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THE FABER–KRAHN INEQUALITY FOR THE FIRST EIGENVALUE OF THE FRACTIONAL DIRICHLET p-LAPLACIAN FOR TRIANGLES AND QUADRILATERALS

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We prove the Faber–Krahn inequality for the first eigenvalue of the fractional Dirichlet p-Laplacian for triangles and quadrilaterals of a given area. The proof is based on a nonlocal Pólya–Szegő inequality under Steiner symmetrization and the continuity of the first eigenvalue of the fractional Dirichlet p-Laplacian with respect to the convergence, in the Hausdorff distance, of convex domains.

1. Introduction and main result

The classical isoperimetric problem reads as follows: "among all domains in $\mathbb{R}^n$ of a given volume with rectifiable boundary, the sphere has the minimum perimeter."

In line with this, various isoperimetric problems have been studied (see [Osserman 1978]). For example, the Faber–Krahn inequality, originally conjectured in [Rayleigh 1894, 339–340], can be stated as follows: "among all open sets of a given volume in Euclidean space the ball minimizes the first eigenvalue of the Dirichlet Laplacian."

The Faber–Krahn inequality for variants of the Laplacian or by restriction to special classes of domains have generated interest in recent years. In fact, inspired by the Faber–Krahn inequality, Pólya and Szegő [1951] conjectured that among all polygons with n sides of fixed area, the regular n-polygon of the same area minimizes the first eigenvalue of the Dirichlet Laplacian. This conjecture is known to hold for $n = 3$ and $n = 4$, but for n-gons with $n \geq 5$ it still remains a conjecture. On the other hand, the Faber–Krahn inequality has been generalized, for example, to the case of the Dirichlet p-Laplacian [Bhattacharya 1999; Ly 2005; Chorwadwala et al. 2015; Toledo Oñate 2012].

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Recently, partial differential equations involving nonlocal versions of the Laplacian and in particular eigenvalue problems involving such operators have generated a lot of interest and have been studied (see [Di Nezza et al. 2012; Lindgren and Lindqvist 2014; Frank et al. 2008; Brasco et al. 2014]).

The first eigenvalue of fractional Dirichlet p-Laplacian is defined as follows:

Definition. Let $n \geq 1,\ 0 < s < 1$ and $1 < p < \infty$. Given an open and bounded set $\Omega \subset \mathbb{R}^n$ we define

$$\lambda_{1,p}^{s}(\Omega) = \inf\left\{ \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy}{\int_{\mathbb{R}^n}|u(x)|^p\,dx} : u \in \widetilde{W}_0^{s,p}(\Omega) \quad \text{and} \quad u \not\equiv 0 \right\}, \tag{1-1}$$

where $\widetilde{W}_0^{s,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$u \mapsto [u]_{W^{s,p}(\mathbb{R}^n)} + \|u\|_p \tag{1-2}$$

where $[u]_{W^{s,p}(\mathbb{R}^n)}$ is defined in (2-1).

Inspired by the nonlocal Faber–Krahn inequality proved in [Brasco et al. 2014] for the fractional Dirichlet p-Laplacian and the Pólya–Szegő conjecture for the usual Laplacian for polygonal domains, we prove a Faber–Krahn inequality for the fractional Dirichlet p-Laplacian in the class of polygonal domains. This is our main result.

Theorem 1.1. *The equilateral triangle has the least first eigenvalue for the fractional Dirichlet p-Laplacian among all triangles of given area. The square has the least first eigenvalue for the fractional Dirichlet p-Laplacian among all quadrilaterals of given area. Moreover, the equilateral triangle and the square are the unique minimizers in the above problems.*

For proving this result we shall study the effect of Steiner symmetrization in nonlocal functionals and the continuity properties of the first eigenvalue of the fractional Dirichlet p-Laplacian with respect to the Hausdorff convergence of convex domains. In particular we will prove the following two results which will be used in the proof of Theorem 1.1:

Proposition 1.2 (nonlocal Pólya–Szegő inequality). *Let $n \geq 1,\ 0 < s < 1,\ 1 \leq p \leq n/s$ and $u \in \widetilde{W}_0^{s,p}(\Omega)$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u^\star(x)-u^\star(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy, \tag{1-3}$$

where $u^\star$ is the Steiner symmetrization of u with respect to a given hyperplane. If $p > 1$, then equality holds if and only if u is proportional to a translate of a function which is symmetric with respect to the hyperplane.

