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Bavard proved a duality theorem between commutator length and quasimorphisms. Burago, Ivanov and Polterovich introduced the notion of a conjugation-invariant norm which is a generalization of commutator length. Entov and Polterovich proved Oh–Schwarz spectral invariants are subset-controlled quasimorphisms, which are generalizations of quasimorphisms. We prove a Bavard-type duality theorem between subset-controlled quasimorphisms on stable groups and conjugation-invariant (pseudo)norms. We also pose a generalization of our main theorem and prove "stably nondisplaceable subsets of symplectic manifolds are heavy" in a rough sense if that generalization holds.

1. Definitions and results

Definitions. Burago, Ivanov and Polterovich defined the notion of conjugation-invariant (pseudo)norms on groups and they gave a number of its applications.

Definition 1.1 [Burago et al. 2008]. Let G be a group. A function $\nu : G \to \mathbb{R}_{\geq 0}$ is *a conjugation-invariant norm* on G if ν satisfies the following axioms:

(1) $\nu(1) = 0$;

(2) $\nu(f) = \nu(f^{-1})$ for every $f \in G$;

(3) $\nu(fg) \leq \nu(f) + \nu(g)$ for every $f, g \in G$;

(4) $\nu(f) = \nu(gfg^{-1})$ for every $f, g \in G$;

(5) $\nu(f) > 0$ for every $f \neq 1 \in G$.

A function $\nu : G \to \mathbb{R}$ is *a conjugation-invariant pseudonorm* on G if ν satisfies axioms (1), (2), (3) and (4) above.

For a conjugation-invariant pseudonorm ν, let $\mathrm{s}\nu$ denote the stabilization of ν, i.e., $\mathrm{s}\nu(g) = \lim_{n\to\infty} \nu(g^n)/n$ (this limit exists by Fekete's Lemma).

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For a perfect group G, the commutator length cl on G is a conjugation-invariant norm. Bavard [1991] proved the following famous theorem (see also [Calegari 2009]):

Theorem 1.2 (Corollary of Bavard's [1991] duality theorem). *Let g be an element of a perfect group G. Then $\mathrm{scl}(g) > 0$ if and only if there exists a homogeneous quasimorphism ϕ such that $\phi(g) > 0$.*

For interesting applications of Bavard's duality theorem, see [Calegari et al. 2014], [Endo and Kotschick 2001] and [Mimura 2010] for example. After Bavard's work, Calegari and Zhuang [2011] proved a Bavard-type duality theorem on W-length which is also conjugation-invariant. In the present paper, we give a Bavard-type duality theorem on general conjugation-invariant (pseudo)norms for some groups which are stable in some sense.

To state our main theorem, we introduce the notion of subset-controlled quasimorphism (partial quasimorphism, prequasimorphism) which is a generalization of quasimorphism:

Definition 1.3. Let G be a group and H a subgroup of G. We define the fragmentation norm ν_H with respect to H for an element f of G, by

$$\nu_H(f) = \min\{k : \text{ there exist } g_1, \dots, g_k \in G, \text{ and } h_1, \dots, h_k \in H \\ \text{such that } f = g_1 h_1 g_1^{-1} \cdots g_k h_k g_k^{-1}\}.$$

If there is no such decomposition of f as above, we put $\nu_H(f) = \infty$.

Definition 1.4. Let H be a subgroup of a group G. A function $\phi : G \to \mathbb{R}$ is called an *H-quasimorphism* if there exists a positive number C such that for any $f, g \in G$,

$$|\phi(fg) - \phi(f) - \phi(g)| < C \min\{\nu_H(f), \nu_H(g)\}.$$

The infimum of such C is called *the defect of ϕ* and we denote it by $D(\phi)$. If $\phi(f^n) = n\phi(f)$ for any element f of G and any integer n, ϕ is called *homogeneous*.

Such generalizations of quasimorphisms appeared first in [Entov and Polterovich 2006]. They proved that Oh–Schwarz spectral invariants (for example, see [Schwarz 2000] and [Oh 2006]) are controlled quasimorphisms.

Remark 1.5. In [Kawasaki 2016], H-quasimorphism is called quasimorphism relative to ν_H. Tomohiko Ishida and Tetsuya Ito pointed out that quasimorphism relative to H usually means quasimorphism which vanishes on H. Thus we use a different notation from that work.

Let K be a subset of a group G. For elements f, g of G, let fKg denote the subset $\{fkg; k \in K\}$ of G.

Definition 1.6. Let H be a subgroup of a group G. If for any element g of G, $\nu_H(g) < \infty$, G is said to be *c-generated* by H.

The author essentially proved the following proposition:

Proposition 1.7 [Kawasaki 2016]. *Let G be a group c-generated by a perfect subgroup H (in particular, G is also perfect). If there exists an H-quasimorphism ϕ with $\lim_{k\to\infty} \phi(g^k)/k > 0$ for some g, then there is a conjugation-invariant norm ν with $s\nu(g) > 0$ (such a norm is called **stably unbounded** [Burago et al. 2008]).*

Our main theorem (Theorem 1.12) is a converse of the Proposition 1.7.

Remark 1.8. The author [Kawasaki 2016] proved that there exists such a $\mathrm{Ham}(\mathbb{B}^{2n})$-quasimorphism μ_K on $\mathrm{Ham}(\mathbb{R}^{2n})$. Here, $\mathrm{Ham}(\mathbb{B}^{2n})$ and $\mathrm{Ham}(\mathbb{R}^{2n})$ are the group of Hamiltonian diffeomorphisms with compact support of the ball and the Euclidean space with the standard symplectic form, respectively. He also proved that $\mu_K(g) > 0$ for some commutator g. Thus, by Proposition 1.7, $[\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{R}^{2n})]$ admits a stably unbounded norm.

Kimura [2016] proved a similar result on the infinite braid group $B_\infty = \bigcup_{k=1}^{\infty} B_k$ (the existence of a stably unbounded norm on $[B_\infty, B_\infty]$ is also proved by Brandenbursky and Kedra [2015]).

