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REGULARITY CONDITIONS FOR SUITABLE WEAK SOLUTIONS OF THE NAVIER–STOKES SYSTEM FROM ITS ROTATION FORM

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We establish new regularity criteria for suitable weak solutions involving Bernoulli (total) pressure $\Pi = \frac{1}{2}|u|^2 + p$. By the rotation form of the Navier–Stokes equations, we also obtain regularity criteria for suitable weak solutions in terms of either $u \times \omega/|\omega|$ or $\omega \times u/|u|$ with sufficiently small local scaled norm, where ω is the vorticity of the velocity. As a consequence, we extend and refine some known interior regularity criteria for suitable weak solutions.

1. Introduction

Consider the initial boundary-value problem for the incompressible time-dependent Navier–Stokes equations:

$$
\begin{cases}
u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \partial\Omega \times [0, T), \\
u|_{t=0} = u_0(x) & \text{on } \Omega \times \{t = 0\},
\end{cases}
\tag{1-1}
$$

where the domain $\Omega \subseteq \mathbb{R}^3$ is a bounded regular domain. Here u describes the velocity of the flow, the scalar function p stands for the pressure of the fluid. The initial data $u_0(x)$ satisfies divergence free. Denote by $\omega = \operatorname{curl} u$ the vorticity of the velocity field.

There have been extensive studies on the regularity of suitable weak solutions to the Navier–Stokes equations since the late 1970s (see, e.g., [Caffarelli et al. 1982; Chae et al. 2007; Dong and Du 2007; Dong and Strain 2012; Chae 2010; Gustafson et al. 2007; Wang and Wu 2014; 2016a; 2016b; Struwe 1988; Seregin 2002; 2007; 2014; Wang et al. 2014; Wang and Zhang 2013; 2014; Scheffer 1976; 1977; 1980; Vasseur 2007; Wolf 2008; Lin 1998; Ladyzhenskaya and Seregin 1999; Tian and Xin 1999]). Suitable weak solutions originated with Scheffer [1976; 1977; 1980] in

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studying the potential singular points of solutions to the Navier–Stokes equations and were later developed by Caffarelli, Kohn, Nirenberg [Caffarelli et al. 1982] and Lin [1998]. For convenience, we recall the definition of suitable weak solutions.

Definition (Suitable weak solutions). A pair (u, p) is a suitable weak solution to the Navier–Stokes equations (1-1), provided the following conditions are satisfied

(i) $u \in L^\infty(t, t'; L^2(\Omega)) \cap L^2(t, t'; W^{1,2}(\Omega))$, $p \in L^{3/2}(t, t'; L^{3/2}(\Omega))$.

(ii) (u, p) solves (1.1) in $\Omega \times (t, t')$ in the sense of distributions.

(iii) (u, p) obeys the local energy inequality

$$\begin{aligned} &\int_\Omega |u(t',x)|^2 \phi \, dx + 2\int_t^{t'}\int_\Omega |\nabla u(s,x)|^2 \phi \, dx \, ds \\ &\le \int_t^{t'}\int_\Omega |u(s,x)|^2 (\partial_s \phi + \Delta \phi) \, dx \, ds + 2\int_t^{t'}\int_\Omega \big(\tfrac{1}{2}|u(s,x)|^2 + p(s,x)\big) u(s,x) \cdot \nabla \phi \, dx \, ds \end{aligned} \tag{1-2}$$

for any nonnegative function $\phi \in C_0^\infty(\Omega \times (t, t'))$.

A point is said to be a regular point of the Navier–Stokes equations (1-1) if one has an L^∞ bound of u in some neighborhood of this point. Otherwise, they are called singular points. In this direction, the milestone work is that the one-dimensional Hausdorff measure of the possible spacetime singular points of suitable weak solutions to the 3D Navier–Stokes equations is zero, which was shown by Caffarelli, Kohn, Nirenberg in [Caffarelli et al. 1982]. This result relies heavily on the following regularity criteria: if there is an absolute constant ε such that

$$\limsup_{\mu \to 0} \frac{1}{\mu} \iint_{Q(\mu)} |\nabla u|^2 \, dx \, dt \le \varepsilon, \tag{1-3}$$

then $(0, 0)$ is a regular point, where $Q(\mu) := B(\mu) \times (-\mu^2, 0)$ and $B(\mu)$ denotes the ball of center 0 and radius μ. Since then, different approaches to show the Caffarelli–Kohn–Nirenberg theorem have been presented. More precisely, based on the blowup method, Lin [1998] provided a simple proof (see also Ladyzenskaja and Seregin [1999] with nonzero external force belonging to parabolic Morrey space). Recently, by means of De Giorgi's iteration technique, Vasseur [2007] provided a constructive proof without external force. In [Wang and Wu 2014], De Giorgi's iteration strategy was applied to the 4D Navier–Stokes equations and the high-dimensional steady Navier–Stokes equations with nonzero external force. In what follows, the local scaled norm of quantity is the one which equips the scale invariant norm similar to (1-3). An alternative proof is offered by Wolf [2008] via establishing a decay estimate of the gradient of the velocity with local scaled norm together with Campanato's Lemma on Hölder continuity. Moreover, notice that regularity condition (1-3) plays a central role in the partial regularity theory of Navier–Stokes. There are a lot of extensions and improvements of (1-3). For instance, Gustafson, Kang and Tsai [Gustafson et al. 2007] obtained the following regularity criteria to suitable weak solutions:

$$\limsup_{\mu\to 0}\mu^{1-\frac{2}{p}-\frac{3}{q}}\|u\|_{L^{p,q}(Q(\mu))}\le\varepsilon,\quad 1\le\tfrac{2}{p}+\tfrac{3}{q}\le 2,\quad 1\le p,q\le\infty;\tag{1-4}$$

$$\limsup_{\mu\to 0}\mu^{2-\frac{2}{p}-\frac{3}{q}}\|\nabla u\|_{L^{p,q}(Q(\mu))}\le\varepsilon,\quad 2\le\tfrac{2}{p}+\tfrac{3}{q}\le 3,\quad 1\le p,q\le\infty;\tag{1-5}$$

$$\limsup_{\mu\to 0}\mu^{2-\frac{2}{p}-\frac{3}{q}}\|\omega\|_{L^{p,q}(Q(\mu))}\le\varepsilon,\quad 2\le\tfrac{2}{p}+\tfrac{3}{q}\le 3,\quad 1\le p,q\le\infty,\tag{1-6}$$

where $(p,q)\ne(1,\infty)$ in (1-6), and where ε is an absolute constant, which extends the work of Tian and Xin [1999]. Employing a blowup procedure, Seregin [2007] improved the regular condition (1-3) to, for any $M>0$, there exists a positive number $\varepsilon(M)$ such that

$$\limsup_{r\to 0}\frac{1}{r}\iint_{Q(r)}|\nabla u|^2dx\,dt\le M\quad\text{and}\quad\liminf_{r\to 0}\frac{1}{r}\iint_{Q(r)}|\nabla_3 u|^2dx\,dt\le\varepsilon(M).\tag{1-7}$$

We also refer the reader to the recent works of Wang and Zhang [2014] and Wang and Wu [2016a; 2016b].

We note that almost all the results mentioned above rest on the Navier–Stokes equations in convective form (1-1). Depending on different expressions of the nonlinear term, the Navier–Stokes equations have several equivalent versions such as the convective form, the skew-symmetric form and the rotation form (see, e.g., [Layton et al. 2009; Zang 1991] and references therein). Thanks to the well-known fact that

$$u\cdot\nabla u=\tfrac{1}{2}\nabla|u|^2+\omega\times u,$$

the 3D Navier–Stokes equations (1-1) can be equivalently reformulated as the rotation form below:

$$\begin{cases}u_t-\Delta u+w\times u+\nabla\Pi=0,\\ \operatorname{div}u=0,\end{cases}\tag{1-8}$$

where $\Pi=\frac{1}{2}|u|^2+p$ is called as the Bernoulli (total) pressure, which can be found in [Prandtl 2004; Heywood et al. 1996; Layton et al. 2009; Olshanskii 2002; Zang 1991] and references therein. By means of the Bernoulli pressure Π, the local energy inequality (1-2) can be rewritten as

$$\begin{aligned}\int_\Omega|u(t',x)|^2\phi dx+2\int_t^{t'}\int_\Omega|\nabla u(s,x)|^2\phi\,dx\,ds&\\ \le\int_t^{t'}\int_\Omega|u|^2(\phi_s+\Delta\phi)\,dx\,ds+2\int_t^{t'}\int_\Omega\Pi u\cdot\nabla\phi\,dx\,ds.&\end{aligned}\tag{1-9}$$

We refer to the above inequality as the local energy inequality with respect to the 3D Navier–Stokes equations in rotation form (1-8).

The goal of this paper is to derive some new regularity criteria for suitable weak solutions from the Navier–Stokes equations in rotation form (1-8). Notice that the Bernoulli pressure Π not only plays important role in the regular theory of the Navier–Stokes equations (see, e.g., [Frehse and Růžička 1994; 1995; Struwe 1995; Seregin and Šverák 2002; Nečas et al. 1996; Chae 2014; Tsai 1998]), but also can be measurable via numerical simulations (see, e.g., [Heywood et al. 1996; Layton et al. 2009; Prandtl 2004; Olshanskii 2002; Zang 1991]). Seregin and Šverák [2002] showed that the weak solutions to the 3D Navier–Stokes equations are regular provided the positive part of the Bernoulli pressure is controlled. Since the pressure p is nonlocal, it seems difficult to obtain regularity criteria via only the pressure p with sufficiently small local scaled norm. One objective of this paper is to establish the regularity criteria in terms of Bernoulli pressure Π with sufficiently small local scaled norm.

