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We compute certain Fourier coefficients of Eisenstein series on the split simple exceptional group G_2, and the result is a product of zeta functions and a finite product of local integrals. The method is via exceptional theta correspondence for $G_2 \times \mathrm{PGL}_3$.

1. Introduction

Let F be a number field with adele ring $\mathbb{A}$. Consider the split simple exceptional group G_2 over F. Let $P = M \cdot N$ be the maximal Heisenberg parabolic subgroup of G_2 associated to the short simple root. Then the Levi subgroup $M \cong \mathrm{GL}_2$ and the unipotent radical N is five-dimensional. For $s \in \mathbb{C}$, let $I_P(s) = \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})} |\det|^{s+3/2}$ (unnormalized induction). For $\Phi_s \in I_P(s)$, consider the Eisenstein series on $G_2(\mathbb{A})$ defined for $\mathrm{Re}(s) \gg 0$ by

$$E(\Phi_s, g) = \sum_{\gamma \in P(F)\backslash G_2(F)} \Phi_s(\gamma g), \quad \text{for all } g \in G_2(\mathbb{A}).$$

If Φ_s is holomorphic, then the Eisenstein series $E(\Phi_s, g)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to $\mathbb{C}$.

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Consider the Fourier coefficient of the Eisenstein series $E(\Phi_s, g)$ with respect to a character $\chi : N(F) \setminus N(\mathbb{A}) \to \mathbb{C}^\times$, which is given by

$$E_\chi(\Phi_s, g) = \int_{N(F)\setminus N(\mathbb{A})} E(\Phi_s, ng)\overline{\chi(n)}\, dn.$$

These Fourier coefficients are interesting objects of study. For example, Dihua Jiang and Stephen Rallis [1997] showed these Fourier coefficients are essentially quotients of Dedekind zeta functions of number fields, under the assumption that the base field F contains the third root of unity. Our goal is to obtain similar results without this assumption. In the following, we give more detailed description.

Write an element in N as $n(x, y, z, u, v)$, where $x, y, z, u, v \in F$. The center Z of N is given by $Z = \{n(0, 0, z, 0, 0) : z \in F\}$. In this paper, we restrict our attention to the following type of characters of N. Fix a nontrivial character $\psi : F \setminus \mathbb{A} \to \mathbb{C}^\times$. Let $b, c \in F$ be such that the cubic polynomial $x^3 + bx - c$ is irreducible over F. Let $\sigma = (1, 0, b, c) \in F^4$, and define a character $\psi_\sigma : N(F) \setminus N(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\psi_\sigma(n(x, y, z, u, v)) = \psi(x + bu + cv).$$

Note that ψ_σ is trivial on $Z(\mathbb{A})$. See [Jiang and Rallis 1997, §2.4] and [Wright 1985, §2] for more details.

Jiang and Rallis [1997] studied the Fourier coefficients of $E(\Phi_s, 1)$ with respect to the character ψ_σ for a standard decomposable section Φ_s. They first showed that if $\Phi_s = \otimes\Phi_{s,v}$ is decomposable, then $E_{\psi_\sigma}(\Phi_s, 1)$ is Eulerian:

$$E_{\psi_\sigma}(\Phi_s, 1) = \prod_v E_{\psi_\sigma,v}(\Phi_{s,v}, 1),$$

where

$$E_{\psi_\sigma,v}(\Phi_{s,v}, 1) = \int_{N(F_v)} \Phi_{s,v}(wn)\psi_{\sigma,v}(n)\, dn,$$

and w is an appropriate Weyl element.

Then Jiang and Rallis evaluated the integrals $E_{\psi_\sigma,v}(\Phi_{s,v}, 1)$ in the unramified case directly, with the help of [Igusa 1988, Lemma 6]. Let E be a cubic field extension of F generated by one of the roots of the polynomial $x^3 + bx - c$. Assuming that F contains the third root of unity, they obtained, after a rather long and complicated computation, that in the unramified case,

$$E_{\psi_\sigma,v}(\Phi_{s,v}, 1) = \frac{\zeta_{E_v}\left(s + \frac{1}{2}\right)}{\zeta_v\left(s + \frac{1}{2}\right)\zeta_v\left(s + \frac{3}{2}\right)\zeta_v(2s + 1)\zeta_v\left(3s + \frac{3}{2}\right)},$$

