Pacific Journal of Mathematics

GAP THEOREMS FOR COMPLETE λ-HYPERSURFACES

HUIJUAN WANG, HONGWEI XU AND ENTAO ZHAO

Volume 288 No. 2 June 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 288, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.288.453

GAP THEOREMS FOR COMPLETE λ-HYPERSURFACES

HUIJUAN WANG, HONGWEI XU AND ENTAO ZHAO

An n-dimensional λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ is the critical point of the weighted area functional $\int_M e^{-\frac{1}{4}|X|^2}\, d\mu$ for weighted volume-preserving variations, which is also a generalization of the self-shrinking solution of the mean curvature flow. We first prove that if the L^n-norm of the second fundamental form of the λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ with $n \geq 3$ is less than an explicit positive constant $K(n, \lambda)$, then M is a hyperplane. Secondly, we show that if the L^n-norm of the trace-free second fundamental form of M with $n \geq 3$ is less than an explicit positive constant $D(n, \lambda)$ and the mean curvature is suitably bounded, then M is a hyperplane. We also obtain similar results for λ-surfaces in $\mathbb{R}^3$ under L^4-curvature pinching conditions.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional immersed smooth hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We call the hypersurface a λ-hypersurface if it satisfies

$$H + \tfrac{1}{2}\langle X, N\rangle = \lambda,$$

where λ is a constant, H is the mean curvature and N is the unit inward normal vector of $X : M \to \mathbb{R}^{n+1}$.

McGonagle and Ross [2015] studied λ-hypersurfaces from the viewpoint of variation. Let $A_\mu(M)$ be the functional defined by $A_\mu(M) = \int_M e^{-\frac{1}{4}|X|^2}\, d\mu$. They showed that the critical points of $\delta A_\mu(u) = 0$ for $u \in C_0^\infty$ satisfying

$$\int_M e^{-\frac{1}{4}|X|^2} u \, d\mu = 0$$

are λ-hypersurfaces. Cheng and Wei [2014a] also introduced λ-hypersurfaces in a different way by investigating the weighted volume-preserving mean curvature flow. Obviously, when $\lambda = 0$, a λ-hypersurface is a self-shrinker of the mean curvature flow. It is well known that self-shrinkers play an important role in the study of mean

Research supported by the National Natural Science Foundation of China, Grant Nos. 11531012, 11371315, 11201416.
MSC2010: 53C42, 53C44.
Keywords: gap theorem, lambda-hypersurfaces, integral curvature pinching.

curvature flow because they describe the singularity models of the mean curvature flow and they arise as tangent flows of mean curvature flow at singularities; see, for example, [Colding and Minicozzi 2012; Huisken 1990; Ilmanen 1995; White 1997].

The rigidity phenomena of self-shrinkers has been studied extensively [Cheng and Peng 2015; Cheng and Wei 2015; Colding et al. 2015; Colding and Minicozzi 2012; Ding and Xin 2013; 2014; Huisken 1990; Le and Sesum 2011]. For example, Le and Sesum [2011] proved that a smooth self-shrinker with polynomial volume growth and satisfying $|A|^2 < \frac{1}{2}$ is a hyperplane. Here A denotes the second fundamental form of the immersion. Cao and Li [2013] generalized this result to arbitrary codimension and proved that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \leq \frac{1}{2}$ is a generalized cylinder. On the other hand, Ding and Xin [2014] showed that a smooth complete self-shrinker satisfying $\big(\int_M |A|^n \,\mathrm{d}\mu\big)^{1/n} < C$ for a certain positive constant C is a linear space. For more curvature pinching theorems for self-shrinkers, see [Cao et al. 2014; Li and Wei 2014; Lin 2016].

The geometric properties of λ-hypersurfaces were recently investigated by Cheng, Wei, Ogata, Guang [Cheng and Wei 2014a; Cheng et al. 2016; Guang 2014]. As generalizations of self-shrinkers of the mean curvature flow, complete λ-hypersurfaces with polynomial area growth and $H - \lambda \geq 0$ were classified by Cheng and Wei [2014a]. They also defined an $\mathcal{F}$-functional and studied $\mathcal{F}$-stability of λ-hypersurfaces. Cheng, Ogata and Wei [Cheng et al. 2016] proved some gap and rigidity theorems for complete λ-hypersurfaces. See [Cheng and Wei 2014b; Guang 2014; Ogata 2015] for more results on the rigidity of λ-hypersurfaces.

We study the integral curvature pinching theorems for λ-hypersurfaces. We first prove the following L^n-pinching theorem of the second fundamental form.

Theorem 1. *Let $X : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be an n-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If*

$$\left(\int_M |A|^n \,\mathrm{d}\mu\right)^{1/n} < K(n, \lambda),$$

where $K(n, \lambda)$ is an explicit positive constant depending only on n and λ, then $|A| \equiv 0$ and M is a hyperplane.

Remark. It is easy to see from the expression of $K(n, \lambda)$ that $\lim_{\lambda\to 0} K(n, \lambda) = K_n$ for a positive constant K_n depending only on n. Hence if $\lambda = 0$, Theorem 1 reduces to the L^n-pinching theorem for self-shrinkers due to Ding and Xin [2014].

Let $\mathring{A}$ denote the trace-free second fundamental form, which is defined by $\mathring{A} = A - (H/n)g$ with g denoting the induced metric on M. We prove an L^n-pinching theorem of the trace-free second fundamental form for λ-hypersurfaces provided that the mean curvature is suitably bounded.

Theorem 2. *Let $X : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be an n-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Suppose the mean curvature satisfies*

$$|H| \leq \sqrt{\tfrac{1}{3}n + \lambda^2} - |\lambda|.$$

If

$$\left(\int_M |\mathring{A}|^n \,\mathrm{d}\mu\right)^{1/n} < D(n, \lambda),$$

where $D(n, \lambda)$ is an explicit positive constant depending on n and λ, then M is a hyperplane.

For the case $n = 2$, we obtain the following results.

Theorem 3. *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^3$. If*

$$\left(\int_M |A|^4 \,\mathrm{d}\mu\right)^{1/2} < K(\lambda),$$

where $K(\lambda)$ is an explicit positive constant depending only on λ, then $|A| \equiv 0$ and M is a hyperplane.

Theorem 4. *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional complete λ-hypersurface in the Euclidean space $\mathbb{R}^3$. Suppose the mean curvature satisfies*

$$|H| \leq \sqrt{\tfrac{2}{3} + \lambda^2} - |\lambda|.$$

If

$$\left(\int_M |\mathring{A}|^4 \,\mathrm{d}\mu\right)^{1/2} < D(\lambda),$$

where $D(\lambda)$ is an explicit positive constant depending on λ, then M is a hyperplane.

The rest of our paper is organized as follows. Some notation and several lemmas are prepared in Section 2. In Section 3, we prove Theorems 1 and 2. Theorems 3 and 4 will be proved in Section 4.

2. Preliminaries

Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional connected hypersurface. Denote by g and $\mathrm{d}\mu$ the induced metric and the volume form on M, respectively. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+1, \quad 1 \leq i, j, k, \ldots \leq n.$$

Choose local orthonormal frame fields $\{e_A\}$ in $\mathbb{R}^{n+1}$ such that, restricted to M, the e_i are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame fields and the connection 1-forms of $\mathbb{R}^{n+1}$, respectively. Then we have the following structure equations:

$$dX = \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij}\omega_j e_{n+1},$$

and

$$de_{n+1} = -\sum_{i,j} h_{ij}\omega_j e_i.$$

Restricting these forms to M, we have

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji},$$

where h_{ij} denotes the components of the second fundamental form of M. $H = \sum_i h_{ii}$ is the mean curvature and $A = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ is the second fundamental form of $X : M^n \to \mathbb{R}^{n+1}$. The trace-free second fundamental form is defined by $\mathring{A} = A - (H/n)g$.