Theorem 1.1. *There exists a constant $\varepsilon_1 > 0$ with the property that if (u, p) is a suitable weak solution of the Navier–Stokes equations such that $\Pi - (\Pi)_{B(\mu)} \in L^{p,q}_{\mathrm{loc}}$ with*

$$\limsup_{\mu\to 0} \mu^{2-\frac{2}{p}-\frac{3}{q}} \left(\int_{-\mu^2}^{0} \left(\int_{B(\mu)} |\Pi - (\Pi)_{B(\mu)}|^q \, dx \right)^{\frac{p}{q}} ds \right)^{\frac{1}{p}} < \varepsilon_1,$$

where $(p, q) \in [1, \infty] \times [1, \infty]$ satisfying

$$2 \le \frac{2}{p} + \frac{3}{q} \le \frac{7}{2} \quad \text{with } 1 \le p \le 2. \tag{1-10}$$

Then u is regular at $(0, 0)$.

Remarks. (1) The range $1 \le p \le 2$ corresponds to the limiting case $2/p+3/q = 7/2$. By means of Hölder's inequality, the range (1-10) can be generalized to

$$\frac{2}{p} + \frac{3}{q} = \begin{cases} \frac{7}{2} - \delta & \text{with } 1 - \delta \le 2/p \le 2 (0 \le \delta \le 1), \\ \hbar \in [2, 5/2] & \text{with } 1 \le p \le \infty. \end{cases}$$

(2) Theorem 1.1 also implies the criteria in terms of the gradient of the Bernoulli pressure. Moreover, Theorem 1.1 holds true for nonzero external force f provided that $f \in L^q_{t,x}$ with $q > \frac{5}{2}$.

(3) The same result is valid if $\Pi - (\Pi)_{B(r)}$ is replaced by Π in Theorem 1.1. As a straightforward consequence, a Serrin-type sufficient regularity condition in terms of Bernoulli pressure can be obtained. More precisely, let (u, p) be a suitable weak solution. Then u is regular on $Q(r/2)$ provided Π belongs to $L^{p,q}(Q(r))$ with $2/p + 3/q = 2$.

The key point for proving the above theorem is how to bound the first term on the right hand side of the local energy inequality (1-9). Generally speaking, the magnitude between $\frac{1}{2}|u|^2$ and $\frac{1}{2}|u|^2 + p$ is not clear. Resorting to the appropriate test function (backward heat kernel) recently adopted in [Dong and Du 2007; Wang et al. 2014; Wang and Zhang 2013], we could circumvent the direct control. This enables us to obtain

$$\begin{aligned}
&\mu^{-1}\|u\|^2_{L^{\infty,2}(Q(\mu))} + \mu^{-1}\|\nabla u\|^2_{L^2(Q(\mu))} \\
&\le C\Big(\frac{\mu}{\rho}\Big)^2 \rho^{-1}\|u\|^2_{L^{\infty,2}(Q(\rho))} \\
&\qquad + C\Big(\frac{\rho}{\mu}\Big)^2 \rho^{-2}\|\Pi - (\Pi)_{B(\rho)}\|_{L^{p,q}(Q(\rho))}\big[\|u\|^2_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|^2_{L^2(Q(\rho))}\big]^{1/2},
\end{aligned}$$

which gives the desired iteration. A slight modification of the latter iteration yields

$$\begin{aligned}
&\mu^{-1}\|u\|^2_{L^{\infty,2}(Q(\mu))} + \mu^{-1}\|\nabla u\|^2_{L^2(Q(\mu))} \\
&\le C\Big(\frac{\mu}{\rho}\Big)^2 \rho^{-1}\|u\|^2_{L^{\infty,2}(Q(\rho))} + C\Big(\frac{\rho}{\mu}\Big)^2 \rho^{-2}\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p^\natural,q^\natural}(Q(\rho))}\big[\|u\|^2_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|^2_{L^2(Q(\rho))}\big].
\end{aligned}$$

This relation leads to the following results:

Theorem 1.2. *There exists a constant $\varepsilon_2 > 0$ with the property that if (u, p) is a suitable weak solution of the Navier–Stokes equations such that $\Pi/|u| \in L^{p^\natural,q^\natural}_{\mathrm{loc}}$ with*

$$\limsup_{\mu\to 0} \mu^{1-\frac{2}{p^\natural}-\frac{3}{q^\natural}} \bigg(\int_{-\mu^2}^{0}\bigg(\int_{B(\mu)}\Big|\frac{\Pi}{|u|}\Big|^{q^\natural} dx\bigg)^{\frac{p^\natural}{q^\natural}} ds\bigg)^{\frac{1}{p^\natural}} < \varepsilon_2,$$

where $(p^\natural, q^\natural) \in [1,\infty] \times [1,\infty]$ satisfy

$$1 \le \frac{2}{p^\natural} + \frac{3}{q^\natural} \le 2, \tag{1-11}$$

then u is regular at $(0,0)$.

Remarks. (1) The statement of Theorem 1.2 remains valid if $\Pi/|u|$ is replaced by $\Pi/(\mu^{-1} + |u|)$. This theorem also means the Serrin-type regular condition in terms of $\Pi/|u|$. This theorem corresponds to Beirão da Veiga's [2000] regularity condition that any weak solution u is regular in $\Omega \times (0, T)$ provided

$$\frac{p}{1+|u|} \in L^p(0, T; L^q(\Omega)) \quad \text{with } \tfrac{2}{p} + \tfrac{3}{q} = 1,\ q > 3.$$

(2) The proofs of Theorems 1.1 and 1.2 also yield the regularity conditions involving $\Pi/|u|^\alpha$ with sufficiently small local scaled norm for $0 \le \alpha \le 1$. Invoking the blowup framework introduced by Seregin [2007], one can improve these results provided $\alpha < 1$ in the sense of (1-7).

In the following, we seek out a quantity which can control the Bernoulli pressure from the equations (1-8). Notice that the Bernoulli pressure is determined by

$$\Delta\Pi = -\operatorname{div}(\omega \times u). \tag{1-12}$$

We find that ω and u may be the apposite candidate. Indeed, by virtue of the split of velocity u, Wolf [2008] established the following criteria: assume that u is a suitable weak solution to (1-1). If there exists an absolute constant ε such that

$$\limsup_{\mu\to 0} \frac{1}{\mu} \iint_{Q(\mu)} \Big|\omega \times \frac{u}{|u|}\Big|^2 \, dx\, ds \leq \varepsilon, \tag{1-13}$$

then (0, 0) is a regular point. The second goal of this paper is to obtain a regular class in terms of $u \times \omega/|\omega|$ and to extend the integral norms with different exponents in space and time in (1-13).

Theorem 1.3. *Let (u, p) be a suitable weak solution to* (1-1) *in $Q(1)$. Then* (0, 0) *is regular point provided one of the following conditions holds*:

(1) *There exists a positive constant ε_3 such that $u \times \omega/|\omega| \in L^{i,j}_{\mathrm{loc}}$ with*

$$\limsup_{\mu\to 0} \mu^{1-\frac{2}{i}-\frac{3}{j}} \bigg(\int_{-\mu^2}^{0} \Big(\int_{B(\mu)} \Big|u \times \frac{\omega}{|\omega|}\Big|^j \, dx\Big)^{\frac{i}{j}} ds\bigg)^{\frac{1}{i}} \leq \varepsilon_3, \tag{1-14}$$

where $(i, j) \in (2, 4) \times (2, 3)$ satisfy

$$1 \leq \frac{2}{i} + \frac{3}{j} \leq 2 \quad \text{with } i < 4. \tag{1-15}$$

(2) *There exists a positive constant ε_3 such that $\omega \times u/|u| \in L^{m,n}_{\mathrm{loc}}$ with*

$$\limsup_{\mu\to 0} \mu^{2-\frac{2}{m}-\frac{3}{n}} \bigg(\int_{-\mu^2}^{0} \Big(\int_{B(\mu)} \Big|\omega \times \frac{u}{|u|}\Big|^n \, dx\Big)^{\frac{m}{n}} ds\bigg)^{\frac{1}{m}} \leq \varepsilon_3, \tag{1-16}$$

where $(m, n) \in (1, 4) \times (6/5, 3)$ satisfy

$$2 \leq \frac{2}{m} + \frac{3}{n} \leq 3 \quad \text{with } m < 4. \tag{1-17}$$

Remarks. (1) As noted in the first remark on page 192, in light of Hölder's inequality, one can extend the range of (1-15) and (1-17) to

$$\frac{2}{i} + \frac{3}{j} = \begin{cases} 2-\delta & \text{with } 1-2\delta < \frac{4}{i} < 2 \ \ \big(0 \leq \delta \leq \frac{1}{2}\big), \\ \ell \in \big[1, \frac{3}{2}\big), & \text{with } 2 < i \leq \infty, \end{cases}$$

and

$$\frac{2}{m} + \frac{3}{n} = \begin{cases} 3-\delta & \text{with } 1-2\delta < \frac{4}{m} < 4 \ \ \big(0 \leq \delta \leq 1/2\big), \\ \ell' \in \big[2, \frac{5}{2}\big), & \text{with } 2 < m \leq \infty, \end{cases}$$

respectively.

(2) Theorem 1.3 is an improvement of corresponding results (1-4) and (1-6) proved by Gustafson, Kang and Tsai [Gustafson et al. 2007]. These extensions of (1-4) and (1-6) include their endpoint cases.

(3) As a corollary of Theorem 1.3, one immediately obtains the Serrin-type regularity conditions via $u \times \omega/|\omega|$ or $\omega \times u/|u|$, which was proved in [Chae 2010].

