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RADIAL LIMITS OF CAPILLARY SURFACES AT CORNERS

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Dedicated to the memory of Amir Entekhabi

Consider a solution $f \in C^2(\Omega)$ of a prescribed mean curvature equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 2H(x, f) \quad \text{in } \Omega \subset \mathbb{R}^2,$$

where Ω is a domain whose boundary has a corner at $\mathcal{O} = (0, 0) \in \partial\Omega$ and the angular measure of this corner is 2α, for some $\alpha \in (0, \pi)$. Suppose $\sup_{x\in\Omega} |f(x)|$ and $\sup_{x\in\Omega} |H(x, f(x))|$ are both finite. If $\alpha > \frac{\pi}{2}$, then the (nontangential) radial limits of f at $\mathcal{O}$, namely

$$Rf(\theta) = \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

were recently proven by the authors to exist, independent of the boundary behavior of f on $\partial\Omega$, and to have a specific type of behavior.

Suppose $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, the contact angle $\gamma(\cdot)$ that the graph of f makes with one side of $\partial\Omega$ has a limit (denoted γ_2) at $\mathcal{O}$ and

$$\pi - 2\alpha < \gamma_2 < 2\alpha.$$

We prove that the (nontangential) radial limits of f at $\mathcal{O}$ exist and the radial limits have a specific type of behavior, independent of the boundary behavior of f on the other side of $\partial\Omega$. We also discuss the case $\alpha \in \left(0, \frac{\pi}{2}\right]$ and the displayed inequalities do not hold.

1. Introduction and statement of main theorems

Let Ω be a domain in $\mathbb{R}^2$ whose boundary has a corner at $\mathcal{O} \in \partial\Omega$. Suppose $H : \Omega \times \mathbb{R} \to \mathbb{R}$ and H satisfies one of the conditions which guarantees that "cusp solutions" (e.g., §5 of [Lancaster and Siegel 1996b]) do not exist; for example, $H(\boldsymbol{x}, t)$ is weakly increasing in t for each $\boldsymbol{x}$ [Echart and Lancaster 2017] or is real-analytic [Lancaster and Siegel 1996a]. We will assume $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball in $\mathbb{R}^2$ of radius δ^* about $\mathcal{O}$. Polar coordinates relative to $\mathcal{O}$ will be denoted by r and θ. We assume that $\partial\Omega$ is

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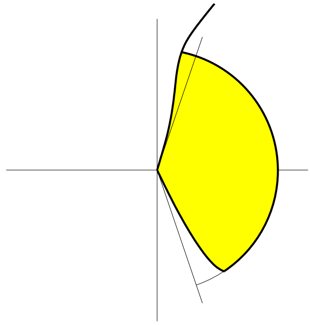

Figure 1. The domain Ω^*.

piecewise smooth and there exists $\alpha \in (0, \pi)$ such that $\partial\Omega \setminus \{\mathcal{O}\} \cap B_{\delta^*}(\mathcal{O})$ consists of two (open) C^1 arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached.

Suppose $\alpha > \frac{\pi}{2}$ and $f \in C^2(\Omega)$ satisfies the prescribed mean curvature equation

$$Nf(x) = 2H(x, f(x)), \quad \text{for } x \in \Omega, \tag{1}$$

where $Nf = \nabla \cdot Tf = \operatorname{div}(Tf)$, $Tf = \nabla f / \sqrt{1 + |\nabla f|^2}$, and

$$\sup_{x\in\Omega} |f(x)| < \infty \quad \text{and} \quad \sup_{x\in\Omega} |H(x, f(x))| < \infty. \tag{2}$$

In [Entekhabi and Lancaster 2016], the authors proved that the radial limits,

$$Rf(\theta) \stackrel{\text{def}}{=} \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

exist for all $\theta \in (-\alpha, \alpha)$, that $Rf(\cdot)$ is a continuous function on $(-\alpha, \alpha)$ and that these radial limits have similar behavior to that observed in Theorem 1 of [Lancaster and Siegel 1996b]. As illustrated in [Lancaster 1989] and in Theorem 3 of [Lancaster and Siegel 1996b], radial limits of nonparametric prescribed mean curvature surfaces do not necessarily exist.

