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CONVEXITY OF THE ENTROPY OF POSITIVE SOLUTIONS TO THE HEAT EQUATION ON QUATERNIONIC CONTACT AND CR MANIFOLDS

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A proof of the monotonicity of an entropy like energy for the heat equation on a quaternionic contact and CR manifolds is given.

1. Introduction

The purpose of this note is to show the monotonicity of the entropy type energy associated to the (subelliptic) heat equation in a sub-Riemannian setting. The result is inspired by the corresponding Riemannian fact related to Perelman's entropy formula for the heat equation on a static Riemannian manifold; see [Ni 2004]. More recently a similar quantity was considered in the CR case [Chang and Wu 2010]. Our goal is to prove the convexity of the entropy of a positive solution to the (subelliptic) heat equation on a quaternionic contact manifold, henceforth abbreviated to QC, and give an alternative proof of the result in that work, more in line with the Riemannian case. We resolve directly the difficulties arising in the sub-Riemannian setting of both quaternionic contact and CR manifolds. Section 3 contains the alternative proof of the result of [Chang and Wu 2010] in the CR case while the remaining parts of the paper focus on the QC case.

To state the problem, let M be a quaternionic contact or a CR manifold of real dimensions $4n+3$ and $2n+1$, respectively, and u be a smooth *positive* solution to the (QC or CR) heat equation

$$\frac{\partial}{\partial t}u = \Delta u. \tag{1-1}$$

Hereafter, $\triangle u = \mathrm{tr}^g(\nabla^2 u)$ denotes the negative sub-Laplacian with the trace taken with respect to an orthonormal basis of the respective horizontal $4n$ or $2n$ dimensional spaces in the QC and CR cases. Associated to such a solution are the

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(Boltzmann–Nash like) entropy

$$\mathcal{N}(t) = \int_M u \ln u \, \mathrm{Vol}_\eta \tag{1-2}$$

and entropy energy functional

$$\mathcal{E}(t) = \int_M |\nabla f|^2 u \, \mathrm{Vol}_\eta, \tag{1-3}$$

where, as usual, $f = -\ln u$ and Vol_η is the naturally associated volume form on M, see (2-4) and (3-3). Exactly as in the Riemannian case, we have that the entropy is decreasing (i.e., nonincreasing) because of the formula

$$\frac{d}{dt}\mathcal{N} = -\mathcal{E}(t).$$

Our goal is the computation of the second derivative of the entropy. In order to state the result in the QC case we consider the Ricci type tensor

$$\mathcal{L}(X, X) \overset{def}{=} 2Sg(X, X) + \alpha_n T^0(X, X) + \beta_n U(X, X), \tag{1-4}$$

where X is any vector from the horizontal distribution, $\alpha_n = 2(2n+3)/(2n+1)$, $\beta_n = 4(2n-1)(n+2)/((2n+1)(n-1))$, and T^0 and U are certain invariant components of the torsion; see Section 2. The tensor (1-4) appeared earlier as a natural assumption in the QC Lichnerowicz–Obata type results; see [Ivanov and Vassilev 2015, Section 8.1] and references therein. In addition, following [Ivanov et al. 2013], we define the P-form of a fixed smooth function f on M by the following equation:

$$\begin{aligned} P_f(X) = \sum_{b=1}^{4n} \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3}\sum_{b=1}^{4n} \nabla^3 f(I_t X, e_b, I_t e_b) \\ -4nS\, df(X) + 4nT^0(X, \nabla f) - \frac{8n(n-2)}{n-1} U(X, \nabla f), \end{aligned} \tag{1-5}$$

which in the case $n = 1$ is defined by formally dropping the last term. The P-function of f is the function $P_f(\nabla f)$. The C-operator of M is the 4th order differential operator

$$f \mapsto Cf = -\nabla^* P_f = \sum_{a=1}^{4n} (\nabla_{e_a} P_f)(e_a).$$

In many respects the C-operator plays a role similar to the Paneitz operator in CR geometry. We say that the P-function of f is nonnegative if

$$\int_M f \cdot Cf \, \mathrm{Vol}_\eta = -\int_M P_f(\nabla f)\, \mathrm{Vol}_\eta \geq 0.$$

If the above holds for any $f \in \mathcal{C}_o^\infty(M)$ we say that the C-operator is nonnegative, $\mathcal{C} \geq 0$.