The idea of proving Theorem 1.3 is to establish an effective iteration scheme via local energy inequality (1-9). Therefore, the main target is devoted to deriving the decay-type estimate of $|u|^2$ and the Bernoulli pressure Π in terms of the rotation term $\omega\times u$. In view of (1-12), one can derive the decay-type estimate of the Bernoulli pressure Π in terms of $\omega\times u$. Since there is no direct relationship between $|u|^2$ and $\omega\times u$, the main difficulty of the proof of this theorem lies in the estimate of the first term on the right hand side of the local energy inequality (1-9). One would want to invoke the backward heat kernel as test function utilized in [Dong and Du 2007; Wang et al. 2014; Wang and Zhang 2013] again, which yields the appearance of $(\rho/\mu)^2 > 1$ in the second term on the right hand side of the local energy inequality. However, this breaks down since now neither Π nor u is assumed to be sufficiently small. Our strategy is to utilize the decomposition introduced by Seregin [2002] for studying the partial regularity of the Navier–Stokes equations near the boundary. Precisely, let (v, p_1) be a unique solution to the following initial boundary value problem:

$$\begin{cases} v_t-\Delta v+\nabla p_1=-w\times u, \operatorname{div} v=0 & \text{in } \ Q(\rho) \\ (p_1)_{B(\rho)}=0 & \text{on } (-\rho^2, 0), \\ v=0 & \text{on } \{t=-\rho^2\}\times B(\rho)\cup[-\rho^2,0]\times\partial B_\rho. \end{cases} \tag{1-18}$$

Then $b=u-v$ and $p_2=\Pi-(\Pi)_{B(\rho/2)}-p_1$ solve the following boundary value problem:

$$\begin{cases} b_t-\Delta b=-\nabla p_2, \operatorname{div} b=0 & \text{in } Q(\rho) \\ b=u & \text{on } \{t=-\rho^2\}\times B(\rho)\cup[-\rho^2, 0]\times\partial B_\rho. \end{cases} \tag{1-19}$$

This allows us to bound the L^2-norm of u in terms of controlling that of v and b separately. On the one hand, applying the L^p-L^q-estimate of solutions to the Stokes system established by Giga and Sohr [1991] to (1-18), we get

$$\|v_t\|_{L^{r,s}(Q(\rho))}+\|A_s v\|_{L^{r,s}(Q(\rho))}+\|\nabla p_1\|_{L^{r,s}(Q(\rho))}\le C\|w\times u\|_{L^{r,s}(Q(\rho))},$$

where $A_s=-\mathbb{P}_s\Delta$ and $\mathbb{P}_s$ is the Leray projection from $L^s(\Omega)^d$ onto $L^s_\sigma(\Omega)$. Then we can apply embedding theorems in mixed norm also shown in the same work to bound $\|v\|_{L^2(Q(\rho))}$ in terms of $\|w\times u\|_{L^{r,s}(Q(\rho))}$. On the other hand, the harmonic function p_2 helps us to get an interior estimate of b below

$$\|b\|^2_{L^2(Q(\mu))}\le C\Big(\frac{\mu}{\rho}\Big)^5\big[\|b\|^2_{L^2(Q(\rho))}+\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big],$$

where $0 < \mu \le \rho/32$. Then we could derive the decay-type estimate

$$\mu^{-3}\|u\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} + C\Big(\frac{\mu}{\rho}\Big)^2\big[\rho^{-3}\|u\|^2_{L^2(Q(\rho))} + \rho^{-3}\|\Pi-(\Pi)_{B(\rho)}\|^2_{L^{r,s'}(Q(\rho))}\big]. \tag{1-20}$$

Remark. The decomposition (1-18)–(1-19) allows us to take full advantage of the structure of the rotation term $\omega\times u$ and the local energy inequality (1-9) to refine regularity criteria (1-4) and (1-6). Roughly speaking, if the rotation term $w\times u$ in (1-18) is replaced by a convective term $u\cdot\nabla u$, then the split (1-18)–(1-19) reduces to Seregin's [2002] original split. However, it seems that, following the pathway of Theorem 1.3, Seregin's original split of the velocity u seems to yield Serrin-type regularity criteria rather than the Caffarelli–Kohn–Nirenberg type regularity conditions via $u\cdot\nabla u/|\nabla u|$ or $u/|u|\cdot\nabla u$.

Finally, we turn our attention to the following stationary Navier–Stokes equations in $\mathbb{R}^d$ for $d = 5, 6$:

$$-\Delta u + u\cdot\nabla u + \nabla p = f, \quad \operatorname{div} u = 0, \quad x\in\Omega. \tag{1-21}$$

First, we also present the definition of suitable weak solutions to the stationary case.

Definition. A pair (u, p) is said to be a suitable weak solution to the stationary Navier–Stokes equations (1-21) if and only if

(1) $u\in W^{1,2}(\Omega)$, $p\in L^{3/2}(\Omega)$.

(2) (u, p) solves (1-21) in the sense of distributions.

(3) (u, p) verifies the local energy inequality

$$2\int_\Omega|\nabla u|^2\psi\,dx \le \int_\Omega|u|^2\Delta\psi\,dx + 2\int_\Omega\big(\tfrac{1}{2}|u|^2+p\big)u\cdot\nabla\psi\,dx + 2\int_\Omega uf\psi\,dx, \tag{1-22}$$

for $\psi\in C_0^\infty(\Omega)$, in the sense of distributions.

According to the dimensional analysis of the Navier–Stokes equations in [Caffarelli et al. 1982], nonstationary Navier–Stokes equations in $\mathbb{R}^d$ may be viewed as stationary Navier–Stokes equations $\mathbb{R}^{d+2}$. The analogue of the Caffarelli–Kohn–Nirenberg criteria (1-3) for suitable weak solutions to the stationary Navier–Stokes equations in $\mathbb{R}^5$ and $\mathbb{R}^6$ were proved by Struwe [1995] and by Dong and Strain [2012], respectively. By means of an observation that both the local energy inequality for the time-dependent Navier–Stokes equations and the stationary case can be dealt with by the unified approach in [Wang and Wu 2014], one can show the analogue theorem of Theorem 1.1 to system (1-21). To make our paper more self-contained and more readable, we outline the proof of the stationary case with the external force f ($\operatorname{div} f = 0$).

Theorem 1.4. *Suppose that* (u, p) *is a suitable weak solution of the Navier–Stokes equations* (1-21) *and the external force* f *belongs to* $L^q(\Omega)$ *with* $q > \frac{1}{2}d$. *There is a constant* ε_4 *such that if the condition*

$$\limsup_{\mu\to 0} \mu^{-\frac{d-2}{2}} \left(\int_{B(\mu)} |\Pi - (\Pi)_{B(\mu)}|^{\frac{2d}{d+2}}\, dx \right)^{\frac{2-d}{2}} < \varepsilon_4, \quad d = 5, 6,$$

holds, *then* u *is regular at origin.*

As a byproduct, Hölder's inequality and absolute continuity of Lebesgue's integral immediately yield the following result:

Corollary 1.5. *Let* (u, p) *be a suitable weak solution of the stationary Navier–Stokes equations* (1-21). *If*

$$\tfrac{1}{2}|u|^2 + p \in L^{d/2}_{\mathrm{loc}}(\Omega), \quad \textit{with } d = 5, 6, \tag{1-23}$$

then one has $u \in L^\infty_{\mathrm{loc}}(\Omega)$.

Remark. Frehse and Růžička [1994] showed that if the weak solutions satisfy

$$\left(\tfrac{1}{2}|u|^2 + p\right)_+ \in L^q_{\mathrm{loc}}(\Omega) \quad \text{with } q > \tfrac{1}{2}d,\ d \ge 5,$$

and the local energy inequality (1-22), then u is regular. Compared with Frehse and Růžička's regularity condition, the regular class (1-23) is scaling-invariant with respect to system (1-21).

The remainder of the paper is organized as follows. In the next section, we recall some helpful results and give some useful auxiliary lemmas such as the decay estimate involving the Bernoulli pressure and $|u|^2$. The last section will be devoted to proving theorems.

Notation. Throughout this paper, we denote

$$B(x,\mu) = \{y \in \mathbb{R}^d \mid |x-y| \le \mu\}, \quad B(\mu) := B(0,\mu),$$
$$Q(x,t,\mu) = B(x,\mu) \times (t-\mu^2, t), \quad Q(\mu) := Q(0,0,\mu).$$

For $p \in [1,\infty]$, the notation $L^p((0,T); X)$ stands for the set of measurable functions f on the interval $(0,T)$ with values in X such that $\|f(t,\cdot)\|_X$ belongs to $L^p(0,T)$. For simplicity, we write

$$\|f\|_{L^{p,q}(Q(\mu))} := \|f\|_{L^p(-\mu^2,0;L^q(B(\mu)))} \quad \text{and} \quad \|f\|_{L^p(Q(\mu))} := \|f\|_{L^{p,p}(Q(\mu))}.$$

Denote by $L^q_\sigma(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^q(\Omega)^d$, where $C^\infty_{0,\sigma}(\Omega)$ denotes the set $\{u \in C^\infty_0(\Omega)^d : \operatorname{div} u = 0\}$. The classical Sobolev space $W^{1,2}(\Omega)$ is equipped with the norm $\|f\|_{W^{1,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}$. We will also use the summation convention on repeated indices. C is an absolute constant which may be different from line to line unless otherwise stated. According to the natural scaling property of the Navier–Stokes equations [Caffarelli et al. 1982], we introduce the following dimensionless quantities for the nonstationary case

$$E(u,\mu)=\mu^{-1}\|u\|^2_{L^{\infty,2}(Q(\mu))},\qquad E_*(u,\mu)=\mu^{-1}\|\nabla u\|^2_{L^2(Q(\mu))},$$
$$U_{p,q}(\times,\mu)=\mu^{1-\frac{2}{p}-\frac{3}{q}}\Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{p,q}(Q(\mu))},\quad E_{p,q}(u,\mu)=\mu^{1-\frac{2}{p}-\frac{3}{q}}\|u\|_{L^{p,q}(Q(\mu))},$$
$$W_{p,q}(\times,\mu)=\mu^{2-\frac{2}{p}-\frac{3}{q}}\Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{p,q}(Q(\mu))},P_{p,q}\Big(\frac{\Pi}{|u|},\mu\Big)=\mu^{1-\frac{2}{p}-\frac{3}{q}}\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p,q}(Q(\mu))},$$
$$P_{p,q}(\Pi-(\Pi)_{B(\mu)},\mu)=\mu^{2-\frac{2}{p}-\frac{3}{q}}\|\Pi-(\Pi)_{B(\mu)}\|_{L^{p,q}(Q(\mu))},$$
$$E_2(u,r)=\mu^{-3}\|u\|^2_{L^2(Q(\mu))},$$

and for the stationary Navier–Stokes equations,

$$\tilde{E}_p(u,\mu)=\mu^{p-d}\|u\|^p_{L^p(B(\mu))},\qquad \tilde{E}_*(u,\mu)=\mu^{4-d}\|\nabla u\|^2_{L^2(B(\mu))},$$
$$\tilde{P}_{\frac{2d}{2+d}}(\Pi-(\Pi),\mu)=\mu^{-\frac{d-2}{2}}\|\Pi-(\Pi)\|_{L^{\frac{2d}{2+d}}(B(\mu))},\ \tilde{F}_q(f,\mu)=\mu^{3q-d}\|f\|^q_{L^q(B(\mu))}.$$

2. Preliminaries and main lemma

Before proceeding further with the decay-type estimate, we shall recall the L^p-L^q-estimate of solutions to the linear Stokes system and an associated interpolation inequality.

