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Let v be a discrete valuation on the function field of a normal projective variety X. Ein, Lazarsfeld, Mustață, Nakamaye, and Popa showed that v induces a nonnegative real-valued continuous function on the big cone of X, which they called the asymptotic order of vanishing along v. The case where v is given by the order of vanishing along a prime divisor was studied earlier by Nakayama, who extended the domain of the function to the pseudoeffective cone and investigated the continuity of the extended function.

Here we generalize Nakayama's results to any discrete valuation v, using an approach inspired by Lazarsfeld and Mustață's construction of the global Okounkov body, which has a quite different flavor from the arguments employed by Nakayama.

A corollary is that the asymptotic order-of-vanishing function can be extended continuously to the pseudoeffective cone PE(X) of X if PE(X) is polyhedral (note that we do *not* require PE(X) to be *rational* polyhedral).

Let X be a normal projective variety over an algebraically closed field k, and let $K(X)$ be the function field of X. Let v be a discrete valuation of $K(X)$ over k, and let Z be the center of v on X. Ein, Lazarsfeld, Mustață, Nakamaye, and Popa gave the following definitions:

Definition 1 [Ein et al. 2006]. Let D be an effective big Cartier divisor on X. We establish the following notation:

(i) $v(D) = v(f)$, where f is a local equation of D at the generic point of Z.

(ii) $v(|D|) = \min\{v(D') : D' \in |D|\} = v(D')$ for general $D' \in |D|$.

(iii) $v(\|D\|) = \lim_{m\to\infty} v(|mD|)/m$. This is called the asymptotic order of vanishing of D along v.

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By taking m to be sufficiently divisible, the definition of $v(\|D\|)$ can also be extended to big $\mathbb{Q}$-divisors D. It is proved in [Ein et al. 2006, Theorem A] that $v(\|D\|)$ depends only on the numerical equivalence class of D, so it induces a function on the set of numerical equivalence classes of big $\mathbb{Q}$-divisors. Moreover, this function extends uniquely to a continuous function on the cone $\mathrm{Big}(X)$ of numerical equivalence classes of big $\mathbb{R}$-divisors. In view of this result, it is natural to ask whether this function can be extended continuously to the pseudoeffective cone $\mathrm{PE}(X)$, the closure of the big cone $\mathrm{Big}(X)$ in the Néron–Severi space $N^1(X)_{\mathbb{R}}$.

The case where v is a divisorial valuation was investigated earlier by Nakayama [2004] during his study of Zariski decomposition in higher dimensions. More precisely, let Γ be a prime divisor on a smooth projective variety X, and let v be the discrete valuation of $K(X)$ given by the order of vanishing at the generic point of Γ. Nakayama used the notation $\sigma_\Gamma(D)$ to denote the asymptotic order of vanishing $v(\|D\|)$ of a big divisor class $D \in \mathrm{Big}(X)$. If $D \in \mathrm{PE}(X)$ is a pseudoeffective class, he defined $\sigma_\Gamma(D)$ by picking an arbitrary ample class $A \in N^1(X)_{\mathbb{R}}$ and setting $\sigma_\Gamma(D)$ to be the limit

$$\sigma_\Gamma(D) = \lim_{\epsilon \to 0^+} \sigma_\Gamma(D + \epsilon A),$$

after establishing that this limit does not depend on the choice of A, in [Nakayama 2004, III.1.5]. In III.1.7 of the same work, Nakayama showed that the function $\sigma_\Gamma : \mathrm{PE}(X) \to \mathbb{R}_{\geq 0}$ is lower semicontinuous, and he gave an example where it is not continuous in IV.2.8. It is interesting to note that in his example $\mathrm{PE}(X)$ is not polyhedral. The goal of this short note is to generalize Nakayama's results to any discrete valuation v of $K(X)/k$, using an approach inspired by Lazarsfeld and Mustață's construction [2009] of the global Okounkov body, which has a quite different flavor from the arguments employed by Nakayama. In addition, we will see that the function $v(\|\cdot\|) : \mathrm{Big}(X) \to \mathbb{R}_{\geq 0}$ can be extended continuously to $\mathrm{PE}(X)$ if $\mathrm{PE}(X)$ is polyhedral.

Theorem 2. *Let X be a normal projective variety over an algebraically closed field k, and let v be a discrete valuation of $K(X)$ over k. If $D \in \mathrm{PE}(X)$ is a pseudoeffective class, then for any ample class $A \in N^1(X)_{\mathbb{R}}$, $\lim_{\epsilon \to 0^+} v(\|D + \epsilon A\|)$ does not depend on the choice of A. Moreover, if we denote this limit by $\sigma_v(D)$, then the function*

$$\sigma_v : \mathrm{PE}(X) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

is lower semicontinuous, and is continuous at every point where $\mathrm{PE}(X)$ *is locally polyhedral.*

A subset S of $\mathbb{R}^n$ is said to be *locally polyhedral* at a point $x \in S$ if there exist a polytope $P \subset \mathbb{R}^n$ and an open subset U of $\mathbb{R}^n$ containing x such that $U \cap S = U \cap P$. It follows from Theorem 2 that the function $v(\|\cdot\|) : \mathrm{Big}(X) \to \mathbb{R}_{\geq 0}$ can be extended continuously to $\mathrm{PE}(X)$ if $\mathrm{PE}(X)$ is polyhedral, which is the case, for example, when the Picard number of X is 2. Note that we do *not* require $\mathrm{PE}(X)$ to be *rational* polyhedral (cf. [Ein et al. 2006, Theorem D]).

