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Dwork's conjecture, now proven by Wan, states that unit root L-functions "coming from geometry" are p-adic meromorphic. In this paper we study the p-adic variation of a family of unit root L-functions coming from a suitable family of toric exponential sums. In this setting, we find that the unit root L-functions each have a unique p-adic unit root. We then study the variation of this unit root over the family of unit root L-functions. Surprisingly, we find that this unit root behaves similarly to the classical case of families of exponential sums, as studied by Adolphson and Sperber (2012). That is, the unit root is essentially a ratio of $\mathcal{A}$-hypergeometric functions.

1. Introduction

Dwork [1973] conjectured that certain L-functions, constructed as Euler products of p-adic unit roots coming from the fibers of an algebraic family of L-functions, are p-adic meromorphic. He proved this in a few cases using the idea of an excellent lifting of Frobenius, but was unable to prove it in general, mainly because excellent lifting in its original form does not always exist. Wan [1999; 2000b; 2000a] proved Dwork's conjecture using a new technique which avoided excellent lifting. In the present paper, we extend Wan's techniques, as established in [Haessig 2014], by constructing a dual theory in which to study the p-adic variation of unit root L-functions.

Let Ψ be a nontrivial additive character on the finite field $\mathbb{F}_q$. Additionally, let $f \in \mathbb{F}_q[\lambda_1^{\pm}, \dots, \lambda_s^{\pm}, x_1^{\pm}, \dots, x_n^{\pm}]$ be a Laurent polynomial, and consider for each $\bar{\lambda} \in (\overline{\mathbb{F}}_q^{\times})^s$ and $m \geq 1$ the exponential sum

$$S_m(f, \bar{\lambda}) := \sum_{\bar{x} \in (\mathbb{F}^{\times}_{q^{m \cdot \deg(\bar{\lambda})}})^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{m \cdot \deg(\bar{\lambda})}} / \mathbb{F}_q} (f(\bar{\lambda}, \bar{x}))$$

where $\deg(\bar{\lambda}) := [\mathbb{F}_q(\bar{\lambda}) : \mathbb{F}_q]$. Define the associated L-function by $L(f, \bar{\lambda}, T) := \exp\left(\sum_{m \geq 1} S_m(f, \bar{\lambda}) T^m / m\right)$. It is known that $L(f, \bar{\lambda}, T)^{(-1)^{n+1}}$ is a rational function

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with a unique p-adic unit root, say $\pi_0(\bar{\lambda})$, which is also a 1-unit. The unit root L-function of this family is defined by

$$L_{\text{unit}}(\kappa, T) := \prod_{\bar{\lambda}\in|\mathbb{G}_m^s/\mathbb{F}_q|} \frac{1}{1-\pi_0(\bar{\lambda})^{\kappa} T^{\deg(\bar{\lambda})}},$$

where κ takes on values in the p-adic integers $\mathbb{Z}_p$. This is a p-adic meromorphic function in T and a p-adic continuous function in κ. As shown in a remark below, $L_{\text{unit}}(\kappa, T)^{(-1)^{s+1}}$ will have a unique p-adic unit root. We conjecture that the unit root will have the following description.

Writing $f(\lambda, x) = \sum a_{\gamma,u}\lambda^{\gamma}x^{u}$ with $a_{\gamma,u} \in \mathbb{F}_q$, let $\hat{a}_{\gamma,u}$ be the Teichmüller lift of $a_{\gamma,u}$ and write $\hat{a} = (\hat{a}_{\gamma,u})_{(\gamma,u)\in\text{supp}(f)}$. Let $\pi \in \overline{\mathbb{Q}}_p$ be such that $\pi^{p-1} = -p$. Define a new polynomial $\tilde{f}$ by replacing the coefficients of f by new variables $\Lambda_{\gamma,u}$ for each monomial $\lambda^{\gamma}x^{u}$, that is, define $\tilde{f}(\Lambda, \lambda, x) := \sum \Lambda_{\gamma,u}\lambda^{\gamma}x^{u}$. Writing $\exp \pi \tilde{f}(\Lambda, \lambda, x) = \sum g_{\gamma,u}(\Lambda)\lambda^{\gamma}x^{u}$, Adolphson and Sperber [2012] have shown that $\mathcal{G}(\Lambda) := g_{0,0}(\Lambda)/g_{0,0}(\Lambda^{p})$ converges on the closed polydisk $|\Lambda_{\gamma,u}|_p \le 1$. Thus, it makes sense to evaluate $\mathcal{G}(\hat{a}) := \mathcal{G}(\Lambda)|_{\Lambda=\hat{a}}$. We conjecture that the unit root of $L_{\text{unit}}(\kappa, T)^{(-1)^{s+1}}$ is of the form $(\mathcal{G}(\hat{a})\mathcal{G}(\hat{a}^{p})\cdots\mathcal{G}(\hat{a}^{p^{a-1}}))^{\kappa}$ where $q = p^{a}$.

Our first main result will be to prove this conjecture when $f(\lambda, x)$ satisfies a lower deformation hypothesis stated below. Our second main result, which explains the paper's length, is the development of a dual theory for L-functions of infinite symmetric powers $L^{(0)}(\kappa, \bar{t}, T)$, defined on page 137. These seem to have a theory similar to that of classical L-functions of exponential sums over finite fields. For example, they display the same type of δ-structure (10) as well as having an attached p-adic cohomology theory (see, e.g., [Haessig 2016]). There is some slight evidence that these may be related to p-adic automorphic forms.

As mentioned above, in this paper we study the p-adic variation of unit root L-functions such as these. The following setup is similar to that of the above family, but more technical for the following reason. As unit root L-functions come from families, and we wish to study a family of unit root L-functions, we need to consider a family of families. The role of the variables in the following is: x denotes the space variables, λ denotes the parameters of the family, and t denotes the parameters defining the family of families.

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^n$. We define the Newton polyhedron of $\mathcal{A}$ at ∞, denoted $\Delta_\infty(\mathcal{A})$, to be the convex closure of $\mathcal{A} \cup 0$ in $\mathbb{R}^n$. We make the simplifying hypothesis that every element $u \in \mathcal{A}$ lies on the Newton boundary at ∞ of $\Delta_\infty(\mathcal{A})$, that is, the union of all faces of $\Delta_\infty(\mathcal{A})$ which do not contain the origin. In other language this is the same as the hypothesis that $w(u) = 1$ for all $u \in \mathcal{A}$, where w is the usual polyhedral weight defined by $\Delta_\infty(\mathcal{A})$ (see the next section for definition). The generic polynomial f, with x-support equal to $\mathcal{A}$, is given by $f(t, x) = \sum t_u x^u \in \mathbb{F}_q[\{t_u\}_{u\in\mathcal{A}}, x_1^{\pm}, \dots, x_n^{\pm}]$, where u runs over $\mathcal{A}$ and $\{t_u\}_{u\in\mathcal{A}}$ are new variables. Let $\Delta_\infty(f)(=\Delta_\infty(\mathcal{A}))$ be the Newton polyhedron at infinity of f. Let $P(\lambda, x) \in \mathbb{F}_q[\lambda_1^{\pm}, \dots, \lambda_s^{\pm}, x_1^{\pm}, \dots, x_n^{\pm}]$ be such that the monomials $\lambda^\gamma x^v$ in the support of $P(\lambda, x)$ all satisfy $0 < w(v) < 1$. Such deformations were studied in [Haessig and Sperber 2014]. It is convenient to assume the origin is not in the set $\mathcal{A}$ and if $\lambda^\gamma x^v$ is in the support of P, then $v \neq 0$ so that neither f nor P have a constant term (with respect to the x-variables). This assumption will be made throughout this work. Let $G(t, \lambda, x) := f(t, x) + P(\lambda, x)$.

We construct a family of L-functions as follows. Let $\bar{t} \in (\overline{\mathbb{F}}_q^*)^{|\mathcal{A}|}$, and denote by $\deg(\bar{t}) = [\mathbb{F}_q(\bar{t}) : \mathbb{F}_q]$ the degree of $\bar{t}$, where $\mathbb{F}_q(\bar{t})$ means we adjoin every coordinate of $\bar{t}$ to $\mathbb{F}_q$. We will often write $d(\bar{t})$ for $\deg(\bar{t})$. For convenience, write $q_{\bar{t}} := q^{d(\bar{t})}$ so that $\mathbb{F}_{q_{\bar{t}}} = \mathbb{F}_q(\bar{t})$. Next, let $\bar{\lambda} \in (\overline{\mathbb{F}}_q^*)^s$. Denote by $\deg_{\bar{t}}(\bar{\lambda})$ or $d_{\bar{t}}(\bar{\lambda})$ the degree $[\mathbb{F}_{q_{\bar{t}}}(\bar{\lambda}) : \mathbb{F}_{q_{\bar{t}}}]$; set $q_{\bar{t},\bar{\lambda}} := q_{\bar{t}}^{d_{\bar{t}}(\bar{\lambda})}$ and $\mathbb{F}_{q_{\bar{t},\bar{\lambda}}} = \mathbb{F}_{q_{\bar{t}}}(\bar{\lambda})$. For each $m \geq 1$, define the exponential sum

$$S_m(\bar{t}, \bar{\lambda}) := \sum_{\bar{x} \in (\mathbb{F}^*_{q^m_{\bar{t},\bar{\lambda}}})^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m_{\bar{t},\bar{\lambda}}}/\mathbb{F}_q}(G(\bar{t}, \bar{\lambda}, \bar{x}))$$

and its associated L-function

$$L(\bar{t}, \bar{\lambda}, T) := \exp\left(\sum_{m=1}^{\infty} S_m(\bar{t}, \bar{\lambda}) \frac{T^m}{m}\right).$$

It is well-known [Adolphson and Sperber 2012] that $L(\bar{t}, \bar{\lambda}, T)^{(-1)^{n+1}}$ has a unique reciprocal p-adic unit root $\pi_0(\bar{t}, \bar{\lambda})$, which is a 1-unit. Let $\kappa \in \mathbb{Z}_p$ be a p-adic integer. For each $\bar{t}$, the unit root L-function is defined by

$$L_{\mathrm{unit}}(\kappa, \bar{t}, T) := \prod_{\bar{\lambda} \in |\mathbb{G}_m^s / \mathbb{F}_{q_{\bar{t}}}|} \frac{1}{1 - \pi_0(\bar{t}, \bar{\lambda})^\kappa T^{d_{\bar{t}}(\bar{\lambda})}},$$

where κ takes values in the p-adic integers $\mathbb{Z}_p$. Wan's theorem tells us that this L-function is p-adic meromorphic and so may be written as a quotient of p-adic entire functions:

$$L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}} = \frac{\prod_{i=1}^{\infty}(1 - \alpha_i(\kappa, \bar{t})T)}{\prod_{j=1}^{\infty}(1 - \beta_j(\kappa, \bar{t})T)}, \quad \alpha_i, \beta_j \to 0 \text{ as } i, j \to \infty.$$

Little is known about the zeros and poles of unit root L-functions. In Theorem 1.1 below we show, for each $\bar{t}$ and κ, that $L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ itself has a unique unit zero (and no unit poles), which is a 1-unit. We then study the variation of this unit root as a function of $\bar{t}$ and κ. We note that the variation of the unit root L-function with respect to the parameter κ has been studied before in Wan's proof of Dwork's conjecture, and is connected to the Gouvêa–Mazur conjecture [1992]. On the other hand, as far as we know, the study of the p-adic analytic variation of the unit root

L-function with respect to $\bar{t}$ is new. To state the main result, first denote by $\pi \in \overline{\mathbb{Q}}_p$ an element satisfying $\pi^{p-1} = -p$. Next, writing

$$\begin{aligned} G(t, \lambda, x) &= f(t, x) + P(\lambda, x) \\ &= \sum t_u x^u + \sum A(\gamma, v)\lambda^\gamma x^v \\ &\in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm, \lambda_1^\pm, \ldots, \lambda_s^\pm, \{t_u\}_{u \in \mathrm{supp}(f)}], \end{aligned}$$

let $\hat{A}(\gamma, v)$ be the Teichmüller lift of $A(\gamma, v)$ in $\mathbb{Q}_q$ for each $(\gamma, v) \in \mathrm{supp}(P)$. We now replace every coefficient $A(\gamma, v)$ of $P(\lambda, x)$ with a new variable Λ: set $\mathcal{P}(\Lambda, \lambda, x) := \sum_{(\gamma, v) \in \mathrm{supp}(P)} \Lambda_{\gamma, v} \lambda^\gamma x^v$ and define

$$H(t, \Lambda, \lambda, x) := f(t, x) + \mathcal{P}(\Lambda, \lambda, x).$$

Note that the series

$$\exp(\pi H(t, \Lambda, \lambda, x)) = \sum_{\substack{\gamma \in \mathbb{Z}^s \\ u \in \mathbb{Z}^n}} K_{\gamma, u}(t, \Lambda) \lambda^\gamma x^u$$

is well-defined, and its coefficients $K_{\gamma,u}(t, \Lambda)$ are themselves elements in the power-series ring $\mathbb{Z}_p[\zeta_p][[\{t_u\}_{u \in \mathcal{A}}, \{\Lambda_{\gamma,v}\}_{(\gamma,v) \in \mathrm{supp}(P)}]]$, and so converge in the open poly-disk $D(0, 1^-)^{|\mathcal{A}| + |\mathrm{supp}(P)|}$ which is defined by the inequalities $|t_u| < 1$ for all $u \in \mathcal{A}$ and $|\Lambda_{\gamma,v}| < 1$ for all $(\gamma, v) \in \mathrm{supp}(P)$. Of particular interest is $K_{0,0}(t)$, a principal p-adic unit for all t and Λ in the polydisk. Define $\mathcal{F}(t, \Lambda) := K_{0,0}(t, \Lambda)/K_{0,0}(t^p, \Lambda^p)$ and set $\mathcal{F}_m(t, \Lambda) := \prod_{i=0}^{m-1} \mathcal{F}(t^{p^i}, \Lambda^{p^i})$. By Adolphson and Sperber [2012], $\mathcal{F}(t, \Lambda)$ analytically continues to the closed polydisc $D(0, 1^+)^{|\mathcal{A}| + |\mathrm{supp}(P)|}$ defined by $|t_u| \le 1$, $u \in \mathcal{A}$ and $|\Lambda_{\gamma,v}| \le 1$, $(\gamma, v) \in \mathrm{supp}(P)$.

Theorem 1.1. *Let $\hat{t}$ be the Teichmüller lift of $\bar{t}$. Then*

$$\mathcal{F}_{ad(\bar{t})}(\hat{t}, \hat{A})^\kappa = \prod_{i=0}^{ad(\bar{t})} \mathcal{F}(\hat{t}^{p^i}, \hat{A}^{p^i})^\kappa$$

is the unique unit root of $L_{\mathrm{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ at each fiber $\bar{t}$ and $\kappa \in \mathbb{Z}_p$, where $\mathcal{F}_{ad(\bar{t})}(\hat{t}, \hat{A})$ means setting each $t_u = \hat{t}_u$ and $\Lambda_{\gamma,v} = \hat{A}(\gamma, v)$.

