Pacific Journal of Mathematics

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Volume 288 No. 2 June 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 288, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.288.307

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We prove a weighted Sobolev estimate of the Bergman projection on the Hartogs triangle, where the weight is some power of the distance to the singularity at the boundary. This method also applies to the *n*-dimensional generalization of the Hartogs triangle.

1. Introduction

Setup and background. Let Ω be a domain in $\mathbb{C}^n$. The set of square integrable holomorphic functions on Ω, denoted by $A^2(\Omega)$, forms a closed subspace of the Hilbert space $L^2(\Omega)$. The Bergman projection associated to Ω is the orthogonal projection

$$\mathcal{B} : L^2(\Omega) \to A^2(\Omega),$$

which has an integral representation

$$\mathcal{B}(f)(z) = \int_\Omega B(z,\zeta) f(\zeta)\, d(\zeta), \tag{1-1}$$

for all $f \in L^2(\Omega)$ and $z \in \Omega$. Here the function $B(z,\zeta)$ defined on $\Omega \times \Omega$ is the Bergman kernel, and $d(\zeta) = dV(\zeta)$ is the usual Euclidean volume form.

The regularity of the Bergman projection $\mathcal{B}$ associated to Ω in $L^p(\Omega)$, $W^{k,p}(\Omega)$, and Hölder spaces are of particular interest. When Ω is bounded, smooth, and pseudoconvex with additional geometric condition on the boundary (e.g., strongly pseudoconvex), the regularity of $\mathcal{B}$ in these spaces has been intensively studied in the literature. See, for example, [Lanzani and Stein 2012] and references therein for details.

When Ω is nonsmooth, there are relatively few results in regard to the regularity of the Bergman projection. Even in $L^p(\Omega)$, we cannot expect the regularity to hold for all $p \in (1,\infty)$. If Ω is a simply connected planar domain, then the interval of p for $\mathcal{B}$ to be L^p-bounded highly depends on the geometry of the boundary; see [Lanzani and Stein 2004]. If Ω is a nonsmooth worm domain, then the interval of p depends on the winding of the domain; see [Krantz and Peloso 2008]. If Ω is an

MSC2010: 32A07, 32A25, 32A50.
Keywords: Hartogs triangle, Bergman projection, Sobolev regularity.

inflation of the unit disc by the norm square of a nonvanishing holomorphic function, then the interval of p depends on the boundary behavior of the holomorphic function on the unit disc; see [Zeytuncu 2013].

Results. In this article, we consider the Sobolev regularity of the Bergman projection $\mathcal{B}$ on the Hartogs triangle $\mathbb{H}$, where the Hartogs triangle is defined as

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}.$$

The Hartogs triangle is a classical nonsmooth domain in $\mathbb{C}^2$. It is well known that the boundary at $(0, 0)$ is not even Lipschitz, and the topological closure of $\mathbb{H}$ does not possess a Stein neighborhood basis. In [Chen 2017a], the L^p regularity of $\mathcal{B}$ on $\mathbb{H}$ was studied: the Bergman projection $\mathcal{B}$ is L^p-bounded if and only if $p \in \left(\frac{4}{3}, 4\right)$. On the other hand, we have $\bar{z}_2 \in W^{k,p}(\mathbb{H})$ for all nonnegative integers k and all $p \in [1, \infty]$, but $\mathcal{B}(\bar{z}_2) = c/z_2 \notin W^{1,p}(\mathbb{H})$ for $p \geq 2$, where c is some nonzero constant. So we cannot expect to obtain regularity in the ordinary Sobolev spaces, nor for all $p \in (1, \infty)$.

A natural way to control the boundary behavior of singularities is the use of weights which measure the distance from the points near the boundary to the singularity at the boundary. Since on the Hartogs triangle we have $|z_2| < |z| < \sqrt{2}|z_2|$, where $z = (z_1, z_2) \in \mathbb{H}$, it is reasonable to consider a weight of the form $|z_2|^s$, for some $s \in \mathbb{R}$. On the other hand, based on the L^p mapping property of the Bergman projection on $\mathbb{H}$ (see [Chakrabarti and Zeytuncu 2016]) and the Sobolev regularity of the weighted canonical solution operator of the $\bar{\partial}$-equation on $\mathbb{H}$ (see [Chakrabarti and Shaw 2013]), it is also reasonable to put a weight of the form $|z_2|^s$ on the target space. Therefore, we consider the following weighted Sobolev spaces:

Definition 1.1. On the Hartogs triangle $\mathbb{H}$, for each $k \in \mathbb{Z}^+ \cup \{0\}$, $s \in \mathbb{R}$, and $p \in (1, \infty)$, we define the *weighted Sobolev space* by

$$W^{k,p}(\mathbb{H}, \delta^s) = \{f \in L^1_{\mathrm{loc}}(\mathbb{H}) \mid \|f\|_{k,p,s} < \infty\},$$

where $\delta(z) = |z_2| \approx |z|$, and the norm is defined as

$$\|f\|_{k,p,s} = \left(\int_{\mathbb{H}} \sum_{|\alpha| \leq k} |D^{\alpha}_{z,\bar{z}}(f)(z)|^p |z_2|^s \, dz\right)^{1/p}.$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the multi-index running over all $|\alpha| \leq k$, and

$$D^{\alpha}_{z,\bar{z}} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \partial \bar{z}_1^{\alpha_3} \partial \bar{z}_2^{\alpha_4}}.$$