where $\zeta_v(s)$ and $\zeta_{E_v}(s)$ are local zeta factors (see [Jiang and Rallis 1997, §5]). Therefore, for a pure tensor $\Phi = \otimes\Phi_v$, the Fourier coefficient $E_{\psi_\sigma}(\Phi_s, 1)$ is of the following form:

$$E_{\psi_\sigma}(\Phi_s, 1) = \frac{\zeta_E\bigl(s+\frac{1}{2}\bigr)}{\zeta_F\bigl(s+\frac{1}{2}\bigr)\zeta_F\bigl(s+\frac{3}{2}\bigr)\zeta_F(2s+1)\zeta_F\bigl(3s+\frac{3}{2}\bigr)} \prod_v E^*_{\psi_\sigma,v}(\Phi_{s,v}, 1),$$

where $\zeta_F(s)$ (resp. $\zeta_E(s)$) is the complete zeta function of F (resp. of E), and the local factors $E^*_{\psi_\sigma,v}(\Phi_{s,v}, 1)$ is equal to 1 for almost all v. See [Jiang and Rallis 1997, Theorem 2 (4)].

The assumption that the base field F contains the third root of unity seems to be nonessential, as observed by Jiang and Rallis [1997]. Note that Gan, Gross and Savin calculated the Fourier coefficients for Eisenstein series on $G_2(\mathbb{Z})$ in [Gan et al. 2002, §9], assuming an extension of Jiang and Rallis's local result.

The purpose of this paper is to try to remove this assumption, and the method of proof, which was suggested to the author by Wee Teck Gan, is to reduce the computation to local integrals over PGL_3 via exceptional theta correspondence for the dual pair (G_2, PGL_3). The main result of this paper is as follows:

Theorem 1.1. *Keep the notation above. If $\Phi_s = \otimes\Phi_{s,v}$ is a pure tensor, then the Fourier coefficient $E_{\psi_\sigma}(\Phi_s, 1)$ is of the following form*:

$$E_{\psi_\sigma}(\Phi_s, 1) = \frac{\zeta_E\bigl(s+\frac{1}{2}\bigr)}{\zeta_F\bigl(s+\frac{1}{2}\bigr)\zeta_F\bigl(s+\frac{3}{2}\bigr)\zeta_F(2s+1)\zeta_F\bigl(3s+\frac{3}{2}\bigr)} \cdot J_\infty(\Phi_s) \cdot \prod_{v\nmid\infty} J_v(\Phi_s),$$

where $J_\infty(\Phi_s)$ and each of the $J_v(\Phi_s)$ are meromorphic functions of s, and $J_v(\Phi_s)$ is equal to 1 for almost all finite v.

Let us give more details on the contents of this paper. Let H be the split simple adjoint group of type E_6 over F. Then $G_2 \times \mathrm{PGL}_3$ forms a dual pair in H. The group $H(\mathbb{A})$ has a distinguished representation Π called the minimal representation, which has an embedding $\theta : \Pi \to \mathcal{A}_2(H)$ into the space of square integrable automorphic forms on H (see [Ginzburg et al. 1997]). For $f \in \Pi$, as in the classical case [Weil 1965; Kudla and Rallis 1988a; 1988b], one may define the theta integral on $G_2(\mathbb{A})$ by

$$I(f)(g) = \int_{\mathrm{PGL}_3(F)\backslash \mathrm{PGL}_3(\mathbb{A})} \theta(f)(gh)\, dh, \quad \text{for all } g \in G_2(\mathbb{A}),$$

but this integral may not converge. Analogous to [Kudla and Rallis 1994; Ichino 2001; 2004], W.T. Gan [2011] found a regularization of the theta integral. For $f \in \Pi$, the regularized theta integral is defined by

$$I^{\mathrm{reg}}(f, s)(g) = \int_{\mathrm{PGL}_3(F)\backslash \mathrm{PGL}_3(\mathbb{A})} \theta(z \cdot f)(gh) E(h, s)\, dh,$$

where z is some element in the Bernstein center of PGL_3 at some finite place of F, and $E(h,s)$ is a spherical Eisenstein series on PGL_3. Note that $\theta(z\cdot f)$ is rapidly decreasing on $\mathrm{PGL}_3(F)\backslash\mathrm{PGL}_3(\mathbb{A})$, so the integral defining $I^{\mathrm{reg}}(f,s)(g)$ is convergent.