Suppose $\alpha \le \frac{\pi}{2}$ (see Figure 1) and $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfies (1) and (2). In [Entekhabi and Lancaster 2016], it is shown that if

$$\lim_{\partial^-\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x}) \quad \text{exists,} \tag{3}$$

then the radial limits of f at $\mathcal{O}$ exist and behave as expected. In this paper, we consider the capillary problem as our model and suppose $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfies (1), (2) and the boundary condition

$$Tf(x) \cdot \nu(x) = \cos\gamma(x) \quad \text{for } x \in \partial^-\Omega^*, \tag{4}$$

where $\nu(x)$ is the exterior unit normal to Ω at $x \in \partial\Omega$ and $\gamma : \partial\Omega \to [0, \pi]$ is the contact angle between the graph of f and $\partial\Omega \times \mathbb{R}$, and

$$\lim_{\partial^-\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} \gamma(\boldsymbol{x}) = \gamma_2. \tag{5}$$

We shall prove

Theorem 1. *Let* $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ *satisfy* (1) *and* (4) *and suppose* (2) *and* (5) *hold*, $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ *and*

$$\pi - 2\alpha < \gamma_2 < 2\alpha. \tag{6}$$

Then (3) *holds*, $Rf(\theta)$ *exists for all* $\theta \in (-\alpha, \alpha)$ *and* $Rf(\cdot)$ *is a continuous function on* $[-\alpha, \alpha)$, *where* $Rf(-\alpha)$ *equals the limit in* (3). *Further*, $Rf(\cdot)$ *behaves in one of the following ways*:

(i) $Rf : [-\alpha, \alpha) \to \mathbb{R}$ *is a constant function*, *hence* f *has a nontangential limit at* $\mathcal{O}$.

(ii) *There exist* α_1 *and* α_2 *so that* $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ *and* Rf *is constant on* $[-\alpha, \alpha_1]$ *and* $[\alpha_2, \alpha)$ *and strictly increasing or strictly decreasing on* $[\alpha_1, \alpha_2)$.

If $\alpha \in \left(0, \frac{\pi}{4}\right]$, then (6) cannot be satisfied. If $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ but $\gamma_2 \ge 2\alpha$ or $\gamma_2 \le \pi - 2\alpha$, then (6) is not satisfied. In both cases, Theorem 1 is not applicable. In these cases, we can prove the existence of $Rf(\cdot)$ if we add an assumption about the behavior of γ on $\partial^+\Omega^*$.

Theorem 2. *Let* $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^* \cup \partial^+\Omega^*)$ *satisfy* (1) *and* (4). *Suppose* (2) *and* (5) *hold*, $\alpha \in \left(0, \frac{\pi}{2}\right]$, *there exist* $\lambda_1, \lambda_2 \in [0, \pi]$ *with* $0 < \lambda_2 - \lambda_1 < 4\alpha$ *such that* $\lambda_1 \le \gamma(x) \le \lambda_2$ *for* $x \in \partial^+\Omega^*$ *and*

$$\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2. \tag{7}$$

Then the conclusions of Theorem 1 hold.

Remarks. (a) Theorem 2 only offers a new result when $\lambda_1 = 0$ or $\lambda_2 = \pi$; Figure 8 of [Shi 2006] illustrates one example in which $\lambda_1 = 0$ or $\lambda_2 = \pi$ occurs. If $0 < \lambda_1 < \lambda_2 < \pi$, then Theorem 2 is a consequence of [Lancaster and Siegel 1996b, Theorem 1]; in this case, the argument given in that reference (and here) implies that $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$.

(b) In [Concus and Finn 1996; Finn 1996] it was proved that, in a neighborhood $\mathcal{U}$ of $\mathcal{O}$ and assuming $\partial^+\Omega^*$ and $\partial^-\Omega^*$ are straight line segments, a solution of a constant mean curvature equation (i.e., H is constant in (1)) with constant contact angles γ_1 on $\mathcal{U} \cap \partial^+\Omega^*$ and γ_2 on $\mathcal{U} \cap \partial^-\Omega^*$ can exist only if $|\pi - \gamma_1 - \gamma_2| \le 2\alpha$. Using this, when $\gamma_1 = 0$, we would obtain a (local) upper bound for f in Theorem 1 when $\pi - 2\alpha < \gamma_2$ and, when $\gamma_1 = \pi$, a (local) lower bound for f when $\gamma_2 < 2\alpha$; these two inequalities are equivalent to (6).

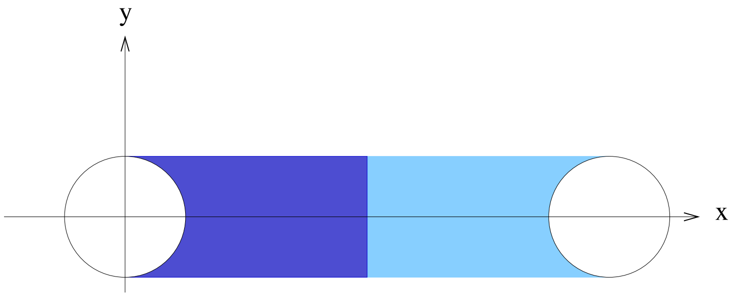

Figure 2. The regions Δ (dark blue) and Δ^R (light blue).

(c) As in [Lancaster and Siegel 1996b], conclusion (3) of Theorems 1 and 2 is a consequence of a general argument; establishing (3) is not a key step in the proof.

