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CHARACTERIZATIONS OF IMMERSED GRADIENT ALMOST RICCI SOLITONS

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Our purpose is to study the geometry of gradient almost Ricci solitons isometrically immersed either in the hyperbolic space $\mathbb{H}^{n+1}$, in the de Sitter space $\mathbb{S}_1^{n+1}$, or in the anti-de Sitter space $\mathbb{H}_1^{n+1}$. In each one of these ambient spaces we obtain extensions of a classical theorem due to Nomizu and Smith. More precisely, we show that the totally umbilical hypersurfaces are the only immersed hypersurfaces of such ambient spaces which admit a structure of gradient almost Ricci soliton via the tangential component of a certain fixed vector, and whose image of the Gauss mapping is also totally umbilical. Furthermore, in the case that the structure of gradient almost Ricci soliton is nontrivial, we conclude that such a hypersurface must be isometric either to $\mathbb{H}^n$, when the ambient space is $\mathbb{H}^{n+1}$ or $\mathbb{H}_1^{n+1}$, or to $\mathbb{S}^n$, when the ambient space is $\mathbb{S}_1^{n+1}$.

1. Introduction

The concept of a Ricci soliton, introduced in the seminal paper [Hamilton 1982], corresponds to a natural generalization of Einstein metrics. We recall that a Riemannian manifold (M^n, g) is called a Ricci soliton if there exist a complete vector field X and a constant λ satisfying the equation

$$\mathrm{Ric} + \tfrac{1}{2}\mathscr{L}_X g = \lambda g, \tag{1-1}$$

where Ric and $\mathscr{L}$ stand for the Ricci tensor and the Lie derivative on M^n.

Ricci solitons also correspond to selfsimilar solutions of Hamilton's Ricci flow [ibid.] and often arise as limits of dilations of singularities in the Ricci flow. They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. For more details on this subject, we recommend the survey [Cao 2010].

Pigola et al. [2011] extended the definition of Ricci solitons by adding the condition that the parameter λ in (1-1) be a smooth real function on M^n; this

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attracted much attention in the mathematical community. Such solitons arise from the Ricci–Bourguignon flow as shown recently in [Catino et al. 2016]. In this more general setting, we refer to (1-1) as being the fundamental equation of an almost Ricci soliton (M^n, g, X, λ). Following the terminology of Ricci solitons, an almost Ricci soliton is called *expanding* or *shrinking* if $\lambda < 0$ or $\lambda > 0$, respectively. When $\lambda = 0$ we have a *steady* Ricci soliton. Otherwise, it will be called *indefinite*.

When the vector field X is a gradient of a smooth function $f : M^n \to \mathbb{R}$, the manifold will be called a gradient almost Ricci soliton. In this case, (1-1) becomes

$$\mathrm{Ric} + \nabla^2 f = \lambda g, \tag{1-2}$$

where $\nabla^2 f$ stands for the Hessian of the potential function f. When either the vector field X is trivial, or the potential f is constant, the almost Ricci soliton will be called *trivial*, otherwise it will be a *nontrivial* almost Ricci soliton.

We notice that when $n \geq 3$ and X is a Killing vector field, an almost Ricci soliton is a genuine a Ricci soliton. Indeed, in this case, (M^n, g) is an Einstein manifold and we can apply Schur's lemma to deduce that λ is constant. Conditions under which a nontrivial almost Ricci soliton structure exists were first investigated in [Pigola et al. 2011]. Subsequently, Barros and Ribeiro [2012] obtained some structural equations and deduced corresponding rigidity theorems; jointly with Batista, they also showed in [Barros et al. 2014b] that any compact nontrivial almost Ricci soliton (M^n, g, X, λ) with constant scalar curvature is isometric to a Euclidean sphere $\mathbb{S}^n$. As a consequence, they concluded that every compact nontrivial almost Ricci soliton with constant scalar curvature must be gradient.

Almost Ricci solitons that are realized as Einstein warped products, with a one-dimensional base and Einstein fibers, were constructed in [Pigola et al. 2011]. By using Lemma 1.1 of that paper, we can prove that the warped product $M = \mathbb{R} \times_\psi \mathbb{H}^m$ with metric $g = dt^2 + \psi^2 g_0$, has a structure of almost Ricci soliton $(M, g, \nabla\tilde{f}, \tilde{\lambda})$, where g_0 is the standard metric of $\mathbb{H}^m$ and the functions involved are the respective lifts of $f(t) = \sinh t$ and $\lambda(t) = \sinh t - m$, whereas the warping function is $\psi(t) = \cosh t$. More generally, a necessary and sufficient condition for a warped product Einstein manifold to support a gradient almost Ricci soliton structure was shown in [Feitosa et al. 2015].

Recall also that there exist manifolds that do not admit an almost Ricci soliton structure. For instance, Pigola et al. [2011] proved that $\mathbb{H}^2 \times \mathbb{H}^2$ has this property. For a locally conformally flat gradient almost Ricci soliton, Catino [2012] proved that, around any regular point of the potential f, such a manifold $(M^n, g, \nabla f, \lambda)$ is locally a warped product with fibers of constant sectional curvature.