Remark. It is worthwhile to compare this result to the result in [Adolphson and Sperber 2012]. To that end, consider the (total) family $H(t, \Lambda, \lambda, x)$ above. For each $\bar{t} \in (\overline{\mathbb{F}}_q^\times)^{|\mathcal{A}|}$ and $m \ge 1$, define the exponential sum

$$S_m(H, \bar{t}) := \sum_{(\bar{\lambda}, \bar{x}) \in (\mathbb{F}_{q^{m \cdot \deg(\bar{t})}}^\times)^s \times (\mathbb{F}_{q^{m \cdot \deg(\bar{t})}}^\times)^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{m \cdot \deg(\bar{t})}} / \mathbb{F}_q}(H(\bar{t}, A, \bar{\lambda}, \bar{x})).$$

Define by $L(H, \bar{t}, T) := \exp\left(\sum_{m \ge 1} S_m(H, \bar{t}) T^m / m\right)$ the associated L-function, a rational function over $\mathbb{Q}(\zeta_p)$. By [Adolphson and Sperber 2012], $L(H, \bar{t}, T)^{(-1)^{s+n+1}}$

has a unique p-adic unit root given by $\mathcal{F}_{ad(\bar{t})}(\hat{t}, \hat{A})$. As mentioned above, this relation should conjecturally hold in greater generality.

Remark. The existence of a unique p-adic unit root is a general result for unit root L-functions defined over the torus $\mathbb{G}_m^s$. This includes the classical case of L-functions of exponential sums defined over the torus; see [Haessig 2014, Section 3] for details.

To give an indication of the proof, we use the language of σ-modules. See [Haessig 2014] as a reference for the following notation. Let K be a finite extension field of $\mathbb{Q}_p$ with uniformizer π, ring of integers R, and residue field $\mathbb{F}_q$. Let (M, ϕ) be a $c \cdot$ log-convergent, nuclear σ-module over R, ordinary at slope zero of rank one ($h_0 = 1$) with basis $\{e_i\}_{i \geq 0}$. Assume further the normalization condition $\phi e_0 \equiv e_0 \bmod(\pi)$ and $\phi e_i \equiv 0 \bmod(\pi)$ for all $i \geq 1$. With this setup, it follows that the associated unit root L-function $L_{\text{unit}}(\kappa, \phi, T)^{(-1)^{s+1}}$ has a unique p-adic unit root (and no unit poles). To see this we first note that, by [Haessig 2014], $L_{\text{unit}}(\kappa, \phi, T)^{(-1)^{s+1}} \equiv \det(1 - F_{B^{[\kappa]}} T) \bmod \pi$. Next, it follows from the normalization condition that the matrix $B^{[\kappa]}$ takes the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \bmod \pi$, and thus $\det(1 - F_{B^{[\kappa]}} T) \equiv 1 - T \bmod \pi$. Hence, the Fredholm determinant $\det(1 - F_{B^{[\kappa]}} T)$ has a unique p-adic unit root proving the result.

2. Lower deformation family

Let $f \in \mathbb{F}_q[\{t_u\}_{u \in \text{supp}(f)}, x_1^{\pm}, \dots, x_n^{\pm}]$ be of the form $f(t, x) = \sum t_u x^u$. In particular, the coefficient of every monomial x^u in f is a new variable t_u. Denote by $\Delta_\infty(f)$ the Newton polytope at infinity of f, defined as the convex closure of $\text{supp}(f) \cup \{0\}$ in $\mathbb{R}^n$. Let $\text{Cone}(f)$ be the union of all rays emanating from the origin and passing through $\Delta_\infty(f)$, and set $M := M(f) := \text{Cone}(f) \cap \mathbb{Z}^n$. We define a weight function w on M as follows. For $u \in M$, let $w(u)$ be the smallest nonnegative rational number such that $u \in w(u)\Delta(f)$. It is convenient to assume $w(u) = 1$ for all u in the x-support of f. In particular this implies that f has no constant term. Let D denote the smallest positive integer such that $w(M) \subset (1/D)\mathbb{Z}_{\geq 0}$. The weight function w satisfies the following norm-like properties:

(1) $w(u) = 0$ if and only if $u = 0$.

(2) $w(cu) = cw(u)$ for every $c \geq 0$.

(3) $w(u + v) \leq w(u) + w(v)$ for every $u, v \in M$, with equality holding if and only if u and v are cofacial.

It is convenient to assume the lower-order deformation $P \in \mathbb{F}_q[\lambda_1^{\pm}, \dots, \lambda_s^{\pm}, x_1^{\pm}, \dots, x_n^{\pm}]$ has no constant term so the origin in $\mathbb{R}^n$ is not in the x-support of P. In fact, if we write $P(\lambda, x) = \sum_{u \in M} P_u(\lambda) x^u$, then $0 < w(u) < 1$. Our lower deformation family then is defined by $G(t,\lambda,x) := f(t,x) + P(\lambda,x)$. Set

(1) $$U := \Big\{ \big(\tfrac{1}{1-w(u)}\big)\gamma \in \mathbb{Q}^s \;\big|\; (\gamma,u) \in \operatorname{supp}(P) \Big\},$$

and let $\Gamma := \Delta_\infty(U) \subset \mathbb{R}^s$. Similarly, define $M(\Gamma) := \operatorname{Cone}(\Gamma) \cap \mathbb{Z}^s$ with associated polyhedral weight function w_Γ. The polyhedral weight makes sense as well on points in $\operatorname{Cone}(\Gamma)$ having real coordinates. Since $0 < w(u) < 1$ for $(\gamma,u) \in \operatorname{supp}(P)$, it follows that $w_\Gamma(\delta) < 1$ for any $\delta = \gamma/(1-w(u)) \in U$. Equivalently, $w_\Gamma(\gamma) < 1$ for any $(\gamma,u) \in \operatorname{supp}(P)$. We call Γ the *relative polytope* of the family $G(x,t)$.

Rings of p-adic analytic functions. Let ζ_p be a primitive p-th root of unity, $\mathbb{Q}_q$ be the unramified extension of $\mathbb{Q}_p$ of degree $a := [\mathbb{F}_q : \mathbb{F}_p]$, and denote by $\mathbb{Z}_q$ its ring of integers. Then $\mathbb{Z}_q[\zeta_p]$ and $\mathbb{Z}_p[\zeta_p]$ are the rings of integers of $\mathbb{Q}_q(\zeta_p)$ and $\mathbb{Q}_p(\zeta_p)$, respectively. Let $\pi \in \overline{\mathbb{Q}}_p$ satisfy $\pi^{p-1} = -p$, and let $\tilde{\pi}$ be an element which satisfies $\operatorname{ord}_p(\tilde{\pi}) = (p-1)/p^2$. We may have occasion to work over a purely ramified extension $\Omega_0 = \mathbb{Q}_p(\hat{\pi})$ of $\mathbb{Q}_p$ with uniformizer $\hat{\pi}$ which contains $\mathbb{Q}_p(\zeta_p, \tilde{\pi})$ and for which $\tilde{\pi}$ is an integral power of $\hat{\pi}$. Let $\Omega = \mathbb{Q}_q(\hat{\pi})$. Denote by R the ring of integers of Ω, and R_0 the ring of integers of Ω_0. Set

$$\mathcal{O}_0 := \Big\{ \sum_{\gamma \in M(\Gamma)} C(\gamma)\tilde{\pi}^{w_\Gamma(\gamma)}\lambda^\gamma \;\Big|\; C(\gamma) \in R,\, C(\gamma) \to 0 \text{ as } \gamma \to \infty \Big\}.$$

(We note that the fractional powers of $\tilde{\pi}$ are to be understood as integral powers of a uniformizer of R.) Then $\mathcal{O}_0$ is a ring with a discrete valuation given by

$$\Big| \sum_{\gamma \in M(\Gamma)} C(\gamma)\lambda^\gamma \tilde{\pi}^{w_\Gamma(\gamma)} \Big| := \sup_{\gamma \in M(\Gamma)} |C(\gamma)|.$$

Define

$$\mathcal{C}_0(\mathcal{O}_0) := \Big\{ \xi = \sum_{\mu \in M(\bar{f})} \xi(\mu)\tilde{\pi}^{w(\mu)}x^\mu \;\Big|\; \xi(\mu) \in \mathcal{O}_0,\, \xi(\mu) \to 0 \text{ as } \mu \to \infty \Big\},$$

an $\mathcal{O}_0$-algebra.

In the following, $q = p^a$ is an arbitrary power of p (including the case $a = 0$), so we can handle the cases of t^q, t^p, and t at the same time. Define

(2) $$\mathcal{O}_{0,q} := \Big\{ \sum_{\gamma \in M(\Gamma)} C(\gamma)\lambda^\gamma \tilde{\pi}^{w_{q\Gamma}(\gamma)} \;\Big|\; C(\gamma) \in R,\, C(\gamma) \to 0 \text{ as } \gamma \to \infty \Big\}.$$

This ring is the same as $\mathcal{O}_0$ except using a weight function defined by the dilation $q\Gamma$ (that is, $w_{q\Gamma}(\gamma) = w_\Gamma(\gamma)/q$). We note that here $\mathcal{O}_{0,1} = \mathcal{O}_0$. A discrete valuation may be defined as follows. If $\xi = \sum_{\gamma \in M(\Gamma)} C(\gamma)\tilde{\pi}^{w_{q\Gamma}(\gamma)}\lambda^\gamma \in \mathcal{O}_{0,q}$ then the valuation on $\mathcal{O}_{0,q}$ is given by

$$|\xi| := \sup_{\gamma \in M(\Gamma)} |C(\gamma)|.$$

We may also define the space

$$\mathcal{C}_0(\mathcal{O}_{0,q}) := \left\{ \sum_{u \in M(f)} \xi_u x^u \tilde{\pi}^{w(u)} \,\middle|\, \xi_u \in \mathcal{O}_{0,q}, \xi_u \to 0 \text{ as } u \to \infty \right\}. \tag{3}$$

For $\eta = \sum_{u \in M(\bar{f})} \xi_u \tilde{\pi}^{w(u)} x^u \in \mathcal{C}_0(\mathcal{O}_{0,q})$, we set

$$|\eta| = \sup_{u \in M(f)} |\xi_u|.$$

Frobenius. At present, we fix $\bar{t} \in (\overline{\mathbb{F}}_q)^{|A|}$, returning to variation in $\bar{t}$ in the last section. Recall the notation $\deg(\bar{t}) = d(\bar{t}) = [\mathbb{F}_q(\bar{t}) : \mathbb{F}_q]$ and $q_{\bar{t}} = q^{d(\bar{t})}$. Now let $\bar{\lambda} \in (\overline{\mathbb{F}}_q)^s$. Similarly, denote by $\deg(\bar{\lambda})$ or $d(\bar{\lambda})$ the degree $[\mathbb{F}_q(\bar{\lambda}, \bar{t}) : \mathbb{F}_q(\bar{t})]$, and $q_{\bar{t},\bar{\lambda}} = q^{d(\bar{t})d(\bar{\lambda})}$.

Dwork defines a splitting function by $\theta(T) := \exp \pi(T - T^p) = \sum_{i=0}^{\infty} \theta_i T^i$. It is well-known that $\mathrm{ord}_p(\theta_i) \geq i(p-1)/p^2$ for all $i \geq 0$. Writing

$$G(\bar{t}, \lambda, x) = f(\bar{t}, x) + P(\lambda, x) = \sum \bar{t}_u x^u + \sum \bar{A}(\gamma, v) \lambda^\gamma x^v$$

in $\mathbb{F}_{q_{\bar{t}}}[x_1^{\pm}, \ldots, x_n^{\pm}, \lambda_1^{\pm}, \ldots, \lambda_s^{\pm}]$, we let

$$\hat{G}(\hat{t}, \lambda, x) := \sum \hat{t}_u x^u + \sum \hat{A}(\gamma, v) \lambda^\gamma x^v \in R[x_1^{\pm}, \ldots, x_n^{\pm}, \lambda_1^{\pm}, \ldots, \lambda_s^{\pm}]$$

be the lifting of G by lifting the coefficients $\bar{A}(\gamma, u)$ and $\bar{t}$ by Teichmüller units. Set

$$F(\hat{t}, \lambda, x) := \prod_{u \in \mathrm{supp}(f)} \theta(\hat{t}_u x^u) \cdot \prod_{(\gamma, v) \in \mathrm{supp}(P)} \theta(\hat{A}(\gamma, v) \lambda^\gamma x^v) \tag{4}$$

and for any $m \geq 1$,

$$F_m(\hat{t}, \lambda, x) := \prod_{i=0}^{m-1} F^{\sigma^i}(\hat{t}, \lambda^{p^i}, x^{p^i}), \tag{5}$$

where σ is the extension of the usual Frobenius generator of $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ to Ω with $\sigma(\hat{\pi}) = \hat{\pi}$. Then, σ acts on a series with coefficients in Ω by acting on these coefficients. Note that if we set

$$F_m(\hat{t}, \lambda, x) = \sum_{u \in M(f)} \mathcal{B}^m(u) x^u = \sum_{\substack{\gamma \in M(\Gamma) \\ u \in M(f)}} \mathcal{B}^m(\gamma, u) \lambda^\gamma x^u,$$

then

$$\mathrm{ord}_p(\mathcal{B}^m(\gamma, u)) \geq \frac{w_\Gamma(\gamma) + w(u)}{p^{m-1}} \cdot \frac{p-1}{p^2}.$$

Define ψ_x by $\sum C(u)x^u \mapsto \sum C(pu)x^u$. Set

$$\alpha_1 := \sigma^{-1} \circ \psi_x \circ F(\hat{t}, \lambda, x).$$

A similar argument to that in [Haessig and Sperber 2014] demonstrates that α_1 is a σ^{-1}-semilinear map of $\mathcal{C}_0(\mathcal{O}_0)$ into $\mathcal{C}_0(\mathcal{O}_{0,p})$. Similarly, for $m \geq 1$, if we define

$$\alpha_m := \sigma^{-m} \circ \psi_x^m \circ F_m(\hat{t}, \lambda, x),$$

then α_m maps $\mathcal{C}_0(\mathcal{O}_0)$ into $\mathcal{C}_0(\mathcal{O}_{0,p^m})$. In particular,

$$\alpha_m(\tilde{\pi}^{w(v)} x^v) = \sum_{u \in M(f)} \tilde{\pi}^{w(v)-w(u)} \mathcal{B}^m(p^m u - v) \tilde{\pi}^{w(u)} x^u,$$

with $\operatorname{ord}_p(\tilde{\pi}^{w(v)-w(u)} \mathcal{B}^m(p^m u - v)) \geq \big((p-1)w(u) + (1-1/p^{m-1})w(v)\big) \operatorname{ord}_p(\tilde{\pi})$. Summarizing, in $\mathcal{C}_0(\mathcal{O}_{0,p^m})$ we have $|\alpha_m(\tilde{\pi}^{w(v)} x^v)| \leq |\tilde{\pi}|^{w(v)(p^{m-1}-1)/p^{m-1}}$.