Definition 1.9. Let G be a group, H a subgroup of G and K a subset of G. We define the set $\mathrm{D}^f_H(K)$ of maps *displacing K far away* by

$$\mathrm{D}^f_H(K) = \{h_0 \in G : \text{ for all } g_1, \dots, g_k \in G, \text{ there exists } h \in G \text{ such that } hh_0h^{-1}K(hh_0h^{-1})^{-1} \text{ commutes with } g_1Hg_1^{-1} \cup \cdots \cup g_kHg_k^{-1}\}.$$

Let ν be a conjugation-invariant pseudonorm on a group G. For a subset K of G, we define *the far away displacement energy* $\mathrm{E}_{H,\nu}(K)$ of K by

$$\mathrm{E}_{H,\nu}(K) = \inf_{g \in \mathrm{D}^f_H(K)} \nu(g).$$

Definition 1.10. Let G be a group and H a subgroup of G. The pair (G, H) satisfies the property FM if G and H satisfy the following conditions.

(1) G is c-generated by H,

(2) For any elements $h_1, \dots, h_k$ of G, $\mathrm{D}^f_H(h_1Hh_1^{-1} \cup \cdots \cup h_kHh_k^{-1}) \neq \varnothing$.

A group G satisfies the property FM if (G, H) satisfies the property FM for some subgroup H.

For a group G, we define the set $\mathsf{FM}(G)$ by

$$\mathsf{FM}(G) = \{H \leq G;\ (G, H) \text{ satisfies the property FM}\}.$$

We give some examples satisfying the property FM.

Proposition 1.11. (1) *For any integer i, the pair (B_∞, B_i) satisfies the property* FM, *and so does the pair* $([B_\infty, B_\infty], [B_i, B_i])$.

(2) *We consider the Riemannian surface $\Sigma_\infty = \bigcup_{k=1}^{\infty} \Sigma_k^1$ where Σ_k^1 is the Riemannian surface which has genus k and 1 puncture. The pair of mapping class groups* $(\mathrm{MCG}(\Sigma_\infty), \mathrm{MCG}(\Sigma_i^1))$ *satisfies the property* FM *for any integer i.*

(3) *The pair* $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ *satisfies the property* FM, *and so does the pair* $([\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{R}^{2n})], [\mathrm{Ham}(\mathbb{B}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n})])$.

Our main theorem is the following one.

Theorem 1.12. *Let G be a group satisfying the property* FM *and ν a conjugation-invariant pseudonorm on G. Then,*

(1) *For any element g of G such that $s\nu(g) > 0$, there exists a function $\phi : G \to \mathbb{R}$ which is a homogeneous H-quasimorphism for any element H of* $\mathsf{FM}(G)$ *such that $\phi(g) > 0$.*

(2) *For any element g of the commutator subgroup $[G, G]$ and any $H \in \mathsf{FM}(G)$,*

$$s\nu(g) \le 8 \sup_{\phi} \frac{\phi(g) \cdot E_{H,\nu}(H)}{D(\phi)},$$

where sup *is taken over the set of homogeneous H-quasimorphisms $\phi : G \to \mathbb{R}$.*

In Section 2, we construct the normed vector space A_ν and prove Theorem 1.12 by applying the Hahn–Banach theorem to A_ν. In Section 3, we prove that A_ν is a normed vector space. In Section 4, we prove Proposition 1.11. In Section 5, we pose a generalization of Theorem 1.12 (Problem 5.6) and give its application to symplectic geometry. There, we prove that "stably nondisplaceable subsets of symplectic manifolds are heavy" in a very rough sense if the positive answer of Problem 5.6 holds.

2. Proof of main theorem

To construct controlled quasimorphisms by using the Hahn–Banach theorem, we consider the normed vector space A_ν which we define here. The idea of our construction comes from [Calegari and Zhuang 2011].

For a group G, we define the set $A_G = \coprod_{k=0}^{\infty} (G \times \mathbb{R})^k$. We denote elements of A_G by $g_1^{s_1} \cdots g_k^{s_k}$, where $g_1, \dots, g_k \in G$ and $s_1, \dots, s_k$ are real numbers.

Let ν be a conjugation-invariant pseudonorm on G. We define the $\mathbb{R}_{\ge 0}$-valued function $\|\cdot\|_\nu : A_G \to \mathbb{R}_{\ge 0}$ by

$$\|g_1^{s_1} \cdots g_k^{s_k}\|_\nu = \lim_{n \to \infty} \frac{1}{n} \cdot \nu(g_1^{[s_1 n]} \dots g_k^{[s_k n]}),$$

where $[\,\cdot\,]$ denotes the integer part. For the trivial element $1 \in (G \times \mathbb{R})^0$ of A_G, we define $\|1\|_\nu = 0$.

Proposition 2.1. *Let ν be a conjugation-invariant pseudonorm on a group G satisfying the property* FM. *Then for any element $g_1^{s_1}\cdots g_k^{s_k}$ of A_G, the above limit $\|g_1^{s_1}\cdots g_k^{s_k}\|_\nu$ exists. Thus $\|\cdot\|_\nu$ is well defined.*

We prove Proposition 2.1 in Section 3. First, we define some operations on A_G. For elements $\mathtt{g}=g_1^{s_1}\cdots g_k^{s_k}$, $\mathtt{h}=h_1^{t_1}\cdots h_l^{t_l}$ of A_G and a real number λ, we define $\mathtt{g}\cdot\mathtt{h}$, $\bar{\mathtt{g}}$ and $\mathtt{g}^{(\lambda)}$ by

$$\mathtt{g}\cdot\mathtt{h}=g_1^{s_1}\cdots g_k^{s_k}h_1^{t_1}\cdots h_l^{t_l},\quad \bar{\mathtt{g}}=g_k^{-s_k}\cdots g_1^{-s_1}\quad\text{and}\quad \mathtt{g}^{(\lambda)}=g_1^{\lambda s_1}\cdots g_k^{\lambda s_k}.$$

