Pacific Journal of Mathematics

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Volume 288 No. 2 June 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 288, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.288.435

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The C^*-algebra of continuous functions on the quantum quaternion sphere H_q^{2n} can be identified with the quotient algebra $C(\mathrm{SP}_q(2n)/\mathrm{SP}_q(2n-2))$. In the commutative case, i.e., for $q=1$, the topological space $\mathrm{SP}(2n)/\mathrm{SP}(2n-2)$ is homeomorphic to the odd-dimensional sphere $\mathcal{S}^{4n-1}$. In this paper, we prove the noncommutative analogue of this result. Using homogeneous C^*-extension theory, we prove that the C^*-algebra $C(H_q^{2n})$ is isomorphic to the C^*-algebra $C(S_q^{4n-1})$. This further implies that for different values of q in $[0, 1)$, the C^*-algebras underlying the noncommutative spaces H_q^{2n} are isomorphic.

1. Introduction

Quantization of Lie groups and their homogeneous spaces has played an important role in linking the theory of compact quantum groups with noncommutative geometry. Many authors (see [Vaksman and Soibelman 1990; Podkolzin and Vainerman 1999; Chakraborty and Pal 2008; Pal and Sundar 2010]) have studied different aspects of the theory of quantum homogeneous spaces. However, in these papers, the main examples have been the quotient spaces of the compact quantum group $\mathrm{SU}_q(n)$. Neshveyev and Tuset [2012] studied quantum homogeneous spaces in a more general setup and gave a complete classification of the irreducible representations of the C^*-algebra $C(G_q/H_q)$ where G_q is the q-deformation of a simply connected semisimple compact Lie group and H_q is the q-deformation of a closed Poisson–Lie subgroup H of G. Moreover, Neshveyev and Tuset [2012] proved that $C(G_q/H_q)$ is KK-equivalent to the classical counterpart $C(G/H)$. In [Saurabh 2017], we studied the quantum symplectic group $\mathrm{SP}_q(2n)$ and its homogeneous space $\mathrm{SP}_q(2n)/\mathrm{SP}_q(2n-2)$, and obtained K-groups of $C(\mathrm{SP}_q(2n)/\mathrm{SP}_q(2n-2))$ with explicit generators.

The C^*-algebra $C(H_q^{2n})$ of continuous functions on the quantum quaternion sphere is defined as the universal C^*-algebra given by a finite set of generators and relations; see [Saurabh 2017]. In the same paper, the isomorphism between

MSC2010: 19K33, 46L80, 58B34.
Keywords: homogeneous extension, quantum double suspension, corona factorization property.

the quotient algebra $C(\mathrm{SP}_q(2n)/\,\mathrm{SP}_q(2n-2))$ and $C(H_q^{2n})$ was established. Now several questions arise about this noncommutative space H_q^{2n}:

(1) Topologically, is H_q^{2n} the same as S_q^{4n-1}, i.e., are the C^*-algebras $C(H_q^{2n})$ and $C(S_q^{4n-1})$ isomorphic?

(2) Are the C^*-algebras $C(H_q^{2n})$ isomorphic for different values of q?

(3) Does the quantum quaternion sphere admit a good spectral triple equivariant under the $\mathrm{SP}_q(2n)$-group action?

We attempt the first two questions in this paper. In the commutative case, that is, for $q = 1$, the quotient space $\mathrm{SP}(2n)/\,\mathrm{SP}(2n-2)$ can be realized as the quaternion sphere H^{2n}. It can be easily verified that the quaternion sphere H^{2n} is homeomorphic to the odd-dimensional sphere $\mathcal{S}^{4n-1}$. One can now expect the quotient algebra $C(\mathrm{SP}_q(2n)/\,\mathrm{SP}_q(2n-2))$, or equivalently, the C^*-algebra $C(H_q^{2n})$, to be isomorphic to the C^*-algebra underlying the odd-dimensional quantum sphere S_q^{4n-1}. Using homogeneous C^*-extension theory, we show that this is indeed the case.

The remarkable work done by L. G. Brown, R. G. Douglas and P. A. Fillmore [Brown et al. 1977] on extensions of commutative C^*-algebras by compact operators has led many authors to extend this theory further in order to provide a tool for analyzing the structure of C^*-algebras. For a nuclear, separable C^*-algebra A and a separable C^*-algebra B, G. G. Kasparov [1979] constructed the group $\mathrm{Ext}(A, B)$ consisting of stable equivalence classes of C^*-algebra extensions of the form

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0.$$

Here E will be called the middle C^*-algebra. One of the important features of this construction is that the group $\mathrm{Ext}(A, B)$ coincides with the group $KK^1(A, B)$. Another important aspect is that it does not demand much. It does not require the extensions to be unital or essential. But at the same time, it does not provide much information about the middle C^*-algebras. Since elements of the group $\mathrm{Ext}(A, B)$ are stable equivalence classes and not strongly unitary equivalence classes of extensions, two elements in the same class may have nonisomorphic middle C^*-algebras. For a nuclear C^*-algebra A and a finite-dimensional compact metric space Y (i.e., a closed subset of $\mathcal{S}^n$ for some $n \in \mathbb{N}$), M. Pimsner, S. Popa and D. Voiculescu [Pimsner et al. 1979] constructed another group $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ consisting of strongly unitary equivalence classes of unital homogeneous extensions of A by $C(Y) \otimes \mathcal{K}$. For $y_0 \in Y$, the subgroup $\mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, A)$ consists of those elements of $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ that split at y_0. For a commutative C^*-algebra A, the group $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ was computed by Schochet [1980]. Further, Rosenberg and Schochet [1981] showed that

$$\mathrm{Ext}_{\mathrm{PPV}}(Y, A^+) = \mathrm{Ext}(A, C(Y)) \quad \text{and} \quad \mathrm{Ext}_{\mathrm{PPV}}(Y^+, +, A^+) = \mathrm{Ext}(A, C(Y)),$$

where Y is a finite-dimensional locally compact Hausdorff space, $+$ is the point at infinity and A^+ is the C^*-algebra obtained by adjoining unity to A.

To prove the claim, our idea is to exhibit two short exact sequences of C^*-algebras in the same equivalence class in the group $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{4n-3}))$ with $C(S_q^{4n-1})$ and $C(H_q^{2n})$ as middle C^*-algebras and then to compare their middle C^*-algebras. First, we prove an isomorphism between groups $\mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, A)$ and $\mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, \Sigma^2 A)$ under certain assumptions on the topological space Y where $\Sigma^2 A$ is the quantum double suspension of A and $y_0 \in Y$. Using this, we describe all elements of the group $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$ explicitly. We compute K-groups of all middle C^*-algebras that occur in all the extensions of the group $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$. Then using the ideal structure of $C(H_q^{2n})$, we show that the extension

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(H_q^{2n}) \to C(S_0^{4n-3}) \to 0$$

is unital and homogeneous. Now by comparing the K-groups of middle C^*-algebras, we prove that the above extension is strongly unitarily equivalent to either the extension

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(S_0^{4n-1}) \to C(S_0^{4n-3}) \to 0,$$

or its inverse in the group $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$, having $C(S_0^{4n-1})$ as a middle C^*-algebra. This proves that the C^*-algebras $C(H_q^{2n})$ and $C(S_0^{4n-1})$ are isomorphic; see [Blackadar 1998, page 147]. For $q = 0$, it follows immediately as the defining relations of $C(H_0^{2n})$ (see [Saurabh 2017]) are exactly the same as those of $C(S_0^{4n-1})$. In [Hong and Szymański 2002], it was proved that for different values of q in [0, 1), the C^*-algebras $C(S_q^{4n-1})$ are isomorphic. As a consequence, the C^*-algebras $C(H_q^{2n})$ and $C(S_q^{4n-1})$ are isomorphic for all q in [0, 1). Also, this establishes the q-invariance of the quantum quaternion spheres, as it shows that the C^*-algebras $C(H_q^{2n})$ are isomorphic for different values of q. Here we must point out that to the best of our knowledge, the group $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ has not been used before to show that two C^*-algebras are isomorphic. In that sense, our idea can be considered as the first of its kind.