We are ready to state our first result.

Proposition 1.1. *Let M be a compact QC manifold of dimension $4n+3$. If $u = e^{-f}$ is a positive solution to heat equation* (1-1), *then we have*

$$\frac{2n+1}{4n}\mathcal{E}'(t) = -\int_M \Big[|(\nabla^2 f)_0|^2 + \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f) + \frac{1}{16n}|\nabla f|^4\Big] u \,\mathrm{Vol}_\eta + \frac{3}{n}\int_M P_F(\nabla F)\,\mathrm{Vol}_\eta,$$

where $u = F^2$ ($f = -2\ln F$) and $(\nabla^2 f)_0$ is the traceless part of horizontal Hessian of f.

Several important properties of the C-operator were found in [Ivanov et al. 2013], most notable of which is the fact that the C-operator is nonnegative for $n > 1$. In dimension seven, $n = 1$, the condition of nonnegativity of the C-operator is nontrivial. However, [Ivanov et al. 2013] showed that on a 7-dimensional compact QC-Einstein manifold with positive QC-scalar curvature the P-function of an eigenfunction of the sub-Laplacian is nonnegative. In particular, this property holds on any 3-Sasakian manifold, see [Ivanov et al. 2014a, Corollary 4.13]. Clearly, these facts together with Proposition 1.1 imply the following theorem:

Theorem 1.2. *Let M be a compact QC manifold of dimension $4n+3$ of nonnegative Ricci type tensor $\mathcal{L}(X, X) \geq 0$. In the case $n = 1$ assume, in addition, that the C-operator is nonnegative. If $u = e^{-f}$ is a positive solution of the heat equation* (1-1), *then the energy $\mathcal{E}(t)$ is monotone decreasing* (*i.e.*, *nonincreasing*).

The proof of Proposition 1.1 follows one of L. Ni's arguments [2004] in the Riemannian case, and thus it relies on Bochner's formula. More precisely, after Ni's initial step, in order to handle the extra terms in Bochner's formula, we will follow the presentation of [Ivanov and Vassilev 2015] where this was done for the QC Lichnerowicz-type lower eigenvalue bound under positive Ricci-type tensor; see [Ivanov et al. 2013; Ivanov et al. 2014b] for the original result. In the QC case, similar to the CR case, the Bochner formula has additional hard to control terms, which include the P-function of f. In our case, since the integrals are with respect to the measure $u\,\mathrm{Vol}_\eta$, rather than Vol_η as in the Lichnerowicz-type estimate, some new estimates are needed. The key is the following proposition which can be considered as an estimates from above of the integral of the P-function of f with respect to the measure $u\,\mathrm{Vol}_\eta$ when the C-operator is nonnegative.

Proposition 1.3. *Let (M, η) be a compact QC manifold of dimension $4n + 3$. If $u = e^{-f}$ is a positive solution to heat equation* (1-1), *then with $f = -2\ln F$ we have the identity*

$$\int_M P_f(\nabla f) u \,\mathrm{Vol}_\eta = \frac{1}{4}\int_M |\nabla f|^4 u \,\mathrm{Vol}_\eta + 4\int_M P_F(\nabla F)\,\mathrm{Vol}_\eta\,. \tag{1-6}$$

In the last section of the paper we apply the same method in the case of a strictly pseudoconvex pseudo-Hermitian manifold and prove the following proposition:

Proposition 1.4. *Let M be a compact strictly pseudoconvex pseudo-Hermitian CR manifold of dimension $2n+1$. If $u = e^{-f}$ is a positive solution to the heat equation* (1-1), *then we have*

$$\frac{n+1}{2n}\mathcal{E}'(t) = -\int_M \Big[|(\nabla^2 f)_0|^2 + \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f) + \frac{1}{8n}|\nabla f|^4\Big] u \operatorname{Vol}_\eta - \frac{6}{n}\int_M F\mathcal{C}(F)\operatorname{Vol}_\eta,$$

where $u = F^2$, $(\nabla^2 f)_0$ is the traceless part of horizontal Hessian of f and $\mathcal{C}$ is the CR-Paneitz operator of M.

