{1}{|Q_j - Q_1|^\eta} = B\Big(\frac{k}{r}\Big)^\eta + O\Big(\frac{k}{|r|^\eta}\Big).$$

Remark A.2. It follows from Lemma A.1 that U_r is bounded.

Lemma A.3 [Wei and Zhao 2013, Lemma 13.1]. *Assume that $0 < m < 3$ and $n > m$. Then*

$$\int_{\mathbb{R}^3} \frac{1}{|y-x|^{3-m}} \frac{1}{(1+|x|)^n}\,dx \le \begin{cases} C(1+|y|)^{m-n} & \textit{if } n < 3,\\ C(1+|y|)^{m-3}\big[1+\log(1+|y|)\big] & \textit{if } n = 3,\\ C(1+|y|)^{m-3} & \textit{if } n > 3.\end{cases}$$

Now, we estimate Φ_{U_r} and $I(U_r)$.

Lemma A.4. *We have*

$$\Phi^t_{U_r}(y) = \frac{a}{r^m} \sum_{j=1}^{k} K(y-Q_j) + O\bigg(\sum_{j=1}^{k} \frac{1}{r^{m+\tau}} \frac{1}{(1+|y-Q_j|)^{3-2t}}\bigg).$$

Proof. For any $\beta > 0$, we get

$$\frac{1}{|y+Q_j|^\beta} = \frac{1}{|Q_j|^\beta}\bigg(1 + O\Big(\frac{|y|}{|Q_j|}\Big)\bigg), \quad y \in B_{\frac{r}{2}}(0).$$

By Lemmas A.1 and A.3, we are led to

(A-1)

$$
\begin{aligned}
&\int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} U_r^2(x)\,dx \\
&= \int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}^2(x)\,dx \\
&\qquad + O\biggl(\int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1} \sum_{j=2}^{k} U_{Q_j}\,dx + \int_{\Omega_1} \frac{V(|x|)}{|y-x|^{3-2t}} \Bigl(\sum_{j=2}^{k} U_{Q_j}\Bigr)^2 dx\biggr) \\
&= \int_{\Omega_1\cap B_{\frac{r}{2}}(0)} \Bigl(\frac{a}{|x+Q_1|^m} + O\Bigl(\frac{1}{|x+Q_1|^{m+\theta}}\Bigr)\Bigr) \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx \\
&\qquad + \int_{\Omega_1\cap B^C_{\frac{r}{2}}(0)} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}^2(x)\,dx \\
&\qquad + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{3+2s} \int_{\Omega_1\cap B_{\frac{r}{2}}(0)} \frac{V(|x+Q_1|)}{|y-x-Q_1|^{3-2t}} U(x)\,dx \\
&\qquad\qquad + \Bigl(\frac{k}{r}\Bigr)^{3+2s} \int_{\Omega_1\cap B^C_{\frac{r}{2}}(0)} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}(x)\,dx\biggr) \\
&\qquad + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{2\eta} \int_{\Omega_1} \frac{1}{|y-x-Q_1|^{3-2t}} \frac{1}{(1+|x|)^{2(3+2s-\eta)}}\,dx\biggr) \\
&= \frac{a}{r^m} \int_{\Omega_1\cap B_{\frac{r}{2}}(0)} \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx \\
&\qquad + O\biggl(\frac{1}{r^{m+\tau}} \int_{\Omega_1\cap B_{\frac{r}{2}}(0)} \frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx\biggr) \\
&\qquad + O\biggl(\frac{1}{r^m} \int_{\Omega_1\cap B^C_{\frac{r}{2}}(0)} \frac{U(x)}{|y-x-Q_1|^{3-2t}} \frac{1}{r^{3+2s}}\,dx\biggr) \\
&\qquad + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{3+2s} \int_{\Omega_1\cap B_{\frac{r}{2}}(0)} \frac{V(|x+Q_1|)}{|y-x-Q_1|^{3-2t}} U(x)\,dx \\
&\qquad\qquad + \Bigl(\frac{k}{r}\Bigr)^{3+2s} \int_{\Omega_1\cap B^C_{\frac{r}{2}}(0)} \frac{V(|x|)}{|y-x|^{3-2t}} U_{Q_1}(x)\,dx\biggr) \\
&\qquad + O\biggl(\Bigl(\frac{k}{r}\Bigr)^{2\eta} \int_{\Omega_1} \frac{1}{|y-x-Q_1|^{3-2t}} \frac{1}{(1+|x|)^{2(3+2s-\eta)}}\,dx\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a}{r^m}\int_{\mathbb{R}^3}\frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx + O\Big(\frac{1}{r^m}\int_{\Omega_1\cap B_{\frac{r}{2}}(0)}\frac{U(x)}{|y-x-Q_1|^{3-2t}}U(x)\,dx\Big)\\
&\quad + O\Big(\frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_1|)^{3-2t}}\Big)\\
&\quad + O\Big(\frac{1}{r^{m+\tau}}\int_{\mathbb{R}^3}\frac{U^2(x)}{|y-x-Q_1|^{3-2t}}\,dx\Big) + O\Big(\frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_1|)^{3-2t}}\Big)\\
&\quad + O\Big(\Big(\frac{k}{r}\Big)^{3+2s}\frac{1}{r^m}\int_{\mathbb{R}^3}\frac{1}{|y-x-Q_1|^{3-2t}}\frac{1}{(1+|x|)^{3+2s}}\,dx\Big)\\
&\quad + O\Big(\Big(\frac{k}{r}\Big)^{2\eta}\frac{1}{(1+|y-Q_1|)^{3-2t}}\Big)\\
&= \frac{a}{r^m}K(y-Q_1) + O\Big(\frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_1|)^{3-2t}}\Big),
\end{aligned}
$$

where $\tau > 0$ is small and we choose $\eta = \frac{1}{2}(3+2s) \in (1, 3+2s]$.

So

$$
\Phi^t_{U_r}(y) = \frac{a}{r^m}\sum_{j=1}^k K(y-Q_j) + O\Big(\sum_{j=1}^k \frac{1}{r^{m+\tau}}\frac{1}{(1+|y-Q_j|)^{3-2t}}\Big). \qquad \square
$$

Lemma A.5. *We have*

$$
\begin{aligned}
&I(U_r)\\
&= kB_3 - kB_4\Big(\frac{k}{r}\Big)^{3+2s} + k\frac{B_5}{r^{2m}} + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^k K(Q_j-Q_1)\int_{\mathbb{R}^3}U^2 + kO\Big(\frac{1}{r^{2m+\tau}}\Big),
\end{aligned}
$$

where $B_3 = \big(\frac{1}{2}-\frac{1}{p+1}\big)\int_{\mathbb{R}^3}U^{p+1}$, $B_4 = \frac{1}{2}B_2$, $B_5 = \frac{a^2}{4}\int_{\mathbb{R}^3}KU^2$ *and* $\tau > 0$ *is small.*

Proof. Recall that

$$
I(U_r) = \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}U_r^2 + \frac{1}{4}\int_{\mathbb{R}^3}V(|x|)\Phi^t_{U_r}U_r^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|U_r|^{p+1}.
$$

By direct computation, we obtain

$$
\begin{aligned}
\text{(A-2)}\quad \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3}U_r^2 &= \frac{1}{2}\sum_{j=1}^k\langle U_{Q_j}, U_{Q_j}\rangle_s + \frac{1}{2}\sum_{i\neq j}\langle U_{Q_j}, U_{Q_j}\rangle_s\\
&\quad + \frac{1}{2}\sum_{j=1}^k\int_{\mathbb{R}^3}U_{Q_j}^2 + \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^3}U_{Q_i}U_{Q_j}\\
&= \frac{k}{2}\int_{\mathbb{R}^3}U^{p+1} + \frac{k}{2}\sum_{j=2}^k\int_{\mathbb{R}^3}U_{Q_1}^pU_{Q_j}.
\end{aligned}
$$

By the result in [Long et al. 2016], we know that

$$\sum_{j=2}^{k}\int_{\mathbb{R}^3} U_{Q_1}^{p} U_{Q_j} = \sum_{j=2}^{k}\frac{B_1}{|Q_1-Q_j|^{3+2s}} + O\biggl(\sum_{j=2}^{k}\frac{1}{|Q_1-Q_j|^{3+2s+\tau}}\biggr), \tag{A-3}$$

where B_1 is a positive constant and $\tau > 0$ is small enough. We also obtain

$$\begin{aligned}
\frac{1}{p+1}\int_{\Omega_1}|U_r|^{p+1}
&= \frac{1}{p+1}\int_{\Omega_1}\biggl(U_{Q_1}+\sum_{j=2}^{k}U_{Q_j}\biggr)^{p+1} \\
&= \frac{1}{p+1}\int_{\Omega_1}|U_{Q_1}|^{p+1} + \int_{\Omega_1}|U_{Q_1}|^{p}\sum_{j=2}^{k}U_{Q_j} \\
&\qquad + O\biggl(\int_{\Omega_1}|U_{Q_1}|^{p-1}\biggl(\sum_{j=2}^{k}U_{Q_j}\biggr)^{2}\biggr) + O\biggl(\int_{\Omega_1}\biggl(\sum_{j=2}^{k}U_{Q_j}\biggr)^{p+1}\biggr) \\
&= \frac{1}{p+1}\int_{\Omega_1}|U_{Q_1}|^{p+1} + \int_{\Omega_1}|U_{Q_1}|^{p}\sum_{j=2}^{k}U_{Q_j} \\
&\qquad + O\biggl(\biggl(\sum_{j=2}^{k}\frac{1}{|Q_j-Q_1|^{\frac{3}{2}+2s}}\biggr)^{2}\biggr) \\
&\qquad + O\biggl(\biggl(\sum_{j=2}^{k}\frac{1}{|Q_j-Q_1|^{3+2s-\frac{3+(p-1)s}{p+1}}}\biggr)^{p+1}\biggr) \\
&= \frac{1}{p+1}\int_{\mathbb{R}^3}|U|^{p+1} + \int_{\Omega_1}|U_{Q_1}|^{p}\sum_{j=2}^{k}U_{Q_j} + O\Bigl(\frac{k}{r}\Bigr)^{3+4s}.
\end{aligned} \tag{A-4}$$

Using (A-3) and Lemma A.4, we see that

$$\begin{aligned}
\int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^{t}U_r^2
&= k\int_{\Omega_1} V(|x|)\Phi_{U_r}^{t}U_r^2 \\
&= k\int_{\Omega_1} V(|x|)\biggl(\frac{a}{r^m}\sum_{j=1}^{k}K(x-Q_j) \\
&\qquad + O\biggl(\sum_{j=1}^{k}\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr)\biggr)\biggl(U_{Q_1}+O\Bigl(\frac{k}{r}\Bigr)^{3+2s}\biggr)^{2}
\end{aligned} \tag{A-5}$$

$$
\begin{aligned}
&= k\int_{\Omega_1} V(|x|)\frac{a}{r^m}\sum_{j=1}^{k} K(x-Q_j)U_{Q_1}^2 \\
&\quad + k\int_{\Omega_1} V(|x|)O\biggl(\sum_{j=1}^{k}\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr)U_{Q_1}^2 \\
&\quad + kO\biggl(\frac{1}{r^{m+\tau}}\int_{\Omega_1} V(|x|)\frac{a}{r^m}\sum_{j=1}^{k} K(x-Q_j)\biggr) \\
&\quad + kO\biggl(\int_{\Omega_1} V(|x|)\sum_{j=1}^{k}\frac{1}{r^{2m+2\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr) \\
&= k\int_{\Omega_1} V(|x|)\frac{a}{r^m}K(x-Q_1)U_{Q_1}^2 + k\int_{\Omega_1} V(|x|)\frac{a}{r^m}\sum_{j=2}^{k} K(x-Q_j)U_{Q_1}^2 \\
&\quad + k\int_{\Omega_1} V(|x|)O\biggl(\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_1|)^{3-2t}}\biggr)U_{Q_1}^2 \\
&\quad + k\int_{\Omega_1} V(|x|)O\biggl(\sum_{j=2}^{k}\frac{1}{r^{m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr)U_{Q_1}^2 \\
&\quad + kO\biggl(\int_{\mathbb{R}^3} V(|x|)\sum_{j=1}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{(1+|x-Q_j|)^{3-2t}}\biggr) \\
&= \frac{ka}{r^m}\int_{\Omega_1\cap B_{\frac{r}{2}}(0)}\biggl(\frac{a}{|x+Q_1|^m} + O\biggl(\frac{1}{|x+Q_1|^{m+\theta}}\biggr)\biggr)K(x)U^2(x) \\
&\quad + kO\biggl(\int_{\Omega_1\cap B^C_{\frac{r}{2}}(0)}\frac{1}{r^{2m}}K(x-Q_j)U_{Q_1}^2\biggr) + \frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j-Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\biggl(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1-Q_j|^{3-2t}}\biggr) + kO\biggl(\frac{1}{r^{2m+\tau}}\biggr) \\
&= \frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{ka^2}{r^{2m}}\sum_{j=2}^{k} K(Q_j-Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\biggl(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1-Q_j|^{3-2t}}\biggr) + kO\biggl(\frac{1}{r^{2m+\tau}}\biggr).
\end{aligned}
$$

Above all, we deduce that

(A-6)

$$
\begin{aligned}
I(U_r) &= \frac{1}{2}\langle U_r, U_r\rangle_s + \frac{1}{2}\int_{\mathbb{R}^3} U_r^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(|x|)\Phi_{U_r}^t U_r^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |U_r|^{p+1} \\
&= \frac{k}{2}\int_{\mathbb{R}^3} U^{p+1} + \frac{k}{2}\sum_{j=2}^k \int_{\mathbb{R}^3} U_{Q_1}^p U_{Q_j} - \frac{k}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} - k\int_{\Omega_1} |U_{Q_1}|^p \sum_{j=2}^k U_{Q_j} \\
&\quad + kO\Big(\frac{k}{r}\Big)^{3+4s} + \frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^k K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\Big(\sum_{j=2}^k \frac{1}{r^{2m+\tau}}\frac{1}{|Q_1 - Q_j|^{3-2t}}\Big) + kO\Big(\frac{1}{r^{2m+\tau}}\Big) \\
&= \frac{k}{2}\int_{\mathbb{R}^3} U^{p+1} + \frac{k}{2}\sum_{j=2}^k \int_{\mathbb{R}^3} U_{Q_1}^p U_{Q_j} - \frac{k}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} - k\int_{\Omega_1} |U_{Q_1}|^p \sum_{j=2}^k U_{Q_j} \\
&\quad + kO\Big(\frac{k}{r}\Big)^{3+4s} + \frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^k K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\Big(\sum_{j=2}^k \frac{1}{r^{2m+\tau}}\frac{1}{|Q_1 - Q_j|^{3-2t}}\Big) + kO\Big(\frac{1}{r^{2m+\tau}}\Big) \\
&= \frac{k}{2}\int_{\mathbb{R}^3} U^{p+1} + \frac{k}{2}\sum_{j=2}^k \int_{\mathbb{R}^3} U_{Q_1}^p U_{Q_j} - \frac{k}{p+1}\int_{\mathbb{R}^3} |U|^{p+1} \\
&\quad - k\int_{\mathbb{R}^3} |U_{Q_1}|^p \sum_{j=2}^k U_{Q_j} + kO\Big(\frac{k}{r}\Big)^{3+2s+\tau} + kO\Big(\frac{k}{r}\Big)^{3+4s} \\
&\quad + \frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3} KU^2 + \frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^k K(Q_j - Q_1)\int_{\mathbb{R}^3} U^2 \\
&\quad + kO\Big(\sum_{j=2}^k \frac{1}{r^{2m+\tau}}\frac{1}{|Q_1 - Q_j|^{3-2t}}\Big) + kO\Big(\frac{1}{r^{2m+\tau}}\Big) \\
&= k\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbb{R}^3} U^{p+1} - \frac{k}{2}\sum_{j=2}^k \frac{B_1}{|Q_1 - Q_j|^{3+2s}} \\
&\quad + kO\Big(\sum_{j=2}^k \frac{1}{|Q_1 - Q_j|^{3+2s+\tau}}\Big) + kO\Big(\frac{k}{r}\Big)^{3+2s+\tau} + kO\Big(\frac{k}{r}\Big)^{3+4s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3}KU^2+\frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k}K(Q_j-Q_1)\int_{\mathbb{R}^3}U^2\\
&+kO\Big(\sum_{j=2}^{k}\frac{1}{r^{2m+\tau}}\frac{1}{|Q_1-Q_j|^{3-2t}}\Big)+kO\Big(\frac{1}{r^{2m+\tau}}\Big)\\
=&\,k\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^3}U^{p+1}-\frac{k}{2}B_2\Big(\frac{k}{r}\Big)^{3+2s}+\frac{1}{4}\frac{ka^2}{r^{2m}}\int_{\mathbb{R}^3}KU^2\\
&+\frac{1}{4}\frac{ka^2}{r^{2m}}\sum_{j=2}^{k}K(Q_j-Q_1)\int_{\mathbb{R}^3}U^2+kO\Big(\frac{1}{r^{2m+\tau}}\Big). \qquad \square
\end{aligned}
$$

Acknowledgements

The author thanks the referee and Professor Robert Finn for helpful discussions and suggestions. This paper was supported by the NSFC (Nos. 11601139, 11301204).

References

[Alves and Souto 2014] C. O. Alves and M. A. S. Souto, "Existence of least energy nodal solution for a Schrödinger–Poisson system in bounded domains", *Z. Angew. Math. Phys.* **65**:6 (2014), 1153–1166. MR Zbl

[Azzollini and Pomponio 2008] A. Azzollini and A. Pomponio, "Ground state solutions for the nonlinear Schrödinger–Maxwell equations", *J. Math. Anal. Appl.* **345**:1 (2008), 90–108. MR Zbl

[Cao and Tang 2006] D. Cao and Z. Tang, "Existence and uniqueness of multi-bump bound states of nonlinear Schrödinger equations with electromagnetic fields", *J. Differential Equations* **222**:2 (2006), 381–424. MR Zbl

[Chen and Wang 2014] S. Chen and C. Wang, "Existence of multiple nontrivial solutions for a Schrödinger–Poisson system", *J. Math. Anal. Appl.* **411**:2 (2014), 787–793. MR Zbl

[D'Aprile and Mugnai 2004] T. D'Aprile and D. Mugnai, "Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations", *Proc. Roy. Soc. Edinburgh Sect. A* **134**:5 (2004), 893–906. MR Zbl

[Di Nezza et al. 2012] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces", *Bull. Sci. Math.* **136**:5 (2012), 521–573. MR Zbl

[Frank and Lenzmann 2013] R. L. Frank and E. Lenzmann, "Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$", *Acta Math.* **210**:2 (2013), 261–318. MR Zbl

[Frank et al. 2016] R. L. Frank, E. Lenzmann, and L. Silvestre, "Uniqueness of radial solutions for the fractional Laplacian", *Comm. Pure Appl. Math.* **69**:9 (2016), 1671–1726. Zbl

[Giammetta 2014] A. R. Giammetta, "Fractional Schrödinger–Poisson–Slater system in one dimension", preprint, 2014. arXiv

[He and Zou 2012] X. He and W. Zou, "Existence and concentration of ground states for Schrödinger–Poisson equations with critical growth", *J. Math. Phys.* **53**:2 (2012), 023702-1–19. MR Zbl

[Ianni 2009] I. Ianni, "Solutions of the Schrödinger–Poisson problem concentrating on spheres, II: Existence", *Math. Models Methods Appl. Sci.* **19**:6 (2009), 877–910. MR Zbl

[Ianni and Vaira 2008] I. Ianni and G. Vaira, "On concentration of positive bound states for the Schrödinger–Poisson problem with potentials", *Adv. Nonlinear Stud.* **8**:3 (2008), 573–595. MR Zbl

[Ianni and Vaira 2009] I. Ianni and G. Vaira, "Solutions of the Schrödinger–Poisson problem concentrating on spheres, I: Necessary conditions", *Math. Models Methods Appl. Sci.* **19**:5 (2009), 707–720. MR Zbl

[Ianni and Vaira 2015] I. Ianni and G. Vaira, "Non-radial sign-changing solutions for the Schrödinger–Poisson problem in the semiclassical limit", *NoDEA Nonlinear Differential Equations Appl.* **22**:4 (2015), 741–776. MR Zbl

[Kim and Seok 2012] S. Kim and J. Seok, "On nodal solutions of the nonlinear Schrödinger–Poisson equations", *Commun. Contemp. Math.* **14**:6 (2012), art. id 1250041. MR Zbl

[Li et al. 2010] G. Li, S. Peng, and S. Yan, "Infinitely many positive solutions for the nonlinear Schrödinger–Poisson system", *Commun. Contemp. Math.* **12**:6 (2010), 1069–1092. MR Zbl

[Liu et al. 2016] Z. Liu, Z.-Q. Wang, and J. Zhang, "Infinitely many sign-changing solutions for the nonlinear Schrödinger–Poisson system", *Ann. Mat. Pura Appl.* (4) **195**:3 (2016), 775–794. MR Zbl

[Long et al. 2016] W. Long, S. Peng, and J. Yang, "Infinitely many positive and sign-changing solutions for nonlinear fractional scalar field equations", *Discrete Contin. Dyn. Syst.* **36**:2 (2016), 917–939. MR Zbl

[Ruiz 2006] D. Ruiz, "The Schrödinger–Poisson equation under the effect of a nonlinear local term", *J. Funct. Anal.* **237**:2 (2006), 655–674. MR Zbl

[Ruiz and Vaira 2011] D. Ruiz and G. Vaira, "Cluster solutions for the Schrödinger–Poisson–Slater problem around a local minimum of the potential", *Rev. Mat. Iberoam.* **27**:1 (2011), 253–271. MR Zbl

[Wei and Zhao 2013] J. Wei and C. Zhao, "Non-compactness of the prescribed Q-curvature problem in large dimensions", *Calc. Var. Partial Differential Equations* **46**:1 (2013), 123–164. MR Zbl

[Zhang 2014] J. Zhang, "The existence and concentration of positive solutions for a nonlinear Schrödinger–Poisson system with critical growth", *J. Math. Phys.* **55**:3 (2014), 031507-1–14. MR Zbl

[Zhang 2015] J. Zhang, "Existence and multiplicity results for the fractional Schrödinger–Poisson systems", preprint, 2015. arXiv

[Zhang et al. 2016] J. Zhang, J. M. do Ó, and M. Squassina, "Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity", *Adv. Nonlinear Stud.* **16**:1 (2016), 15–30. MR Zbl

[Zhao et al. 2013] L. Zhao, H. Liu, and F. Zhao, "Existence and concentration of solutions for the Schrödinger–Poisson equations with steep well potential", *J. Differential Equations* **255**:1 (2013), 1–23. MR Zbl

Received January 31, 2016. Revised June 7, 2016.