Proposition 1.3. *Let B be a fixed compact set in $\mathbb{R}^n$ and Ω_n be a family of convex open subsets of B which converges, for the Hausdorff distance, to a set Ω. Furthermore, assume that there exist $r > 0$ such that $B(0,r) \subset \Omega_n$ and $B(0,r) \subset \Omega$. Then $\lambda^s_{1,p}(\Omega) = \lim_{n\to\infty} \lambda^s_{1,p}(\Omega_n)$.*

The basic definitions, notions and results which will be used in this paper are to be given in the next section.

2. Tools

Fractional Sobolev spaces and the first eigenvalue. Let $p \in [1,\infty)$ and $s \in (0,1)$. Then

$$[u]_{W^{s,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\right)^{1/p} \tag{2-1}$$

denotes the (s,p)-Gagliardo seminorm in $\mathbb{R}^n$ of a measurable function u. The Gagliardo seminorm satisfies the following Poincaré-type inequality:

Proposition 2.1 (Poincaré-type inequality). *Let $1 \le p < \infty$ y $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be an open and bounded set. There then exists a constant $C_{n,s,p}$, depending only on n, s, p and Ω, so that, for every function $u \in C_0^\infty(\Omega)$ we have*

$$\|u\|_p^p \le C_{n,s,p}(\Omega)[u]^p_{W^{s,p}(\mathbb{R}^n)}$$

Proof. See Lemma 2.4, [Brasco et al. 2014]. □

Proposition 2.1 shows that for an open and bounded set $\Omega \subset \mathbb{R}^n$ the space $\widetilde{W}_0^{s,p}(\Omega)$ can be equivalently defined as the closure of $C_0^\infty(\Omega)$ with respect to the Gagliardo seminorm. The space $\widetilde{W}_0^{s,p}(\Omega)$ is a reflexive Banach space for $1 < p < \infty$.

Theorem 2.2 (Rellich–Kondrachov theorem). *Let $p \in [1,\infty)$ and $s \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be a open and bounded set. Let $\{u_n\}_{n=1}^\infty \subset \widetilde{W}_0^{s,p}(\Omega)$ be a bounded sequence. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ which converges strongly in $L^p(\Omega)$ to a function u. Moreover, if $p > 1$ then $u \in \widetilde{W}_0^{s,p}(\Omega)$.*

Proof. See Theorem 2.7, [Brasco et al. 2014]. □

Remark 1. Following Theorem 2.2, it can be shown that the infimum in (1-1) is a minimum and by the homogeneity of the Rayleigh quotient, the expression (1-1) can be written as

$$\lambda^s_{1,p}(\Omega) = \min\left\{\|u\|^p_{\widetilde{W}_0^{s,p}(\Omega)} : u \in \widetilde{W}_0^{s,p}(\Omega),\ \|u\|_p = 1\right\}. \tag{2-2}$$

Observe also that $\lambda^s_{1,p}(\Omega)$ equals the inverse of the best constant in the Poincaré inequality (Proposition 2.1).

The minimizer in (1-1) satisfies the following Euler–Lagrange equation

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+ps}}\,dx\,dy = \lambda_{1,p}^s(\Omega)\int_{\mathbb{R}^n}|u(x)|^{p-2}u(x)\phi(x)\,dx, \tag{2-3}$$

for all $\phi \in \widetilde{W}_0^{s,p}(\Omega)$ (see Theorem 5, [Lindgren and Lindqvist 2014]).

One can easily check that the following properties hold:

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set.*

(1) (*Homothety law*) $\lambda_{1,p}^s(t\Omega) = t^{-sp}\lambda_{1,p}^s(\Omega)$ *for* $t > 0$.

(2) (*Translation invariance*) $\lambda_{1,p}^s(\Omega) = \lambda_{1,p}^s(\Omega + x)$ *for all* $x \in \mathbb{R}^n$.

(3) (*Invariance under orthonormal transformations*) $\lambda_{1,p}^s(\Omega) = \lambda_{1,p}^s(T(\Omega))$ *for every orthonormal transformation* T.

(4) (*Domain monotony*) *If* $A \subset B$ *are open sets, then* $\lambda_{1,p}^s(B) \le \lambda_{1,p}^s(A)$.