We also denote the usual norm in the (unweighted) Sobolev space $W^{k,p}(\mathbb{H})$ by

$$\|f\|_{k,p} = \left(\int_{\mathbb{H}} \sum_{|\alpha|\le k} |D^{\alpha}_{z,\bar z}(f)(z)|^p \, dz\right)^{1/p}.$$

With the definition above, we can state our main result:

Theorem 1.2. *The Bergman projection $\mathcal{B}$ on the Hartogs triangle $\mathbb{H}$ maps continuously from $W^{k,p}(\mathbb{H})$ to $W^{k,p}(\mathbb{H}, \delta^{kp})$ for $p \in \left(\frac{4}{3}, 4\right)$.*

That is, for each $k \in \mathbb{Z}^+ \cup \{0\}$ and $p \in \left(\frac{4}{3}, 4\right)$, there exists a constant $C_{k,p} > 0$, such that

$$\|\mathcal{B}(f)\|_{k,p,kp} \le C_{k,p} \|f\|_{k,p} \quad \textit{for any} \quad f \in W^{k,p}(\mathbb{H}).$$

Remark 1.3. It is clear that $\mathcal{B}$ doesn't lose any derivatives away from the singular point of the Hartogs triangle. If we put a suitable power of the weight δ around the singularity on the target space, then there is no loss of differentiability of $\mathcal{B}(f)$ around the singular point (see also the result in [Chakrabarti and Shaw 2013]).

Remark 1.4. Note that we have $\mathcal{B}(\bar z_2) = c/z_2 \notin W^{k,p}(\mathbb{H}, \delta^{kp})$ for $p \ge 4$, where c is some nonzero constant. So we cannot obtain regularity for $p \ge 4$, unless we use more weights on the target space. Conversely, we can only obtain regularity for fewer values of p, if we use less weights on the target space.

Organization and outline. The idea of the proof of the main result is the following. In Section 2, we start with an idea from [Chakrabarti and Shaw 2013] to transfer $\mathbb{H}$ to the product model $\mathbb{D} \times \mathbb{D}^*$, as well as to transfer the differential operators D^α to the ones in new variables. From this, we focus on the integration over the punctured disc $\mathbb{D}^*$ in Section 3. We then use an idea from [Straube 1986] to convert D^α acting on the Bergman kernel in the holomorphic component to the ones acting on the kernel in the antiholomorphic part. The resulting differential operators can be written as a combination of tangential operators, and therefore, integration by parts applies to the smooth functions. Finally, in Section 4, we apply the weighted L^p estimates in [Chen 2017b] to our integral, and the resulting integral is majorized by the weighted L^p norm of $D^\alpha(f)$. To complete the proof, we approximate the weighted Sobolev functions by smooth functions and transfer the product model back to $\mathbb{H}$.

2. Transfer to the product model

Transfer $\mathbb{H}$ to $\mathbb{D} \times \mathbb{D}^*$. In view of Definition 1.1, we adopt the following notation.

Definition 2.1. Let $\beta = (\beta_1, \beta_2)$ be a multi-index, we use the notations below to denote the differential operators

$$D^{\beta}_{z} = \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \partial z_2^{\beta_2}} \quad \text{and} \quad D^{\beta}_{z_j,\bar z_j} = \frac{\partial^{|\beta|}}{\partial z_j^{\beta_1} \partial \bar z_j^{\beta_2}} \quad \text{for} \quad j = 1, 2.$$

From the result in [Chen 2017a], we see that $\mathcal{B}(f) \in A^p(\mathbb{H})$ (the set of L^p functions that are holomorphic), whenever $p \in \left(\frac{4}{3}, 4\right)$ and $f \in L^p(\mathbb{H})$. So we can rewrite the weighted L^p Sobolev norm of $\mathcal{B}(f)$ as

$$\|\mathcal{B}(f)\|^p_{k,p,kp} = \sum_{|\beta|\le k} \int_{\mathbb{H}} \left|D^\beta_z(\mathcal{B}(f))(z)\right|^p |z_2|^{kp}\, dz, \tag{2-1}$$

where β and D^β_z are as in Definition 2.1.

In order to transfer $\mathbb{H}$ to the product model, we first recall the transformation formula for the Bergman kernels.