Remark 3. If v is divisorial, the limit $\lim_{\epsilon\to 0^+} v(\|D+\epsilon A\|)$ in Theorem 2 is finite [Nakayama 2004, III.1.5]. We do not know if this is true for all discrete valuations v, which is why we include $+\infty$ in the target of σ_v. In case the value of σ_v is $+\infty$ at a point of $\mathrm{PE}(X)$, the (semi)continuity of σ_v should be interpreted with respect to the usual order topology on $\mathbb{R}_{\geq 0}\cup\{+\infty\}$.

Let us introduce some notions from convex analysis which will be useful in the proof of Theorem 2. Let $f : S \to \mathbb{R}\cup\{+\infty\}$ be a function on a convex subset S of $\mathbb{R}^n$. We say that f is *convex* if

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $x_1, x_2 \in S$ and all $0 \leq \lambda \leq 1$. The *epigraph* of f is the set

$$\{(x, y) \in S \times \mathbb{R} : y \geq f(x)\}.$$

A convex function f is said to be *closed* if its epigraph is a closed subset of $\mathbb{R}^{n+1}$. It is not difficult to show that if f is a closed convex function, then f is lower semicontinuous.

Proof of Theorem 2. As mentioned earlier, our approach is inspired by the construction of the global Okounkov body due to [Lazarsfeld and Mustață 2009]. The strategy is to construct the epigraph of the asymptotic order-of-vanishing function as the closed convex cone spanned by a certain lattice semigroup. To see how this works for one big divisor D, let $\mathbb{N}$ denote the set of nonnegative integers, and let

$$S(D) = \{(m, y) \in \mathbb{N}^2 : y \geq v(|mD|)\},$$

which is a subsemigroup of $\mathbb{N}^2$. Let $C(D) = \overline{\mathrm{cone}}(S(D))$ be the closed convex cone spanned by $S(D)$ in $\mathbb{R}^2$. Then $C(D)$ is the epigraph of the function $x \mapsto v(\|xD\|)$. In order to get the epigraph of the function $v(\|\cdot\|) : \mathrm{Big}(X) \to \mathbb{R}_{\geq 0}$, pick a $\mathbb{Z}$-basis $D_1, \dots, D_n$ for $N^1(X)$ such that, after identifying $N^1(X)_{\mathbb{R}}$ with $\mathbb{R}^n$ by this basis, we have $\mathrm{PE}(X) \subseteq \mathbb{R}^n_{\geq 0}$. Let

$$S(X) = \{(m_1, \dots, m_n, y) \in \mathbb{N}^n \times \mathbb{N} : y \geq v(|m_1 D_1 + \cdots + m_n D_n|)\},$$

and let

$$C(X) = \overline{\mathrm{cone}}(S(X)) \subseteq \mathbb{R}^n_{\geq 0} \times \mathbb{R}_{\geq 0}$$

be the closed convex cone spanned by $S(X)$. Let

$$f : \mathrm{PE}(X) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

be the function whose epigraph is $C(X)$. Then f is a closed convex function since $C(X)$ is a closed convex cone. Moreover, on the big cone $\mathrm{Big}(X)$, f coincides with the function $v(\|\cdot\|)$ by [Lazarsfeld and Mustaţă 2009, Proposition 4.9].

To see what $f(D)$ is if D is on the boundary of $\mathrm{PE}(X)$, we invoke a theorem of Gale, Klee, and Rockafellar, which states that a closed convex function is continuous at every point where its domain is locally polyhedral ([Gale et al. 1968, Theorem 2]; see also the introduction of [Ernst 2013]). It follows that for any ample $A \in N^1(X)_{\mathbb{R}}$, the restriction of f to the half-line $D + \mathbb{R}_{\geq 0} A$ is continuous. Hence

$$f(D) = \lim_{\epsilon \to 0^+} f(D + \epsilon A) = \lim_{\epsilon \to 0^+} v(\|D + \epsilon A\|).$$

This shows that the limit on the right does not depend on the choice of A, and that in fact $f = \sigma_v$. Since σ_v is a closed convex function, it is lower semicontinuous, and is continuous at every point where $\mathrm{PE}(X)$ is locally polyhedral by the theorem of Gale, Klee and Rockafellar. $\square$

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