We now set up some notation. The standard basis of the Hilbert space $L_2(\mathbb{N})$ will be denoted by $\{e_n : n \in \mathbb{N}\}$. We denote the left shift operator on $L_2(\mathbb{N})$ and $L_2(\mathbb{Z})$ by the same notation S. For $m < 0$, $(S^*)^m$ denotes the operator S^{-m}. Let p_i be the rank-1 projection sending e_i to e_i. The operator p_0 will be denoted by p. We write $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ for the sets of all bounded linear operators and compact operators on $\mathcal{H}$, respectively. We denote by $\mathcal{K}$ the C^*-algebra of compact operators. For a C^*-algebra A, $\Sigma^2 A$ and $M(A)$ are used to denote the quantum double suspension (see [Hong and Szymański 2002; 2008]) of A and multiplier algebra of A, respectively. The map π will denote the canonical homomorphism from $M(A)$ to $Q(A) := M(A)/A$ and for $a \in M(A)$, $[a]$ stands for the image of a

under the map π. For a locally compact Hausdorff space Y, we write Y^+ to denote one-point compactification of Y. For a C^*-algebra A, A^+ is the C^*-algebra obtained by adjoining unity to A. The symbol $\mathcal{S}^n$ will be reserved for the n-dimensional sphere. However, sometimes we will use $\mathbb{T}$ in place of $\mathcal{S}^1$ to denote the circle. Unless otherwise stated, q will denote a real number in the interval (0, 1).

2. C^*-algebra extensions

In this section, we briefly recall some notions related to C^*-extension theory. For a detailed treatment, we refer the reader to [Blackadar 1998]. Let A be a unital separable nuclear C^*-algebra. Let B be a stable C^*-algebra. An extension of A by B is a short exact sequence

$$0 \to B \xrightarrow{i} E \xrightarrow{j} A \to 0.$$

In such cases there exists a unique homomorphism $\sigma : E \to M(B)$ such that $\sigma(i(b)) = b$ for all $b \in B$. We can now define the Busby invariant for the extension $0 \to B \xrightarrow{i} E \xrightarrow{j} A \to 0$ by the homomorphism $\tau : A \to M(B)/B$ given by $\tau(a) = \pi \circ \sigma(e)$, where e is a preimage of a and π is the quotient map from $M(B)$ to $M(B)/B$. It is easy to see that τ is well-defined. Up to strong isomorphism, an extension can be identified with its Busby invariant. In this paper, we will not distinguish between an extension and its Busby invariant, as all the equivalence relations given here are weaker than the strong isomorphism relation.

An extension $\tau : A \to M(B)/B$ is called essential if τ is injective or, equivalently, the image of B is an essential ideal of E. We call an extension unital if it is a unital homomorphism or, equivalently, E is a unital C^*-algebra. An extension τ is called a trivial (or split) extension if there exists a homomorphism $\lambda : A \to M(B)$ such that $\tau = \pi \circ \lambda$. Extensions τ_1 and τ_2 are said to be unitarily equivalent if there exists a unitary u in $Q(B)$ such that $u\tau_1(a)u^* = \tau_2(a)$ for all $a \in A$. The two extensions are said to be strongly unitarily equivalent if there exists a unitary U in $M(B)$ such that $\pi(U)\tau_1(a)\pi(U^*) = \tau_2(a)$ for all $a \in A$. We denote a strongly unitary equivalence relation by $\sim_{su}$. Let $\mathrm{Ext}_{\sim_{su}}(A, B)$ denote the set of strongly unitary equivalence classes of extensions of A by B. One can put a binary operation $+$ on $\mathrm{Ext}_{\sim_{su}}(A, B)$ as follows. Since $M(B)$ is a stable C^*-algebra, we can get two isometries ν_1 and ν_2 in $M(B)$ such that $\nu_1\nu_1^* + \nu_2\nu_2^* = 1$. Let $[\tau_1]_{su}$ and $[\tau_2]_{su}$ be two elements in $\mathrm{Ext}_{\sim_{su}}(A, B)$. Define the extension $\tau_1 + \tau_2 : A \to Q(B)$ by $(\tau_1 + \tau_2)(a) := \pi(\nu_1)\tau_1(a)\pi(\nu_1^*) + \pi(\nu_2)\tau_2(a)\pi(\nu_2^*)$. The binary operation $+$ on $\mathrm{Ext}_{\sim_{su}}(A, B)$ can now be defined as

$$[\tau_1]_{su} + [\tau_2]_{su} := [\tau_1 + \tau_2]_{su}. \tag{2-1}$$

This makes $\mathrm{Ext}_{\sim_{su}}(A, B)$ a commutative semigroup. Moreover, the set of trivial extensions forms a subsemigroup of $\mathrm{Ext}_{\sim_{su}}(A, B)$. We denote the quotient of

$\mathrm{Ext}_{\sim_{su}}(A, B)$ with the set of trivial extensions by $\mathrm{Ext}(A, B)$. For a separable nuclear C^*-algebra A, the set $\mathrm{Ext}(A, B)$ under the operation $+$ is a group; see [Blackadar 1998]. Two extensions τ_1 and τ_2 represent the same element in $\mathrm{Ext}(A, B)$ if there exist two trivial extensions ϕ_1 and ϕ_2 such that $\tau_1 + \phi_1 \sim_{su} \tau_2 + \phi_2$. We denote an equivalent class in the group $\mathrm{Ext}(A, B)$ of an extension τ by $[\tau]_s$. One can show that for a stable C^*-algebra B, $\mathrm{Ext}(A, B) = \mathrm{Ext}(A, B \otimes \mathcal{K})$. Now for an arbitrary C^*-algebra B, define $\mathrm{Ext}(A, B) := \mathrm{Ext}(A, B \otimes \mathcal{K})$. For $B = \mathbb{C}$, we denote the group $\mathrm{Ext}(A, \mathbb{C})$ by $\mathrm{Ext}(A)$. Note that in this case, two unital essential extensions τ_1 and τ_2 are in the same equivalence class (i.e., $[\tau_1]_s = [\tau_2]_s$) if and only if they are strongly unitarily equivalent. Suppose that Y is a finite-dimensional compact metric space, i.e., a closed subset of $\mathcal{S}^n$ for some $n \in \mathbb{N}$. Let $M(Y)$, $Q(Y)$ and Q be the C^*-algebras $M(C(Y) \otimes \mathcal{K})$, $M(C(Y) \otimes \mathcal{K})/C(Y) \otimes \mathcal{K}$ and $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ (Calkin algebra) respectively. One can easily see that $M(Y)$ is the set of all $*$-strong continuous functions from Y to $\mathcal{L}(\mathcal{H})$. We call an extension τ of A by $C(Y) \otimes \mathcal{K}$ homogeneous if for all $y \in Y$, the map $ev_y \circ \tau : A \to Q$ is injective where $ev_y : Q(Y) \to Q$ is the evaluation map at y. Let $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ be the set of strongly unitary equivalence classes of unital homogeneous extensions of A by $C(Y) \otimes \mathcal{K}$. For a nuclear C^*-algebra A, Pimsner, Popa and Voiculescu [Pimsner et al. 1979] showed that $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ is a group with the additive operation defined as in (2-1). We denote the equivalence class in the group $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ of an extension τ by $[\tau]_{su}$. For $y_0 \in Y$, define the set

$$\mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, A) = \big\{[\tau]_{su} \in \mathrm{Ext}_{\mathrm{PPV}}(Y, A) : ev_{y_0} \circ \tau \text{ is split}\big\}.$$

The set $\mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, A)$ is a subgroup of $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$.