As in the classical case [Kudla and Rallis 1994], Gan showed that regularized theta integral $I^{\mathrm{reg}}(f,s)(g)$ is essentially an Eisenstein series $E(\Phi(f,s),g)$ associated with a meromorphic section $\Phi(f,s)$ in $I_P(s)$. Thus the Fourier coefficients of the regularized theta integral contains information of the Fourier coefficients of Eisenstein series.

The contents of this paper are as follows. In Section 2 we give notation and preliminaries. In Section 3, we study the Fourier coefficient of the regularized theta integral $I^{\mathrm{reg}}(f,s)(1)$ with respect to ψ_σ, which turns out to be equal to an integral over $\mathrm{PGL}_3(\mathbb{A})$ and can be decomposed as the product of an archimedean part and a finite part. In Section 4, we compute the local unramified factors in the finite part with the help of the higher dimensional Hensel's lemma. In Section 5, we give the proof of Theorem 1.1.

2. Notation and preliminaries

Groups and principal series. In this paper (except in Section 4), F is a number field with adele ring $\mathbb{A}$, b and c are two elements in F such that the cubic polynomial x^3+bx-c is irreducible over F, and E is a cubic field extension of F generated by one of the roots of the polynomial x^3+bx-c. For a place v of F, let F_v be the corresponding local field. If v is a finite place of F, let $\mathcal{O}_v$ be the ring of integers in F_v, ϖ_v a uniformizer of $\mathcal{O}_v$, $k_v=\mathcal{O}_v/\varpi_v\mathcal{O}_v$ the residue field, and q_v the cardinality of k_v.

Fix a nontrivial character $\psi: F\backslash\mathbb{A}\to\mathbb{C}^\times$. For each place v of F, take the Haar measure dx_v on F_v which is self-dual with respect to ψ_v. For a finite place v, if ψ_v is unramified (i.e., ψ_v is trivial on $\mathcal{O}_v$ but nontrivial on $\varpi_v^{-1}\mathcal{O}_v$), then $\mathrm{Vol}(\mathcal{O}_v)=1$.

For an algebraic group G over F, denote $[G]=G(F)\backslash G(\mathbb{A})$.

Let G be an algebraic group over F, and let $P=NM$ be a parabolic F-subgroup of G with Levi subgroup M and unipotent radical N. Let $K=\prod K_v$ be a maximal compact subgroup of $G(\mathbb{A})$, where each K_v is a maximal compact subgroup of $G(F_v)$ such that $G(F_v)=P(F_v)K_v$ for every place v of F. Then there is an Iwasawa decomposition $G(\mathbb{A})=P(\mathbb{A})K$. Let δ_P be the modulus character of P. For $s\in\mathbb{C}$ and a character $\chi: M(F)\backslash M(\mathbb{A})\to\mathbb{C}^\times$, the *principal series representation* $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\chi\cdot\delta_P^s$ is the space of all smooth functions $\Phi_s: G(\mathbb{A})\to\mathbb{C}$ such that $\Phi_s(nmg)=\chi(m)\delta_P(m)^s\Phi_s(g)$ for all $n\in N(\mathbb{A})$, $m\in M(\mathbb{A})$ and $g\in G(\mathbb{A})$. An element in $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\chi\cdot\delta_P^s$ is called a *section*. Because of the Iwasawa decomposition, a section is determined by its restriction to K. A section Φ_s is called *holomorphic* (resp. *meromorphic*) if $s\mapsto\Phi_s(g)$ is a holomorphic (resp. meromorphic) function in s for every $g\in G(\mathbb{A})$. A *standard* section is a holomorphic section whose restriction to K is independent of s. For a place v of F, $s \in \mathbb{C}$ and a character $\chi_v : M(F_v) \to \mathbb{C}^\times$, one can define the local *principal series representation* $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \chi_v \delta_P^s$ similarly. For a place v of F, the *spherical vector* $\Phi_{s,v}^0$ in $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \chi_v \delta_P^s$ is the one whose restriction to K_v is equal to 1. Then $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi \cdot \delta_P^s$ is the restricted tensor product of $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \chi_v \delta_P^s$ with respect to $\Phi_{s,v}^0$, as defined in [Bump 1997, pp. 300–301]. A section is called *decomposable* if it is of the form $\Phi_s = \bigotimes \Phi_{s,v}$ with each $\Phi_{s,v} \in I_{P,v}(s)$. A *pure tensor* is a decomposable section $\Phi_s = \bigotimes \Phi_{s,v}$ where almost all $\Phi_{s,v}$ are spherical vectors.