Steiner symmetrizations of sets and functions. Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a measurable set. We denote by Ω' the projection of Ω in the x_n-direction:

$$\Omega' := \{x' \in \mathbb{R}^{n-1} : \text{ there exists } x_n \text{ such that } (x', x_n) \in \Omega\},$$

and, for $x' \in \mathbb{R}^{n-1}$, we denote by $\Omega(x')$ the section of Ω in x':

$$\Omega(x') := \{x_n \in \mathbb{R} : (x', x_n) \in \Omega\},\ x' \in \Omega'.$$

Definition. Let $\Omega \subset \mathbb{R}^n$ be a measurable set. The set

$$\Omega^\star := \left\{x = (x', x_n) : -\tfrac{1}{2}|\Omega(x')| < x_n < \tfrac{1}{2}|\Omega(x')|,\ x' \in \Omega'\right\} \tag{2-4}$$

is the Steiner symmetrization of Ω with respect to the hyperplane $x_n = 0$. In the above, $|\Omega(x')|$ denotes the one-dimensional Lebesgue measure of $\Omega(x')$.

The Steiner symmetrization of a convex set with respect to a given hyperplane can be similarly defined.

A convex body is a compact convex set. For a convex body A in $\mathbb{R}^n$, the inradius $r(A)$ is the maximum of the radii of balls contained in A and the circumradius $R(A)$ is the minimum of the radii of balls containing A.

The Steiner symmetrization of sets has the following properties:

Proposition 2.4. *Let A, B be convex bodies. Then*

(1) $A^\star \subseteq B^\star$ *for* $A \subseteq B$.

(2) $r(A) \le r(A^\star)$.

(3) $R(A^\star) \le R(A)$.

(4) $V(A) = V(A^\star)$ *where* $V(A)$ *denotes the volume of* A.

Proof. See Proposition 9.1, page 169–171 of [Gruber 2007]. □

Definition. Let f be a nonnegative measurable function defined on Ω, which vanishes on $\partial\Omega$. The Steiner symmetrization of f is the function $f^\star$ defined on $\Omega^\star$ by

$$f^\star(x) = \sup\{c : x \in \{y \in \Omega : f(y) \geq c\}^\star\}. \tag{2-5}$$

The Steiner symmetrization of functions has the following properties.

Proposition 2.5. (1) *The definitions of $A^\star$ and $f^\star$ are consistent, i.e.,*

$$\chi_{A^\star} = (\chi_A)^\star \quad \textit{and} \quad \{x : f(x) \geq t\}^\star = \{x : f^\star(x) \geq t\}.$$

(2) *Let f and g be two nonnegative measurable functions such that $f(x) \leq g(x)$. Then $f^\star(x) \leq g^\star(x)$.*

(3) *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function. Then $(\Phi \circ f)^\star = \Phi \circ f^\star$.*

(4) *Let f be a nonnegative measurable function defined on Ω vanishing on $\partial\Omega$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a measurable function. Then,*

$$\int_\Omega F(f(x))\,dx = \int_{\Omega^\star} F(f^\star(x))\,dx.$$

(5) *Let f, g and h be nonnegative measurable functions on $\mathbb{R}^n$. Then with $I(f, g, h) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)\,dx\,dy$, we have*

$$I(f, g, h) \leq I(f^\star, g^\star, h^\star). \tag{2-6}$$

Moreover, if g is strictly symmetric decreasing, then there is equality in (2-6) *if only if $f(x) = f^\star(x-y)$ and $h(x) = h^\star(x-y)$ almost everywhere for some $y \in \mathbb{R}^n$.*

Proof. The proof of (1)–(4) is straightforward. For the proof of (5), we refer to Theorem 3.7, page 87 and Theorem 3.9, page 93 of [Lieb and Loss 2001] and [Brascamp et al. 1974] . □

For J a nonnegative, convex function on $\mathbb{R}$ with $J(0) = 0$ and k a nonnegative measurable function on $\mathbb{R}^n$, we let

$$E[u] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(u(x) - u(y))k(x-y)\,dx\,dy.$$

Following the same ideas given in Lemma A.2. of [Frank and Seiringer 2008], using principally part (5) of Proposition 2.5 for Steiner symmetrization instead of symmetric decreasing rearrangement, we get the following lemma:

Lemma 2.6. *Let J be a nonnegative, convex function on $\mathbb{R}$ with $J(0) = 0$ and let $k \in L_1(R^n)$ be a nonnegative function which is symmetric and decreasing. Then for* *all nonnegative measurable u with $E[u]$ and $|\{u > \tau\}|$ finite for all $\tau > 0$ one has*

$$E[u] \geq E[u^\star],$$

with $u^\star$ the Steiner symmetrization of u with respect a hyperplane. If, in addition, J is strictly convex and k is strictly decreasing, then equality holds if and only if u is a translate of a function which is symmetric with respect to the hyperplane.