Let $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_l \nabla_k h_{ij}$, where ∇ is the Levi-Civita connection on M. Gauss equations, Codazzi equations and Ricci formulas are given by

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \quad h_{ijk} = h_{ikj},$$
$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im}R_{mjkl} + \sum_{m=1}^{n} h_{mj}R_{mikl}.$$

For λ-hypersurfaces, an elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L} = \Delta - \tfrac{1}{2}\langle X, \nabla(\,\cdot\,)\rangle = e^{\frac{1}{4}|X|^2}\operatorname{div}\big(e^{-\frac{1}{4}|X|^2}\nabla(\,\cdot\,)\big),$$

where Δ and div denote the Laplacian and divergence on the λ-hypersurface, respectively. The $\mathcal{L}$ operator was introduced by Colding and Minicozzi [2012] when they investigated self-shrinkers. They showed that $\mathcal{L}$ is self-adjoint with respect to the measure $e^{-\frac{1}{4}|X|^2}\,d\mu$. We set $\rho = e^{-\frac{1}{4}|X|^2}$ and the volume form $d\mu$ might be omitted in the integrations for notational simplicity.

The following lemma, which was proved in [Cheng and Wei 2014a], is needed in order to prove our results. For convenience, we also include the proof here.

Lemma 5. *Let $X : M \to \mathbb{R}^{n+1}$ be a λ-hypersurface satisfying $H + \frac{1}{2}\langle X, N\rangle = \lambda$. Then*

$$\tfrac{1}{2}\mathcal{L}H^2 = |\nabla H|^2 + \tfrac{1}{2}H^2 + |A|^2(\lambda - H)H, \tag{1}$$

$$\tfrac{1}{2}\mathcal{L}|A|^2 = |\nabla A|^2 + \big(\tfrac{1}{2} - |A|^2\big)|A|^2 + \lambda f_3, \tag{2}$$

where $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$.

Proof. Since $H + \frac{1}{2}\langle X, N\rangle = \lambda$, one has

$$\nabla_i H = \frac{1}{2}\sum_j h_{ij}\langle X, e_j\rangle,$$

and

$$\nabla_k\nabla_i H = \frac{1}{2}\sum_j h_{ijk}\langle X, e_j\rangle + \tfrac{1}{2}h_{ik} + \sum_j h_{ij}h_{jk}(\lambda - H).$$

Hence,

$$\Delta H = \sum_i \nabla_i\nabla_i H = \frac{1}{2}\sum_i \nabla_i H\langle X, e_i\rangle + \tfrac{1}{2}H + |A|^2(\lambda - H),$$

and

$$\mathcal{L}H = \Delta H - \frac{1}{2}\sum_i \nabla_i H\langle X, e_i\rangle = \tfrac{1}{2}H + |A|^2(\lambda - H).$$

Therefore, we obtain

$$\tfrac{1}{2}\mathcal{L}H^2 = \tfrac{1}{2}\Delta H^2 - \frac{1}{4}\sum_i \nabla_i H^2\langle X, e_i\rangle = |\nabla H|^2 + \tfrac{1}{2}H^2 + |A|^2(\lambda - H)H.$$

By using the Ricci formulas, the Gauss equations and the Codazzi equations, we have

$$\begin{aligned}\mathcal{L}h_{ij} &= \Delta h_{ij} - \frac{1}{2}\sum_k \langle X, e_k\rangle h_{ijk}\\ &= \sum_k h_{ijkk} - \frac{1}{2}\sum_k\langle X, e_k\rangle h_{ijk}\\ &= \big(\tfrac{1}{2} - |A|^2\big)h_{ij} + \lambda\sum_k h_{ik}h_{kj}.\end{aligned}$$

Then it follows that

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}|A|^2 &= \tfrac{1}{2}\Delta\Big(\sum_{ij} h_{ij}^2\Big) - \frac{1}{4}\sum_k\langle X, e_k\rangle\nabla_k\Big(\sum_{ij} h_{ij}^2\Big)\\ &= \sum_{i,j,k} h_{ijk}^2 + \big(\tfrac{1}{2} - |A|^2\big)\sum_{ij} h_{ij}^2 + \lambda\sum_{i,j,k} h_{ik}h_{kj}h_{ji}\\ &= |\nabla A|^2 + \big(\tfrac{1}{2} - |A|^2\big)|A|^2 + \lambda f_3,\end{aligned}$$

where $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$. □

We need the following Sobolev inequality for submanifolds in the Euclidean space.

Lemma 6 [Xu and Gu 2007a; Hoffman and Spruck 1974]. *Let M^n $(n \geq 3)$ be an n-dimensional complete submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Let f be a* *nonnegative C^1 function with compact support. Then we have*

$$\|f\|^2_{2n/(n-2)} \leq D^2(n)\left[\frac{4(n-1)^2(1+s)}{(n-2)^2}\|\nabla f\|_2^2 + \left(1+\frac{1}{s}\right)\frac{1}{n^2}\big\||H|f\big\|_2^2\right],$$

where

$$D(n) = 2^n(1+n)^{(n+1)/n}(n-1)^{-1}\sigma_n^{-1/n},$$

and σ_n denotes the volume of the unit ball in $\mathbb{R}^n$.

3. Gap theorems for λ-hypersurfaces

Proof of Theorem 1. It follows from (2) and the inequality $|\nabla A|^2 \geq \big|\nabla|A|\big|^2$, which is an easy consequence of the Schwartz inequality, that

$$\begin{aligned}\mathcal{L}|A|^2 &= 2|\nabla A|^2 + 2\big(\tfrac{1}{2} - |A|^2\big)|A|^2 + 2\lambda f_3 \\ &\geq 2\big|\nabla|A|\big|^2 + 2\big(\tfrac{1}{2} - |A|^2\big)|A|^2 - 2|\lambda||A|^3.\end{aligned}$$

Let η be a smooth function with compact support on M. Multiplying $\eta^2|A|^{n-2}$ on both sides of the inequality above and integrating by parts with respect to the measure $\rho\,d\mu$ on M yields that for any $\tau > 0$

$$\begin{aligned}0 &\geq 2\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\int_M |A|^{n+1}\eta^2\rho - \int_M \eta^2|A|^{n-2}\rho\mathcal{L}|A|^2 \\ &= 2\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\int_M |A|^{n+1}\eta^2\rho + 2\int_M \rho|A|\nabla|A|\cdot\nabla(|A|^{n-2}\eta^2) \\ &= 2(n-1)\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\int_M |A|^{n+1}\eta^2\rho + 4\int_M (\nabla|A|\cdot\nabla\eta)|A|^{n-1}\eta\rho \\ &\geq 2(n-1)\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \int_M |A|^n\eta^2\rho - 2\int_M |A|^{n+2}\eta^2\rho \\ &\quad - 2|\lambda|\left(\frac{\tau}{2}\int_M |A|^n\eta^2\rho + \frac{1}{2\tau}\int_M |A|^{n+2}\eta^2\rho\right) + 4\int_M (\nabla|A|\cdot\nabla\eta)|A|^{n-1}\eta\rho.\end{aligned}$$

By the Cauchy inequality, for any $\varepsilon > 0$, we have

$$\left(\frac{|\lambda|}{\tau}+2\right)\int_M |A|^{n+2}\eta^2\rho+(|\lambda|\tau-1)\int_M |A|^n\eta^2\rho+\frac{2}{\varepsilon}\int_M |A|^n|\nabla\eta|^2\rho \geq 2(n-1-\varepsilon)\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho. \tag{3}$$

Set $f=|A|^{n/2}\rho^{1/2}\eta$. Integrating by parts, we obtain

$$\begin{aligned}\int_M|\nabla f|^2 &= \int_M \big|\nabla(|A|^{n/2}\eta)\big|^2\rho+\int_M |A|^n\eta^2|\nabla\rho^{1/2}|^2+\frac{1}{2}\int_M \nabla(|A|^n\eta^2)\nabla\rho\\ &= \int_M \big|\nabla(|A|^{n/2}\eta)\big|^2\rho+\frac{1}{16}\int_M |A|^n\eta^2|X^T|^2\rho-\frac{1}{2}\int_M |A|^n\eta^2\Delta\rho.\end{aligned} \tag{4}$$