Fibers. Define

$$\alpha_{\bar{t},\bar{\lambda}} := \psi_x^{ad(\bar{t})d(\bar{\lambda})} \circ F_{ad(\bar{t})d(\bar{\lambda})}(\hat{t}, \hat{\lambda}, x),$$

where $\hat{t}$ and $\hat{\lambda}$ are the Teichmüller representatives of $\bar{t}$ and $\bar{\lambda}$, respectively. Notice that $\alpha_{\bar{t},\bar{\lambda}}$ is an endomorphism of $\mathcal{C}_0(\hat{\lambda})$, where $\mathcal{C}_0(\hat{\lambda})$ denotes the space obtained from $\mathcal{C}_0(\mathcal{O}_0)$ by applying the map on $\mathcal{O}_0$ which sends λ to $\hat{\lambda}$.

To relate the L-function $L(\bar{t}, \bar{\lambda}, T)$ to the operator $\alpha_{\bar{t},\bar{\lambda}}$ it is convenient to introduce the following operation: for any function $g(T)$, define $g(T)^{\delta_q} := g(T)/g(qT)$. Set $q_{\bar{t},\bar{\lambda}} := q^{d(\bar{t})d(\bar{\lambda})}$. Dwork's trace formula states

$$(q_{\bar{t},\bar{\lambda}}^m - 1)^n \operatorname{Tr}(\alpha_{\bar{t},\bar{\lambda}}^m \mid \mathcal{C}_0(\hat{\lambda})) = \sum_{\bar{x} \in (\mathbb{F}^*_{q^m_{\bar{t},\bar{\lambda}}})^n} \Psi \circ \operatorname{Tr}_{\mathbb{F}_{q^m_{\bar{t},\bar{\lambda}}}/\mathbb{F}_q}(G(\bar{t}, \bar{\lambda}, \bar{x}))$$

Equivalently,

$$L(\bar{t}, \bar{\lambda}, T)^{(-1)^{n+1}} = \det\big(1 - \alpha_{\bar{t},\bar{\lambda}} T \mid \mathcal{C}_0(\hat{\lambda})\big)^{\delta^n_{q_{\bar{t},\bar{\lambda}}}}.$$

This is a rational function, and it is well-known that $L(\bar{t}, \bar{\lambda}, T)^{(-1)^{n+1}}$ has a unique unit (reciprocal) root $\pi_0(\bar{t}, \bar{\lambda})$ (see [Adolphson and Sperber 2012], for example). This unit root is a 1-unit, so it makes sense to define, for any p-adic integer κ, the unit root L-function at the fiber $\bar{t}$:

$$L_{\text{unit}}(\kappa, \bar{t}, T) := \prod_{\bar{\lambda} \in |\mathbb{G}_m^s/\mathbb{F}_q(\bar{t})|} \frac{1}{1 - \pi_0(\bar{t}, \bar{\lambda})^\kappa T^{\deg(\bar{\lambda})}}. \tag{6}$$

Denote the roots of $\det(1-\alpha_{\bar{t},\bar{\lambda}}T \mid \mathcal{C}_0(\hat{\lambda}))$ by $\pi_i(\bar{t},\bar{\lambda})$, and order them such that $\operatorname{ord}_p \pi_i(\bar{t},\bar{\lambda}) \le \operatorname{ord}_p \pi_{i+1}(\bar{t},\bar{\lambda})$ for $i \ge 0$. For each $m \ge 0$, define

$$L^{(m)}(\kappa,\bar{t},T) := \prod_{\bar{\lambda}\in|\mathbb{G}_m^s/\mathbb{F}_{q_{\bar{t}}}|} \prod \bigl(1-\pi_0(\bar{t},\bar{\lambda})^{\kappa-r-m}\pi_{i_1}(\bar{t},\bar{\lambda})\cdots\pi_{i_r}(\bar{t},\bar{\lambda})\pi_{j_1}(\bar{t},\bar{\lambda})\cdots\pi_{j_m}(\bar{t},\bar{\lambda})T^{\deg(\bar{\lambda})}\bigr)^{-1},$$

where the inner product runs over all $r \ge 0$, $1 \le i_1 \le i_2 \le \cdots$, and $0 \le j_1 \le j_2 < \cdots < j_m$. Note that the factors indexed by i_k are allowed to repeat, whereas the factors indexed by j_l are distinct. Intuitively, the inner product is $\det(1-\mathrm{Sym}^{\kappa-m}\,\alpha_{\bar{t},\bar{\lambda}} \otimes \wedge^m \alpha_{\bar{t},\bar{\lambda}} T)$. From [Haessig 2014, Lemma 2.1],

$$L_{\text{unit}}(\kappa,\bar{t},T) = \prod_{i=0}^{\infty} L^{(i)}(\kappa,\bar{t},T)^{(-1)^{i-1}(i-1)} = L^{(0)}(\kappa,\bar{t},T)\prod_{i\ge 2} L^{(i)}(\kappa,\bar{t},T)^{(-1)^{i-1}(i-1)}. \tag{7}$$

In the next section, we will show each $L^{(i)}$ with $i \ge 1$ has no unit root or pole, whereas $L^{(0)}$ will. This will show $L_{\text{unit}}(\kappa,\bar{t},T)^{(-1)^{s+1}}$ has a unique unit root.

3. Infinite symmetric powers

Denote by $\mathcal{S}(\hat{\lambda}) := R[\hat{\lambda}][\![\{e_u\}_{u\in M\setminus\{0\}}]\!]$ the formal power series ring over $R[\hat{\lambda}]$ in the variables $\{e_u\}_{u\in M\setminus\{0\}}$ which are formal symbols indexed by $M\setminus\{0\}$. We equip this ring with the sup-norm on coefficients (in $R[\hat{\lambda}]$). This ring will play the role of the formal infinite symmetric power of $\mathcal{C}_0(\hat{\lambda})$ over $R[\hat{\lambda}]$ in a way we describe below. It is convenient to write the monomials of degree r in the variables $\{e_u\}$ using the notation $e_{\boldsymbol{u}} := e_{u_1}\cdots e_{u_r}$, where $u_1,\dots,u_r \in M(f)\setminus\{0\}$ for $r \ge 0$. To fix ideas, it helps to assume we have a linear order on $M(f)\setminus\{0\}$ with the property that if $w(u) \le w(v)$ for $u, v \in M(f)\setminus\{0\}$, then $u \le v$. We may extend this to all of $M(f)$ by taking 0 as the least element. We then agree that in this notation we have $0 < u_1 \le u_2 \le \cdots \le u_r$ (equality indicating repeated variables). When $r = 0$ we understand there is only the monomial 1 of degree 0. We extend the weight function w to such monomials by defining, for $e_{\boldsymbol{u}} := e_{u_1}\cdots e_{u_r}$, the weight $w(\boldsymbol{u}) := w(u_1)+\cdots+w(u_r)$. Denote by $\mathcal{S}(M)$ the set of all indices $\boldsymbol{u}$ corresponding to monomials $e_{\boldsymbol{u}}$. We emphasize that we will often equate elements $\boldsymbol{u} \in \mathcal{S}(M)$ with the monomials $e_{\boldsymbol{u}}$; it should be clear from the context which meaning is desired. We may assume $\mathcal{S}(M)$ has a linear order defined on it such that the weight $w(\boldsymbol{u})$ is nondecreasing and such that the restriction of this linear order to $M(f)$ is our earlier linear order.

We may identify $\mathcal{C}_0(\hat{\lambda})$ as an $R[\hat{\lambda}]$-submodule of $\mathcal{S}(\hat{\lambda})$ by defining an $R[\hat{\lambda}]$-linear map

$$\Upsilon : \mathcal{C}_0(\hat{\lambda}) \to \mathcal{S}(\hat{\lambda}) \quad \text{via} \quad \sum_{u \in M(f)} \xi_u \tilde{\pi}^{w(u)} x^u \longmapsto \xi_0 + \sum_{u \in M(f) \setminus \{0\}} \xi_u e_u.$$

That is, the image $\Upsilon(\mathcal{C}_0(\hat{\lambda}))$ consists of the powers series with support in the monomials of $\mathcal{S}(\hat{\lambda})$ of degree ≤ 1 with coefficients $\{\xi_u\}_{u \in M(f)} \subset R[\hat{\lambda}]$ satisfying $\xi_u \to 0$ as $u \to \infty$. Note that $\Upsilon(\tilde{\pi}^{w(u)} x^u) = e_u$ for $u \in M \setminus \{0\}$, and $\Upsilon(1) := 1$. Define the $R[\hat{\lambda}]$-subalgebra of $\mathcal{S}(\hat{\lambda})$

$$\mathcal{S}_0(\hat{\lambda}) := \left\{ \xi = \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi(\boldsymbol{u}) e_{\boldsymbol{u}} \,\middle|\, \xi(\boldsymbol{u}) \in R[\hat{\lambda}], \xi(\boldsymbol{u}) \to 0 \text{ as } w(\boldsymbol{u}) \to \infty \right\}.$$

Hence, $\Upsilon(\mathcal{C}_0(\hat{\lambda})) \subset \mathcal{S}_0(\hat{\lambda})$. Note that we may write $\alpha_{\bar{t},\bar{\lambda}}(1) = 1 + \eta(x)$ for some element $\eta \in \mathcal{C}_0(\hat{\lambda})$ satisfying $|\eta| < 1$ with support of η in $M(f) \setminus \{0\}$. For $\xi = \sum \xi(\boldsymbol{u}) e_{\boldsymbol{u}} \in \mathcal{S}_0(\hat{\lambda})$, define $|\xi| := \sum_{\boldsymbol{u} \in \mathcal{S}(M)} |\xi(\boldsymbol{u})|$, which makes $\mathcal{S}_0(\hat{\lambda})$ a p-adic Banach algebra over $R[\hat{\lambda}]$. Then for any $\zeta \in \mathcal{C}_0(\hat{\lambda})$, $|\Upsilon(\zeta)| = |\zeta|$. It follows that $(\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(1))^\tau$ is defined and belongs to $\mathcal{S}_0(\hat{\lambda})$ for any $\tau \in \mathbb{Z}_p$. Define $[\alpha_{\bar{t},\bar{\lambda}}]_\kappa : \mathcal{S}_0(\hat{\lambda}) \to \mathcal{S}_0(\hat{\lambda})$ by extending linearly over $R[\hat{\lambda}]$ the action on monomials of degree r:

$$[\alpha_{\bar{t},\bar{\lambda}}]_\kappa (e_{u_1} \cdots e_{u_r}) := (\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(1))^{\kappa - r} (\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(\tilde{\pi}^{w(u_1)} x^{u_1})) \cdots (\Upsilon \circ \alpha_{\bar{t},\bar{\lambda}}(\tilde{\pi}^{w(u_r)} x^{u_r})).$$

By a similar argument to [Haessig 2014, Corollary 2.4, part 2],

$$\det\bigl(1 - [\alpha_{\bar{t},\bar{\lambda}}]_\kappa T \mid \mathcal{S}_0(\hat{\lambda})\bigr) = \prod_{r=0}^{\infty} \prod \bigl(1 - \pi_0(\bar{t}, \bar{\lambda})^{\kappa - r} \pi_{i_1}(\bar{t}, \bar{\lambda}) \cdots \pi_{i_r}(\bar{t}, \bar{\lambda}) T\bigr),$$

where the inner product runs over all multisets $\{i_1, \ldots, i_r\}$ of positive integers of cardinality r satisfying $1 \leq i_1 \leq i_2 \leq \cdots$.

Infinite symmetric power on the family. Denote by $\mathcal{S}(\mathcal{O}_0) := \mathcal{O}_0[\![\{e_u\}_{u \in M \setminus \{0\}}]\!]$, the formal power series ring supported by the monomials $\mathcal{S}(M)$, with coefficients in the ring $\mathcal{O}_0$. As in the constant fiber case above, this ring is equipped with the sup-norm on coefficients. Define the p-adic Banach algebra over $\mathcal{O}_0$,

$$\begin{aligned}\mathcal{S}_0(\mathcal{O}_0) &:= \left\{ \xi = \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi(\boldsymbol{u}) e_{\boldsymbol{u}} \,\middle|\, \xi(\boldsymbol{u}) \in \mathcal{O}_0, \xi(\boldsymbol{u}) \to 0 \text{ as } w(\boldsymbol{u}) \to \infty \right\} \\ &= \left\{ \xi = \sum_{\substack{\gamma \in M(\Gamma) \\ \boldsymbol{u} \in \mathcal{S}(M)}} C(\gamma, \boldsymbol{u}) \tilde{\pi}^{w_\Gamma(\gamma)} \lambda^\gamma e_{\boldsymbol{u}} \,\middle|\, C(\gamma, \boldsymbol{u}) \in R, C(\gamma, \boldsymbol{u}) \to 0 \text{ as } w_\Gamma(\gamma) + w(\boldsymbol{u}) \to \infty \right\},\end{aligned}$$

and similarly, for any $q = p^a$ an arbitrary power of p (including the case when $a = 0$),

$$\mathcal{S}_0(\mathcal{O}_{0,q}) := \left\{ \sum_{\boldsymbol{u}\in\mathcal{S}(M)} \xi(\boldsymbol{u})e_{\boldsymbol{u}} \,\middle|\, \xi(\boldsymbol{u}) \in \mathcal{O}_{0,q}, \xi(\boldsymbol{u}) \to 0 \text{ as } w(\boldsymbol{u}) \to \infty \right\}.$$

Note that $\mathcal{S}_0(\mathcal{O}_{0,q})$ is a p-adic Banach algebra over $\mathcal{O}_{0,q}$ with $\mathcal{S}(M)$ an orthonormal basis. We embed $\mathcal{C}_0(\mathcal{O}_{0,q}) \hookrightarrow \mathcal{S}_0(\mathcal{O}_{0,q})$ via a map Υ defined in the same way as on the fibers. Again, $(\Upsilon \circ \alpha_m(1))^\tau \in \mathcal{S}_0(\mathcal{O}_{0,p^m})$ for any $\tau \in \mathbb{Z}_p$. We define a map $[\alpha_m]_\kappa : \mathcal{S}_0(\mathcal{O}_0) \to \mathcal{S}_0(\mathcal{O}_{0,p^m})$ as follows. On a basis element $e_{\boldsymbol{u}} = e_{u_1} \cdots e_{u_r}$ with $r > 0$ and $0 < u_1 \le \cdots \le u_r$,

$$\begin{aligned}[]
[\alpha_m](e_{\boldsymbol{u}}) &:= [\alpha_m]_\kappa(e_{u_1} \cdots e_{u_r}) \\
&:= (\Upsilon \circ \alpha_m(1))^{\kappa - r}(\Upsilon \circ \alpha_m(\tilde{\pi}^{w(u_1)}x^{u_1})) \cdots (\Upsilon \circ \alpha_m(\tilde{\pi}^{w(u_r)}x^{u_r})).
\end{aligned}$$