We refer to Section 3 for the relevant notation and definitions. As a consequence of Proposition 1.4 we recover the monotonicity of the entropy energy shown previously in [Chang and Wu 2010]. We note that one of the motivations to consider the problem was the application of the CR version of the monotonicity of the entropy-like energy (see their Lemma 3.3) in obtaining (nonoptimal) estimate on the bottom of the L^2 spectrum of the CR sub-Laplacian. However, the proof of their Corollary 1.9 and Section 6 is not fully justified since their Lemma 3.3 is proved for a compact manifold. It should be noted that a proof of S.-Y. Cheng's type (even nonoptimal) estimate in a sub-Riemannian setting, such as CR or QC manifold, is an interesting problem in particular because of the lack of general comparison theorems.

We conclude by mentioning another proof of the monotonicity of the energy in the recent preprint [Ivanov and Petkov 2016], which was the result of a past collaborative work with Ivanov and Petkov. Remarkably, [Chang and Wu 2010] is also not acknowledged in [Ivanov and Petkov 2016] despite the line for line substantial overlap of their Section 3 with Chang and Wu's [2010, Lemma 3.3] proof. In this paper we give an independent direct approach to the problem.

2. Proofs of the propositions

Some preliminaries. Throughout this section M will be a QC manifold of dimension $4n+3$, [Biquard 1999], with horizontal space H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, and Biquard connection ∇ with torsion T. Below we record some of the properties needed for this paper, see also [Biquard 2000] and [Ivanov and Vassilev 2011] for a more expanded exposition.

The $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ structure on H is fixed by a positive definite symmetric tensor g and a rank-three bundle $\mathbb{Q}$ of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions and also the conditions

$$g(I_s\cdot, I_s\cdot) = g(\cdot,\cdot) \quad \text{and} \quad 2g(I_sX, Y) = d\eta_s(X, Y).$$

Associated with the Biquard connection is the vertical space V, which is complementary to H in TM. In the case $n=1$ we shall make the usual assumption of existence of Reeb vector fields ξ_1, ξ_2, ξ_3, so that the connection is defined following D. Duchemin [2006]. The fundamental 2-forms ω_s of the fixed QC structure will be denoted by ω_s,

$$2\omega_{s|H} = d\eta_{s|H}, \quad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

In order to give some idea of the involved quantities we list a few more essential for us details. Recall that ∇ preserves the decomposition $H \oplus V$ and the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ structure on H,

$$\nabla g = 0, \quad \nabla \Gamma(\mathbb{Q}) \subset \Gamma(\mathbb{Q})$$

and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$. Furthermore, for a vertical field $\xi \in V$, the endomorphism $T_\xi \equiv T(\xi, \cdot)_{|H}$ of H belongs to the space $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$ hence $T(\xi, X, Y) = g(T_\xi X, Y)$ is a well defined tensor field. The two $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant trace-free symmetric 2-tensors

$$T^0(X, Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y) \quad \text{and} \quad U(X, Y) = g(uX, Y)$$

on H, introduced in [Ivanov et al. 2014a], satisfy

$$\begin{aligned} T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) &= 0, \\ U(X,Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) &= U(I_3X, I_3Y). \end{aligned} \tag{2-1}$$