***The groups* $\mathbf{Ext_{PPV}(Y, A)}$ *and* $\mathbf{Ext_{PPV}(Y, \Sigma^2 A)}$.** Here we will show that for a separable nuclear C^*-algebra A and a finite-dimensional compact metric space Y such that K-groups of $C(Y)$ are finitely generated, the groups $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ and $\mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A)$ are isomorphic. Let us recall some definitions. We say that two elements a and b in $Q(B)$ are strongly unitarily equivalent if there exists a unitary $U \in M(B)$ such that $[U]a[U^*] = b$. Two elements a and b in $Q(B)$ are said to be unitarily equivalent if there exists unitary $u \in Q(B)$ such that $uau^* = b$. We call an element a in a C^*-algebra B norm-full if it is not contained in any proper closed ideal in B. Suppose that A and B are separable C^*-algebras. An extension $\tau : A \to Q(B \otimes \mathcal{K})$ is said to be norm-full if for every nonzero element $a \in A$, $\tau(a)$ is norm-full element of $Q(B \otimes \mathcal{K})$.

Definition 2.1 [Lin 2009]. Let B be a separable σ-unital C^*-algebra. We say $Q(B \otimes \mathcal{K})$ has property (P) if for any norm-full element $b \in Q(B \otimes \mathcal{K})$, there exist $x, y \in Q(B \otimes \mathcal{K})$ such that $xby = 1$.

Definition 2.2 [Kucerovsky and Ng 2006]. Let B be a separable C^*-algebra. Then $B \otimes \mathcal{K}$ is said to have the corona factorization property if every norm-full projection in $M(B \otimes \mathcal{K})$ is Murray–von Neumann equivalent to the unit element of $M(B \otimes \mathcal{K})$.

One can show that in a C^*-algebra $B \otimes \mathcal{K}$ with the corona factorization property, any norm-full projection in $Q(B \otimes \mathcal{K})$ is Murray–von Neumann equivalent to the unit element of $Q(B \otimes \mathcal{K})$; see [Kucerovsky and Ng 2006]. Also note that the fact that $Q(B \otimes \mathcal{K})$ has property (P) implies that $B \otimes \mathcal{K}$ has the corona factorization property. It is proved in [Lin 2007] that for a finite-dimensional compact metric space Y, $Q(C(Y) \otimes \mathcal{K})$ has property (P) and hence $C(Y) \otimes \mathcal{K}$ has the corona factorization property. We will see that these properties play important roles in proving the isomorphism between the groups $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ and $\mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A)$. But for that, we need the following proposition that says that for a C^*-algebra with certain properties, the group $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ can be viewed as a subgroup of the group $KK^1(A, C(Y))$.

Proposition 2.3. *Let A be a unital separable nuclear C^*-algebra which satisfies the universal coefficient theorem. Assume that $K_0(A) = G \oplus \mathbb{Z}$ with $[1_A] = (0, 1)$. Suppose that Y is a finite-dimensional compact metric space. Then the map*

$$i : \mathrm{Ext}_{\mathrm{PPV}}(Y, A) \to KK^1(A, C(Y)), \qquad [\tau]_{su} \mapsto [\tau]_s$$

is an injective homomorphism.

Proof. Since strongly unitary equivalence implies stable equivalence, the map i is well-defined. Any unital homogeneous extension is a purely large extension and hence a norm-full extension; see [Elliott and Kucerovsky 2001, page 19]. Therefore, from [Lin 2009, Theorem 2.4 and Corollary 3.9], it follows that i is injective. □

From now on, without loss of generality, we will assume that the Hilbert space $\mathcal{H}$ is $L_2(\mathbb{N})$. Let τ be a unital homogeneous extension of A by $C(Y) \otimes \mathcal{K}(\mathcal{H})$. Define $\tilde{\tau} : A \to Q\big(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H})\big)$ by $\tilde{\tau}(a) = [\tau_a \otimes p]$ where $[\tau_a] = \tau(a)$. By the universal property of quantum double suspension (see [Hong and Szymański 2008, Proposition 2.2]), there exists a unique homomorphism

$$\Sigma^2\tau : \Sigma^2 A \to Q\big(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H})\big) \tag{2-2}$$

such that $\Sigma^2\tau(a \otimes p) = \tilde{\tau}(a) = [\tau_a \otimes p]$ and $\Sigma^2\tau(1 \otimes S) = [1 \otimes 1 \otimes S]$. Clearly $\Sigma^2\tau$ is a unital extension. Since τ is homogeneous, the map $ev_y \circ \tau$ is injective for all $y \in Y$. Therefore the map $ev_y \circ \Sigma^2\tau$ is injective on the C^*-algebra $A \otimes p$ as $ev_y \circ \Sigma^2\tau(a \otimes p) = [(ev_y \circ \tau)_a \otimes p]$ where $[(ev_y \circ \tau)_a] = ev_y \circ \tau(a)$. Making use of the fact that $(1 \otimes p)A \otimes \mathcal{K}(1 \otimes p) = A \otimes p$, one can prove that the map $ev_y \circ \Sigma^2\tau$ is injective on $A \otimes \mathcal{K}$. Since $A \otimes \mathcal{K}$ is an essential ideal of $\Sigma^2 A$, we conclude that the map $ev_y \circ \Sigma^2\tau$ is injective on $\Sigma^2 A$ and hence $\Sigma^2\tau$ is a homogeneous extension. Moreover, if τ_1 and τ_2 are strongly unitarily equivalent by a unitary $U \in M\big(C(Y) \otimes \mathcal{K}(\mathcal{H})\big)$ then so are $\Sigma^2\tau_1$ and $\Sigma^2\tau_2$ by the unitary $U \otimes 1 \in M(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}))$. This gives a well-defined map:

$$\beta : \mathrm{Ext}_{\mathrm{PPV}}(Y, A) \to \mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A), \qquad [\tau]_{su} \mapsto [\Sigma^2\tau]_{su}. \tag{2-3}$$

Proposition 2.4. *The map $\beta : \mathrm{Ext}_{\mathrm{PPV}}(Y, A) \to \mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A)$ given above is an injective group homomorphism.*

Proof. Let τ be a unital homogeneous extension of A by $C(Y) \otimes \mathcal{K}$ such that $\Sigma^2\tau$ is a split extension. In this case, there exists a homomorphism $\lambda : \Sigma^2 A \to M(Y)$ such that $\pi \circ \lambda = \Sigma^2\tau$. Define $\alpha : A \to M(Y)$ by $\alpha(a) := \lambda(a \otimes p)$ for $a \in A$. It is easy to check that $\pi \circ \alpha = \tau$ which implies that τ is a split extension. This proves that the map β is injective. □

To get surjectivity of the map β, we need to put certain assumptions on the topological space Y.