Proposition 2.2. *Let Ω_j be a domain in $\mathbb{C}^n$ and B_j be its Bergman kernel on $\Omega_j \times \Omega_j$, $j = 1, 2$. Suppose $\Psi : \Omega_1 \to \Omega_2$ is a biholomorphism, then for $(w, \eta) \in \Omega_1 \times \Omega_1$ we have*

$$\det J_{\mathbb{C}}\Psi(w) B_2(\Psi(w), \Psi(\eta)) \det \overline{J_{\mathbb{C}}\Psi(\eta)} = B_1(w, \eta).$$

Proof. See, for example, [Krantz 1992, Proposition 1.4.12]. □

Now let us consider the biholomorphism $\Phi : \mathbb{H} \to \mathbb{D} \times \mathbb{D}^*$ with its inverse $\Psi : \mathbb{D} \times \mathbb{D}^* \to \mathbb{H}$, where

$$\Phi(z_1, z_2) = \left(\frac{z_1}{z_2}, z_2\right) \quad \text{and} \quad \Psi(w_1, w_2) = (w_1 w_2, w_2).$$

A simple computation shows $\det J_{\mathbb{C}}\Psi(w) = w_2$, for $w = (w_1, w_2) \in \mathbb{D} \times \mathbb{D}^*$. Therefore, by the proposition above, we have

$$B(\Psi(w), \Psi(\eta)) = \frac{1}{w_2\bar{\eta}_2} \cdot \frac{1}{(1 - w_1\bar{\eta}_1)^2} \cdot \frac{1}{(1 - w_2\bar{\eta}_2)^2}, \tag{2-2}$$

where B is the Bergman kernel on $\mathbb{H} \times \mathbb{H}$ as in (1-1) and $(w, \eta) \in \mathbb{D} \times \mathbb{D}^* \times \mathbb{D} \times \mathbb{D}^*$.

Transfer the differential operators. We next need to transfer the differential operators D^β_z to the ones in the new variable w. We need a lemma.

Lemma 2.3. *Under the biholomorphism $\Phi(z) = w$, for each β let $m = |\beta|$. Then*

$$D^\beta_z = \sum_{a+b\le m} \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \tag{2-3}$$

where $p_{a,b,\beta}(w_1)$ is a polynomial of degree at most m in variable w_1. In addition, if $|\beta| \le k$ for some $k \in \mathbb{Z}^+ \cup \{0\}$, then $|p_{a,b,\beta}(w_1)| \le C_k$ on $\mathbb{D}$ uniformly in β, a, and b, for some constant $C_k > 0$ depending only on k.

Proof. We prove (2-3) by induction on $m = |\beta|$. The case $m = 0$ is trivial. When $m = 1$, a direct computation shows

$$\frac{\partial}{\partial z_1} = \frac{1}{w_2} \cdot \frac{\partial}{\partial w_1} \quad \text{and} \quad \frac{\partial}{\partial z_2} = -\frac{w_1}{w_2} \cdot \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}.$$

So both $\partial/\partial z_1$ and $\partial/\partial z_2$ are of the form in (2-3).

Suppose for all β with $|\beta| = m$, the D_z^β are of the form in (2-3). We now check the case $|\beta'| = m+1$. Note that $D_z^{\beta'} = (\partial/\partial z_1) \circ D_z^\beta$ or $D_z^{\beta'} = (\partial/\partial z_2) \circ D_z^\beta$ for some β. By the inductive assumption, we have

$$\begin{aligned}
\frac{\partial}{\partial z_1} \circ D_z^\beta &= \frac{1}{w_2} \cdot \frac{\partial}{\partial w_1} \circ \sum_{a+b \le m} \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} \\
&= \sum_{a+b \le m} \frac{p'_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^{a+1} \partial w_2^b} \\
&= \sum_{a+b \le m+1} \frac{p_{a,b,\beta'}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z_2} \circ D_z^\beta &= \Big(-\frac{w_1}{w_2} \cdot \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}\Big) \circ \sum_{a+b \le m} \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} \\
&= \sum_{a+b \le m} \frac{-w_1 p'_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{-w_1 p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^{a+1} \partial w_2^b} \\
&\qquad + \frac{(b-m) p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^a \partial w_2^{b+1}} \\
&= \sum_{a+b \le m+1} \frac{p_{a,b,\beta'}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}.
\end{aligned}$$

We see that $p_{a,b,\beta'}(w_1)$ is a polynomial of degree at most $m+1$ and $D_z^{\beta'}$ has the form in (2-3).