Note that when $n = 1$, the tensor U vanishes. The tensors T^0 and U determine completely the torsion endomorphism due to the identity [Ivanov and Vassilev 2010, Proposition 2.3]

$$4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y),$$

which in view of (2-1) implies

$$\sum_{s=1}^{3} T(\xi_s, I_s X, Y) = T^0(X, Y) - 3U(X, Y). \tag{2-2}$$

The curvature of the Biquard connection is $R = [\nabla, \nabla] - \nabla_{[\,,\,]}$ with QC-Ricci tensor and *normalized* QC-scalar curvature, defined by respectively by

$$\mathrm{Ric}(X, Y) = \sum_{a=1}^{4n} g(R(e_a, X)Y, e_a), \qquad 8n(n+2)S = \sum_{a=1}^{4n} \mathrm{Ric}(e_a, e_a).$$

According to [Biquard 2000] the Ricci tensor restricted to H is a symmetric tensor. Remarkably, the torsion tensor determines the QC-Ricci tensor of the Biquard connection on M in view of the formula, [Ivanov et al. 2014a],

$$\mathrm{Ric}(X,Y) = (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + \frac{S}{4n}g(X,Y). \tag{2-3}$$

Finally, Vol_η will denote the volume form, see [Ivanov et al. 2014a, Chapter 8],

$$\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \Omega^n, \tag{2-4}$$

where $\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$ is the fundamental 4-form. We note the integration by parts formula

$$\int_M (\nabla^* \sigma)\, \mathrm{Vol}_\eta = 0, \tag{2-5}$$

where the (horizontal) divergence of a horizontal vector field $\sigma \in \Lambda^1(H)$ is given by $\nabla^*\sigma = -\operatorname{tr}|_H \nabla\sigma = -\nabla\sigma(e_a, e_a)$ for an orthonormal frame $\{e_a\}_{a=1}^{4n}$ of the horizontal space.

Proof of Proposition 1.3. We start with a formula for the change of the dependent function in the P-function of f. To this effect, with $f = f(F)$, a short calculation shows the next identity

$$\begin{aligned}\nabla^3 f(Z,X,Y) = f'\nabla^3 F(Z,X,Y) + f'''dF(Z)dF(X)dF(Y) + f''\nabla^2 F(Z,X)dF(Y)\\ + f''\nabla^2 F(Z,Y)dF(X) + f''\nabla^2 F(X,Y)dF(Z).\end{aligned}$$

Recalling definition (1-5) we obtain

$$\begin{aligned}P_f(Z) = f'P_F(Z) + f'''|\nabla F|^2 dF(Z) + 2f''^2 F(Z,\nabla F)\\ + f''(\Delta F)dF(Z) + f''\sum_{s=1}^{3} g(\nabla^2 F, \omega_s)dF(I_s Z),\end{aligned} \tag{2-6}$$

which implies the identity

$$P_f(\nabla f) = f'^2 P_F(\nabla F) + f'f'''|\nabla F|^4 + 2f'f''\nabla^2 F(\nabla F,\nabla F) + f'f''|\nabla F|^2 \Delta F. \tag{2-7}$$

In our case, since we are interested in expressing the integral of $uP_f(\nabla f) = e^{-f}P_f(\nabla f)$ in terms of the integral of a P -function of some function, equation (2-7) leads to the ordinary differential equation $u(-u'/u)^2 = constant$. Therefore, we let $u = F^2$ and find

$$uP_f(\nabla f) = 4P_F(\nabla F) + 8F^{-2}|\nabla F|^4 - 8F^{-1}\nabla^2 F(\nabla F,\nabla F) - 4F^{-1}|\nabla F|^2 \Delta F. \tag{2-8}$$

Now, the last three terms will be expressed back in the variable f which gives

$$uP_f(\nabla f) = 4P_F(\nabla F) + \big[-\tfrac14|\nabla f|^4 + \tfrac12|\nabla f|^2\Delta f + \nabla^2 f(\nabla f,\nabla f)\big]u. \tag{2-9}$$