Proposition 2.5. *Let Y be a finite-dimensional compact metric space. Assume that $K_0(C(Y))$ and $K_1(C(Y))$ are finitely generated abelian groups. Then, letting $V \in Q\big(C(Y) \otimes \mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H})\big)$ be an isometry such that VV^* and $1 - VV^*$ both are norm-full projections, V is unitarily equivalent to* $[1 \otimes 1 \otimes S^*]$.

Proof. Let $G_n := \mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^{2n} C(\mathbb{T}))$. Since

$$KK^1(\Sigma^{2n} C(\mathbb{T}), C(Y)) \equiv K_0(C(Y)) \oplus K_1(C(Y)),$$

one can consider the groups G_n as subgroups of $K_0(C(Y)) \oplus K_1(C(Y))$ thanks to Proposition 2.3. This implies that the G_n are finitely generated abelian groups. For $n \in \mathbb{N}$, define the map

$$\beta_n : \mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^{2n} C(\mathbb{T})) \to \mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^{2n+2} C(\mathbb{T})), \qquad [\tau]_{su} \mapsto [\Sigma^2\tau]_{su}, \tag{2-4}$$

where $\Sigma^2\tau$ is as in (2-2). From Proposition 2.4, it follows that the maps β_n are injective homomorphisms. Assume that V is not unitarily equivalent to $[1 \otimes 1 \otimes S^*]$. For each $n \in \mathbb{N}$, the isometry V will induce an isometry $V_n \in Q(C(Y) \otimes \mathcal{K}(\mathcal{H})^{\otimes n+1})$ (where $^{\otimes k}$ means the tensor product of k copies) such that $V_n V_n^*$ and $1 - V_n V_n^*$ both are norm-full projections and V_n is not unitarily equivalent to $[1^{\otimes n+1} \otimes S^*]$. Since $C(Y) \otimes \mathcal{K}$ has the corona factorization property, it follows that $V_n V_n^*$ and $1 - V_n V_n^*$ both are Murray–von Neumann equivalent to $[1]$. Also, one can easily verify that $[1^{\otimes n+1} \otimes p]$ and $[1 - 1^{\otimes n+1} \otimes p] = [1^{\otimes n+1} \otimes (1 - p)]$ are Murray–von Neumann equivalent to $[1]$. This implies that $V_n V_n^*$ is unitarily equivalent to $1 - [1^{\otimes n+1} \otimes p]$. So, without loss of generality, we can assume that V_n has final projection $1 - [1^{\otimes n+1} \otimes p]$. Take a split unital homogeneous extension τ of $C(\mathbb{T})$ by $C(Y) \otimes \mathcal{K}(\mathcal{H})$. Clearly $\Sigma^{2n}\tau$ is a split unital homogeneous extension of

$\Sigma^{2n}C(\mathbb{T})$ by $C(Y)\otimes\mathcal{K}(\mathcal{H})^{\otimes n+1}$. Let $\Sigma_V^2(\Sigma^{2n}\tau)$ be a unital homogeneous extension of $\Sigma^{2n+2}C(\mathbb{T})$ by $C(Y)\otimes\mathcal{K}(\mathcal{H})^{\otimes n+2}$ given by

$$\Sigma_V^2(\Sigma^{2n}\tau)(a\otimes p)=[\Sigma^{2n}\tau_a\otimes p] \quad\text{and}\quad \Sigma_V^2(\Sigma^{2n}\tau)(1\otimes S^*)=V_{n+1},$$

where $[\Sigma^{2n}\tau_a]=\Sigma^{2n}\tau(a)$. From [Lin 2009, Corollary 3.8] and the fact that V_{n+1} is not unitarily equivalent to $[1^{\otimes n+2}\otimes S^*]$, it follows that $[\Sigma_V^2(\Sigma^{2n}\tau)]_{su}$ is not in the image of the map β_n defined as in (2-4). Let $m[\Sigma_V^2(\Sigma^{2n}\tau)]_{su}=\beta_n([\phi]_{su})$ for some $m\in\mathbb{Z}-\{0\}$ and for some unital homogeneous extension ϕ of $\Sigma^{2n}C(\mathbb{T})$ by $C(Y)\otimes\mathcal{K}$. It is easy to see that ϕ must be split and in that case $m[\Sigma_V^2(\Sigma^{2n}\tau)]$ is the class of split extensions. This shows that for all $n\in\mathbb{N}$, the group G_{n+1} has either one more free generator or one more element of finite order than the group G_n. Since $K_0(C(Y))\oplus K_1(C(Y))$ is a finitely generated group for all $n\in\mathbb{N}$, $G_n\subset K_0(C(Y))\oplus K_1(C(Y))$, and we reach a contradiction. This proves that V is unitarily equivalent to $[1\otimes 1\otimes S^*]$. □

Remark 2.6. Here we should point out that the above proposition may hold for a more general finite-dimensional compact metric space Y. But since we could not find any general result along this direction in literature, we prove the proposition under certain assumptions on Y.

Corollary 2.7. *Let Y and V be as in the above proposition. Then V is strongly unitarily equivalent to* $[1\otimes 1\otimes S^*]$.

Proof. Consider the unital extension $\Sigma_V^2\tau$ constructed in Proposition 2.5 where τ is a split unital homogeneous extension of $C(\mathbb{T})$ by $C(Y)\otimes\mathcal{K}(\mathcal{H})$. Using Proposition 2.5, one can show that $\Sigma_V^2\tau$ is unitarily equivalent to $\Sigma^2\tau$ defined in (2-2) with $A=C(\mathbb{T})$. Therefore, by [Lin 2009, Corollary 3.10], it follows that $\Sigma_V^2\tau$ is strongly unitarily equivalent to $\Sigma^2\tau$. Hence V is strongly unitarily equivalent to $[1\otimes 1\otimes S^*]$. □

Lemma 2.8 establishes the isomorphism between the groups $\mathrm{Ext}_{\mathrm{PPV}}(Y, A)$ and $\mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A)$ under certain assumptions on the space Y.