If $r = 0$,

$$[\alpha_m]_\kappa(1) := \Upsilon(\alpha_m(1))^\kappa.$$

We may calculate an estimate for $\alpha_m(\tilde{\pi}^{w(u)}x^u)$, where we recall that $\alpha_m := \sigma^{-m} \circ \psi_x^m \circ F_m(\bar{t}, \lambda, x)$. As noted earlier, we may write

$$F_m(\hat{t}, \lambda, x) = \sum_{\gamma\in M(\Gamma), v\in M(f)} B(\gamma, v)\tilde{\pi}^{(w_\Gamma(\gamma)+w(v))/p^{m-1}}\lambda^\gamma x^v, \tag{8}$$

with $\operatorname{ord}_p B(\gamma, r) \ge 0$, and set $\mathcal{B}^m(\gamma, v) = B(\gamma, v)\tilde{\pi}^{(w_\Gamma(\gamma)+w(v))/p^{m-1}}$. So

$$\begin{aligned}
\alpha_m(\tilde{\pi}^{w(u)}x^u) &= \psi_x^m(F_m(\hat{t}, \lambda, x) \cdot \tilde{\pi}^{w(u)}x^u) \\
&= \sum \Big(\tilde{\pi}^{(w_\Gamma(\gamma)+w(p^m v-u))/p^{m-1}+w(u)-w_\Gamma(\gamma)/p^{m-1}-w(v)} \\
&\qquad\qquad \times B(\gamma, p^m v - u) \cdot \tilde{\pi}^{w_\Gamma(\gamma)/p^{m-1}}\lambda^\gamma \cdot \tilde{\pi}^{w(v)}x^v\Big).
\end{aligned}$$

We note that

$$\begin{aligned}
\tfrac{w(p^m v-u)}{p^{m-1}} + w(u) - w(v) &\ge pw(v) - \tfrac{w(u)}{p^{m-1}} + w(u) - w(v) \\
&\ge (p-1)w(v) + \tfrac{p^{m-1}-1}{p^{m-1}}w(u).
\end{aligned}$$

Hence,

$$|\Upsilon(\alpha_m(\tilde{\pi}^{w(u)}x^u))| \le |\tilde{\pi}|^{w(u)(p^{m-1}-1)/p^{m-1}}. \tag{9}$$

The R-linear map $\psi_\lambda : \mathcal{S}_0(\mathcal{O}_{0,p}) \to \mathcal{S}_0(\mathcal{O}_0)$ is defined by

$$\psi_\lambda : \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} A(\gamma, \boldsymbol{u})\lambda^\gamma e_{\boldsymbol{u}} \longmapsto \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} A(p\gamma, \boldsymbol{u})\lambda^\gamma e_{\boldsymbol{u}}.$$

We may in the usual manner view $\mathcal{S}_0(\mathcal{O}_0)$ as a p-adic Banach space over R with orthonormal basis $\{\tilde{\pi}^{w_\Gamma(\gamma)}\lambda^\gamma e_{\boldsymbol{u}} \mid \gamma \in M(\Gamma), \boldsymbol{u} \in \mathcal{S}(M)\}$. Then

$$\beta_{\kappa,\bar{t}} := \psi_\lambda^{ad(\bar{t})} \circ [\alpha_{ad(\bar{t})}]_\kappa : \mathcal{S}_0(\mathcal{O}_0) \to \mathcal{S}_0(\mathcal{O}_0)$$

is a completely continuous operator (over R). Set $\mathcal{B} := \{e_{\boldsymbol{u}} \mid \boldsymbol{u} \in \mathcal{S}(M)\}$. Let $B_{\bar{t}}^{[\kappa]}(\lambda)$ be the matrix of $[\alpha_{ad(\bar{t})}]_\kappa$ with respect to $\mathcal{B}$, the basis of $\mathcal{S}_0(\mathcal{O}_0)$ over $\mathcal{O}_0$ (as well as $\mathcal{S}_0(\mathcal{O}_{0,p^m})$ over $\mathcal{O}_{0,p^m}$). The entries of $B_{\bar{t}}^{[\kappa]}(\lambda)$ are series with support in $\mathcal{B}$ and coefficients in $\mathcal{O}_{0,p^m}$ (which tend to 0 as $w(\boldsymbol{u}) \to \infty$). We may write $B_{\bar{t}}^{[\kappa]}(\lambda) = \sum_{\gamma \in M(\Gamma)} b_\gamma^{[\kappa]}\lambda^\gamma$, where $b_\gamma^{[\kappa]}$ is a matrix with rows and columns indexed by $M(\Gamma)$ and entries in R. We define the matrix $F_{B_{\bar{t}}^{[\kappa]}} := (b_{q_{\bar{t}}\gamma-\mu}^{[\kappa]})_{(\gamma,\mu)}$ indexed by $\gamma, \mu \in M(\Gamma)$, and we set $b_{q_{\bar{t}}\gamma-\mu}^{[\kappa]} := 0$ if $q_{\bar{t}}\gamma - \mu \notin M(\Gamma)$. Note that $F_{B_{\bar{t}}^{[\kappa]}}$ is a matrix with entries in R whose (γ, μ) entry is again a matrix in R with rows and columns indexed by $M(\Gamma)$. As we showed in [Haessig and Sperber 2014, §2.3], $F_{B_{\bar{t}}^{[\kappa]}}$ is the matrix of the completely continuous operator $\beta_{\kappa,\bar{t}}$, and as such it has a well-defined Fredholm determinant. In particular, the Dwork trace formula gives

$$\begin{aligned}(q_{\bar{t}}^m - 1)^s \operatorname{Tr}(\beta_{\kappa,\bar{t}}^m) &= (q_{\bar{t}}^m - 1)^s \operatorname{Tr}(F_{B_{\bar{t}}^{[\kappa]}}^m) \\ &= \sum_{\lambda^{q_{\bar{t}}^m-1}=1} \operatorname{Tr}\big(B_{\bar{t}}^{[\kappa]}(\hat{\lambda}^{q_{\bar{t}}^{m-1}}) \cdots B_{\bar{t}}^{[\kappa]}(\hat{\lambda}^{q_{\bar{t}}}) B^{[\kappa]}\bar{t}(\hat{\lambda})\big) \\ &= \sum_{\substack{\bar{\lambda} \in (\mathbb{F}_{q_{\bar{t}}^m}^*)^s \\ \hat{\lambda} = \mathrm{Teich}(\bar{\lambda})}} \operatorname{Tr}\big([\alpha_{\bar{t},\bar{\lambda}}]_\kappa^m \mid \mathcal{S}_0(\hat{\lambda})\big).\end{aligned}$$

Using an argument similar to that succeeding [Haessig 2014, (8)], it follows that

$$L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}} = \det(1 - \beta_{\kappa,\bar{t}}T)^{\delta_{q_{\bar{t}}}^s}. \tag{10}$$

Since the Fredholm determinant $\det(1-\beta_{\kappa,\bar{t}}T)$ is p-adically entire, this demonstrates the meromorphic continuation of $L^{(0)}(\kappa, \bar{t}, T)$. Since the matrix of $\beta_{\kappa,\bar{t}}$ shows that $\det(1-\beta_{\kappa,\bar{t}}T)$ has a unique unit root, it follows that $L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ has a unique unit root equal in fact to the unique unit root of $\det(1 - \beta_{\kappa,\bar{t}}T)$.

In a similar way, define on the space $\mathcal{S}_0(\mathcal{O}_0) \otimes \wedge^m \mathcal{C}_0(\mathcal{O}_0)$, the operator $\beta_{\kappa,\bar{t}}^{(m)} := \psi_\lambda^{ad(\bar{t})} \circ \big([\alpha_{ad(\bar{t})})]_{\kappa-m} \otimes \wedge^m \alpha_{ad(\bar{t})}\big)$. Then

$$L^{(m)}(\kappa, \bar{t}, T)^{(-1)^{s+1}} = \det(1 - \beta_{\kappa,\bar{t}}^{(m)}T)^{\delta_{q_{\bar{t}}}^s}.$$

In particular, for $m \geq 2$, due to the wedge product, $L^{(m)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ has no zeros or poles on the closed unit disk. Hence, by (7), we have:

Theorem 3.1. *$L_{\text{unit}}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$ has a unique p-adic unit root which in fact is the unique unit root of $L^{(0)}(\kappa, \bar{t}, T)^{(-1)^{s+1}}$.*

4. Dual theory

In this section, we define a dual theory for the operator $\beta_{\kappa,\bar{t}}$ acting on $\mathcal{S}_0(\mathcal{O}_0)$. We begin by defining a dual map to $\alpha_{ad(\bar{t})}$. For $q = p^a$ an arbitrary power of p (including the case $a = 0$) define the $\mathcal{O}_{0,q}$-module

$$\mathcal{C}_0^*(\mathcal{O}_{0,q}) := \left\{ \sum_{u \in M(f)} \xi(u)\tilde{\pi}^{-w(u)} x^{-u} \,\middle|\, \xi(u) \in \mathcal{O}_{0,q} \right\},$$

equipped with the sup-norm on the set of coefficients $\{\xi(u)\}_{u \in M(f)}$. Define the projection (or truncation) map

$$\mathrm{pr}_{M(f)}: \quad \sum_{u \in \mathbb{Z}^n} A(u) x^{-u} \longmapsto \sum_{u \in M(f)} A(u) x^{-u}.$$

For each $m \geq 1$, define

$$\alpha_m^* := \mathrm{pr}_{M(f)} \circ F_m(\hat{t}, \lambda, x) \circ \Phi_x^m \circ \sigma^m,$$

where $\sigma \in \mathrm{Gal}(\Omega/\Omega_0)$ acts on coefficients (as mentioned above), and Φ_x acts on monomials by $\Phi_x(x^u) := x^{pu}$.

Lemma 4.1. *$\alpha_m^* : \mathcal{C}_0^*(\mathcal{O}_{0,p^m}) \to \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ is a linear map over $\mathcal{O}_{0,p^m}$. Furthermore, writing*

$$\alpha_m^*(\tilde{\pi}^{-w(v)} x^{-v}) = \sum_{z \in M(f)} C_v(z) \tilde{\pi}^{-w(z)} x^{-z},$$

with $C_v(z) \in \mathcal{O}_{0,p^m}$, then $C_v(z) \to 0$ in $\mathcal{O}_{0,p^m}$ as $w(v) \to \infty$. In addition, we may write $\alpha_m^*(1) = 1 + \eta_m^*(\lambda, x)$, with $\eta_m^*(\lambda, x) \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ having $|\eta_m^*| \leq |\tilde{\pi}|$.

Proof. We consider $\alpha_m^*(\tilde{\pi}^{-w(v)} x^{-v})$ with $v \in M(f)$. Using (8), we may write this as

$$\alpha_m^*(\tilde{\pi}^{-w(v)} x^{-v}) = \sum_{\substack{z \in M(f) \\ \gamma \in M(\Gamma)}} \Big(B(\gamma, -z + p^m v) \tilde{\pi}^{w_\Gamma(\gamma)/p^{m-1}} \lambda^\gamma \times \tilde{\pi}^{-w(v)+w(z)+(w(-z+p^m v)/p^{m-1})} \tilde{\pi}^{-w(z)} x^{-z} \Big).$$

Since

$$-w(v) + w(z) + \tfrac{1}{p^{m-1}} w(-z + p^m v) \geq \tfrac{p^{m-1}-1}{p^{m-1}} w(z) + (p-1) w(v),$$

we see that

$$\alpha_m^*(\tilde{\pi}^{-w(v)} x^{-v}) = \tilde{\pi}^{(p-1)w(v)} \zeta_v^*(\lambda, x), \tag{11}$$

where $\zeta_v^*(\lambda, x) \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$.

If $\xi^* \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ with $\xi^* = \sum_{v \in M(f)} A_v(\lambda) \tilde{\pi}^{-w(v)} x^{-v}$, then

$$\alpha_m^*(\xi^*) = \sum_{v \in M(f)} \tilde{\pi}^{(p-1)w(v)} A_v(\lambda) \zeta_v^*(\lambda, x) \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m}).$$

Finally, note that by the above,

$$\alpha_m^*(1) = 1 + \sum_{\gamma \in M(\Gamma)\setminus\{0\}} B(\gamma, 0)\tilde{\pi}^{w(\gamma)/p^{m-1}} \lambda^\gamma + \sum_{\substack{z \in M(f)\setminus\{0\} \\ \gamma \in M(\Gamma)}} B(\gamma, -z)\tilde{\pi}^{w(z)+(w(-z)/p^{m-1})} (\tilde{\pi}^{w(\gamma)/p^{m-1}} \lambda^\gamma)(\tilde{\pi}^{w(-z)} x^{-z}).$$

This proves the lemma. □

Define

$$\mathcal{A}_0 := \left\{ \sum_{\gamma \in M(\Gamma)} A(\gamma)\lambda^\gamma \,\middle|\, A(\gamma) \in R \text{ and } A(\gamma) \to 0 \text{ as } w(\gamma) \to \infty \right\}.$$

For q_1 and q_2 any two powers of the prime p, define a pairing

$$(\cdot\,,\cdot) : \mathcal{C}_0(\mathcal{O}_{0,q_1}) \times \mathcal{C}_0^*(\mathcal{O}_{0,q_2}) \to \mathcal{A}_0$$

by

$$(\xi, \xi^*) := \text{ the constant term with respect to } x \text{ of the product } \xi \cdot \xi^*.$$

This product is well-defined since if $\{\eta_1(v)\}_{v \in M(\Gamma)} \subset \mathcal{O}_{0,q_1}$ with $\eta_1(v) \to 0$ as $w(v) \to \infty$, and $\{\eta_2(v)\}_{v \in M(\Gamma)} \subset \mathcal{O}_{0,q_2}$, then $\sum_{v \in M(\Gamma)} \eta_1(v)\eta_2(v) \in \mathcal{A}_0$. Next let $\xi \in \mathcal{C}_0(\mathcal{O}_0)$, $\xi^* \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$. Writing F_m for $F_m(\hat{t}, \lambda, x)$, observe that

$$((\psi_x^m \circ F_m)\xi, \xi^*) = (F_m\xi, \Phi_x^m \xi^*) = (\xi, (\mathrm{pr}_{M(f)} \circ F_m \circ \Phi_x^m)(\xi^*)). \tag{12}$$

Symmetric powers. We construct in a now familiar manner formal k-th symmetric powers of $\mathcal{C}_0(\mathcal{O}_0)$ and $\mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ over $\mathcal{O}_0$. Similar to the construction used above, we consider a linear order on $\{u \in M(f)\}$ under which the weight is nondecreasing, say $0 = u_0 \leq u_1 \leq \cdots$. We will for convenience of notation write the "basis" as $\{E_u := \tilde{\pi}^{w(u)} x^u \mid u \in M(f)\}$, and the k-th symmetric power of the basis as

$$E_{\boldsymbol{u}} := E_{u_{j_1}} E_{u_{j_2}} \cdots E_{u_{j_k}} \quad (0 \leq j_1 \leq j_2 \leq \cdots \leq j_k),$$

where $\boldsymbol{u}$ runs over multisets of indices of cardinality k, say

$$\{\boldsymbol{u} = (u_{j_1}, u_{j_2}, \ldots, u_{j_k}) \mid 0 \leq u_{j_1} \leq u_{j_2} \leq \cdots \leq u_{j_k}\}.$$