When $|\beta| \le k$, all the possible combinations of derivatives in D_z^β are finite. So there are finitely many different coefficients in all of the $p_{a,b,\beta}(w_1)$. Since $|w_1| \le 1$ on $\mathbb{D}$ and $a, b \le m \le k$, we obtain $|p_{a,b,\beta}(w_1)| \le C_k$ on $\mathbb{D}$ as desired. □

Now we can transfer $\mathbb{H}$ to the product model $\mathbb{D} \times \mathbb{D}^*$ by the biholomorphism Φ. Combining (2-2) and (2-3), we see that the right hand side of (2-1) becomes

$$\sum_{|\beta| \le k} \int_{\mathbb{D}\times\mathbb{D}^*} \bigg| \sum_{a+b \le |\beta|} \int_{\mathbb{D}\times\mathbb{D}^*} K_{a,b,\beta}(w,\eta) f(\Psi(\eta)) |\eta_2|^2 \, d\eta \bigg|^p |w_2|^{kp+2} \, dw, \tag{2-4}$$

where

$$K_{a,b,\beta}(w,\eta) = \frac{p_{a,b,\beta}(w_1)}{w_2^{|\beta|-b}} \cdot \frac{\partial^a}{\partial w_1^a} \bigg(\frac{1}{(1 - w_1 \bar\eta_1)^2} \bigg) \cdot \frac{\partial^b}{\partial w_2^b} \bigg(\frac{1}{w_2 \bar\eta_2} \cdot \frac{1}{(1 - w_2 \bar\eta_2)^2} \bigg).$$

3. Convert the differential operators on $\mathbb{D}^*$

Convert to the antiholomorphic part. Since $\mathbb{D}^*$ is a Reinhardt domain, by using the idea in [Straube 1986], we can convert the differential operators as follows.

Lemma 3.1. *As in* (2-4), *for the last factor in* $K_{a,b,\beta}(w,\eta)$, *we have*

$$\frac{\partial^b}{\partial w_2^b}\left(\frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2}\right)=\frac{\bar{\eta}_2^b}{w_2^b}\cdot\frac{\partial^b}{\partial\bar{\eta}_2^b}\left(\frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2}\right). \tag{3-1}$$

Proof. The kernel in (3-1) is the weighted Bergman kernel associated to $\mathbb{D}^*$ with the weight $|z|^2$; see [Chen 2017b]. It has the expansion

$$\frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2}=\sum_{j=0}^{\infty}(j+1)(w_2\bar{\eta}_2)^{j-1},$$

which converges uniformly on every compact subset $K\times K\subset\mathbb{D}^*\times\mathbb{D}^*$. Differentiate the series term by term, and we see that

$$\begin{aligned}w_2^b\cdot\frac{\partial^b}{\partial w_2^b}\left(\frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2}\right)&=\sum_{j=0}^{\infty}(j+1)w_2^b\cdot\frac{\partial^b}{\partial w_2^b}(w_2\bar{\eta}_2)^{j-1}\\&=\sum_{j=0}^{\infty}(j+1)\bar{\eta}_2^b\cdot\frac{\partial^b}{\partial\bar{\eta}_2^b}(w_2\bar{\eta}_2)^{j-1}\\&=\bar{\eta}_2^b\cdot\frac{\partial^b}{\partial\bar{\eta}_2^b}\left(\frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2}\right).\end{aligned}\qquad\square$$

Integration by parts. Now we focus on the integration over $\mathbb{D}^*$ in (2-4). We first define a “tangential” operator.

Definition 3.2. Let $S_w=w(\partial/\partial w)$ be the *complex normal differential operator* on a neighborhood of $\partial\mathbb{D}$. We define the *tangential operator* by

$$T_w=\Im(S_w)=\frac{1}{2i}\left(w\frac{\partial}{\partial w}-\bar{w}\frac{\partial}{\partial\bar{w}}\right).$$

Remark 3.3. Indeed, T_w is well defined on a neighborhood of $\bar{\mathbb{D}}$. Moreover, for any disc $\mathbb{D}_\rho=\{|w|<\rho\}$ of radius $\rho<1$ with defining function $r_\rho(w)=|w|^2-\rho^2$, we have

$$T_w(r_\rho)=0 \tag{3-2}$$

on $\partial\mathbb{D}_\rho$. That is, T_w is tangential on $\partial\mathbb{D}_\rho$ for all $\rho<1$.

In order to make use of integration by parts, we need the following lemma:

Lemma 3.4. *Let T_w be as above. For $b \in \mathbb{Z}^+ \cup \{0\}$, we have*

$$T_w^b \equiv \sum_{j=0}^{b} c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} \quad \Big(\mathrm{mod}\ \frac{\partial}{\partial w}\Big), \tag{3-3}$$

where the c_j are constants, $c_b \neq 0$, and T_w^b is the composition of b copies of T_w.