Jointly with Barros and Ribeiro, the third author studied in [Barros et al. 2011] isometric immersions of an almost Ricci soliton (M^n, g, X, λ) into a Riemannian manifold $\widetilde{M}^{n+p}$. In this context, they presented some obstruction results in order to obtain a minimal immersion under conditions on the sectional curvature of $\widetilde{M}^{n+p}$. In particular, when $\widetilde{M}^{n+p}$ has nonpositive sectional curvature, they proved that if (M^n, g, X, λ) is a traditional Ricci soliton and X has integrable norm on M^n, then M^n cannot be minimal. Moreover, they showed that if (M^n, g, X, λ) is a shrinking Ricci soliton and X has bounded norm on M^n, then M^n must be compact. Hence, when $\widetilde{M}^{n+p}$ is a space-form of nonpositive sectional curvature, such an immersion cannot be minimal. We refer to [Mastrolia et al. 2013] for further discussions.

On the other hand, it is well known that the study of the behavior of the Gauss mapping gives deep information on the geometry of an isometric immersion. For instance, Nomizu and Smyth [1969] showed that a compact connected orientable manifold M^n immersed in the sphere $\mathbb{S}^{n+1}$ with constant mean curvature is a hypersphere if the Gauss image of M^n lies in a closed hemisphere of $\mathbb{S}^{n+1}$. More recently, the first and second authors jointly with Barros [Barros et al. 2014a] showed that a constant mean curvature complete hypersurface of the hyperbolic space $\mathbb{H}^{n+1}$, whose image of the Gauss mapping lies in a totally umbilical spacelike hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$, must be totally umbilical.

In the Lorentzian setting, Xin [1991] and Aiyama [1992], working independently, characterized spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in the Lorentz–Minkowski space $\mathbb{R}_1^{n+1}$ whose image of the Gauss mapping is contained in a geodesic ball of $\mathbb{H}^n$; see also [Palmer 1990] for a weaker first version of this result. When the ambient space is $\mathbb{S}_1^{n+1}$, Aledo and Alías [2002] showed that the spacelike geodesic round spheres are the only complete constant mean curvature hypersurfaces in $\mathbb{S}_1^{n+1}$ having the image of its Gauss mapping contained in a geodesic ball of $\mathbb{H}^{n+1}$. The first and second authors [Aquino and de Lima 2014] established another rigidity results showing that a complete spacelike hypersurface immersed with constant mean curvature either in the de Sitter space $\mathbb{S}_1^{n+1}$ or in the anti-de Sitter space $\mathbb{H}_1^{n+1}$ must be totally umbilical, provided that its Gauss mapping has some suitable behavior.

Here, motivated by the works previously described, we apply suitable formulas for the covariant and Lie derivatives of the scalar curvature (see Lemmas 1 and 2, respectively) in order to study the geometry of gradient almost Ricci solitons isometrically immersed either in the hyperbolic space $\mathbb{H}^{n+1}$ or in the de Sitter space $\mathbb{S}_1^{n+1}$ or in the anti-de Sitter space $\mathbb{H}_1^{n+1}$. In this setting, we show that the totally umbilical hypersurfaces of such ambient spaces are the only immersed hypersurfaces which admit a structure of gradient almost Ricci soliton via the tangential component of a certain fixed vector, and whose image of the corresponding Gauss mapping is also totally umbilical (see Theorems 4, 6, and 8). Furthermore, if in addition we impose that the structure of gradient almost Ricci soliton must be nontrivial, then we conclude that such a hypersurface is isometric either to $\mathbb{H}^n$, when the ambient space is $\mathbb{H}^{n+1}$ or $\mathbb{H}_1^{n+1}$, or to $\mathbb{S}^n$, when the ambient space is $\mathbb{S}_1^{n+1}$ (see Corollaries 5, 7, and 9).

To close this introductory section, we also observe that the existence of a Ricci soliton structure on hypersurfaces of the Euclidean space whose potential vector is given by the tangential component of the position vector was recently investigated by Chen and Deshmukhin [2014].

2. Preliminaries

Let $\mathbb{R}^{n+2}_\nu$ denote the $(n+2)$-dimensional semi-Euclidean space of index $\nu \geq 1$, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the semi-Riemannian metric

$$\langle\cdot,\cdot\rangle = -\sum_{i=1}^{\nu} dx_i^2 + \sum_{j=\nu+1}^{n+2} dx_j^2,$$

where $x = (x_1, \ldots, x_{n+2})$ denote the usual coordinates in $\mathbb{R}^{n+2}$. When $\nu = 1$, $\mathbb{R}^{n+2}_1$ is the so-called Lorentz–Minkowski space.

For a vector field X in $\mathbb{R}^{n+2}_\nu$, let $\varepsilon_X = \langle X, X\rangle$. We say that X is a *unit* vector field if $\varepsilon_X = \pm 1$, and *timelike* if $\varepsilon_X = -1$.

The $(n+1)$-dimensional hyperbolic space is the following hyperquadric of $\mathbb{R}^{n+2}_1$

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1;\ \langle x, x\rangle = -1,\ x_{n+2} \geq 1\}.$$

Let us consider a connected and oriented isometrically immersed hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ and let us denote by $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the Weingarten operator associated to the vector field N as well as $H = \frac{1}{n}\operatorname{tr}(A)$ stands for mean curvature of Σ^n.