Defining

$$\mathrm{Sym}^k_{\mathcal{O}_0} \mathcal{C}_0(\mathcal{O}_0) := \left\{ \xi = \sum_{|\boldsymbol{u}|=k} \xi_{\boldsymbol{u}}(\lambda) E_{\boldsymbol{u}} \,\middle|\, \xi_{\boldsymbol{u}}(\lambda) \in \mathcal{O}_0, \xi_{\boldsymbol{u}}(\lambda) \to 0 \text{ as } w(\boldsymbol{u}) \to +\infty \right\},$$

we then define the map

$$\mathrm{Sym}^k \alpha_m : \mathrm{Sym}^k_{\mathcal{O}_0} \mathcal{C}_0(\mathcal{O}_0) \to \mathrm{Sym}^k_{\mathcal{O}_{0,p^m}} \mathcal{C}_0(\mathcal{O}_{0,p^m})$$

as follows. Let

$$\alpha_m(\tilde{\pi}^{w(u)}x^u) = \sum_{v\in M(f)} \mathcal{A}^m_{v,u}(\lambda)\tilde{\pi}^{w(v)}x^v$$
$$= \sum_{v\in M(f)} \mathcal{A}^m_{v,u}(\lambda)E_v.$$

We know from Section 2 that

$$\mathcal{A}^m_{u,v} = \sum_{\gamma\in M(\Gamma),v\in M(f)} \tilde{\pi}^{w(u)-w(v)}\mathcal{B}^m(\gamma, p^m v - u)\lambda^\gamma.$$

Then

$$\mathrm{Sym}^k\,\alpha_m(E_{u_{j_1}}E_{u_{j_2}}\cdots E_{u_{j_k}}) = \sum \mathcal{A}^m_{v_{l_1},u_{j_1}}(\lambda)\cdots\mathcal{A}^m_{v_{l_k},u_{j_k}}(\lambda)E_{v_{l_1}}\cdots E_{v_{l_k}},$$

where the sum runs over all $v_{l_i}\in M(f)$ for each i, $1\le i\le k$. Since, by above, $|\alpha_m(\tilde{\pi}^{w(u)}x^u)|\le|\tilde{\pi}|^{w(u)(p^{m-1}-1)/p^{m-1}}$ therefore $\mathrm{Sym}^k(\alpha_m)$ is a completely continuous map. The map Υ may be extended to $\mathrm{Sym}^k_{\mathcal{O}_0}(\mathcal{C}_0(\mathcal{O}_0))\hookrightarrow\mathcal{S}_0(\mathcal{O}_0)$ as follows. For $\boldsymbol{u}=(u_{j_1},\ldots,u_{j_k})$ an ordered multiset of cardinality k with elements in $M(f)$, set

$$\Upsilon(E_{\boldsymbol{u}}) = \begin{cases} e_{\boldsymbol{u}} & \text{if } j_1>0,\\ e_{u_{j_{r+1}}}e_{u_{j_{r+2}}}\cdots e_{u_{j_k}} & \text{if } j_1=j_2=\cdots=j_r=0.\end{cases}$$

Thus $\Upsilon(\mathrm{Sym}^k_{\mathcal{O}_0}\mathcal{C}_0(\mathcal{O}_0))$ consists of all power-series with coefficients in $\mathcal{O}_0$ and support in monomials $e_{\boldsymbol{u}}$ of degree $\le k$, with coefficients going to 0 as $w(\boldsymbol{u})=w(u_1)+\cdots+w(u_r)\to\infty$.

We have as well a dual variant

$$\mathrm{Sym}^k_{\mathcal{O}_{0,p^m}}\mathcal{C}^*_0(\mathcal{O}_{0,p^m}) := \left\{\sum_{|\boldsymbol{u}|=k} A_{\boldsymbol{u}}(\lambda)E^*_{\boldsymbol{u}}\,\middle|\, A_{\boldsymbol{u}}(\lambda)\in\mathcal{O}_{0,p^m}\right\},$$

where we denote $E^*_u := \tilde{\pi}^{-w(u)}x^{-u}$ for each $u\in M(f)$, and using the linear order above write for each multiset $\boldsymbol{u}=(u_{j_1},\ldots,u_{j_k})$ of cardinality k of indices, with $j_1\le\cdots\le j_k$ we set $E^*_{\boldsymbol{u}} := E^*_{u_1}\cdots E^*_{u_k}$. Then

$$\mathrm{Sym}^k_{\mathcal{O}_{0,p^m}}\mathcal{C}^*_0(\mathcal{O}_{0,p^m}) = \left\{\sum_{|\boldsymbol{u}|=k}\xi(\boldsymbol{u})E^*_{\boldsymbol{u}}\,\middle|\,\xi(\boldsymbol{u})\in\mathcal{O}_{0,p^m}\right\},$$

there being no requirement here that the coefficients tend to 0 as $w(\boldsymbol{u})\to\infty$. Since $\alpha^*_m:\mathcal{C}^*_0(\mathcal{O}_{0,p^m})\to\mathcal{C}^*_0(\mathcal{O}_{0,p^m})$, we may define for $\boldsymbol{u}=(u_{j_1},\ldots,u_{j_k})$,

$$\mathrm{Sym}^k(\alpha^*_m)(E^*_{\boldsymbol{u}}) = \sum \mathcal{A}^*_{v_{l_1},u_{j_1}}(\lambda)\mathcal{A}^*_{v_{l_2},u_{j_2}}(\lambda)\cdots\mathcal{A}^*_{v_{l_k},u_{j_k}}(\lambda)E^*_{\boldsymbol{v}},$$

where $\boldsymbol{v} = (v_{l_1}, \dots v_{l_k})$, the sum runs over $v_{l_i} \in \{\tilde{\pi}^{-w(u)} x^{-u} \mid u \in M(f)\}$, and where $\alpha_m^*(\tilde{\pi}^{-w(u)} x^{-u}) = \sum_{v \in M(f)} \mathcal{A}_{u,v}^*(\lambda) \tilde{\pi}^{-w(v)} x^{-v}$. The map $\mathrm{Sym}^k(\alpha_m^*)$ then is defined on $\mathrm{Sym}^k_{\mathcal{O}_{0,p^m}}$ since, as we noted earlier in (11), $|\alpha_m^*(\tilde{\pi}^{-w(u)} x^{-u})| \leq |\tilde{\pi}|^{w(u)(p-1)}$.

We extend the pairing above to these symmetric power spaces by "linearly" extending the following: for decomposable elements $\xi = \xi_1 \cdots \xi_k \in \mathrm{Sym}^k_{\mathcal{O}_{0,q_1}} \mathcal{C}_0(\mathcal{O}_{0,q_1})$ and $\xi^* = \xi_1^* \cdots \xi_k^* \in \mathrm{Sym}^k_{\mathcal{O}_{0,q_2}} \mathcal{C}_0^*(\mathcal{O}_{0,q_2})$,

$$(\xi, \xi^*) := (\xi_1 \cdots \xi_k, \xi_1^* \cdots \xi_k^*)_k := \tfrac{1}{k!} \sum_{\sigma \in S_k} \prod_{i=1}^{k} (\xi_i, \xi_{\sigma(i)}^*), \tag{13}$$

where S_k denotes the symmetric group on k letters. This pairing $(\cdot, \cdot)_k$ is well-defined since $\mathcal{A}_0$ is a ring. It follows from (12) that, for $\xi \in \mathrm{Sym}^k \mathcal{C}_0(\mathcal{O}_0)$ and $\xi^* \in \mathrm{Sym}^k \mathcal{C}_0^*(\mathcal{O}_{0,q_{\bar{t}}})$,

$$(\mathrm{Sym}^k \alpha_{ad(\bar{t})} \xi, \xi^*)_k = (\xi, \mathrm{Sym}^k \alpha_{ad(\bar{t})}^* \xi^*)_k. \tag{14}$$

Infinite symmetric powers. Denote by $\mathcal{S}_0^*(\mathcal{O}_0) := \mathcal{O}_0[\![e_u^* : u \in M \setminus \{0\}]\!]$ the formal power series ring over $\mathcal{O}_0$ in the variables $\{e_u^*\}_{u \in M \setminus \{0\}}$, a set of formal symbols indexed by $M \setminus \{0\}$. We endow $\mathcal{S}_0^*(\mathcal{O}_0)$ with the sup-norm on coefficients. Monomials in $\mathcal{S}_0^*(\mathcal{O}_0)$ have the form $e_{\boldsymbol{u}}^* := e_{u_1}^* e_{u_2}^* \cdots e_{u_r}^*$, where $u_1, \dots, u_r \in M(f) \setminus \{0\}$ for $r > 0$, and $e_0^* := 1$ when $r = 0$. Thus, elements in the ring may be described by

$$\mathcal{S}_0^*(\mathcal{O}_0) := \left\{ \xi^* = \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi^*(\boldsymbol{u}) e_{\boldsymbol{u}}^* \;\middle|\; \xi^*(\boldsymbol{u}) \in \mathcal{O}_0 \right\}.$$

Using the same notation as before, define the embedding $\Upsilon : \mathcal{C}_0^*(\mathcal{O}_0) \hookrightarrow \mathcal{S}_0^*(\mathcal{O}_0)$ by $\Upsilon(\tilde{\pi}^{-w(u)} x^{-u}) := e_u^*$ for $u \in M \setminus \{0\}$, and $\Upsilon(1) := e_0^* = 1$. For each $m \geq 1$, recall from Lemma 4.1 that $\alpha_m^*(1) = 1 + \eta_m^*(\lambda, x)$ for some element $\eta_m^* \in \mathcal{C}_0^*(\mathcal{O}_{0,p^m})$ satisfying $|\eta_m^*| < 1$. It follows that $(\Upsilon \circ \alpha_m^*(1))^\tau \in \mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ for any $\tau \in \mathbb{Z}_p$. For $m \geq 1$, we define the map $[\alpha_m^*]_\kappa : \mathcal{S}_0^*(\mathcal{O}_{0,p^m}) \to \mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ by

$$\begin{aligned} &[\alpha_m^*]_\kappa (e_{u_1}^* \cdots e_{u_r}^*) \\ &\quad := (\Upsilon(\alpha_m^*(1)))^{\kappa - r} (\Upsilon(\alpha_m^*(\tilde{\pi}^{-w(u_1)} x^{-u_1}))) \cdots (\Upsilon(\alpha_m^*(\tilde{\pi}^{-w(u_r)} x^{-u_r}))). \end{aligned} \tag{15}$$

The product on the right side makes sense and lives in $\mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ since $\mathcal{S}_0^*(\mathcal{O}_{0,p^m})$ is a ring and each factor is clearly in $\mathcal{S}_0^*(\mathcal{O}_{0,p^m})$. Furthermore,

$$|[\alpha_m^*]_\kappa (e_{\boldsymbol{u}}^*)| \leq |\tilde{\pi}^{(p-1)w(\boldsymbol{u})}|. \tag{16}$$

Define the R-module

$$\mathcal{O}_{0,q}^* := \left\{ \zeta^* = \sum_{\gamma \in M(\Gamma)} \zeta^*(\gamma) \tilde{\pi}^{-w_{q\Gamma}(\gamma)} \lambda^{-\gamma} \;\middle|\; \zeta^*(\gamma) \in R \right\}.$$

Here we do not insist that the coefficients go to 0 and we do not claim $\mathcal{O}_{0,q}^*$ is a ring. As usual we define an absolute value on $\mathcal{O}_{0,q}^*$ by $|\zeta^*| := \sup_{\gamma\in M(\Gamma)} |\zeta^*(\gamma)|$. For series in λ, we define a projection (or truncation) map

$$\mathrm{pr}_{M(\Gamma)} : \quad \sum_{\gamma\in\mathbb{Z}^s} A(\gamma)\lambda^{-\gamma} \longmapsto \sum_{\gamma\in M(\Gamma)} A(\gamma)\lambda^{-\gamma}.$$

Note that for any q a power of the prime p, if γ, γ', and δ all belong to $M(\Gamma)$ with $\gamma-\gamma'=-\delta$ then $w_{q\Gamma}(\gamma)-w_{q\Gamma}(\gamma')\geq -w_{q\Gamma}(\delta)$. It follows that, for $\xi\in\mathcal{O}_{0,q}$ and $\xi^*\in\mathcal{O}_{0,q}^*$,

$$\mathrm{pr}_{M(\Gamma)}(\xi\cdot\xi^*)\in\mathcal{O}_{0,q}^*. \tag{17}$$

Define the R module

$$\mathcal{S}_0^*(\mathcal{O}_0^*) := \left\{\omega^* = \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} \omega^*(\gamma,\boldsymbol{u})\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma}e_{\boldsymbol{u}}^* \,\middle|\, \omega^*(\gamma,\boldsymbol{u})\in R\right\}.$$

Define the map Φ_λ by $\lambda\mapsto\lambda^p$. We define an R-linear map

$$\beta_{\kappa,\bar{t}}^* := \mathrm{pr}_{M(\Gamma)}\circ[\alpha_{ad(\bar{t})}^*]_\kappa\circ\Phi_\lambda^{ad(\bar{t})}$$

by "linearly" extending over R the action

$$\beta_{\kappa,\bar{t}}^*(\lambda^{-\gamma}e_{\boldsymbol{u}}^*) = \mathrm{pr}_{M(\Gamma)}(\lambda^{-q_{\bar{t}}\gamma}\cdot[\alpha_{ad(\bar{t})}^*]_\kappa(e_{\boldsymbol{u}}^*)).$$

Lemma 4.2. *$\beta_{\kappa,\bar{t}}^*$ is an R-linear endomorphism of $\mathcal{S}_0^*(\mathcal{O}_0^*)$.*