Since

$$\Delta|X|^2=2|\nabla X|^2+2\langle X,\Delta X\rangle=2n+2H\langle X,N\rangle=2n+2\lambda\langle X,N\rangle-|X^N|^2,$$

where X^N is the normal part of X, we have

$$\begin{aligned}\Delta\rho &= -\tfrac{1}{4}\rho\Delta|X|^2+\tfrac{1}{16}\rho\big|\nabla|X|^2\big|^2=-\tfrac{1}{4}\rho\big(2n+2\lambda\langle X,N\rangle-|X^N|^2\big)+\tfrac{1}{4}\rho|X^T|^2\\ &= -\tfrac{1}{2}n\rho-\tfrac{1}{2}\lambda\rho\langle X,N\rangle+\tfrac{1}{4}\rho|X|^2.\end{aligned}$$

From (4), we get

$$\begin{aligned}\int_M|\nabla f|^2 &= \int_M\big|\nabla(|A|^{n/2}\eta)\big|^2\rho-\frac{1}{16}\int_M|A|^n\eta^2|X^T|^2\rho-\frac{1}{8}\int_M|A|^n\eta^2|X^N|^2\rho\\ &\quad+\frac{n}{4}\int_M|A|^n\eta^2\rho+\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho.\end{aligned} \tag{5}$$

Combining the Sobolev inequality in Lemma 6 and (5), we have

$$\begin{aligned}&\left(\int_M|f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\ &\quad\leq D^2(n)\cdot\left[\frac{4(n-1)^2(1+s)}{(n-2)^2}\int_M|\nabla f|^2+\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M H^2f^2\right]\\ &\quad=\frac{4D^2(n)(n-1)^2(1+s)}{(n-2)^2}\left[\int_M\big|\nabla(|A|^{n/2}\eta)\big|^2\rho-\frac{1}{16}\int_M|A|^n\eta^2|X^T|^2\rho\right.\\ &\qquad\qquad-\frac{1}{8}\int_M|A|^n\eta^2|X^N|^2\rho+\frac{n}{4}\int_M|A|^n\eta^2\rho\\ &\qquad\qquad\left.+\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\right]\\ &\qquad+D^2(n)\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}\int_M|A|^n\eta^2\big(\lambda-\tfrac{1}{2}\langle X,N\rangle\big)^2\rho.\end{aligned}$$

We choose

$$s=\frac{(n-2)^2}{2n^2(n-1)^2}\in\mathbb{R}^+$$

such that

$$\frac{4(n-1)^2(1+s)}{(n-2)^2}\cdot\frac{1}{8}=\frac{1}{4}\left(1+\frac{1}{s}\right)\cdot\frac{1}{n^2}.$$

Hence

$$\begin{aligned}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}&\\ \le \frac{2D^2(n)[(n-2)^2+2n^2(n-1)^2]}{n^2(n-2)^2}&\left[\int_M\big|\nabla(|A|^{n/2}\eta)\big|^2\rho\right.\\ &\left.+\frac{n}{4}\int_M|A|^n\eta^2\rho+\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\right]\\ +\frac{D^2(n)[(n-2)^2+2n^2(n-1)^2]}{n^2(n-2)^2}&\left[\int_M\lambda^2|A|^n\eta^2\rho-\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\right].\end{aligned}$$

Now we put

$$\kappa=\frac{2D^2(n)[(n-2)^2+2n^2(n-1)^2]}{n^2(n-2)^2}.$$

It follows from the inequality above that

$$\begin{aligned}(6)\qquad \kappa^{-1}&\left(\int_M|f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\ &\le\int_M\big|\nabla(|A|^{n/2}\eta)\big|^2\rho+\frac{n}{4}\int_M|A|^n\eta^2\rho+\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\\ &\quad+\frac{1}{2}\left(\int_M\lambda^2|A|^n\eta^2\rho-\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\right)\\ &=\int_M\big|\nabla(|A|^{n/2}\eta)\big|^2\rho+\left(\frac{n+2\lambda^2}{4}\right)\int_M|A|^n\eta^2\rho-\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho\\ &=\int_M\left(\frac{n^2}{4}\big|\nabla|A|\big|^2|A|^{n-2}\eta^2+n|A|^{n-1}\eta\nabla|A|\cdot\nabla\eta+|A|^n|\nabla\eta|^2\right)\rho\\ &\quad+\left(\frac{n+2\lambda^2}{4}\right)\int_M|A|^n\eta^2\rho-\frac{1}{4}\int_M|A|^n\eta^2\lambda\langle X,N\rangle\rho.\end{aligned}$$

On the other hand, for any $\theta > 0$, we have

$$
\begin{aligned}
(7)\quad -\frac{1}{2}\int_M |A|^n\eta^2\lambda\langle X, N\rangle\rho &= -\int_M |A|^n\eta^2\lambda(\lambda - H)\rho \\
&= -\int_M |A|^n\eta^2\lambda^2\rho + \int_M |A|^n\eta^2\lambda H\rho \\
&\le -\lambda^2\int_M |A|^n\eta^2\rho + |\lambda|\int_M |A|^n\eta^2\Big(\frac{\theta}{2}H^2 + \frac{1}{2\theta}\Big)\rho \\
&\le \Big(\frac{|\lambda|}{2\theta} - \lambda^2\Big)\int_M |A|^n\eta^2\rho + \frac{|\lambda|\theta}{2}\int_M |A|^n\eta^2H^2\rho \\
&\le \Big(\frac{|\lambda|}{2\theta} - \lambda^2\Big)\int_M |A|^n\eta^2\rho + \frac{n|\lambda|\theta}{2}\int_M |A|^{n+2}\eta^2\rho.
\end{aligned}
$$

Combining (6) and (7), we get

$$
\begin{aligned}
(8)\quad &\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\
&\le \int_M \Big(\frac{n^2}{4}\big|\nabla|A|\big|^2|A|^{n-2}\eta^2 + n|A|^{n-1}\eta\nabla|A|\cdot\nabla\eta + |A|^n|\nabla\eta|^2\Big)\rho \\
&\quad + \Big(\frac{|\lambda|}{4\theta} + \frac{n}{4}\Big)\int_M |A|^n\eta^2\rho + \frac{n\theta|\lambda|}{4}\int_M |A|^{n+2}\eta^2\rho.
\end{aligned}
$$

Combining the Cauchy inequality, (3) and (8), we have for any $\delta > 0$

$$
\begin{aligned}
&\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\
&\le (1+\delta)\frac{n^2}{4}\int_M \big|\nabla|A|\big|^2|A|^{n-2}\eta^2\rho + \Big(1+\frac{1}{\delta}\Big)\int_M |A|^n|\nabla\eta|^2\rho \\
&\quad + \Big(\frac{|\lambda|}{4\theta} + \frac{n}{4}\Big)\int_M |A|^n\eta^2\rho + \frac{n\theta|\lambda|}{4}\int_M |A|^{n+2}\eta^2\rho \\
&\le \frac{(1+\delta)n^2}{8(n-1-\varepsilon)}\Big[\Big(\frac{|\lambda|}{\tau} + 2\Big)\int_M |A|^{n+2}\eta^2\rho \\
&\qquad + (|\lambda|\tau - 1)\int_M |A|^n\eta^2\rho + \frac{2}{\varepsilon}\int_M |A|^n|\nabla\eta|^2\rho\Big] \\
&\quad + \Big(1+\frac{1}{\delta}\Big)\int_M |A|^n|\nabla\eta|^2\rho + \Big(\frac{|\lambda|}{4\theta} + \frac{n}{4}\Big)\int_M |A|^n\eta^2\rho + \frac{n\theta|\lambda|}{4}\int_M |A|^{n+2}\eta^2\rho.
\end{aligned}
$$