2. Preliminary remarks

Let $f \in C^2(\Omega)$ satisfy (1) and suppose (2) holds. Throughout the remainder of the article, let us assume that $M_1 \in (0, \infty)$, $M_2 \in [0, \infty)$,

$$\sup_{x\in\Omega} |f(x)| \le M_1 \quad \text{and} \quad \sup_{x\in\Omega} |H(x, f(x))| \le M_2. \tag{8}$$

2.1. *A specific torus.* We will use portions of tori and comparison function arguments as, for instance, in Examples 2 and 3 of [Lancaster and Siegel 1996b] and the Courant–Lebesgue lemma [Courant 1950, Lemma 3.1] to obtain upper and lower bounds on f near $\mathcal{O}$ in specific subsets of Ω and prove Theorems 1 and 2. Let us discuss the construction of a particular torus.

Set

$$r_0 = \begin{cases} 1 & \text{if } M_2 = 0, \\ 1/M_2 + 1 - \sqrt{(1/M_2)^2 + 1} & \text{if } M_2 > 0. \end{cases}$$

Let

$$\Delta = \left\{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : |\boldsymbol{x}| \ge r_0, 0 \le x_1 \le 2, |x_2| \le r_0 \right\},$$

$$\Delta^R = \{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : (4 - x_1, x_2) \in \Delta \}, \text{ and}$$

$$\mathcal{T} = \left\{ \big(2 + (2 + r_0 \cos v) \cos u, r_0 \sin v, (2 + r_0 \cos v) \sin u\big) : u \in [0, 2\pi], v \in \left[\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right] \right\}.$$

$\mathcal{T}$ is the inner half of a torus of revolution with axis of symmetry $\{(2, y, 0) : y \in \mathbb{R}\}$, major radius $R_0 = 2$ and minor radius r_0; recall that the mean curvature of $\mathcal{T}$ (with respect to the exterior normal) at $\big(2+(2+r_0 \cos v) \cos u, r_0 \sin v, (2+r_0 \cos v) \sin u\big)$

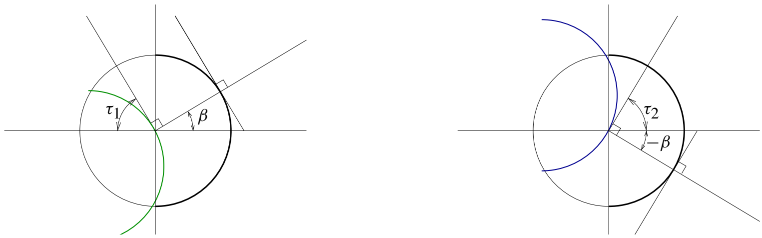

Figure 3. Left: $\beta+\tau_1=\frac{\pi}{2}$. Right: $-\beta+\tau_2=\frac{\pi}{2}$. In both cases, $\beta\ge 0$.

is given by

$$H_T=-\frac{2+2r_0\cos v}{2r_0(2+r_0\cos v)}.$$

A calculation shows that

$$-\Big(\frac{1}{r_0}+\frac{1}{2+r_0}\Big)\le 2H_T\le -\Big(\frac{1}{r_0}-\frac{1}{2-r_0}\Big)=-M_2. \tag{9}$$

Set

$$\mathcal{T}^+=\{(\boldsymbol{x},z)\in\mathcal{T}:\boldsymbol{x}\in\Delta,z\ge 0\}\quad\text{and}\quad \mathcal{T}^-=\{(\boldsymbol{x},z)\in\mathcal{T}:\boldsymbol{x}\in\Delta,z\le 0\}.$$

Let $h^+,h^-:\Delta\to\mathbb{R}$ be functions whose graphs satisfy

$$\{(\boldsymbol{x},h^+(\boldsymbol{x})):\boldsymbol{x}\in\Delta\}=\mathcal{T}^+\quad\text{and}\quad\{(\boldsymbol{x},h^-(\boldsymbol{x})):\boldsymbol{x}\in\Delta\}=\mathcal{T}^-.$$

Then, from (9), we have

$$\operatorname{div}\frac{h^+}{\sqrt{1+|\nabla h^+|^2}}\ge M_2\quad\text{and}\quad\operatorname{div}\frac{h^-}{\sqrt{1+|\nabla h^-|^2}}\le -M_2. \tag{10}$$

For each $\beta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ let $\Delta_\beta=\mathcal{R}_\alpha\circ T_\beta(\Delta)$, where $\mathcal{R}_\alpha:\mathbb{R}^2\to\mathbb{R}^2$, given by

$$(x_1,x_2)\mapsto\big(\cos(\alpha)x_1+\sin(\alpha)x_2,-\sin(\alpha)x_1+\cos(\alpha)x_2\big),$$

is the rotation about (0, 0) through the angle $-\alpha$ and $T_\beta:\mathbb{R}^2\to\mathbb{R}^2$, given by

$$(x_1,x_2)\mapsto(x_1-r_0\cos\beta,x_2-r_0\sin\beta),$$

is the translation taking $(r_0\cos\beta,r_0\sin\beta)\in\partial\Delta$ to $(0,0)$. We will let τ_1 denote the angle that the upward tangent ray to $T_\beta(C)$ makes with the negative x_1-axis and let τ_2 denote the angle that the upward tangent ray to $T_{-\beta}(C)$ makes with the positive x_1-axis, where $C=\big\{\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}^2:|\boldsymbol{x}|=r_0,x_1\ge 0\big\}$. (Figure 3 illustrates this when $\beta>0$.) Let $h^\pm_\beta:\Delta_\beta\to\mathbb{R}$ be defined by $h^\pm_\beta=h^\pm\circ T_\beta^{-1}\circ\mathcal{R}_\alpha^{-1}$, see Figure 4.