Proof. We prove (3-3) by induction on b. The case $b = 0$ is trivial. When $b = 1$, it is easy to see that

$$T_w \equiv -\frac{1}{2i} \bar{w} \frac{\partial}{\partial \bar{w}} \quad \Big(\mathrm{mod}\ \frac{\partial}{\partial w}\Big).$$

Suppose (3-3) holds for some b. Then we see that

$$T_w^b = \sum_{j=0}^{b} c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} + A \circ \frac{\partial}{\partial w},$$

for some operator A. So for the case $b+1$, we have

$$\begin{aligned} T_w \circ T_w^b &= \frac{1}{2i}\Big(w \frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{w}}\Big) \circ \Big(\sum_{j=0}^{b} c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} + A \circ \frac{\partial}{\partial w}\Big) \\ &= \frac{1}{2i}\Big(\sum_{j=0}^{b} c_j w \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j}\frac{\partial}{\partial w} - j c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} - c_j \bar{w}^{j+1} \frac{\partial^{j+1}}{\partial \bar{w}^{j+1}}\Big) + T_w \circ A \circ \frac{\partial}{\partial w} \\ &= \sum_{j=0}^{b+1} c'_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} + A' \circ \frac{\partial}{\partial w}, \end{aligned}$$

for some constants c'_j with $c'_{b+1} = -(1/2i)c_b \neq 0$ and some operator A'. Therefore, (3-3) holds for T_w^{b+1}. □

Combine (3-1) and (3-3). Since the kernel in (3-1) is antiholomorphic in η_2, the inside integration over $\mathbb{D}^*$ with regard to variable η_2 in (2-4) denoted by I becomes

$$\begin{aligned} I &= \int_{\mathbb{D}^*} \frac{\partial^b}{\partial w_2^b}\Big(\frac{1}{w_2\bar{\eta}_2} \cdot \frac{1}{(1-w_2\bar{\eta}_2)^2}\Big) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2 \\ &= \int_{\mathbb{D}^*} \frac{\bar{\eta}_2^b}{w_2^b} \cdot \frac{\partial^b}{\partial \bar{\eta}_2^b}\Big(\frac{1}{w_2\bar{\eta}_2} \cdot \frac{1}{(1-w_2\bar{\eta}_2)^2}\Big) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2 \\ &= \frac{1}{w_2^b} \int_{\mathbb{D}^*} \sum_{j=0}^{b} c_j T_{\eta_2}^j \Big(\frac{1}{w_2\bar{\eta}_2} \cdot \frac{1}{(1-w_2\bar{\eta}_2)^2}\Big) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2 \\ &= \frac{1}{w_2^b} \sum_{j=0}^{b} c_j \lim_{\epsilon \to 0^+} \int_{\mathbb{D}-\mathbb{D}_\epsilon} T_{\eta_2}^j \Big(\frac{1}{w_2\bar{\eta}_2} \cdot \frac{1}{(1-w_2\bar{\eta}_2)^2}\Big) f(\Psi(\eta)) |\eta_2|^2 \, d\eta_2. \end{aligned}$$

Let us assume in addition for a moment that $f(\Psi(\eta))$ belongs to $C^\infty(\bar{\mathbb{D}}-\{0\})$ in variable η_2. Then by (3-2) we obtain

$$I = \frac{1}{w_2^b}\sum_{j=0}^{b} c_j \lim_{\epsilon\to 0^+}\int_{\mathbb{D}-\mathbb{D}_\epsilon} T^j_{\eta_2}\biggl(\frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2}\biggr) f(\Psi(\eta))|\eta_2|^2\,d\eta_2 \tag{3-4}$$
$$= \frac{1}{w_2^b}\sum_{j=0}^{b} c_j(-1)^j \lim_{\epsilon\to 0^+}\int_{\mathbb{D}-\mathbb{D}_\epsilon} \frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2} T^j_{\eta_2}\bigl(f(\Psi(\eta))|\eta_2|^2\bigr)\,d\eta_2$$
$$= \frac{1}{w_2^b}\sum_{j=0}^{b} (-1)^j c_j \int_{\mathbb{D}^*} \frac{1}{w_2\bar{\eta}_2}\cdot\frac{1}{(1-w_2\bar{\eta}_2)^2} T^j_{\eta_2}\bigl(f(\Psi(\eta))\bigr)|\eta_2|^2\,d\eta_2,$$

where the last line follows from the fact that $T_{\eta_2}(|\eta_2|^2)=0$.

Definition 3.5. We use the following notation:

$$F_j(\eta)=T^j_{\eta_2}\bigl(f(\Psi(\eta))\bigr)\cdot\eta_2 \quad\text{and}\quad \mathcal{B}_{1,a}(g)(w_1)=\int_{\mathbb{D}}\frac{\partial^a}{\partial w_1^a}\biggl(\frac{1}{(1-w_1\bar{\eta}_1)^2}\biggr)g(\eta_1)\,d\eta_1,$$

for any g whenever the integral is well defined, and

$$\mathcal{B}_2(h)(w_2)=\int_{\mathbb{D}^*}\frac{h(\eta_2)}{(1-w_2\bar{\eta}_2)^2}\,d\eta_2,$$

for any h whenever the integral is well defined.

By (3-4) and the notation above (Definition 3.5), we see that (2-4) becomes

$$\sum_{|\beta|\le k}\int_{\mathbb{D}\times\mathbb{D}^*}\Biggl|\sum_{a+b\le|\beta|}\frac{p_{a,b,\beta}(w_1)}{w_2^{|\beta|+1}}\sum_{j=0}^{b}(-1)^j c_j\mathcal{B}_{1,a}\bigl(\mathcal{B}_2(F_j)\bigr)(w)\Biggr|^p |w_2|^{kp+2}\,dw. \tag{3-5}$$

4. Proof of the main theorem

L^p boundedness. To finish the proof, we first need two lemmas.