By the definition of conjugation-invariant pseudonorms, we can confirm that the function $\|\cdot\|_\nu : A_G\to\mathbb{R}$ satisfies the following properties easily. For any $\mathtt{g},\mathtt{h}\in A_G$,

$$\|\mathtt{g}\cdot\mathtt{h}\|_\nu\le\|\mathtt{g}\|_\nu+\|\mathtt{h}\|_\nu,\quad \|\mathtt{h}\cdot\mathtt{g}\cdot\bar{\mathtt{h}}\|_\nu=\|\mathtt{g}\|_\nu\quad\text{and}\quad\|\bar{\mathtt{g}}\|_\nu=\|\mathtt{g}\|_\nu.$$

We define the equivalence relation $\sim$ by $\mathtt{g}\sim\mathtt{h}$ if and only if $\|\mathtt{g}\cdot\bar{\mathtt{h}}\|_\nu=0$. We denote the set $A_G/\sim$ by A_ν and the function $\|\cdot\|_\nu : A_G\to\mathbb{R}$ on A_G induces the function $\|\cdot\|_\nu : A_\nu\to\mathbb{R}$ on A_ν.

In the present paper, we want to consider A_ν as an $\mathbb{R}$-vector space with the norm $\|\cdot\|_\nu$. We define a sum operation, an inverse operation and an $\mathbb{R}$-action on A_ν. For elements $\mathsf{g}=[\mathtt{g}]$, $\mathsf{h}=[\mathtt{h}]$ of A_ν and a real number λ, we define $\mathsf{g}+\mathsf{h}$ and $\lambda\mathsf{g}$ by

$$\mathsf{g}+\mathsf{h}=[\mathtt{g}\cdot\mathtt{h}]\quad\text{and}\quad\lambda\mathsf{g}=[\mathtt{g}^{(\lambda)}].$$

Proposition 2.2. *Assume that G satisfies the property* FM. *Then the above operations are well defined.*

To use the Hahn–Banach theorem, we prove that A_ν is a normed vector space.

Proposition 2.3. *Assume that G satisfies the property* FM. *Then $(A_\nu,\|\cdot\|_\nu)$ is a normed vector space with respect to the above operations.*

We prove Proposition 2.2 and 2.3 in Section 3.

Let G be a group and ν a conjugation-invariant pseudonorm on G. Let $L(G,\nu)$ denote the set of Lipschitz continuous (linear) homomorphisms from A_ν to $\mathbb{R}$. By the Hahn–Banach theorem, Proposition 2.3 implies the following proposition.

Proposition 2.4. *Assume that G satisfies the property* FM. *Then for any $\mathsf{g}\in A_\nu$,*

$$\|\mathsf{g}\|_\nu=\sup_{\hat{\phi}\in L(G,\nu)}\frac{\hat{\phi}(\mathsf{g})}{l(\hat{\phi})},$$

where $l(\hat{\phi})$ is the optimal Lipschitz constant of $\hat{\phi}$.

For an element $\hat{\phi}$ of $L(G,\nu)$, we define the map $\phi : G\to\mathbb{R}$ by $\phi(g)=\hat{\phi}([g^1])$.

Proposition 2.5. *Let H be an element of* $\mathsf{FM}(G)$. *For any element $\hat{\phi}$ of $L(G,\nu)$, ϕ is a homogeneous H-quasimorphism. Moreover, $D(\phi)\le 8l(\hat{\phi})\cdot E_{H,\nu}(H)$.*

To prove Proposition 2.5, we use the following lemmas:

Lemma 2.6. *Let G be a group and H, K subgroups of G. Assume (G, H) satisfies the property* FM. *Then for any $g \in G$ and any element $f \in K$, $\nu([g, f]) \leq 4E_{H,\nu}(K)$.*

Proof. Let f, g and h_0 be elements of K, G and $\mathrm{D}_H^f(K)$, respectively. Since G is c-generated by H and the set $\{f, g\}$ is a finite set, there exist elements $h_1, \ldots, h_k$ of G such that $f, g \in \langle h_1 H h_1^{-1}, \ldots, h_k H h_k^{-1}\rangle$.

Then, by the definition of $\mathrm{D}_H^f(K)$, there exists an element h of G such that $(hh_0h^{-1})K(hh_0h^{-1})^{-1}$ commutes with $\langle h_1 H h_1^{-1}, \ldots, h_k H h_k^{-1}\rangle$. Since $f \in K$ and $f, g \in \langle h_1 H h_1^{-1}, \ldots, h_k H h_k^{-1}\rangle$, $(hh_0h^{-1})f(hh_0h^{-1})^{-1}$ commutes with both of f and g and thus $[g, f] = [g, [f, hh_0h^{-1}]]$ holds.

Since ν is a conjugation-invariant pseudonorm,

$$\begin{aligned}\nu([g, f]) &\leq \nu(g[f, hh_0h^{-1}]g^{-1}) + \nu([f, hh_0h^{-1}]^{-1}) = 2\nu([f, hh_0h^{-1}])\\ &\leq 2(\nu(f(hh_0h^{-1})f^{-1}) + \nu((hh_0h^{-1})^{-1}))\\ &= 4\nu(hh_0h^{-1}) = 4\nu(h_0).\end{aligned}$$

By taking the infimum, $\nu([g, f]) \leq 4E_{H,\nu}(K)$. ☐

Lemma 2.7 [Entov and Polterovich 2006],[Kimura 2016]. *Let G be a group, H a subgroup of G and C a positive real number. Assume that a map $\phi : G \to \mathbb{R}$ satisfies $|\phi(f) + \phi(g) - \phi(fg)| \leq C$ for any elements f, g of G with $\nu_H(f) = 1$. Then ϕ is an H-quasimorphism. Moreover, $D(\phi) \leq 2C$.*