Put

$$
\delta = \frac{2(|\lambda| + n\theta)(n-1+\varepsilon)}{(1-|\lambda|\tau)\theta n^2} - 1 > 0,
$$

where $\varepsilon, \theta, \tau$ are positive constants such that $|\lambda|\tau - 1 < 0$. Then

$$
\begin{aligned}
(9)\qquad \kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}
&\le \left[\frac{n\theta+|\lambda|}{4\theta(1-|\lambda|\tau)}\cdot\Big(\frac{|\lambda|}{\tau}+2\Big)\frac{n-1+\varepsilon}{n-1-\varepsilon}+\frac{n\theta|\lambda|}{4}\right]\int_M |A|^{n+2}\eta^2\rho \\
&\quad + \left[\frac{n\theta+|\lambda|}{2\theta\varepsilon(1-|\lambda|\tau)}\frac{n-1+\varepsilon}{n-1-\varepsilon}+1+\frac{1}{\delta}\right]\int_M |A|^n|\nabla\eta|^2\rho \\
&\le \frac{(n\theta+|\lambda|)(|\lambda|+2\tau)+n\tau\theta^2(1-|\lambda|\tau)|\lambda|}{4\tau\theta(1-|\lambda|\tau)}\cdot\frac{n-1+\varepsilon}{n-1-\varepsilon} \\
&\qquad\qquad \times\Big(\int_M |A|^{2\cdot\frac{n}{2}}\Big)^{\frac{2}{n}}\cdot\Big(\int_M (|A|^n\eta^2\rho)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \\
&\quad + \left(\frac{n\theta+|\lambda|}{2\theta\varepsilon(1-|\lambda|\tau)}\cdot\frac{n-1+\varepsilon}{n-1-\varepsilon}+1+\frac{1}{\delta}\right)\int_M |A|^n|\nabla\eta|^2\rho.
\end{aligned}
$$

Set

$$K(n,\lambda,\theta,\tau)=\sqrt{\frac{4\tau\theta(1-|\lambda|\tau)}{\big[(n\theta+|\lambda|)(|\lambda|+2\tau)+n\tau\theta^2(1-|\lambda|\tau)|\lambda|\big]\kappa}}.$$

By a direct computation, $K(n,\lambda,\theta,\tau)$ achieves its maximum

$$K(n,\lambda)=\sqrt{\frac{2(\sqrt{\lambda^2+2}-|\lambda|)}{\big(n|\lambda|+2\sqrt{n}|\lambda|+n\sqrt{\lambda^2+2}\big)\kappa}}$$

when

$$\tau=\tfrac{1}{2}\big(\sqrt{\lambda^2+2}-|\lambda|\big),\qquad \theta=\sqrt{\frac{|\lambda|+2\tau}{n\tau-n|\lambda|\tau^2}}=\frac{2}{\sqrt{n}\big(\sqrt{\lambda^2+2}-|\lambda|\big)}=\frac{1}{\sqrt{n}\tau}.$$

Since

$$\Big(\int_M |A|^n\,d\mu\Big)^{1/n} < K(n,\lambda),$$

we have from (9) that there exists $0<\varepsilon_0<1$ such that

$$\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le \frac{n-1+\varepsilon}{n-1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\kappa}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+C(\varepsilon,\lambda)\int_M |A|^n|\nabla\eta|^2\rho,$$

namely,

$$(10)\qquad \frac{(n-1+\varepsilon)\varepsilon_0-2\varepsilon}{(n-1-\varepsilon)\kappa}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le C(\varepsilon,\lambda)\int_M |A|^n|\nabla\eta|^2\rho.$$

Let $\eta(X)=\eta_r(X)=\phi(|X|/r)$ for any $r>0$, where ϕ is a nonnegative function on $[0,+\infty)$ satisfying

$$\phi(x)=\begin{cases}1 & \text{if } x\in[0,1),\\ 0 & \text{if } x\in[2,+\infty),\end{cases}$$

and $|\phi'|\le C$ for some absolute constant. Let $\varepsilon=\frac{1}{2}\varepsilon_0$. Since $\int_M|A|^n\,d\mu$ is bounded, the right-hand side of (10) approaches zero as $r\to+\infty$, which implies $|A|\equiv 0$. Hence M is a hyperplane of $\mathbb{R}^{n+1}$. This completes the proof of Theorem 1. □

Setting $\mathring{A}=\sum_{i,j}\mathring{h}_{ij}\omega_i\otimes\omega_j$, we have $\mathring{h}_{ij}=h_{ij}-(H/n)g_{ij}$. Choose $\{e_i\}$ such that $h_{ij}=\lambda_i\delta_{ij}$ at a point p. Then $\mathring{h}_{ij}=\mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i=\lambda_i-H/n$, and

$$f_3=\sum_i\lambda_i^3=\sum_i\left(\mathring{\lambda}_i+\frac{H}{n}\right)^3=B_3+\frac{3}{n}H|\mathring{A}|^2+\frac{1}{n^2}H^3,$$

where $|\mathring{A}|^2=\sum_i\mathring{\lambda}_i^2=|A|^2-H^2/n$ and $B_3=\sum_i\mathring{\lambda}_i^3$. Thus, from (1) and (2) we have

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}|\mathring{A}|^2&=\tfrac{1}{2}\mathcal{L}|A|^2-\tfrac{1}{2}\mathcal{L}\Big(\frac{H^2}{n}\Big)\\&=|\nabla A|^2-\frac{1}{n}|\nabla H|^2+\big(\tfrac{1}{2}-|A|^2\big)|A|^2+\lambda f_3-\frac{H^2}{2n}-|A|^2(\lambda-H)\frac{H}{n}\\&=|\nabla\mathring{A}|^2+\big(\tfrac{1}{2}-|\mathring{A}|^2\big)|\mathring{A}|^2-\frac{1}{n}H^2|\mathring{A}|^2+\lambda B_3+\frac{2}{n}\lambda H|\mathring{A}|^2.\end{aligned}$$

By using an algebraic inequality in [Okumura 1974], we have

$$|B_3|\le\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3,$$

and the equality holds if and only if at least $n-1$ of the $\mathring{\lambda}_i$ are equal. Then we get

(11)
$$\begin{aligned}\tfrac{1}{2}\mathcal{L}|\mathring{A}|^2&\ge|\nabla\mathring{A}|^2+\big(\tfrac{1}{2}-|\mathring{A}|^2\big)|\mathring{A}|^2-\frac{1}{n}H^2|\mathring{A}|^2-|\lambda|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3+\frac{2}{n}\lambda H|\mathring{A}|^2\\&\ge\big|\nabla|\mathring{A}|\big|^2+\big(\tfrac{1}{2}-|\mathring{A}|^2\big)|\mathring{A}|^2-\frac{1}{n}\big(\lambda-\tfrac{1}{2}\langle X,N\rangle\big)^2|\mathring{A}|^2\\&\qquad-|\lambda|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3+\frac{2}{n}\lambda\big(\lambda-\tfrac{1}{2}\langle X,N\rangle\big)|\mathring{A}|^2\\&=\big|\nabla|\mathring{A}|\big|^2+\left(\frac{1}{2}+\frac{\lambda^2}{n}\right)|\mathring{A}|^2-\frac{1}{4n}|\mathring{A}|^2|X^N|^2-|\lambda|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3-|\mathring{A}|^4.\end{aligned}$$

By using (11), we give the proof of Theorem 2 as follows.

Proof of Theorem 2. Let η be a smooth function with compact support on M. Multiplying $|\mathring{A}|^{n-2}\eta^2$ on both sides of the inequality (11) above and integrating by parts with respect to the measure $\rho\,\mathrm{d}\mu$ on M yields

$$
\begin{aligned}
0 &\geq 2\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left(1+\frac{2\lambda^2}{n}\right)\int_M |\mathring{A}|^n\eta^2\rho - \frac{1}{2n}\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho \\
&\quad - 2|\lambda|\frac{n-2}{\sqrt{n(n-1)}}\int_M |\mathring{A}|^{n+1}\eta^2\rho - 2\int_M |\mathring{A}|^{n+2}\eta^2\rho - \int_M |\mathring{A}|^{n-2}\eta^2\mathcal{L}|\mathring{A}|^2\rho \\
&= 2\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left(1+\frac{2\lambda^2}{n}\right)\int_M |\mathring{A}|^n\eta^2\rho - \frac{1}{2n}\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho \\
&\quad - 2|\lambda|\frac{n-2}{\sqrt{n(n-1)}}\int_M |\mathring{A}|^{n+1}\eta^2\rho - 2\int_M |\mathring{A}|^{n+2}\eta^2\rho \\
&\quad + 2\int_M \rho|\mathring{A}|\nabla|\mathring{A}|\cdot\nabla(|\mathring{A}|^{n-2}\eta^2) \\
&\geq 2(n-1)\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho + \left[\left(1+\frac{2\lambda^2}{n}\right) - |\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}}\right]\int_M |\mathring{A}|^n\eta^2\rho \\
&\quad - \left(2+\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho - \frac{1}{2n}\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho \\
&\quad + 4\int_M (\nabla|\mathring{A}|\cdot\nabla\eta)|\mathring{A}|^{n-1}\eta\rho
\end{aligned}
$$

with constant $\zeta > 0$.