Lemma 4.1. *The operator $\mathcal{B}_{1,a}$ defined as in Definition 3.5 is bounded from $W^{a,p}(\mathbb{D})$ to $L^p(\mathbb{D})$ for $p\in(1,\infty)$.*

Proof. This follows from the well-known result that the Bergman projection on $\mathbb{D}$ is bounded from $W^{k,p}(\mathbb{D})$ to itself for $p\in(1,\infty)$ and all $k\in\mathbb{Z}^+\cup\{0\}$. □

Lemma 4.2. *The integral operator $\mathcal{B}_2$ defined as in Definition 3.5 is bounded from $L^p\bigl(\mathbb{D}^*,|w|^{2-p}\bigr)$ to itself for $p\in\bigl(\frac{4}{3},4\bigr)$, where $L^p\bigl(\mathbb{D}^*,|w|^{2-p}\bigr)$ is the weighted L^p space with $w\in\mathbb{D}^*$.*

Proof. This is equivalent to the statement that the weighted Bergman projection associated to $\mathbb{D}^*$ with the weight $|w|^2$ is bounded from $L^p\bigl(\mathbb{D}^*,|w|^2\bigr)$ to itself for $p\in\bigl(\frac{4}{3},4\bigr)$. For a proof, see [Chen 2017b]. □

Proof under the additional assumption. With Lemma 4.1 and Lemma 4.2, we can prove Theorem 1.2 under the additional assumption $f(\Psi(\eta)) \in C^\infty(\bar{\mathbb{D}} - \{0\})$ in variable η_2.

Proof of Theorem 1.2 under the additional assumption. By (2-1), (2-4), (3-5) and Lemma 2.3, we obtain

$$\begin{aligned}\|\mathcal{B}(f)\|^p_{k,p,kp} &\le \sum_{|\beta|\le k}\sum_{a+b\le|\beta|}\sum_{j=0}^{b} C_{k,p}\int_{\mathbb{D}\times\mathbb{D}^*} |\mathcal{B}_{1,a}(\mathcal{B}_2(F_j))(w)|^p |w_2|^{kp+2-p(|\beta|+1)}\,dw \\ &\le C_{k,p}\sum_{a+b\le k}\int_{\mathbb{D}\times\mathbb{D}^*} |\mathcal{B}_{1,a}(\mathcal{B}_2(F_b))(w)|^p |w_2|^{2-p}\,dw.\end{aligned}$$

By Lemma 4.1, for $p \in (1,\infty)$ we have

$$\begin{aligned}\|\mathcal{B}(f)\|^p_{k,p,kp} &\le C_{k,p}\sum_{a+b\le k}\int_{\mathbb{D}^*}\biggl(\int_{\mathbb{D}}\sum_{|\beta|\le a}|D^\beta_{w_1,\bar{w}_1}(\mathcal{B}_2(F_b))(w)|^p\,dw_1\biggr)|w_2|^{2-p}\,dw_2 \\ &\le C_{k,p}\sum_{|\beta|+b\le k}\int_{\mathbb{D}}\biggl(\int_{\mathbb{D}^*}|\mathcal{B}_2(D^\beta_{w_1,\bar{w}_1}(F_b))(w)|^p|w_2|^{2-p}\,dw_2\biggr)dw_1.\end{aligned}$$

Similarly, by Lemma 4.2, for $p \in \left(\frac{4}{3}, 4\right)$ we have

$$\begin{aligned}\|\mathcal{B}(f)\|^p_{k,p,kp} &\le C_{k,p}\sum_{|\beta|+b\le k}\int_{\mathbb{D}}\biggl(\int_{\mathbb{D}^*}\bigl|D^\beta_{w_1,\bar{w}_1}(F_b)(w)\bigr|^p|w_2|^{2-p}\,dw_2\biggr)dw_1 \\ &= C_{k,p}\sum_{|\beta|+b\le k}\int_{\mathbb{D}\times\mathbb{D}^*}\bigl|D^\beta_{w_1,\bar{w}_1}T^b_{w_2}\bigl(f(\Psi(w))\bigr)\cdot w_2\bigr|^p|w_2|^{2-p}\,dw \\ &= C_{k,p}\sum_{|\beta|+b\le k}\int_{\mathbb{D}\times\mathbb{D}^*}\bigl|D^\beta_{w_1,\bar{w}_1}T^b_{w_2}\bigl(f(\Psi(w))\bigr)\bigr|^p|w_2|^{2}\,dw \\ &\le C_{k,p}\sum_{|\beta|+|\beta'|\le k}\int_{\mathbb{D}\times\mathbb{D}^*}\bigl|D^\beta_{w_1,\bar{w}_1}D^{\beta'}_{w_2,\bar{w}_2}\bigl(f(\Psi(w))\bigr)\bigr|^p|w_2|^{2}\,dw,\end{aligned}\tag{4-1}$$

where the last line follows from $T_{w_2} = (1/2i)(w_2(\partial/\partial w_2) - \bar{w}_2(\partial/\partial\bar{w}_2))$, $|w_2| < 1$ for $w_2 \in \mathbb{D}^*$, and a similar equation as (3-3).