Proof. We have remarked already that $[\alpha_{ad(\bar{t})}^*]_\kappa$ is a well-defined endomorphism of $\mathcal{S}_0^*(\mathcal{O}_{0,q_{\bar{t}}})$. As such, we may write for each $\boldsymbol{u}\in\mathcal{S}(M)$,

$$[\alpha_{ad(\bar{t})}^*]_\kappa(e_{\boldsymbol{u}}^*) = \sum_{\substack{\sigma\in M(\Gamma)\\ \boldsymbol{v}\in\mathcal{S}(M)}} B_{\boldsymbol{u}}(\sigma,\boldsymbol{v})\tilde{\pi}^{w_{q_{\bar{t}}\Gamma}(\sigma)}\lambda^\sigma e_{\boldsymbol{v}}^*\in\mathcal{S}_0^*(\mathcal{O}_{0,q_{\bar{t}}}),$$

with $B_{\boldsymbol{u}}(\sigma,\boldsymbol{v})\in R$, and $B_{\boldsymbol{u}}(\sigma,\boldsymbol{v})\to 0$ as $w_{q_{\bar{t}}\Gamma}(\sigma)+w(\boldsymbol{v})\to\infty$ using (16). For $\omega^* = \sum_{\gamma\in M(\Gamma),\boldsymbol{u}\in\mathcal{S}(M)}\omega^*(\gamma,\boldsymbol{u})\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma}e_{\boldsymbol{u}}^*\in S_0^*(\mathcal{O}_0^*)$, we have

$$\begin{aligned}
\beta_{\kappa,\bar{t}}^*(\omega^*) &= \mathrm{pr}_{M(\Gamma)}\Bigg(\sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} \omega^*(\gamma,u)\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-q_{\bar{t}}\gamma}\cdot[\alpha_{ad(\bar{t})}^*]_\kappa(e_{\boldsymbol{u}}^*)\Bigg)\\
&= \mathrm{pr}_{M(\Gamma)}\Bigg(\sum_{\gamma\in M(\Gamma)}\lambda^{-q_{\bar{t}}\gamma}\sum_{\boldsymbol{u}\in\mathcal{S}(M)}\omega^*(\gamma,\boldsymbol{u})\sum_{\substack{\sigma\in M(\Gamma)\\ \boldsymbol{v}\in\mathcal{S}(M)}} B_{\boldsymbol{u}}(\sigma,\boldsymbol{v})\tilde{\pi}^{-w_{q_{\bar{t}}\Gamma}(\sigma)}\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^\sigma e_{\boldsymbol{v}}^*\Bigg)\\
&= \sum_{\substack{\tau\in M(\Gamma)\\ \boldsymbol{v}\in\mathcal{S}(M)}} C(\tau,\boldsymbol{v})\tilde{\pi}^{-w_\Gamma(\tau)}\lambda^{-\tau}e_{\boldsymbol{v}}^*,
\end{aligned}$$

where

$$C(\tau, \boldsymbol{v}) := \sum_{\boldsymbol{u}\in\mathcal{S}(M)} \sum_{\substack{\gamma,\sigma\in M(\Gamma)\\ q_{\bar{t}}\gamma-\sigma=\tau}} \omega^*(\gamma, \boldsymbol{u}) B_{\boldsymbol{u}}(\sigma, \boldsymbol{v}) \tilde{\pi}^{-w_\Gamma(\gamma)+w_{q_{\bar{t}}\Gamma}(\sigma)+w_\Gamma(\tau)}.$$

Observe that the exponent of $\tilde{\pi}$ satisfies

$$-w_\Gamma(\gamma) + w_{q_{\bar{t}}\Gamma}(\sigma) + w_\Gamma(\tau) \geq \left(1 - \tfrac{1}{q_{\bar{t}}}\right) w_\Gamma(\tau),$$

so that the term $\tilde{\pi}^{-w_\Gamma(\gamma)+w_{q_{\bar{t}}\Gamma}(\sigma)+w_\Gamma(\tau)}$ is bounded in norm by 1 since $w(\tau) \geq 0$, and $\omega^*(\gamma, \boldsymbol{u})$ and $B_{\boldsymbol{u}}(\sigma, v) \in R$. On the other hand, $B_{\boldsymbol{u}}(\sigma, \boldsymbol{v}) \to 0$ as $w_\Gamma(\sigma) + w(\boldsymbol{v}) \to \infty$ so that the coefficient $C(\tau, \boldsymbol{v})$ is defined, in R, and $\beta_\kappa^*(\omega^*) \in \mathcal{S}_0^*(\mathcal{O}_0^*)$. Clearly it is R-linear. □

Estimation using finite symmetric powers. It is useful to estimate $\beta_{\kappa,\bar{t}}$ and $\beta_{\kappa,\bar{t}}^*$ using finite symmetric powers. For monomials $e_{\boldsymbol{u}}$ or $e_{\boldsymbol{u}}^*$, with $\boldsymbol{u} \in \mathcal{S}(M)$, $\boldsymbol{u} = (u_1, \ldots, u_r) \in (M(f)\setminus 0)^r$, we say as usual that the degree or length of $e_{\boldsymbol{u}}$ or $e_{\boldsymbol{u}}^*$ is r. For $\xi \in \mathcal{S}_0(\mathcal{O}_0)$, define length($\xi$) as the supremum of the lengths of the monomials $e_{\boldsymbol{u}}$ in the support of ξ (i.e., those terms appearing with nonzero coefficients). In the case length(ξ) $= r$, we may write $\xi = \sum_{|\boldsymbol{u}|\leq r} \xi(\boldsymbol{u}) e_{\boldsymbol{u}}$, and ξ may be a series (not a polynomial), since $M(f)$ and the set of monomials of degree $\leq r$ are infinite in general. Similarly for $\xi_{\boldsymbol{u}}^*$.

Let k be a positive integer. Define $\mathcal{S}_0^{(k)}(\mathcal{O}_0) := \{\xi \in \mathcal{S}_0(\mathcal{O}_0) \mid \text{length}(\xi) \leq k\}$. Then the map

$$E_0^{k-r} E_{u_1} \cdots E_{u_r} \longmapsto e_{u_1} e_{u_2} \cdots e_{u_r}$$

identifies $\mathrm{Sym}^k\, \mathcal{C}_0(\mathcal{O}_0)$ with $\mathcal{S}_0^{(k)}(\mathcal{O}_0)$ as $\mathcal{O}_0$-submodules in $\mathcal{S}_0(\mathcal{O}_0)$. Similarly, we identify $\mathrm{Sym}^k\, \mathcal{C}_0^*(\mathcal{O}_0)$ in $\mathcal{S}_0^*(\mathcal{O}_0)$ as the $\mathcal{O}_0$-submodule $\mathcal{S}_0^{*(k)}(\mathcal{O}_0)$ of power series in $\{e_{\boldsymbol{u}}^* \mid |\boldsymbol{u}| \leq k\}$ with coefficients in $\mathcal{O}_0$. By transfer of structure, we have a pairing $(\cdot\,,\cdot)_k : \mathcal{S}_0^{(k)}(\mathcal{O}_0) \times \mathcal{S}_0^{*(k)}(\mathcal{O}_0) \to \mathcal{O}_0$.

We now work over R and define a new pairing $\langle\cdot\,,\cdot\rangle_k : \mathcal{S}_0^{(k)}(\mathcal{O}_0) \times \mathcal{S}_0^{*(k)}(\mathcal{O}_0^*) \to \Omega$ as follows. (Here again $\mathcal{S}_0^{*(k)}(\mathcal{O}_0^*)$ is the R-submodule of $\mathcal{S}_0^*(\mathcal{O}_0^*)$ of series with support in monomials of degree $\leq k$, namely $\{e_{\boldsymbol{u}}^* \mid |\boldsymbol{u}| \leq k\}$, with coefficients in $\mathcal{O}_0^*$.) Let

$$\xi := \sum_{\gamma\in M(\Gamma), \boldsymbol{u}\in\mathcal{S}(M)} \xi(\gamma, \boldsymbol{u}) \tilde{\pi}^{w_\Gamma(\gamma)} \lambda^\gamma e_{\boldsymbol{u}} \in \mathcal{S}_0^{(k)}(\mathcal{O}_0),$$

$$\xi^* := \sum_{\sigma\in M(\Gamma), \boldsymbol{v}\in\mathcal{S}(M)} \xi^*(\sigma, \boldsymbol{v}) \tilde{\pi}^{-w_\Gamma(\sigma)} \lambda^{-\sigma} e_{\boldsymbol{v}}^* \in \mathcal{S}_0^{*(k)}(\mathcal{O}_0^*),$$

and set

$$\langle \xi, \xi^* \rangle_k := \sum_{\substack{\gamma\in M(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M)}} \xi(\gamma, \boldsymbol{u}) \xi^*(\gamma, \boldsymbol{u}) (e_{\boldsymbol{u}}, e_{\boldsymbol{u}}^*)_k,$$

where $(\cdot\,,\cdot)_k$ was defined above. (Observe that as defined, a denominator $k!$ is introduced, so $(e_{\boldsymbol{u}}, e^*_{\boldsymbol{u}})_k$ is a rational number with p-adic valuation bounded below by $-k/(p-1)$. This is independent of $\boldsymbol{u}$, so $\langle \xi, \xi^* \rangle_k$ is well-defined and takes values in the R-submodule of Ω consisting of elements with $\operatorname{ord}_p c \geq -k/(p-1)$.) It is useful to think of $\langle \xi, \xi^* \rangle_k$ as the constant term with respect to λ and the $e_{\boldsymbol{u}}$ and $e^*_{\boldsymbol{u}}$ of the product $\xi \cdot \xi^*$, where the product $e_{\boldsymbol{u}} \cdot e^*_{\boldsymbol{v}}$ is defined to be zero if $\boldsymbol{u} \neq \boldsymbol{v}$ and $(e_{\boldsymbol{u}}, e^*_{\boldsymbol{u}})_k$ if $\boldsymbol{u} = \boldsymbol{v}$.

Let k_m be a sequence of positive integers which tend to infinity (in the usual archimedean sense) and such that $\lim_{m\to\infty} k_m = \kappa$ p-adically. For each m we have a Frobenius map $\operatorname{Sym}^{k_m}(\alpha_{ad(\bar{t})})$ on $\operatorname{Sym}^{k_m} \mathcal{C}_o(\mathcal{O}_0)$, as well as a Frobenius map $\operatorname{Sym}^{k_m}(\alpha^*_{ad(\bar{t})})$ on $\operatorname{Sym}^{k_m} \mathcal{C}^*_0(\mathcal{O}_{0,q_{\bar{t}}})$. By transport of structure, we have then a Frobenius map $[\alpha_{ad(\bar{t})}]_{(\kappa;m)}$ on $\mathcal{S}^{(k_m)}_0(\mathcal{O}_0)$ and a dual Frobenius $[\alpha^*_{ad(\bar{t})}]_{(\kappa;m)}$ on $\mathcal{S}^{*(k_m)}_0(\mathcal{O}_{0,q_{\bar{t}}})$. We extend by zero these maps to all of $\mathcal{S}_0(\mathcal{O}_0)$ and $\mathcal{S}^*_0(\mathcal{O}_{0,q_{\bar{t}}})$, respectively. That is, we define

$$[\alpha_{ad(\bar{t})}]_{(\kappa;m)}(e_{\boldsymbol{u}}) := \begin{cases} [\alpha_{ad(\bar{t})}]_{k_m}(e_{\boldsymbol{u}}) & \text{if } |\boldsymbol{u}| \leq k_m, \\ 0 & \text{otherwise.} \end{cases}$$

To avoid any possible confusion, we note

$$\begin{aligned} &[\alpha_{ad(\bar{t})}]_{(\kappa;m)}(e_{u_1} \cdots e_{u_r}) \\ &\qquad = (\Upsilon \circ \alpha_{ad(\bar{t})}(1))^{k_m - r} (\Upsilon \circ \alpha_{ad(\bar{t})} \tilde{\pi}^{w(u_1)} x^{u_1}) \cdots (\Upsilon \circ \alpha_{ad(\bar{t})} \tilde{\pi}^{w(u_r)} x^{u_r}) \\ &\qquad \cong (\operatorname{Sym}^{k_m} \alpha_{ad(\bar{t})})(E_0^{k_m - r} E_{u_1} \cdots E_{u_r}), \end{aligned}$$

when $r \leq k_m$. Similarly

$$[\alpha^*_{ad(\bar{t})}]_{(\kappa;m)}(e^*_{\boldsymbol{u}}) := \begin{cases} [\alpha^*_{ad(\bar{t})}]_{k_m}(e^*_{\boldsymbol{u}}) & \text{if } |\boldsymbol{u}| \leq k_m, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3. $\lim_{m\to\infty} [\alpha_{ad(\bar{t})}]_{(\kappa;m)} = [\alpha_{ad(\bar{t})}]_\kappa$ *as maps from* $\mathcal{S}_0(\mathcal{O}_0) \to \mathcal{S}_0(\mathcal{O}_{0,q_{\bar{t}}})$.

Proof. Write

$$\begin{aligned} (18)\quad &([\alpha_{ad(\bar{t})}]_{(\kappa;m)} - [\alpha_{ad(\bar{t})}]_\kappa)(e_{u_1} e_{u_2} \cdots e_{u_r}) \\ &\qquad = \big(\Upsilon(\alpha_{ad(\bar{t})}(1))^{k_m - r} - \Upsilon(\alpha_{ad(\bar{t})}(1))^{\kappa - r}\big) \\ &\qquad\qquad \times (\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_1)} x^{u_1}))) \cdots (\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_r)} x^{u_r}))). \end{aligned}$$

If $r \leq k_m$, then the first factor on the right may itself be factored into

$$-\Upsilon(\alpha_{ad(\bar{t})}(1))^{\kappa - r}(1 - (\Upsilon(\alpha_{ad(\bar{t})}(1))^{k_m - \kappa})).$$

If $\kappa = k_m + p^{\tau(m)} \sigma_m$ (with $\tau(m) \to \infty$ and $\sigma_m \in \mathbb{Z}_p$) then

$$\big|1 - (\Upsilon(\alpha_{ad(\bar{t})}(1))^{k_m - \kappa}\big| \leq \big|\tilde{\pi}^{\tau(m)+1}\big|,$$

as in the proof of [Haessig 2014, Lemma 2.2], and using the estimate (9). If $r > k_m$ then (18) becomes

$$\begin{aligned}\big([\alpha_{ad(\bar{t})}]_{(\kappa;m)} - [\alpha_{ad(\bar{t})}]_\kappa\big)(e_{\boldsymbol{u}}) \\ &= -[\alpha_{ad(\bar{t})}]_\kappa e_{\boldsymbol{u}} \\ &= -\Upsilon(\alpha_{ad(\bar{t})}(1))^{\kappa-r}(\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_1)}x^{u_1})))\cdots(\Upsilon(\alpha_{ad(\bar{t})}(\tilde{\pi}^{w(u_r)}x^{u_r}))).\end{aligned}$$

Applying (9) to the r rightmost factors we see that

$$\big|\big([\alpha_{ad(\bar{t})}]_{(\kappa;m)} - [\alpha_{ad(\bar{t})}]_\kappa\big)e_{\boldsymbol{u}}\big| \leq |\tilde{\pi}|^{w(\boldsymbol{u})(p^{ad(\bar{t})-1}-1)/p^{ad(\bar{t})-1}}.$$

But $w(\boldsymbol{u}) \geq r w_0 > k_m w_0$ (where $w_0 := \min\{w(\boldsymbol{u}) \mid \boldsymbol{u} \in M(f) \setminus \{0\}\}$). In terms of the operator norm,

$$\|[\alpha_{ad(\bar{t})}]_\kappa - [\alpha_{ad(\bar{t})}]_{(\kappa;m)}\| \leq |\tilde{\pi}|^{\min\{\tau(m)+1,\, k_m w_0(p^{ad(\bar{t})-1}-1)/p^{ad(\bar{t})-1}\}}.$$

As k_m and $\tau(m)$ both tend to infinity as m grows, we see that

$$\lim_{m\to\infty}[\alpha_{ad(\bar{t})}]_{(\kappa;m)} = [\alpha_{ad(\bar{t})}]_\kappa. \qquad \square$$