At this point, we integrate the above identity and then apply the (integration by parts) divergence formula (2-5) in order to show

$$\int_M \nabla^2 f(\nabla f, \nabla f) u \,\mathrm{Vol}_\eta = \frac{1}{2}\int_M \big[|\nabla f|^4 - |\nabla f|^2 \Delta f\big] u \,\mathrm{Vol}_\eta,$$

which leads to (1-6). The proof of Proposition 1.3 is complete. □

Proof of Proposition 1.1. The initial steps are identical to the Riemannian case [Ni 2004] thus we skip the detailed computations and only sketch the common steps. Let $w = 2\Delta f - |\nabla f|^2$. Using the heat equation, exactly as in the Riemannian case, we have the identities

$$\partial_t f = \Delta f - |\nabla f|^2, \quad u\Delta f = u|\nabla f|^2 - \Delta u, \quad \text{and} \quad \Delta u = (|\nabla f|^2 - \Delta f)u, \tag{2-10}$$

which imply

$$\mathcal{E}'(t) = \int_M (\partial_t - \Delta)(uw) \,\mathrm{Vol}_\eta \tag{2-11}$$

and also

$$(\partial_t - \Delta)(uw) = [2g(\nabla(\Delta f), \nabla f) - \Delta|\nabla f|^2]u. \tag{2-12}$$

Next, we apply the QC Bochner formula [Ivanov et al. 2013; Ivanov et al. 2014b]

$$\begin{aligned}\tfrac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 + g(\nabla(\triangle f), \nabla f) + 2(n+2)S|\nabla f|^2 \\ + 2(n+2)T^0(\nabla f, \nabla f) + 4(n+1)U(\nabla f, \nabla f) + 4R_f(\nabla f),\end{aligned}$$

where

$$R_f(Z) = \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s Z).$$

Therefore,

$$\begin{aligned}\tfrac{1}{2}(\partial_t - \Delta)(uw) = [-|\nabla^2 f|^2 - 2(n+2)S|\nabla f|^2 - 2(n+2)T^0(\nabla f, \nabla f) \\ - 4(n+1)U(\nabla f, \nabla f) - 4R_f(\nabla f)]u.\end{aligned} \tag{2-13}$$

The next step is the computation of $\int_M R_f(\nabla f) u \,\mathrm{Vol}_\eta$ in two ways as was done in [Ivanov et al. 2013; Ivanov et al. 2014b] for the Lichnerowicz-type first eigenvalue lower bound but integrating with respect to Vol_η rather than $u\,\mathrm{Vol}_\eta$ as we need to do here. For ease of reading we will follow closely [Ivanov and Vassilev 2015, Section 8.1.1] but notice the opposite convention of the sub-Laplacian in their Section 8.1.1. First with the help of the P-function, working similarly to [Ivanov et al. 2013, Lemma 3.2], where the integration was with respect to Vol_η, we have

$$\begin{aligned}(2\text{-}14)\quad \int_M R_f(\nabla f)u\,\mathrm{Vol}_\eta = \int_M\Big[-\frac{1}{4n}P_f(\nabla f) - \frac{1}{4n}(\triangle f)^2 - S|\nabla f|^2 \\ + \frac{n+1}{n-1}U(\nabla f,\nabla f)\Big]u\,\mathrm{Vol}_\eta + \frac{1}{4n}\int_M |\nabla f|^2(\Delta f)u\,\mathrm{Vol}_\eta,\end{aligned}$$

with the convention that in the case $n=1$ the formula is understood by formally dropping the term involving (the vanishing) tensor U. Notice the appearance of a "new" term in the last integral in comparison to the analogous formula in [Ivanov and Vassilev 2015, Section 8.1.1, p. 310]. Indeed, taking into account the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ invariance of $R_f(\nabla f)$ and Ricci's identities we have, cf., [Ivanov et al. 2013, Lemma 3.2],

$$R_f(X) = -\frac{1}{4n}\sum_{s=1}^{3}\sum_{a=1}^{4n}\nabla^3 f(I_sX, e_a, I_se_a) + [T^0(X,\nabla f) - 3U(X,\nabla f)]$$

hence (1-5) gives

$$uR_f(\nabla f) = \Big[-\frac{1}{4n}P_n(\nabla f) - S|\nabla f|^2 + \frac{n+1}{n-1}U(\nabla f,\nabla f)\Big]u + \frac{1}{4n}\sum_{a=1}^{4n}\nabla^3 f(\nabla f, e_a, e_a)u.$$

An integration by parts shows the validity of (2-14).