Associated to A we have its traceless operator $\Phi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by

$$\Phi(X) = AX - HX,$$

for every $X \in \mathfrak{X}(\Sigma)$. It is easily checked that the Hilbert–Schmidt norm of Φ (that is, $|\Phi|^2 = \operatorname{tr}(\Phi^*\Phi)$, where Φ^* stands for the adjoint of Φ) satisfies

$$|\Phi|^2 = |A|^2 - nH^2.$$

Consequently, we have that $|\Phi|^2 = 0$ if, and only, if Σ^n is a totally umbilical hypersurface.

Recall that, if ∇^0, $\overline{\nabla}$, and ∇ stands for the Levi–Civita connections in $\mathbb{R}^{n+2}_1$, $\mathbb{H}^{n+1}$, and Σ^n, respectively, then the Gauss and Weingarten formulas for a hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N \tag{2-1}$$

and

$$AX = -\overline{\nabla}_X N = -\nabla^0{}_X N, \tag{2-2}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Consequently, from Gauss equations we have that the Ricci curvature of Σ^n is given by

$$\mathrm{Ric}_\Sigma(X, Y) = (1-n)\langle X, Y\rangle + nH\langle AX, Y\rangle - \langle AX, AY\rangle. \tag{2-3}$$

In addition, for a fixed arbitrary vector $a \in \mathbb{R}^{n+2}_1$, let us consider the *height* and the *angle* functions, defined respectively by, $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. A direct computation allows us to conclude that the gradient of such functions are given by $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that this

$$a^\top = a - f_a N + l_a \psi. \tag{2-4}$$

Taking into account that $\nabla^0 a = 0$ and using the Gauss and Weingarten formulas concerning a vector field X tangent to Σ^n,

$$\nabla_X a^\top = f_a AX + l_a X. \tag{2-5}$$

We use (2-5) and the Codazzi equation to deduce

$$\nabla_X A(a^\top) = f_a A^2 X + l_a AX + (\nabla_{a^\top} A)(X). \tag{2-6}$$

Thus, it follows from (2-5) and (2-6) that

$$\Delta l_a = nHf_a + nl_a \tag{2-7}$$

and

$$\Delta f_a = -|A|^2 f_a - nHl_a - n\langle \nabla H, a^\top\rangle. \tag{2-8}$$

See also [Rosenberg 1993].

Now, we deal with hypersurfaces isometrically immersed into two classes of simply connected Lorentzian space-forms. The first one is the $(n+1)$-dimensional de Sitter space

$$\mathbb{S}^{n+1}_1 = \{x \in \mathbb{R}^{n+2}_1;\ \langle x, x\rangle = 1\},$$

a hyperquadric of $\mathbb{R}^{n+2}_1$ with sectional curvature equal to 1. The second one is the $(n+1)$-dimensional anti-de Sitter space

$$\mathbb{H}^{n+1}_1 = \{x \in \mathbb{R}^{n+2}_2;\ \langle x, x\rangle = -1\},$$

a hyperquadric of $\mathbb{R}^{n+2}_2$ with sectional curvature equal to -1. Topologically, $\mathbb{H}^{n+1}_1$ is $\mathbb{S}^1 \times \mathbb{R}^n$ and the semi-Euclidean metric on $\mathbb{R}^{n+2}_2$ induces a Lorentzian metric of constant sectional curvature -1 on $\mathbb{H}^{n+1}_1$. Moreover, the universal covering manifold $\widetilde{\mathbb{H}}^{n+1}_1$ of $\mathbb{H}^{n+1}_1$ is topologically $\mathbb{R}^{n+1}$ (that is, $\widetilde{\mathbb{H}}^{n+1}_1$ is simply connected) and is thus a Lorentzian analogue of the usual Riemannian hyperbolic space of negative curvature -1, which is called the *universal anti-de Sitter spacetime*; see, for instance, [Beem et al. 1996, Section 5.3] or [O'Neill 1983, Section 8.6].

In order to simplify our notation, we will denote by $\mathbb{M}_c^{n+1}$ either the de Sitter space or the anti-de Sitter space, according to whether $c = 1$ or $c = -1$, respectively. In this setting, let $\psi : \Sigma^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_\nu^{n+2}$ be a connected spacelike hypersurface immersed into $\mathbb{M}_c^{n+1}$ (that is, the induced metric via ψ is a Riemannian metric on Σ^n). Let us consider $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the Weingarten operator of Σ^n with respect to a choice of timelike orientation N for Σ^n. We will denote by ∇^0, $\overline{\nabla}$, and ∇ the Levi–Civita connections of $\mathbb{R}_\nu^{n+2}$, $\mathbb{M}_c^{n+1}$, and Σ^n, respectively. Then, the Gauss and Weingarten formulas corresponding to Σ^n are given, respectively, by

$$\nabla^0{}_X Y = \nabla_X Y - \langle AX, Y\rangle N - c\langle X, Y\rangle\psi \quad \text{and} \quad AX = -\overline{\nabla}_X N = -\nabla^0{}_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Thus, from Gauss equation we have that the Ricci curvature of Σ^n is given by

$$\mathrm{Ric}_\Sigma(X, Y) = c(n-1)\langle X, Y\rangle + nH\langle AX, Y\rangle + \langle AX, AY\rangle, \tag{2-9}$$

where $H = -\frac{1}{n}\,\mathrm{tr}(A)$ is the mean curvature of Σ^n.

At this point, we observe that the choice of the sign in our definition of H is motivated by the fact that in that case the mean curvature vector is given by $\overrightarrow{H} = HN$. Hence, $H(p) > 0$ at a point $p \in \Sigma^n$ if and only if $\overrightarrow{H}(p)$ is in the same time-orientation as $N(p)$ (in the sense that $\langle \overrightarrow{H}, N\rangle_p < 0$).