From the assumption $|H| \leq \sqrt{\frac{1}{3}n+\lambda^2} - |\lambda| \triangleq C$, we have

$$
\int_M |\mathring{A}|^n|X^N|^2\eta^2\rho = 4\int_M |\mathring{A}|^n(\lambda-H)^2\eta^2\rho \leq 4(\lambda^2+C^2+2C|\lambda|)\int_M |\mathring{A}|^n\eta^2\rho.
$$

This implies

$$
\begin{aligned}
0 &\geq 2(n-1)\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho \\
&\quad + \left[\left(1+\frac{2\lambda^2}{n}\right) - |\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}} - \frac{2}{n}(\lambda^2+C^2+2C|\lambda|)\right]\int_M |\mathring{A}|^n\eta^2\rho \\
&\quad - \left(2+\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho + 4\int_M (\nabla|\mathring{A}|\cdot\nabla\eta)|\mathring{A}|^{n-1}\eta\rho.
\end{aligned}
$$

By using the Cauchy inequality, for any $\varepsilon > 0$ we obtain

$$
\begin{aligned}
&\left(\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}+2\right)\int_M |\mathring{A}|^{n+2}\eta^2\rho \\
&\quad + \left[|\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}} + \frac{2}{n}(C^2+2C|\lambda|) - 1\right]\int_M |\mathring{A}|^n\eta^2\rho + \frac{2}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\rho \\
&\qquad\qquad \geq 2(n-1-\varepsilon)\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho.
\end{aligned}
\tag{12}
$$

Set $f=|\mathring{A}|^{n/2}\rho^{1/2}\eta$. Using the same argument as in the proof of Theorem 1, for any $\delta>0$ we get

$$
\begin{aligned}
(13)\quad \kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}
&\le (1+\delta)\frac{n^2}{4}\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho+\Big(1+\frac{1}{\delta}\Big)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho\\
&\quad+\frac{n+2\lambda^2}{4}\int_M |\mathring{A}|^n\eta^2\rho-\frac14\int_M |\mathring{A}|^n\eta^2\lambda\langle X,N\rangle\rho.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
(14)\qquad -\int_M |\mathring{A}|^n\eta^2\lambda\langle X,N\rangle\rho &= -2\int_M |\mathring{A}|^n\eta^2\lambda(\lambda-H)\rho\\
&=-2\int_M |\mathring{A}|^n\eta^2\lambda^2\rho+2\int_M |\mathring{A}|^n\eta^2\lambda H\rho\\
&\le 2(C|\lambda|-\lambda^2)\int_M |\mathring{A}|^n\eta^2\rho.
\end{aligned}
$$

Combining (12), (13) and (14), we have

$$
\begin{aligned}
&\kappa^{-1}\Big(\int_M |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\
&\quad\le (1+\delta)\frac{n^2}{4}\int_M \big|\nabla|\mathring{A}|\big|^2|\mathring{A}|^{n-2}\eta^2\rho+\Big(1+\frac{1}{\delta}\Big)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho\\
&\qquad+\frac{n+2C|\lambda|}{4}\int_M |\mathring{A}|^n\eta^2\rho\\
&\quad\le \frac{(1+\delta)n^2}{8(n-1-\varepsilon)}\bigg\{\Big(\frac{|\lambda|}{\zeta}\frac{n-2}{\sqrt{n(n-1)}}+2\Big)\int_M |\mathring{A}|^{n+2}\eta^2\rho\\
&\qquad+\Big[|\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}}+\frac{2}{n}(C^2+2C|\lambda|)-1\Big]\int_M |\mathring{A}|^n\eta^2\rho\\
&\qquad+\frac{2}{\varepsilon}\int_M |\mathring{A}|^n|\nabla\eta|^2\rho\bigg\}+\Big(1+\frac{1}{\delta}\Big)\int_M |\mathring{A}|^n|\nabla\eta|^2\rho+\frac{n+2C|\lambda|}{4}\int_M |\mathring{A}|^n\eta^2\rho.
\end{aligned}
$$

Let

$$
|\lambda|\zeta\frac{n-2}{\sqrt{n(n-1)}}+\frac{2}{n}(C^2+2C|\lambda|)-1<0,
$$

i.e.,

$$
0<\zeta<\frac{\big[n-2(C^2+2C|\lambda|)\big]\sqrt{n(n-1)}}{n(n-2)|\lambda|}.
$$

Putting

$$\delta = \frac{2(n+2C|\lambda|)\sqrt{n-1}\cdot(n-1+\varepsilon)}{n\big[n\sqrt{n-1}-(n-2)\sqrt{n}\,|\lambda|\,\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)\big]}-1>0$$

for some $\varepsilon > 0$ to be defined later, we have

$$\begin{aligned}
(15)\quad & \kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\le \left\{\frac{\sqrt{n}(n+2C|\lambda|)[(n-2)|\lambda|+2\zeta\sqrt{n(n-1)}]}{4\zeta[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)]}\right\} \\
&\qquad\qquad \times\frac{n-1+\varepsilon}{n-1-\varepsilon}\int_M |\mathring{A}|^{n+2}\eta^2\rho \\
&\quad+\left\{\frac{n(n+2C|\lambda|)\sqrt{n-1}\cdot(n-1+\varepsilon)}{2\varepsilon[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)]}+1+\frac{1}{\delta}\right\} \\
&\qquad\qquad \times\int_M |\mathring{A}|^n\,|\nabla\eta|^2\rho \\
&\le \left\{\frac{\sqrt{n}(n+2C|\lambda|)[(n-2)|\lambda|+2\zeta\sqrt{n(n-1)}]}{4\zeta[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)]}\right\} \\
&\qquad\qquad \times\frac{n-1+\varepsilon}{n-1-\varepsilon}\cdot\left(\int_M |\mathring{A}|^{2\cdot\frac{n}{2}}\right)^{\frac{2}{n}}\left(\int_M (|\mathring{A}|^n\eta^2\rho)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\quad+\widetilde{C}(\varepsilon,\lambda,n,\zeta,C)\int_M |\mathring{A}|^n\,|\nabla\eta|^2\rho.
\end{aligned}$$

Set

$$D(n,\lambda,\zeta,C)=\sqrt{\frac{4\zeta\big[n\sqrt{n-1}-(n-2)\sqrt{n}|\lambda|\zeta-2\sqrt{n-1}(C^2+2C|\lambda|)\big]}{\sqrt{n}(n+2C|\lambda|)\big[(n-2)|\lambda|+2\zeta\sqrt{n(n-1)}\big]\kappa}}.$$

We choose

$$\zeta=\frac{\sqrt{(n-2)^2\lambda^2+2(n-1)\big[n-2(C^2+2C|\lambda|)\big]}-(n-2)|\lambda|}{2\sqrt{n(n-1)}}$$

such that $D(n,\lambda,\zeta,C)$ achieves its maximum $D(n,\lambda)$ with

$$\begin{aligned}
D(n,\lambda)&=\frac{\sqrt{(n-2)^2\lambda^2+2(n-1)\big[n-2(C^2+2C|\lambda|)\big]}-(n-2)|\lambda|}{\sqrt{n(n-1)(n+2C|\lambda|)\kappa}} \\
&=\frac{\sqrt{(n-2)^2\lambda^2+\frac{2}{3}n(n-1)}-(n-2)|\lambda|}{\sqrt{n(n-1)\Big(n+2|\lambda|\sqrt{\frac{1}{3}n+\lambda^2}-2\lambda^2\Big)\kappa}}.
\end{aligned}$$