WEIMING LIU
SCHOOL OF MATHEMATICS AND STATISTICS
HUBEI NORMAL UNIVERSITY
HUANGSHI, 435002
CHINA
whu.027@163.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.465

A GAUSSIAN UPPER BOUND OF THE CONJUGATE HEAT EQUATION ALONG RICCI-HARMONIC FLOW

XIAN-GAO LIU AND KUI WANG

We mainly study the Ricci-harmonic flow. Using the monotonicity formulae of entropies, we show a uniform Sobolev inequality along Ricci-harmonic flow. Furthermore, we obtain a Gaussian upper bound for the fundamental solutions of the conjugate heat equation via Moser iteration and Sobolev inequality.

1. Introduction

Let M be a closed manifold of dimension n. List [2008] studied the following Ricci flow, coupled with a harmonic flow:

$$\begin{cases} \partial_t g(x,t) = -2\operatorname{Ric}_{g(x,t)} + 4\, d\phi(x,t)\otimes d\phi(x,t), \\ \partial_t \phi(x,t) = \Delta_{g(x,t)}\phi, \end{cases} \tag{1-1}$$

where $g(x,t)$ is a family of Riemannian metrics, and $\phi(x,t)$ is a scalar function on $M\times\mathbb{R}$. This flow is called Ricci-harmonic flow (see also [List 2008; Müller 2012; Zhu 2013]). If ϕ is a constant, the system (1-1) degenerates to Hamilton's Ricci flow, which has been discussed widely recently; see for example the book [Chow et al. 2006] and celebrated papers [Hamilton 1982; 1986; 1993; Li 2007; Ni 2006; Perelman 2002]. The stationary solutions of (1-1) satisfy the static Einstein vacuum system

$$\begin{cases} \operatorname{Ric} = 2\, d\phi\otimes d\phi, \\ \Delta\phi = 0. \end{cases}$$

Similarly to Ricci flow, corresponding theories for Ricci-harmonic flow have been established; see for instance [List 2008].

For the sake of convenience, we denote as in [List 2008] the symmetric tensor field $Sy\in\operatorname{Sym}_2(M)$ and its trace by

$$S_{ij} := R_{ij} - 2\partial_i\phi\partial_j\phi \quad\text{and}\quad S := R - 2|d\phi|^2,$$

Wang is the corresponding author.

MSC2010: 35B40, 53C44, 35K05.

Keywords: Ricci-harmonic flow, Sobolev inequality, Gaussian upper bound.

where R denotes the scalar curvature of the Riemannian manifold (M, g). Then the Ricci-harmonic flow can be written simply as

$$\begin{cases} \partial_t g = -2Sy, \\ \partial_t \phi = \Delta\phi. \end{cases}$$

It is well known that Sobolev inequality contains a host of analytical and geometric information (e.g., [Carrillo and Ni 2009; Chau et al. 2011; Hebey 1996; Saloff-Coste 2002]), including noncollapsing properties, isoperimetric inequalities and so on. Sobolev inequality is also an important tool in studying elliptic and parabolic differential equations on manifolds (see for example [Saloff-Coste 2002]). Via the monotonicity of Perelman's W entropy, some uniform Sobolev inequalities were proven in Ricci flow, see [Carrillo and Ni 2009; Chau et al. 2011; Kuang and Zhang 2008; Zhang 2006; 2007; 2011]. Zhang [2007] showed a global upper bound for the fundamental solution of the heat equation along the backward Ricci flow

$$\begin{cases} \partial_t g = -2\,\mathrm{Ric}, \\ \Delta u + \partial_t u - Ru = 0, \end{cases}$$

providing Ricci curvature is nonnegative and the injective radius is bounded from below.

Along flow (1-1), we consider the conjugate heat equation

$$\partial_t u(x,t) + \Delta u(x,t) - S(x,t)u(x,t) = 0. \tag{1-2}$$

In [Zhu 2013], some pointwise gradient estimates for the positive solutions of (1-1) were proven, which can be viewed as Li–Yau estimates for the parabolic kernel of the Schrödinger operator in [Chau et al. 2011; Li and Yau 1986; Ni 2004; 2006].

The main goal of this paper is to establish certain Sobolev inequalities under system (1-1) and a global upper bound for the fundamental solutions of heat equation (1-2). Via the monotonicity of the entropies, we obtain the following Sobolev inequality.

Theorem 1.1. *Let $(M, g(x,t), \phi(x,t))$ be a solution of the system (1-1) for $t \in [0, T_0)$ with initial metric g_0, where $T_0 \le \infty$ is the life span of (1-1). Let A_0 and B_0 be positive numbers such that the following L^2 Sobolev inequality holds initially, i.e., for each $v \in W^{1,2}(M, g_0)$,*

$$\left(\int_M v^{2n/(n-2)}\,d\mu_{g_0}\right)^{(n-2)/n} \le A_0\int_M \left(|\nabla v|^2 + \tfrac14 Sv^2\right)d\mu_{g_0} + B_0\int_M v^2\,d\mu_{g_0}.$$

Then for all $v \in W^{1,2}(M, g(t))$, we have

$$\left(\int_M v^{2n/(n-2)}\,d\mu_{g(t)}\right)^{(n-2)/n} \le A(t)\int_M \left(|\nabla v|^2 + \tfrac14 Sv^2\right)d\mu_{g(t)} + B(t)\int_M v^2\,d\mu_{g(t)}, \tag{1-3}$$

where $A(t)$ and $B(t)$ are positive constants depending on A_0, $(1+t)B_0$, n and S_0^-. Here $S_0^- = \sup_{x\in M} S^-(x,0)$, and $S^-(x,0)$ denotes the negative part of $S(x,0)$.

Via Sobolev inequality (1-3), combined with Morse iteration and Davies' heat kernel estimates, we prove the following Gaussian-type upper bound for the fundamental solutions of (1-2), with constants not depending on the lower bound of injective radius but on the first eigenvalue of the entropy, different from Zhang's result [2007]. More precisely:

Theorem 1.2. *Let $(M, g(x,t), \phi(x,t))$ be a smooth solution of system (1-1) in $M\times[0,T]$ and $G(x,t;y,T)$ be a fundamental solution of the following backward conjugate heat equation (1-2); that is,*

$$\begin{cases} \Delta_x G(x,t;y,T) + \partial_t G(x,t;y,T) - S(x,t)G(x,t;y,T) = 0 & \text{if } 0 \le t < T; \\ G(x,t;y,T) = \delta(x,y) & \text{if } t = T. \end{cases}$$

Assume that $Sy \ge 0$ and the first eigenvalue λ_0 of $\inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + Sv^2)\, d\mu_{g_0}$ is positive. Then for each $t \in (0,T)$, and $x, y \in M$, we have the following estimates:

$$G(x,t;y,T) \le \frac{c}{|B(y,\sqrt{T-t},T)|_T} \exp\frac{-c_1 d^2(x,y,T)}{T-t}, \tag{1-4}$$

where c_1 is a constant depending only on the dimension n, and c is a constant depending on n, λ_0 and the initial metric g_0. Here $d(x,y,T)$ denotes the distance between x and y with respect to metric $g(T)$, $B(y,\sqrt{T-t},T)$ denotes the geodesic ball centered at y with radius $\sqrt{T-t}$, and $|B(y,\sqrt{T-t},T)|_T$ denotes the volume of the ball $B(y,\sqrt{T-t},T)$ with respect to the metric $g(T)$.

The rest of the paper is organized as follows. We give the evolution equations of entropies under system (1-1) in Section 2. We prove Sobolev inequalities along Ricci-harmonic flow in Section 3. In Section 4, we prove Theorem 1.2.

2. Entropies of Ricci-harmonic flow

In this section, we recall the definitions of entropies via corresponding conjugate heat equation, as Perelman's [2002] entropy in Ricci flow. Through direct computations, we obtain the monotonicity of the entropies. Although the monotonicity of the entropies were proven in [List 2008] via the entropies' invariance under diffeomorphism. But here for the completeness, we give a direct computation.

Let $u(x,t)$ be a positive solution to the conjugate heat equation (1-2):

$$H^*u = \Delta u - Su + \partial_t u = 0.$$

Note by (1-2) and equation (1-1) that

$$\frac{d}{dt}\int_M u(x,t)\,d\mu_{g(t)} = \int_M (\partial_t - S)u\,d\mu_{g(t)} = \int_M H^* u\,d\mu_{g(t)} = 0,$$

where we used the closure of M. Hereafter we always assume that $u(x,t)$ satisfies

$$\int_M u(x,t)\,d\mu_{g(t)} = 1 \tag{2-1}$$

for each $t \in [0, T]$.

Via the positive solution u, the entropies are defined (see, e.g., [List 2008]) as follows.

Definition 2.1. F entropy is defined as the following integration:

$$F(t) = \int_M \Big(Su + \frac{|\nabla u|^2}{u}\Big)\,d\mu_{g(t)}, \tag{2-2}$$

and W entropy is defined by

$$W(t) = \int_M \Big[\tau\Big(Su + \frac{|\nabla u|^2}{u}\Big) - u\ln u - \frac{n}{2}\ln(4\pi\tau)u - nu\Big]\,d\mu_{g(t)}, \tag{2-3}$$

where $d\tau/dt = -1$.

In order to simplify computations, we introduce a potential function $f(x,t)$ via

$$u(x,t) = \frac{e^{-f}}{(4\pi\tau)^{n/2}},$$

i.e.,

$$f = -\ln u - \frac{n}{2}(\ln 4\pi\tau). \tag{2-4}$$

With the above preparations, we now give a direct calculation of the following monotonicity formulae.

Proposition 2.2 [List 2008, Theorem 6.1]. *Let (M, g, ϕ) be a solution of (1-1) and $u(x,t)$ be a positive solution of (1-2). Then both F entropy and W entropy are nondecreasing in t. Moreover, we have*

$$\frac{d}{dt}F(t) = 2\int_M \big(|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\big)u\,d\mu_{g(t)} \geq 0, \tag{2-5}$$

and

$$\frac{d}{dt}W(t) = \int_M \Big(2\tau\Big|Sy + \nabla^2 f - \frac{g}{2\tau}\Big|^2 + 4\tau|\Delta\phi - d\phi(\nabla f)|^2\Big)u\,d\mu_{g(t)} \geq 0. \tag{2-6}$$

Proof. To start, we have by direct calculations that

$$H^*(u\ln u)=\frac{|\nabla u|^2}{u}+Su, \tag{2-7}$$

and

$$H^*\Big(\frac{|\nabla u|^2}{u}+Su\Big)=\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u}+4\langle\nabla u,\nabla S\rangle+2u\Delta S+2(|Sy|^2+2|\Delta\phi|^2)u. \tag{2-8}$$

Here we used the well-known equation (see [List 2008, Lemma 3.2])

$$\partial_t S=\Delta S+2|Sy|^2+4|\Delta\phi|^2.$$

Note that

$$\frac{d}{dt}F=\frac{d}{dt}\int_M\Big(S+\frac{|\nabla u|^2}{u}\Big)\,d\mu=\int_M(\partial_t-S)\Big(Su+\frac{|\nabla u|^2}{u}\Big)\,d\mu=\int_M H^*\Big(Su+\frac{|\nabla u|^2}{u}\Big)\,d\mu,$$

and substituting (2-8) into the above equality we have

$$\frac{d}{dt}F=\int_M\Big[\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u}+4\langle\nabla u,\nabla S\rangle+2u\Delta S+2\big(|Sy|^2+2|\Delta\phi|^2\big)u\Big]\,d\mu. \tag{2-9}$$

By integration by parts and the contracted second Bianchi identity, we see

$$\begin{aligned}\int_M\langle\nabla u,\nabla S\rangle\,d\mu&=\int_M\langle\nabla u,\nabla(R-2|d\phi|^2)\rangle\,d\mu\\&=\int_M(2u_i\nabla_jR_{ij}-4u_i\phi_j\phi_{ij})\,d\mu\\&=\int_M(-2u_{ij}R_{ij}+4u_{ij}\phi_j\phi_i+4u_i\phi_i\Delta\phi)\,d\mu\\&=\int_M(-2u_{ij}S_{ij}+4u_i\phi_i\Delta\phi)\,d\mu;\end{aligned} \tag{2-10}$$

Then substituting (2-10) into (2-9) yields

$$\begin{aligned}\frac{d}{dt}F&=\int_M\Big[\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u}+2\langle\nabla u,\nabla S\rangle+2\big(|Sy|^2+2|\Delta\phi|^2\big)u\Big]\,d\mu\\&=\int_M\Big[\frac{2}{u}\Big(\frac{u_iu_j}{u}-u_{ij}\Big)^2+\frac{2S_{ij}u_iu_j}{u}+\frac{2R_{ij}u_iu_j}{u}-4u_{ij}S_{ij}+8u_i\phi_i\Delta\phi+2\big(|Sy|^2+2|\Delta\phi|^2\big)u\Big]\,d\mu.\end{aligned}$$

Replacing u with f in the above equality yields

$$\begin{aligned}\frac{d}{dt}F &= \int_M \big[|f_{ij}|^2 + 2S_{ij}f_{ij} + |S_{ij}|^2 + 2|\Delta\phi|^2 + 2|2\,d\phi(\nabla f)|^2 - 4\Delta\phi(d\phi(\nabla f))\big]\,d\mu \\ &= 2\int_M \big(|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\big)u\,d\mu,\end{aligned}$$

proving formula (2-5).

From the definition of W entropy, it follows that

$$\frac{d}{dt}W(t) = \int_M H^*\Big(\tau\Big(\frac{|\nabla u|^2}{u} + Su\Big)\Big) - H^*(u\ln u) - \frac{n}{2}H^*(u\ln\tau)\,d\mu.$$

Substituting (2-7) and (2-8) to the above equation yields

$$\frac{d}{dt}W = \tau\frac{d}{dt}F - 2F + \frac{n}{2\tau} = \tau\frac{d}{dt}F - 2\int_M \big(Su + |\nabla f|^2 u\big)\,d\mu(g(t)) + \frac{n}{2\tau}. \tag{2-11}$$

From the definition of f and integrations by parts, we deduce

$$\int_M \Big(Su + \frac{|\nabla u|^2}{u}\Big)d\mu(g(t)) = \int_M (Su - \langle\nabla u, \nabla f\rangle)\,d\mu = \int_M (Su + \Delta f u)\,d\mu.$$

Substituting the above equality and equality (2-5) into (2-11), we have

$$\begin{aligned}\frac{d}{dt}W &= 2\tau\int_M \big(|Sy + \nabla^2 f|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\big)u\,d\mu - 2\int_M (S + \Delta f)u\,d\mu + \frac{n}{2\tau} \\ &= \int_M 2\tau\Big(\Big|Sy + \mathrm{Hess}(f) - \frac{g}{2\tau}\Big|^2 + 2|\Delta\phi - d\phi(\nabla f)|^2\Big)u\,d\mu,\end{aligned}$$

completing the proof. □

Similarly to the Ricci flow, one can define a family of generalized W entropy along the Ricci-harmonic flow by

$$\begin{aligned}W(a,t) &= \int_M \Big(\frac{a^2\tau}{2\pi}\Big(Su + \frac{|\nabla u|^2}{u}\Big) - u\ln u - \frac{n}{2}\ln(4\pi\tau)u - nu\Big)d\mu_{g(t)} \\ &= \int_M \Big(\frac{a^2\tau}{2\pi}(S + |\nabla f|^2) + f - n\Big)u\,d\mu_{g(t)}.\end{aligned} \tag{2-12}$$

Here the second equality is due to the relations between u and f given in (2-4). For applications of generalized entropy, we refer to the paper [Li 2007]. Using the calculations in [Kuang and Zhang 2008], one can easily show the following monotonicity formula of generalized W entropy along Ricci-harmonic flow.

Proposition 2.3. *Let (M, g, ϕ) be a solution of (1-1) and $u(x, t)$ be a positive solution of (1-2). Then the generalized entropy $W(a, t)$ is nondecreasing in t and* *we have*

$$\frac{d}{dt}W(a,t) \geq \frac{a^2\tau}{\pi}\int_M\Big(\Big|Sy+\mathrm{Hess}(f)-\frac{g}{2\tau}\Big|^2+2|\Delta\phi-d\phi(\nabla f)|^2\Big)u\,d\mu.$$

Since the proof is similar to that in Ricci flow (see for example [Kuang and Zhang 2008, Theorem 4.1]), we omit it.

3. Sobolev inequalities in Ricci-harmonic flow

In this section, we mainly use the monotonicity of W entropy to derive a uniform Sobolev inequality along system (1-1), which will be useful in Section 4.

To prove Theorem 1.1, we need the following lemma, giving the equivalence of the logarithmic Sobolev inequality, the $W^{1,2}$ Sobolev inequality and the so-called ultracontractivity of the heat semigroup of the associated Schrödinger operator. The proof of this lemma is more or less standard.

Lemma 3.1 [Zhang 2011, Theorem 4.2.1]. *Let (M^n, g) be a closed Riemannian manifold $(n \geq 3)$. Then the following inequalities are equivalent up to constants.*

(I) *Sobolev inequality: there exists positive constants A and B such that for $v \in W^{1,2}(M)$*

$$\Big(\int_M v^{2n/(n-2)}\,d\mu\Big)^{(n-2)/n} \leq A\int_M|\nabla v|^2\,d\mu+B\int_M v^2\,d\mu;$$

(II) *Log-Sobolev inequality: for $v \in W^{1,2}(M)$ with $\|v\|_2 = 1$ and $\epsilon > 0$,*

$$\int_M v^2\ln v^2\,d\mu \leq \epsilon^2\int_M|\nabla v|^2\,d\mu-\frac{n}{2}\ln\epsilon^2+BA^{-1}\epsilon^2+\frac{n}{2}\ln\frac{nA}{2e};$$

(III) *Heat kernel upper bound: for $t > 0$,*

$$G(x,t;y) \leq \frac{(nA)^{n/2}}{t^{n/2}}e^{A^{-1}Bt}.$$

By Lemma 3.1, to prove Theorem 1.1 it suffices to show some log-Sobolev inequalities or heat kernel estimates for each $t \in [0, T_0)$. By the monotonicity of W entropy, we obtain the following log-Sobolev inequality.