Let q denote the be modulus of continuity of h^-, so that $|h^-_\beta(\boldsymbol{x}_1) - h^-_\beta(\boldsymbol{x}_2)| \le q(|\boldsymbol{x}_1 - \boldsymbol{x}_2|$. Notice that q is also the modulus of continuity of h^+, as well as for h^-_β and h^+_β for each $\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2.2. *Parametric framework.* Since $f \in C^0(\Omega)$, we may assume that f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| > \delta\}$ for each $\delta \in (0, \delta^*)$; if this is not true, we may replace Ω with a subset $U \subset \Omega$, such that $\partial\Omega \cap \partial U = \{\mathcal{O}\}$ and $\partial U \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+ U$ and $\partial^- U$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached. Set

$$S_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \Omega^*\} \quad \text{and} \quad \Gamma_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \partial\Omega^* \setminus \{\mathcal{O}\}\};$$

the points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled $A \in \partial^-\Omega^*$ and $B \in \partial^+\Omega^*$. From the calculation on page 170 of [Lancaster and Siegel 1996b], we see that the area of S_0^* is finite; let M_0 denote this area. For $\delta \in (0, 1)$, set

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(1/\delta)}}.$$

Let $E = \{(u, v) : u^2 + v^2 < 1\}$. As in [Elcrat and Lancaster 1986; Lancaster and Siegel 1996b], there is a parametric description of the surface S_0^*,

$$Y(u, v) = (a(u, v), b(u, v), c(u, v)) \in C^2(E : \mathbb{R}^3), \tag{11}$$

which has the following properties:

(a_1) Y is a diffeomorphism of E onto S_0^*.

(a_2) Set $G(u, v) = (a(u, v), b(u, v))$, $(u, v) \in E$. Then $G \in C^0(\bar{E} : \mathbb{R}^2)$.

(a_3) Let $\sigma = G^{-1}(\partial\Omega^* \setminus \{\mathcal{O}\})$; then σ is a connected arc of ∂E and Y maps σ strictly monotonically onto Γ_0^*. We may assume the endpoints of σ are $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ and there exist points $\boldsymbol{a}, \boldsymbol{b} \in \sigma$ such that $G(\boldsymbol{a}) = A$, $G(\boldsymbol{b}) = B$, G maps the (open) arc $\boldsymbol{o}_1\boldsymbol{b}$ onto $\partial^+\Omega$, and G maps the (open) arc $\boldsymbol{o}_2\boldsymbol{a}$ onto $\partial^-\Omega$. (Note that $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ are not assumed to be distinct at this point; Figures 4a and 4b of [Lancaster and Siegel 1997] illustrate this situation.)

(a_4) Y is conformal on E: $Y_u \cdot Y_v = 0$, $Y_u \cdot Y_u = Y_v \cdot Y_v$ on E.

(a_5) $\triangle Y := Y_{uu} + Y_{vv} = H(Y) Y_u \times Y_v$ on E.

Here by the (open) arcs $\boldsymbol{o}_1\boldsymbol{b}$ and $\boldsymbol{o}_2\boldsymbol{a}$ are meant the component of $\partial E \setminus \{\boldsymbol{o}_1, \boldsymbol{b}\}$ which does not contain $\boldsymbol{a}$ and the component of $\partial E \setminus \{\boldsymbol{o}_2, \boldsymbol{a}\}$ which does not contain $\boldsymbol{b}$, respectively. Let $\sigma_0 = \partial E \setminus \sigma$.

There are two cases we will need to consider during the proofs of Theorem 1 and Theorem 2:

$$\text{(A) } \boldsymbol{o}_1 = \boldsymbol{o}_2 \quad \text{or} \quad \text{(B) } \boldsymbol{o}_1 \neq \boldsymbol{o}_2.$$

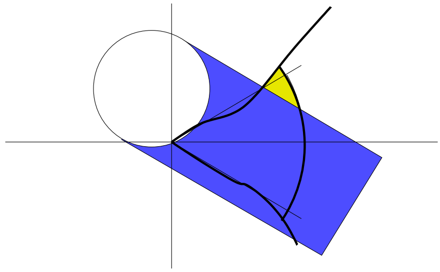

Figure 4. The domain (in blue) of a toroidal function $h^{\pm}_{\beta}$, $\alpha < \frac{\pi}{4}$.