By the biholomorphism $\Psi(w) = z$ defined in Section 2, we have

$$\frac{\partial}{\partial w_1} = w_2\frac{\partial}{\partial z_1} \quad\text{and}\quad \frac{\partial}{\partial \bar{w}_1} = \bar{w}_2\frac{\partial}{\partial \bar{z}_1},$$

and also

$$\frac{\partial}{\partial w_2} = w_1\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \quad\text{and}\quad \frac{\partial}{\partial \bar{w}_2} = \bar{w}_1\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}.$$

Again, since $(w_1, w_2) \in \mathbb{D} \times \mathbb{D}^*$, we have $|w_1|, |w_2| < 1$. Therefore, by (4-1) and transferring $\mathbb{D} \times \mathbb{D}^*$ back to $\mathbb{H}$, we finally arrive at

$$\|\mathcal{B}(f)\|^p_{k,p,kp} \le C_{k,p} \sum_{|\alpha|\le k} \int_{\mathbb{H}} |D^{\alpha}_{z,\bar{z}}(f)(z)|^p \, dz. \qquad \square$$

Remove the additional assumption. To remove the additional assumption that $f(\Psi(\eta)) \in C^\infty(\bar{\mathbb{D}} - \{0\})$ in variable η_2, we need the following lemma.

Lemma 4.3. *The subspace $C^\infty(\bar{\mathbb{D}}-\{0\}) \bigcap W^{k,p}(\mathbb{D}^*, |w|^2)$ is dense in $W^{k,p}(\mathbb{D}^*, |w|^2)$ with regard to the weighted norm in $W^{k,p}(\mathbb{D}^*, |w|^2)$.*

Proof. The argument is based on [Evans 1998, §5.3 Theorem 2 and Theorem 3].

Given any $g \in W^{k,p}(\mathbb{D}^*, |w|^2)$, fix $\varepsilon > 0$. On $V_0 = \mathbb{D} - \overline{\mathbb{D}_{1/2}}$, the weighted norm $W^{k,p}(V_0, |w|^2)$ is equivalent to the unweighted norm $W^{k,p}(V_0)$. Arguing as in the proof of [Evans 1998, §5.3 Theorem 3], we see that there is a $g_0 \in C^\infty(\overline{V_0})$ such that

$$\|g_0 - g\|_{W^{k,p}(V_0,|w|^2)} < \varepsilon.$$

Define $U_j = \mathbb{D}_{\rho-1/j} - \overline{\mathbb{D}_{1/j}}$ for some $1 > \rho > \frac{1}{2}$ and for $j \in \mathbb{Z}^+$ ($U_1 = \varnothing$). Let $V_j = U_{j+3} - \overline{U_{j+1}}$, then we see $\bigcup_{j=1}^\infty V_j = \mathbb{D}_\rho - \{0\}$. Arguing as in the proof of [Evans 1998, §5.3 Theorem 2], we can find a smooth partition of unity $\{\psi_j\}_{j=1}^\infty$ subordinate to $\{V_j\}_{j=1}^\infty$, so that $\sum_{j=1}^\infty \psi_j = 1$ on $\mathbb{D}_\rho - \{0\}$. Moreover, for each j, the support of $\psi_j g$ lies in V_j (so $|w| > 1/(j+3)$), and hence $\psi_j g \in W^{k,p}(\mathbb{D}_\rho - \{0\})$. Therefore, we can find a smooth function g_j with support in $U_{j+4} - \overline{U_j}$ such that

$$\|g_j - \psi_j g\|_{W^{k,p}(\mathbb{D}_\rho-\{0\})} \le \frac{\varepsilon}{2^j};$$

see [Evans 1998, §5.3 Theorem 2] for details. Write $\tilde{g}_0 = \sum_{j=1}^\infty g_j$. It is easy to see that $\tilde{g}_0 \in C^\infty(\mathbb{D}_\rho - \{0\})$ and

$$\|\tilde{g}_0 - g\|_{W^{k,p}(\mathbb{D}_\rho-\{0\},|w|^2)} \le \|\tilde{g}_0 - g\|_{W^{k,p}(\mathbb{D}_\rho-\{0\})} \le \varepsilon,$$

since $|w| < 1$ on $\mathbb{D}_\rho - \{0\}$.