Proof of Proposition 2.5. Let $\hat{\phi}$ be an element of $L(G, \nu)$ and f, g elements of G with $\nu_H(f) = 1$. Since H is a subgroup, $\nu_H(f^i) = 1$ for any nonzero integer i. Since ν is a conjugation-invariant pseudonorm, by Lemma 2.6,

$$\begin{aligned}&|\phi(g) + \phi(f) - \phi(fg)|\\ &\quad= |\hat{\phi}([g^1]) + \hat{\phi}([f^1]) - \hat{\phi}([(fg)^1])|\\ &\quad= |\hat{\phi}([g^1] + [f^1] + (-1)[(fg)^1])|\\ &\quad\leq l(\hat{\phi}) \cdot \lim_m m^{-1} \cdot \nu(g^m f^m (g^{-1}f^{-1})^m)\\ &\quad= l(\hat{\phi}) \cdot \lim_m m^{-1} \cdot \nu((g^{m-1}[g, f^m]g^{-m+1})(g^{m-2}[g, f^{m-1}]g^{-m+2}) \cdots (g^0[g, f]g^0))\\ &\quad\leq l(\hat{\phi}) \cdot \liminf_m m^{-1} \cdot \sum_{i=1}^{m-1} \nu([g, f^i])\\ &\quad\leq l(\hat{\phi}) \cdot \liminf_m m^{-1} \cdot (m-1) \cdot 4E_{H,\nu}(H)\\ &\quad= 4l(\hat{\phi}) \cdot E_{H,\nu}(H).\end{aligned}$$

Thus, by Lemma 2.7, ϕ is an H-quasimorphism and $D(\phi) \leq 8l(\hat{\phi}) \cdot E_{H,\nu}(H)$. Since $\hat{\phi}$ is a homomorphism, $\phi : G \to \mathbb{R}$ is a homogeneous H-quasimorphism. □

Proof of Theorem 1.12. Note that $\|[g^1]\|_\nu = \mathrm{s}\nu(g)$ for any element g of G. Then (1) follows from Proposition 2.4 and 2.5. To prove (2), it is sufficient to prove it for an element g of $[G, G]$ with $\mathrm{s}\nu(g) > 0$. Then, by Proposition 2.4 and $\|[g^1]\|_\nu = \mathrm{s}\nu(g)$, there exists an element $\hat{\phi}$ of $L(G, \nu)$ satisfying $\phi(g) = \hat{\phi}([g^1]) \neq 0$. Since $g \in [G, G]$, $D(\phi) > 0$. Thus Proposition 2.5 implies $8l(\hat{\phi})^{-1} \leq D(\phi)^{-1} \cdot E_{H,\nu}(H)$. Therefore Proposition 2.4 implies

$$\mathrm{s}\nu(g) \leq 8 \sup_{\phi} \frac{\phi(g) \cdot E_{H,\nu}(H)}{D(\phi)}. \qquad \square$$

3. Proof of being a normed vector space

Definition 3.1. Let H be a subgroup of a group G and ν a conjugation-invariant pseudonorm on G. For elements $g_1, \dots, g_k$ of G, we define the far away displacement energy $E_{H,\nu}[g_1, \dots, g_k]$ of $(g_1, \dots, g_k)$ by

$$E_{H,\nu}[g_1, \dots, g_k] = \inf E_{H,\nu}(\langle h_1 H h_1^{-1}, \dots, h_l H h_l^{-1} \rangle),$$

where inf is taken over $h_1, \dots, h_l$ such that $g_1, \dots, g_k \in \langle h_1 H h_1^{-1}, \dots, h_l H h_l^{-1} \rangle$. If (G, H) satisfies the property FM, $E_{H,\nu}[g_1, \dots, g_k] < \infty$ for any $g_1, \dots, g_k \in G$.

To prove Proposition 2.1, 2.2 and 2.3, we use the following lemma:

Lemma 3.2 [Calegari and Zhuang 2011]. *Let ν a conjugation-invariant pseudonorm on a group G. For any elements $g_1, \dots, g_k$ of G and integers $s_1, \dots, s_k, t_1, \dots, t_k$,*

$$\nu((g_1^{s_1} \cdots g_k^{s_k})^{-1}(g_1^{t_1} \cdots g_k^{t_k})) \leq \sum_{i=1}^{k} |t_i - s_i| \cdot \nu(g_i).$$

Proof. By using a graphical calculus argument (for example, see 2.2.4 of [Calegari 2009]), there exist elements $h_1, \dots, h_k$ of $\langle g_1, \cdots, g_k \rangle$ such that

$$(g_1^{s_1} \cdots g_k^{s_k})^{-1}(g_1^{t_1} \cdots g_k^{t_k}) = h_k^{-1} g_k^{t_k - s_k} h_k \cdots h_1^{-1} g_1^{t_1 - s_1} h_1.$$

Since ν is a conjugation-invariant pseudonorm,

$$\nu((g_1^{s_1} \cdots g_k^{s_k})^{-1}(g_1^{t_1} \cdots g_k^{t_k})) \leq \sum_{i=1}^{k} \nu(h_i^{-1} g_i^{t_i - s_i} h_i) \leq \sum_{i=1}^{k} |t_i - s_i| \cdot \nu(g_i). \qquad \square$$

Proof of Proposition 2.1. Fix an element $\mathrm{g} = [g_1^{s_1} \cdots g_k^{s_k}]$ of A_ν. Define a function $F : \mathbb{Z}_{>0} \to \mathbb{R}$ by $F(m) = \nu(g_1^{[s_1 m]} \cdots g_k^{[s_k m]})$. By Fekete's Lemma, it is sufficient to prove that there exists a positive real number C such that $F(m+n) \leq F(m) + F(n) + C$ for any positive integers m, n. By Lemma 3.2,

$$\begin{aligned} F(m+n) &= \nu(g_1^{[s_1(m+n)]} \cdots g_k^{[s_k(m+n)]}) \\ &\leq \nu(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) \\ &\qquad + \nu((g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]})^{-1}(g_1^{[s_1(m+n)]} \cdots g_k^{[s_k(m+n)]})) \\ &\leq \nu(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) + \sum_{i=1}^{k} \nu(g_i). \end{aligned}$$