These correspond to Cases 5 and 3 respectively in Step 1 of the proof of Theorem 1 of [Lancaster and Siegel 1996b].

3. Proof of Theorem 1

Since $\pi - 2\alpha < \gamma_2 < 2\alpha$, we can choose $\tau_1 \in (\pi - 2\alpha, \gamma_2)$ and $\tau_2 \in (\gamma_2, 2\alpha)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \frac{\pi}{2} - (\pi - \tau_2) = \tau_2 - \frac{\pi}{2}$. With these choices of β_1 and β_2, notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos \tau_1 > \cos \gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0,$$
$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos \tau_2 < \cos \gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0.$$

This implies that, for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough,

$$(12) \qquad T(h^-_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos \gamma(\boldsymbol{x}) \quad \text{and} \quad T(h^+_{\beta_2})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos \gamma(\boldsymbol{x}),$$

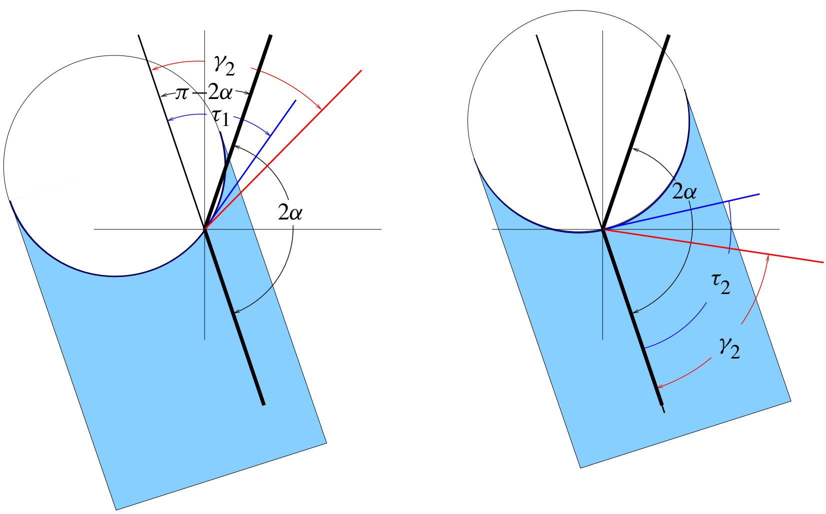

Figure 5. Left: Δ_{β_1}, the domain of $h^-_{\beta_1}$. Right: Δ_{β_2}, the domain of $h^+_{\beta_2}$.

for $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, where $\vec{\nu}(\boldsymbol{x})$ is the exterior unit normal to Ω at $\boldsymbol{x} \in \partial\Omega$. (See Figure 5.) (We may also assume $\nu(\boldsymbol{x}) \cdot (1, 1) < 0$, for $\boldsymbol{x} \in \partial^+\Omega$ with $|\boldsymbol{x}| < \delta_1$ and $\nu(\boldsymbol{x}) \cdot (1, -1) < 0$, for $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, since $\alpha > \frac{\pi}{4}$.)

Let $\mu \in (0, \min\{\gamma_2 - (\pi - 2\alpha), 2\alpha - \gamma_2\})$ and set $\tau_1(\mu) = \pi - 2\alpha + \mu$ and $\tau_2(\mu) = 2\alpha - \mu$, so that $\beta_1 = \beta_2$. Let us write $\delta_1(\mu)$ for $\delta_1(\beta_1, \beta_2)$, h^+_μ for $h^+_{\beta_2}$, h^-_μ for $h^-_{\beta_1}$ and Δ_μ for $\Delta_{\beta_1} = \Delta_{\beta_2}$. Since $\beta_1, \beta_2 \neq \pm\frac{\pi}{2}$, there exists a positive $R = R(\mu)$ such that $B(\mathcal{O}, R(\mu)) \cap \Omega^* \subset \Delta_\mu$ (where $B(\mathcal{O}, R) = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < R\}$).

Let us first assume that (A) holds and set $\boldsymbol{o} = \boldsymbol{o}_1 = \boldsymbol{o}_2$.

Claim. *f is uniformly continuous on Ω_0, where $\Omega_0 = \Omega^* \cap \Delta_\mu$.*

Proof. For $r > 0$, set $B_r = \{\boldsymbol{u} \in \bar{E} : |\boldsymbol{u} - \boldsymbol{o}| < r\}$, $C_r = \{\boldsymbol{u} \in \bar{E} : |\boldsymbol{u} - \boldsymbol{o}| = r\}$ and let l_r be the length of the image curve $Y(C_r)$; also let $C'_r = G(C_r)$ and $B'_r = G(B_r)$. From the Courant–Lebesgue lemma (e.g., Lemma 3.1 in [Courant 1950]), we see that for each $\delta \in (0, 1)$, there exists a $\rho = \rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength l_ρ of $Y(C_\rho)$ is less than $p(\delta)$. For $\delta > 0$, let $k(\delta) = \inf_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \inf_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$ and $m(\delta) = \sup_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \sup_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$; notice that $m(\delta) - k(\delta) \leq l_\rho < p(\delta)$.