Minimal representations. Let H be the split simple adjoint group of type E_6 over F, which is explicitly described in [Magaard and Savin 1997, §3] . We also simply write $H = E_6$. Then $H(\mathbb{A})$ has a distinguished representation $\Pi = \bigotimes_v \Pi_v$ called the *minimal representation*, where each Π_v is the local minimal representation of $H(F_v)$. Each Π_v is an irreducible spherical representation of $H(F_v)$ with spherical vector f_v^0, and Π is the restricted tensor product of Π_v with respect to f_v^0. See [Kazhdan and Savin 1990; Ginzburg et al. 1997; Gan and Savin 2005] for more details.

Let $P_H = N_H M_H$ be the Heisenberg parabolic subgroup of H, where M_H is the Levi subgroup and N_H is the unipotent radical, so that N_H is a Heisenberg group with one-dimensional center Z_H. For a finite place v of F, let Ω_v be the minimal nontrivial orbit of $M_H(F_v)$ on the set of unitary characters of $N_H(F_v)$, which can be noncanonically identified with $\bar{V}_H(F_v) = \bar{N}_H(F_v)/\bar{Z}_H(F_v)$, where $\bar{N}_H$ is the opposite of N_H and $\bar{Z}_H$ is the center of $\bar{N}_H$. Then there is a $P_H(F_v)$-equivariant embedding

$$i_v : (\Pi_v)_{Z_H} \hookrightarrow C^\infty(\Omega_v),$$

where $(\Pi_v)_{Z_H} = \Pi_v / \langle \Pi_v(z) f - f \mid z \in Z_H(F_v),\ f \in \Pi_v \rangle$ is the maximal Z_H-invariant quotient of Π_v, and $C^\infty(\Omega_v)$ is the space of locally constant functions on Ω_v (see [Gan 2011, §2.3]).

For a finite place v of F, let $\bar{f}_v^0$ be the image of the spherical vector f_v^0 in $(\Pi_v)_{Z_H}$. Then the action of $\bar{f}_v^0$ on Ω_v is easily described as follows. For each $n \in \mathbb{Z}$, let $\Omega_v(n) = \Omega_v \cap (\varpi_v^n \Lambda_v \setminus \varpi_v^{n+1} \Lambda_v)$, where $\Lambda_v = \bar{V}_H(\mathcal{O}_v)$ is the $\mathcal{O}_v$-lattice in $\bar{V}_H(F_v)$. Then $\bar{f}_v^0$ is constant on each $\Omega_v(n)$; more precisely, it is zero on $\Omega_v(n)$ if $n < 0$, and it takes the value $(q_v^{n+1} - 1)(q_v - 1)^{-1}$ on $\Omega_v(n)$ for $n \geq 1$. See [Kazhdan and Polishchuk 2004, Theorem 1.1.3] or [Gan 2011, §2] for more details.

3. Reduction to PGL_3

Exceptional theta correspondence. Let $H = E_6$ be the split simple adjoint group of type E_6 over F. Let $\Pi = \bigotimes_v \Pi_v$ be the minimal representation of $H(\mathbb{A})$. There is an $H(\mathbb{A})$-equivariant embedding $\theta : \Pi \hookrightarrow \mathcal{A}_2(H)$, where $\mathcal{A}_2(H)$ is the space of square integrable automorphic forms on H (see [Ginzburg et al. 1997]).

Consider the dual pair $G_2 \times \mathrm{PGL}_3$ in $H = E_6$ as in [Magaard and Savin 1997], where G_2 denotes the split simple group of type G_2 over F. See [Carter 1972] for more details on the structure of exceptional groups.