Proposition 2.1 [Giga and Sohr 1991]. *Let Ω be a bounded domain and $r, s\in(1,\infty)$. Then for every $f\in L^r(0,T;L^s(\Omega))$, there exists a unique solution $(v,\nabla p_1)$ to the Stokes system below*:

$$\begin{cases} v_t-\Delta v+\nabla p_1=f, \operatorname{div} v=0 & \textit{in } (0,T)\times\Omega,\\ v|_{\partial\Omega}=0, & \\ (p_1)_\Omega=0, & t\in(0,T),\\ v|_{t=0}=0. & \end{cases}$$

satisfying

$$\|v_t\|_{L^r(0,T;L^s(\Omega))}+\|A_sv\|_{L^r(0,T;L^s(\Omega))}+\|\nabla p_1\|_{L^r(0,T;L^s(\Omega))}\le C\|f\|_{L^r(0,T;L^s(\Omega))},$$

where $C=C(q,s,\Omega)$.

Lemma 2.2 [Giga and Sohr 1991]. *Let $D(A_s)=\{v\in L^s_\sigma(\Omega);\ \partial_l\partial_kv\in L^s_\sigma(\Omega)^d;\ 1\le l,\ k\le d,\ v|_{\partial\Omega}=0\}$. Suppose that $1<s<3/2$, $s<h^*<\infty$, and $1<r\le\rho<\infty$. Assume that*

$$\frac{2}{r}+\frac{3}{s}=2+\frac{3}{h^*}+\frac{2}{\rho}.$$

Then there are constants C such that

$$\|v\|_{L^{\rho}(0,T;L^{h^*}(\Omega))} \le C(\|v_t\|_{L^r(0,T;L^s(\Omega))} + \|A_s v\|_{L^r(0,T;L^s(\Omega))}),$$

for all $v \in L^r(0, T; D(A_s))$ satisfying $v_t, A_s v \in L^r(0, T; L^s(\Omega))$, and $v(0) = 0$.

Applying Proposition 2.1 to system (1-18), we immediately get, by Lemma 2.2,

$$\|v\|^2_{L^2(Q(\rho))} \le C(\|v_t\|^2_{L^{r,s}(Q(\rho))} + \|A_s v\|^2_{L^{r,s}(Q(\rho))}) \le C\|\omega \times u\|^2_{L^{r,s}(Q(\rho))}, \tag{2-1}$$

provided that r, s satisfy

$$\frac{2}{r} + \frac{3}{s} = \frac{9}{2}, \quad \text{with } 1 < s < \frac{6}{5}.$$

We recall a well-known interpolation inequality, which will be frequently used later. For every $2 \le \kappa \le \infty$ and $2 \le \tau \le 6$ satisfying $(2/\kappa) + (3/\tau) = \frac{3}{2}$, by Hölder's inequality, Sobolev's inequality and Young's inequality, we see that

$$\begin{aligned} \|u\|_{L^{\kappa,\tau}(Q(\mu))} &\le C\|u\|^{1-2/\kappa}_{L^{\infty,2}(Q(\mu))}\|u\|^{2/\kappa}_{L^{2,6}(Q(\mu))} \\ &\le C\|u\|^{1-2/\kappa}_{L^{\infty,2}(Q(\mu))}(\|u\|_{L^{\infty,2}(Q(\mu))} + \|\nabla u\|_{L^2(Q(\mu))})^{2/\kappa} \\ &\le C(\|u\|_{L^{\infty,2}(Q(\mu))} + \|\nabla u\|_{L^2(Q(\mu))}). \end{aligned} \tag{2-2}$$

The following lemma will play a crucial role in the proof of Theorem 1.1.

Lemma 2.3. *For $\mu \le \frac{1}{2}\rho$, there exists a constant C independent of μ and ρ such that*

$$\begin{aligned} E(u,\mu) + E_*(u,\mu) \le{}& C\Big(\frac{\mu}{\rho}\Big)^2 E(u,\rho) \\ &+ \Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho)[E(u,\rho) + E_*(u,\rho)]^{1/2}, \end{aligned} \tag{2-3}$$

$$\begin{aligned} E(u,\mu) + E_*(u,\mu) \le{}& C\Big(\frac{\mu}{\rho}\Big)^2 E(u,\rho) \\ &+ C\Big(\frac{\rho}{\mu}\Big)^2 P_{p^\natural,q^\natural}\Big(\frac{\Pi}{|u|}, \rho\Big)[E(u,\rho) + E_*(u,\rho)], \end{aligned} \tag{2-4}$$

where (p, q) and $(p^\natural, q^\natural)$ satisfy

$$\frac{2}{p} + \frac{3}{q} = \frac{7}{2} \quad \text{and} \quad \frac{2}{p^\natural} + \frac{3}{q^\natural} = 2 \quad \text{with } 1 \le p \le 2,\ 1 \le p^\natural \le \infty. \tag{2-5}$$

Proof. Consider the following smooth cutoff function

$$\psi(x,t) = \begin{cases} 1, & (x,t) \in Q(\rho/2), \\ 0, & (x,t) \in Q^c(\rho); \end{cases}$$

which satisfies $0 \le \psi(x,t) \le 1$, $|\psi_t(x,t)| + |\Delta\psi(x,t)| \le C/\rho^2$ and $|\nabla\psi(x)| \le C/\rho$. We denote the backward heat kernel

$$\Gamma(x,t) = \frac{1}{4\pi(\mu^2-t)^{3/2}} e^{-\frac{|x|^2}{4(\mu^2-t)}}.$$

Plugging $\phi = \psi(x,t)\Gamma(x,t)$ into the local energy inequality (1-9) and using that $\Gamma_t + \Delta\Gamma = 0$, we know that

$$\begin{aligned}(2\text{-}6)\quad &\sup_{-\rho^2\le t\le 0}\int_{B(\rho)} |u(x,t)|^2\Gamma(t,x)\psi(x,t)\,dx + 2\iint_{Q(\rho)} |\nabla u|^2\Gamma(x,s)\psi(x,s)\,dx\,ds\\ &\le \iint_{Q(\rho)} |u|^2[\Gamma(x,s)\psi_s(x,s) + \Gamma(x,s)\Delta\psi(x,s) + 2\nabla\psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds\\ &\quad + \iint_{Q(\rho)} (\Pi - (\Pi)_{B(\rho)})u\cdot[\Gamma\nabla\psi(x,s) + \psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds.\end{aligned}$$

This inequality in turn implies

$$\begin{aligned}(2\text{-}7)\quad &\sup_{-\mu^2\le t\le 0}\int_{B(\mu)} |u(x,s)|^2\Gamma(x,t)\,dx + 2\iint_{Q(\mu)} |\nabla u|^2\Gamma(x,s)\,dx\,ds\\ &\le \iint_{Q(\rho)\setminus Q(\rho/2)} |u|^2[\Gamma(x,s)\psi_s(x,s) + \Gamma(x,s)\Delta\psi(x,s) + 2\nabla\psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds\\ &\quad + \iint_{Q(\rho)} (\Pi - (\Pi)_{B(\rho)})u\cdot[\Gamma\nabla\psi(x,s) + \psi(x,s)\nabla\Gamma(x,s)]\,dx\,ds,\end{aligned}$$

where we have used the fact that $\mathrm{supp}(\psi_s, \partial_i\psi) \subset Q(2\rho)\setminus Q(\rho)$.

To proceed further, we list some properties of the test function $\phi(x,t)$ whose deduction rests on elementary calculations.

(i) There is a constant $c > 0$ independent of μ such that, for any $(x,t) \in Q(\mu)$,

$$\Gamma(x,t) \ge c\mu^{-3}.$$

(ii) It is clear that, for any $(x,t) \in Q(\rho)$,

$$|\Gamma(x,t)\psi(x,t)| \le C\mu^{-3}, \ |\nabla\psi(x,t)\Gamma(x,t)| \le C\mu^{-4},$$

and

$$\partial_i\Gamma(x,t) = -\frac{1}{4\pi(\mu^2-t)^{3/2}} e^{-\frac{|x|^2}{4(\mu^2-t)}} \frac{2x_i}{4(\mu^2-t)},$$

which in turn yields

$$|\psi(x,t)\nabla\Gamma(x,t)| \le C\mu^{-4}.$$

(iii) For any $(x,t) \in Q(\rho)\setminus Q(\rho/2)$, one can deduce that

$$\Gamma(x,t) \le C\rho^{-3}, \quad \partial_i\Gamma(x,t) \le C\rho^{-4},$$

which leads to

$$|\Gamma(x,t)\partial_t\psi(x,t)|+|\Gamma(x,t)\Delta\psi(x,t)|+|\nabla\psi(x,t)\nabla\Gamma(x,t)|\le C\rho^{-5}.$$

Take $1/\kappa=1-1/p$ and $1/\tau=1-1/q$. Then, in light of (2-7), the Hölder inequality, (2-2) and (2-5), we see that

$$\begin{aligned}E(u,\mu)+E_*(u,\mu)&\le C\Big(\frac{\mu}{\rho}\Big)^2E_2(u,\rho)+\Big(\frac{\rho}{\mu}\Big)^2P_{p,q}(\Pi-(\Pi)_{B(\rho)},\rho)E_{\kappa,\tau}(u,\rho)\\&\le C\Big(\frac{\mu}{\rho}\Big)^2E(u,\rho)+\Big(\frac{\rho}{\mu}\Big)^2P_{p,q}(\Pi-(\Pi)_{B(\rho)},\rho)[E(u,\rho)+E_*(u,\rho)]^{1/2},\end{aligned}$$

which means (2-3).