Lemma 3.2 (log-Sobolev inequality). *Under the assumptions of Theorem 1.1, for each $t \in [0, T_0)$, $v \in W^{1,2}(M, g(t))$ with $\int_M v^2\,d\mu_{g(t)} = 1$ and $\epsilon > 0$, we have*

$$\int_M v^2\ln v^2\,d\mu_{g(t)}\leq\epsilon^2\int_M\big(4|\nabla v|^2+Sv^2\big)\,d\mu_{g(t)}-n\ln\epsilon+(t+\epsilon^2)B_0A_0^{-1}+\frac{n}{2}\ln\frac{nA_0}{2e}. \tag{3-1}$$

Proof. For $t_0 \in [0, T_0)$ and $\epsilon > 0$, we set

$$\tau(t) = \epsilon^2+t_0-t.$$

Recall that W entropy is defined by

$$W(g,\,f,t)=\int_M\big(\tau(S+|\nabla f|^2)+f-n\big)u\,d\mu_{g(t)}.$$

Then from the monotonicity of W entropy in Proposition 2.2, we deduce

$$\inf_{\int_M u\,d\mu_{g(t_0)}=1}W(g(t_0),\,f,\epsilon^2)\geq\inf_{\int_M u_0\,d\mu_{g(0)}=1}W(g(0),\,f_0,t_0+\epsilon^2).\tag{3-2}$$

One can find a more detailed proof of this property in Section 3 of [Perelman 2002]. Here f_0 and f are given via the formulae

$$u_0=\frac{e^{-f_0}}{\big(4\pi(t_0+\epsilon^2)\big)^{n/2}}\quad\text{and}\quad u=\frac{e^{-f}}{(4\pi\epsilon^2)^{n/2}}.$$

Using this notation we rewrite (3-2) as

$$\begin{aligned}&\inf_{\int u\,d\mu_{g(t_0)}=1}\int_M\Big(\epsilon^2\big(S+|\nabla\ln u|^2\big)-\ln u-\frac{n}{2}\ln 4\pi\epsilon^2\Big)u\,d\mu_{g(t_0)}\\&\quad\geq\inf_{\int u_0\,d\mu_{g(0)}=1}\int_M\Big((\epsilon^2+t_0)\big(S+|\nabla\ln u_0|^2\big)-\ln u_0-\frac{n}{2}\ln 4\pi(t_0+\epsilon^2)\Big)u_0\,d\mu_{g(0)}.\end{aligned}$$

Let $v=\sqrt{u}$ and $v_0=\sqrt{u_0}$, and the above inequality gives

$$\begin{aligned}&\inf_{\int v^2\,d\mu_{g(t_0)}=1}\int_M\big[\epsilon^2(Sv^2+4|\nabla v|^2)-v^2\ln v^2\big]\,d\mu_{g(t_0)}-\frac{n}{2}\ln\epsilon^2\\&\quad\geq\inf_{\int v_0^2\,d\mu_{g(0)}=1}\int_M\big((\epsilon^2+t_0)(Sv_0^2+4|\nabla v_0|^2)-v_0^2\ln v_0^2\big)\,d\mu_{g(0)}-\frac{n}{2}\ln(t_0+\epsilon^2)\end{aligned}\tag{3-3}$$

Since $\ln x$ is a concave function and $\int_M v_0^2\,d\mu_{g(0)}=1$, then applying Jensen's inequality we derive

$$\int_M v_0^2\ln v_0^{q-2}\,d\mu_{g(0)}\leq\ln\int v_0^{q-2}v_0^2\,d\mu_{g(0)},$$

i.e.,

$$\int_M v_0^2\ln v_0^2 d\mu_{g(0)}\leq\tfrac{1}{2}n\ln\|v_0\|_q^2,$$

where $q=2n/(n-2)$. By the assumption that the Sobolev inequality holds for the initial time $t=0$, we have

$$\int_M v_0^2\ln v_0^2\,d\mu_{g(0)}\leq\tfrac{1}{2}n\ln\Big(A_0\int_M(4|\nabla v_0|^2+Sv_0^2)\,d\mu_{g(0)}+B_0\Big).$$

From the elementary inequality

$$\ln z\leq yz-\ln y-1,$$

we deduce that for any $y, z > 0$

$$\int_M v_0^2 \ln v_0^2 \, d\mu_{g(0)} \le \frac{n}{2} y \Big(A_0 \int_M \big(4|\nabla v_0|^2 + S v_0^2\big) \, d\mu_{g(0)} + B_0 \Big) - \frac{n}{2} \ln y - \frac{n}{2}.$$

Letting $y = 2(t_0 + \epsilon^2)/(nA_0)$ in the above inequality, we get

$$\int_M v_0^2 \ln v_0^2 \, d\mu_{g(0)} \le (t_0 + \epsilon^2) \int_M \big(4|\nabla v_0|^2 + S v_0^2\big) \, d\mu_{g(0)} + \frac{(t_0+\epsilon^2) B_0}{A_0} - \frac{n}{2} \ln \frac{2(t_0+\epsilon^2)}{nA_0} - \frac{n}{2}.$$

Substituting the above inequality to the right-hand side of (3-3), we arrive at

$$\int_M v^2 \ln v^2 \, d\mu_{g(t_0)} \le \epsilon^2 \int_M \big(4|\nabla v|^2 + S v^2\big) \, d\mu_{g(t_0)} - n \ln \epsilon + (t_0+\epsilon^2) B_0 A_0^{-1} + \frac{n}{2} \ln \frac{nA_0}{2e}.$$

Thus the log-Sobolev inequality (3-1) holds. □

Proof of Theorem 1.1. As the right-hand side of inequality (1-3) has an extra term S, we can not use Lemma 3.1 directly. Instead, we use Zhang's [2007] trick to obtain the estimates of the fundamental solutions of the heat equation, and then use Lemma 3.1 to derive the Sobolev inequality. More precisely, we consider the following heat equation:

$$\Delta_{g(t_0)} u(x,t) - \tfrac{1}{4} S(x,t_0) u(x,t) - S_0^- u(x,t) - u_t(x,t) = 0,$$

where $S_0^- = \sup_{x \in M} S^-(x,0)$ and the metric is fixed at t_0. Then following the same process as in [Zhang 2007]; we see the fundamental solution $p(x,T;y)$ is contractive and satisfies the estimates

$$p(x,T;y) \le \frac{C_1}{t^{n/2}} \quad \text{for} \quad t > 0,$$

where C_1 is a constant depending on n, A_0, $(1+t_0)B_0$ and S_0^- Then from Lemma 3.1, we conclude that the Sobolev inequality (1-3) at $t = t_0$ holds with constants $A(t_0)$ and $B(t_0)$ (depending only on n, A_0, $(1+t_0)B_0$ and S_0^-). Thus the theorem is true by the arbitrariness of t_0. □

Since (M, g_0) is a closed Riemannian manifold, the Sobolev inequality holds as described in Section 4.1 in [Zhang 2011]. That is, for any $v \in W^{1,2}(M)$, there exist positive constants A_0 and B_0 depending only on n and the initial metric g_0 such that

$$\left(\int_M v^{2n/(n-2)} \, d\mu_{g_0} \right)^{(n-2)/n} \le A_0 \int_M |\nabla v|^2 \, d\mu_{g_0} + B_0 \int_M v^2 \, d\mu_{g_0}. \tag{3-4}$$

Recall that λ_0 is the first eigenvalue of F entropy as characterized in (2-2), that is,

$$\lambda_0 = \inf_{\|v\|_2=1} \int_M (4|\nabla v|^2 + Sv^2)\, d\mu_{g_0}. \tag{3-5}$$

This eigenvalue has been studied widely and is a very powerful tool for understanding Riemannian manifolds [Li 2007].

Note that

$$\int_M |\nabla v|^2\, d\mu_{g_0} \le \int_M \Big(|\nabla v|^2 + \frac{S}{4}v^2\Big)\, d\mu_{g_0} + \frac{S_0^-}{4} \int_M v^2\, d\mu_{g_0}.$$

Then if $\lambda_0 > 0$, we conclude from the inequality (3-4) that the assumption of Sobolev inequality in Theorem 1.1 holds initially as follows:

$$\Big(\int_M v^{2n/(n-2)}\, d\mu_{g_0}\Big)^{(n-2)/n} \le \Big[A_0 + \Big(\frac{S_0^-}{4} + B_0\Big)\frac{4}{\lambda_0}\Big] \int_M \big(|\nabla v|^2 + \tfrac{1}{4}Sv^2\big)\, d\mu_{g_0}.$$

That is, the log-Sobolev inequality (3-1) in Lemma 3.2 holds with constant $B_0 = 0$. Therefore we conclude

Corollary 3.3. *Let (M, g, ϕ) be a solution of the system (1-1). Assume further that $\lambda_0 > 0$. Then for all $v \in W^{1,2}(M, g(t))$, $t \in [0, T_0)$, it holds that*

$$\Big(\int_M v^{2n/(n-2)}\, d\mu_{g(t)}\Big)^{(n-2)/n} \le \tilde{A}_0 \int_M \big(|\nabla v|^2 + \tfrac{1}{4}Sv^2\big)\, d\mu_{g(t)},$$

where $\tilde{A}_0$ depends on initial Sobolev constants A_0 and B_0, and λ_0 and S_0^- are independent of t.

4. Proof of Theorem 1.2

In this section, we prove a Gaussian-type upper bound for fundamental solutions of the conjugate heat equation. The Gaussian upper bound in Ricci flow was proven in [Zhang 2006] with the assumption on the lower bound of injectivity, via Sobolev inequality by Heybey [1996]. Here using the uniform Sobolev inequality in Corollary 3.3, we derive a similar Gaussian upper bound without the assumption on the lower bound of injectivity. To prove the theorem, we need the following interpolation theorem.

Theorem 4.1. *Let (M, g, ϕ) be a solution of (1-1) and $u(x, t)$ be a positive solution to heat equation*

$$\Delta u - \partial_t u = 0 \tag{4-1}$$

for $t \in [0, T]$. *Then it holds that*

$$\frac{|\nabla u(x,t)|}{u(x,t)} \le \sqrt{\frac{1}{t}}\sqrt{\ln\frac{A}{u(x,t)}} \tag{4-2}$$

for $(x,t) \in M \times [0,T]$. *Here* $A = \sup_{M\times[0,T]} u$.

Moreover, for each $\delta > 0$, $x, y \in M$ *and* $0 < t < T$, *the following interpolation inequality holds*:

$$u(y,t) \le A^{\delta/(1+\delta)} u^{1/(1+\delta)}(x,t)\exp\Big(\frac{d^2(x,y,t)}{4t\delta}\Big). \tag{4-3}$$

Proof. The proof is based on maximum principles, see also [Li and Yau 1986; Ni 2006; Zhu 2013]. Using (4-1), we compute

$$\begin{aligned}(\Delta-\partial_t)\Big(u\ln\frac{A}{u}\Big) &= \Delta u\ln\frac{A}{u} + u\Delta\Big(\ln\frac{A}{u}\Big) + 2\nabla u\nabla\ln\frac{A}{u}\\ &\qquad -\partial_t u\ln\frac{A}{u} - u\partial_t\Big(\ln\frac{A}{u}\Big)\\ &= \Delta u\ln\frac{A}{u} + u\Big(-\frac{\Delta u}{u}+\frac{|\nabla u|^2}{u^2}\Big) - 2\frac{|\nabla u|^2}{u} - \Delta u\ln\frac{A}{u} + \partial_t u\\ &= -\frac{|\nabla u|^2}{u},\end{aligned} \tag{4-4}$$

$$\begin{aligned}\Delta\Big(\frac{|\nabla u|^2}{u}\Big) &= \frac{\Delta|\nabla u|^2}{u} + \Delta\Big(\frac{1}{u}\Big)|\nabla u|^2 + 2\nabla|\nabla u|^2\nabla\Big(\frac{1}{u}\Big)\\ &= \frac{\Delta|\nabla u|^2}{u} + \Big(\frac{2|\nabla u|^2}{u^3} - \frac{\Delta u}{u^2}\Big)|\nabla u|^2 - 4\frac{u_iu_ju_{ij}}{u^2},\end{aligned} \tag{4-5}$$

and

$$\partial_t\Big(\frac{|\nabla u|^2}{u}\Big) = \frac{\partial_t|\nabla u|^2}{u} - \frac{|\nabla u|^2}{u^2}\partial_t u = \frac{2\langle\nabla u,\nabla\Delta u\rangle + 2S_{ij}u_iu_j}{u} - \frac{|\nabla u|^2}{u^2}\Delta u. \tag{4-6}$$

Putting (4-5) and (4-6) together, we get

$$\begin{aligned}(\Delta-\partial_t)\Big(\frac{|\nabla u|^2}{u}\Big) &= \frac{\Delta|\nabla u|^2}{u} + \frac{2|\nabla u|^4}{u^3} - 4\frac{u_iu_ju_{ij}}{u^2} - \frac{2\langle\nabla u,\nabla\Delta u\rangle + 2S_{ij}u_iu_j}{u}\\ &= \frac{2u_{ij}^2 + 4u_iu_j\phi_i\phi_j}{u} + \frac{2|\nabla u|^4}{u^3} - 4\frac{u_iu_ju_{ij}}{u^2}\\ &= \frac{4}{u}|d\phi(\nabla u)|^2 + \frac{2}{u}\Big|u_{ij} - \frac{u_iu_j}{u}\Big|^2.\end{aligned} \tag{4-7}$$

Combining (4-4) and (4-7), we have

$$(\Delta - \partial_t)\left(\frac{t|\nabla u|^2}{u} - u\ln\frac{A}{u}\right) = -\frac{|\nabla u|^2}{u} + \frac{4}{u}|d\phi(\nabla u)|^2 + \frac{2}{u}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2 + \frac{|\nabla u|^2}{u} = \frac{4}{u}|d\phi(\nabla u)|^2 + \frac{2}{u}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2 \geq 0. \tag{4-8}$$

By $A = \sup_{M\times[0,T]} u$, we know at $t = 0$

$$\frac{t|\nabla u|^2}{u} - u\ln\frac{A}{u} = -u\ln\frac{A}{u} \leq 0.$$

Then from (4-8), the maximum principle implies that

$$\frac{t|\nabla u|^2}{u} - u\ln\frac{A}{u} \leq 0,$$

giving (4-2).

Set $\ell(x,t) = \ln(A/u(x,t))$. Then inequality (4-2) yields

$$|\nabla\sqrt{\ell(x,t)}| = \frac{1}{2}\left|\frac{\nabla u}{u\sqrt{\ell}}\right| \leq \frac{1}{\sqrt{4t}}.$$

For each $x, y \in M$, integrating the above inequality along a minimizing geodesic joining x and y yields

$$\sqrt{\ln\frac{A}{u(x,t)}} \leq \sqrt{\ln\frac{A}{u(y,t)}} + \frac{d(x,y,t)}{\sqrt{4t}}.$$

Then for any $\delta > 0$ it follows

$$\begin{aligned}\ln\frac{A}{u(x,t)} &\leq \ln\frac{A}{u(y,t)} + \frac{d^2(x,y,t)}{4t} + \sqrt{\ln\frac{A}{u(y,t)}}\,\frac{d(x,y,t)}{\sqrt{t}}\\ &\leq \ln\frac{A}{u(y,t)} + \frac{d^2(x,y,t)}{4t} + \delta\ln\frac{A}{u(y,t)} + \frac{d^2(x,y,t)}{4t\delta},\end{aligned}$$

proving (4-3). □

Now we turn to proving Theorem 1.2. With the uniform Sobolev inequality in Corollary 3.3 and the interpolation theorem, we establish a mean value inequality via Moser iteration, and a weighted estimate in the spirit of Davies [1989], and then give the full proof of Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into two steps.

Step 1. Using Morse iteration, we prove a mean value inequality for the positive solution u of the conjugate equation (1-2).

For $p \geq 1$, it follows that

$$\Delta u^p - pSu^p + \partial_t u^p \geq 0. \tag{4-9}$$

Define

$$Q_{\sigma r} := \{(y, s) \mid y \in M, t \leq s \leq t + (\sigma r)^2, d(x, y, s) \leq \sigma r\},$$

with $r > 0$, $1 < \sigma \leq 2$. Let $\varphi(\rho) : [0, +\infty) \to [0, 1]$ be a smooth function satisfying:

$$|\varphi'| \leq \frac{2}{(\sigma - 1)r},$$

$\varphi' \leq 0$, $\varphi \geq 0$, $\varphi(\rho) = 1$ when $0 \leq \rho \leq r$, and $\varphi(\rho) = 0$ when $\rho \geq \sigma r$. Let $\eta(s) : [0, +\infty) \to [0, 1]$ be a smooth function satisfying:

$$|\eta'| \leq \frac{2}{(\sigma - 1)^2 r^2},$$

$\eta' \leq 0$, $\eta \geq 0$, $\eta(s) = 1$ when $s \leq t + r^2$, and $\eta(s) = 0$ when $t + (\sigma r)^2 \leq s \leq T$. Define a cutoff function $\psi(y, s)$ by

$$\psi(y, s) = \varphi(d(y, x, s))\eta(s).$$

Writing $\omega = u^p$, multiplying $\omega\psi^2$ to (4-9) and integrating by parts yield

$$\begin{aligned} \int_{Q_{\sigma r}} \nabla(\omega\psi^2)\nabla\omega \, dg(y, s)\, ds + p\int_{Q_{\sigma r}} S\omega^2\psi^2 \, dg(y, s)\, ds \\ \leq \int_{Q_{\sigma r}} (\partial_s\omega)\omega\psi^2 \, dg(y, s)\, ds. \end{aligned} \tag{4-10}$$

Integrating by parts, the right-hand side of (4-10) gives

$$\begin{aligned} &\int_{Q_{\sigma r}} (\partial_s\omega)\omega\psi^2 \, dg(y, s)\, ds \\ &= -\int_{Q_{\sigma r}} \omega^2\psi\,\partial_s\psi \, dg(y, s)\, ds + \frac{1}{2}\int_{Q_{\sigma r}} (\psi\omega)^2 S \, dg(y, s)\, ds - \frac{1}{2}\int_{B_{\sigma r}(t)} (\psi\omega)^2 \, dg(y, t). \end{aligned}$$

By the nonnegativity of Sy and the identity (see [Chow et al. 2006; List 2008])

$$\partial_s d(x, y, s) = -\int_0^{d(x,y,s)} Sy(\gamma'(\tau), \gamma'(\tau))\, d\tau \leq 0,$$

we have

$$\partial_s\psi = \eta(s)\varphi'(d(y, x, s))\partial_s d(x, y, s) + \varphi(d(y, x, s))\eta'(s) \geq \varphi(d(y, x, s))\eta'(s).$$

Hence

$$\int_{Q_{\sigma r}} (\partial_s\omega)\omega\psi^2\,dg(y,s)\,ds \le \int_{Q_{\sigma r}} \omega^2\psi\varphi(d(y,x,s))|\eta'(s)|\,dg(y,s)\,ds + \frac{1}{2}\int_{Q_{\sigma r}} (\psi\omega)^2 S\,dg(y,s)\,ds - \frac{1}{2}\int_{B_{\sigma r}(t)} (\psi\omega)^2\,dg(y,t). \tag{4-11}$$

Also, note that

$$\int_{Q_{\sigma r}} \nabla(\omega\psi^2)\nabla\omega\,dg(y,s)\,ds = \int_{Q_{\sigma r}} |\nabla(\omega\psi)|^2\,dg(y,s)\,ds - \int_{Q_{\sigma r}} |\nabla\psi|^2\omega^2\,dg(y,s)\,ds. \tag{4-12}$$

Then from (4-10), (4-11) and (4-12), we deduce

$$\begin{aligned}\int_{Q_{\sigma r}} |\nabla(\omega\psi)|^2\,dg(y,s)\,ds + \frac{1}{2}\int_{Q_{\sigma r}} S(\omega\psi)^2\,dg(y,s)\,ds + \frac{1}{2}\int_{B_{\sigma r}(t)} (\psi\omega)^2\,dg(y,t)\\ \le \int_{Q_{\sigma r}} \omega^2\psi\varphi(d(y,x,s))|\eta'(s)|\,dg(y,s)\,ds + \int_{Q_{\sigma r}} |\nabla\psi|^2\omega^2\,dg(y,s)\,ds\\ \le \frac{c}{(\sigma-1)^2r^2}\int_{Q_{\sigma r}} \omega^2\,dg(y,t).\end{aligned} \tag{4-13}$$

Using Hölder's inequality one finds

$$\int (\psi\omega)^{2(1+2/n)}\,dg \le \Big(\int (\psi\omega)^{2n/(n-2)}\,dg\Big)^{(n-2)/n}\Big(\int (\psi\omega)^2\,dg\Big)^{2/n}, \tag{4-14}$$

and using Corollary 3.3, we see that for each $t \in (0, T)$

$$\Big(\int (\psi\omega)^{2n/(n-2)}\,dg(s)\Big)^{(n-2)/n} \le A_0\int \big(|\nabla(\psi\omega)|^2 + S(\psi\omega)^2\big)\,dg(s), \tag{4-15}$$

where A_0 depends only on the dimension n, λ_0 and the initial metric g_0.