By using a graphical calculus argument, there exists an integer $l(k)$ which depends only on k and elements $f_1, \ldots, f_{l(k)}, f_1', \ldots, f_{l(k)}'$ of $\langle g_1, \ldots, g_k \rangle$ such that

$$(g_1^{[s_1m]} \cdots g_k^{[s_km]})^{-1}(g_1^{[s_1n]} \cdots g_k^{[s_kn]})^{-1}(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) = [f_1, f_1'] \cdots [f_{l(k)}, f_{l(k)}'].$$

Fix an element H of $\mathsf{FM}(G)$. Then $E_{H,\nu}[g_1, \ldots, g_k] < \infty$. Thus, by Lemma 2.6,

$$\begin{aligned} &F(m+n) - F(m) - F(n) \\ &\leq \nu(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]}) + \sum_{i=1}^{k} \nu(g_i) - \nu(g_1^{[s_1m]} \cdots g_k^{[s_km]}) - \nu(g_1^{[s_1n]} \cdots g_k^{[s_kn]}) \\ &\leq \nu((g_1^{[s_1m]} \cdots g_k^{[s_km]})^{-1}(g_1^{[s_1n]} \cdots g_k^{[s_kn]})^{-1}(g_1^{[s_1m]+[s_1n]} \cdots g_k^{[s_km]+[s_kn]})) + \sum_{i=1}^{k} \nu(g_i) \\ &\leq \nu([f_1, f_1'] \cdots [f_{l(k)}, f_{l(k)}']) + \sum_{i=1}^{k} \nu(g_i) \\ &\leq \sum_{j=1}^{l(k)} \nu([f_j, f_j']) + \sum_{i=1}^{k} \nu(g_i) \\ &\leq 4l(k) E_{H,\nu}[g_1, \ldots, g_k] + \sum_{i=1}^{k} \nu(g_i). \end{aligned}$$

Thus we can apply Fekete's Lemma. □

To prove Proposition 2.2 and 2.3, we use the following lemmas.

Lemma 3.3. *Let G be a group satisfying the property* FM *and ν any conjugation-invariant pseudonorm on G. Then for any $\mathrm{g} \in A_G$ and any real numbers λ_1, λ_2,*

$$\|\bar{\mathrm{g}}^{(\lambda_1+\lambda_2)} \cdot \mathrm{g}^{(\lambda_1)} \cdot \mathrm{g}^{(\lambda_2)}\|_\nu = 0.$$

Proof. Assume that g is represented by $g_1^{s_1} g_2^{s_2} \cdots g_k^{s_k} \in A_G$. For any integer n, by using a graphical calculus argument, there exist elements $f_{n,1}, \ldots, f_{n,l(k)}$ and $f_{n,1}', \ldots, f_{n,l(k)}'$ of $\langle g_1, \ldots, g_k \rangle$ such that

$$\begin{aligned} &(g_1^{[n\lambda_1s_1]+[n\lambda_2s_1]} g_2^{[n\lambda_1s_2]+[n\lambda_2s_2]} \cdots g_k^{[n\lambda_1s_k]+[n\lambda_2s_k]})^{-1} \\ &(g_1^{[n\lambda_1s_1]} g_2^{[n\lambda_1s_2]} \cdots g_k^{[n\lambda_1s_k]})(g_1^{[n\lambda_2s_1]} g_2^{[n\lambda_2s_2]} \cdots g_k^{[n\lambda_2s_k]}) = [f_{n,1}, f_{n,1}'] \cdots [f_{n,l(k)}, f_{n,l(k)}']. \end{aligned}$$

Fix $H \in \mathsf{FM}(G)$. Then $E_{H,\nu}[g_1, \dots, g_k] < \infty$. Thus, by Lemma 3.2 and Lemma 2.6,

$$
\begin{aligned}
&\|\bar{\mathrm{g}}^{(\lambda_1+\lambda_2)} \cdot \mathrm{g}^{(\lambda_1)} \cdot \mathrm{g}^{(\lambda_2)}\|_\nu \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \nu\big((g_1^{[n\lambda_1 s_1 + n\lambda_2 s_1]} \cdots g_k^{[n\lambda_1 s_k + n\lambda_2 s_k]})^{-1} \\
&\qquad\qquad (g_1^{[n\lambda_1 s_1]} \cdots g_k^{[n\lambda_1 s_k]})(g_1^{[n\lambda_2 s_1]} \cdots g_k^{[n\lambda_2 s_k]})\big) \\
&\le \lim_{n\to\infty} \frac{1}{n} \cdot \Bigg(\nu((g_1^{[n\lambda_1 s_1]+[n\lambda_2 s_1]} \cdots g_k^{[n\lambda_1 s_k]+[n\lambda_2 s_k]})^{-1} \\
&\qquad\qquad (g_1^{[n\lambda_1 s_1]} \cdots g_k^{[n\lambda_1 s_k]})(g_1^{[n\lambda_2 s_1]} \cdots g_k^{[n\lambda_2 s_k]})) + \sum_{i=1}^{k} \nu(g_i)\Bigg) \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \Bigg(\nu([f_{n,1}, f'_{n,1}] \cdots [f_{n,l(k)}, f'_{n,l(k)}]) + \sum_{i=1}^{k} \nu(g_i)\Bigg) \\
&\le \lim_{n\to\infty} \frac{1}{n} \cdot \Bigg(\sum_{j=1}^{l(k)} \nu([f_{n,j}, f'_{n,j}]) + \sum_{i=1}^{k} \nu(g_i)\Bigg) \\
&\le \lim_{n\to\infty} \frac{1}{n} \cdot \Bigg(4l(k) E_{H,\nu}[g_1, \dots, g_k] + \sum_{i=1}^{k} \nu(g_i)\Bigg) \\
&= 0. \qquad \square
\end{aligned}
$$