For $f \in \Pi$, the associated theta integral on $G_2(\mathbb{A})$ is defined by

$$I(f)(g) = \int_{[\mathrm{PGL}_3]} \theta(f)(gh)\, dh, \quad \text{for all } g \in G_2(\mathbb{A}),$$

where $[\mathrm{PGL}_3] = \mathrm{PGL}_3(F) \backslash \mathrm{PGL}_3(\mathbb{A})$. This integral may not converge. Similar to [Kudla and Rallis 1994; Ichino 2001; 2004], W.T. Gan [2011] found a regularization of the theta integral by using an element z of the Bernstein center of PGL_3 at some finite place v_0 of F. Precisely, z belongs to the component of the Bernstein center associated to the trivial representation of $\mathrm{PGL}_3(F_{v_0})$, which is equal to $\mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]^{S_3}/(x_1x_2x_3 - 1)$, and $z = \prod_{i=1}^{3}(x_i - q)(x_i^{-1} - q)$, where q is the order of the residue field of F at v_0. For any $f \in \Pi$, the function $\theta(z \cdot f)$ is rapidly decreasing on $\mathrm{PGL}_3(F) \backslash \mathrm{PGL}_3(\mathbb{A})$ (see [Gan 2011, Proposition 5.2]). The regularized theta integral is defined as

$$I^{\mathrm{reg}}(f, s)(g) = \int_{[\mathrm{PGL}_3]} \theta(z \cdot f)(gh) E(h, s)\, dh,$$

where $E(h, s)$ is the spherical Eisenstein series associated to the spherical vector φ_s^0 in $I_Q(s) = \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{PGL}_3(\mathbb{A})} |\det|^{s+1/2}$, and $Q = UL$ is the parabolic subgroup of PGL_3 with Levi subgroup $L \cong \mathrm{GL}_2$ and unipotent radical $U \cong F^2$.

The regularized theta integral is essentially an Eisenstein series. The following results are proved in [Gan 2011, §6].

Proposition 3.1. (i) *For $f \in \Pi$ and* $\mathrm{Re}(s) \gg 0$,

$$I^{\mathrm{reg}}(f, s)(g) = P_z(s) E(\Phi(f, s), g),$$

where $\Phi(f, s)$ is a meromorphic section in $I_P(s) = \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})} |\det|^{s+3/2}$, and

$$P_z(s) = \big(q^{-s-\frac{1}{2}} - q\big)\big(q^{s+\frac{1}{2}} - q\big)\big(q^{-s+\frac{1}{2}} - q\big)\big(q^{s-\frac{1}{2}} - q\big)\big(q^{2s} - q\big)\big(q^{-2s} - q\big),$$

where q is the order of the residue field of F at v_0.

(ii) *If $f = \bigotimes f_v$ is decomposable, then $\Phi(f, s) = \bigotimes \Phi_v(f_v, s)$ is also decomposable; if v is a finite place and f_v^0 is the spherical vector in Π_v, then*

$$\Phi_v(f_v^0, s) = \zeta_v\big(s + \tfrac{1}{2}\big)\zeta_v\big(s + \tfrac{3}{2}\big)\zeta_v(2s + 1)\Phi_v^0(s),$$

where $\Phi_v^0(s) \in I_{P,v}(s)$ is the spherical vector.

We see from this proposition that the Fourier coefficients of the regularized theta integral are closely related to those of the Eisenstein series. Next we study them.

Fourier coefficient of regularized theta integral. For $f \in \Pi$, consider the Fourier coefficient of $I^{\mathrm{reg}}(f,s)(1)$ with respect to the character ψ_σ, which is given by

$$\begin{aligned} I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) &= \int_{[N]} \overline{\psi_\sigma(n)}\, I^{\mathrm{reg}}(f,s)(n)\,dn \\ &= \int_{[N]} \overline{\psi_\sigma(n)} \int_{[\mathrm{PGL}_3]} \theta(z\cdot f)(nh) E(h,s)\,dh\,dn. \end{aligned}$$

We want to interchange the order of integration in the above iterated integral. By Fubini's theorem, this is possible if the integral

$$\int_{[N]} \int_{[\mathrm{PGL}_3]} \theta(z\cdot f)(nh) E(h,s)\,dh\,dn$$

is absolutely convergent. The author of this paper did not know how to prove this statement. Luckily, the referee communicated a note to the author in which the following inequality is proved:

For every integer r, there exist an integer m and a constant c such that

$$|\theta(z\cdot f)(gh)| \le c\|g\|^m \|h\|^{-r} \tag{1}$$

for all $g \in G_2(\mathbb{A})$ and $h \in S$, where $\|\cdot\|$ is the height function as defined in [Mœglin and Waldspurger 1995, page 20] and S is a Siegel domain in $\mathrm{PGL}_3(\mathbb{A})$ with $\mathrm{PGL}_3(\mathbb{A}) = \mathrm{PGL}_3(F)S$.