By (4-14) and (4-15), we obtain

$$\int_{B_{\sigma r}(s)} (\psi\omega)^{2(1+2/n)}\,dg(s) \le A_0\Big(\int_{B_{\sigma r}(s)} \big(|\nabla(\psi\omega)|^2 + S(\psi\omega)^2\big)\,dg\Big)\Big(\int_{B_{\sigma r}(s)} (\psi\omega)^2\,dg\Big)^{2/n}.$$

Setting $\theta = 1+2/n$, integrating the above inequality with respect to s on $[t, t+(\sigma r)^2]$ and using (4-13), we reach

$$\int_{Q_{\sigma r}} (\psi\omega)^{2\theta}\,dg(y,s)\,ds \le A_0\Big(\frac{1}{(\sigma-1)^2r^2}\int_{Q_{\sigma r}(x,t)} \omega^2\,dg(y,s)\,ds\Big)^{\theta},$$

which implies

$$\int_{Q_r} \omega^{2\theta}\,dg(y,s)\,ds \le A_0\Big(\frac{1}{(\sigma-1)^2r^2}\int_{Q_{\sigma r}(x,t)} \omega^2\,dg(y,s)\,ds\Big)^{\theta}. \tag{4-16}$$

Now we choose the sequences of σ_i and p_i as $\sigma_0 = 2$, $\sigma_i = 2 - \sum_{j=1}^{i} 2^{-j}$, $p_i = \theta^i$. Then inequality (4-16) gives that

$$\|u^2\|_{L^{\theta^{i+1}}(\sigma_{i+1}r)} \le A_0^{1/\theta^{i+1}} \left(\frac{\sigma_{i+1}^2}{(\sigma_i - \sigma_{i+1})^2 r^2} \right)^{1/\theta^i} \|u^2\|_{L^{\theta^i}(\sigma_i r)},$$

which gives an L^2 mean value inequality

$$\sup_{Q_{r/2}(x,t)} u^2 \le \frac{c}{r^{2+n}} \int_{Q_r(x,t)} u^2 \, dg(y,s)\, ds, \tag{4-17}$$

where c depends on the dimension n, λ_0 and the initial metric g_0. Then by a generic trick of Li and Schoen (see [Li 2012, Section 32]) we arrive at an L^1 mean value inequality: for $r > 0$,

$$\sup_{Q_{r/2}(x,t)} u \le \frac{c}{r^{2+n}} \int_{Q_r(x,t)} u \, dg(y,s)\, ds. \tag{4-18}$$

For $y \in M$ and $s > t$, applying (4-18) on $u = G(\cdot\,,\cdot : y, T)$ with $r = \sqrt{\frac{1}{2}(T-t)}$ and from the fact $\int_M u(z,\tau)\, dg(z,\tau)\, d\tau = 1$, we conclude

$$G(x,t;y,T) \le \frac{c}{(T-t)^{n/2}}. \tag{4-19}$$

Step 2. Using methods of the exponential weight due to Davies [1989], we prove the bound with the exponential term.

It is clear that we only have to deal with the case $d(x_0, y_0, T) \ge 2\sqrt{T-t}$. Otherwise, by (4-19) the Gaussian-type upper bound (1-4) holds obviously. Pick a point $x_0 \in M$, a number $\lambda < 0$ which is determined later and a function $f \in L^2(M, g(T))$. Consider the functions $u(x,t)$ and $H(x,t)$ defined by

$$u(x,t) = \int_M G(x,t;y,T) e^{-\lambda d(y,x_0,T)} f(y)\, dg(y,T),$$
$$H(x,t) = e^{\lambda d(x,x_0,t)} u(x,t).$$

It is clear that u is a solution of (1-2) with initial data

$$u(x,T) = e^{-\lambda d(x,x_0,T)} f(x).$$

Direct calculation shows

$$\begin{aligned}\partial_t\int_M H^2(x,t)\,dg(x,t) &= \partial_t\int_M e^{2\lambda d(x.x_0,t)}u^2(x,t)\,dg(x,t)\\ &= 2\lambda\int_M e^{2\lambda d(x.x_0,t)}\partial_t d(x,x_0,t)u^2(x,t)\,dg(x,t)\\ &\qquad -\int_M e^{2\lambda d(x.x_0,t)}u^2(x,t)S(x,t)\,dg(x,t)\\ &\qquad\quad -2\int_M e^{2\lambda d(x.x_0,t)}u(x,t)(\Delta u-Su)\,dg(x,t)\\ &\geq -2\int_M e^{2\lambda d(x.x_0,t)}u(x,t)\Delta u\,dg(x,t),\end{aligned}$$

where the last inequality holds due to $Sy\geq 0$, $\lambda<0$, and $\partial_t\, d(x,x_0,t)\leq 0$.

By integration by parts, we obtain

$$\begin{aligned}\partial_t\int_M H^2(x,t)\,dg(x,t) \geq 4\lambda\int_M e^{2\lambda d(x.x_0,t)}u(x,t)\langle\nabla u,\nabla d(x,x_0,t)\rangle\,dg(x,t)\\ +2\int_M e^{2\lambda d(x.x_0,t)}|\nabla u|^2\,dg(x,t),\end{aligned}$$

and also

$$\begin{aligned}&\int_M|\nabla H(x,t)|^2\,dg(x,t)\\ &=\int_M|\nabla(u(x,t)e^{\lambda d(x.x_0,t)})|^2\,dg(x,t)\\ &=\int_M e^{2\lambda d(x.x_0,t)}|\nabla u|^2\,dg(x,t)+2\lambda\int_M e^{2\lambda d(x.x_0,t)}u(x,t)\langle\nabla u,\nabla d(x,x_0,t)\rangle\,dg(x,t)\\ &\qquad\qquad +\lambda^2\int_M e^{2\lambda d(x.x_0,t)}|\nabla d|^2u^2\,dg(x,t).\end{aligned}$$

Combining the above two expressions, we conclude

$$\partial_t\int_M H^2(x,t)\,dg(x,t)\geq 2\int_M|\nabla H(x,t)|^2\,dg(x,t)-2\lambda^2\int_M e^{2\lambda d(x.x_0,t)}u^2\,dg(x,t),$$

which implies

$$\partial_t\int_M H^2(x,t)\,dg(x,t)\geq -2\lambda^2\int_M H^2(x,t)\,dg(x,t).$$

Integrating on $[t, T]$, we arrive at the L^2 estimate

$$\int_M H^2(x,t)\,dg(x,t) \le e^{2\lambda^2(T-t)}\int_M H^2(x,T)\,dg(x,T) = e^{2\lambda^2(T-t)}\int_M f^2(x)\,dg(x,T). \tag{4-20}$$

Therefore, by the mean value inequality (4-17) with $r=\sqrt{\frac{1}{2}(T-t)}$, it holds that

$$\begin{aligned}u^2(x,t) &\le \frac{c}{(T-t)^{1+n/2}}\int_t^{(T+t)/2}\int_{B(x,\sqrt{(T-t)/2},\tau)} u^2(z,\tau)\,dg(z,\tau)\,d\tau\\ &\le \frac{c}{(T-t)^{1+n/2}}\int_t^{(T+t)/2}\int_{B(x,\sqrt{(T-t)/2},\tau)} e^{-2\lambda d(z,x_0,\tau)}H^2(z,\tau)\,dg(z,\tau)\,d\tau.\end{aligned}$$

Particularly, at $x=x_0$, we get

$$u^2(x_0,t) \le \frac{ce^{-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{1+n/2}}\int_t^{(T+t)/2}\int_{B(x_0,\sqrt{(T-t)/2},\tau)} H^2(z,\tau)\,dg(z,\tau)\,d\tau.$$

From (4-20), it follows that

$$u^2(x_0,t) \le \frac{ce^{2\lambda^2(T-t)-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{n/2}}\int_M f^2(y)\,dg(y,T),$$

i.e.,

$$\left(\int_M G(x_0,t;z,T)e^{-\lambda d(z,x_0,T)}f(z)\,dg(z,T)\right)^2 \le \frac{ce^{2\lambda^2(T-t)-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{n/2}}\int_M f^2(y)\,dg(y,T). \tag{4-21}$$

Now we fix y_0 such that $d(y_0,x_0,T)^2 \ge 4(T-t)$. Then it follows from the triangle inequality that

$$-\lambda d(z,x_0,T) \ge -\tfrac{1}{2}\lambda d(x_0,y_0,T),$$

provided by $d(z,y_0,T)\le\sqrt{T-t}$. Then (4-21) implies

$$\left(\int_{B(y_0,\sqrt{T-t},T)} G(x_0,t;z,T)f(z)\,dg(T)\right)^2 \le \frac{ce^{\lambda d(x_0,y_0,T)+2\lambda^2(T-t)-2\lambda\sqrt{(T-t)/2}}}{(T-t)^{n/2}}\int_M f^2(y)\,dg(T).$$

Note that by the Cauchy–Schwartz inequality

$$2\lambda^2(T-t)-2\lambda\sqrt{\tfrac{1}{2}(T-t)} \le 3\lambda^2(T-t)+\tfrac{1}{2},$$

and letting $\lambda = -d(x_0, y_0, T)/(b(T-t))$, we obtain

$$\left(\int_{B(y_0,\sqrt{T-t},T)} G(x_0,t;z,T)f(z)\,dg(z,T)\right)^2 \le \frac{ce^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}}}{(T-t)^{n/2}}\int_M f^2(y)\,dg(y,T)$$

with $b > 0$ sufficiently large, and c_1 is an absolute constant. Then by the arbitrariness of f, we derive

$$\int_{B(y_0,\sqrt{T-t},T)} G^2(x_0,t;z,T)\,dg(z,T) \le \frac{ce^{-c_1\frac{d^2(x_0,y_0,T)}{T-t}}}{(T-t)^{n/2}}.$$

Hence, there exists $z_0 \in B(y_0, \sqrt{T-t}, T)$ such that

$$G^2(x_0,t;z_0,T) \le \frac{ce^{-c_1\frac{d^2(x_0,y_0,T)}{T-t}}}{(T-t)^{n/2}|B(y_0,\sqrt{T-t},T)|_T}. \tag{4-22}$$

Let us recall that in [Guenther 2002] the adjoint property of the $G(x_0, t : \cdot, \cdot)$ is obtained, thus

$$\Delta_z G(x,t;z,\tau) - \partial_\tau G(x,t;z,\tau) = 0$$

along Ricci-harmonic flow (1-1).

Choosing $\delta = 1$ in Theorem 4.1 and $z_0 \in B(y_0, \sqrt{T-t}, T)$, it then follows that

$$\begin{aligned} G(x_0,t;y_0,T) &\le \sqrt{G(x_0,t;z_0,T)}\sqrt{A}e^{d^2(y_0,z_0,T)/4(T-t)} \\ &\le e^{1/4}\sqrt{G(x_0,t;z_0,T)}\sqrt{A}, \end{aligned} \tag{4-23}$$

where $A = \sup_{M\times[(t+T)/2,T]} G(x_0, t; \cdot, \cdot)$.

Since (4-19) implies

$$A \le \frac{c}{(T-t)^{n/2}},$$

then combining with (4-22) and (4-23) we have

$$G(x_0,t;y_0,T)^2 \le \frac{c}{(T-t)^{n/2}}\frac{1}{(T-t)^{n/4}\sqrt{|B(y_0,\sqrt{T-t},T)|_T}}e^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}}.$$

Therefore by the Cauchy–Schwartz inequality, we get

$$\begin{aligned} G(x_0,t;y_0,T) &\le c\left(\frac{1}{(T-t)^{n/2}} + \frac{1}{|B(y_0,\sqrt{T-t},T)|_T}\right)e^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}} \\ &\le \frac{c}{|B(y_0,\sqrt{T-t},T)|_T}e^{-\frac{c_1 d^2(x_0,y_0,T)}{T-t}}, \end{aligned}$$

where c depends on the dimension n, λ_0 and the initial metric g_0, and c_1 depends only on dimension n. In the last inequality, we used the volume comparison theorem with the nonnegative Ricci curvature. By the arbitrariness of x_0 and y_0, we complete the proof. □

Acknowledgements

Liu's research is partially supported by NSFC 11131005. Wang's research is partially supported by China Postdoctoral Science Foundation grant 2016M591900, Ph.D. Programs Foundation (China) grant 20133201110001, Natural Science Foundation of Jiangsu Province grant BK20160301, and Natural Science Foundation of Education Committee of Jiangsu Province grant 16KJB110018. The authors are very grateful to the anonymous referees for the careful reading and the valuable suggestions.

References

[Carrillo and Ni 2009] J. A. Carrillo and L. Ni, "Sharp logarithmic Sobolev inequalities on gradient solitons and applications", *Comm. Anal. Geom.* **17**:4 (2009), 721–753. MR Zbl

[Chau et al. 2011] A. Chau, L.-F. Tam, and C. Yu, "Pseudolocality for the Ricci flow and applications", *Canad. J. Math.* **63**:1 (2011), 55–85. MR Zbl

[Chow et al. 2006] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics **77**, American Mathematical Society, Providence, RI, 2006. MR Zbl

[Davies 1989] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics **92**, Cambridge University Press, 1989. MR Zbl

[Guenther 2002] C. M. Guenther, "The fundamental solution on manifolds with time-dependent metrics", *J. Geom. Anal.* **12**:3 (2002), 425–436. MR Zbl

[Hamilton 1982] R. S. Hamilton, "Three-manifolds with positive Ricci curvature", *J. Differential Geom.* **17**:2 (1982), 255–306. MR Zbl

[Hamilton 1986] R. S. Hamilton, "Four-manifolds with positive curvature operator", *J. Differential Geom.* **24**:2 (1986), 153–179. MR Zbl

[Hamilton 1993] R. S. Hamilton, "The Harnack estimate for the Ricci flow", *J. Differential Geom.* **37**:1 (1993), 225–243. MR Zbl

[Hebey 1996] E. Hebey, "Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius", *Amer. J. Math.* **118**:2 (1996), 291–300. MR Zbl

[Kuang and Zhang 2008] S. Kuang and Q. S. Zhang, "A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow", *J. Funct. Anal.* **255**:4 (2008), 1008–1023. MR Zbl

[Li 2007] J.-F. Li, "Eigenvalues and energy functionals with monotonicity formulae under Ricci flow", *Math. Ann.* **338**:4 (2007), 927–946. MR Zbl

[Li 2012] P. Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics **134**, Cambridge University Press, 2012. MR Zbl

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR Zbl

[List 2008] B. List, "Evolution of an extended Ricci flow system", *Comm. Anal. Geom.* **16**:5 (2008), 1007–1048. MR Zbl

[Müller 2012] R. Müller, "Ricci flow coupled with harmonic map flow", *Ann. Sci. Éc. Norm. Supér.* (4) **45**:1 (2012), 101–142. MR Zbl

[Ni 2004] L. Ni, "The entropy formula for linear heat equation", *J. Geom. Anal.* **14**:1 (2004), 87–100. MR Zbl

[Ni 2006] L. Ni, "A note on Perelman's LYH-type inequality", *Comm. Anal. Geom.* **14**:5 (2006), 883–905. MR Zbl

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl arXiv

[Saloff-Coste 2002] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series **289**, Cambridge University Press, 2002. MR Zbl

[Zhang 2006] Q. S. Zhang, "Some gradient estimates for the heat equation on domains and for an equation by Perelman", *Int. Math. Res. Not.* **2006**:15 (2006), art. id. 92314. MR Zbl

[Zhang 2007] Q. S. Zhang, "A uniform Sobolev inequality under Ricci flow", *Int. Math. Res. Not.* **2007**:17 (2007), art. id. rnm056. Corrected in art. id. rnm096 (2007). MR Zbl

[Zhang 2011] Q. S. Zhang, *Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture*, CRC Press, Boca Raton, FL, 2011. MR Zbl

[Zhu 2013] A. Zhu, "Differential Harnack inequalities for the backward heat equation with potential under the harmonic-Ricci flow", *J. Math. Anal. Appl.* **406**:2 (2013), 502–510. MR Zbl

Received August 21, 2015. Revised June 8, 2016.

XIAN-GAO LIU
INSTITUTE OF MATHEMATICS
FUDAN UNIVERSITY
200433 SHANGHAI
CHINA
xgliu@fudan.edu.cn

KUI WANG
SCHOOL OF MATHEMATICAL SCIENCES
SOOCHOW UNIVERSITY
215006 SUZHOU
CHINA
kuiwang@suda.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 287, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.287.485

APPROXIMATION TO AN EXTREMAL NUMBER, ITS SQUARE AND ITS CUBE

JOHANNES SCHLEISCHITZ

We study rational approximation properties for successive powers of extremal numbers defined by Roy. For $n \in \{1, 2\}$, the classic approximation constants $\lambda_n(\zeta)$, $\hat{\lambda}_n(\zeta)$, $w_n(\zeta)$, $\widehat{w}_n(\zeta)$ connected to an extremal number ζ have been established and in fact much more is known. However, so far almost nothing had been known for $n \geq 3$. In this paper we determine all classic approximation constants as above for $n = 3$. Our methods will more generally provide detailed information on the combined graph defined by Schmidt and Summerer assigned to an extremal number, its square and its cube. We provide some results for $n = 4$ as well. In the course of the proofs of the main results we establish a very general connection between Khintchine's transference inequalities and uniform approximation.

1. Approximation constants and extremal numbers

Let ζ be a real transcendental number and $n \geq 1$ be an integer. For $1 \leq j \leq n+1$ we define the approximation constants $\lambda_{n,j}(\zeta)$ as the supremum of $\eta \in \mathbb{R}$ such that the system

$$|x| \leq X, \qquad \max_{1\leq i\leq n} |\zeta^i x - y_i| \leq X^{-\eta} \tag{1}$$

has (at least) j linearly independent solutions $(x, y_1, y_2, \ldots, y_n) \in \mathbb{Z}^{n+1}$ for arbitrarily large values of X. Moreover, let $\hat{\lambda}_{n,j}(\zeta)$ be the supremum of η such that (1) has (at least) j linearly independent solutions for all sufficiently large X. In the case of $j = 1$ we also only write $\lambda_n(\zeta)$ and $\hat{\lambda}_n(\zeta)$ respectively, which are just the classical approximation constants defined by Bugeaud and Laurent [2005]. By Dirichlet's theorem for all transcendental real ζ and $n \geq 1$ these exponents satisfy the estimate

$$\lambda_n(\zeta) \geq \hat{\lambda}_n(\zeta) \geq \frac{1}{n}. \tag{2}$$

Supported by the Austrian Science Fund FWF grant P24828.
MSC2010: primary 11J13; secondary 11H06.
Keywords: extremal numbers, Diophantine approximation constants, geometry of numbers, lattices.

Moreover from the definition we see that

$$\lambda_1(\zeta) \geq \lambda_2(\zeta) \geq \cdots \quad \text{and} \quad \hat{\lambda}_1(\zeta) \geq \hat{\lambda}_2(\zeta) \geq \cdots .$$

Similarly, let $w_{n,j}(\zeta)$ and $\widehat{w}_{n,j}(\zeta)$ be the supremum of $\eta \in \mathbb{R}$ such that the system

$$H(P) \leq X, \qquad 0 < |P(\zeta)| \leq X^{-\eta} \tag{3}$$

has (at least) j linearly independent polynomial solutions $\sum_{i=0}^{n} a_i T^i$ of degree at most n with integers a_j for arbitrarily large X and all large X respectively, where $H(P) = \max_{0 \leq j \leq n} |a_j|$. Again for $j = 1$ we also write $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$ which coincide with classical exponents. Again by Dirichlet's theorem we have

$$w_n(\zeta) \geq \widehat{w}_n(\zeta) \geq n. \tag{4}$$

Moreover it is obvious that

$$w_1(\zeta) \leq w_2(\zeta) \leq \cdots \quad \text{and} \quad \widehat{w}_1(\zeta) \leq \widehat{w}_2(\zeta) \leq \cdots .$$

The exponents defined above are connected via Khintchine's [1926] transference inequalities.

$$\frac{w_n(\zeta)}{(n-1)w_n(\zeta)+n} \leq \lambda_n(\zeta) \leq \frac{w_n(\zeta)-n+1}{n}. \tag{5}$$

Similarly thanks to German [2012] we know that the uniform exponents are connected via

$$\frac{\widehat{w}_n(\zeta)-1}{(n-1)\widehat{w}_n(\zeta)} \leq \hat{\lambda}_n(\zeta) \leq \frac{\widehat{w}_n(\zeta)-n+1}{\widehat{w}_n(\zeta)}. \tag{6}$$

We point out that the estimates (5) and (6) hold more generally for the analogue exponents concerning vectors $\underline{\zeta} \in \mathbb{R}^n$ whose coordinates are $\mathbb{Q}$-linearly independent together with $\{1\}$; see for example [Schmidt and Summerer 2009]. This will be of some importance in Remark 3.2. Moreover in this case all estimates in (5) and (6) are known to be optimal.