As before, it is also convenient to consider the traceless operator associated to the second fundamental form of Σ^n, $\Phi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which, in the Lorentzian setting, is given by $\Phi(X) = AX + HX$, for all $X \in \mathfrak{X}(\Sigma)$. It is easy to verify that Σ^n is a totally umbilical hypersurface if and only if Φ vanishes identically on Σ^n.

As in the case of immersions in the hyperbolic space, associated to a fixed arbitrary vector $a \in \mathbb{R}_\nu^{n+2}$, let us consider the height function $l_a = \langle \psi, a\rangle$ and the angle function $f_a = \langle N, a\rangle$. A direct computation allows us to conclude that the gradients of such functions are given by $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^\top = a + f_a N - c\, l_a \psi. \tag{2-10}$$

Taking into account that $\nabla^0 a = 0$ and using the Gauss and Weingarten formulas concerning a vector field X tangent to Σ^n, it is not difficult to verify that

$$\nabla_X \nabla l_a = -f_a AX - c l_a X. \tag{2-11}$$

Now, we use (2-11) jointly with the Codazzi equation to deduce

$$\nabla_X \nabla f_a = f_a A^2 X + c l_a AX - (\nabla_{a^\top} A)(X). \tag{2-12}$$

Then, it follows from (2-11) and (2-12) that

$$\Delta l_a = nHf_a - cnl_a \tag{2-13}$$

and

$$\Delta f_a = |A|^2 f_a - cnHl_a + n\langle \nabla H, a^\top \rangle. \tag{2-14}$$

To close this section, we will quote three key lemmas, which will be essential in the proofs of our results. The first one corresponds to item (2) of Proposition 1 in [Barros and Ribeiro 2012].

Lemma 1. If Σ^n is a gradient almost Ricci soliton with potential function f, then

$$\nabla R = 2\,\mathrm{Ric}_\Sigma(\nabla f) + 2(n-1)\nabla\lambda, \tag{2-15}$$

where R stands for the scalar curvature of Σ^n.

The second auxiliary lemma is a well known formula of the theory of conformal vector fields in Riemannian geometry; see, for instance, Yano [1970].

Lemma 2. If X is a conformal vector field on a Riemannian manifold Σ^n with metric g such that $\mathscr{L}_X g = 2\sigma g$, then

$$\mathscr{L}_X R = -2(n-1)\Delta\sigma - 2R\sigma, \tag{2-16}$$

where R stands for the scalar curvature of Σ^n.

The third key lemma gives a suitable characterization of totally umbilical hypersurfaces in a semi-Riemannian space-form due to Kim et al. [2002], which can be regarded as a converse of a theorem due to Sharma and Duggal [1985].

Lemma 3. Let Σ^n be a connected semi-Riemannian hypersurface immersion in a semi-Riemannian space-form $\mathbb{M}_c^{n+1}$. Suppose that $\mathbb{M}_c^{n+1}$ carries a conformal vector field V whose tangential component $V^\top$ on Σ^n becomes a conformal vector field. Then, one of the following holds:

(a) Σ^n is a totally umbilical hypersurface;

(b) the restriction of V to Σ^n reduces to a tangent vector field on Σ^n.

3. Characterizing gradient almost Ricci solitons in $\mathbb{H}^{n+1}$

We recall that the totally umbilical hypersurfaces of L_σ of $\mathbb{H}^{n+1}$ can be realized in the Lorentzian model as

$$L_\sigma = \{x \in \mathbb{H}^{n+1};\ \langle x, a\rangle = \sigma\},$$

where $a \in \mathbb{R}^{n+2}_1$ is a fixed vector, and $\sigma^2 + \langle a, a\rangle > 0$; see [López and Montiel 1999]. Furthermore, from a straightforward computation, we see that the Gauss mapping of such hypersurfaces is given by

$$N(x) = \frac{1}{\sqrt{\sigma^2 + \langle a, a\rangle}}(a + \sigma x) \in \mathbb{S}^{n+1}_1.$$

Consequently, from previous expression we obtain that the angle function f_a of a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ satisfies

$$f_a = \langle N, a\rangle = \sqrt{\sigma^2 + \langle a, a\rangle} = \tau = \text{constant}.$$

Hence, it follows from [Montiel 1988, Example 1] that $N(L_\sigma)$ is a totally umbilical spacelike hypersurface of $\mathbb{S}^{n+1}_1$ which is isometric to:

(1) an n-dimensional hyperbolic space of constant sectional curvature $-\frac{1}{\tau^2-1}$, when a is a unit spacelike vector;

(2) the n-dimensional Euclidean space, when a is a nonzero null vector; or

(3) an n-dimensional sphere of constant sectional curvature $\frac{1}{\tau^2+1}$, when a is a unit timelike vector.