Choose $1/p^\sharp=1-1/p^\natural$ and $1/q^\sharp=1-1/q^\natural$. Then we derive from (2-5) that

$$\frac{2}{2p^\sharp}+\frac{3}{2q^\sharp}=\frac{3}{2}.$$

This together with Hölder's inequality and interpolation inequality (2-2) yields that

$$\begin{aligned}\iint_{Q(\rho)}|\Pi||u|\,dx\,dt&\le\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p^\natural,q^\natural}(Q(\rho))}\|u\|^2_{L^{2p^\sharp,2q^\sharp}(Q(\rho))}\\&\le\Big\|\frac{\Pi}{|u|}\Big\|_{L^{p^\natural,q^\natural}}(\|u\|^2_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|^2_{L^2(Q(\rho))}).\end{aligned}$$

Collecting these estimates leads to (2-4). This completes the proof. □

Next, we derive the decay estimate of the Bernoulli pressure.

Lemma 2.4. *Let $0<4\mu\le\rho$ and i,j,m,n be defined as the limiting case of* (1-15) *and* (1-17). *There exists an absolute constant C independent of μ and ρ such that*

$$P_{r,s'}(\Pi-(\Pi)_{B(\mu)},\mu)\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}U_{i,j}(\times,\rho)E_*^{1/2}(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^{3-\frac{2}{r}}P_{r,s'}(\Pi-(\Pi)_{B(\rho)},\rho),\tag{2-8}$$

$$P_{r,s'}(\Pi-(\Pi)_{B(\mu)},\mu)\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}W_{m,n}(\times,\rho)[E_*(u,\rho)+E(u,\rho)]^{1/2}+C\Big(\frac{\mu}{\rho}\Big)^{3-\frac{2}{r}}P_{r,s'}(\Pi-(\Pi)_{B(\rho)},\rho),\tag{2-9}$$

where the pair (r,s') satisfies

$$\frac{2}{r}+\frac{3}{s'}=\frac{7}{2}\quad\textit{with } 1<r<\frac{4}{3},\ \frac{3}{2}<s'<2.$$

Proof. Utilizing that $(p_1)_{B(\rho)}=0$ and the Poincaré–Sobolev inequality and applying Proposition 2.1 to system (1-18), we get

$$\|p_1\|_{L^{r,s'}(Q(\rho))}\le C\|\nabla p_1\|_{L^{r,s}(Q(\rho))}\le C\|\omega\times u\|_{L^{r,s}(Q(\rho))}, \tag{2-10}$$

where

$$\frac{3}{s}=1+\frac{3}{s'}. \tag{2-11}$$

Since $\Delta p_2=0$ on $B(\rho/4)$, then, by the interior estimate of harmonic functions and Hölder's inequality, we see that, for every $x_0\in B(\rho/4)$,

$$|\nabla p_2(x_0)|\le\frac{C}{\rho^{3+1}}\|p_2\|_{L^1(B_{x_0}(\rho/4))}\le\frac{C}{\rho^{3+1}}\|p_2\|_{L^1(B(\rho/2))}\le\frac{C}{\rho^{3+1}}\rho^{3(1-1/s')}\|p_2\|_{L^{s'}(B(\rho/2))},$$

which in turn implies

$$\|\nabla p_2\|^{s'}_{L^\infty(B(\rho/4))}\le C\rho^{-3-s'}\|p_2\|^{s'}_{L^{s'}(B(\rho/2))}.$$

The latter inequality together with the mean value theorem leads to

$$\begin{aligned}\|p_2-(p_2)_{B(\mu)}\|^{s'}_{L^{s'}(B(\mu))}\le&C\mu^3\|p_2-(p_2)_{B(\mu)}\|^{s'}_{L^\infty(B(\mu))}\\ \le&C\mu^3(2\mu)^{s'}\|\nabla p_2\|^{s'}_{L^\infty(B(\rho/4))}\\ \le&C\Big(\frac{\mu}{\rho}\Big)^{3+s'}\|p_2\|^{s'}_{L^{s'}(B(\rho/2))}.\end{aligned}$$

Integrating this inequality in time, we obtain

$$\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}\le C\Big(\frac{\mu}{\rho}\Big)^{\frac{3}{s'}+1}\|p_2\|_{L^{r,s'}(Q(\rho/2))}.$$

With the help of the triangle inequality and (2-10), we infer that

$$\begin{aligned}\|p_2\|_{L^{r,s'}(Q(\rho/2))}&\le\|\Pi-(\Pi)_{B_{\rho/2}}\|_{L^{r,s'}(Q(\rho/2))}+\|p_1\|_{L^{r,s'}(Q(\rho/2))}\\ &\le\|\Pi-(\Pi)_{B_\rho}\|_{L^{r,s'}(Q(\rho/2))}+\|p_1\|_{L^{r,s'}(Q(\rho))}\\ &\le\|\Pi-(\Pi)_{B_\rho}\|_{L^{r,s'}(Q(\rho))}+\|\omega\times u\|_{L^{r,s}(Q(\rho))},\end{aligned} \tag{2-12}$$

which in turns yields

$$\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}\le C\Big(\frac{\mu}{\rho}\Big)^{\frac{3}{s'}+1}(\|\Pi-(\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}+\|\omega\times u\|_{L^{r,s}(Q(\rho))}).$$

It follows from (2-10) and the last estimate that

$$\begin{aligned}
(2\text{-}13)\quad \|\Pi-(\Pi)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} &\le \|p_1-(p_1)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}+\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}\\
&\le \|p_1\|_{L^{r,s'}(Q(\mu))}+\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}\\
&\le \|p_1\|_{L^{r,s'}(Q(\rho))}+\|p_2-(p_2)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))}\\
&\le C\|\omega\times u\|_{L^{r,s}(Q(\rho))}+C\Big(\frac{\mu}{\rho}\Big)^{\frac{3}{s'}+1}\|\Pi-(\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}.
\end{aligned}$$

Now, we bound $\omega\times u$ in two different ways.

Case I: The Hölder inequality and hypothesis (1-15) in Theorem 1.3 ensure that

$$\begin{aligned}
(2\text{-}14)\qquad \|\omega\times u\|_{L^{r,s}(Q(\rho))} &\le \Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{i,j}(Q(\rho))}\|\omega\|_{L^2(Q(\rho))}\\
&\le C\Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{i,j}(Q(\rho))}\|\nabla u\|_{L^2(Q(\rho))},
\end{aligned}$$

where the pair (r,s) satisfies

$$\frac{2}{r}+\frac{3}{s}=\frac{9}{2},$$

and

$$(2\text{-}15)\qquad \frac12<\frac1r=\frac12+\frac1i<1,\quad \frac23<\frac1s=\frac12+\frac1j<1,$$

which guarantees that Proposition 2.1 and Lemma 2.2 work. Substituting (2-14) into (2-13), we conclude that

$$\begin{aligned}
\mu^{-3/2}\|\Pi-(\Pi)_{B(\mu)}\|_{L^{r,s'}(Q(\mu))} &\le C\Big(\frac{\rho}{\mu}\Big)^{3/2}\rho^{-1}\Big\|u\times\frac{\omega}{|\omega|}\Big\|_{L^{i,j}(Q(\rho))}\rho^{-1/2}\|\nabla u\|_{L^2(Q(\rho))}\\
&\quad+C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}\rho^{-3/2}\|\Pi-(\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}.
\end{aligned}$$

where we have used the fact $2/r+3/s'=7/2$.

Case II: Using Hölder's inequality, (1-16) and (2-2), we see that

$$\begin{aligned}
(2\text{-}16)\quad \|\omega\times u\|_{L^{r,s}(Q(\rho))} &\le \Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{m,n}(Q(\rho))}\|u\|_{L^{\kappa,\tau}(Q(\rho))}\\
&\le \Big\|\omega\times\frac{u}{|u|}\Big\|_{L^{m,n}(Q(\rho))}(\|u\|_{L^{\infty,2}(Q(\rho))}+\|\nabla u\|_{L^2(Q(\rho))}),
\end{aligned}$$

where the pair (r,s) satisfies

$$\frac{2}{r}+\frac{3}{s}=\frac{9}{2}.$$

Just as (2-15), it suffices to verify that

$$\frac{1}{2} < \frac{1}{r} = \frac{1}{m} + \frac{1}{\kappa} < 1 \quad \text{and} \quad \frac{2}{3} < \frac{1}{s} = \frac{1}{n} + \frac{1}{\tau} < 1. \tag{2-17}$$

Indeed, for $1 < m \le 2$, we choose

$$\kappa = \frac{3m}{2m-2} \quad \text{and} \quad \tau = \frac{18m}{m+8}.$$

For $2 < m < 4$, we pick up $\kappa = 2$, $\tau = 6$.

Inserting (2-16) into (2-13), we know that

$$\begin{aligned}
\mu^{-3/2} \|\Pi - (\Pi)_{B(\mu)}\|_{L^{r,s'}(B(\mu))} \\
\le C\Big(\frac{\rho}{\mu}\Big)^{3/2} \rho^{-1} \Big\|\omega \times \frac{u}{|u|}\Big\|_{L^{m,n}(Q(\rho))} \rho^{-1/2} (\|u\|_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|_{L^2(Q(\rho))}) \\
+ C\Big(\frac{\mu}{\rho}\Big)^{3-2/r} \rho^{-3/2} \|\Pi - (\Pi)_{B(\rho)}\|_{L^{r,s'}(Q(\rho))}.
\end{aligned}$$

This finishes the proof. □

Taking full advantage of the interior estimate of harmonic functions, we can extend Lemma 2.1 in [Wolf 2008] and present its proof arguing as with the heat equation.