For each $\delta \in (0, 1)$ with $\sqrt{\delta} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$, there are two points in $C_{\rho(\delta)} \cap \partial E$; we denote these points as $\boldsymbol{e}_1(\delta) \in \boldsymbol{ob}$ and $\boldsymbol{e}_2(\delta) \in \boldsymbol{oa}$ and set $\boldsymbol{y}_1(\delta) = G(\boldsymbol{e}_1(\delta))$ and $\boldsymbol{y}_2(\delta) = G(\boldsymbol{e}_2(\delta))$. Notice that $C'_{\rho(\delta)}$ is a curve in $\bar{\Omega}$ which joins $\boldsymbol{y}_1 \in \partial^+\Omega^*$ and $\boldsymbol{y}_2 \in \partial^-\Omega^*$ and $\partial\Omega \cap C'_{\rho(\delta)} \setminus \{\boldsymbol{y}_1, \boldsymbol{y}_2\} = \varnothing$; therefore there exists $\eta = \eta(\delta) > 0$ such that $B_{\eta(\delta)}(\mathcal{O}) = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x}| < \eta(\delta)\} \subset B'_{\rho(\delta)}$ (see Figure 6).

Let $\epsilon > 0$. Choose $\delta > 0$ such that $\sqrt{\delta} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$, $p(\delta) < \delta_1(\mu)$, $p(\delta) < R(\mu)$, and $p(\delta) + q(p(\delta)) < \frac{1}{2}\epsilon$. Pick a point $\boldsymbol{w} \in C'_{\rho(\delta)}$ and define $b^\pm_j : \Delta_\mu \to \mathbb{R}$ by

$$b^\pm(\boldsymbol{x}) = f(\boldsymbol{w}) \pm p(\delta) \pm h^\mp_\mu(\boldsymbol{x}), \quad \boldsymbol{x} \in \Delta_\mu.$$

Recalling that $Tb^+ \cdot \eta_1 = 1$ on $C_1 = R_\alpha \circ T_{\beta_1}(C)$ and $Tb^- \cdot \eta_2 = -1$ on $C_2 = R_\alpha \circ T_{\beta_2}(C)$, where $\eta_j(\boldsymbol{x})$ is the interior unit normal to C_j at $\boldsymbol{x} \in C_j$ (and $C =$

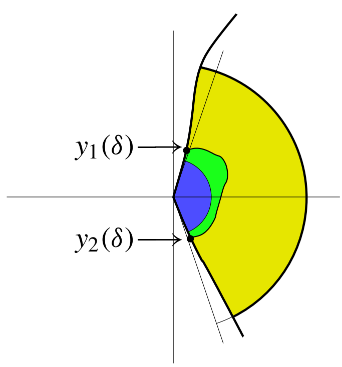

Figure 6. $B_{\eta(\delta)}(\mathcal{O})$ (blue region) and $B'_{\rho(\delta)}$ (blue and green regions).

$\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| = r_0, x_1 \geq 0\})$, it follows from (10), (12) and the general comparison principle (e.g., [Finn 1986, Theorem 5.1]) that

$$b^-(\boldsymbol{x}) < f(\boldsymbol{x}) < b^+(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in B'_{\rho(\delta)} \cap \Delta_\mu.$$

Thus if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ satisfy $|\boldsymbol{x}_1| < \eta(\delta)$, $|\boldsymbol{x}_2| < \eta(\delta)$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \eta(\delta)$, then

$$|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < 2p(\delta) + 2q(p(\delta)) < \epsilon. \tag{13}$$

Since f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| \geq \frac{1}{2}\eta(\delta)\}$, there exists a $\lambda > 0$ such that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega^*$ satisfy $|\boldsymbol{x}_1| \geq \frac{1}{2}\eta(\delta)$, $\boldsymbol{x}_2| \geq \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \lambda$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Now set $d = d(\epsilon) = \min\{\lambda, \frac{1}{2}\eta(\delta)\}$. If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \leq \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1| < \frac{1}{2}\eta(\delta)$, then $|\boldsymbol{x}_1| < \eta(\delta)$ and $|\boldsymbol{x}_2| < \eta(\delta)$; hence $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$ by (13). Next, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \leq \lambda$, $|\boldsymbol{x}_1| \geq \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_2| \geq \frac{1}{2}\eta(\delta)$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Therefore, for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ with $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon)$, we have $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. The claim is proven. □

Notice that if $\theta(\mu) = \alpha - \mu$ $(= \tau_2(\mu) - \alpha = \pi - \alpha - \tau_1(\mu))$, then

$$\{(r\cos\theta(\mu), r\sin\theta(\mu)) : r \geq 0\}$$

is the tangent ray to $\partial\Omega_0$ at $\mathcal{O}$ and it follows from the claim that $f \in C^0(\overline{\Omega_0})$; hence the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ exist for $\theta \in [-\alpha, \theta(\mu)]$ and the radial limits are identical (i.e., $Rf(\theta) = f(\mathcal{O})$ for all $\theta \in [-\alpha, \theta(\mu)]$, where $f(\mathcal{O})$ is the value at $\mathcal{O}$ of the restriction of f to $\overline{\Omega_0}$). Since $\lim_{\mu\downarrow 0}\theta(\mu) = \alpha$, Theorem 1 is proven in this case.