On the other hand, given $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$, a totally umbilical hypersurface and $a \in \mathbb{R}^{n+2}_1$ a nonzero fixed vector, a straightforward computation yields that $a^\top$ is a conformal vector field on Σ^n. Indeed, after a choice of an orientation on Σ^n by the unit vector field N, we use (2-5) to deduce that

$$\nabla^2 l_a(X, Y) = (Hf_a + l_a)\langle X, Y\rangle. \tag{3-1}$$

Thus, from (2-7) and (3-1) we conclude that the Lie derivative of the Riemannian metric g of Σ^n in the direction of $a^\top$ satisfies

$$\mathcal{L}_{a^\top} g = \frac{2}{n}(\Delta l_a)g. \tag{3-2}$$

On the other hand, since Σ^n is totally umbilical, we obtain from (2-3), that the Ricci curvature of Σ^n satisfies

$$\mathrm{Ric}_\Sigma(X, Y) = (1 - n)(H^2 - 1)\langle X, Y\rangle. \tag{3-3}$$

Hence, from (3-2) and (3-3) we arrive at

$$\mathrm{Ric}_\Sigma + \tfrac{1}{2}\mathcal{L}_{a^\top} g = \big((1 - n)(H^2 - 1) + \tfrac{1}{n}\Delta l_a\big)g. \tag{3-4}$$

Therefore, from (3-4) we conclude that, with an appropriate choice of a nonzero vector $a \in \mathbb{R}^{n+2}_1$, the vector field $a^\top$ provides on Σ^n a nontrivial structure of a gradient almost Ricci soliton.

Motivated by the previous digression, we establish the following characterization concerning gradient almost Ricci solitons immersed in the hyperbolic space, which can also be regarded as a version of the rigidity theorem for hyperbolic hypersurfaces in [Barros et al. 2014a].

Theorem 4. Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a hypersurface immersed in $\mathbb{H}^{n+1}$. Suppose that for some nonzero vector $a \in \mathbb{R}^{n+2}_1$ the vector field $a^\top$ provides the structure of a gradient almost Ricci soliton for Σ^n. If the image of the Gauss mapping of Σ^n lies in a totally umbilical spacelike hypersurface of $\mathbb{S}^{n+1}_1$ determined by a, then Σ^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$.

Proof. Initially, we note that our hypothesis under the image of the Gauss mapping N of Σ^n amounts to the fact that the angle function f_a of Σ^n satisfies $f_a = \langle N, a\rangle = \tau$ on Σ^n, for some constant τ satisfying $\tau^2 > \langle a, a\rangle$. Now, since $a^\top = \nabla l_a$ provides the structure of a gradient almost Ricci soliton for Σ^n, from (1-2)

$$\operatorname{Ric}_\Sigma(\nabla l_a) + \nabla^2 l_a(\nabla l_a) = \lambda \nabla l_a, \tag{3-5}$$

for some smooth function $\lambda : \Sigma^n \to \mathbb{R}$.

On the other hand, from (2-3) we have that

$$\operatorname{Ric}_\Sigma(X) = (1-n)X + nHAX - A^2X.$$

Hence, since f_a is a constant function, we conclude from the above expression that

$$\operatorname{Ric}_\Sigma(\nabla l_a) = (1-n)\nabla l_a. \tag{3-6}$$

Now, we use the (2-5), (3-5), and (3-6) to conclude that

$$(1-n+l_a-\lambda)\nabla l_a = 0, \tag{3-7}$$

on Σ^n. Observe that, from (2-4), we arrive at

$$|\nabla l_a|^2 + \tau^2 - l_a^2 = \langle a, a\rangle. \tag{3-8}$$

Since $\tau^2 > \langle a, a\rangle$, we obtain from (3-8) that the height function l_a has strict sign on Σ^n. Moreover, from (3-7) we have that $l_a - \lambda$ is constant on the open set where $\nabla l_a \neq 0$ and, consequently,

$$|\nabla l_a|^2 \nabla(l_a - \lambda) = 0, \tag{3-9}$$

on Σ^n. Hence, from (3-9) we deduce that

$$\langle \nabla l_a, \nabla\lambda\rangle = |\nabla l_a|^2. \tag{3-10}$$

From Lemma 1, we can use equations (3-6) and (3-10) to deduce that

$$\langle \nabla R, \nabla l_a\rangle = 0. \tag{3-11}$$

Contracting (1-2), we have $\Delta l_a = n\lambda - R$; so,

$$\begin{aligned}\langle \nabla\Delta l_a, \nabla l_a\rangle &= n\langle\nabla\lambda, \nabla l_a\rangle - \langle\nabla R, \nabla l_a\rangle \\ &= n|\nabla l_a|^2. \end{aligned} \tag{3-12}$$

On the other hand, from (2-7) we have that

$$\langle \nabla\Delta l_a, \nabla l_a\rangle = n\tau\langle\nabla H, \nabla l_a\rangle + n|\nabla l_a|^2. \tag{3-13}$$

Thus, from (3-12) and (3-13) it follows immediately that $\langle\nabla H, \nabla l_a\rangle = 0$, when $\tau \neq 0$. In this case, observing that $a^\top = \nabla l_a$, we have from formula (2-8) that

$$|A|^2 = -\frac{nH}{\tau} l_a. \tag{3-14}$$

Since the scalar curvature R of Σ^n is given by

$$R = n(1-n) + n^2H^2 - |A|^2 \tag{3-15}$$

and we have $\langle\nabla H, \nabla l_a\rangle = \langle\nabla R, \nabla l_a\rangle = 0$ on Σ^n, we obtain from (3-15) after a simple computation that $a^\top(|A|^2) = 0$ on Σ^n. Thus, from (3-14) we deduce

$$H\nabla l_a = 0. \tag{3-16}$$

Hence, taking into account that $\langle\nabla H, \nabla l_a\rangle = 0$, we can use once more formula (2-7) jointly with (3-16) to obtain that

$$nH^2 = -\frac{nH}{\tau} l_a. \tag{3-17}$$

Therefore, the equations (3-14) and (3-17) allows us to conclude that $|A|^2 = nH^2$ and this means that Σ^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$.