Lemma 2.5. *Assume that b is the solution of* (1-19). *Then, for $\mu \le \rho/32$, there is a constant C independent of μ and ρ such that*

$$\mu^{-3} \|b\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^2 \big(\rho^{-3} \|b\|^2_{L^2(Q(\rho/2))} + C\rho^{-3} \|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big), \tag{2-18}$$

where the pair (r, s') has been defined as in Lemma 2.4.

Proof. Consider the following smooth cutoff functions:

$$\xi(t) = \begin{cases} 1, & t \ge -(\rho/8)^2, \\ 0, & t \le -(\rho/4)^2; \end{cases} \quad \text{and} \quad \eta(x) = \begin{cases} 1, & x \in B(\rho/8), \\ 0, & x \in B^c(\rho/4), \end{cases}$$

which satisfy

$$0 \le \xi(t), \eta(x) \le 1, \quad |\xi'(t)| \le \frac{C}{\rho^2} \quad \text{and} \quad |\nabla \eta(x)| \le \frac{C}{\rho}.$$

Taking the inner product of (1-19) with $\xi^2\eta^2 b$ over $(-(\rho/4)^2, t) \times B(\rho/4)$, $(t \le 0)$, we arrive at

$$\int_{-(\rho/4)^2}^{t} \int_{B(\rho/4)} \xi^2\eta^2 b b_s \, dx\, ds - \int_{-(\rho/4)^2}^{t} \int_{B(\rho/4)} \xi^2\eta^2 b \Delta b \, dx\, ds = -\int_{-(\rho/4)^2}^{t} \int_{B(\rho/4)} \xi^2\eta^2 b \nabla p_2 \, dx\, ds.$$

Integrating by parts and the Cauchy–Schwarz inequality, we infer that

$$
\begin{aligned}
&\frac{1}{2}\int_{B(\rho/4)}\xi^2(t)\eta^2(x)b^2(t,x)\,dx+\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi^2(s)\eta^2(x)|\nabla b|^2\,dx\,ds\\
&=\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi'\xi\eta^2b^2\,dx\,ds-2\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi^2\nabla\eta\eta b\nabla b\,dx\,ds\\
&\qquad\qquad-\int_{-(\rho/4)^2}^{t}\int_{B(\rho/4)}\xi^2\eta^2b\nabla p_2\,dx\,ds\\
&\le C\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}\xi(\xi|\nabla\eta|^2+|\xi'|\eta^2)b^2dx\,ds+\frac{1}{2}\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}\xi^2(s)\eta^2(x)|\nabla b|^2dx\,ds\\
&\qquad+C\biggl(\int_{-(\rho/4)^2}^{0}\biggl(\int_{B(\rho/4)}\xi^2\eta^2|\nabla p_2|^2\,dx\biggr)^{1/2}ds\biggr)^2+\frac{1}{4}\|\xi\eta b\|^2_{L^{\infty,2}(Q(\rho/4))},
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{-(\rho/4)^2\le t<0}\frac{1}{2}\int_{B(\rho/4)}\xi^2(t)\eta^2(x)b^2(t,x)\,dx+\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}\xi^2(s)\eta(x)^2|\nabla b|^2dx\,ds\\
&\le\frac{C}{\rho^2}\int_{-(\rho/4)^2}^{0}\int_{B(\rho/4)}|b|^2dx\,ds+C\biggl(\int_{-(\rho/4)^2}^{0}\biggl(\int_{B(\rho/4)}|\nabla p_2|^2dx\biggr)^{1/2}ds\biggr)^2+\frac{1}{4}\|\xi\eta b\|^2_{L^{\infty,2}(Q(\rho/4))}.
\end{aligned}
$$

Consequently,

$$
\|b\|^2_{L^{\infty,2}(Q(\rho/8))}+\|\nabla b\|^2_{L^2(Q(\rho/8))}\le C\rho^{-2}\|b\|^2_{L^2(Q(\rho/4))}+C\|\nabla p_2\|^2_{L^{1,2}(Q(\rho/4))}. \tag{2-19}
$$

Notice that the system (1-19) is linear, thus, a slight variant of the proof above provides the estimates

$$
\|\nabla b\|^2_{L^{\infty,2}(Q(\rho/16))}+\|\nabla^2 b\|^2_{L^2(Q(\rho/16))}\le C\rho^{-2}\|\nabla b\|^2_{L^2(Q(\rho/8))}+\|\nabla^2 p_2\|^2_{L^{1,2}(Q(\rho/8))},
$$

and

$$
\|\nabla^2 b\|^2_{L^{\infty,2}(Q(\rho/32))}+\|\nabla^3 b\|^2_{L^2(Q(\rho/32))}\le C\rho^{-2}\|\nabla^2 b\|^2_{L^2(Q(\rho/16))}+\|\nabla^3 p_2\|^2_{L^{1,2}(Q(\rho/16))}.
$$

Collecting the above estimates, we find

$$
\begin{aligned}
&\|\nabla^2 b\|^2_{L^{\infty,2}(Q(\rho/32))}+\|\nabla^3 b\|^2_{L^2(Q(\rho/32))}\\
&\quad\le C\rho^{-2}\bigl\{C\rho^{-2}\|\nabla b\|^2_{L^2(Q(\rho/8))}+\|\nabla^2 p_2\|^2_{L^{1,2}(Q(\rho/8))}\bigr\}+C\|\nabla^3 p_2\|^2_{L^{1,2}(Q(\rho/16))}\\
&\quad\le C\rho^{-2}\bigl\{C\rho^{-2}\bigl[C\rho^{-2}\|b\|^2_{L^2(Q(\rho/4))}+\|\nabla p_2\|^2_{L^{1,2}(Q(\rho/4))}\bigr]+C\|\nabla^2 p_2\|^2_{L^{1,2}(Q(\rho/8))}\bigr\}\\
&\qquad\qquad+C\|\nabla^3 p_2\|^2_{L^{1,2}(Q(\rho/16))}.
\end{aligned}\tag{2-20}
$$

By virtue of the interior estimate of harmonic functions, for every $k\in\mathbb{N}^+$, we have

$$
|\nabla^k p_2(x_0)|\le C\rho^{-3-k}\|p_2\|_{L^1(B_{x_0}(\rho/4))}\le\rho^{-3-k}\|p_2\|_{L^1(B(\rho/2))},
$$

for any $x_0 \in B(\rho/4)$, from which it follows that

$$\begin{aligned}\|\nabla^{k+1} p_2\|_{L^2(B(\rho/4))} &\le C\rho^{\frac{3}{2}}\|\nabla^{k+1} p_2\|_{L^\infty(B(\rho/4))}\\ &\le C\rho^{\frac{3}{2}}\rho^{-(k+1+3)}\|p_2\|_{L^1(B(\rho/2))}\\ &\le C\rho^{-(k+1)}\rho^{\frac{3}{2}-\frac{3}{s'}}\|p_2\|_{L^{s'}(B(\rho/2))}.\end{aligned}$$

Integrating the last inequality in time yields

$$\|\nabla^{k+1} p_2\|_{L^{r,2}(Q(\rho/4))} \le C\rho^{-(k+1)}\rho^{\frac{3}{2}-\frac{3}{s'}}\|p_2\|_{L^{r,s'}(Q(\rho/2))}.$$

Utilizing Hölder's inequality, we discover

$$\|\nabla^{k+1} p_2\|_{L^{1,2}(Q(\rho/4))} \le C\rho^{-(k+1)}\|p_2\|_{L^{r,s'}(Q(\rho/2))},$$

where we have used the fact $2/r+3/s'=7/2$. Plugging this inequality into bounds (2-19) and (2-20) gives

$$\|b\|^2_{L^{\infty,2}(Q(\rho/8))}+\|\nabla b\|^2_{L^2(Q(\rho/8))} \le \frac{C}{\rho^2}\|b\|^2_{L^2(Q(\rho/4))}+\frac{C}{\rho^2}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))},$$

and

$$\|\nabla^2 b\|^2_{L^{\infty,2}(Q(\rho/32))}+\|\nabla^3 b\|^2_{L^2(Q(\rho/32))} \le \frac{C}{\rho^6}\|b\|^2_{L^2(Q(\rho/4))}+\frac{C}{\rho^6}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}.$$

By the Gagliardo–Nirenberg inequality and the latter inequalities, we infer that

$$\begin{aligned}\|b\|^2_{L^2(Q(\mu))} &\le C\mu^5\|b\|^2_{L^\infty(Q(\rho/32))}\\ &\le C\mu^5\big(\|b\|^{2\cdot(1/4)}_{L^{\infty,2}Q(\rho/32)}\|\nabla^2 b\|^{2\cdot(3/4)}_{L^{\infty,2}(Q(\rho/32))}+\frac{C}{\rho^3}\|b\|^2_{L^{\infty,2}(Q(\rho/32))}\big)\\ &\le C\mu^5\big(\rho^{-2}\|b\|^2_{L^2(Q(\rho/4))}+C\rho^{-2}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big)^{1/4}\\ &\quad\times\big(\rho^{-6}\|b\|^2_{L^2(Q(\rho/4))}+C\rho^{-6}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big)^{3/4}\\ &\qquad+C\mu^5\frac{C}{\rho^3}\big(\rho^{-2}\|b\|^2_{L^2(Q(\rho/4))}+C\rho^{-2}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big)\\ &\le C\Big(\frac{\mu}{\rho}\Big)^5\big(\|b\|^2_{L^2(Q(\rho/4))}+C\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big),\end{aligned}$$

which means that

$$\mu^{-3}\|b\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^2\big(\rho^{-3}\|b\|^2_{L^2(Q(\rho/2))}+C\rho^{-3}\|p_2\|^2_{L^{r,s'}(Q(\rho/2))}\big),$$

which is the desired result. □

This lemma entails the desired decay estimate (1-20), that is,

$$
\begin{aligned}
\text{(2-21)}\quad E_2(u,\mu) \le C\Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} \\
+ C\Big(\frac{\mu}{\rho}\Big)^2 [E_2(u,\rho) + P^2_{r,s'}(\Pi - (\Pi)_{B_\rho},\rho)].
\end{aligned}
$$