Lemma 3.4. *Let G be a group satisfying the property* FM *and ν a conjugation-invariant pseudonorm on G. For $g_1, \dots, g_k \in G$ and real numbers $\lambda, s_1, \dots, s_k$,*

$$
\lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[\lambda s_1 n]} \cdots g_k^{[\lambda s_k n]}) = |\lambda| \lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[s_1 n]} \cdots g_k^{[s_k n]}).
$$

Proof. We first prove for the case when λ is a positive rational number, i.e., $\lambda = q/p$ where p, q are positive integers. By the existence of the limits (Proposition 2.1), since the limit of any subsequence equals that of the original sequence,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[\lambda s_1 n]} \cdots g_k^{[\lambda s_k n]}) &= \lim_{n\to\infty} \frac{1}{pn} \cdot \nu(g_1^{[q s_1 n]} \cdots g_k^{[q s_k n]}) \\
&= \lim_{n\to\infty} \frac{q}{pn} \cdot \nu(g_1^{[s_1 n]} \cdots g_k^{[s_k n]}) \\
&= \lambda \lim_{n\to\infty} \frac{1}{n} \cdot \nu(g_1^{[s_1 n]} \cdots g_k^{[s_k n]}).
\end{aligned}
$$

We prove for the case $\lambda = -1$.

Let g denote the element $g_1^{s_1} g_2^{s_2} \cdots g_k^{s_k}$ of A_G. By Lemma 3.3, $[\mathrm{g}^{(-1)} \cdot \mathrm{g}] = [\mathrm{g}^{(0)}] = [1]$. Recall that $1 \in (G \times \mathbb{R})^0$ is the trivial element of A_G. Thus $[\mathrm{g}^{(-1)}] = [\mathrm{g}^{(-1)} \cdot \mathrm{g} \cdot \bar{\mathrm{g}}] = [1 \cdot \bar{\mathrm{g}}] = [\bar{\mathrm{g}}]$. Therefore $\|(-1)\mathrm{g}\|_\nu = \|\bar{\mathrm{g}}\|_\nu = \|\mathrm{g}\|_\nu$ and we have completed the proof for the case when λ is a rational number.

Since Lemma 3.2 implies that the function $\mathbb{R} \to \mathbb{R}$, $\lambda \mapsto \lim_{n\to\infty}(1/n) \cdot \nu(g_1^{[\lambda s_1 n]} \cdots g_k^{[\lambda s_k n]})$ is continuous, we have completed the proof. $\square$

Proof of Proposition 2.2. Assume that elements $\mathtt{f}_1, \mathtt{f}_2, \mathtt{g}_1, \mathtt{g}_2$ of A_G satisfy $[\mathtt{f}_1] = [\mathtt{f}_2]$ and $[\mathtt{g}_1] = [\mathtt{g}_2]$. Then

$$\begin{aligned}\|(\mathtt{f}_1 \cdot \mathtt{g}_1) \cdot \overline{(\mathtt{f}_2 \cdot \mathtt{g}_2)}\|_\nu &= \|\mathtt{f}_1 \cdot \mathtt{g}_1 \cdot \bar{\mathtt{g}}_2 \cdot \bar{\mathtt{f}}_2\|_\nu \\ &\le \|\mathtt{f}_1 \cdot \mathtt{g}_1 \cdot \bar{\mathtt{g}}_2 \cdot \bar{\mathtt{f}}_1\|_\nu + \|\mathtt{f}_1 \cdot \bar{\mathtt{f}}_2\|_\nu \\ &= \|\mathtt{g}_1 \cdot \bar{\mathtt{g}}_2\|_\nu + \|\mathtt{f}_1 \cdot \bar{\mathtt{f}}_2\|_\nu = 0.\end{aligned}$$

Thus $[\mathtt{f}_1 \cdot \mathtt{g}_1] = [\mathtt{f}_2 \cdot \mathtt{g}_2]$.

Assume $\mathtt{g}_1, \mathtt{g}_2 \in A_G$ satisfy $[\mathtt{g}_1] = [\mathtt{g}_2]$. For any $\lambda \in \mathbb{R}$, Lemma 3.4 implies $\|\bar{\mathtt{g}}_1^{(\lambda)} \cdot \mathtt{g}_2{}^{(\lambda)}\|_\nu = \|(\bar{\mathtt{g}}_1 \cdot \mathtt{g}_2)^{(\lambda)}\|_\nu = |\lambda| \cdot \|(\bar{\mathtt{g}}_1 \cdot \mathtt{g}_2)\|_\nu = 0$. Thus $[\mathtt{g}_1^{(\lambda)}] = [\mathtt{g}_2^{(\lambda)}]$. □

Lemma 3.5. *Let G be a group satisfying the property* FM *and ν a conjugation-invariant pseudonorm on G. Then for any elements* f, g *of A_ν,*

$$\mathsf{f} + \mathsf{g} = \mathsf{g} + \mathsf{f}.$$

Proof. Assume f, g are represented by $[\mathtt{f}] = [f_1^{s_1} f_2^{s_2} \cdots f_k^{s_k}]$, $[\mathtt{g}] = [g_1^{t_1} g_2^{t_2} \cdots g_l^{t_l}]$, respectively. Fix an element H of $\mathsf{FM}(G)$. Then $E_{H,\nu}[g_1, \dots, g_l] < \infty$. Since $g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]} \in \langle g_1, \dots, g_l\rangle$ for any n, Lemma 2.6 implies