When $\tau = 0$, it follows from (2-5) that the Hessian of the height function l_a satisfies $\nabla^2 l_a = l_a g$, where g stands for the Riemannian metric of Σ^n. Consequently, we conclude that $\nabla l_a = a^\top$ is a conformal vector field on Σ^n. Thus, from Lemma 3 we have that either Σ^n is a totally umbilical hypersurface or $a = a^\top$ on Σ^n. But, since l_a has strict sign on Σ^n, from (2-4) we see that this last situation cannot occur. Hence, we conclude that Σ^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$.

Moreover, from Lemma 2 we obtain that

$$\mathscr{L}_{\nabla l_a} R = -2(n-1)\Delta l_a - 2Rl_a,$$

Now, combining this latter with formula (2-7) and (3-11) we deduce that

$$(n(n-1) + R)l_a = 0.$$

By using once more that the height function l_a has strict sign, we conclude from the previous equality that the scalar curvature of Σ^n satisfies $R = n(1-n)$. Consequently, since the umbilicity of Σ^n implies that $|A|^2 = nH^2$, from (3-15) we get $H = 0$ on Σ^n. Therefore, Σ^n must be, in fact, a totally geodesic hypersurface of $\mathbb{H}^{n+1}$. □

From the proof of Theorem 4 we also get the following:

Corollary 5. If Σ^n is a complete hypersurface of $\mathbb{H}^{n+1}$ such that, for some nonzero vector $a \in \mathbb{R}^{n+2}_1$, the vector field $a^\top$ provides on it the nontrivial structure of a gradient almost Ricci soliton and the image of its Gauss mapping lies in a totally umbilical spacelike hypersurface of $\mathbb{S}^{n+1}_1$ determined by a, then Σ^n is isometric to $\mathbb{H}^n$.

4. Characterizing gradient almost Ricci solitons in $\mathbb{S}^{n+1}_1$ and $\mathbb{H}^{n+1}_1$

We start by recalling the description of the totally umbilical hypersurfaces of $\mathbb{M}^{n+1}_c \subset \mathbb{R}^{n+2}_\nu$; see [Abe et al. 1987, Section 4], and also [Montiel 1988, Example 1; Lucas and Ramírez-Ospina 2013, Example 2]. Let $a \in \mathbb{R}^{n+2}_\nu$ be a fixed nonzero vector with $\langle a, a\rangle \in \{-1, 0, 1\}$ and consider the smooth function $h_a : \mathbb{M}^{n+1}_c \to \mathbb{R}$ defined by $h_a(x) = \langle x, a\rangle$. A straightforward computation allows us to conclude that for every real number ϱ, with $\langle a, a\rangle - c\varrho^2 \neq 0$, the level set

$$L_\varrho = h_a^{-1}(\varrho) = \{x \in \mathbb{M}^{n+1}_c : \langle x, a\rangle = \varrho\},$$

is a totally umbilical hypersurface in $\mathbb{M}^{n+1}_c$, with Gauss mapping

$$N(x) = \frac{1}{\sqrt{|\langle a, a\rangle - c\varrho^2|}}(a - c\varrho x). \tag{4-1}$$

Consequently, the corresponding angle function f_a of Σ^n satisfies

$$|f_a| = |\langle N, a\rangle| = \sqrt{|\langle a, a\rangle - c\varrho^2|}. \tag{4-2}$$

It follows from (4-1) that if Σ^n is a totally umbilical hypersurface in $\mathbb{M}^{n+1}_c$, then the image of its Gauss mapping lies in a totally umbilical hypersurface of the hyperbolic space, in the case $c = 1$, and in a totally umbilical spacelike hypersurface of the anti-de Sitter space $\mathbb{H}^{n+1}_1$, in the case $c = -1$. Furthermore, from (4-2) we conclude that f_a must be a constant function on Σ^n and, consequently, we have the following possibilities.

When $c = 1$:

(I.1) if a is a unit spacelike vector, then either $|\varrho| > 1$ and L_ϱ is isometric to an n-dimensional hyperbolic space of constant sectional curvature $-1/(\varrho^2 - 1)$, or $|\varrho| < 1$ and L_ϱ is isometric to an n-dimensional de Sitter space of constant sectional curvature $1/(1 - \varrho^2)$;

(I.2) if a is a nonzero null vector, then $\varrho \neq 0$ and L_ϱ is isometric to an n-dimensional Euclidean space;

(I.3) if a is a unit timelike vector, then L_ϱ is isometric to an n-dimensional Euclidean sphere of constant sectional curvature $1/(1+\varrho^2)$.

When $c=-1$:

(II.1) if a is a unit spacelike vector, then L_ϱ is isometric to the n-dimensional anti-de Sitter space of constant sectional curvature $-1/(\varrho^2+1)$;

(II.2) if a is a nonzero null vector, then $\varrho \neq 0$ and L_ϱ is isometric to the n-dimensional Lorentz–Minkowski space;

(II.3) if a is a unit timelike vector, then either $|\varrho|>1$ and L_ϱ is isometric to an n-dimensional de Sitter space of constant sectional curvature $1/(\varrho^2-1)$, or $|\varrho|<1$ and L_ϱ is isometric to an n-dimensional hyperbolic space of constant sectional curvature $-1/(1-\varrho^2)$.