Indeed, it is enough to bound the right hand of the following inequality:

$$
\begin{aligned}
\mu^{-3}\|u\|^2_{L^2(Q(\mu))} &\le \mu^{-3}\|v\|^2_{L^2(Q(\mu))} + \mu^{-3}\|b\|^2_{L^2(Q(\mu))} \\
&\le \Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|v\|^2_{L^2(Q(\rho))} + \mu^{-3}\|b\|^2_{L^2(Q(\mu))} \\
&\le C\Big(\frac{\rho}{\mu}\Big)^3 \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} + \mu^{-3}\|b\|^2_{L^2(Q(\mu))},
\end{aligned}
$$

where we have used (2-1). To end this, first, by triangle inequality and (2-1) again, we see that

$$
\begin{aligned}
\rho^{-3}\|b\|^2_{L^2(Q(\rho))} &\le \rho^{-3}\|u\|^2_{L^2(Q(\rho))} + \rho^{-3}\|v\|^2_{L^2(Q(\rho))} \\
&\le \rho^{-3}\|u\|^2_{L^2(Q(\rho))} + C\rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))}.
\end{aligned}
$$

Then, we insert the latter estimate and (2-12) into (2-18) to obtain

$$
\begin{aligned}
\mu^{-3}\|b\|^2_{L^2(Q(\mu))} \le C\Big(\frac{\mu}{\rho}\Big)^2 \big(\rho^{-3}\|u\|^2_{L^2(Q(\rho))} + \rho^{-3}\|\omega\times u\|^2_{L^{r,s}(Q(\rho))} \\
+ C\rho^{-3}\|\Pi - (\Pi)_{B_\rho}\|^2_{L^{r,s'}(Q(\rho))}\big).
\end{aligned}
$$

This inequality yields the desired estimate (2-21).

Before we state the auxiliary results to the stationary Navier–Stokes equations, we first recall the Caffarelli–Kohn–Nirenberg regular condition below to the steady Navier–Stokes equations.

Proposition 2.6 [Struwe 1995; Dong and Strain 2012; Wang and Wu 2014]. *Suppose (u, p) is a suitable weak solution to* (1-21) *and the external force $f \in L^q(\Omega)$ with $q > \frac{1}{2}d$. Then the origin* 0 *is a regular point for $u(x)$ if the following condition holds*:

$$
\text{(2-22)}\qquad \limsup_{\mu\to 0} \frac{1}{\mu^{d-4}} \int_{B(\mu)} |\nabla u|^2\,dx < \varepsilon, \quad d = 5, 6,
$$

for a universal constant $\varepsilon > 0$.

To show Theorem 1.4, we need to prove the following lemma:

Lemma 2.7. *Let* $0 < 2r < \rho$. *It holds that*

$$\begin{aligned}
(2\text{-}23)\quad & \tilde{E}(u,\mu)+\tilde{E}_*(u,\mu) \\
&\le C\Big(\frac{\mu}{\rho}\Big)^2\tilde{E}(u,\rho) \\
&+C\Big(\frac{\rho}{\mu}\Big)^{d-3}\big(\tilde{P}_{\frac{2d}{d+2}}(\Pi-(\Pi)_{B(\rho)},\rho)+\tilde{F}_{\frac{2d}{d+2}}(f,\rho)\big)[\tilde{E}(u,\rho)+\tilde{E}_*(u,\rho)]^{1/2},
\end{aligned}$$

where the constant C *is independent of* μ *and* ρ.

Proof. The conclusion can be derived by a slight change of the proof of Lemma 2.3 as follows. In the spirit of the backward heat kernel for the time-dependent case, we modify slightly the fundamental solution of Laplace equations to set

$$\Gamma(x)=\frac{1}{(\mu^2+|x|^2)^{(d-2)/2}},\qquad d=5,6.$$

An easy computation gives

$$\partial_i\Gamma(x)=-\frac{(d-2)x_i}{(\mu^2+|x|^2)^{d/2}}\quad\text{and}\quad\Delta\Gamma(x)=\frac{-d(d-2)\mu^2}{(\mu^2+|x|^2)^{(d+2)/2}}.$$

Consider the smooth cutoff function

$$\eta(x)=\begin{cases}1, & x\in B(\rho/2),\\ 0, & x\in B^c(\rho),\end{cases}$$

which satisfies

$$0\le\eta(x)\le 1,\quad |\nabla\eta(x)|\le\frac{C}{\rho},\quad\text{and}\quad|\Delta\eta(x)|\le\frac{C}{\rho^2}.$$

The desired estimate turns out to be a consequence of the following properties of the test function $\eta(x)\Gamma(x)$:

(i) For every $x\in B(\mu)$, straightforward calculations yield

$$-\Delta\Gamma\ge C\mu^{-d},\qquad\Gamma\ge C\mu^{-(d-2)}.$$

(ii) For every $x\in B(\rho)$, it is easy to verify that

$$|\eta(x)\Gamma|\le C\mu^{-(d-2)},\qquad|\eta(x)\nabla\Gamma|+|\Gamma\nabla\eta(x)|\le C\mu^{-(d-1)},$$

(iii) For every $\rho/2\le|x|\le\rho$, we know that

$$|\Gamma\Delta\eta(x)|+|\nabla\eta(x)\cdot\nabla\Gamma|\le C\rho^{-d}.$$

Inserting $\phi = \eta(x)\Gamma(x)$ into the local energy inequality (1-22), we see that

$$-\int_{B(\rho)} |u|^2 \eta \Delta\Gamma \, dx + 2\int_{B(\rho)} |\nabla u|^2 \eta \Gamma \, dx \le \int_{B(\rho)} |u|^2 (\Gamma \Delta\eta + 2\nabla\eta \cdot \nabla\Gamma) \, dx + 2\int_{B(\rho)} (\Pi - (\Pi)_{B(\rho)}) u \cdot (\eta\nabla\Gamma + \Gamma\nabla\eta) + 2\int_{B(\rho)} f \cdot u\eta\Gamma \, dx.$$

This inequality implies

$$-\int_{B(\mu)} |u|^2 \Delta\Gamma \, dx + 2\int_{B(\mu)} |\nabla u|^2 \Gamma \, dx \le \int_{B(\rho)\setminus B(\rho/2)} |u|^2 (\Gamma \Delta\eta + 2\nabla\eta \cdot \nabla\Gamma) \, dx + 2\int_{B(\rho)} (\Pi - (\Pi)_{B(\rho)}) u \cdot (\eta\nabla\Gamma + \Gamma\nabla\eta) \, dx + 2\int_{B(\rho)} f \cdot u\eta\Gamma \, dx.$$

The property of test functions and Hölder's inequality yield that

$$\begin{aligned} &\frac{C}{\mu^{d-2}} \int_{B(\mu)} |u|^2 \, dx + \frac{C}{\mu^{d-4}} \int_{B(\mu)} |\nabla u|^2 \, dx \\ &\le \Big(\frac{\mu}{\rho}\Big)^2 \frac{1}{\rho^{d-2}} \int_{B(\rho)\setminus B(\rho/2)} |u|^2 \, dx \\ &\quad + C\Big(\frac{\rho}{\mu}\Big)^{d-3} \Big(\frac{1}{\rho^{\frac{d^2-2d}{d+2}}} \int_{B(\rho)} |(\Pi - (\Pi)_{B(\rho)})|^{\frac{2d}{d+2}} \, dx\Big)^{\frac{d+2}{2d}} \Big(\frac{1}{\rho^{\frac{d^2-4d}{d-2}}} \int_{B(\rho)} |u|^{\frac{2d}{d-2}} \, dx\Big)^{\frac{d-2}{2d}} \\ &\quad + C\Big(\frac{\rho}{\mu}\Big)^{d-4} \Big(\frac{1}{\rho^{\frac{d^2-4d}{d+2}}} \int_{B(\rho)} |f|^{\frac{2d}{d+2}} \, dx\Big)^{\frac{d+2}{2d}} \Big(\frac{1}{\rho^{\frac{d^2-4d}{d-2}}} \int_{B(\rho)} |u|^{\frac{2d}{d-2}} \, dx\Big)^{\frac{d-2}{2d}}. \end{aligned}$$

Combining this estimate with the Sobolev embedding

$$\|u\|_{L^{2d/(d-2)}(B(\rho))} \le C(\|\nabla u\|_{L^2(B(\rho))} + \rho^{-1}\|u\|_{L^2(B(\rho))}), \quad x \in \mathbb{R}^d \tag{2-24}$$

with $d = 5, 6$, we derive the desired estimate (2-23). □

3. Proofs of theorems

This section is devoted to the proofs of Theorem 1.1–1.4.