On the other hand, we have

$$\begin{aligned}(2\text{-}15)\quad &\int_M R_f(\nabla f)u\,\mathrm{Vol}_\eta \\ &= -\int_M\Big[\frac{1}{4n}\sum_{s=1}^{3}g(\nabla^2 f,\omega_s)^2 + T^0(\nabla f,\nabla f) - 3U(\nabla f,\nabla f)\Big]u\,\mathrm{Vol}_\eta,\end{aligned}$$

which other than using different volume forms is identical to the second formula in [Ivanov and Vassilev 2015, Section 8.1.1, p. 310]. Indeed, following [Ivanov et al. 2014b, Lemma 3.4], using Ricci's identity

$$\nabla^2 f(X,\xi_s) - \nabla^2 f(\xi_s, X) = T(\xi_s, X, \nabla f)$$

and (2-2), we have

$$R_f(\nabla f) = \Big(\sum_{s=1}^{3}\nabla^2 f(I_s\nabla f,\xi_s)\Big) - [T^0(\nabla f,\nabla f) - 3U(\nabla f,\nabla f)]$$

An integration by parts gives (2-15), noting that $\sum_{s=1}^{3} df(\xi_s)\, df(I_s\nabla f) = 0$ and taking into account that by Ricci's identity

$$\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = -2\sum_{s=1}^{3} \omega_s(X, Y)\, df(\xi_s)$$

we have $g(\nabla^2 f, \omega_s) = \sum_{a=1}^{4n} \nabla^2 f(e_a, I_s e_a) = -4n\, df(\xi_s)$.

Now, working as in [Ivanov and Vassilev 2015, Section 8.1.1, p. 310], we subtract (2-15) and three times formula (2-14) from (2-13) which brings us to the identity

$$\text{(2-16)}\quad \frac{1}{2}\mathcal{E}'(t) = \int_M \Big[-|(\nabla^2 f)_0|^2 - \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f)\Big] u \operatorname{Vol}_\eta + \frac{1}{4n}\int_M [3P_f(\nabla f) + 2(\Delta f)^2 - 3|\nabla f|^2 \Delta f] u \operatorname{Vol}_\eta,$$

where $|(\nabla^2 f)_0|^2$ is the square of the norm of the traceless part of the horizontal Hessian

$$|(\nabla^2 f)_0|^2 = |\nabla^2 f|^2 - \frac{1}{4n}\Big[(\triangle f)^2 + \sum_{s=1}^{3}[g(\nabla^2 f, \omega_s)]^2\Big].$$

Next, we consider $\int_M [2(\Delta f)^2 - 3|\nabla f|^2 \Delta f] u \operatorname{Vol}_\eta$. Using the heat equation we have the relation, identical to the Riemannian case, see (2-10),

$$\mathcal{E}'(t) = \frac{d}{dt}\int_M w\,\Delta u \operatorname{Vol}_\eta = \int_M (-2(\Delta f)^2 + 3|\nabla f|^2 \Delta f - |\nabla f|^4) u \operatorname{Vol}_\eta,$$

hence

$$\text{(2-17)}\quad \int_M (2(\Delta f)^2 - 3|\nabla f|^2 \Delta f) u \operatorname{Vol}_\eta = -\frac{d}{dt}\mathcal{E}(t) - \int_M |\nabla f|^4 u \operatorname{Vol}_\eta .$$