It is known due to Davenport and Schmidt [1969] that $\widehat{w}_2(\zeta) \leq \frac{3+\sqrt{5}}{2}$ for all real transcendental ζ. Roy [2004a] proved that there exist countably many real transcendental numbers for which equality holds, and called such numbers extremal numbers. Their approximation properties have been intensely studied in dimensions $n \in \{1, 2\}$. We gather below some of the known facts which will be of importance for this paper. Throughout the paper let

$$\rho = 2+\sqrt{5}, \qquad \tau = \frac{3+\sqrt{5}}{2}, \qquad \nu = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \gamma = \frac{\sqrt{5}-1}{2}.$$

These values are linked via $\tau=\nu^2$, $\rho=\nu^3$ and $\gamma=\nu^{-1}$. Moreover $\tau=\nu+1$ and $\nu^2-\nu-1=0$. It is known that for ζ an extremal number, the identities

$$(7)\qquad w_1(\zeta)=\lambda_1(\zeta)=\lambda_2(\zeta)=1,\quad \hat{\lambda}_2(\zeta)=\gamma,\quad w_2(\zeta)=\rho,\quad \widehat{w}_2(\zeta)=\tau$$

hold. Concerning the higher successive minima functions it is immediate by Roy's results that any extremal number satisfies

$$(8)\qquad w_{2,2}(\zeta)=\tau,\qquad w_{2,3}(\zeta)=\nu,\qquad \lambda_{2,2}(\zeta)=\gamma,\qquad \lambda_{2,3}(\zeta)=\gamma^2,$$

$$(9)\qquad \widehat{w}_{2,2}(\zeta)=\nu,\qquad \widehat{w}_{2,3}(\zeta)=1,\qquad \hat{\lambda}_{2,2}(\zeta)=\gamma^2,\qquad \hat{\lambda}_{2,3}(\zeta)=\gamma^3.$$

In fact even more detailed approximation properties are known for $n=2$. There is concise information on the integral approximation vectors inducing very good approximations in (1) such as for the polynomials inducing very good approximations in (3). We will concretely utilize the following consequence of Roy's results, which is part of the claim of [Roy 2004b, Theorem 7.2]. See also [Roy 2004a, Proposition 8.1, Theorem 8.2]. As usual $a\asymp b$ means both $a\ll b$ and $b\ll a$ are satisfied everywhere it occurs in the sequel.

Theorem 1.1 (Roy). *For any extremal number ζ there exists a sequence of irreducible polynomials $(P_k)_{k\geq 1}\in\mathbb{Z}[T]$ of degree precisely two such that*

$$H(P_{k+1})\asymp H(P_k)^{\nu}\quad \textit{and}\quad |P_k(\zeta)|\asymp H(P_k)^{-\rho}.$$

Moreover we have

$$(10)\qquad |P_k'(\zeta)|\asymp H(P_k).$$

All the implied constants depend on ζ only.

For the irreducibility and (10), see [Roy 2004a, Proposition 8.1, Theorem 8.2]. The other claims are part of the claims of [Roy 2004b, Theorem 7.2]. In fact the irreducibility is easily deduced from $\lambda_1(\zeta)=1$ in (7) and (42) below. Indeed these relations imply that P_k in the theorem cannot have a rational root at least for large k and are thus indeed irreducible. In context of (8), (9) we finally mention that for $n=2$, extremal numbers induce the regular graph defined by Schmidt and Summerer [2013b].

This paper aims to provide a better understanding of the classic approximation constants for extremal numbers in higher dimension $n>2$. More generally we will provide a description of the behavior of the approximation functions $L_j(q)$ and $L_j^*(q)$ defined by Schmidt and Summerer [2009] in the course of their study of parametric geometry of numbers, for $n=3$ and partially for $n=4$. We recall basic facts on parametric geometry of numbers at the start of Section 3. Our results will arise as a combination of the known results on extremal numbers for $n\in\{1,2\}$ recalled above with estimates from parametric geometry of numbers. So far only a few nontrivial quantitative results on classical approximation constants for extremal numbers in dimension $n > 2$ exist. The estimates

$$w_n(\zeta) \leq \exp\{c(\zeta) \cdot (\log(3n))^2 (\log\log(3n))^2\}$$

for all $n \geq 1$ and some constant $c(\zeta) > 0$ are due to Adamczewski and Bugeaud [2010]. It was recently proved [Bugeaud and Schleischitz 2016] that $\widehat{w}_3(\zeta) \leq 4$ for extremal numbers ζ, which improves the upper bound $3 + \sqrt{2}$ valid for all transcendental real ζ from the same paper (which in turn improved the bound $2n - 1 = 5$ of Davenport and Schmidt [1969, Theorem 2b]). However, we will determine the precise value of $\widehat{w}_3(\zeta)$ in Theorem 2.1. Besides approximation to extremal numbers by cubic algebraic integers has been investigated. Roy [2004a] showed that for extremal number ζ and any algebraic integer α of degree three we have

$$|\zeta - \alpha| \gg H(\alpha)^{-\tau-1}.$$

Moreover in [Roy 2003, Theorem 1.1] he showed that for some extremal numbers the exponent $-1-\tau$ can be replaced by $-\tau$. The exponent $-\tau$ is optimal since

$$|\zeta - \alpha| \ll H(\alpha)^{-\tau}$$

has solutions in algebraic integers α of degree at most three and arbitrarily large height $H(\alpha)$ for any given real number ζ, as shown by Davenport and Schmidt [1969]. It follows that for any real ζ there are monic polynomials of degree at most three and arbitrarily large height $H(P)$ such that

$$|P(\zeta)| \ll H(P)^{-\nu}.$$

It follows from [Roy 2004a] that the exponent ν is optimal as well, since again the reverse inequality holds at least for some class of extremal numbers and arbitrarily large $H(P)$.

2. New results

The case n = 3. The first major result of the paper is the following.

Theorem 2.1. *Let ζ be an extremal number. Then we have*

$$w_3(\zeta) = w_2(\zeta) = \rho, \qquad \lambda_3(\zeta) = \frac{1}{\sqrt{5}}, \tag{11}$$

and

$$\widehat{w}_3(\zeta) = 3, \qquad \hat{\lambda}_3(\zeta) = \tfrac{1}{3}. \tag{12}$$

See the comments subsequent to Lemma 3.3 below for additional information on the dynamic behavior of the successive minima as parametric functions. This dynamical point of view will also enable us to derive the following Theorem 2.2 from Theorem 2.1. As usual for an algebraic number α we write $H(\alpha)=H(P)$ where $P\in\mathbb{Z}[T]$ is the irreducible minimal polynomial of α over $\mathbb{Z}[T]$ with coprime coefficients.

Theorem 2.2. *Let ζ be an extremal number and $\epsilon>0$. Then the estimate*

$$|Q(\zeta)|\le H(Q)^{-3-\epsilon} \tag{13}$$

has only finitely many irreducible solutions $Q\in\mathbb{Z}[T]$ of degree precisely three. In particular

$$|\zeta-\alpha|\le H(\alpha)^{-4-\epsilon} \tag{14}$$

has only finitely many algebraic solutions α of degree precisely three. On the other hand the estimates

$$|Q(\zeta)|\le H(Q)^{-3+\epsilon} \quad \textit{and} \quad |\zeta-\alpha|\le H(\alpha)^{-4+\epsilon} \tag{15}$$

have solutions in irreducible polynomials Q of degree precisely three and algebraic α of degree precisely three of arbitrarily large heights $H(Q)$ and $H(\alpha)$. Moreover there are arbitrarily large X such that

$$H(Q)\le X, \qquad |Q(\zeta)|\le X^{-\sqrt{5}-\epsilon} \tag{16}$$

has no irreducible solution $Q\in\mathbb{Z}[T]$ of degree precisely three. In particular for arbitrarily large X the system

$$H(\alpha)\le X, \qquad |\zeta-\alpha|\le H(\alpha)^{-1}X^{-\sqrt{5}-\epsilon} \tag{17}$$

has no algebraic solution α of degree precisely three.

We strongly expect that the exponents in (16) and (17) are optimal as well. See the comments below the proof of Theorem 2.2 for a heuristic argument that supports this belief. Compare Theorem 2.2 with the estimates concerning approximation by algebraic integers α at the end of Section 1.

The case $n=4$. We want to establish a lower bound for the exponent $\lambda_4(\zeta)$. Our result, based on parametric geometry of numbers, is the following.

Theorem 2.3. *Let ζ be an extremal number. Then*

$$\lambda_4(\zeta)\ge\frac{\gamma}{2}=\frac{\sqrt{5}-1}{4}. \tag{18}$$

If $w_4(\zeta)=w_2(\zeta)=\rho$, then there is equality in (18) and moreover

$$\widehat{w}_4(\zeta)=4, \qquad \hat{\lambda}_4(\zeta)=\tfrac{1}{4}. \tag{19}$$

Observe that $\rho \approx 4.2361 > 4$, so the assumption of the conditioned results are natural and thus we believe that there is actually equality in (18) and that (19) holds. Theorems 2.1 and 2.2 also support this belief. On the other hand (4) prohibits $w_n(\zeta) = \rho$ for $n \geq 5$, which in general prohibits the methods of the paper from working for $n \geq 5$.

The constant in (18) is approximately $\gamma/2 \approx 0.3090$. Observe that this improves the lower bound derived from $w_4(\zeta) \geq w_2(\zeta) = \rho$ in combination with Khintchine's transference inequalities (5), which turns out to be $\frac{2+\sqrt{5}}{10+3\sqrt{5}} \approx 0.2535$, only slightly larger than the trivial bound $\frac{1}{4}$ from (2).

3. Preparatory results

Parametric geometry of numbers. For the proofs of the new results we introduce some concepts of the parametric geometry of numbers following Schmidt and Summerer [2009, 2013a], where we develop the theory only as far as it is needed for our purposes and slightly deviate from their notation. In particular we restrict to the case of successive powers. Some more specific properties will be carried out in Section 4 for immediate application to preliminary results. Let $\zeta \in \mathbb{R}$ be given and $Q > 1$ a parameter. For $n \geq 1$ and $1 \leq j \leq n+1$, define $\psi_{n,j}(Q)$ as the minimum of $\eta \in \mathbb{R}$ such that

$$|x| \leq Q^{1+\eta}, \qquad \max_{1 \leq j \leq n} |\zeta^j x - y_j| \leq Q^{-(1/n)+\eta}$$

has (at least) j linearly independent solutions $(x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}$. The functions $\psi_{n,j}(Q)$ can be equivalently defined via a lattice point problem, see [Schmidt and Summerer 2009]. They have the properties

$$-1 \leq \psi_{n,j}(Q) \leq \frac{1}{n} \quad \text{and} \quad Q > 1, \quad 1 \leq j \leq n+1.$$

Let

$$\underline{\psi}_{n,j} = \liminf_{Q \to \infty} \psi_{n,j}(Q) \quad \text{and} \quad \overline{\psi}_{n,j} = \limsup_{Q \to \infty} \psi_{n,j}(Q).$$

These values clearly all lie in the interval $[-1, 1/n]$. From Dirichlet's theorem it follows that $\psi_{n,1}(Q) \leq 0$ for all $Q > 1$ and hence $\overline{\psi}_{n,1} \leq 0$. For our purposes, even more important will be the functions $\psi_{n,j}^*(Q)$ from [Schmidt and Summerer 2009]. For $1 \leq j \leq n+1$ and a parameter $Q > 1$, define the value $\psi_{n,j}^*(Q)$ as the minimum of $\eta \in \mathbb{R}$ such that

$$|H(P)| \leq Q^{(1/n)+\eta}, \quad |P(\zeta)| \leq Q^{-1+\eta}$$

has (at least) j linearly independent solutions in polynomials $P \in \mathbb{Z}[T]$ of degree at most n. See the same work for the connection of the functions $\psi_{n,j}^*$ to a related lattice point problem, similarly as for simultaneous approximation. Again put

$$\underline{\psi}^*_{n,j} = \liminf_{Q\to\infty} \psi^*_{n,j}(Q) \quad \text{and} \quad \overline{\psi}^*_{n,j} = \limsup_{Q\to\infty} \psi^*_{n,j}(Q).$$

For transcendental ζ Schmidt and Summerer [2013a, (1.11)] established the inequalities

$$j\underline{\psi}_{n,j} + (n+1-j)\overline{\psi}_{n,n+1} \geq 0 \quad \text{and} \quad j\overline{\psi}_{n,j} + (n+1-j)\underline{\psi}_{n,n+1} \geq 0,$$

for $1 \leq j \leq n+1$. The dual inequalities

(20) $$j\underline{\psi}^*_{n,j} + (n+1-j)\overline{\psi}^*_{n,n+1} \geq 0 \quad \text{and} \quad j\overline{\psi}^*_{n,j} + (n+1-j)\underline{\psi}^*_{n,n+1} \geq 0,$$

hold as well for the same reason. As pointed out in [Schmidt and Summerer 2009] Mahler's inequality implies

(21) $$|\psi_{n,j}(Q) + \psi^*_{n,n+2-j}(Q)| \ll \frac{1}{\log Q} \quad \text{for} \quad 1 \leq j \leq n+1.$$

In particular we have

(22) $$\underline{\psi}_{n,j} = -\overline{\psi}^*_{n,n+2-j} \quad \text{and} \quad \overline{\psi}_{n,j} = -\underline{\psi}^*_{n,n+2-j} \quad \text{for} \quad 1 \leq j \leq n+1.$$

In particular all values $\underline{\psi}^*_{n,j}$, $\overline{\psi}^*_{n,j}$ lie in the interval $[-1/n, 1]$, and $\overline{\psi}^*_{n,1} \leq 0$ follows again from Dirichlet's theorem. The constants $\underline{\psi}_{n,j}$, $\overline{\psi}_{n,j}$, $\underline{\psi}^*_{n,j}$, $\overline{\psi}^*_{n,j}$ relate to the classical approximation constants $\lambda_{n,j} = \lambda_{n,j}(\zeta)$, $w_{n,j} = w_{n,j}(\zeta)$ assigned to real ζ via

(23) $$(1+\lambda_{n,j})(1+\underline{\psi}_{n,j}) = (1+\hat{\lambda}_{n,j})(1+\overline{\psi}_{n,j}) = \frac{n+1}{n} \quad \text{for} \quad 1 \leq j \leq n+1,$$

and

(24) $$(1+w_{n,j})\left(\frac{1}{n}+\underline{\psi}^*_{n,j}\right) = (1+\widehat{w}_{n,j})\left(\frac{1}{n}+\overline{\psi}^*_{n,j}\right) = \frac{n+1}{n} \quad \text{for} \quad 1 \leq j \leq n+1.$$

See [Schmidt and Summerer 2009, Theorem 1.4] for a proof of $j = 1$ which can be readily extended to the case of arbitrary $1 \leq j \leq n+1$ as noticed in [Schleischitz 2013]. From repeated application of (22), (23) and (24) one can deduce

(25) $$\lambda_{n,j}(\zeta) = \frac{1}{\widehat{w}_{n,n+2-j}(\zeta)} \quad \text{and} \quad \hat{\lambda}_{n,j}(\zeta) = \frac{1}{w_{n,n+2-j}(\zeta)},$$

for $1 \leq j \leq n+1$, already noticed in [Schleischitz 2014]. For $q > 0$ we also define the functions

(26) $$L_{n,j}(q) = q\psi_{n,j}(Q) \quad \text{and} \quad L^*_{n,j}(q) = q\psi^*_{n,j}(Q),$$

where $Q = e^q$. They are piecewise linear with slopes among $\{-1, 1/n\}$ and $\{-1/n, 1\}$ respectively. More precisely locally any $L_{n,j}$ coincides with some

$$L_{\underline{x}}(q) = \max\left\{\log|x| - q,\ \max_{1\le j\le n}\log|\zeta^j x - y_j| + \frac{q}{n}\right\} \tag{27}$$

where $\underline{x} = (x, y_1, \dots, y_n) \in \mathbb{Z}^{n+1}$ for y_j the closest integer to $\zeta^j x$, see [Schmidt and Summerer 2009, page 75]. Similarly any $L^*_{n,j}$ coincides locally with

$$L^*_P(q) = \max\left\{\log H(P) - \frac{q}{n},\ \log|P(\zeta)| + q\right\} \tag{28}$$

for some $P \in \mathbb{Z}[T]$ of degree at most n. Observe that for fixed P the left expression in (28) decays with slope $-1/n$ whereas the right expression rises with slope 1 in the parameter q. Consequently, at a local maximum of some $L^*_{n,j}$, the rising right expression of some $L^*_P(q)$ meets the falling left expression of some $L^*_Q(q)$ with $H(Q) > H(P)$, and similarly for local maxima of $L_{n,j}$. On the other hand, at any local minimum q of some $L^*_{n,j}$ there is either equality in the expressions in (28) for some P, or the rising phase of some L^*_P meets the falling phase of some L^*_Q for some Q with $H(Q) > H(P)$. In the first case, which always applies for $j = 1$, the function $L^*_{n,j}$ coincides with L^*_P in a neighborhood of q. The situation is again very similar for $L_{n,j}$. The identity (24) has a parametric version in the sense that for any $(Q, \psi^*_{n,j}(Q))$ in the graph of some function $\psi^*_{n,j}$ there exist j linearly independent polynomials $P_1, \dots, P_j \in \mathbb{Z}[T]$ of degree at most n such that

$$(1 + w_n^{(j)})\Big(\frac{1}{n} + \psi^*_{n,j}(Q)\Big) = \frac{n+1}{n} + o(1), \qquad Q \to \infty, \tag{29}$$

holds where

$$w_n^{(j)} := \frac{\min_{1\le i\le j}(-\log|P_i(\zeta)|)}{\max_{1\le i\le j}\log H(P_i)},$$

and vice versa. Very similarly a dual parametric version of (23) for the functions $\psi_{n,j}(Q)$ can be obtained. Both versions are basically inherited from the proof of [Schmidt and Summerer 2009, Theorem 1.4]. A crucial observation for the parametric geometry of numbers developed in [Schmidt and Summerer 2009, 2013a] is that Minkowski's second lattice point theorem translates into

$$\left|\sum_{j=1}^{n+1} L_{n,j}(q)\right| \ll 1 \quad \text{and} \quad \left|\sum_{j=1}^{n+1} L^*_{n,j}(q)\right| \ll 1. \tag{30}$$

This implies that in any interval $I = (q_1, q_2)$, the sum of the differences $L_{n,j}(q_2) - L_{n,j}(q_1)$ and $L^*_{n,j}(q_2) - L^*_{n,j}(q_1)$ over $1 \le j \le n+1$ are bounded in absolute value as well by a fixed constant independent of I. We will implicitly use this fact in the proof of Theorem 2.1. This argument is widely used in [Schmidt and Summerer 2013a].

Two technical lemmas. For the conditioned result (19) we need parts of Lemma 3.1 below, which is of some interest on its own. For its proof we will use that every local maximum of $L_{n,1}$ is a local minimum of $L_{n,2}$ (note: the analogue is in general false for $L_{n,j}, L_{n,j+1}$ when $j > 1$). This follows from the elementary fact that for any vector $\underline{x} = (x, y_1, \dots, y_n) \in \mathbb{Z}^{n+1}$ clearly any integral multiple $N\underline{x}$ cannot lead to a smaller value in (1). Hence if two functions $L_{\underline{x}_1}, L_{\underline{x}_2}$ as in (27) induce two (successive) falling slopes -1 of $L_{n,1}$, with some rising phase of $L_{n,1}$ of slope $1/n$ in between, then the corresponding vectors $\underline{x}_1, \underline{x}_2$ are linearly independent, and the claim follows. Moreover we use $L_{n,1}(q) < 0$ for all $q > 0$, which is equivalent to Dirichlet's theorem.