On the other hand, reasoning as in Section 3, we can verify that if Σ^n is a totally umbilical spacelike hypersurface of $\mathbb{M}_c^{n+1}$, then for an arbitrary fixed vector $a \in \mathbb{R}_\nu^{n+2}$ we have

$$\mathrm{Ric}_\Sigma + \tfrac{1}{2}\mathscr{L}_{a^\top} g = \big((1-n)(H^2+c)+\tfrac{1}{n}\Delta l_a\big)g, \tag{4-3}$$

where g stands for the Riemannian metric of Σ^n. Now (4-3) allows us to conclude that, for a suitable choice of a fixed vector $a \in \mathbb{R}_\nu^{n+2}$, the vector field $a^\top$ provides on Σ^n the nontrivial structure of a gradient almost Ricci soliton.

In a similar way to that of Theorem 4, the previous discussion allows us to establish the following characterization concerning gradient almost Ricci solitons immersed in the de Sitter space:

Theorem 6. Let $\psi : \Sigma^n \to \mathbb{S}_1^{n+1}$ be a spacelike hypersurface immersed in $\mathbb{S}_1^{n+1}$. Suppose that for some nonzero vector $a \in \mathbb{R}_1^{n+2}$ the vector field $a^\top$ provides the structure of a gradient almost Ricci soliton for Σ^n. If the image of the Gauss mapping of Σ^n lies in a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ determined by a, then Σ^n is a totally umbilical hypersurface of $\mathbb{S}_1^{n+1}$.

Proof. Observe that the hypothesis on the image of the Gauss mapping N of Σ^n implies that the angle function f_a of Σ^n satisfies $f_a = \langle N, a\rangle = \tau$ on Σ^n, for some constant τ, with $\tau^2 + \langle a, a\rangle > 0$. Since $a^\top = \nabla l_a$ provides the structure of a gradient almost Ricci soliton for Σ^n, from (1-2), the Ricci curvature of Σ^n satisfies

$$\mathrm{Ric}_\Sigma(\nabla l_a) = \lambda \nabla l_a - \nabla^2 l_a(\nabla l_a), \tag{4-4}$$

for some smooth function $\lambda : \Sigma^n \to \mathbb{R}$, where $\nabla^2 l_a$ stands for the Hessian of the height function $l_a = \langle \psi, a\rangle$.

On the other hand, if we denote by A the Weingarten operator of Σ^n with respect to the normal vector field N and taking into account that f_a is a constant function on Σ^n, we have from Gauss equation that

$$\mathrm{Ric}_\Sigma(\nabla l_a) = (n-1)\nabla l_a. \tag{4-5}$$

Now, from the expression of the Hessian of the height function $l_a = \langle \psi, a\rangle$ and using once more that f_a is constant, we conclude from (4-4) and (4-5) that

$$(n-1-l_a-\lambda)\nabla l_a = 0. \tag{4-6}$$

From (4-6), $l_a - \lambda$ is constant on the open set where $\nabla l_a \neq 0$ and, consequently,

$$|\nabla l_a|^2 \nabla(l_a+\lambda) = 0.$$

This equality allows us to conclude that

$$l_a\langle \nabla l_a, \nabla(l_a+\lambda)\rangle = 0. \tag{4-7}$$

We observe from (2-10) that the height function l_a can be sign changing on Σ^n, since $\tau^2 + \langle a, a\rangle > 0$. However, (4-7) provides us the following identity:

$$l_a\langle \nabla l_a, \nabla\lambda\rangle = -l_a|\nabla l_a|^2. \tag{4-8}$$

Now, from (4-4),

$$l_a\langle \nabla\Delta l_a, \nabla l_a\rangle = nl_a\langle \nabla\lambda, \nabla l_a\rangle - l_a\langle \nabla R, \nabla l_a\rangle.$$

From Lemma 1 and (4-7) we conclude that $l_a\langle \nabla R, \nabla l_a\rangle = 0$ on Σ^n. Furthermore, we use the (4-8) to rewrite the above expression as

$$l_a\langle \nabla\Delta l_a, \nabla l_a\rangle = -nl_a|\nabla l_a|^2. \tag{4-9}$$

On the other hand, since $\Delta l_a = nH\tau - nl_a$, we deduce

$$l_a\langle \nabla\Delta l_a, \nabla l_a\rangle = nl_a\tau\langle \nabla H, \nabla l_a\rangle - nl_a|\nabla l_a|^2. \tag{4-10}$$