Let us next assume that (B) holds. This part of the proof is essentially the same as the proof of case (B) in Theorem 1 of [Entekhabi and Lancaster 2016]. As in that paper, and taking into account the hypothesis $\alpha \leq \frac{\pi}{2}$, we see that

(i) $c \in C^0(\overline{E} \setminus \{\boldsymbol{o}_1, \boldsymbol{o}_2\})$,

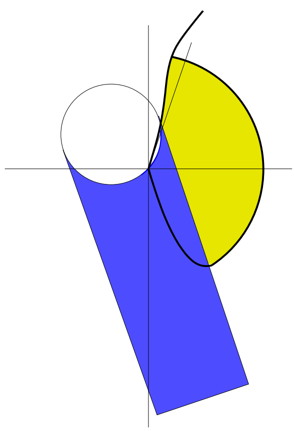

Figure 7. The domain (in blue) of the toroidal functions $h^{\pm}_{\mu}$, $\alpha > \frac{\pi}{4}$.

(ii) there exist $\alpha_1, \alpha_2 \in [-\alpha, \alpha]$ with $\alpha_1 < \alpha_2$ such that $Rf(\theta)$ exists when $\theta \in (\alpha_1, \alpha_2)$, and

(iii) Rf is strictly increasing or strictly decreasing on (α_1, α_2).

Taking hypothesis (5) into account and using cylinders as in case 3 of step 1 in the proof of Theorem 1 of [Lancaster and Siegel 1996b] (see Figure 2b in [Lancaster and Siegel 1997]) or using $h^{\pm}_{\mu}$ (see Figure 7), we see that in addition to (i)–(iii), we have

(iv) $c \in C^0(\bar{E} \setminus \{\boldsymbol{o}_1\})$ and

(v) $Rf(\theta)$ exists when $\theta \in [-\alpha, \alpha_2)$.

If $\alpha_2 = \alpha$, then Theorem 1 is proven. Otherwise, suppose $\alpha_2 < \alpha$ and fix $\delta_0 \in (0, \delta^*)$ and $\Omega_0 = \Omega^* \cap \Delta_\mu$ as before.

Claim. *Suppose $\alpha_2 < \alpha$. Then f is uniformly continuous on Ω_0^+, where*

$$\Omega_0^+ \stackrel{\text{def}}{=} \{(r\cos\theta, r\sin\theta) \in \Omega_0 : 0 < r < \delta^*, \alpha_2 < \theta < \pi\}.$$

Notice that the restriction of Y to $G^{-1}(\overline{\Omega_0^+})$ maps only one point, $\boldsymbol{o}_1$, to $\mathcal{O} \times \mathbb{R}$ and so the proof of this claim is the same as the proof of the previous claim. Thus $f \in C^0(\overline{\Omega_0^+})$; since $\lim_{\mu \downarrow 0} \theta(\mu) = \alpha$, we see that

$$Rf(\theta) = \lim_{\tau \uparrow \alpha_2} Rf(\tau) \quad \text{for all } \theta \in [\alpha_2, \alpha).$$

Thus Theorem 1 is proven. □

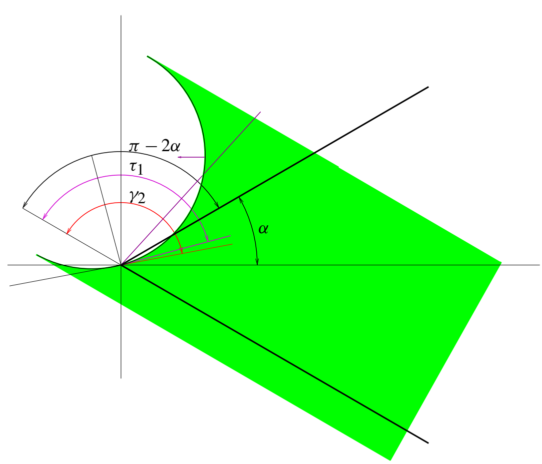

Figure 8. $\alpha = \frac{\pi}{6}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\pi}{2}$, $\gamma_2 = \frac{7\pi}{9}$, and $\tau_1 = \frac{27\pi}{36}$. The domain of $h^-_{\beta_1}$ is the green region.