Lemma 2.8. *Let Y be a finite-dimensional compact metric space. Assume that the groups $K_0(C(Y))$ and $K_1(C(Y))$ are finitely generated abelian groups. Then the map $\beta:\mathrm{Ext}_{\mathrm{PPV}}(Y, A)\to\mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A)$ given above is an isomorphism.*

Proof. We only need to show that β is surjective thanks to Proposition 2.4. Let ϕ be a unital homogeneous extension of $\Sigma^2 A$ by $C(Y)\otimes\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(\mathcal{H})$. Let $\phi(1\otimes S^*)=V$. Since ϕ is a unital homogeneous extension and hence a norm-full extension, it follows that VV^* and $1-VV^*$ are norm-full projections. Therefore, by Corollary 2.7, there exists a unitary $U\in M(C(Y)\otimes\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(\mathcal{H}))$ such that $[U]V[U^*]=[1\otimes 1\otimes S^*]$. So without loss of generality, we can assume that ϕ maps $1\otimes S^*$ to $[1\otimes 1\otimes S^*]$. This implies that $\phi(1\otimes p)=[1\otimes 1\otimes p]$. But then $\phi(A\otimes p)\subset(1\otimes 1\otimes p)\phi(A\otimes p)(1\otimes 1\otimes p)\subset Q(C(Y)\otimes\mathcal{K}(\mathcal{H}))\otimes p$ which induces a map $\tau : A \to Q(C(Y) \otimes \mathcal{K}(\mathcal{H}))$ by omitting the projection p. Therefore, we get a unital homogeneous extension τ of A such that $\beta([\tau]_u) = [\phi]_u$. This proves that the map β is surjective. □

Corollary 2.9. *For $y_0 \in Y$, the map*

$$\beta_{|\,\mathrm{Ext}_{\mathrm{PPV}}(Y,y_0,A)} : \mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, A) \to \mathrm{Ext}_{\mathrm{PPV}}(Y, y_0, \Sigma^2 A)$$

is an isomorphism.

Proof. It is easy to check that if $ev_{y_0} \circ \tau$ is split then so is $ev_{y_0} \circ \Sigma^2\tau$ and vice versa. Now the claim will follow from Lemma 2.8. □

3. Elements of $\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C\big(S_0^{2\ell+1}\big)\big)$

In this section, we will write down all elements of the groups $\mathrm{Ext}(C(S_0^{2\ell+1}))$ and $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$ explicitly in terms of their Busby invariants. We start with the definition of $C(S_0^{2\ell+1})$. The C^*-algebra $C(S_0^{2\ell+1})$ is defined as the C^*-subalgebra of $\mathcal{L}\big(L_2(\mathbb{N})^{\otimes\ell}\big) \otimes C(\mathbb{T})$ generated by the following operators:

$$\begin{gathered}
S^* \otimes 1 \otimes \cdots \otimes 1,\\
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1,\\
\vdots\\
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1,\\
p \otimes p \otimes \cdots \otimes p \otimes p \otimes t.
\end{gathered}$$

Hong and Szymański [2002] showed that for $q \in (0, 1)$, the C^*-algebra $C(S_q^{2\ell+1})$ of continuous functions on the odd-dimensional quantum sphere $S_q^{2\ell+1}$ is isomorphic to the C^*-algebra $C(S_0^{2\ell+1})$. Since for calculation purposes, the generators of $C(S_0^{2\ell+1})$ given above are easier to deal with in comparison to those of $C(S_q^{2\ell+1})$, we will, without loss of generality, take the C^*-algebra $C(S_0^{2\ell+1})$. Define the $*$-homomorphisms φ_m as follows:

$$\begin{aligned}
\varphi_m : C(S_0^{2\ell+1}) &\to Q\big(\mathcal{K}\big(L_2(\mathbb{N})^{\otimes\ell+1}\big)\big),\\
S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto S^* \otimes 1 \otimes \cdots \otimes 1,\\
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1,\\
&\vdots\\
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 &\mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1,\\
p \otimes p \otimes \cdots \otimes t &\mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m.
\end{aligned}$$

The following proposition says that for each $m \in \mathbb{Z}$, the homomorphism $[\varphi_m]_s$ is an element of the group $\mathrm{Ext}(C(S_0^{2\ell+1}))$.

Proposition 3.1. *For each $m \in \mathbb{Z}$, the extension φ_m is an essential unital extension of $C(S_0^{2\ell+1})$ by compact operators.*

Proof. Clearly the φ_m are unital extensions of $C(S_0^{2\ell+1})$ by compact operators. We need to show that the φ_m are injective homomorphisms. Let $C_t(\mathbb{T})$ be the set of all continuous functions on $\mathbb{T}$ vanishing at t. Using irreducible representations of $C(S_0^{2\ell+1})$, it is easy to see that

(1) $\{\mathcal{K}(L_2(\mathbb{N}))^{\otimes \ell} \otimes C_t(\mathbb{T})\}_{t \in \mathbb{T}}$ are primitive ideals of $C(S_0^{2\ell+1})$,

(2) all other primitive ideals contain $p \otimes p \otimes \cdots \otimes p \otimes p \otimes t$ and $\mathcal{K}(L_2(\mathbb{N}))^{\otimes \ell} \otimes C_t(\mathbb{T})$ for all $t \in \mathbb{T}$.

Since $\ker \varphi_m$ is the intersection of all primitive ideals that contain $\ker \varphi_m$ and since $p \otimes p \otimes \cdots \otimes t \notin \ker \varphi_m$, we conclude that $\ker \varphi_m = \mathcal{K}(L_2(\mathbb{N}))^{\otimes \ell} \otimes C_F(\mathbb{T})$ for some closed subset F of $\mathbb{T}$ where $C_F(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$ vanishing on F. Consider the function $\chi : C(\mathbb{T}) \to Q$ such that $\chi(t) = [(S^*)^m]$. Since $[(S^*)^m]$ is unitary in Q with spectrum equal to $\mathbb{T}$, it follows that the map χ is injective. This shows that for any nonzero continuous complex valued function f on $\mathbb{T}$, $\varphi_m(p \otimes p \otimes \cdots \otimes f(t)) \neq 0$. Hence $F = \mathbb{T}$ and $\ker \varphi_m = \{0\}$. □

We shall show that each element in the group $\mathrm{Ext}(C(S_0^{2\ell+1}))$ is of the form $[\varphi_m]_s$ for some $m \in \mathbb{Z}$. Let $\mathcal{H}_0$ be the Hilbert space $L_2(\mathbb{N})^{\otimes \ell} \otimes L_2(\mathbb{Z})$. For $m \in \mathbb{Z}$, let ϑ_m be the representation of $C(S_0^{2\ell+1})$ given by

$$\begin{aligned}
\vartheta_m : C(S_0^{2\ell+1}) &\to \mathcal{L}(\mathcal{H}_0),\\
S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto S^* \otimes 1 \otimes \cdots \otimes 1,\\
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1,\\
&\ \vdots\\
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 &\mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1,\\
p \otimes p \otimes \cdots \otimes t &\mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m.
\end{aligned}$$

Let P be the self-adjoint projection in $\mathcal{L}(\mathcal{H}_0)$ on the subspace spanned by the basis elements $\{e_{n_1} \otimes \cdots \otimes e_{n_{\ell+1}} : n_i \in \mathbb{N} \text{ for all } i \in \{1, 2, \ldots, \ell+1\}\}$. One can check that $\mathcal{F}_m := \big(C(S_0^{2\ell+1}), \mathcal{H}_0, 2P-1\big)$ with the underlying representation ϑ_m is a Fredholm module. By [Blackadar 1998, Proposition 17.6.5, page 157], the group $\mathrm{Ext}(C(S_0^{2\ell+1}))$ is isomorphic to the group $K^1(C(S_0^{2\ell+1}))$. Under this identification, one can easily show that the equivalence class of the Fredholm module $\mathcal{F}_m$ corresponds to the equivalence class $[\varphi_m]_s$.