Hausdorff distance.

Definition. Let K_1 and K_2 be two nonempty compact sets in $\mathbb{R}^n$. Taking $d(x, K_2) := \inf\{|y - x| : y \in K_2\}$ for $x \in \mathbb{R}^n$, we set

$$\rho(K_1, K_2) := \sup\{d(x, K_2) : x \in K_1\}.$$

Let C^n be the family of compact subsets of $\mathbb{R}^n$. It is a metric space when equipped with the Hausdorff distance

$$d^H(K_1, K_2) := \max(\rho(K_1, K_2), \rho(K_2, K_1)). \tag{2-7}$$

For open sets inside a fixed compact set, we define the Hausdorff distance through their complement.

Definition. Let O_1, O_2 be two open sets of a compact set B. Then their Hausdorff distance is defined by

$$d_H(O_1, O_2) = d^H(B \setminus O_1, B \setminus O_2). \tag{2-8}$$

The Minkowski addition and Minkowski difference.

Definition. The Minkowski addition of two sets $A, B \subset \mathbb{R}^n$ can be defined by

$$A \oplus B := \bigcup_{b \in B} (A + b). \tag{2-9}$$

Definition. The Minkowski difference of two sets $A, B \subset \mathbb{R}^n$ can be defined by

$$A \ominus B := \bigcap_{b \in B} (A - b). \tag{2-10}$$

Clearly, we may also write $A \ominus B := \{x \in \mathbb{R}^n : B + x \subset A\}$. If $B = -B$, then

$$A \ominus B := \bigcap_{b \in B} (A + b).$$

The following proposition can be obtained without much difficulty using the above definition:

Proposition 2.7. *Let A, B and C be subsets of $\mathbb{R}^n$ such that $B = -B$, $A \subset C$ and $B \subset C$. Then*

$$A \ominus B \subseteq C \setminus ((C \setminus A) \oplus B).$$

Recall that, $K \ominus B(0,\epsilon)$ is the inner parallel body of K at distance ϵ. The main tool in the proof of Proposition 1.2 is the following lemma, which states that a suitable contraction of a convex body is contained in the inner parallel body of the convex body.

Lemma 2.8. *Let K be a convex body in $\mathbb{R}^n$, with $B(0,r) \subset K \subset B(0,R)$ for some numbers $r > 0$ and $R > 0$. If $0 < \epsilon < r^2/4R$, then*

$$\Big(1 - 4\frac{R\epsilon}{r^2}\Big)K \subset K \ominus B(0,\epsilon) \subset K. \tag{2-11}$$

Proof. See Lemma 2.3.6, page 93 of [Schneider 2014]. □

3. Proofs

The proof of Proposition 1.2 is given in Theorem A.1 of [Frank and Seiringer 2008] for the symmetric decreasing rearrangement. We sketch the proof of the adaptation to the case of Steiner symmetrization for the sake of completeness.

Proof of Proposition 1.2. Since $u^\star(x)$ is nonnegative and $\big||u(x)| - |u(y)|\big| \leq |u(x) - u(y)|$, it suffices to prove the theorem for nonnegative functions. By definition of the Gamma function and following a change of variables we obtain

$$\frac{1}{\Gamma(\frac{n+ps}{2})}\int_0^\infty \alpha^{\frac{n+ps}{2}-1} \mathrm{e}^{-\alpha|x-y|^2}\, d\alpha = \frac{1}{|x-y|^{n+ps}}. \tag{3-1}$$

Using (3-1) and Tonelli's theorem for nonnegative integrands and we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dx\, dy &= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty \alpha^{\frac{n+ps}{2}-1}\mathrm{e}^{-\alpha|x-y|^2}|u(x)-u(y)|^p d\alpha\, dx\, dy \\ &= C\int_0^\infty I_\alpha[u]\alpha^{\frac{n+ps}{2}-1}\, d\alpha\end{aligned}$$

with

$$I_\alpha[u] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^p \mathrm{e}^{-\alpha|x-y|^2}\, dx\, dy \quad \text{and} \quad C = \frac{1}{\Gamma(\frac{n+ps}{2})}.$$