Combining the assumption

$$\left(\int_M |\mathring{A}|^n \, \mathrm{d}\mu\right)^{1/n} < D(n, \lambda)$$

and (15) implies that there exists $0 < \varepsilon_0 < 1$ such that

$$\kappa^{-1}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{n-1+\varepsilon}{n-1-\varepsilon} \cdot \frac{1-\varepsilon_0}{\kappa}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \widetilde{C}(\varepsilon, \lambda, n)\int_M |\mathring{A}|^n |\nabla\eta|^2 \rho,$$

namely,

$$\frac{(n-1+\varepsilon)\varepsilon_0 - 2\varepsilon}{(n-1-\varepsilon)\kappa}\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \widetilde{C}(\varepsilon, \lambda, n)\int_M |\mathring{A}|^n |\nabla\eta|^2 \rho.$$

Let $\varepsilon = \frac{1}{2}\varepsilon_0$ and choose η as in the proof of Theorem 1. Since $\int_M |\mathring{A}|^n \, \mathrm{d}\mu$ is bounded, by using a similar argument we obtain $\mathring{A} \equiv 0$. Therefore, M is totally umbilical, i.e., M is $\mathbb{S}^n(\sqrt{\lambda^2 + 2n} - \lambda)$ or $\mathbb{R}^n$. Since we have assumed that

$$|H| \leq \sqrt{\tfrac{1}{3}n + \lambda^2} - |\lambda|,$$

the first case is excluded. This completes the proof of Theorem 2. □

Remark. In fact, we can prove that if $\sup_M |H| < \sqrt{\frac{1}{2}n + \lambda^2} - |\lambda|$ and if

$$\left(\int_M |\mathring{A}|^n \, \mathrm{d}\mu\right)^{1/n} < D(n, \lambda, \sup_M |H|),$$

then M is a hyperplane. Here $D(n, \lambda, \sup_M |H|)$ is a positive constant depending on n, λ and $\sup_M |H|$.

Remark. In particular, if $\lambda = 0$, Theorem 2 reduces to the rigidity result for self-shrinkers in [Lin 2016]. For the higher codimension case, Cao, Xu and Zhao [Cao et al. 2014] proved some L^n-pinching theorems of $\mathring{A}$ for self-shrinkers.

4. Gap theorems in dimension 2

We need another Sobolev-type inequality in dimension 2, which was proved by Xu and Gu [2007b]:

$$\tilde{c}^{-1}\left(\int f^4 \, \mathrm{d}\mu\right)^{1/2} \leq \frac{1}{t}\int |\nabla f|^2 \, \mathrm{d}\mu + t\int f^2 \, \mathrm{d}\mu + \frac{1}{2}\int |H| f^2 \, \mathrm{d}\mu \tag{16}$$

for all $f \in C_c^\infty(M)$ and for all $t \in \mathbb{R}^+$, where $\tilde{c} = 12\sqrt{3\pi}/\pi$.

Proof of Theorem 3. As in the proof of Theorem 1, for any $0<\varepsilon<1$, we have

$$\left(\frac{|\lambda|}{\tau}+2\right)\int_M |A|^4\eta^2\rho+(|\lambda|\tau-1)\int_M |A|^2\eta^2\rho+\frac{2}{\varepsilon}\int_M |A|^2|\nabla\eta|^2\rho \geq 2(1-\varepsilon)\int_M \big|\nabla|A|\big|^2\eta^2\rho. \tag{17}$$

Setting $f=|A|\eta\rho^{1/2}$, we get

$$\int_M |\nabla f|^2=\int_M \big|\nabla(|A|\eta)\big|^2\rho-\frac{1}{16}\int_M |A|^2\eta^2|X^T|^2\rho-\frac{1}{8}\int_M |A|^2\eta^2|X^N|^2\rho +\frac{1}{2}\int_M |A|^2\eta^2\rho+\frac{1}{4}\int_M |A|^2\eta^2\lambda\langle X,N\rangle\rho. \tag{18}$$

Combining the Sobolev-type inequality (16) and (18), we have

$$\begin{aligned}
\tilde{c}^{-1}&\left(\int_M |f|^4\right)^{1/2}\\
&\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho-\frac{1}{16}\int_M |A|^2\eta^2|X^T|^2\rho\\
&\qquad -\frac{1}{8}\int_M |A|^2\eta^2|X^N|^2\rho+\frac{1}{2}\int_M |A|^2\eta^2\rho+\frac{1}{4}\int_M |A|^2\eta^2\lambda\langle X,N\rangle\rho\bigg]\\
&\quad +t\int_M |A|^2\eta^2\rho+\frac{1}{2}\int |H||A|^2\eta^2\rho\\
&\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho+\frac{1}{2}\int_M |A|^2\eta^2\rho+\frac{\lambda^2}{8}\int_M |A|^2\eta^2\rho\bigg]\\
&\quad +t\int_M |A|^2\eta^2\rho+\frac{1}{2}\int |H||A|^2\eta^2\rho.
\end{aligned}$$

By the Cauchy inequality, for any $\theta>0$, we get

$$\begin{aligned}
\tilde{c}^{-1}\left(\int_M |f|^4\right)^{1/2} &\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho+\frac{1}{2}\int_M |A|^2\eta^2\rho+\frac{\lambda^2}{8}\int_M |A|^2\eta^2\rho\bigg]\\
&\quad +t\int_M |A|^2\eta^2\rho+\frac{1}{2}\int \left(\frac{\theta}{2}H^2+\frac{1}{2\theta}\right)|A|^2\eta^2\rho\\
&\leq \frac{1}{t}\bigg[\int_M \big|\nabla(|A|\eta)\big|^2\rho+\frac{1}{2}\int_M |A|^2\eta^2\rho+\frac{\lambda^2}{8}\int_M |A|^2\eta^2\rho\bigg]\\
&\quad +t\int_M |A|^2\eta^2\rho+\frac{1}{2}\int \left(\theta|A|^2+\frac{1}{2\theta}\right)|A|^2\eta^2\rho
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&= \frac{1}{t}\int_M \big(\big|\nabla|A|\big|^2\eta^2 + 2|A|\eta\nabla|A|\cdot\nabla\eta + |A|^2|\nabla\eta|^2\big)\rho \\
&\qquad + \left(t + \frac{1}{4\theta} + \frac{1}{2t} + \frac{\lambda^2}{8t}\right)\int |A|^2\eta^2\rho + \frac{\theta}{2}\int |A|^4\eta^2\rho \\
&\le \frac{1}{t}\left[(1+\delta)\int_M \big|\nabla|A|\big|^2\eta^2\rho + \left(1+\frac{1}{\delta}\right)\int_M |A|^2|\nabla\eta|^2\rho\right] \\
&\qquad + \left(t + \frac{1}{4\theta} + \frac{1}{2t} + \frac{\lambda^2}{8t}\right)\int |A|^2\eta^2\rho + \frac{\theta}{2}\int |A|^4\eta^2\rho.
\end{aligned}
$$

Combining (17) and (19), we have

$$
\begin{aligned}
&\tilde{c}^{-1}\left(\int_M |A|^4\eta^4\rho^2\right)^{1/2} \\
&\le \frac{1}{t}(1+\delta)\cdot\frac{1}{2(1-\varepsilon)}\left[\left(\frac{|\lambda|}{\tau}+2\right)\int_M |A|^4\eta^2\rho + (|\lambda|\tau - 1)\int_M |A|^2\eta^2\rho\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{2}{\varepsilon}\int_M |A|^2|\nabla\eta|^2\rho\right] \\
&\quad + \frac{1}{t}\left(1+\frac{1}{\delta}\right)\int_M |A|^2|\nabla\eta|^2\rho + \left(t + \frac{1}{4\theta} + \frac{1}{2t} + \frac{\lambda^2}{8t}\right)\int |A|^2\eta^2\rho + \frac{\theta}{2}\int |A|^4\eta^2\rho.
\end{aligned}
$$