Proposition 3.2. *For $\ell \in \mathbb{N}$, one has*

$$\mathrm{Ext}\big(C(S_0^{2\ell+1})\big) = \big\{[\varphi_m]_s : m \in \mathbb{Z}\big\}.$$

Proof. To prove the claim, we will use the index pairing between the groups $K_1(C(S_0^{2\ell+1}))$ and $K^1(C(S_0^{2\ell+1}))$ which is given by the Kasparov product. The group $K_1(C(S_0^{2\ell+1}))$ is generated by the unitary $u := p^{\otimes\ell} \otimes t + 1 - p^{\otimes\ell} \otimes 1$. For $m \in \mathbb{Z}$, let $R_m : P\mathcal{H}_0 \to P\mathcal{H}_0$ be the operator $P\vartheta_m(u)P$. Hence we get

$$\langle u, \mathcal{F}_m\rangle = \mathrm{Index}(R_m) = m. \qquad \square$$

To describe all elements of $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$, define the $*$-homomorphisms ϕ_m:

$$\begin{aligned}
\phi_m : C(S_0^{2\ell+1}) &\to Q\big(\mathcal{K}\big(L_2(\mathbb{N})^{\otimes\ell+1}\big) \otimes C(\mathbb{T})\big),\\
S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto S^* \otimes 1 \otimes \cdots \otimes 1,\\
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1,\\
&\vdots\\
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 &\mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1 \otimes 1,\\
p \otimes p \otimes \cdots \otimes t &\mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m \otimes 1.
\end{aligned}$$

It follows from Proposition 3.1 that the ϕ_m are essential unital extensions. Since the last component is 1, these extensions are homogeneous. Let A_m be the C^*-subalgebra of $C(S_0^{2\ell+3})$ generated by the operators

$$\begin{gathered}
S^* \otimes 1 \otimes \cdots \otimes 1 \otimes 1,\\
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 \otimes 1,\\
\vdots\\
p \otimes p \otimes \cdots \otimes S^* \otimes 1 \otimes 1,\\
p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m \otimes 1
\end{gathered}$$

and $\mathcal{K}\big(L_2(\mathbb{N})^{\otimes\ell+1}\big) \otimes C(\mathbb{T})$. Then for each $m \in \mathbb{Z}$, we have the exact sequence

$$0 \to \mathcal{K}\big(L_2(\mathbb{N})^{\otimes\ell+1}\big) \otimes C(\mathbb{T}) \to A_m \to C(S_0^{2\ell+1}) \to 0$$

with the Busby invariant ϕ_m. By using the six-term exact sequence, one can show

$$K_0(A_m) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}, \qquad K_1(A_m) = \mathbb{Z}. \tag{3-1}$$

Lemma 3.3. *For $\ell \in \mathbb{N}$ and $t_0 \in \mathbb{T}$, one has*

$$\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, t_0, C(S_0^{2\ell+1})\big) = \{0\}, \qquad \mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C(S_0^{2\ell+1})\big) = \mathbb{Z}.$$

Proof. It follows from Theorem 1.5 in [Rosenberg and Schochet 1981] that

$$\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, t_0, C(\mathbb{T})\big) = \mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{R}^+, t_0, C_0(\mathbb{R})^+\big) = \mathrm{Ext}\big(C_0(\mathbb{R}), C_0(\mathbb{R})\big) = \{0\}.$$

The C^*-algebra $C(S_0^{2\ell+1})$ can be obtained by applying quantum double suspension on $C(\mathbb{T})$ repeatedly; see [Hong and Szymański 2002]. Therefore, from Corollary 2.9, we have

$$\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, t_0, C(S_0^{2\ell+1})\big) = \mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, t_0, C(\mathbb{T})\big) = \{0\}.$$

Further, from Theorem 1.4 in [Rosenberg and Schochet 1981], we get

$$\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C(\mathbb{T})\big) = \mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C_0(\mathbb{R})^+\big) = \mathrm{Ext}\big(C_0(\mathbb{R}), C(\mathbb{T})\big) = \mathbb{Z}.$$

Hence by applying Lemma 2.8, we get the claim. □

The following lemma says that each element of the group $\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C(S_0^{2\ell+1})\big)$ is of the form $[\phi_m]_{su}$ for some $m \in \mathbb{Z}$.

Lemma 3.4. *For $\ell \in \mathbb{N}$, one has*

$$\mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C(S_0^{2\ell+1})\big) = \big\{[\phi_m]_{su} : m \in \mathbb{Z}\big\}.$$

Proof. Fix $t_0 \in \mathbb{T}$. Define a homomorphism Ψ as follows:

$$\Psi : \mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, C(S_0^{2\ell+1})\big) \to \mathrm{Ext}\big(C(S_0^{2\ell+1})\big), \qquad [\tau]_{su} \mapsto [ev_{t_0} \circ \tau]_s.$$

Clearly $\ker \Psi = \mathrm{Ext}_{\mathrm{PPV}}\big(\mathbb{T}, t_0, C(S_0^{2\ell+1})\big) = \{0\}$. Therefore, Ψ is an injective group homomorphism. Since $ev_{t_0} \circ \phi_m = \varphi_m$, for all $m \in \mathbb{Z}$, it follows that the homomorphism Ψ is surjective. This proves the claim. □

4. Quantum quaternion sphere

We first recall the definition and representation theory of the C^*-algebra $C(H_q^{2n})$ of continuous functions on the quantum quaternion sphere. Then we prove our main result that the C^*-algebra $C(H_q^{2n})$ is isomorphic to the C^*-algebra $C(S_q^{4n-1})$.

Definition 4.1. Let $i' = 2n+1-i$. The C^*-algebra $C(H_q^{2n})$ of continuous functions on the quantum quaternion sphere is defined as the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_{2n}$ satisfying the following relations:

$$z_i z_j = q z_j z_i \qquad \text{for } i > j,\ i+j \neq 2n+1, \tag{4-1}$$

$$z_i z_{i'} = q^2 z_{i'} z_i - (1-q^2) \sum_{k>i} q^{i-k} z_k z_{k'} \qquad \text{for } i > n, \tag{4-2}$$

$$z_i^* z_{i'} = q^2 z_{i'} z_i^* \tag{4-3}$$

$$z_i^* z_j = q z_j z_i^* \qquad \text{for } i+j > 2n+1,\ i \neq j, \tag{4-4}$$

$$z_i^* z_j = q z_j z_i^* + (1-q^2)\epsilon_i \epsilon_j q^{\rho_i+\rho_j} z_{i'} z_{j'}^* \qquad \text{for } i+j < 2n+1,\ i \neq j, \tag{4-5}$$

$$z_i^* z_i = z_i z_i^* + (1-q^2) \sum_{k>i} z_k z_k^* \qquad \text{for } i > n, \tag{4-6}$$

$$z_i^* z_i = z_i z_i^* + (1-q^2)\Big(q^{2\rho_i} z_{i'} z_{i'}^* + \sum_{k>i} z_k z_k^*\Big) \qquad \text{for } i \leq n, \tag{4-7}$$

$$\sum_{i=1}^{2n} z_i z_i^* = 1. \tag{4-8}$$

In [Saurabh 2017], we showed that the C^*-algebra $C(H_q^{2n})$ is isomorphic to the quotient algebra $C(\mathrm{SP}_q(2n)/\mathrm{SP}_q(2n-2))$ that can also be described as the C^*-subalgebra of $C(\mathrm{SP}_q(2n))$ generated by $\{u_m^1, u_m^{2n} : m \in \{1, 2, \ldots, 2n\}\}$, i.e., elements of the first and last row of the fundamental matrix of the quantum symplectic group $\mathrm{SP}_q(2n)$. Here we briefly describe all irreducible representations of $C(H_q^{2n})$. For a detailed treatment on this, we refer the reader to [Saurabh 2017]. Let N be the number operator given by $N : e_n \mapsto ne_n$ and S be the shift operator given by $S : e_n \mapsto e_{n-1}$ on $L_2(\mathbb{N})$. We denote by $\mathscr{T}$ the Toeplitz algebra. Let $E_{i,j} \in M_n(\mathbb{R})$ be the $n \times n$ matrix with the only nonzero entry at the ij-th place and equal to 1. Define

$$\begin{aligned} s_i = I - E_{i,i} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}, &\qquad \text{for } i = 1, 2, \ldots, n-1, \\ s_n = I - 2E_{n,n}, &\qquad \text{for } i = n. \end{aligned}$$