The absolute convergence of the integral thus follows from the above inequality, since $E(h,s)$, as an automorphic form on PGL_3, is of moderate growth, and the height function is bounded on compact sets.

So we have

$$I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) = \int_{[\mathrm{PGL}_3]} E(h,s) \int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn\;dh.$$

Next we study the Fourier coefficient $\int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn$ of the function $\theta(z\cdot f)$.

Let $P_H = N_H \cdot M_H$ be the Heisenberg parabolic subgroup of $H = E_6$. Let Z_H be the center of N_H. Then $Z_H \cong F$ and $V_H := N_H/Z \cong F \oplus M_3(F) \oplus M_3(F) \oplus F$ ([Magaard and Savin 1997, p. 114]). Furthermore, $P = P_H \cap G_2$, $N = N_H \cap G_2$ and $Z = Z_H$.

Let $V = N/Z$, which is abelian. Then

$$\int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn = \int_{[V]} \overline{\psi_\sigma(v)}\theta(z\cdot f)_Z(vh)\,dv,$$

where

$$\theta(z\cdot f)_Z(vh) = \int_{[Z]} \theta(z\cdot f)(zvh)\,dz.$$

Let Ω be the minimal nontrivial orbit of $M_H(F)$ on the set of unitary characters of $N_H(F)\setminus N_H(\mathbb{A})$, which can be noncanonically identified with $\overline{V}_H(F)=\overline{N}_H(F)/\overline{Z}_H(F)$, where $\overline{N}_H$ is the opposite of N_H and $\overline{Z}_H$ is the center of $\overline{N}_H$.

By [Gan and Savin 2003, Proposition 5.2], the term $\theta(z\cdot f)_Z$ has the following Fourier expansion along N_H:

$$\theta(z\cdot f)_Z(h)=\sum_{\chi\in\bar{\Omega}}\theta(z\cdot f)_{N_H,\chi}(h),$$

where $\bar{\Omega}$ is the union of Ω with the trivial character, and the term $\theta(z\cdot f)_{N_H,\chi}$ is given by

$$\theta(z\cdot f)_{N_H,\chi}(h)=\int_{[N_H]}\theta(z\cdot f)(nh)\overline{\chi(n)}\,dn.$$

Moreover, since $\theta(z\cdot f)$ is rapidly decreasing on $\mathrm{PGL}_3(F)\setminus\mathrm{PGL}_3(\mathbb{A})$, this Fourier expansion converges absolutely and uniformly on compact subsets of $\mathrm{PGL}_3(\mathbb{A})$.

So,

$$\begin{aligned}\int_{[N]}\overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn&=\int_{[V]}\overline{\psi_\sigma(v)}\theta(z\cdot f)_Z(vh)\,dv\\&=\int_{[V]}\overline{\psi_\sigma(v)}\Bigl(\sum_{\chi\in\bar{\Omega}}\theta(z\cdot f)_{N_H,\chi}(vh)\Bigr)\,dv\\&=\sum_{\chi\in\bar{\Omega}}\int_{[V]}\overline{\psi_\sigma(v)}\theta(z\cdot f)_{N_H,\chi}(vh)\,dv,\end{aligned}$$

where the order change of the integration and summation is justified by the uniform convergence of the Fourier expansion. Then

$$\begin{aligned}\int_{[N]}\overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn&=\sum_{\chi\in\bar{\Omega}}\int_{[V]}\overline{\psi_\sigma(v)}\int_{[N_H]}\theta(z\cdot f)(nvh)\overline{\chi(n)}\,dn\,dv\\&=\sum_{\chi\in\bar{\Omega}}\int_{[V]}\overline{\psi_\sigma(v)}\chi(v)\int_{[N_H]}\theta(z\cdot f)(nvh)\overline{\chi(nv)}\,dn\,dv\\&=\sum_{\chi\in\bar{\Omega}}\int_{[V]}\overline{\psi_\sigma(v)}\chi(v)\theta(z\cdot f)_{N_H,\chi}(h)\,dv\\&=\sum_{\chi\in\bar{\Omega}}\theta(z\cdot f)_{N_H,\chi}(h)\int_{[V]}\overline{\psi_\sigma(v)}\chi(v)\,dv.\end{aligned}$$