$$\begin{aligned}\|\mathtt{f} \cdot \mathtt{g} \cdot \overline{(\mathtt{g} \cdot \mathtt{f})}\|_\nu &= \|\mathtt{f} \cdot \mathtt{g} \cdot \bar{\mathtt{f}} \cdot \bar{\mathtt{g}}\|_\nu \\ &= \lim_{n\to\infty} \frac{1}{n} \cdot \nu\big((f_1^{[s_1 n]} f_2^{[s_2 n]} \cdots f_k^{[s_k n]})(g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]}) \\ &\qquad (f_1^{[s_1 n]} f_2^{[s_2 n]} \cdots f_k^{[s_k n]})^{-1} (g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]})^{-1}\big) \\ &= \lim_{n\to\infty} \frac{1}{n} \cdot \nu\big([f_1^{[s_1 n]} f_2^{[s_2 n]} \cdots f_k^{[s_k n]}, g_1^{[t_1 n]} g_2^{[t_2 n]} \cdots g_l^{[t_l n]}]\big) \\ &= \lim_{n\to\infty} \frac{1}{n} \cdot 4E_{H,\nu}[g_1, \dots, g_l] = 0.\end{aligned}$$

Thus $\mathsf{f} + \mathsf{g} = [\mathtt{f} \cdot \mathtt{g}] = [\mathtt{g} \cdot \mathtt{f}] = \mathsf{g} + \mathsf{f}$. □

Proof of Proposition 2.3. By Lemma 3.3, 3.4 and 3.5, for any elements f, g of A_ν and real numbers λ_1, λ_2,

$$(\lambda_1 + \lambda_2)\mathsf{g} = \lambda_1 \mathsf{g} + \lambda_2 \mathsf{g}, \quad \|\lambda_1 \mathsf{g}\|_\nu = |\lambda_1| \cdot \|\mathsf{g}\|_\nu, \quad \text{and} \quad \mathsf{f} + \mathsf{g} = \mathsf{g} + \mathsf{f}.$$

We can confirm the other axioms of a normed vector space easily. Thus we complete the proof of Proposition 2.3. □

4. Proof that examples satisfy the property FM

In the present section, we prove that $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ satisfies the property FM. We can prove other parts of Proposition 1.11 similarly.

We use the following notations. For a diffeomorphism g on a manifold M, let $\mathrm{Supp}(g)$ denote the support of g. For a point p of $\mathbb{R}^{2n}$ and a positive real number R, let $\mathbb{B}^{2n}(p, R)$ denote a subset $\{x \in \mathbb{R}^{2n}; \|x - p\| < R\}$ of $\mathbb{R}^{2n}$.

Proof. For simplicity, let $\mathcal{B}$ denote the subgroup $\mathrm{Ham}(\mathbb{B}^{2n})$ and p_0 denote the point $(3, 0, \dots, 0)$ of $\mathbb{R}^{2n}$.

Let f_0 be a Hamiltonian diffeomorphism on $\mathbb{R}^{2n}$ such that $f_0(\mathbb{B}^{2n}) = \mathbb{B}^{2n}(p_0, 1)$. Fix Hamiltonian diffeomorphisms $g_1, \dots, g_k$ on $\mathbb{R}^{2n}$. Then there exists a positive real number R such that $\mathrm{Supp}(g_1) \cup \cdots \cup \mathrm{Supp}(g_k) \subset \mathbb{B}^{2n}(0, R)$. Since $f_0(\mathbb{B}^{2n}) = \mathbb{B}^{2n}(p_0, 1)$ and $\mathbb{B}^{2n}(p_0, 1) \cap \mathbb{B}^{2n} = \varnothing$, we can take a Hamiltonian diffeomorphism f such that $f(\mathbb{B}^{2n}) = \mathbb{B}^{2n}$ and $ff_0(\mathbb{B}^{2n}) \cap \mathbb{B}^{2n}(0, R) = \varnothing$. Since $(ff_0f^{-1})\mathcal{B}(ff_0f^{-1})^{-1} = \mathrm{Ham}(ff_0f^{-1}(\mathbb{B}^{2n})) = \mathrm{Ham}(ff_0(\mathbb{B}^{2n}))$ and

$$g_1\mathcal{B}g_1^{-1} \cup \cdots \cup g_k\mathcal{B}g_k^{-1} = \mathrm{Ham}(g_1(\mathbb{B}^{2n}) \cup \cdots \cup g_k(\mathbb{B}^{2n})) \subset \mathrm{Ham}(\mathbb{B}^{2n}(0, R)),$$

$ff_0(\mathbb{B}^{2n}) \cap \mathbb{B}^{2n}(0, R) = \varnothing$ implies that $(ff_0f^{-1})\mathcal{B}(ff_0f^{-1})^{-1}$ commutes with $g_1\mathcal{B}g_1^{-1} \cup \cdots \cup g_k\mathcal{B}g_k^{-1}$. Thus $f_0 \in \mathrm{D}^f_{\mathcal{B}}(\mathcal{B})$.

Note that Banyaga's [1978] fragmentation lemma states that for any Hamiltonian diffeomorphism g, there exist Hamiltonian diffeomorphisms $f_1, \dots, f_k$ such that $g \in \langle f_1\mathcal{B}f_1^{-1}, \dots, f_k\mathcal{B}f_k^{-1}\rangle$. Thus $\mathrm{Ham}(\mathbb{R}^{2n})$ is c-generated by $\mathcal{B}$ and the proof is complete. □

5. Are stably nondisplaceable subsets heavy? Bavard's duality in Hofer's geometry

We have considered subgroups which are displaceable far away. We now pose a problem on displaceable subgroups and give its application to symplectic geometry.

On notions related to symplectic geometry, we follow [Entov 2014].

Definition 5.1. Let G be a group, H a subgroup of G and $\mu : G \to \mathbb{R}$ an H-quasimorphism on G. If $\mu(g^n) = n\mu(g)$ for any element g of G and any nonnegative integer n, μ is called *semihomogeneous*.