Lemma 3.1. *Let $n \geq 1$ be an integer and ζ be a real transcendental number. Assume there is equality in either inequality of (5), that is, either*

$$n\lambda_n(\zeta) + n - 1 = w_n(\zeta) \tag{31}$$

or

$$\lambda_n(\zeta) = \frac{w_n(\zeta)}{(n-1)w_n(\zeta) + n} \tag{32}$$

holds. Then $\hat{\lambda}_n(\zeta) = 1/n$ and $\widehat{w}_n(\zeta) = n$.

Proof. Assume there is equality in the right inequality, that is $n\lambda_n(\zeta)+n-1 = w_n(\zeta)$. In case of $\lambda_n(\zeta) = \infty$ we have $\hat{\lambda}_n(\zeta) = 1/n$ and $\widehat{w}_n(\zeta) = n$ anyway by [Schleischitz 2016, Theorem 1.12 and Theorem 5.1]. Hence we can assume $\lambda_n(\zeta) < \infty$, which will simplify the estimates. It suffices to show $\hat{\lambda}_n(\zeta) = 1/n$ since the two claims are well known to be equivalent, which follows for example from (6). It was shown by Schmidt and Summerer [2009, remark on page 80, after the proof of Theorem 1.4] that the right inequality in (5) is equivalent to $\underline{\psi}_{n,1} + n\overline{\psi}_{n,n+1} \geq 0$. It follows directly from their deduction of the mentioned remark that more generally the identity (31) implies that for any $\varepsilon > 0$ there exist arbitrarily large parameters Q such that

$$|\psi_{n,1}(Q) + n\psi_{n,j}(Q)| < \varepsilon \quad \text{for} \quad 2 \leq j \leq n+1,$$

where Q can be chosen so that simultaneously $\psi_{n,1}(Q)$ is arbitrarily close to $\underline{\psi}_{n,1}$ and $\psi_{n,j}(Q)$ is arbitrarily close to $\overline{\psi}_{n,j}$ for $2 \leq j \leq n+1$. In particular, the identity (31) implies

$$\underline{\psi}_{n,1} = -n\overline{\psi}_{n,2} = -n\overline{\psi}_{n,3} = \cdots = -n\overline{\psi}_{n,n+1} \tag{33}$$

and that for any $\epsilon > 0$ and the (arbitrarily large) parameters Q as above the estimate

$$0 < \psi_{n,n+1}(Q) - \psi_{n,2}(Q) < \epsilon \tag{34}$$

is satisfied. Moreover, since $\psi_{n,1}(Q)$ is close to $\underline{\psi}_{n,1}$, we may assume that at such Q the function $\psi_{n,1}$ has a local minimum, or equivalently $L_{n,1}$ has a local minimum at $\log Q$ (otherwise we get a contradiction to the definition of $\underline{\psi}_1$ either for some $\tilde{Q} < Q$ or some $\tilde{Q} > Q$ dependent on whether $\psi_{n,1}$ rises in some interval $(Q-\delta, Q)$ or decays in some interval $(Q, Q+\delta)$). Let $\epsilon > 0$ and Q_1 be any fixed large value as above that in particular satisfies (34). Further let $q_1 = \log Q_1$. The estimate (34) can be written in terms of the functions $L_{n,.}$ as

$$0 < L_{n,n+1}(q_1) - L_{n,2}(q_1) < \epsilon \cdot q_1. \tag{35}$$

From (30) we know that $L_{n,1}(q_1)$ approximately equals $-\sum_{j=2}^{n+1} L_{n,j}(q_1)$ up to addition of some constant, that is

$$\left| L_{n,1}(q_1) + \sum_{j=2}^{n+1} L_{n,j}(q_1) \right| \le C.$$

Since all $L_{n,2}(q_1), \dots, L_{n,n+1}(q_1)$ are roughly equal by (35), we further deduce

$$\begin{aligned} |L_{n,1}(q_1) + nL_{n,2}(q_1)| &= \left| \left(L_{n,1}(q_1) + \sum_{j=2}^{n+1} L_{n,j}(q_1) \right) + \sum_{j=2}^{n+1} (L_{n,2}(q_1) - L_{n,j}(q_1)) \right| \\ &\le C + n\epsilon q_1, \end{aligned}$$

and hence in particular

$$L_{n,2}(q_1) \ge -\frac{L_{n,1}(q_1)}{n} - \epsilon q_1 - \tilde{C}, \tag{36}$$

where $\tilde{C} = C/n$ is another constant. Now let q_0 be the largest value smaller than q_1 at which the function $L_{n,1}(q)$ has a local maximum. Then by the assumption that q_1 is a local minimum of $L_{n,1}$ justified above, the function $L_{n,1}$ decays in the interval $[q_0, q_1]$ with slope -1 so that

$$L_{n,1}(q_1) - L_{n,1}(q_0) = q_0 - q_1. \tag{37}$$

On the other hand

$$L_{n,2}(q_1) - L_{n,2}(q_0) \le \frac{q_1 - q_0}{n}, \tag{38}$$

since the function $L_{n,2}(q)$ has slope at most $1/n$. Moreover, since any local maximum of $L_{n,1}(q)$ is a local minimum of $L_{n,2}(q)$, we have

$$L_{n,1}(q_0) = L_{n,2}(q_0).$$

Combining this with (37) and (38) yields

$$L_{n,2}(q_1) - L_{n,1}(q_1) \le \left(1 + \frac{1}{n}\right)(q_1 - q_0).$$

Together with (36) we obtain

$$L_{n,1}(q_1) \geq L_{n,2}(q_1) - \Big(1+\frac{1}{n}\Big)(q_1-q_0) \geq -\frac{L_{n,1}(q_1)}{n} - \epsilon q_1 - \tilde{C} - \Big(1+\frac{1}{n}\Big)(q_1-q_0),$$

which yields

$$L_{n,1}(q_1) \geq -\frac{n\epsilon}{n+1} q_1 - \tilde{C} - (q_1 - q_0).$$

Together with (37) we infer

$$L_{n,1}(q_0) \geq -\frac{n\epsilon}{n+1} q_1 - \tilde{C}.$$

Now the assumption $\lambda_n(\zeta) < \infty$ implies with (23) that $\underline{\psi}_{n,1} > -1$ and from this it is not hard to see that $q_1 \ll q_0$ for all q_0, q_1 as above with a constant depending only on $\lambda_n(\zeta)$ or equivalently $\underline{\psi}_{n,1}$. Hence, for $q_0 > 1$, we have

$$0 > L_{n,1}(q_0) \gg -\epsilon q_0.$$

Since by the transcendence of ζ the values q_0 induced from q_1 as above clearly tend to infinity as q_1 does, we infer $\overline{\psi}_{n,1} = 0$ as we may choose ϵ arbitrarily small. By (23) this is again equivalent to $\hat{\lambda}_n(\zeta) = 1/n$. The proof in case of equality in the right inequality is finished.

We only sketch the deduction of the dual result. Assume the identity (32) holds. The dual characterization $\underline{\psi}^*_{n,1} + n\overline{\psi}^*_{n,n+1} \geq 0$ from [Schmidt and Summerer 2009] for the related left inequality in (5) yields the dual characterization for the equality (32) for the same reasons. Proceeding as above yields very similarly as above $0 < \psi^*_{n,n+1}(Q) - \psi^*_{n,2}(Q) < \epsilon$ for large Q for which $\log Q$ are local minima of $L^*_{n,1}$ and such that $\psi^*_{n,1}(Q)$ is close to $\underline{\psi}^*_{n,1}$, dual to (34). For such Q we now look at the smallest local maximum of $L^*_{n,1}$ greater than $\log Q$. Since all $L^*_{n,j}$ have slope within $\{-1/n, 1\}$, the claim $\widehat{w}_n(\zeta) = n$ follows very similarly incorporating that any local maximum of $L^*_{n,1}$ is a local minimum of $L^*_{n,2}$ again. $\square$

Remark 3.2. We point out that the proof of Lemma 3.1 does not require that the point lies on the Veronese curve defined as $\{(t, t^2, \ldots, t^k) : t \in \mathbb{R}\}$. The only point where we used the special form of successive powers was for $\lambda_n(\zeta) = \infty$, and in this case more concise estimates show the claim as well. Hence the claim extends naturally to the analogue exponents assigned to $\underline{\zeta} \in \mathbb{R}^k$ whose coordinates are linearly independent together with $\{1\}$.

It will be convenient to utilize the following Lemma 3.3 for the proof of Theorem 2.1. Roughly speaking, it shows that multiplication of a polynomial P with a polynomial Q for which $|Q(\zeta)| \approx H(Q)^{-1}$ holds induces an increase of the corresponding function $L^*_{3,.}$ by $\frac{1}{3}$ in some interval. For fixed real ζ we will say a polynomial $P \in \mathbb{Z}[T]$ of degree at most 3 *induces a point* $(q, L^*_P(q))$ *in the*

3-*dimensional Schmidt–Summerer diagram* if $(q, L_P^*(q))$ is the local minimum of L_P^* implicitly defined via $H(P)$, $P(\zeta)$ by

$$L_P^*(q) = \log H(P) - \frac{q}{3} = \log|P(\zeta)| + q, \tag{39}$$

consistent with (28). Recall that any local minimum of some successive minimum function $L_{3,\cdot}^*$ is obtained as in (39) for some $P \in \mathbb{Z}[T]$.

Lemma 3.3. *Let $P, Q, R \in \mathbb{Z}[T]$ be of large heights and such that $R = PQ$ and R has degree at most three. Assume P induces the point $(q_1, L_P^*(q_1))$ and R induces the point $(q_2, L_R^*(q_2))$ in the 3-dimensional Schmidt–Summerer diagram. Further assume*

$$|Q(\zeta)| = H(Q)^{-1+\delta} \tag{40}$$

for δ of small absolute value, and that $(\log H(Q))^{-1} = O(\delta)$. Then

$$\frac{L_R^*(q_2) - L_P^*(q_1)}{q_2 - q_1} = \frac{1}{3} + O(\delta). \tag{41}$$

Proof. From (39) we calculate

$$q_1 = \tfrac{3}{4}\cdot(\log H(P) - \log|P(\zeta)|) \quad \text{and} \quad L_P^*(q_1) = \tfrac{3}{4}\cdot\log H(P) + \tfrac{1}{4}\cdot\log|P(\zeta)|.$$

Similarly, we infer

$$\begin{aligned} q_2 &= \tfrac{3}{4}\cdot(\log H(R) - \log|R(\zeta)|) \\ &= \tfrac{3}{4}\cdot(\log H(P) + \log H(Q) + \Delta - (\log|P(\zeta)| + \log|Q(\zeta)|)), \end{aligned}$$

and

$$L_R^*(q_2) = \tfrac{3}{4}\cdot(\log H(P) + \log H(Q) + \Delta) + \tfrac{1}{4}\cdot(\log|P(\zeta)| + \log|Q(\zeta)|),$$

where Δ is bounded by virtue of (42) below. Inserting yields

$$\frac{L_R^*(q_2) - L_P^*(q_1)}{q_2 - q_1} = \frac{\frac{3}{4}\log H(Q) + \frac{1}{4}\log|Q(\zeta)| + \frac{3}{4}\Delta}{\frac{3}{4}\log H(Q) - \frac{3}{4}\log|Q(\zeta)| + \frac{3}{4}\Delta},$$

and with the assumption (40) further

$$\frac{L_R^*(q_2) - L_P^*(q_1)}{q_2 - q_1} = \frac{\left(\frac{1}{2} + \frac{1}{4}\delta\right)\log H(Q) + \frac{3}{4}\Delta}{\left(\frac{3}{2} - \frac{3}{4}\delta\right)\log H(Q) + \frac{3}{4}\Delta}.$$

The claim follows by simply rearranging and assuming $(\log H(Q))^{-1} = O(\delta)$. $\square$

Conversely (41) implies that $\log|Q(\zeta)|/\log H(Q) + 1$ is small by a very similar argument, but we will not use this. Again the proposition did not use the fact that we deal with successive powers of a number, and can be generalized to any dimension.

4. Proofs of Theorems 2.1, 2.2 and 2.3

Apart from Theorem 1.1 and the concepts of the parametric geometry of numbers discussed on pages 490–492, we will use that for any polynomials Q_1, Q_2 with integral coefficients of degree bounded by n we have

$$H(Q_1Q_2) \asymp_n H(Q_1)H(Q_2). \tag{42}$$

See [Wirsing 1961, Hilfssatz 3]. As in our applications the dimensions n are fixed we can assume absolute constants in (42). We will sometimes implicitly use the consequence that if $Q = Q_1Q_2$ then $|Q(\zeta)| \leq H(Q)^{-z}$ implies that either $|Q_1(\zeta)| \ll H(Q_1)^{-z}$ or $|Q_2(\zeta)| \ll H(Q_2)^{-z}$ must be satisfied, which was essentially used by Wirsing [1961]. We start with the proof of Theorem 2.3 since it is the least technical one.

Proof of Theorem 2.3. We will prove that any extremal number ζ satisfies

$$w_{4,4}(\zeta) \geq \rho. \tag{43}$$

Assume we have already shown (43). Then the unconditional claim (18) follows from iterated use of results from parametric geometry of numbers. Indeed, from (43) applying (24) with $n = j = 4$ we first obtain

$$\underline{\psi}^*_{4,4} \leq \frac{2-\sqrt{5}}{4(3+\sqrt{5})}. \tag{44}$$

In view of (22) and (20) applied with $n = j = 4$, we obtain

$$\underline{\psi}_{4,1} = -\overline{\psi}^*_{4,5} \leq 4 \cdot \underline{\psi}^*_{4,4} \leq \frac{2-\sqrt{5}}{3+\sqrt{5}}. \tag{45}$$

Eventually computing the corresponding value of λ_4 by applying (23) with $n = 4$, $j = 1$ leads precisely to the lower bound $\gamma/2$ in the theorem.

We are left to prove (43). For this we use the characterization of the polynomials $P_k \in \mathbb{Z}[T]$ of degree 2 for $n = 2$ from Theorem 1.1. Consider for fixed large k three successive polynomials P_{k-2}, P_{k-1}, P_k. Then we know from Theorem 1.1 that

$$|P_j(\zeta)| \asymp H(P_j)^{-\rho} \quad \text{for} \quad j \in \{k-2, k-1, k\}. \tag{46}$$

Applied with $j = k$ it is obvious that the polynomials $R_k(T) = TP_k(T)$ and $S_k(T) = T^2P_k(T)$ have degrees 3 and 4, heights $H(P_k) = H(R_k) = H(S_k)$, and satisfy

$$|P_k(\zeta)| \asymp_\zeta |R_k(\zeta)| \asymp_\zeta |S_k(\zeta)| \asymp_\zeta H(P_k)^{-\rho}$$

as well. The polynomials P_k, R_k, S_k are obviously linearly independent and hence $w_{4,3}(\zeta) \geq \rho$. As the fourth polynomial T_k we take the product of P_{k-1} and P_{k-2}. First we show that $\{P_k, R_k, S_k, T_k\}$ are linearly independent. Otherwise

$T_k = P_{k-1}P_{k-2}$ would lie in the 3-dimensional space spanned by P_k, R_k, S_k, which by the special form of R_k, S_k means $T_k = P_k Z$ for some polynomial $Z(T) \in \mathbb{Q}[T]$ of degree 2. However we know from Theorem 1.1 that the best approximating polynomials P_j are irreducible over $\mathbb{Z}[T]$ for all large j. Hence by the unique factorization in $\mathbb{Z}[T]$ the polynomial P_k must equal (up to sign) either P_{k-1} or P_{k-2}, which is clearly false, so we have a contradiction.

Moreover from (46) and the characterization in Theorem 1.1 it is known that $H(P_{k-2})^{\nu^2} \asymp H(P_{k-1})^{\nu} \asymp H(P_k)$. Since $\nu^{-1}+\nu^{-2}=1$ and $H(T_k) \asymp H(P_{k-1})H(P_{k-2})$ by (42), we deduce $H(T_k) \asymp H(P_k)$. Together with property (46) for $j = k-1$ and $j = k-2$ we infer

$$|T_k(\zeta)| = |P_{k-1}(\zeta)P_{k-2}(\zeta)| \asymp_\zeta H(P_{k-1})^{-\rho}H(P_{k-2})^{-\rho} \asymp H(P_{k-1}P_{k-2})^{-\rho} \asymp H(P_k)^{-\rho}.$$

Summing up, we have found four linearly independent polynomials P_k, R_k, S_k, T_k with the properties

$$H(P_k) \asymp H(R_k) \asymp H(S_k) \asymp H(T_k)$$

and

$$|P_k(\zeta)| \asymp_\zeta |R_k(\zeta)| \asymp_\zeta |S_k(\zeta)| \asymp_\zeta |T_k(\zeta)| \asymp_\zeta H(P_k)^{-\rho}.$$

Since this holds for any large k we have established (43).

Finally we show the conditioned results. The equality $\lambda_4(\zeta) = \gamma/2$ follows immediately from Khintchine's inequalities (5) since the upper bound for $\lambda_4(\zeta)$ that arises from $n = 4$, $w_4(\zeta) = \rho$, coincides with the lower bound $\gamma/2$ established above (the argument essentially used the characterization (33), (34) for equality (31) from [Schmidt and Summerer 2009] used in the proof of Lemma 3.1). Finally (19) follows from Lemma 3.1 since we have just shown that $w_4(\zeta) = \rho$ implies the identity (31) for any extremal number ζ and $n = 4$. □

Remark 4.1. It was essentially shown in the proof of [Bugeaud 2010, Theorem 2] that the condition

$$w_1(\zeta) = w_2(\zeta) = \cdots = w_n(\zeta) \tag{47}$$

implies (31). If the hypothesis $w_4(\zeta) = \rho$ of Theorem 2.3 holds then its assertion and (7) show that extremal numbers provide counterexamples for the reverse implication for $n = 4$. In this context note that if $\lambda_n(\zeta) > 1$ the claims (47) and (31) are indeed equivalent by [Schleischitz 2016, Theorem 5.4]. Note also that from Lemma 3.1 and the above implication we could deduce that (47) implies $\hat{\lambda}_n(\zeta) = 1/n$ and $\widehat{w}_n(\zeta) = n$. However, the weaker condition $w_1(\zeta) \geq n$ already implies $\hat{\lambda}_n(\zeta) = 1/n$ and $\widehat{w}_n(\zeta) = n$ as established in [Schleischitz 2016, Theorem 5.1].

The proof of Theorem 2.3 in fact provides upper bounds for the frequency of good simultaneous rational approximations to $(\zeta, \zeta^2, \zeta^3, \zeta^4)$. More precisely the proof shows that there exists a sequence $(x_k)_{k\geq 1}$ of positive integers that satisfy

$$x_{k+1} \ll x_k^{\nu} \quad \text{and} \quad \max_{1\leq j\leq 4} \|x_k\zeta^j\| \ll x_k^{-\gamma/2}.$$

In case of the conjectured equality in (18) we even have

$$x_{k+1} \asymp x_k^{\nu} \quad \text{and} \quad \max_{1\leq j\leq 4} \|x_k\zeta^j\| \asymp x_k^{-\gamma/2}. \tag{48}$$

Here as usual $\|\cdot\|$ denotes the distance to the nearest integer. We briefly sketch how to deduce these facts from the proof above. The polynomials P_k, R_k, S_k, T_k in the proof which induce the bound for the value $\underline{\psi}^*_{4,4}$ in (44) appear with frequency $H(P_{k+1}) \asymp H(P_k)^{\nu}$ (and very similarly for R_k, S_k, T_k). The last minimum $\psi^*_{4,5}(Q)$ at the corresponding positions Q in the Schmidt–Summerer diagram is asymptotically bounded below as in (45) and the corresponding polynomials appear with the same logarithmic asymptotic height frequency ν. We now flip the diagram along the horizontal axis according to (21) to obtain (roughly) the dual problem of simultaneous approximation. Thereby with simple geometric considerations involving (27) and reinterpreting to classical exponents $\lambda_{4,\cdot}$ we see that the first coordinates of best approximations related to the bound for $\underline{\psi}_{4,1}$ in (45) appear with frequency $x_{k+1} \ll x_k^{\nu}$ as well (with a technical proof it possible to show that a single x_k cannot induce the good approximations for two consecutive values of Q obtained this way). In case of equality in (18) the functions $\psi_{4,1}(Q)$ must have a local minimum at such places Q and (48) follows. It is tempting to further conjecture that for the corresponding approximation vectors $(x_k, y_{k,1}, \ldots, y_{k,4})_{k\geq 1}$, where x_k is as in (48) and $y_{k,j}$ is the closest integer to $\zeta^j x_k$, similar general recursive patterns as for $n = 2$ noticed in [Roy 2004a] exist. However, we do not further investigate this topic here.