Proof of Theorem 1.1. In the light of Hölder's inequality, it suffices to deal with the case $2/p + 3/q = 7/2$. According to the hypothesis of Theorem 1.1, we know that there exists a constant $r_0 > 0$ such that

$$P_{p,q}(\Pi - (\Pi)_{B(\mu)}, \mu) \le \varepsilon_1, \quad \text{for any } \mu \le r_0.$$

Before going further, we set

$$G_1(\mu) = E(u, \mu) + E_*(u, \mu) \quad \text{and} \quad \lambda = \mu/\rho \quad (\lambda \le 1/4).$$

By (2-3) in Lemma 2.3 and Young's inequality, we derive that

$$\begin{aligned} G_1(\mu) &\le C\Big(\frac{\mu}{\rho}\Big)^2 E(u, \rho) + C\Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho)[E(u, \rho) + E_*(u, \rho)]^{1/2} \\ &\le C\Big(\frac{\mu}{\rho}\Big)^2 G_1(\rho) + C\Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho) G_1(\rho) + \Big(\frac{\rho}{\mu}\Big)^2 P_{p,q}(\Pi - (\Pi)_{B(\rho)}, \rho) \\ &\le C_2\lambda^2 G_1(\rho) + C_1\lambda^{-2}\varepsilon_1 G_1(\rho) + \lambda^{-2}\varepsilon_1. \end{aligned}$$

Choosing λ, ε_1 such that $q = 2C_2\lambda^2 < 1$ and $\varepsilon_1 = \min\{q\lambda^2/(2C_1), (1-q)\lambda^3\varepsilon/2\}$, we obtain

$$G_1(\lambda\rho) \le q G_1(\rho) + \lambda^{-2}\varepsilon_1$$

Iterating the latter inequality, we deduce that

$$G_1(\lambda^k\rho) \le q^k G_1(\rho) + \tfrac{1}{2}\lambda\varepsilon.$$

From the definition of $G_1(\mu)$, there exists a positive number K_0 such that

$$q^{K_0} G_1(r_0) \le 2\frac{C(\|u\|_{L^\infty L^2}, \|\nabla u\|_{L^2})}{r_0} q^{K_0} \le \frac{1}{2}\lambda\varepsilon.$$

Let $r_2 := \lambda^{K_0} r_0$. For every $0 < r \le r_2$, there exists $k \ge K_0$ such that $\lambda^{k+1} r_0 \le r \le \lambda^k r_0$. An easy computation yields that

$$E_*(r) \le \frac{1}{\lambda^{k+1} r_0} \iint_{Q(\lambda^k r_0)} |\nabla u|^2 \, dx\, dt \le \frac{1}{\lambda} G_1(\lambda^k r_0) \le \frac{1}{\lambda}\Big(q^{k-K_0} q^{K_0} G_1(r_0) + \frac{1}{2}\lambda\varepsilon\Big) \le \varepsilon.$$

This together with (1-3) completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Thanks to Hölder's inequality, without loss of generality, we just consider the endpoint case $2/p^\natural + 3/q^\natural = 2$. With the estimate (2-4) in hand, arguing as with the iteration method above, we can finish the proof. □

Proof of Theorem 1.4. It follows from Hölder's inequality that

$$\tilde{F}_p(f, \mu) = \mu^{3p-d} \int_{B(\mu)} |f(x)|^p \, dx \le \mu^{3p - \frac{p}{q} d} \Big(\int_\Omega |f(x)|^q \, dx \Big)^{p/q}, \tag{3-1}$$

which together with the integrability hypothesis on the force f implies that

$$\tilde{F}_p(f, \mu) \text{ tends to } 0 \text{ as } \mu \to 0,$$

where $p < \frac{1}{2}d < q$. Therefore, we see that there is a constant r_1 such that for any $\mu \le r_1$, $\tilde{F}_{2d/(d+2)}(f, \mu) \le \varepsilon_2$. Owing to the assumption, there exists a constant $r_2 \le r_1$ such that

$$\tilde{P}_{2d/(d+2)}(\Pi - (\Pi)_{B(\mu)}, \mu) \le \varepsilon_2, \quad \text{for any } \mu \le r_2.$$

Based on this inequality and (2-23) in Lemma 2.7, we complete the proof in the same way as in the proof of Theorem 1.1. □

Proof of Theorem 1.3. This will occupy the remainder of the section. We start with some preliminaries. Recall the symbols r, s' defined ias n Lemma 2.4, which correspond to the borderline cases of (1-15) and (1-17). Set

$$\frac{1}{r^\sharp} = 1 - \frac{1}{r} \quad \text{and} \quad \frac{1}{s^\sharp} = 1 - \frac{1}{s'}.$$

Then it is obvious that

$$\frac{2}{r^\sharp} + \frac{3}{s^\sharp} = \frac{3}{2} \quad \text{with } r^\sharp \in [2, \infty),\ s^\sharp \in (2, 6).$$

It follows from (2-2) that

$$\|u\|_{L^{r^\sharp, s^\sharp}(Q(\rho))} \le C(\|u\|_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|_{L^2(Q(\rho))}).$$

Consider the usual cutoff function $\varphi(x, t) \in C_0^\infty(Q(2\mu))$ satisfying $\varphi \equiv 1$ in $Q(\mu)$, $0 \le \varphi \le 1$, $|\nabla\varphi| \le C\mu^{-1}$ and $|\partial_t\varphi| + |\Delta\varphi| \le C\mu^{-2}$. By the divergence-free condition div $= 0$, Hölder's inequality and the latter inequality, for $32\mu \le \rho$, we infer that

$$\begin{aligned}\iint_{Q(2\mu)} u\cdot\nabla\varphi\Pi\, dx\, ds = & \iint_{Q(2\mu)} u\cdot\nabla\varphi(\Pi - (\Pi)_{B(2\mu)})\, dx\, ds \\ \le & C\mu^{-1}\|\Pi - (\Pi)_{B(2\mu)}\|_{L^{r,s'}(Q(2\mu))} \|u\|_{L^{r^\sharp, s^\sharp}(Q(2\mu))} \\ \le & C\mu^{-1}\|\Pi - (\Pi)_{B(2\mu)}\|_{L^{r,s'}(Q(2\mu))}(\|u\|_{L^{\infty,2}(Q(\rho))} + \|\nabla u\|_{L^2(Q(\rho))}).\end{aligned}$$

Choosing $\varphi(x, t)$ as the test function in (1-9) and using the latter relation, we see that

$$\begin{aligned}&E(u, \mu) + E_*(u, \mu) \\ &\qquad \le E_2(u, 2\mu) + P_{r,s'}(\Pi - (\Pi), 2\mu)\Big(\frac{\rho}{\mu}\Big)^{1/2}(E(u, \rho) + E_*(u, \rho))^{1/2}.\end{aligned} \tag{3-2}$$

This concludes the preliminaries. The proof proper is divided into two steps.

(1) Substituting (2-14) into (2-21), we have

$$E_2(u,\mu) \le C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)E_*(u,\rho) + C\Big(\frac{\mu}{\rho}\Big)^2[E_2(u,\rho) + P_{r,s'}^2(\Pi - (\Pi)_{B_\rho}, \rho)]. \tag{3-3}$$

Plugging (3-3) and (2-8) into (3-2), we infer that

$$\begin{aligned}E(u,\mu)+E_*(u,\mu)\le C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)E_*(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^2[E(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)]\\ +\Big[C\Big(\frac{\rho}{\mu}\Big)^{3/2}U_{i,j}(\times,\rho)E_*^{1/2}(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}P_{r,s'}(\Pi-(\Pi)_{B_\rho},\rho)\Big]\\ \times\Big(\frac{\rho}{\mu}\Big)^{1/2}[E(u,\rho)+E_*(u,\rho)]^{1/2}.\end{aligned}$$

We define $G_2(\mu)=E(u,\mu)+E_*(u,\mu)+P_{r,s'}^2(\Pi-(\Pi)_{B_\mu},\mu)$. Then the last inequality and (2-8) in Lemma 2.4 lead to

$$\begin{aligned}G_2(\mu)\le{}& C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)G_2(\rho)+C\Big(\frac{\mu}{\rho}\Big)^2 G_2(\rho)\\ &+C\Big(\frac{\rho}{\mu}\Big)^2 U_{i,j}(\times,\rho)G_2(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r}G_2(\rho)\\ &+C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}^2(\times,\rho)E_*(u,\rho)+C\Big(\frac{\mu}{\rho}\Big)^{6-4/r}P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)\\ \le{}& C\Big(\frac{\rho}{\mu}\Big)^3 U_{i,j}(\times,\rho)G_2(\rho)+C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r}G_2(\rho).\end{aligned}\tag{3-4}$$

Now, by an argument completely analogous to that in the proof of Theorem 1.1, we can complete the first part of the proof of Theorem 1.3.

(2) Substituting (2-16) into (2-21), we get

$$\begin{aligned}E_2(u,\mu)\le{}&\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}^2(\times,\rho)[E(u,\rho)+E_*(u,\rho)]\\ &+C\Big(\frac{\mu}{\rho}\Big)^2[E(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)].\end{aligned}\tag{3-5}$$

Plugging (3-5) and (2-9) into (3-2), we infer that

$$\begin{aligned}&E(u,\mu)+E_*(u,\mu)\\ &\le\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}^2(\times,\rho)[E(u,\rho)+E_*(u,\rho)]+C\Big(\frac{\mu}{\rho}\Big)^2[E(u,\rho)+P_{r,s'}^2(\Pi-(\Pi)_{B_\rho},\rho)]\\ &+\Big\{C\Big(\frac{\rho}{\mu}\Big)^{3/2}W_{m,n}(\times,\rho)[E(u,\rho)+E_*(u,\rho)]^{1/2}+C\Big(\frac{\mu}{\rho}\Big)^{3-2/r}P_{r,s'}(\Pi-(\Pi)_{B_\rho},\rho)\Big\}\\ &\qquad\times\Big(\frac{\rho}{\mu}\Big)^{1/2}[E(u,\rho)+E_*(u,\rho)]^{1/2}.\end{aligned}$$

Let

$$G_3(\mu)=E(u,\mu)+E_*(u,\mu)+P_{r,s'}^2(\Pi-(\Pi)_{B_\mu},\mu).$$

Then the latter relation and (2-9) allow us to obtain

$$
\begin{aligned}
\text{(3-6)}\qquad G_3(\mu) \leq{} & C\Big(\frac{\rho}{\mu}\Big)^3 W^2_{m,n}(\times,\rho)G_3(\rho) + C\Big(\frac{\mu}{\rho}\Big)^2 G_3(\rho)\\
& + C\Big(\frac{\rho}{\mu}\Big)^2 W_{m,n}(\times,\rho)G_3(\rho) + C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r} G_3(\rho)\\
& + C\Big(\frac{\rho}{\mu}\Big)^3 W^2_{m,n}(\times,\rho)G_3(\rho) + C\Big(\frac{\mu}{\rho}\Big)^{6-\frac{4}{r}} G_3(\rho)\\
\leq{} & C\Big(\frac{\rho}{\mu}\Big)^3 W_{m,n}(\times,\rho)G_3(\rho) + C\Big(\frac{\mu}{\rho}\Big)^{5/2-2/r} G_3(\rho).
\end{aligned}
$$

Combining equations (3-4) and (3-6) and iterating as in the proof of Theorem 1.1 completes the second part of the proof of Theorem 1.3. □

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