Let (M, ω) be a $2m$-dimensional closed symplectic manifold. A subset X of (M, ω) is called *displaceable* if $\bar{X} \cap \phi^1_F(X) = \varnothing$ for some Hamiltonian function $F : S^1 \times M \to \mathbb{R}$ where ϕ_F is the Hamiltonian diffeomorphism generated by F and $\bar{X}$ is the topological closure of X. Otherwise, X is *nondisplaceable*. Let $\mathsf{DO}(M)$ denote the set of displaceable open subsets of (M, ω). A subset X of a symplectic manifold M is *stably displaceable* if $X \times S^1$ is displaceable in $M \times T^*S^1$. Otherwise, X is *stably nondisplaceable*.

Entov and Polterovich [2006] defined for an idempotent a of the quantum homology $QH_*(M, \omega)$, the asymptotic spectral invariant $\mu_a : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ on the universal covering $\widetilde{\mathrm{Ham}}(M)$ of the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms in terms of Oh–Schwarz spectral invariants and proved that μ_a is a semihomogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$. Here $\widetilde{\mathrm{Ham}}_U(M)$ is the set of elements of $\widetilde{\mathrm{Ham}}(M)$ which are generated by Hamiltonian functions with support in $S^1 \times U$.

A Hamiltonian function $F : S^1 \times M \to \mathbb{R}$ is *normalized* if $\int_M F_t \omega^m = 0$ for any $t \in S^1$.

Definition 5.2 [Entov and Polterovich 2009]. Let (M, ω) be a closed symplectic manifold and a an idempotent of $QH_*(M, \omega)$. A compact subset X of (M, ω) is *a-heavy* if for any normalized Hamiltonian function $F : S^1 \times M \to \mathbb{R}$,

$$-\mu_a(\phi_F) \geq \mathrm{vol}(M) \cdot \inf_{S^1 \times X} F,$$

where $\mathrm{vol}(M) = \int_M \omega^m$.

In particular, if X is a-heavy, $\mu_a(\phi_F) < 0$ for any normalized Hamiltonian function F with $F|_{S^1 \times X} > 0$.

Remark 5.3. The above definition of heaviness is different from the one of [Entov and Polterovich 2009] and [Entov 2014] (in their definition, they consider only autonomous Hamiltonian functions). However, as remarked in [Seyfaddini 2014], the above definition is known to be equivalent.

Entov and Polterovich [2009] also proved that heavy subsets are stably nondisplaceable. In the present section, we consider the converse problem, "are stably nondisplaceable subsets heavy?"

Definition 5.4. Let G be a group, H a subgroup of G and K a subset of G. We define the set $\mathrm{D}_H(K)$ of maps *displacing* K by

$$\mathrm{D}_H(K) = \{h_0 \in G;\ h_0 K (h_0)^{-1} \text{ commutes with } H\}.$$

Definition 5.5. Let G be a group and H a subgroup of G. The pair (G, H) satisfies the property FD if G and H satisfy the following conditions:

(1) G is c-generated by H,

(2) $\mathrm{D}_H(H) \neq \varnothing$.

A group G satisfies the property FD if (G, H) satisfies the property FD for some subgroup H.

For a group G which satisfies the property FD, we define the set $\mathsf{FD}(G)$ by

$$\mathsf{FD}(G) = \{H \leq G;\ (G, H) \text{ satisfies the property } \mathsf{FD}\}.$$

We pose the following problem.

Problem 5.6. Let G be a group satisfying the property FD, H an element of $\mathsf{FD}(G)$ and ν a conjugation-invariant pseudonorm on G. Prove that for any element g of G such that $s\nu(g) > 0$, there exists a function $\mu : G \to \mathbb{R}$ which is a semihomogeneous H-quasimorphism for any element H of $\mathsf{FD}(G)$ such that $\mu(g) > 0$.

Here, we give an application of Problem 5.6 to symplectic geometry.

Proposition 5.7. *Assume that the positive answer of Problem 5.6 holds.*

Let X be a stably nondisplaceable compact subset of a closed symplectic manifold (M,ω). For any normalized Hamiltonian function $F : S^1\times M \to \mathbb{R}$ with $F|_{S^1\times X} > 0$, there exists a function $\mu_F : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ which is a semihomogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$ such that $\mu_F(\phi_F) < 0$.

Proposition 5.7 states that "stably nondisplaceable subsets are heavy" in a very rough sense if the positive answer of Problem 5.6 holds.

To prove Proposition 5.7, we use the following theorem, due to Polterovich:

Theorem 5.8 [Polterovich 1998, 2001]. *Let X be a stably nondisplaceable subset of a closed symplectic manifold (M,ω). For any normalized Hamiltonian function $F : S^1\times M \to \mathbb{R}$ with $F|_{S^1\times X} \geq p$ for some positive number p, $\|\phi_F\|_H \geq p$. Here $\|\cdot\|_H : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ is the Hofer norm which is known to be a conjugation-invariant pseudonorm.*

Proof of Proposition 5.7. Since X is compact, there exists some positive number p with $F|_{S^1\times X} \geq p$. For any positive integer n, we define a Hamiltonian function $F^{(n)} : S^1\times M \to \mathbb{R}$ by $F^{(n)}(t,x) = n\cdot F(nt,x)$. Note that $\phi_{F^{(n)}} = (\phi_F)^n$. Then, by $F^{(n)}|_{S^1\times X} \geq np$ and Theorem 5.8, $\|(\phi_F)^n\|_H \geq np$ for any positive integer n. Since $\widetilde{\mathrm{Ham}}_U(M) \in \mathsf{FD}(\widetilde{\mathrm{Ham}}(M))$ for any element U of $\mathsf{DO}(M)$, by the positive answer of Problem 5.6, there exists a function $\mu'_F : \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ which is a semihomogeneous $\widetilde{\mathrm{Ham}}_U(M)$-quasimorphism for any element U of $\mathsf{DO}(M)$ such that $\mu'_F(\phi_F) > 0$. Then setting $\mu_F = -\mu'_F$ completes the proof. □

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