In an altogether similar manner, for $u \neq 0$ we have, by Lemma 4.1, that $\alpha^*_m(\tilde{\pi}^{-w(u)}x^{-u})$ belongs to $\mathcal{C}^*_0(\mathcal{O}_{0,p^m})$, and (recalling (11))

$$|\alpha^*_m(\tilde{\pi}^{-w(u)}x^{-u})| \leq |\tilde{\pi}|^{(p-1)w(u)}.$$

Also $\alpha^*_m(1) = 1 + \eta^*(\lambda)$ with $\eta^*(\lambda) \in \mathcal{O}_{0,p^m}$ and $|\eta^*(\lambda)| \leq |\tilde{\pi}|$. With these observations, an entirely similar argument shows $\lim_{m\to\infty}[\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} = [\alpha^*_{ad(\bar{t})}]_\kappa$ as maps from $\mathcal{S}^*_0(\mathcal{O}_{0,q_{\bar{t}}}) \to \mathcal{S}^*_0(\mathcal{O}_{0,q_{\bar{t}}})$. Define

$$\begin{aligned}\beta_{(\kappa;m),\bar{t}} &:= \psi_\lambda^{ad(\bar{t})} \circ [\alpha_{ad(\bar{t})}]_{(\kappa;m)}, \\ \beta^*_{(\kappa;m),\bar{t}} &:= \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} \circ \Phi_\lambda^{ad(\bar{t})}.\end{aligned}$$

As ψ_λ and Φ_λ are bounded maps, it follows that as operators on $\mathcal{S}_0(\mathcal{O}_0)$ and $\mathcal{S}^*_0(\mathcal{O}^*_0)$, respectively,

$$\lim_{m\to\infty}\beta_{(\kappa;m),\bar{t}} = \beta_{\kappa,\bar{t}} \quad \text{and} \quad \lim_{m\to\infty}\beta^*_{(\kappa;m),\bar{t}} = \beta^*_{\kappa,\bar{t}}. \tag{19}$$

Lemma 4.4. *For $\xi \in \mathcal{S}_0^{(k_m)}(\mathcal{O}_0)$ and $\xi^* \in \mathcal{S}_0^{*(k_m)}(\mathcal{O}^*_0)$,*

$$\langle \beta_{(\kappa;m),\bar{t}}\xi, \xi^*\rangle_{k_m} = \langle \xi, \beta^*_{(\kappa;m),\bar{t}}\xi^*\rangle_{k_m}. \tag{20}$$

Proof. With $\xi \in \mathcal{S}_0^{(k_m)}(\mathcal{O}_0)$ and $\xi^* \in \mathcal{S}_0^{*(k_m)}(\mathcal{O}_{0,q_{\bar{t}}})$, we may rewrite (14) as

$$([\alpha_{ad(\bar{t})}]_{(\kappa,m)}\xi, \xi^*)_{k_m} = (\xi, [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)}\xi^*)_{k_m}. \tag{21}$$

By linearity we only need consider $\xi = \lambda^{\gamma} e_{\boldsymbol{u}}$ and $\xi^* = \lambda^{-\sigma} e_{\boldsymbol{v}}^*$ where $\gamma, \sigma \in M(\Gamma)$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{S}(M)$. We may write

$$(e_{\boldsymbol{u}}, [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} e^*_{\boldsymbol{v}})_{k_m} = \sum_{\tau \in M(\Gamma)} C(\tau)\lambda^{\tau}.$$

Next, observe that

$$\langle \psi_\lambda \xi, \xi^* \rangle_{k_m} = \langle \xi, \Phi_\lambda \xi^* \rangle_{k_m}.$$

Hence, in the case $\xi = \lambda^{\gamma} e_{\boldsymbol{u}}$ and $\xi^* = \lambda^{-\sigma} e^*_{\boldsymbol{v}}$,

$$\begin{aligned}
\langle \beta_{(\kappa;m)}\xi, \xi^* \rangle_{k_m} &= \langle [\alpha_{ad(\bar{t})}]_{(\kappa;m)}\xi, \Phi_\lambda^{ad(\bar{t})}\xi^* \rangle_{k_m} \\
&= \text{the constant term of } [\lambda^{\gamma - q_{\bar{t}}\sigma}([\alpha_{ad(\bar{t})}]_{(\kappa;m)} e_{\boldsymbol{u}}, e^*_{\boldsymbol{v}})_{k_m}] \\
&= \text{the constant term of } [\lambda^{\gamma - q_{\bar{t}}\sigma}(e_{\boldsymbol{u}}, [\alpha^*_{ad(\bar{t})}]_{(\kappa;m)} e^*_{\boldsymbol{v}})_{k_m}] \quad \text{(by (21))} \\
&= \text{the constant term of } \left[\lambda^{\gamma - q_{\bar{t}}\sigma} \sum_{\tau \in M(\Gamma)} C(\tau)\lambda^{\tau}\right] \\
&= \begin{cases} C(q_{\bar{t}}\sigma - \gamma) & \text{if } q_{\bar{t}}\sigma - \gamma \in M(\Gamma), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In the other direction, again setting $\xi = \lambda^{\gamma} e_{\boldsymbol{u}}$ and $\xi^* = \lambda^{-\sigma} e^*_{\boldsymbol{v}}$,

$$\begin{aligned}
\langle \xi, \beta^*_{(\kappa;m)}\xi^* \rangle_{k_m} &= \text{the constant term of } [\lambda^{\gamma} \cdot \mathrm{pr}_{M(\Gamma)}(\lambda^{-q_{\bar{t}}\sigma}(e_{\boldsymbol{u}}, [\alpha^*_{ad(\bar{t}}]_{(\kappa;m)} e^*_{\boldsymbol{v}})_{k_m}))] \\
&= \text{the constant term of } \left[\lambda^{\gamma} \cdot \mathrm{pr}_{M(\Gamma)}\left(\sum_{\tau \in M(\Gamma)} C(\tau)\lambda^{-(q_{\bar{t}}\sigma - \tau)}\right)\right] \\
&= \text{the constant term of } \left[\lambda^{\gamma} \cdot \sum_{\substack{\tau \in M(\Gamma) \text{ such that} \\ q_{\bar{t}}\sigma - \tau \in M(\Gamma)}} C(\tau)\lambda^{-(q_{\bar{t}}\sigma - \tau)}\right] \\
&= \begin{cases} C(q_{\bar{t}}\sigma - \gamma) & \text{if } q_{\bar{t}}\sigma - \gamma \in M(\Gamma), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Observe that $\beta_{(\kappa;m),\bar{t}}$ and $\beta_{\kappa,\bar{t}}$ are completely continuous operators on the p-adic Banach R-algebra $\mathcal{S}_0(\mathcal{O}_0)$ (viewed as R-algebra) with orthonormal basis $\{\tilde{\pi}^{w_\Gamma(\gamma)}\lambda^{\gamma} e_{\boldsymbol{u}} \mid \gamma \in M(\Gamma), \boldsymbol{u} \in \mathcal{S}(M)\}$. Let $\mathcal{T}_0(R)$ be $\mathcal{S}_0(\mathcal{O}_0)$ viewed in this way as an R-algebra. Similarly, write $\mathcal{T}_0^*(R)$ for the $b(I)$-space (over R) in Serre's terminology with "basis" $I := \{\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma} e^*_{\boldsymbol{u}} \mid \gamma \in M(\Gamma), \boldsymbol{u} \in \mathcal{S}(M)\}$ with coefficients in R. Again, $\mathcal{T}_0^*(R)$ is just $\mathcal{S}_0^*(\mathcal{O}_0^*)$ viewed over R. Then

$$\lim_{m \to \infty} \det(1 - \beta_{(\kappa;m),\bar{t}}T) = \det(1 - \beta_{\kappa,\bar{t}}T).$$

Similarly, $\beta^*_{(\kappa;m),\bar{t}}$ is a continuous R-linear endomorphism of $\mathcal{T}_0^*(R)$ to itself. We may consider a matrix $\mathfrak{B}^{*(\kappa;m),\bar{t}}$ with entries in R defined by

$$\beta^*_{(\kappa;m),\bar{t}}(\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma}e^*_{\boldsymbol{u}}) = \sum \mathfrak{B}^{*(\kappa;m),\bar{t}}_{(\delta,\boldsymbol{v}),(\gamma,\boldsymbol{u})}\tilde{\pi}^{-w_\Gamma(\delta)}\lambda^{-\delta}e^*_{\boldsymbol{v}}.$$

Using the matrix $\mathfrak{B}^{*(\kappa;m),\bar{t}}$, we define in the usual way the Fredholm determinant $\det(1-\beta^*_{(\kappa;m),\bar{t}}T) = \sum_{j\geq 0}(-1)^{j+1}C_j(\beta^*_{(\kappa;m),\bar{t}})T^j$ where $C_0 = 1$ and C_j is the series of all principal $j \times j$ subdeterminants of the matrix $\mathfrak{B}^{*(\kappa;m),\bar{t}}$. The $\langle\cdot\,,\cdot\rangle_{k_m}$-adjointness of $\beta_{(\kappa;m),\bar{t}}$ and $\beta^*_{(\kappa;m),\bar{t}}$ implies $C_j(\beta_{(\kappa;m),\bar{t}}) = C_j(\beta^*_{(\kappa;m),\bar{t}})$, so that

$$\det(1-\beta^*_{(\kappa;m),\bar{t}}T) = \det(1-\beta_{(\kappa;m),\bar{t}}T).$$

The uniform convergence $\lim_{m\to\infty}\mathfrak{B}^{*(\kappa;m),\bar{t}} =: \mathfrak{B}^*_{\kappa,\bar{t}}$ over the entries implies that the series $\sum_{j\geq 0}(-1)^{j+1}C_j(\mathfrak{B}^*_{\kappa,\bar{t}})T^j$ is well-defined, and is the coefficient-wise limit of $\det(1-\mathfrak{B}^*_{(\kappa;m),\bar{t}}T)$ as $m\to\infty$. If we define

$$\det(1-\beta^*_{\kappa,\bar{t}}T) := \sum_{j\geq 0}(-1)^{j+1}C_j(\mathfrak{B}^*_{\kappa,\bar{t}})T^j,$$

then we have shown:

Theorem 4.5. $\det(1-\beta_{\kappa,\bar{t}}T) = \det(1-\beta^*_{\kappa,\bar{t}}T)$, *and thus from* (10),

$$L^{(0)}(\kappa,\bar{t},T)^{(-1)^{s+1}} = \det(1-\beta^*_{\kappa,\bar{t}}T)^{\delta^s_{q_{\bar{t}}}}. \tag{22}$$

5. Eigenvector

Recall that

$$G(t,\lambda,x) = f(t,x) + P(\lambda,x) = \sum t_u x^u + \sum A(\gamma,v)\lambda^\gamma x^v$$

in $\mathbb{F}_q[x_1^\pm,\ldots,x_n^\pm,\lambda_1^\pm,\ldots,\lambda_s^\pm,\{t_u\}_{u\in\mathrm{supp}(f)}]$. Let $\hat{A}(\gamma,v)$ be the Teichmüller lift in $\mathbb{Q}_q$ for each $(\gamma,v)\in\mathrm{supp}(P)$, and denote the lifting of G by

$$\hat{G}(t,\lambda,x) := \hat{f}(t,x) + \hat{P}(\lambda,x) = \sum t_u x^u + \sum \hat{A}(\gamma,v)\lambda^\gamma x^v$$

in $\mathbb{Q}_q[x_1^\pm,\ldots,x_n^\pm,\lambda_1^\pm,\ldots,\lambda_s^\pm,\{t_u\}_{u\in\mathrm{supp}(f)}]$. We now replace every coefficient of G (with respect to the variables x and λ) with a new variable Λ:

$$\boldsymbol{f}(\Lambda,x) = \sum_{u\in\mathrm{supp}(f)}\Lambda_u x^u,$$
$$\mathcal{P}(\Lambda,\lambda,x) = \sum_{(\gamma,v)\in\mathrm{supp}(P)}\Lambda_{\gamma,v}\lambda^\gamma x^v,$$
$$H(\Lambda,\lambda,x) := \boldsymbol{f}(\Lambda,x) + \mathcal{P}(\Lambda,\lambda,x).$$

As before, let $\Delta_\infty(H)$ denote the Newton polytope of H at infinity in $\mathbb{R}^{s+n}$ (in λ and x variables). Let $\mathrm{Cone}(H)$ be the cone in $\mathbb{R}^{s+n}$ over $\Delta_\infty(H)$ and $M(H) = \mathrm{Cone}(H) \cap \mathbb{Z}^{s+n}$ be the relevant monoid. Clearly $M(H) \subset M(\Gamma) \times M(f)$. By our hypothesis that the x-support of P is contained in $\Delta_\infty(f)$ we have that the polyhedral weight function on this polytope w_H dominates the total weight $w_\Gamma + w$ relative to the polyhedron $\Gamma \times \Delta_\infty$; more precisely

$$w_\Gamma(\gamma) + w(u) \le w_H(\gamma, u),$$

for all $(\gamma, u) \in M(H)$.

The following definitions extend those in Section 4, by replacing R with the (multivariable) formal power series ring $\mathcal{K} := R[\![\Lambda]\!]$. We equip $\mathcal{K}$ with the sup-norm. Denote by $\mathcal{K}_0$ the subring of $\mathcal{K}$ of power series which converge on the closed unit polydisk $|\Lambda| \le 1$. For q any power of the prime p, define

$$\mathcal{O}_{0,q}(\mathcal{K}) := \left\{ \sum_{\gamma \in M(\Gamma)} C(\gamma)\lambda^\gamma \tilde{\pi}^{w_{q\Gamma}(\gamma)} \,\middle|\, C(\gamma) \in \mathcal{K}, C(\gamma) \to 0 \text{ as } \gamma \to \infty \right\},$$

$$\mathcal{C}_0^*(\mathcal{O}_{0,q}(\mathcal{K})) := \left\{ \sum_{u \in M(f)} \xi(u)\tilde{\pi}^{-w(u)} x^{-u} \,\middle|\, \xi(u) \in \mathcal{O}_{0,q}(\mathcal{K}) \right\},$$

$$\mathcal{S}_0^*(\mathcal{O}_{0,q}(\mathcal{K})) := \left\{ \sum_{\boldsymbol{u} \in \mathcal{S}(M)} \xi^*(\boldsymbol{u}) e_{\boldsymbol{u}}^* \,\middle|\, \xi^*(\boldsymbol{u}) \in \mathcal{O}_{0,q}(\mathcal{K}) \right\},$$

$$\mathcal{O}_{0,q}^*(\mathcal{K}) := \left\{ \sum_{\gamma \in M(\Gamma)} \zeta^*(\gamma)\tilde{\pi}^{-w_{q\Gamma}(\gamma)} \lambda^{-\gamma} \,\middle|\, \zeta^*(\gamma) \in \mathcal{K} \right\},$$

$$\mathcal{S}_0^*(\mathcal{O}_{0,q}^*(\mathcal{K})) := \left\{ \sum_{\substack{\gamma \in M(\Gamma) \\ \boldsymbol{u} \in \mathcal{S}(M)}} \omega^*(\gamma, \boldsymbol{u})\tilde{\pi}^{-w_{q\Gamma}(\gamma)} \lambda^{-\gamma} e_{\boldsymbol{u}}^* \,\middle|\, \omega^*(\gamma, \boldsymbol{u}) \in \mathcal{K} \right\}.$$