Put

$$
\delta = \frac{(4\theta + 2t + 8\theta t^2 + \theta\lambda^2)(1+\varepsilon)}{4\theta(1-|\lambda|\tau)} - 1 > 0,
$$

where $\varepsilon, \theta, \tau, t$ are positive constants such that $|\lambda|\tau - 1 < 0$. Then

$$
\begin{aligned}
\text{(20)}\quad &\tilde{c}^{-1}\left(\int_M |A|^4\eta^4\rho^2\right)^{1/2} \\
&\le \left[\frac{1}{t}\cdot\frac{(1+\varepsilon)}{2(1-\varepsilon)}\cdot\frac{(4\theta + 2t + 8\theta t^2 + \theta\lambda^2)}{4\theta(1-|\lambda|\tau)}\cdot\left(\frac{|\lambda|}{\tau}+2\right) + \frac{\theta}{2}\right]\int_M |A|^4\eta^2\rho \\
&\quad + \frac{1}{t}\left[\frac{(1+\delta)}{\varepsilon(1-\varepsilon)} + \left(1+\frac{1}{\delta}\right)\right]\int_M |A|^2|\nabla\eta|^2\rho \\
&\le \frac{(4\theta + 2t + 8\theta t^2 + \theta\lambda^2)(|\lambda| + 2\tau) + 4\theta^2 t\tau(1-|\lambda|\tau)}{8\theta t\tau(1-|\lambda|\tau)}\cdot\frac{1+\varepsilon}{1-\varepsilon} \\
&\qquad\qquad\qquad \times\left(\int_M |A|^4\eta^4\rho^2\right)^{1/2}\cdot\left(\int_M |A|^4\right)^{1/2} \\
&\quad + \frac{1}{t}\left[\frac{(1+\delta)}{\varepsilon(1-\varepsilon)} + \left(1+\frac{1}{\delta}\right)\right]\int_M |A|^2|\nabla\eta|^2\rho.
\end{aligned}
$$

Set

$$K(t,\lambda,\theta,\tau)=\frac{8\theta t\tau(1-|\lambda|\tau)}{\big[(4\theta+2t+8\theta t^2+\theta\lambda^2)(|\lambda|+2\tau)+4\theta^2 t\tau(1-|\lambda|\tau)\big]\tilde{c}},$$

where $\tilde{c}=12\sqrt{3\pi}/\pi$. By a direct computation, $K(t,\lambda,\theta,\tau)$ achieves its maximum

$$K(\lambda)=\frac{\sqrt{2}\big(\lambda^2+1-\sqrt{\lambda^2+2}|\lambda|\big)}{\big(2\sqrt{4+\lambda^2}+\sqrt{\lambda^2+2}-|\lambda|\big)\tilde{c}}$$

when

$$t=\sqrt{\tfrac{1}{8}(4+\lambda^2)},\quad \tau=\tfrac{1}{2}\big(\sqrt{\lambda^2+2}-|\lambda|\big),\quad \theta=\sqrt{\frac{|\lambda|+2\tau}{2\tau(1-|\lambda|\tau)}}=\frac{1}{\sqrt{2}\tau}.$$

Since

$$\left(\int_M |A|^4\right)^{1/2}<K(\lambda),$$

we have from (20) that there exists $0<\varepsilon_0<1$ such that

$$\tilde{c}^{-1}\left(\int_M |A|^4\eta^4\rho^2\right)^{1/2}\le\frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\tilde{c}}\left(\int_M |A|^4\eta^4\rho^2\right)^{1/2}+C(\varepsilon,\lambda)\int_M |A|^2|\nabla\eta|^2\rho.$$

Let $\varepsilon=\frac{1}{2}\varepsilon_0$. Since $\int_M |A|^4\,\mathrm{d}\mu$ is bounded, we choose η as in the proof of Theorem 1 and a similar argument implies $|A|\equiv 0$. □

Using a similar argument, we give the proof of Theorem 4.

Proof of Theorem 4. For $n=2$, we have

$$\tfrac{1}{2}\mathcal{L}|\mathring{A}|^2\ge\big|\nabla|\mathring{A}|\big|^2+\frac{1+\lambda^2}{2}|\mathring{A}|^2-\tfrac{1}{8}|\mathring{A}|^2|X^N|^2-|\mathring{A}|^4,$$

and

$$\text{(21)}\quad 2\int_M |\mathring{A}|^4\eta^2\rho+(C^2+2C|\lambda|-1)\int_M |\mathring{A}|^2\eta^2\rho+\frac{2}{\varepsilon}\int_M |\mathring{A}|^2|\nabla\eta|^2\rho \ge 2(1-\varepsilon)\int_M \big|\nabla|\mathring{A}|\big|^2\eta^2\rho$$

with $0<\varepsilon<1$.

Set $f=|\mathring{A}|\rho^{1/2}\eta$. By (16) and the hypothesis $|H|\le\sqrt{\frac{2}{3}+\lambda^2}-|\lambda|\triangleq C$, we have

$$\begin{aligned}(22)\qquad \tilde{c}^{-1}\left(\int_M|f|^4\right)^{1/2} &\le \frac{1}{t}\left[\int_M\left|\nabla(|\mathring{A}|\eta)\right|^2\rho+\frac{1}{2}\int_M|\mathring{A}|^2\eta^2\rho+\frac{\lambda^2}{8}\int_M|\mathring{A}|^2\eta^2\rho\right]\\ &\quad+t\int_M|\mathring{A}|^2\eta^2\rho+\frac{1}{2}\int|H|\,|\mathring{A}|^2\eta^2\rho\\ &\le\frac{1}{t}\int_M\left(\left|\nabla|\mathring{A}|\right|^2\eta^2+2|\mathring{A}|\eta\nabla|\mathring{A}|\cdot\nabla\eta+|\mathring{A}|^2|\nabla\eta|^2\right)\rho\\ &\quad+\left(t+\frac{C}{2}+\frac{1}{2t}+\frac{\lambda^2}{8t}\right)\int_M|\mathring{A}|^2\eta^2\rho.\end{aligned}$$

Combining the Cauchy inequality, (21) and (22), we have for any $\delta>0$

$$\begin{aligned}\tilde{c}^{-1}\left(\int_M|f|^4\right)^{1/2} &\le\frac{1}{t}\left[(1+\delta)\int_M\left|\nabla|\mathring{A}|\right|^2\eta^2\rho+\left(1+\frac{1}{\delta}\right)\int_M|\mathring{A}|^2|\nabla\eta|^2\rho\right]\\ &\quad+\left(t+\frac{C}{2}+\frac{1}{2t}+\frac{\lambda^2}{8t}\right)\int_M|\mathring{A}|^2\eta^2\rho\\ &\le\frac{1+\delta}{t}\frac{1}{2(1-\varepsilon)}\left[2\int_M|\mathring{A}|^4\eta^2\rho+(C^2+2C|\lambda|-1)\int_M|\mathring{A}|^2\eta^2\rho\right.\\ &\qquad\qquad\left.+\frac{2}{\varepsilon}\int_M|\mathring{A}|^2|\nabla\eta|^2\rho\right]\\ &\quad+\frac{1}{t}\left(1+\frac{1}{\delta}\right)\int_M|\mathring{A}|^2|\nabla\eta|^2\rho+\left(t+\frac{C}{2}+\frac{1}{2t}+\frac{\lambda^2}{8t}\right)\int_M|\mathring{A}|^2\eta^2\rho.\end{aligned}$$

Put

$$\delta=\frac{(4+\lambda^2+8t^2+4tC)(1+\varepsilon)}{4[1-(C^2+2C|\lambda|)]}-1>0.$$

Then we get

$$\begin{aligned}(23)\qquad \tilde{c}^{-1}&\left(\int_M|f|^4\right)^{1/2}\\ &\le\frac{1}{t}\cdot\frac{4+\lambda^2+8t^2+4tC}{4[1-(C^2+2C|\lambda|)]}\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot\int_M|\mathring{A}|^4\eta^2\rho\\ &\quad+\frac{1}{t}\left[\frac{1+\delta}{\varepsilon(1-\varepsilon)}+1+\frac{1}{\delta}\right]\int_M|\mathring{A}|^2|\nabla\eta|^2\rho\\ &\le\frac{1}{t}\cdot\frac{4+\lambda^2+8t^2+4tC}{4[1-(C^2+2C|\lambda|)]}\cdot\frac{1+\varepsilon}{1-\varepsilon}\cdot\left(\int_M|f|^4\right)^{1/2}\cdot\left(\int_M|\mathring{A}|^4\right)^{1/2}\\ &\quad+\frac{1}{t}\left[\frac{1+\delta}{\varepsilon(1-\varepsilon)}+1+\frac{1}{\delta}\right]\int_M|\mathring{A}|^2|\nabla\eta|^2\rho.\end{aligned}$$