Let V_0' be an open set such that $\partial\mathbb{D} \subset V_0'$ and $V_0' \bigcap \mathbb{D} = V_0$, then $V_0' \bigcup \mathbb{D}_\rho$ cover $\bar{\mathbb{D}}$. Take a smooth partition of unity $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ on $\bar{\mathbb{D}}$ subordinate to $\{V_0', \mathbb{D}_\rho\}$. Then $h = \tilde{\psi}_1 g_0 + \tilde{\psi}_2 \tilde{g}_0$ belongs to $C^\infty(\bar{\mathbb{D}} - \{0\})$, and

$$\|h - g\|_{W^{k,p}(\mathbb{D}^*,|w|^2)} \le C\big(\|g_0 - g\|_{W^{k,p}(V_0,|w|^2)} + \|\tilde{g}_0 - g\|_{W^{k,p}(\mathbb{D}_\rho-\{0\},|w|^2)}\big) < 2C\varepsilon$$

as desired $\square$

Now we are ready to remove the extra assumption and prove our main result.

Proof of Theorem 1.2.

For any $f \in W^{k,p}(\mathbb{H})$, we have $f(\Psi(w)) \in W^{k,p}(\mathbb{D}^*, |w_2|^2)$ in variable w_2. Then by Lemma 4.3, we can find a sequence $\{h_j(w)\} \subset C^\infty(\bar{\mathbb{D}} - \{0\})$ tending to $f(\Psi(w))$

in variable w_2 with regard to the norm in $W^{k,p}(\mathbb{D}^*, |w_2|^2)$. We have already seen that (4-1) holds for each $h_j(w)$ replacing $f(\Psi(w))$. Indeed, if we focus on the integration over $\mathbb{D}^*$, by comparing with (2-4), we see that (4-1) is just the following: for each $b = 0, 1, \dots, k$

$$\int_{\mathbb{D}^*} \left| w_2^b \frac{\partial^b}{\partial w_2^b} (\mathcal{B}_3(h_j)) \right|^p |w_2|^2 \, dw_2 \le C_{k,p} \|h_j\|_{W^{k,p}(\mathbb{D}^*, |w_2|^2)}, \tag{4-2}$$

where $\mathcal{B}_3$ is the weighted Bergman projection associated to $\mathbb{D}^*$ with the weight $|w_2|^2$.

Now letting $j \to \infty$, in view of the boundedness of $\mathcal{B}_3$ (Lemma 4.2), we see that $w_2^b(\partial^b/\partial w_2^b)(\mathcal{B}_3(h_j))$ indeed tends to $w_2^b(\partial^b/\partial w_2^b)(\mathcal{B}_3(f(\Psi)))$ in $L^p(\mathbb{D}^*, |w_2|^2)$ for each $b = 0, 1, \dots, k$. Therefore, (4-2) is valid for general $f(\Psi(w)) \in W^{k,p}(\mathbb{D}^*, |w_2|^2)$, which completes the proof for any general $f \in W^{k,p}(\mathbb{H})$. □

Remark 4.4. The method also applies to the n-dimensional generalization of the Hartogs triangle, see [Chen 2017a]. To be precise, for $j = 1, \dots, l$, let Ω_j be a bounded smooth domain in $\mathbb{C}^{m_j}$ with a biholomorphic mapping $\phi_j : \Omega_j \to \mathbb{B}^{m_j}$ between Ω_j and the unit ball $\mathbb{B}^{m_j}$ in $\mathbb{C}^{m_j}$. We use the notation $\tilde{z}_j$ to denote the j-th m_j-tuple in $z \in \mathbb{C}^{m_1+\cdots+m_l}$, that is, $z = (\tilde{z}_1, \dots, \tilde{z}_l)$. Let $n = m_1 + \cdots + m_l + n'$, $n - n' \ge 1$, and $n' \ge 1$, we define the *n-dimensional Hartogs triangle* by

$$\mathbb{H}^n_{\phi_j} = \left\{ (z, z') \in \mathbb{C}^{m_1+\cdots+m_l+n'} \;\middle|\; \max_{1\le j\le l} |\phi_j(\tilde{z}_j)| < |z'_1| < |z'_2| < \cdots < |z'_{n'}| < 1 \right\}.$$

Following the same idea, we see that the Bergman projection $\mathcal{B}$ on $\mathbb{H}^n_{\phi_j}$ is bounded from $W^{k,p}(\mathbb{H}^n_{\phi_j})$ to $W^{k,p}(\mathbb{H}^n_{\phi_j}, |z'_1|^{kp})$ for $p \in (2n/(n+1), 2n/(n-1))$. However, the weight $|z'_1|$ is no longer comparable to $|(z, z')|$, the distance from points near the boundary to the singularity at the boundary.

Acknowledgements

The content of this paper is a part of the author's Ph.D. thesis at Washington University in St. Louis; see [Chen 2015]. The author would like to thank his thesis advisor Prof. S. G. Krantz for giving him a very interesting problem to work on and lots of guidance on his research. The author also wants to thank Prof. E. J. Straube for very helpful comments and suggestions on his work.

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Received November 10, 2015. Revised October 9, 2016.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 288 No. 2 June 2017

0030-8730(201706)288:2;1-R