A substitution of the above formula in (2-16) gives

$$\frac{2n+1}{4n}\frac{d}{dt}\mathcal{E}(t) = \int_M \Big[-|(\nabla^2 f)_0|^2 - \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f)\Big] u \operatorname{Vol}_\eta + \frac{1}{4n}\int_M [3P_f(\nabla f) - |\nabla f|^4] u \operatorname{Vol}_\eta .$$

Finally, we invoke Proposition 1.3 in order to complete the proof. □

3. The CR case

In this section, following the method we employed in the QC case, we prove the monotonicity formula in the CR case stated in Proposition 1.4. This implies the monotonicity of the entropy-like energy which was proved earlier in [Chang and Wu 2010].

Throughout the section M will be a $(2n+1)$-dimensional strictly pseudoconvex (integrable) CR manifold with a fixed pseudo-Hermitian structure defined by a contact form η and a complex structure J on the horizontal space $H = \operatorname{Ker}\eta$. The fundamental 2-form is defined by $\omega = \frac{1}{2}d\eta$ and the Webster metric is $g(X, Y) = -\omega(JX, Y)$ which is extended to a Riemannian metric on M by declaring that the Reeb vector field associated to η is of length one and orthonormal to the horizontal space. We shall denote by ∇ the associated Tanaka–Webster connection [Tanaka 1975] and [Webster 1975; 1978], while $\triangle u = \operatorname{tr}^g(\nabla^2 u)$ will be the negative sub-Laplacian with the trace taken with respect to an orthonormal basis of the horizontal $2n$-dimensional space. Finally, we define the Ricci-type tensor

$$\mathcal{L}(X, Y) = \rho(JX, Y) + 2nA(JX, Y), \tag{3-1}$$

recalling that on a CR manifold we have

$$\operatorname{Ric}(X, Y) = \rho(JX, Y) + 2(n-1)A(JX, Y), \tag{3-2}$$

where ρ is the $(1, 1)$-part of the pseudo-Hermitian Ricci tensor (the Webster Ricci tensor) while the $(2, 0) + (0, 2)$-part is the Webster torsion A; see [Ivanov and Vassilev 2011, Chapter 7] for the expressions in real coordinates of these known formulas [Webster 1975; 1978]; see also [Dragomir and Tomassini 2006].

With the above convention in place, as in [Chang and Wu 2010], for a positive solution of (1-1) we consider the entropy (1-2) and energy (1-3), where

$$\operatorname{Vol}_\eta = \eta \wedge (d\eta)^{2n}. \tag{3-3}$$

For a function f we define the one-form,

$$P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f) \tag{3-4}$$

so that the fourth order CR-Paneitz operator is given by

$$\begin{aligned} C(f) = -\nabla^* P &= (\nabla_{e_a} P)(e_a) \\ &= \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b) \\ &\qquad - 4n\nabla^* A(J\nabla f) - 4ng(\nabla^2 f, JA). \end{aligned} \tag{3-5}$$

By [Graham and Lee 1988], when $n > 1$ a function $f \in \mathcal{C}^3(M)$ satisfies the equation $Cf = 0$ if and only if f is CR-pluriharmonic. Furthermore, the CR-Paneitz operator is nonnegative,

$$\int_M f \cdot Cf \operatorname{Vol}_\eta = -\int_M P_f(\nabla f) \operatorname{Vol}_\eta \geq 0.$$

On the other hand, in the three dimensional case the positivity condition is a CR invariant since it is independent of the choice of the contact form by the conformal invariance of C proven in [Hirachi 1993]. The nonnegativity of the

CR-Paneitz operator is relevant in the embedding problem for a three dimensional strictly pseudoconvex CR manifold. As shown in [Chanillo et al. 2012], if the pseudo-Hermitian scalar curvature of M is positive and C is nonnegative, then M is embeddable in some $\mathbb{C}^n$