One can prove that the Weyl group W_n of sp_{2n} is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$ generated by $s_1, s_2, \ldots, s_n$. We refer the reader to [Fulton and Harris 1991] for a proof of this fact. For $i = 1, 2, \ldots, n-1$, let π_{s_i} denote the following representation of $C(\mathrm{SP}_q(2n))$,

$$\pi_{s_i}(u_l^k) = \begin{cases} \sqrt{1-q^{2N+2}}S & \text{if } (k,l) = (i,i) \text{ or } (2n-i, 2n-i), \\ S^*\sqrt{1-q^{2N+2}} & \text{if } (k,l) = (i+1,i+1) \text{ or } (2n-i+1, 2n-i+1), \\ -q^{N+1} & \text{if } (k,l) = (i, i+1), \\ q^N & \text{if } (k,l) = (i+1, i), \\ q^{N+1} & \text{if } (k,l) = (2n-i, 2n-i+1), \\ -q^N & \text{if } (k,l) = (2n-i+1, 2n-i), \\ \delta_{kl} & \text{otherwise.} \end{cases}$$

For $i = n$,

$$\pi_{s_n}(u_l^k) = \begin{cases} \sqrt{1-q^{4N+4}}S & \text{if } (k,l) = (n,n), \\ S^*\sqrt{1-q^{4N+4}} & \text{if } (k,l) = (n+1,n+1), \\ -q^{2N+2} & \text{if } (k,l) = (n, n+1), \\ q^{2N} & \text{if } (k,l) = (n+1, n), \\ \delta_{kl} & \text{otherwise.} \end{cases}$$

Each π_{s_i} is an irreducible representation and is called an elementary representation of $C(\mathrm{SP}_q(2n))$. For any two representations φ and ψ of $C(\mathrm{SP}_q(2n))$, define $\varphi * \psi$ to be $(\varphi \otimes \psi) \circ \Delta$, where Δ is the comultiplication map of $C(\mathrm{SP}_q(2n))$. Let ϑ be an element of W_n such that $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced expression for ϑ. Then $\pi_\vartheta = \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots * \pi_{s_{i_k}}$ is an irreducible representation which is independent of the reduced expression. Now for $t = (t_1, t_2, \dots, t_n) \in \mathbb{T}^n$, define the map $\tau_t : C(\mathrm{SP}_q(2n) \to \mathbb{C}$ by

$$\tau_t(u^i_j) = \begin{cases} \overline{t_i}\delta_{ij} & \text{if } i \le n, \\ t_{2n+1-i}\delta_{ij} & \text{if } i > n. \end{cases}$$

Then τ_t is a $*$-algebra homomorphism. For $t \in \mathbb{T}^n$, $\vartheta \in W$, let $\pi_{t,\vartheta} = \tau_t * \pi_\vartheta$. Define the representation $\eta_{t,\vartheta}$ of $C(H^{2n}_q)$ as the representation $\pi_{t,\vartheta}$ restricted to $C(H^{2n}_q)$. Denote by ω_k the following reduced word of Weyl group of sp_{2n},

$$\omega_k = \begin{cases} I & \text{if } k = 1, \\ s_1 s_2 \cdots s_{k-1} & \text{if } 2 \le k \le n, \\ s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_{2n-k+1} & \text{if } n < k \le 2n. \end{cases}$$

For $k=1$, define $\eta_{t,I} : C(H^{2n}_q) \to \mathbb{C}$ such that $\eta_{t,I}(z_j) = t\delta_{1j}$. The set $\{\eta_{t,I} : t \in T\}$ gives all one-dimensional irreducible representations of $C(H^{2n}_q)$.

Theorem 4.2 [Saurabh 2017]. *The set $\{\eta_{t,\omega_k} : 1 \le k \le 2n, t \in \mathbb{T}\}$ gives a complete list of irreducible representations of $C(H^{2n}_q)$.*

Define
$$\eta_{\omega_k} : C(H^{2n}_q) \to C(\mathbb{T}) \otimes \mathscr{T}^{\otimes k-1}$$
such that $\eta_{\omega_k}(a)(t) = \eta_{t,\omega_k}(a)$ for all $a \in C(H^{2n}_q)$. Let $C^{2n}_1 = C(\mathbb{T})$ and for $2 \le k \le 2n$, $C^{2n}_k = \eta_{\omega_k}(C(H^{2n}_q))$.

Corollary 4.3. *The set $\{\eta_{t,\omega_l} : 1 \le l \le k, t \in \mathbb{T}\}$ gives a complete list of irreducible representations of C^{2n}_k.*

By Corollary 4.3, one can find all primitive ideals, i.e., kernels of irreducible representations of C^{2n}_k. Define $y^k_l := \eta_{\omega_k}(z_l)$ and $I^k_{t,l} := \ker(\eta_{t,\omega_l})$ for $1 \le l \le k$ and $t \in \mathbb{T}$. Then

$$\{I^k_{t,k} = C_t(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)}\}_{t\in\mathbb{T}}, \{I^k_{t,k-1}\}_{t\in\mathbb{T}}, \dots, \{I^k_{t,1}\}_{t\in\mathbb{T}} \tag{4-9}$$

is a complete list of primitive ideals of C^{2n}_k. Moreover for $t, t' \in \mathbb{T}$ and $1 \le l \le k-1$, we have $C_t(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)} \subset I^k_{t',l}$ and $y^k_k \in I^k_{t',l}$. In Lemma 5.1 of [Saurabh 2017], we established the exact sequence

$$0 \to C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k)} \to C^{2n}_{k+1} \xrightarrow{\sigma_{k+1}} C^{2n}_k \to 0,$$

where σ_{k+1} is the restriction of $1 \otimes 1^{\otimes(k-1)} \otimes \sigma$ to C^{2n}_{k+1} and the map $\sigma : \mathscr{T} \to \mathbb{C}$ is the homomorphism such that $\sigma(S) = 1$. The following lemma says that this exact sequence is a unital homogeneous extension of C^{2n}_k by $C(\mathbb{T}) \otimes \mathcal{K}$:

Lemma 4.4. *For $1 \le k \le 2n$, the exact sequence*

$$0 \to C(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k)} \to C^{2n}_{k+1} \xrightarrow{\sigma_{k+1}} C^{2n}_k \to 0$$

is a unital homogeneous extension of C^{2n}_k by $C(\mathbb{T}) \otimes \mathcal{K}$.