The function $J : \mathbb{R} \to \mathbb{R},\ x \mapsto |x|^p$ is strictly convex and nonnegative with $J(0) = 0$. The function $k : \mathbb{R}^n \to \mathbb{R},\ x \mapsto \mathrm{e}^{-|x|^2}$ is a strictly decreasing symmetric function and $k \in L_1(\mathbb{R}^n)$. Applying Lemma 2.6 to the functional I_α we obtain the desired result. □

Proof of Proposition 1.3. Since by hypothesis the sequence of sets $\{\Omega_n\}_{n=1}^\infty$ converges in the Hausdorff distance to Ω, then for any $\epsilon > 0$ there exist n_ϵ such that

$$B \setminus \Omega_n \subset (B \setminus \Omega) \oplus B(0,\epsilon) \quad \text{for all} \quad n \geq n_\epsilon \tag{3-2}$$

and

$$B \setminus \Omega \subset (B \setminus \Omega_n) \oplus B(0,\epsilon) \quad \text{for all} \quad n \geq n_\epsilon. \tag{3-3}$$

By Proposition 2.7, we have

$$\Omega \ominus B(0,\epsilon) \subseteq B \setminus ((B \setminus \Omega) \oplus B(0,\epsilon)). \tag{3-4}$$

It is clear that

$$\overline{\Omega} \ominus B(0,\epsilon) = \Omega \ominus B(0,\epsilon). \tag{3-5}$$

By using (3-2) (after taking the complement), (3-4), and (3-5) we obtain

$$\overline{\Omega} \ominus B(0,\epsilon) \subseteq B \setminus ((B \setminus \Omega) \oplus B(0,\epsilon)) \subset \Omega_n. \tag{3-6}$$

Using Lemma 2.8 and (3-6) we get

$$\Big(1 - 4\frac{R\epsilon}{r^2}\Big)\Omega \subset \Big(1 - 4\frac{R\epsilon}{r^2}\Big)\overline{\Omega} \subset \overline{\Omega} \ominus B(0,\epsilon) \subset \Omega_n. \tag{3-7}$$

Then applying parts (1) and (4) of Proposition 2.3 to (3-7) we obtain:

$$\Big(1 - 4\frac{R\epsilon}{r^2}\Big)^{sp} \lambda^s_{1,p}(\Omega_n) \leq \lambda^s_{1,p}(\Omega). \tag{3-8}$$

Taking the upper limit in (3-8) gives:

$$\Big(1 - 4\frac{R\epsilon}{r^2}\Big)^{sp} \varlimsup_{n\to\infty} \lambda^s_{1,p}(\Omega_n) \leq \lambda^s_{1,p}(\Omega). \tag{3-9}$$

Now, taking the limit as ϵ goes to 0 in (3-9) we get

$$\varlimsup_{n\to\infty} \lambda^s_{1,p}(\Omega_n) \leq \lambda^s_{1,p}(\Omega). \tag{3-10}$$

Similarly, applying (3-4) and (3-5) in (3-3), and arguing as above, we can get

$$\lambda^s_{1,p}(\Omega) \leq \varliminf_{n\to\infty} \lambda^s_{1,p}(\Omega_n). \tag{3-11}$$

The result follows immediately from (3-10) and (3-11). □

Proof of Theorem 1.1. Since $\lambda^s_{1,p}$ is translation and rotation invariant (see parts (2) and (3) of Proposition 2.3), to prove Theorem 1.1 for triangles, it is sufficient to find one equilateral triangle T' such that $\lambda^s_{1,p}(T') \leq \lambda^s_{1,p}(T)$.