We turn to the case $n = 3$. For a real number ζ we define the sequence of 1-dimensional *best approximation polynomials* $(E_l)_{l\geq 1}$ attached to ζ. They are given by linear polynomials $E_l(T) = a_l T + b_l$ with $a_l, b_l \in \mathbb{Z}$ defined by $E_1(T) = T - \lfloor\zeta\rfloor$ and E_{l+1} is recursively defined via E_l as the linear polynomial of least height for which $0 < |E_{l+1}(\zeta)| < |E_l(\zeta)|$. These polynomials obviously satisfy $H(E_1) < H(E_2) < \cdots$ and

$$E_l(\zeta) = \min\{|Q(\zeta)| : Q \in \mathbb{Z}[T], \quad \deg(Q) = 1, \quad 1 \leq H(Q) \leq H(E_l)\}.$$

It follows from the theory of continued fractions that the rational numbers b_l/a_l are precisely the convergents to ζ. Moreover by Dirichlet's theorem the best approximating polynomials satisfy

$$|E_l(\zeta)| \ll_\zeta H(E_l)^{-1} \quad \text{for} \quad l \geq 1. \tag{49}$$

Furthermore it is well known and follows from elementary results on the theory of continued fractions that $|E_l(\zeta)| \asymp_\zeta H(E_{l+1})^{-1}$ for all irrational ζ, which readily implies

$$1 \leq \liminf_{l\to\infty} \frac{\log H(E_{l+1})}{\log H(E_l)} \leq \limsup_{l\to\infty} \frac{\log H(E_{l+1})}{\log H(E_l)} = \lambda_1(\zeta). \tag{50}$$

In view of the rather technical proof of (11), for the convenience of the reader we give a brief outline of some facts we will show in the course of the proof. We will establish a rather precise description of the functions $L_{3,1}^*(q), \dots, L_{3,4}^*(q)$ on $q \in (0, \infty)$ induced by an extremal number, its square and its cube. Denote by $|I|$ the length of an interval I. We will show there exists a partition of the positive real numbers in successive intervals $I_1, J_1, I_2, J_2, \dots$ with the following properties:

- $\lim_{k\to\infty} |I_k|/|J_k| = 1$.
- $\lim_{k\to\infty} |I_{k+1}|/|I_k| = \lim_{k\to\infty} |J_{k+1}|/|J_k| = \nu$.
- At the beginning of every I_k all $L_{3,i}^*(q)$ are all small (more precisely $o(q)$ as $q \to \infty$) by absolute value. Then in I_k the functions $L_{3,1}^*(q)$, $L_{3,2}^*(q)$ basically decay with slope $-1/3$, whereas $L_{3,3}^*(q)$, $L_{3,4}^*(q)$ basically rise with slope $\frac{1}{3}$ in any not too short subinterval of I_k (clearly not in too short intervals, since the $L_{3,.}^*$ have slope within $\left\{-\frac{1}{3}, 1\right\}$).
- At the end of I_k and beginning of J_k the opposite behavior appears; that is, $L_{3,1}^*(q)$, $L_{3,2}^*(q)$ basically rise with slope $\frac{1}{3}$ on any not too short subinterval of J_k, whereas $L_{3,3}^*(q)$, $L_{3,4}^*(q)$ basically decay with slope $-\frac{1}{3}$ until the functions $L_{3,1}^*, \dots, L_{3,4}^*$ asymptotically meet again at the end of J_k which is the beginning of I_{k+1}.
- The functions $|L_{3,1}^*(q) - L_{3,2}^*(q)|$ such as $|L_{3,3}^*(q) - L_{3,4}^*(q)|$ are bounded uniformly in q.

All above is basically true for the simultaneous approximation functions $L_{3,j}(q)$ as well by (22). Observe that by the last point above in particular

$$w_{3,1}(\zeta) = w_{3,2}(\zeta), \quad w_{3,3}(\zeta) = w_{3,4}(\zeta), \quad \widehat{w}_{3,1}(\zeta) = \widehat{w}_{3,2}(\zeta), \quad \widehat{w}_{3,3}(\zeta) = \widehat{w}_{3,4}(\zeta), \tag{51}$$

$$\lambda_{3,1}(\zeta) = \lambda_{3,2}(\zeta), \quad \lambda_{3,3}(\zeta) = \lambda_{3,4}(\zeta), \quad \hat{\lambda}_{3,1}(\zeta) = \hat{\lambda}_{3,2}(\zeta), \quad \hat{\lambda}_{3,3}(\zeta) = \hat{\lambda}_{3,4}(\zeta), \tag{52}$$

which extends the claim of Theorem 2.1. See also Remark 4.2 below. We point out that roughly speaking the decay phases of $L_{3,.}^*$ are induced by the polynomials P_k from Theorem 1.1. The rising phases are induced by products $P_k E_l$ for fixed P_k and suitable successive best approximating polynomials E_l defined above, which indeed lead to asymptotic increase by $\frac{1}{3}$ as stated in the description above, basically in view of Lemma 3.3.

Proof of Theorem 2.1. First we prove (12). We show that

$$w_{3,4}(\zeta) \geq 3. \tag{53}$$

Provided this is true it follows immediately that $w_{4,3}(\zeta) = \widehat{w}_3(\zeta) = 3$, since $w_{3,4}(\zeta) = \hat{\lambda}_3(\zeta)^{-1} \leq 3$ by (25) and (2). This argument in fact utilizes parametric geometry of numbers. Actually it is well known and follows, for example, from (6) that both claims in (12) are equivalent.

For (12) it remains to be shown that (53) holds. Let k be fixed large and consider the polynomials $P_k, P_{k+1}, \ldots$ from Theorem 1.1, and let $R_j(T) = TP_j(T)$ for $j \geq k$. Further let $X = H(P_{k+1})$. Then obviously $P_{k+1}(T)$ and $R_{k+1}(T) = TP_{k+1}$ satisfy

$$H(P_{k+1}) = H(R_{k+1}) = X \quad \text{and} \quad |P_{k+1}(\zeta)| \asymp_\zeta |R_{k+1}(\zeta)| \asymp_\zeta X^{-\rho} < X^{-3}. \tag{54}$$

Let $\epsilon > 0$. We shall construct polynomial multiples

$$Q_{k,1} = R_{k,1} \cdot P_k \quad \text{and} \quad Q_{k,2} = R_{k,2} \cdot P_k \tag{55}$$

of P_k with $R_{k,i} \in \mathbb{Z}[T]$ polynomials of degree one such that $\{R_{k,1}, R_{k,2}\}$ and hence also $\{Q_{k,1}, Q_{k,2}\}$ are linearly independent and satisfy

$$H(Q_{k,i}) \ll X \quad \text{and} \quad |Q_{k,i}(\zeta)| \ll X^{-3+\epsilon} \quad \text{for} \quad i \in \{1,2\}. \tag{56}$$

One readily verifies that $\{Q_{k,1}, Q_{k,2}\}$ span the same space as $\{P_k, TP_k\}$ regardless of which linear polynomials $R_{k,i}$ we choose. Observe that the space spanned by $\{P_{k+1}, R_{k+1}, Q_{k,1}, Q_{k,2}\}$ consequently has dimension 4. Indeed, otherwise the polynomial identity $P_k(T)Y_1(T) = P_{k+1}(T)Y_2(T)$ would have linear integer polynomial solutions Y_1, Y_2, which is a contradiction since P_k, P_{k+1} have degree two and are irreducible and not proportional and $\mathbb{Z}[T]$ has unique factorization. Hence from (54) and (56) indeed the claim (53) follows by considering $\{P_{k+1}, R_{k+1}, Q_{k,1}, Q_{k,2}\}$ as ϵ can be chosen arbitrarily small. To finally prove (56), for the given $X = H(P_{k+1})$ we let $R_{k,1} = E_l$ and $R_{k,2} = E_{l+1}$ be two successive best approximating polynomials in dimension $n = 1$ as introduced before the proof with l chosen largest possible such that still $H(R_{k,i})H(P_k) \leq X$ for $i \in \{1,2\}$. It follows from (42) and (55) that

$$H(Q_{k,i}) \ll X \quad \text{for} \quad i \in \{1,2\}. \tag{57}$$

On the other hand, since extremal numbers satisfy $\lambda_1(\zeta) = 1$ as mentioned in (7), by (50) the sequence $(E_l(T))_{l \geq 1}$ of best approximating polynomials in dimension 1 satisfies

$$\lim_{l\to\infty} \frac{\log H(E_{l+1})}{\log H(E_l)} = 1 \quad \text{and} \quad \lim_{l\to\infty} -\frac{\log |E_l(\zeta)|}{H(E_l)} = 1. \tag{58}$$

Since $R_{k,1} = E_l,\ R_{k,2} = E_{l+1}$ and by our maximal choice of l, it is easy to see that

$$H(Q_{k,i}) \geq X^{1-\epsilon} \quad \text{for} \quad i \in \{1,2\}.$$

It further follows from (42) and the fact that $H(P_{k+1}) \asymp H(P_k)^{\nu}$, or equivalently $H(P_k) \asymp H(P_{k+1})^{\gamma}$ in view of Theorem 1.1, that we have

$$H(R_{k,i}) \gg H(Q_{k,i})H(P_k)^{-1} \gg X^{1-\epsilon}H(P_k)^{-1} \gg X^{1-\gamma-\epsilon} \quad \text{for} \quad i \in \{1,2\}.$$

Together with (49) this leads to

$$|R_{k,i}(\zeta)| \ll_{\zeta} X^{-1+\gamma+\epsilon} \quad \text{for} \quad i \in \{1,2\}.$$

Hence

$$|Q_{k,i}(\zeta)| = |P_k(\zeta)| \cdot |R_{k,i}(\zeta)| \ll_{\zeta} X^{-\rho\gamma} \cdot X^{-1+\gamma+\epsilon} = X^{-3+\epsilon} \quad \text{for} \quad i \in \{1,2\},$$

where we used $\rho\gamma + 1 - \gamma = 3$, which can be readily checked. Thus recalling (57) we have proved (56) and hence together with (54) finally (12).

Now we prove the more technical identities (11). In the proof of (12) above we have shown that for any large k, with $X = H(P_{k+1})$ we have four linearly independent polynomials $\{T_1, \dots, T_4\} = \{P_{k+1}, R_{k+1}, Q_{k,1}, Q_{k,2}\}$ with $H(T_i) \ll X$ and $|T_i(\zeta)| \le X^{-3+\epsilon}$. Following the proof of (24), this means that for arbitrarily small $\varepsilon > 0$, any large k induces $q_k > 0$ such that all

$$|L_{3,i}^*(q_k)| \le \varepsilon q_k \quad \text{for} \quad 1 \le i \le 4, \tag{59}$$

where $\lim_{k\to\infty} q_k / \log H(P_{k+1}) = 3$ in view of (28). Since by Theorem 1.1 any polynomial P_{k+1} induces an approximation of quality

$$-\frac{\log |P_{k+1}(\zeta)|}{\log H(P_{k+1})} = \rho + o(1) > 3, \qquad k \to \infty,$$

and so does $R_{k+1}(T) = TP_{k+1}(T)$, it follows that $L_{3,1}^*$ and $L_{3,2}^*$ decay with asymptotic slope $-1/3$ in some interval (q_k, b_k) and (q_k, c_k) respectively, for b_k and c_k local minima of $L_{3,1}^*$ and $L_{3,2}^*$ respectively. More precisely, the local minima $(d_k, L_{P_{k+1}}^*(d_k))$ and $(e_k, L_{R_{k+1}}^*(e_k))$ of the functions $L_{P_{k+1}}^*$ and $L_{R_{k+1}}^*$ as in (28), almost coincide with local minima $(b_k, L_{3,1}^*(b_k))$ and $(c_k, L_{3,2}^*(c_k))$, respectively. By this more precisely we mean that all differences

$$|b_k - d_k|, \quad |b_k - e_k|, \quad |c_k - d_k|, \quad |c_k - e_k|$$

as well as the corresponding differences of the L^* evaluations

$$\begin{aligned} &|L_{3,1}^*(b_k) - L_{P_{k+1}}^*(d_k)|, \qquad &|L_{3,1}^*(b_k) - L_{R_{k+1}}^*(e_k)|, \\ &|L_{3,2}^*(c_k) - L_{P_{k+1}}^*(d_k)|, \qquad &|L_{3,2}^*(c_k) - L_{R_{k+1}}^*(e_k)|, \end{aligned}$$

at these points are bounded by a fixed constant for all k. Very similarly it is obvious from the fact that $P_{k+1}(\zeta)$ and $R_{k+1}(\zeta)$ differ only by the factor ζ that b_k and c_k are asymptotically equal, by which we mean their ratio b_k/c_k tends to one (in fact their difference $|b_k - c_k|$ is again bounded) as $k \to \infty$. Hence with the parametric formula (29) for the parameter $w_3^{(1)} = w_3^{(2)} = \rho$, with

$$Q_k := e^{b_k} \quad \text{for} \quad k \geq 1,$$

(not to confuse with the polynomials $Q_{k,i}$) we calculate

$$\lim_{k\to\infty} \psi_{3,1}^*(Q_k) = \lim_{k\to\infty} \psi_{3,2}^*(Q_k) = \frac{1-\sqrt{5}}{3(3+\sqrt{5})}. \tag{60}$$

Since $L_{3,1}^*$ and $L_{3,2}^*$ both decay with asymptotic slope $-\frac{1}{3}$ in intervals $I_k := (q_k, b_k)$, that is,

$$L_{3,1}^*(b_k) - L_{3,1}^*(q_k) = (b_k - q_k)\left(-\tfrac{1}{3} + \varepsilon\right) \quad \text{and} \quad L_{3,2}^*(b_k) - L_{3,2}^*(q_k) = (b_k - q_k)\left(-\tfrac{1}{3} + \varepsilon\right),$$

it follows from (30) that the sum $L_{3,3}^* + L_{3,4}^*$ asymptotically increases with constant slope $\frac{2}{3}$ in I_k, that is,

$$L_{3,3}^*(b_k) + L_{3,4}^*(b_k) - L_{3,3}^*(q_k) - L_{3,4}^*(q_k) = (b_k - q_k)\left(\tfrac{2}{3} + \varepsilon\right).$$

Consequently, if we can show that both $L_{3,3}^*$ and $L_{3,4}^*$ increase at most by $\frac{1}{3}$ in any large subinterval of I_k, that is, for any $q_k \leq a < b \leq b_k$, we have

$$L_{3,3}^*(b) - L_{3,3}^*(a) \leq (b-a)\left(\tfrac{1}{3} + \varepsilon\right) \quad \text{and} \quad L_{3,4}^*(b) - L_{3,4}^*(a) \leq (b-a)\left(\tfrac{1}{3} + \varepsilon\right), \tag{61}$$

then both must have asymptotically constant increase by precisely $\frac{1}{3}$ in the entire interval I_k, i.e., equality in (61). We more precisely show the following claims:

Claim A: For any parameter $\tilde{X} \in (H(P_k), \infty)$, let

$$U_{k,\tilde{X}} = P_k \cdot E_t \quad \text{and} \quad V_{k,\tilde{X}} = P_k \cdot E_{t+1},$$

with $t = t(k, \tilde{X})$ chosen as the largest integer such that $\max\{H(U_{k,\tilde{X}}), H(V_{k,\tilde{X}})\} \leq \tilde{X}$. Then the functions $L_{3,\cdot}^*(q)$ arising from the succession (equals the pointwise minimum) of the $L_{U_{k,\tilde{X}}}^*$, $L_{V_{k,\tilde{X}}}^*$ as $\tilde{X}$ runs through $(H(P_k), \infty)$ via (28) have asymptotically constant slope $\frac{1}{3}$ in (b_{k-1}, ∞). By this more precisely we mean that for any $b_{k-1} \leq \tilde{X} < \tilde{Y}$ if $(a, L_{U_{k,\tilde{X}}}(a))$ or $(a, L_{V_{k,\tilde{X}}}(a))$ lies in the graph of $L_{U_{k,\tilde{X}}}$ or $L_{V_{k,\tilde{X}}}$, respectively, and similarly for $(b, L_{U_{k,\tilde{X}}}(b))$ or $(b, L_{V_{k,\tilde{X}}}(b))$, then we have

$$L_{U_{k,\tilde{Y}}}^*(b) - L_{U_{k,\tilde{X}}}^*(a) = (b-a)\left(\tfrac{1}{3} + \varepsilon\right) \quad \text{and} \quad L_{V_{k,\tilde{Y}}}^*(b) - L_{V_{k,\tilde{X}}}^*(a) = (b-a)\left(\tfrac{1}{3} + \varepsilon\right).$$

Claim B: Moreover if we restrict to $\tilde{X} \in (H(P_{k+1}), H(P_{k+2}))$, then the functions $L_{U_{k,\tilde{X}}}^*$ and $L_{V_{k,\tilde{X}}}^*$ induce $L_{3,3}^*$ and $L_{3,4}^*$ on I_k, respectively (remark: as we will see later on they induce $L_{3,1}^*$ and $L_{3,2}^*$ in intervals (b_{k-1}, q_k) if we let $\tilde{X} \in (H(P_k), H(P_{k+1}))$).

First recall that at the beginning q_k of the interval I_k the successive minima are induced basically by $\{P_k, TP_k, P_{k+1}, TP_{k+1}\}$. Claim A follows basically directly from Lemma 3.3, where E_t and E_{t+1} respectively play the role of Q and P_k the role of P. Note also that δ from Lemma 3.3 tends to 0 in our context in view of (58), which also implies that the minima (in fact the entire functions) of consecutive functions of the form $L^*_{U_{k,\tilde{X}}}$ or $L^*_{V_{k,\tilde{X}}}$ do not differ much. Finally, it should be pointed out that the condition that $1/\log H(Q) = O(\delta)$ does not cause problems since for any fixed $\delta > 0$ and smaller heights $H(Q)$ only minor changes of the function $L^*_{3,\cdot}(q)$ can appear in intervals $(b_{k-1}, b_{k-1} + O(1))$, so that the global behavior of the function is not affected. For Claim B further observe that $\{U_{k,\tilde{X}}, V_{k,\tilde{X}}\}$ span the same space as $\{P_k, TP_k\}$ for all $\tilde{X} \in (H(P_k), \infty)$, and we have already noticed that polynomials in the space $\{P_{k+1}, TP_{k+1}\}$ induce the first two successive minima in I_k and $\{P_k, TP_k, P_{k+1}, TP_{k+1}\}$ are linearly independent. Hence $L^*_{3,3}$ and $L^*_{3,4}$ are bounded above by $L^*_{U_{k,\tilde{X}}}$ and $L^*_{V_{k,\tilde{X}}}$ in I_k respectively, and thus each increase at most by $\frac{1}{3}$. As noticed above, we may conclude $L^*_{3,3}$ and $L^*_{3,4}$ must actually coincide with the functions induced by $L^*_{U_{k,\tilde{X}}}$ and $L^*_{V_{k,\tilde{X}}}$, respectively.