In all cases, the spaces above have versions (with obvious modification of notation), where the ring of coefficients $\mathcal{K}$ is replaced by the subring $\mathcal{K}_0$. Define the maps

$$\mathrm{pr}_{M(f)} : \sum_{u \in \mathbb{Z}^n} C(u) x^{-u} \longmapsto \sum_{u \in M(f)} C(u) x^{-u},$$

and $\Upsilon : \mathcal{C}_0^*(\mathcal{O}_0(\mathcal{K})) \hookrightarrow \mathcal{S}_0^*(\mathcal{O}_0(\mathcal{K}))$ by $\tilde{\pi}^{-w(u)} x^{-u} \mapsto e_u^*$ for $u \in M \backslash \{0\}$ and $\Upsilon(1) := 1$. Next, define a relative Frobenius map as follows. First, set

$$F(\Lambda, \lambda, x) := \prod_{u \in \mathrm{supp}(f)} \theta(\Lambda_u x^u) \cdot \prod_{(\gamma, v) \in \mathrm{supp}(P)} \theta(\Lambda_{\gamma,v} \lambda^\gamma x^v),$$

$$F_m(\Lambda, \lambda, x) := \prod_{i=0}^{m-1} F(\Lambda^{p^i}, \lambda^{p^i}, x^{p^i}),$$

and note that, similar to before,

$$F_m(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M(H)} B_{\gamma,u}(\Lambda)\tilde{\pi}^{w_H(\gamma,u)/p^{m-1}}\lambda^\gamma x^u,$$

with $|B_{\gamma,u}(\Lambda)| \leq 1$. It follows that, if we set

$$\alpha^*_{m,\Lambda} := \mathrm{pr}_{M(f)} \circ F_m(\Lambda, \lambda, x) \circ \Phi^m_x,$$

where Φ_x sends $x^u \mapsto x^{pu}$, then an argument similar to Lemma 4.1 shows

$$\alpha^*_{m,\Lambda} : \mathcal{C}^*_0(\mathcal{O}_{0,p^m}(\mathcal{K})) \to \mathcal{C}^*_0(\mathcal{O}_{0,p^m}(\mathcal{K})).$$

For any $\kappa \in \mathbb{Z}_p$, we define $[\alpha^*_{m,\Lambda}]_\kappa : \mathcal{S}^*_0(\mathcal{O}_{0,p^m}(\mathcal{K})) \to \mathcal{S}^*_0(\mathcal{O}_{0,p^m}(\mathcal{K}))$ using (15). By an argument similar to Lemma 4.2, the map

$$\beta^*_{\kappa,\bar{t},\Lambda} : \mathcal{S}^*_0(\mathcal{O}^*_0(\mathcal{K})) \to \mathcal{S}^*_0(\mathcal{O}^*_0(\mathcal{K}))$$

defined by

$$\beta^*_{\kappa,\bar{t},\Lambda} := \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \circ \Phi^{ad(\bar{t})}_\lambda,$$

is an endomorphism over $\mathcal{K}$.

Eigenvector. In the following, we will define an eigenvector $\Upsilon(\eta)^\kappa$ of $\beta^*_{\kappa,\bar{t},\Lambda}$ whose eigenvalue is $\mathcal{F}_{ad(\bar{t})}(\Lambda)^\kappa$. We will then specialize Λ, proving Theorem 1.1. We start by defining the groups

$$M_0(\Gamma) = M(\Gamma) \cap (-M(\Gamma)) \quad \text{and} \quad M_0(f) = M(f) \cap (-M(f)).$$

Define the projection map

$$\mathrm{pr}_0 : \sum_{\substack{\gamma\in\mathbb{Z}^s \\ u\in\mathbb{Z}^n}} C(\gamma, u)\lambda^\gamma x^u \longmapsto \sum_{\substack{\gamma\in M_0(\Gamma) \\ u\in M_0(f)}} C(\gamma, u)\lambda^\gamma x^u,$$

and write

$$\mathrm{pr}_0 \circ \exp \pi H(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M_0(\Gamma)\times M_0(f)} J_{\gamma,u}(\Lambda)\lambda^\gamma x^u, \tag{23}$$

with $J_{\gamma,u} \in R[\![\Lambda]\!]$. Observe that $J_{0,0} \in 1 + \Lambda R[\![\Lambda]\!]$, and so we may define

$$\eta(\Lambda, \lambda, x) := \tfrac{1}{J_{0,0}(\Lambda)} \mathrm{pr}_0(\exp \pi H(\Lambda, \lambda, x)).$$

We will eventually need to specialize Λ to Teichmüller units. The following lemma demonstrates that this is possible.

Lemma 5.1. *$J_{\gamma,u}(\Lambda)/J_{0,0}(\Lambda) \in \mathcal{K}_0$ for each $(\gamma, u) \in M_0(\Gamma) \times M_0(f)$. Also, $J_{0,0}(\Lambda)/J_{0,0}(\Lambda^p) \in \mathcal{K}_0$.*

Proof. This result is essentially a version of the main result, Proposition 2.15 and its corollaries, in [Adolphson and Sperber 2012]. The proof of the version here necessitates only some minor modifications from that in the above reference. The key difference is that the setup here uses total weight, $w_{\text{tot}} = w_\Gamma + w$ based on $\Gamma \times \Delta_\infty(f)$ rather than the straightforward polyhedral weight w_H based on $\Delta_\infty(H)$. The argument of [Adolphson and Sperber 2012] works here as well. □

Next, we will show that the $\Upsilon(\eta)^\kappa$ is a well-defined element of our dual space.

Lemma 5.2. $\Upsilon(\eta(\Lambda, \lambda, x))^\kappa \in S_0^*(\mathcal{O}_0^*(\mathcal{K}_0))$.

Proof. First, write

$$\eta(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M_0(\Gamma)\times M_0(f)} C_{\gamma,u}(\Lambda)\lambda^\gamma x^u,$$

with $|C_{\gamma,u}| \le 1$ and $C_{\gamma,u} \in \mathcal{K}_0$. Since $u \in M_0(f)$, we may write

$$\eta(\Lambda, \lambda, x) = \sum_{(\gamma,u)\in M_0(\Gamma)\times M_0(f)} (C_{\gamma,-u}(\Lambda)\tilde{\pi}^{w(u)})\lambda^\gamma \tilde{\pi}^{-w(u)} x^{-u},$$

and so $\Upsilon(\eta(\Lambda, \lambda, x)) = \sum_{\gamma,u} \tilde{C}_{\gamma,u}(\Lambda)\lambda^\gamma e_u^*$, with $\tilde{C}_{\gamma,u}(\Lambda) := C_{\gamma,-u}(\Lambda)\tilde{\pi}^{w(u)}$. Next, since $\tilde{C}_{0,0}$ is a unit, we may write

$$\begin{aligned}\Upsilon(\eta)^\kappa &= \Bigl(\tilde{C}_{0,0}(\Lambda) + \sum_{(\gamma,u)\in M\setminus\{0\}} \tilde{C}_{\gamma,u}(\Lambda)\lambda^\gamma e_u^*\Bigr)^\kappa \\ &= \sum_{l=0}^{\infty} \binom{\kappa}{l} \tilde{C}_{0,0}(\Lambda)^{\kappa-l} \Bigl(\sum_{(\gamma,u)\in M\setminus\{0\}} \tilde{C}_{\gamma,u}(\Lambda)\lambda^\gamma e_u^*\Bigr)^l \\ &= \sum_{\substack{\gamma\in M_0(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M_0(f))}} D_{\gamma,\boldsymbol{u}}(\Lambda)\lambda^\gamma e_{\boldsymbol{u}}^*,\end{aligned}$$

with $D_{\gamma,\boldsymbol{u}} \in \mathcal{K}_0$. Lastly, since $\gamma \in M_0(\Gamma)$, we may rewrite this as

$$\Upsilon(\eta)^\kappa = \sum_{\substack{\gamma\in M_0(\Gamma)\\ \boldsymbol{u}\in\mathcal{S}(M_0(f))}} (D_{-\gamma,\boldsymbol{u}}(\Lambda)\tilde{\pi}^{w_\Gamma(\gamma)})\tilde{\pi}^{-w_\Gamma(\gamma)}\lambda^{-\gamma} e_{\boldsymbol{u}}^*,$$

and thus $\Upsilon(\eta)^\kappa \in S_0^*(\mathcal{O}_0^*(\mathcal{K}_0))$. □

We now consider the action of $\alpha^*_{1,\Lambda}$ on η. Set $\mathcal{F}(\Lambda) := J_{0,0}(\Lambda)/J_{0,0}(\Lambda^p)$. Observe that

$$\begin{aligned}
&\alpha^*_{1,\Lambda}(\eta(\Lambda^p, \lambda^p, x)) \\
&= \mathrm{pr}_{M(f)}\big(F(\Lambda, \lambda, x)\,\mathrm{pr}_0\big(\exp \pi H(\Lambda^p, \lambda^p, x^p)/J_{0,0}(\Lambda^p)\big)\big) \\
&= \mathrm{pr}_{M(f)}\big(F(\Lambda, \lambda, x)\big(\exp \pi H(\Lambda^p, \lambda^p, x^p)/J_{0,0}(\Lambda^p) + \hat{\omega}(\Lambda, \lambda, x) + \epsilon(\Lambda.\lambda, x)\big)\big) \\
&= \mathcal{F}(\Lambda)\big(\mathrm{pr}_{M(f)}\big(\exp \pi H(\Lambda, \lambda, x)/J_{0,0}(\Lambda) + \omega^*(\Lambda, \lambda, x)\big)\big) \\
&= \mathcal{F}(\Lambda)\big(\eta(\Lambda, \lambda, x) + \tilde{\omega}(\Lambda, \lambda, x)\big),
\end{aligned}$$

where each $\lambda^\gamma x^u$ appearing in $\hat{\omega}$ (and ω^* and $\tilde{\omega}$) has γ in $M(\Gamma) \setminus M_0(\Gamma)$, and every $\lambda^\gamma x^u$ appearing in ϵ has u in $M(f) \setminus M_0(f)$. Iterating this, if we set

$$\mathcal{F}_m(\Lambda) := \prod_{i=0}^{m-1} \mathcal{F}(\Lambda^{p^i}),$$

then we have

$$\alpha^*_{ad(\bar{t}),\Lambda}\eta(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x) = \mathcal{F}_{ad(\bar{t})}(\Lambda)(\eta(\Lambda, \lambda, x) + \omega(\Lambda, \lambda, x)), \tag{24}$$

where each λ^γ appearing in ω lies in $M(\Gamma) \setminus M_0(\Gamma)$.

For the calculation of the eigenvalue, we will need the following. First, as every λ^γ appearing in $\Upsilon(\omega)$ (from Equation (24)) satisfies $\gamma \in M(\Gamma) \setminus M_0(\Gamma)$, it follows that the same is true for $\Upsilon(\eta)^{\kappa-r}\Upsilon(\omega)^r$ for every $r \in \mathbb{Z}_{\geq 1}$. Hence,

$$\mathrm{pr}_{M(\Gamma)}(\Upsilon(\eta) + \Upsilon(\omega))^\kappa = \mathrm{pr}_{M(\Gamma)} \sum_{r=0}^{\infty} \tbinom{\kappa}{r} \Upsilon(\eta)^{\kappa-r}\Upsilon(\omega)^r = \Upsilon(\eta)^\kappa. \tag{25}$$

We may now finish the proof of Theorem 1.1. For convenience, write $\eta(\Lambda, \lambda, x) = 1 + h(\Lambda, \lambda, x)$ so that $\Upsilon(\eta)^\kappa = (1 + \Upsilon(h))^\kappa = \sum_{l=0}^{\infty} \binom{\kappa}{l} \Upsilon(h)^l$. Observe that

$$\begin{aligned}
&\beta^*_{\kappa,\bar{t},\Lambda}\Upsilon(\eta(\Lambda^{q_{\bar{t}}}, \lambda, x))^\kappa \\
&\qquad= \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \circ \Phi_\lambda^{ad(\bar{t})} \Upsilon(\eta(\Lambda^{q_{\bar{t}}}, \lambda, x))^\kappa \\
&\qquad= \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \Upsilon(\eta(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^\kappa \\
&\qquad= \mathrm{pr}_{M(\Gamma)} \circ [\alpha^*_{ad(\bar{t}),\Lambda}]_\kappa \sum_{l=0}^{\infty} \binom{\kappa}{l} \Upsilon(h(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^l \\
&\qquad= \mathrm{pr}_{M(\Gamma)} \sum_{l=0}^{\infty} \binom{\kappa}{l} (\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} \cdot 1)^{\kappa-l} (\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} h(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^l \qquad \text{(by (15))} \\
&\qquad= \mathrm{pr}_{M(\Gamma)}\big(\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} \cdot 1 + \Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} h(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x)\big)^\kappa \\
&\qquad= \mathrm{pr}_{M(\Gamma)}(\Upsilon \circ \alpha^*_{ad(\bar{t}),\Lambda} \eta(\Lambda^{q_{\bar{t}}}, \lambda^{q_{\bar{t}}}, x))^\kappa
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{pr}_{M(\Gamma)}\, \mathcal{F}_{ad(\bar{t})}(\Lambda)^{\kappa} (\Upsilon(\eta(\Lambda,\lambda,x)) + \Upsilon(\omega(\Lambda,\lambda,x)))^{\kappa} \quad \text{(by (24))} \\
&= \mathrm{pr}_{M(\Gamma)}\, \mathcal{F}_{ad(\bar{t})}(\Lambda)^{\kappa} \Upsilon(\eta(\Lambda,\lambda,x))^{\kappa} \Big(1 + \frac{\Upsilon(\omega(\Lambda,\lambda,x))}{\Upsilon(\eta(\Lambda,\lambda,x))}\Big)^{\kappa} \\
&= \mathcal{F}_{ad(\bar{t})}(\Lambda)^{\kappa} \Upsilon(\eta(\Lambda,\lambda,x))^{\kappa} \quad \text{(by (25))}.
\end{aligned}
$$

Finally, we may specialize this equality by taking Λ at the Teichmüller unit coefficients of $\hat{G}(\hat{t},\lambda,x)$,

$$\Lambda_u = \hat{t}_u \quad \text{and} \quad \Lambda_{\gamma,v} = \hat{A}(\gamma, v)$$

for all u and γ, v in the support of H. Setting

$$\eta_{\mathrm{sp}}(\lambda,x) := \big(\eta(\Lambda,\lambda,x) \text{ specialized at } \Lambda_u = \hat{t}_u \text{ and } \Lambda_{\gamma,v} = \hat{A}(\gamma,v)\big),$$

we see that

$$\beta^{*}_{\kappa,\bar{t}} \Upsilon(\eta_{\mathrm{sp}}(\lambda,x))^{\kappa} = \mathcal{F}_{ad(\bar{t})}(\hat{t})^{\kappa} \Upsilon(\eta_{\mathrm{sp}}(\lambda,x))^{\kappa}. \tag{26}$$

This demonstrates that $\mathcal{F}_{ad(\bar{t})}(\hat{t})^{\kappa}$ is the unique unit root of $L^{(0)}(\kappa,\bar{t},T)^{(-1)^{s+1}}$ by (22), which, together with Theorem 3.1, completes the proof of Theorem 1.1.

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