Proof. Since C^{2n}_{k+1} is unital, the given extension is unital. Let $\tau : C^{2n}_k \to Q(\mathbb{T})$ be the Busby invariant corresponding to this extension. For $t_0 \in \mathbb{T}$, let $\tau_{t_0} : C^{2n}_k \to Q$ be the map $ev_{t_0} \circ \tau$ where $ev_{t_0} : Q(\mathbb{T}) \to Q$ is the evaluation map at t_0. Assume that $J_{t_0} = \ker(\tau_{t_0})$. To show that the given short exact sequence is a homogeneous extension, we need to prove that $J_{t_0} = \{0\}$ for all $t_0 \in \mathbb{T}$.

Case 1: $n < k < 2n$. We have

$$\begin{aligned}\tau_{t_0}(y^k_k) &= \tau_{t_0}\big(t \otimes q^{N\otimes(n-1)} \otimes q^{2N} \otimes q^{N\otimes(k-n-1)}\big) \\ &= t_0\big[q^{N\otimes(n-1)} \otimes q^{2N} \otimes q^{N\otimes(k-n-1)} \otimes \sqrt{1-q^{2N}}S^*\big] \neq 0.\end{aligned} \tag{4-10}$$

This shows $y^k_k \notin J_{t_0}$. Since J_{t_0} is the intersection of all primitive ideals that contain J_{t_0}, and $y^k_k \in I^k_{t',l}$ and $C_t(\mathbb{T}) \otimes \mathcal{K}(L_2(\mathbb{N}))^{\otimes(k-1)} \subset I^k_{t',l}$ for $t, t' \in \mathbb{T}$ and $1 \le l \le k-1$, we conclude that J_{t_0} is equal to $C_F(\mathbb{T}) \otimes \mathcal{K}$ for some closed subset F of $\mathbb{T}$ where $C_F(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$ vanishing on F. From (4-10), we get

$$\begin{aligned}\tau_{t_0}((y^k_k)(y^k_k)^*) &= \big[q^{2N\otimes(n-1)} \otimes q^{4N} \otimes q^{2N\otimes(k-n-1)} \otimes (1-q^{2N})\big] \\ &= \big[q^{2N\otimes(n-1)} \otimes q^{4N} \otimes q^{2N\otimes(k-n-1)} \otimes 1\big].\end{aligned}$$

Therefore,

$$\tau_{t_0}\big(1 \otimes p^{\otimes(k-1)}\big) = \big[p^{\otimes(k-1)} \otimes 1\big].$$

Hence,

$$\begin{aligned}\tau_{t_0}(t \otimes p^{\otimes(k-1)}) &= t_0\big[p^{\otimes(k-1)} \otimes \sqrt{1-q^{2N}}S^*\big] \\ &= t_0\big[p^{\otimes(k-1)} \otimes S^*\big].\end{aligned}$$

Since the function $\chi : C(\mathbb{T}) \to Q$ such that $\chi(t) = [S^*]$ is an injective homomorphism as shown in Proposition 3.1, it follows that for any nonzero continuous function f on $\mathbb{T}$,

$$\tau_{t_0}\big(f(t) \otimes p^{\otimes(k-1)}\big) \neq 0.$$

This proves that $F = \mathbb{T}$ and $J_{t_0} = \{0\}$.

Case 2: $1 \le k \le n$. For $k = n$,

$$\tau_{t_0}(y^n_n) = t_0\big[q^{N\otimes(n-1)} \otimes \sqrt{1-q^{4N}}S^*\big].$$

For $1 \le k < n$,

$$\tau_{t_0}(y^k_k) = t_0\big[q^{N\otimes(k-1)} \otimes \sqrt{1-q^{2N}}S^*\big].$$

Similar calculations to those in Case 1 show that $J_{t_0} = \{0\}$. This proves the claim. □

We now state the main result of this paper.

Theorem 4.5. *For all $n \geq 2$ and $1 \leq k \leq 2n$, the C^*-algebra C_k^{2n} is isomorphic to the C^*-algebra $C(S_0^{2k-1})$ of continuous functions on odd-dimensional quantum spheres. In particular, $C(H_q^{2n})$ is isomorphic to $C(S_0^{4n-1})$ or, equivalently, to $C(S_q^{4n-1})$.*

Proof. Fix n. To prove the theorem, we use induction on k. For $k = 1$, $C_1^{2n} = C(\mathbb{T})$. So the claim is true for $k = 1$. Assume that the claim is true for k, i.e., C_k^{2n} is isomorphic to $C(S_0^{2k-1})$. From Lemma 4.4, it follows that the short exact sequence

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C_{k+1}^{2n} \to C_k^{2n} \to 0 \tag{4-11}$$

is a unital homogeneous extension. Therefore, it can be viewed as an element of the group $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2k-1}))$. It follows from Lemma 3.4 that it is strongly unitarily equivalent to ϕ_m or, equivalently, to the following exact sequence

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to A_m \to C(S_0^{2k-1}) \to 0,$$

for some $m \in \mathbb{Z}$. From Theorem 5.3 in [Saurabh 2017] and equation (3-1), we have

$$K_0(C_{k+1}^{2n}) = \mathbb{Z}, \qquad K_0(A_m) = \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}.$$

Since strongly unitary equivalence gives an isomorphism of the middle C^* algebras and hence an isomorphism of the K-groups of middle C^*-algebras, it follows that the exact sequence (4-11) is strongly unitarily equivalent to ϕ_1 or ϕ_{-1}. This implies that C_{k+1}^{2n} is isomorphic to A_1 or A_{-1}. Since $A_1 = A_{-1} = C(S_0^{2k+1})$, it follows that C_{k+1}^{2n} is isomorphic to $C(S_0^{2k+1})$. Hence by induction, it follows that $C(H_q^{2n})$ is isomorphic to $C(S_0^{4n-1})$. From Theorem 4.4 in [Hong and Szymański 2002], it follows that the C^*-algebra $C(S_q^{4n-1})$ is isomorphic to $C(S_0^{4n-1})$, for $q \in (0, 1)$. This proves that $C(H_q^{2n})$ is isomorphic to $C(S_q^{4n-1})$. □

Remark 4.6. In the case where $q = 0$, we need to be slightly careful to get the defining relations of $C(H_0^{2n})$. In the relation (4-2), we first start with $i = 2n$. This gives the relation $z_{2n} z_1 = 0$. Then we take $i = 2n - 1$ and so on and get the relation $z_i z_{i'} = 0$ for $i < n$. Further, in the relation (4-5), it is easy to check that for $i + j < 2n + 1$, $\rho_i + \rho_j > 0$. Now by putting $q = 0$ into the relations (4-3), (4-4) and (4-4), we get $z_i^* z_j = 0$ for $i \neq j$. The other relations are obtained by putting $q = 0$ in the remaining relations. By looking at the relations, one can see that the defining relations of $C(H_0^{2n})$ are exactly the same as those of $C(S_0^{4n-1})$. These facts together with Theorem 4.5 prove that for different values of $q \in [0, 1)$, the C^*-algebras $C(H_q^{2n})$ are isomorphic.

Acknowledgement

I would like to thank Arup Kumar Pal, my supervisor, for his constant support and for his valuable suggestions.

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Received June 22, 2016. Revised December 12, 2016.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 288 No. 2 June 2017

0030-8730(201706)288:2;1-R