4. Proof of Theorem 2

Suppose (6) does not hold. Since $\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2$, we can choose $\tau_1, \tau_2 \in (0, \pi)$ such that $\tau_1 \in (\pi - 2\alpha - \lambda_1, \gamma_2)$ and $\tau_2 \in (\gamma_2, \pi + 2\alpha - \lambda_2)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. (See Figures 8 and 9.) With these choices of β_1 and β_2, notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos\tau_1 > \cos\gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0,$$
$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos\tau_2 < \cos\gamma_2, \quad \text{for } 0 < x_1 < 2 - r_0.$$

This implies that for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough,

$$T(h^-_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos\gamma(\boldsymbol{x}) \quad \text{and} \quad T(h^+_{\beta_2})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos\gamma(\boldsymbol{x}), \tag{14}$$

for $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, where $\vec{\nu}(\boldsymbol{x})$ is the exterior unit normal to Ω at $\boldsymbol{x} \in \partial\Omega$. (See Figures 5, 8 and 9.)

Notice that the tangent plane at (0, 0, 0) to the surface $\{(\boldsymbol{x}, h^-_{\beta_1}(\boldsymbol{x})) : \boldsymbol{x} \in \Delta_{\beta_1}\}$ is a vertical plane with (downward oriented) unit normal

$$\vec{n} = (-\sin(\tau_1 + \alpha), -\cos(\tau_1 + \alpha), 0)$$

and

$$\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{\nu}(\boldsymbol{x}) = (-\sin\alpha, \cos\alpha, 0).$$

Suppose $\tau_1 + 2\alpha \le \pi$. Then

$$\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{n} \cdot \vec{\nu}(\boldsymbol{x}) = -\cos(\tau_1 + 2\alpha) > -\cos(\pi - \lambda_1) = \cos\lambda_1,$$

since $\tau_1 + 2\alpha > \pi - \lambda_1$; since $\liminf_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \gamma(\boldsymbol{x}) \ge \lambda_1$, this implies that for some $\delta_2 > 0$ small enough,

$$T(h^-_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos\gamma(\boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in \partial^+\Omega \text{ with } |\boldsymbol{x}| < \delta_2. \tag{15}$$

If $\tau_1 + 2\alpha > \pi$, then λ_1 doesn't matter and we argue as in the proof of Theorem 1; see Figure 8 for an illustration of this case.

Now the tangent plane at (0, 0, 0) to the surface $\{(\boldsymbol{x}, h^+_{\beta_2}(\boldsymbol{x})) : \boldsymbol{x} \in \Delta_{\beta_2}\}$ is a vertical plane with (downward oriented) unit normal $\vec{m} = (\sin(\tau_2 - \alpha), -\cos(\tau_2 - \alpha), 0)$ and $\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{\nu}(\boldsymbol{x}) = (-\sin\alpha, \cos\alpha, 0)$.

Suppose $\tau_2 \ge 2\alpha$. Then

$$\lim_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \vec{m} \cdot \vec{\nu}(\boldsymbol{x}) = -\cos(\tau_2 - 2\alpha) < -\cos(\pi - \lambda_2) = \cos\lambda_2,$$

since $\tau_2 - 2\alpha < \pi - \lambda_2$; since $\limsup_{\partial^+\Omega \ni \boldsymbol{x} \to \mathcal{O}} \gamma(\boldsymbol{x}) \le \lambda_2$, this implies that for some $\delta_3 > 0$ small enough,

$$T(h^+_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos\gamma(\boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in \partial^+\Omega \text{ with } |\boldsymbol{x}| < \delta_3. \tag{16}$$

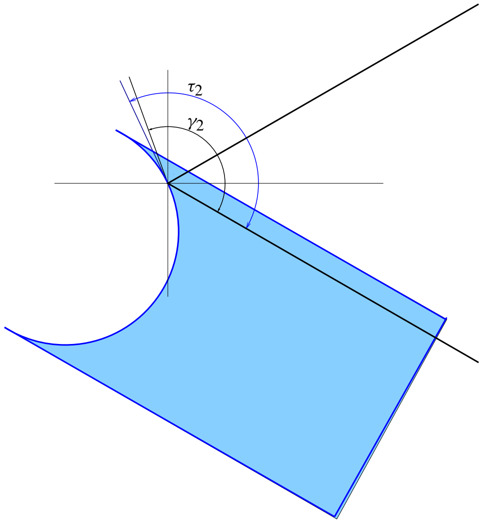

Figure 9. $\alpha = \frac{\pi}{6}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\pi}{2}$, $\gamma_2 = \frac{7\pi}{9}$, and $\tau_2 = \frac{29\pi}{36}$. The domain of $h^+_{\beta_2}$ is the blue region.

If $\tau_2 < 2\alpha$, then λ_2 doesn't matter and we argue as in the proof of Theorem 1.

Now set $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. The proof of Theorem 2 now follows essentially as in the proof of Theorem 1. □

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