Thus together with (60) we have proved

$$\begin{aligned} \lim_{k\to\infty} \psi^*_{3,1}(Q_k) &= \lim_{k\to\infty} \psi^*_{3,2}(Q_k) \\ &= \lim_{k\to\infty} -\psi^*_{3,3}(Q_k) = \lim_{k\to\infty} -\psi^*_{3,4}(Q_k) = \frac{1-\sqrt{5}}{3(3+\sqrt{5})}. \end{aligned} \tag{62}$$

We show next that in the interval $J_k := (b_k, q_{k+1})$ the functions $L^*_{3,1}, L^*_{3,2}$ have slope $-\frac{1}{3}$ whereas the functions $L^*_{3,3}, L^*_{3,4}$ have (asymptotic) slope $\frac{1}{3}$ until they all meet (asymptotically) at q_{k+1}. More precisely

$$L^*_{3,1}(q_{k+1}) - L^*_{3,1}(b_k) = (q_{k+1} - b_k)\left(-\tfrac{1}{3} + \varepsilon\right),$$
$$L^*_{3,2}(q_{k+1}) - L^*_{3,2}(b_k) = (q_{k+1} - b_k)\left(-\tfrac{1}{3} + \varepsilon\right)$$

whereas

$$L^*_{3,3}(q_{k+1}) - L^*_{3,3}(b_k) = (q_{k+1} - b_k)\left(\tfrac{1}{3} + \varepsilon\right),$$
$$L^*_{3,4}(q_{k+1}) - L^*_{3,4}(b_k) = (q_{k+1} - b_k)\left(\tfrac{1}{3} + \varepsilon\right)$$

and

$$L^*_{3,4}(q_{k+1}) - L^*_{3,1}(q_{k+1}) \le \varepsilon q_{k+1}.$$

Again by (59) with index shift k to $k+1$ we know that for arbitrarily small ε and all large $k \ge k_0(\varepsilon)$ we indeed have

$$|L^*_{3,i}(q_{k+1})| \le \varepsilon q_{k+1} \quad \text{for} \quad 1 \le i \le 4. \tag{63}$$

Since we have shown that $L^*_{3,1}$ and $L^*_{3,2}$ decay in I_k with slope $-\frac{1}{3}$ and (62) holds it suffices to show that J_k has asymptotically equal length to I_k, that is, $\lim_{k\to\infty} |J_k|/|I_k| = 1$, to conclude that $L^*_{3,3}$ and $L^*_{3,4}$ must decay with the minimum possible slope $-\frac{1}{3}$ in the entire interval J_k and more precisely

(64)
$$\lim_{k\to\infty} -\frac{L^*_{3,1}(b_k)}{q_{k+1}-b_k} = \lim_{k\to\infty} -\frac{L^*_{3,2}(b_k)}{q_{k+1}-b_k} = \lim_{k\to\infty} \frac{L^*_{3,3}(b_k)}{q_{k+1}-b_k} = \lim_{k\to\infty} \frac{L^*_{3,4}(b_k)}{q_{k+1}-b_k} = \frac{1}{3}.$$

We show the claim that I_k and J_k have asymptotically equal length, that is, $|I_k|/|J_k| = 1+o(1)$ as $k\to\infty$. By construction this is equivalent to b_k being asymptotically equal to $(q_k+q_{k+1})/2$, that is, $b_k = (q_k+q_{k+1})/2+o(q_k)$. Since $\lim_{k\to\infty} \log H(P_{k+1})/\log H(P_k) = \nu$ and $L^*_{3,\cdot}(q_k) = o(q_k)$ and $L^*_{3,\cdot}(q_{k+1}) = o(q_{k+1})$ as $k\to\infty$. Further notice that $L^*_{3,1}, L^*_{3,2}$ decay in (q_k, b_k) induced by P_{k+1}, TP_{k+1} and thus by (28) we have $L^*_{3,j}(q_i) = \log H(P_{i+1}) - q_i/3 + O(1)$ for $1\le j\le 4$ and all $i\ge 1$. Putting all this together leads to

$$\lim_{k\to\infty} \frac{q_{k+1}}{q_k} = \nu. \tag{65}$$

Thus the claimed asymptotic relation $b_k = (q_k+q_{k+1})/2+o(q_k)$ is equivalent to $b_k = q_k\cdot(1+\nu)/2+o(q_k)$. We know that at $Q_k = e^{b_k}$. We have asymptotically

$$\psi^*_{3,3}(Q_k) = \frac{b_k-q_k}{3} + o(q_k), \qquad k\to\infty, \tag{66}$$

since $L^*_{3,3}$ and $L^*_{3,3}$ are small at q_k by (59) and rise with slope $\frac{1}{3}$ in I_k. We remark that the asymptotic (66) holds for $\psi^*_{3,4}(Q_k)$ as well. On the other hand (62) provides an asymptotic formula for $\psi^*_{3,3}(Q_k)$ and $\psi^*_{3,4}(Q_k)$. It follows directly from the definition of $L^*_{3,j}$ via $\psi^*_{3,j}$ in (26) that $\psi^*_{3,3}(Q_k)$ is the slope from the origin to $(b_k, L^*_{3,3}(b_k))$ of $L^*_{3,3}$ in the Schmidt–Summerer diagram (and similarly for $L^*_{3,4}$). Hence asymptotically

$$\psi^*_{3,3}(Q_k) = \psi^*_{3,4}(Q_k) = \frac{\sqrt{5}-1}{3(3+\sqrt{5})} b_k + o(b_k), \qquad k\to\infty. \tag{67}$$

Again the asymptotic (67) holds for $\psi^*_{3,4}(Q_k)$ as well. Comparing the two expressions for $\psi^*_{3,3}(Q_k)$ in (66) and (67), with a short computation, indeed we verify $b_k = q_k\cdot(1+\nu)/2+o(q_k)$, so we have proved that I_k and J_k have asymptotically equal length.

Since consequently $L^*_{3,3}$ and $L^*_{3,4}$ both asymptotically decay with slope $-\frac{1}{3}$ in J_k, from (30) again we deduce that the sum $L^*_{3,1}+L^*_{3,2}$ must asymptotically increase by $\frac{2}{3}$ in J_k. Now recall in Claim A we showed that $L^*_{U_{k,\tilde{X}}}, L^*_{V_{k,\tilde{X}}}$ asymptotically induce an increase with slope at most $\frac{1}{3}$ in the entire interval (b_{k-1},∞) if we let $\tilde{X}$ run through $(H(P_k),\infty)$. Hence if we restrict to $\tilde{X}\in(H(P_k), H(P_{k+1}))$, by a very similar argument as in Claim B, in the interval (b_{k-1}, q_k) they induce $L^*_{3,1}$ and $L^*_{3,2}$ such that they both asymptotically increase precisely with slope $\frac{1}{3}$. By index shift the analogous claim is clearly also true for $(b_k, q_{k+1}) = J_k$. Hence indeed both $L^*_{3,1}$ and $L^*_{3,2}$ must asymptotically increase with slope precisely $\frac{1}{3}$ in the entire interval J_k.

Observe that the end of J_k is the beginning of I_{k+1}, so that we have basically established a complete description of all functions $L_{3,1}^*, \dots, L_{3,4}^*$ on $(0,\infty)$. The characterizations of the graphs of $L_{3,i}^*(q)$ established above show that asymptotically at the values $q = b_k$ both the smallest local minima of $L_{3,1}^*(q)$, $L_{3,2}^*(q)$ (in the sense of minimal values of $\psi_{3,1}^*(Q)$, $\psi_{3,2}^*(Q)$) and the largest local maxima of $L_{3,3}^*(q)$, $L_{3,4}^*(q)$ (in the sense of maximal values of $\psi_{3,3}^*(Q)$, $\psi_{3,4}^*(Q)$) are attained. Moreover both $|L_{3,1}^*(b_k) - L_{3,2}^*(b_k)|$ and $|L_{3,3}^*(b_k) - L_{3,4}^*(b_k)|$ are bounded uniformly in k, in fact more generally $|L_{3,1}^*(q) - L_{3,2}^*(q)|$ and $|L_{3,3}^*(q) - L_{3,4}^*(q)|$ are uniformly bounded for $q \in (0,\infty)$. Thus with (62) we have

$$\underline{\psi}_{3,1}^* = \underline{\psi}_{3,2}^* = \frac{1-\sqrt{5}}{3(3+\sqrt{5})} \quad \text{and} \quad \overline{\psi}_{3,3}^* = \overline{\psi}_{3,4}^* = \frac{\sqrt{5}-1}{3(3+\sqrt{5})}.$$

With (22), (23) and (24) we derive

$$w_3(\zeta) = w_{3,2}(\zeta) = \rho \quad \text{and} \quad \lambda_3(\zeta) = \lambda_{3,2}(\zeta) = \frac{1}{\sqrt{5}}. \tag{68}$$

This contains in particular the claims in (11). □

Remark 4.2. We can also determine the remaining constants $w_{3,i}, \lambda_{3,i}, \widehat{w}_{3,i}, \hat{\lambda}_{3,i}$ for extremal numbers. From (25) and (68) we deduce

$$\widehat{w}_{3,3}(\zeta) = \widehat{w}_{3,4}(\zeta) = \sqrt{5}, \qquad \hat{\lambda}_{3,3}(\zeta) = \hat{\lambda}_{3,4}(\zeta) = \frac{1}{\rho}. \tag{69}$$

Moreover the above characterizations of the functions $L_{3,i}^*$ imply

$$\overline{\psi}_{3,1}^* = \overline{\psi}_{3,2}^* = \underline{\psi}_{3,1}^* = \underline{\psi}_{3,2}^* = 0.$$

With (24) and (25) this is equivalent to

$$\begin{aligned} w_{3,3}(\zeta) = w_{3,4}(\zeta) = \widehat{w}_3(\zeta) = \widehat{w}_{3,2}(\zeta) = 3, \\ \lambda_{3,3}(\zeta) = \lambda_{3,4}(\zeta) = \hat{\lambda}_3(\zeta) = \hat{\lambda}_{3,2}(\zeta) = \tfrac{1}{3}. \end{aligned} \tag{70}$$

The description of the combined graph of the functions $L_{3,j}^*(q)$ and the information on the structure of the polynomials inducing them from the proof of Theorem 2.1 allows one to estimate the approximation to an extremal number by algebraic numbers of degree precisely three.

Proof of Theorem 2.2. It follows from the proof of Theorem 2.1 and the description above that the first two successive minima functions of the linear form problem related to $\psi_{3,1}^*$, $\psi_{3,2}^*$ are induced by polynomial multiples of P_k from Theorem 1.1, and for each k these multiples span the same space as $\{P_k, TP_k\}$. Since P_k have degree two, there is no irreducible polynomial of degree three which lies in the space spanned by $\{P_k, TP_k\}$ for some k. Thus the optimal exponent in (13) is not larger than $w_{3,3}(\zeta)$. On the other hand it was shown in the proof of Theorem 2.1

that $w_{3,3}(\zeta) = 3$, see (70). Thus, combining these facts, we see that indeed (13) has only finitely many solutions in $Q \in \mathbb{Z}[T]$ an irreducible polynomial of degree precisely three. From (13) we infer (14) by a standard argument. Indeed if R is the minimal polynomial of some α then $|R(\zeta)| = |R(\zeta) - R(\alpha)| = |\zeta - \alpha| \cdot R'(z)$ for some z between α and ζ by the intermediate theorem of differentiation. On the other hand $|R'(z)| \ll H(R)$ for bounded z is easy to see, and the claim (14) follows from (13).

Next we show (16) and (17). By essentially the argument from the proof of (13) again $\widehat{w}_{3,3}(\zeta)$ is an upper bound for the exponent in (16) for some large X. On the other hand we have noticed in (69) that $\widehat{w}_{3,3}(\zeta) = \widehat{w}_{3,4}(\zeta) = \sqrt{5}$. Combining these yields (16) and we deduce (17) from it very similarly as (14) from (13).

For (15) recall that in the proof of Theorem 2.1 we showed that for any large k there exists a linear polynomial E_l such that, with $X := H(P_{k+1})$ and $Q_{k,1} := P_k E_l$, we have

$$
\begin{aligned}
&H(Q_{k,1}) \asymp H(P_{k+1}) = X, \\
&|Q_{k,1}(\zeta)| \le X^{-3+\epsilon}, \qquad |P_{k+1}(\zeta)| \le X^{-3+\epsilon}.
\end{aligned}
\tag{71}
$$

Since $Q_{k,1}$ is not irreducible by construction and P_{k+1} has degree only 2, we consider the polynomials $S_{k,j}(T) := Q_{k,1}(T) + jT \cdot P_{k+1}(T)$ for $j \in \{1, 2\}$. We show that at least one of these two polynomials has the desired properties (in fact we need the distinction only for the right hand side of (15); the left follows for both $j = 1$ and $j = 2$). The polynomials $S_{k,j}(T)$ obviously have degree three and height $H(S_{k,j}) \ll X$. Moreover with (71) we infer

$$
|S_{k,j}(\zeta)| = |Q_{k,1}(\zeta) + j\zeta P_{k+1}(\zeta)| \le |Q_{k,1}(\zeta)| + j|\zeta| \cdot |P_{k+1}(\zeta)| \ll_\zeta X^{-3+\epsilon},
\tag{72}
$$

for $1 \le j \le 2$. Next we check that $S_{k,j}$ are irreducible for large k and $1 \le j \le 2$. Consider j fixed and suppose $S_{k,j}$ is reducible. Then we may write $S_{k,j}(T) = M(T)N(T)$ for $M, N \in \mathbb{Z}[T]$ each of degree one or two. Then $|S_{k,j}(\zeta)| = |M(\zeta)| \cdot |N(\zeta)|$ and it follows from (42) and (72) that at least one of the inequalities

$$
|M(\zeta)| \le H(M)^{-3+2\epsilon} \quad \text{or} \quad |N(\zeta)| \le H(N)^{-3+2\epsilon}
$$

must be satisfied; see also the remark subsequent to (42). Without loss of generality say this holds for M. However, since $w_{2,2}(\zeta) \le \tau < 3$, see (8), and M has degree at most two, it follows from Theorem 1.1 that the inequality can only be satisfied if M is some P_l from Theorem 1.1. However, by construction of $S_{k,j}$ we clearly cannot have $P_k | S_{k,j}$ or $P_{k+1} | S_{k,j}$. Thus $M = P_l$ for some $l \le k-1$. Theorem 1.1 further implies

$$
H(M) \ll H(P_{k-1}) \ll H(P_{k+1}) \cdot \frac{H(P_{k-1})}{H(P_{k+1})} = X \cdot \frac{H(P_{k-1})}{H(P_{k+1})} \ll X^{1/\nu^2} = X^{1/\tau}
$$

and it follows further that

$$(73)\qquad |M(\zeta)| \asymp H(M)^{-\rho} \gg X^{-\rho/\tau} = X^{-\nu}.$$

Since $M = P_l$ has degree two and $S_{k,j}$ degree three, the polynomial N must have degree one such that by $\lambda_1(\zeta) = 1$ from (7) we have

$$(74)\qquad |N(\zeta)| \gg H(N)^{-1-\epsilon} \gg X^{-1-\epsilon}.$$

Combining (73) and (74) yields

$$|S_{k,j}(\zeta)| = |M(\zeta)| \cdot |N(\zeta)| \gg X^{-\nu-1-\epsilon} = X^{-\tau-\epsilon}.$$

Again from $\tau < 3$ we obtain a contradiction to (72) for small ϵ. Hence the assumption was wrong and indeed $S_{k,j}$ must be irreducible for $j \in \{1, 2\}$, and in view of (72) we have finished the proof of the left-hand side of (15).

For the right-hand side of (15) suppose we have already shown that for all large k and some $j = j(k) \in \{1, 2\}$ we have

$$(75)\qquad |S'_{k,j}(\zeta)| \gg X^{1-\epsilon}.$$

Then the claim follows together with (72) from the left-hand side for α some root of the corresponding $S_{k,j}$ by a similar standard argument as in the deduction of (14) from (13). Indeed it is well known that any polynomial $U \in \mathbb{Z}[T]$ has a root β that satisfies $|\beta - \zeta| \ll |U(\zeta)|/H(U)$, see for example [Roy 2004a]. The claim follows with $U = S_{k,j}$. It remains to be checked that (75) holds, for which we use (10). First note that the derivative of $S_{k,j}$ can be written

$$(76)\qquad |S'_{k,j}(\zeta)| = |Q'_{k,1}(\zeta) + jP_{k+1}(\zeta) + j\zeta P'_{k+1}(\zeta)|, \qquad 1 \le j \le 2.$$

Obviously the term $jP_{k+1}(\zeta)$ in the sum is negligible since it is very small. Hence (76) can be small only if $Q'_{k,1}(\zeta)$ is of the same order (and reverse sign) as $j\zeta P'_{k+1}(\zeta)$. On the other hand (10) implies for all large k the estimate

$$|j\zeta P'_{k+1}(\zeta)| \ge j|\zeta| H(P_{k+1})^{1-\epsilon} \gg_\zeta X^{1-\epsilon} \quad \text{for} \quad j \in \{1, 2\},$$

and very similarly the difference between the right-hand sides in (76) for $j = 2$ and $j = 1$ is at least of order $X^{1-\epsilon}$ as well. It follows that (75) can be violated for at most one index $j \in \{1, 2\}$, and for the other index (75) must be satisfied. This finishes the proof of (15). □

We finish by giving a heuristic argument why the exponents in (16) and (17) should be optimal as well. For any $\tilde{X}$ we can again consider linear combinations $S_{k,j}(T) = jTP_{k+1}(T) + P_k(T)E_t(T)$ for $k = k(\tilde{X})$ largest possible such that $H(P_{k+1}) \le \tilde{X}$ and some E_t of degree one from the proof of Theorem 2.1 such that (16) is satisfied for $Q(T) = Q_{k,1}(T) = P_k(T)E_t(T)$. Given the irreducibility of

$S_{k,j}$ for all large k and j rather small, we can again basically proceed as in the proof of (15). However, the method from the proof of (15) to guarantee the irreducibility of some of the arising $S_{k,j}(T)$ does not work here.

Acknowledgement

The author warmly thanks the anonymous referee for the careful reading and for pointing out inaccuracies.

References

[Adamczewski and Bugeaud 2010] B. Adamczewski and Y. Bugeaud, "Mesures de transcendance et aspects quantitatifs de la méthode de Thue–Siegel–Roth–Schmidt", *Proc. Lond. Math. Soc.* (3) **101**:1 (2010), 1–26. MR Zbl

[Bugeaud 2010] Y. Bugeaud, "On simultaneous rational approximation to a real number and its integral powers", *Ann. Inst. Fourier* (*Grenoble*) **60**:6 (2010), 2165–2182. MR Zbl

[Bugeaud and Laurent 2005] Y. Bugeaud and M. Laurent, "Exponents of Diophantine approximation and Sturmian continued fractions", *Ann. Inst. Fourier* (*Grenoble*) **55**:3 (2005), 773–804. MR Zbl

[Bugeaud and Schleischitz 2016] Y. Bugeaud and J. Schleischitz, "On uniform approximation to real numbers", *Acta Arith.* **175** (2016), 255–268.

[Davenport and Schmidt 1969] H. Davenport and W. M. Schmidt, "Approximation to real numbers by algebraic integers", *Acta Arith.* **15** (1969), 393–416. MR Zbl

[German 2012] O. N. German, "On Diophantine exponents and Khintchine's transference principle", *Mosc. J. Comb. Number Theory* **2**:2 (2012), 22–51. MR Zbl

[Khintchine 1926] A. Khintchine, "Über eine Klasse linearer diophantischer Approximationen", *Rend. Circ. Mat. Palermo* **50**:2 (1926), 170–195. Zbl

[Roy 2003] D. Roy, "Approximation to real numbers by cubic algebraic integers, II", *Ann. of Math.* (2) **158**:3 (2003), 1081–1087. MR Zbl

[Roy 2004a] D. Roy, "Approximation to real numbers by cubic algebraic integers, I", *Proc. London Math. Soc.* (3) **88**:1 (2004), 42–62. MR Zbl

[Roy 2004b] D. Roy, "Diophantine approximation in small degree", pp. 269–285 in *Number theory*, edited by H. Kisilevsky and E. Z. Goren, CRM Proc. Lecture Notes **36**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[Schleischitz 2013] J. Schleischitz, "Diophantine approximation and special Liouville numbers", *Commun. Math.* **21**:1 (2013), 39–76. MR Zbl

[Schleischitz 2014] J. Schleischitz, "Two estimates concerning classical diophantine approximation constants", *Publ. Math. Debrecen* **84**:3-4 (2014), 415–437. MR Zbl

[Schleischitz 2016] J. Schleischitz, "On the spectrum of Diophantine approximation constants", *Mathematika* **62**:1 (2016), 79–100. MR Zbl

[Schmidt and Summerer 2009] W. M. Schmidt and L. Summerer, "Parametric geometry of numbers and applications", *Acta Arith.* **140**:1 (2009), 67–91. MR Zbl

[Schmidt and Summerer 2013a] W. M. Schmidt and L. Summerer, "Diophantine approximation and parametric geometry of numbers", *Monatsh. Math.* **169**:1 (2013), 51–104. MR Zbl

[Schmidt and Summerer 2013b] W. M. Schmidt and L. Summerer, "Simultaneous approximation to three numbers", *Mosc. J. Comb. Number Theory* **3**:1 (2013), 84–107. MR Zbl

[Wirsing 1961] E. Wirsing, "Approximation mit algebraischen Zahlen beschränkten Grades", *J. Reine Angew. Math.* **206** (1961), 67–77. MR Zbl

Received March 9, 2016. Revised August 22, 2016.

JOHANNES SCHLEISCHITZ
INSTITUTE OF MATHEMATICS
UNIVERSITÄT FÜR BODENKULTUR WIEN
GREGOR-MENDEL-STRASSE 33
1180 VIENNA
AUSTRIA
johannes.schleischitz@boku.ac.at

CONTENTS

Volume 287, no. 1 and no. 2

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 287 No. 2 April 2017