Let T_1 be an arbitrary triangle. We define recursively T_{n+1} to be the Steiner symmetrization of T_n with respect to the perpendicular bisector of one side (a side with respect to which there is no symmetry). Let u_n be a normalized function for the fractional Dirichlet p-Laplacian on T_n. Then, by Proposition 1.2 we have,

$$\lambda^s_{1,p}(T_n) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_n^\star(x)-u_n^\star(y)|^p}{|x-y|^{n+ps}}\,dx\,dy,$$

and by part (4) of Proposition 2.5 we have $\|u_n\|_p = \|u_n^\star\|_p = 1$. Therefore, using the definition on page 426, we obtain

$$\lambda^s_{1,p}(T_{n+1}) \le \lambda^s_{1,p}(T_n) \quad \text{for each} \quad n. \tag{3-12}$$

Now, recall the fact that the sequence of Steiner symmetrizations T_n of the arbitrary initial triangle T_1 converges to an equilateral triangle T with respect to the Hausdorff distance (see page 158 of [Pólya and Szegő 1951]). Then, using part (2) of Proposition 2.4, and if necessary a translation, we can show that there is a fixed ball contained in all the triangles T_n. Using part (3) of Proposition 2.4 we also conclude that all the triangles T_n are contained in a fixed ball. This allows us to apply Proposition 1.3, and we get

$$\lambda^s_{1,p}(T) = \lim_{n\to\infty} \lambda^s_{1,p}(T_n) \le \lambda^s_{1,p}(T_1)\,.$$

In the case of quadrilaterals, a similar argument can be used. In fact, a sequence of Steiner symmetrizations of a given quadrilateral, done alternatingly, with respect to the perpendicular bisector of a side and the diagonal, converges in the Hausdorff distance to a square (see page 158–159 of [Pólya and Szegő 1951]). This fact together with a reasoning as in the case of triangles leads to the Faber–Krahn inequality for quadrilaterals.

We now turn to the question of uniqueness. Suppose that T is any triangle for which the minimum is attained in the Faber–Krahn inequality. We can assume without loss of generality that T is not an equilateral triangle. Then T is not symmetric respect to the perpendicular bisector L to at least one side l. Let $T^\star$ the Steiner symmetrization of T respect to L. Let u be a normalized eigenfunction of $\lambda^s_{1,p}(T)$. Applying Proposition 1.2 and $\|u\|_p = \|u^\star\|_p = 1$, we get

$$\lambda^s_{1,p}(T^\star) \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u^\star(x)-u^\star(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy = \lambda^s_{1,p}(T).$$

Since, $\lambda^s_{1,p}(T)$ is minimum, we obtain $\lambda^s_{1,p}(T^\star) = \lambda^s_{1,p}(T)$. This means that there is equality in the nonlocal Pólya–Szegő inequality and so, by the uniqueness part of Proposition 1.2, we get that u is a translate of $u^\star$. This is possible only if the triangles T and $T^\star$ are translates of each other. However, $T^\star$ is symmetric with respect to the L and T and $T^\star$ being translates of each other, T would have to be symmetric with respect to L. This gives a contradiction. So, the only minimizers are equilateral triangles.

The uniqueness in the case of quadrilaterals is completely analogous to case of the triangles. □

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Volume 288 No. 2 June 2017

Order on the homology groups of Smale spaces 257
MASSOUD AMINI, IAN F. PUTNAM and SARAH SAEIDI GHOLIKANDI

Characterizations of immersed gradient almost Ricci solitons 289
CÍCERO P. AQUINO, HENRIQUE F. DE LIMA and JOSÉ N. V. GOMES

Weighted Sobolev regularity of the Bergman projection on the Hartogs triangle 307
LIWEI CHEN

Knots of tunnel number one and meridional tori 319
MARIO EUDAVE-MUÑOZ and GRISSEL SANTIAGO-GONZÁLEZ

On bisectional nonpositively curved compact Kähler–Einstein surfaces 343
DANIEL GUAN

Effective lower bounds for $L(1, \chi)$ via Eisenstein series 355
PETER HUMPHRIES

Asymptotic order-of-vanishing functions on the pseudoeffective cone 377
SHIN-YAO JOW

Augmentations and rulings of Legendrian links in $\#^k(S^1 \times S^2)$ 381
CAITLIN LEVERSON

The Faber–Krahn inequality for the first eigenvalue of the fractional Dirichlet p-Laplacian for triangles and quadrilaterals 425
FRANCO OLIVARES CONTADOR

Topological invariance of quantum quaternion spheres 435
BIPUL SAURABH

Gap theorems for complete λ-hypersurfaces 453
HUIJUAN WANG, HONGWEI XU and ENTAO ZHAO

Bach-flat h-almost gradient Ricci solitons 475
GABJIN YUN, JINSEOK CO and SEUNGSU HWANG

A sharp height estimate for the spacelike constant mean curvature graph in the Lorentz–Minkowski space 489
JINGYONG ZHU

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