Set

$$D(\lambda, C, t) = \frac{4t[1-(C^2+2C|\lambda|)]}{(4+\lambda^2+8t^2+4tC)\tilde{c}}.$$

We choose $t = \sqrt{\frac{1}{8}(4+\lambda^2)}$ such that $D(\lambda, C, t)$ achieves its maximum

$$D(\lambda) = \frac{1}{3\Bigl(\sqrt{8+2\lambda^2}+\sqrt{\frac{2}{3}+\lambda^2}-|\lambda|\Bigr)\tilde{c}}.$$

Since

$$\biggl(\int_M |\mathring{A}|^4\biggr)^{1/2} < D(\lambda),$$

we have from (23) that there exists $0 < \varepsilon_0 < 1$ such that

$$\tilde{c}^{-1}\biggl(\int_M |f|^4\biggr)^{1/2} \le \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{1-\varepsilon_0}{\tilde{c}}\cdot\biggl(\int_M |f|^4\biggr)^{1/2} + C(\varepsilon,\lambda)\int_M |\mathring{A}|^2|\nabla\eta|^2\rho.$$

Let $\varepsilon = \frac{1}{2}\varepsilon_0$. Since $\int_M |\mathring{A}|^4\,d\mu$ is bounded, we choose η as above and a similar argument implies $\mathring{A} \equiv 0$. Therefore, M is totally umbilical, i.e., M is $\mathbb{S}^2(\sqrt{\lambda^2+4}-\lambda)$ or $\mathbb{R}^2$. Since we have assumed that

$$|H| \le \sqrt{\tfrac{2}{3}+\lambda^2}-|\lambda|,$$

the first case is excluded. This completes the proof of Theorem 4. □

Remark. Similarly, it is seen from the proof of Theorem 4 that we can prove that if $\sup_M |H| < \sqrt{1+\lambda^2}-|\lambda|$ and if $\bigl(\int_M |\mathring{A}|^4\,d\mu\bigr)^{1/2} < D(\lambda, \sup_M |H|)$, then M is a hyperplane. Here $D(\lambda, \sup_M |H|)$ is a positive constant depending on λ and $\sup_M |H|$.

Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestions.

References

[Cao and Li 2013] H.-D. Cao and H. Li, "A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension", *Calc. Var. Partial Differential Equations* **46**:3-4 (2013), 879–889. MR Zbl

[Cao et al. 2014] S.-J. Cao, H.-W. Xu, and E.-T. Zhao, "Pinching theorems for self-shrinkers of higher codimensions", preprint, 2014, available at http://www.cms.zju.edu.cn/upload/file/20170320/1489994331903839.pdf.

[Cheng and Peng 2015] Q.-M. Cheng and Y. Peng, "Complete self-shrinkers of the mean curvature flow", *Calc. Var. Partial Differential Equations* **52**:3-4 (2015), 497–506. MR Zbl

[Cheng and Wei 2014a] Q.-M. Cheng and G. Wei, "Complete λ-hypersurfaces of weighted volume-preserving mean curvature flow", preprint, 2014. arXiv

[Cheng and Wei 2014b] Q.-M. Cheng and G. Wei, "The Gauss image of λ-hypersurfaces and a Bernstein type problem", preprint, 2014. arXiv

[Cheng and Wei 2015] Q.-M. Cheng and G. Wei, "A gap theorem of self-shrinkers", *Trans. Amer. Math. Soc.* **367**:7 (2015), 4895–4915. MR Zbl

[Cheng et al. 2016] Q.-M. Cheng, S. Ogata, and G. Wei, "Rigidity theorems of λ-hypersurfaces", *Comm. Anal. Geom.* **24**:1 (2016), 45–58. MR Zbl

[Colding and Minicozzi 2012] T. H. Colding and W. P. Minicozzi, II, "Generic mean curvature flow, I: Generic singularities", *Ann. of Math.* (2) **175**:2 (2012), 755–833. MR Zbl

[Colding et al. 2015] T. H. Colding, T. Ilmanen, and W. P. Minicozzi, II, "Rigidity of generic singularities of mean curvature flow", *Publ. Math. Inst. Hautes Études Sci.* **121** (2015), 363–382. MR Zbl

[Ding and Xin 2013] Q. Ding and Y. L. Xin, "Volume growth, eigenvalue and compactness for self-shrinkers", *Asian J. Math.* **17**:3 (2013), 443–456. MR Zbl

[Ding and Xin 2014] Q. Ding and Y. L. Xin, "The rigidity theorems of self-shrinkers", *Trans. Amer. Math. Soc.* **366**:10 (2014), 5067–5085. MR Zbl

[Guang 2014] Q. Guang, "Gap and rigidity theorems of λ-hypersurfaces", preprint, 2014. arXiv

[Hoffman and Spruck 1974] D. Hoffman and J. Spruck, "Sobolev and isoperimetric inequalities for Riemannian submanifolds", *Comm. Pure Appl. Math.* **27** (1974), 715–727. MR Zbl

[Huisken 1990] G. Huisken, "Asymptotic behavior for singularities of the mean curvature flow", *J. Differential Geom.* **31**:1 (1990), 285–299. MR Zbl

[Ilmanen 1995] T. Ilmanen, "Singularities of mean curvature flow of surfaces", preprint, 1995, available at https://people.math.ethz.ch/~ilmanen/papers/sing.ps.

[Le and Sesum 2011] N. Q. Le and N. Sesum, "Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers", *Comm. Anal. Geom.* **19**:4 (2011), 633–659. MR Zbl

[Li and Wei 2014] H. Li and Y. Wei, "Classification and rigidity of self-shrinkers in the mean curvature flow", *J. Math. Soc. Japan* **66**:3 (2014), 709–734. MR Zbl

[Lin 2016] H. Lin, "Some rigidity theorems for self-shrinkers of the mean curvature flow", *J. Korean Math. Soc.* **53**:4 (2016), 769–780. MR Zbl

[McGonagle and Ross 2015] M. McGonagle and J. Ross, "The hyperplane is the only stable, smooth solution to the isoperimetric problem in Gaussian space", *Geom. Dedicata* **178** (2015), 277–296. MR Zbl

[Ogata 2015] S. Ogata, "A global pinching theorem of complete λ-hypersurfaces", preprint, 2015. arXiv

[Okumura 1974] M. Okumura, "Hypersurfaces and a pinching problem on the second fundamental tensor", *Amer. J. Math.* **96** (1974), 207–213. MR Zbl

[White 1997] B. White, "Stratification of minimal surfaces, mean curvature flows, and harmonic maps", *J. Reine Angew. Math.* **488** (1997), 1–35. MR Zbl

[Xu and Gu 2007a] H.-W. Xu and J.-R. Gu, "A general gap theorem for submanifolds with parallel mean curvature in $\mathbb{R}^{n+p}$", *Comm. Anal. Geom.* **15**:1 (2007), 175–193. MR Zbl

[Xu and Gu 2007b] H.-W. Xu and J.-R. Gu, "L^2-isolation phenomenon for complete surfaces arising from Yang–Mills theory", *Lett. Math. Phys.* **80**:2 (2007), 115–126. MR Zbl

Received February 25, 2016. Revised September 22, 2016.

HUIJUAN WANG
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
ZHEJIANG 310027
CHINA
whjuan@zju.edu.cn

HONGWEI XU
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
ZHEJIANG 310027
CHINA
xuhw@zju.edu.cn

ENTAO ZHAO
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
ZHEJIANG 310027
CHINA
zhaoet@zju.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 288 No. 2 June 2017

0030-8730(201706)288:2;1-R