We turn to the proof of Proposition 1.4. Taking into account (2-12) and the CR Bochner formula [Greenleaf 1985],

$$\tfrac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 + g(\nabla(\triangle f),\nabla f) + \mathrm{Ric}(\nabla f,\nabla f) + 2A(J\nabla f,\nabla f) + 4R_f(\nabla f), \tag{3-6}$$

where $R_f(Z) = \nabla\, df(\xi, JZ)$, see [Ivanov and Vassilev 2015, Section 7.1] and references therein but note the opposite sign of the sub-Laplacian, we obtain the next identity:

$$\tfrac{1}{2}(\partial_t - \Delta)(uw) = [-|\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f) - 2A(\nabla f, \nabla\nabla f) - 4R_f(\nabla f)]u. \tag{3-7}$$

Since (2-11) still holds, working as in the QC case we compute $\int_M R_F(\nabla f)u\,\mathrm{Vol}_\eta$ in two ways [Greenleaf 1985, Lemma 4] and [Ivanov and Vassilev 2012, Lemma 8.7] following the exposition [Ivanov and Vassilev 2015].

From Ricci's identity $\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = -2\omega(X, Y)\, df(\xi)$, it follows that $df(\xi) = -\frac{1}{2n} g(\nabla^2 f, \omega)$. Hence

$$\nabla^2 f(JZ, \xi) = -\frac{1}{2n}\sum_{b=1}^{2n} \nabla^3 f(JZ, e_b, Je_b),$$

where $\{e_b\}_{b=1}^{2n}$ is an orthonormal basis of the horizontal space. Applying Ricci's identity $\nabla^2 f(X, \xi) - \nabla^2 f(\xi, X) = A(X, \nabla f)$, it follows that

$$R_f(Z) = \nabla^2 f(\xi, JZ) = -\frac{1}{2n}\sum_{b=1}^{2n} \nabla^3 f(JZ, e_b, Je_b) - A(JZ, \nabla f). \tag{3-8}$$

Taking into account (3-4), the last formula gives

$$R_f(Z) = -\frac{1}{2n} P_f(Z) + A(JZ, \nabla f) + \frac{1}{2n}\sum_{b=1}^{2n} \nabla^3 f(Z, e_b, e_b).$$

Now, an integration by parts shows the next identity

$$\begin{aligned} &\int_M R_f(\nabla f) u \mathrm{Vol}_\eta \\ &\quad = \int_M \Big[-\frac{1}{2n} P_f(\nabla f) + A(J\nabla f,\nabla f) - \frac{1}{2n}(\Delta f)^2 + \frac{1}{2n}|\nabla f|^2(\Delta f)\Big] u \mathrm{Vol}_\eta. \end{aligned} \tag{3-9}$$

On the other hand, using again (3-8) but now integrating and then using integration by parts, we have

$$\int_M R_f(\nabla f)u\,\mathrm{Vol}_\eta = \int_M\Big[-\frac{1}{2n}g(\nabla^2 f,\omega)^2 - A(J\nabla f,\nabla f)\Big]u\,\mathrm{Vol}_\eta\,. \tag{3-10}$$

At this point, exactly as in the QC case, we subtract (3-10) and three times formula (3-9) from (3-7), which gives

$$\mathcal{E}'(t) = -\int_M\big[|(\nabla^2 f)_0|^2+\mathcal{L}(\nabla f,\nabla f)\big]u\,\mathrm{Vol}_\eta+\frac{1}{2n}\int_M[3P_f(\nabla f)+2(\Delta f)^2-3|\nabla f|^2\Delta f]u\,\mathrm{Vol}_\eta,$$

where $|(\nabla^2 f)_0|^2$ is the square of the norm of the traceless part of the horizontal Hessian

$$|(\nabla^2 f)_0|^2 = |\nabla^2 f|^2 - \frac{1}{2n}[(\triangle f)^2 + g(\nabla^2 f,\omega)^2].$$

Taking into account that the formulas in Proposition 1.3 and (2-17) hold unchanged, we complete the proof. □

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