From (4-9) and (4-10) it follows that $l_a\langle \nabla H, \nabla l_a\rangle = 0$, when $\tau \neq 0$. Thus, in this case, we obtain from formula (2-14) that

$$|A|^2 = \frac{nH}{\tau} l_a. \tag{4-11}$$

Now, we recall that $a^\top = \nabla l_a$ and that the scalar curvature R of Σ^n is given by $R = n(n-1) - n^2H^2 + |A|^2$. Furthermore, since we have $l_a\langle \nabla H, \nabla l_a\rangle = 0$ and $l_a\langle \nabla R, \nabla l_a\rangle = 0$ on Σ^n, it follows from (4-11) that

$$l_a a^\top(|A|^2) = l_a\langle \nabla|A|^2, \nabla l_a\rangle = l_a\langle \nabla R, \nabla l_a\rangle + 2n^2Hl_a\langle \nabla H, \nabla l_a\rangle = 0$$

on Σ^n. On the other hand, by using once more the Equation (4-11) we arrive at

$$l_a a^\top(|A|^2) = \frac{nHl_a}{\tau}|\nabla l_a|^2.$$

Hence, these previous identities allow us to conclude that

$$Hl_a\nabla l_a = 0.$$

Therefore, since $\Delta l_a = nH\tau - nl_a$, we use the above equality to obtain

$$nH^2 l_a^2 = \frac{nH}{\tau} l_a^3. \tag{4-12}$$

From (4-11) and (4-12) it follows that $|A|^2 l_a^2 = nH^2 l_a^2$ and hence,

$$|\Phi|^2 l_a^2 = 0. \tag{4-13}$$

Now, after a simple algebraic argument, we can write

$$R = n(n-1)(1-H^2) + |\Phi|^2. \tag{4-14}$$

Thus, by using once more that $l_a\langle\nabla H, \nabla l_a\rangle = 0$ and $l_a\langle\nabla R, \nabla l_a\rangle = 0$ on Σ^n it follows from (4-14) that

$$l_a a^\top(|\Phi|^2) = 0. \tag{4-15}$$

On the other hand, from (4-13) we obtain that $|\Phi|^2 l_a = 0$ and, consequently, we deduce from (4-15) that

$$|\Phi|^2|\nabla l_a|^2 = l_a a^\top(|\Phi|^2) + |\Phi|^2 a^\top(l_a) = a^\top(|\Phi|^2 l_a) = 0. \tag{4-16}$$

Therefore, we obtain that $|\Phi|^2|\nabla l_a|^2 = 0$ on Σ^n. Thus, since we also have $|\Phi|^2 l_a^2 = 0$ on Σ^n it follows from (2-10) that $|\Phi|^2 = 0$ on Σ^n, because $\tau^2 + \langle a, a\rangle > 0$. This means that Σ^n is a totally umbilical hypersurface of $\mathbb{S}_1^{n+1}$.

When $\tau = 0$, it follows from (2-11) that the Hessian of the height function l_a satisfies $\nabla^2 l_a = -l_a g$, where g is the Riemannian metric of Σ^n. Consequently, we conclude that $\nabla\, l_a = a^\top$ is a conformal vector field on Σ^n. Hence, from Lemma 3, either Σ^n is a totally umbilical hypersurface or $a = a^\top$ on Σ^n. From (2-10), we see that this last situation implies that $l_a = 0$ on Σ^n. On the other hand, from (2-10),

$$|\nabla\, l_a|^2 + l_a^2 = \langle a\,, a\rangle. \tag{4-17}$$

Thus, taking into account that $a = a^\top$ implies that $\langle a, a\rangle > 0$, from (4-17) we reach a contradiction. Hence, Σ^n is a totally umbilical hypersurface.

Now, from Lemma 2,

$$\mathscr{L}_{\nabla l_a} R = -2(n-1)\Delta l_a - 2Rl_a.$$

From (2-13) and using that $l_a\langle\nabla R, \nabla l_a\rangle = 0$ on Σ^n, we deduce from above that

$$(n(n-1) - R)l_a^2 = 0. \tag{4-18}$$

We claim that we must have $R = n(n-1)$. Indeed, otherwise (4-18) implies that $l_a = 0$ on Σ^n. But, as before, this cannot occur. Therefore, reasoning as in the last part of the proof of Theorem 4, we conclude that Σ^n must be, in fact, a totally geodesic hypersurface of $\mathbb{S}^{n+1}_1$. □

From the proof of Theorem 6, we also obtain the following result:

Corollary 7. If Σ^n is a complete spacelike hypersurface of $\mathbb{S}^{n+1}_1$ such that, for some nonzero vector $a \in \mathbb{R}^{n+2}_1$, the vector field $a^\top$ provides on it a nontrivial structure of a gradient almost Ricci soliton and the image of its Gauss mapping lies in a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ determined by a, then Σ^n is isometric to $\mathbb{S}^n$.

Finally, we can reason in an analogous way to the proof of Theorem 6 in order to establish corresponding versions of Theorem 6 and Corollary 7 for the case that the ambient space is the anti-de Sitter space $\mathbb{H}^{n+1}_1$. More precisely:

Theorem 8. Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}_1$ be a spacelike hypersurface immersed in $\mathbb{H}^{n+1}_1$. Suppose that for some nonzero vector $a \in \mathbb{R}^{n+2}_2$ the vector field $a^\top$ provides the structure of a gradient almost Ricci soliton for Σ^n. If the image of the Gauss mapping of Σ^n lies in a totally umbilical hypersurface of $\mathbb{H}^{n+1}_1$ determined by a, then Σ^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}_1$.

Corollary 9. If Σ^n is a complete spacelike hypersurface of $\mathbb{H}^{n+1}_1$ such that, for some nonzero vector $a \in \mathbb{R}^{n+2}_2$, the vector field $a^\top$ provides on it a nontrivial structure of a gradient almost Ricci soliton and the image of its Gauss mapping lies in a totally umbilical hypersurface of $\mathbb{H}^{n+1}_1$ determined by a, then Σ^n is isometric to $\mathbb{H}^n$.

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