Note that

$$\int_{[V]}\overline{\psi_\sigma(v)}\chi(v)\,dv=\begin{cases}\mathrm{Vol}([V])=1 & \text{if } \chi|_{V(\mathbb{A})}=\psi_\sigma,\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn = \sum_{\chi\in\bar{\Omega}:\chi|_{V(\mathbb{A})}=\psi_\sigma} \theta(z\cdot f)_{N_H,\chi}(h) = \sum_{\chi\in\Omega:\chi|_{V(\mathbb{A})}=\psi_\sigma} \theta(z\cdot f)_{N_H,\chi}(h),$$

since ψ_σ is nontrivial.

Let $\Omega^0 = \{\chi \in \Omega : \chi|_{V(\mathbb{A})} = \psi_\sigma\}$. Then

$$\int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)dn = \sum_{\chi\in\Omega^0} \theta(z\cdot f)_{N_H,\chi}(h).$$

It follows that the Fourier coefficient of $I^{\mathrm{reg}}(f,s)(1)$ with respect to ψ_σ is given by

$$I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) = \int_{[\mathrm{PGL}_3]} E(h,s) \sum_{\chi\in\Omega^0} \theta(z\cdot f)_{N_H,\chi}(h)\,dh.$$

Note that $\Omega \subset \overline{V}_H = \overline{N}_H/\bar{Z}_H = F \oplus M_3(F) \oplus M_3(F) \oplus F$ can be described as

$$\Omega = \{(a,x,y,b) : y^\sharp = bx,\ x^\sharp = ay,\ xy = abI_3\},$$

where I_3 is the identity matrix in $M_3(F)$, and $x^\sharp$ is the adjoint of $x \in M_3(F)$.

Also $\overline{V} = \overline{N}/\bar{Z} \subset \overline{V}_H = \overline{N}_H/\bar{Z}_H$ can be described as

$$\overline{V} = F \oplus F \oplus F \oplus F.$$

The restriction from $\overline{V}_H$ to $\overline{V}$ is given by

$$(\alpha,x,y,\beta) \mapsto (\alpha, \mathrm{Tr}(x), \mathrm{Tr}(y), \beta).$$

For $\sigma = (1,0,b,c) \in \overline{V} = \overline{N}/\bar{Z}$, we have

$$\begin{aligned}\Omega^0 &= \{\chi \in \Omega : \chi|_{N(\mathbb{A})} = \psi_\sigma\}\\ &= \{(1,x,x^\sharp,\det(x)) : x \in M_3(F) \text{ has characteristic polynomial } \lambda^3 + b\lambda - c\}.\end{aligned}$$

Compare [Gan 2008, Lemma 2.1].

Note that there is an action of $\mathrm{PGL}_3(F)$ on Ω given by conjugation:

$$h\cdot(\alpha,x,y,\beta) = (\alpha, hxh^{-1}, hyh^{-1}, \beta).$$

The following result is easy to verify.

Lemma 3.2. $\mathrm{PGL}_3(F)$ *acts on* Ω^0 *transitively with representative*

$$\chi_0 = (1, A_0, A_0^\sharp, \det(A_0)),$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & -b \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $T(F)$ be the stabilizer of χ_0 in $\mathrm{PGL}_3(F)$. Then $T(F) = E^\times/F^\times$, where E is the cubic field generated by one of the roots of the irreducible polynomial $x^3 + bx - c$; and

$$T(F) \backslash \mathrm{PGL}_3(F) \cong \Omega^0 \quad \text{via } T(F)h \mapsto h \cdot \chi_0.$$

So

$$\begin{aligned}
I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) &= \int_{[\mathrm{PGL}_3]} E(h, s) \sum_{\gamma \in T(F)\backslash \mathrm{PGL}_3(F)} \theta(z \cdot f)_{N_H, \gamma \cdot \chi_0}(h)\, dh \\
&= \int_{[\mathrm{PGL}_3]} E(h, s) \sum_{\gamma \in T(F)\backslash \mathrm{PGL}_3(F)} \theta(z \cdot f)_{N_H, \chi_0}(\gamma h)\, dh \\
&= \int_{T(F)\backslash \mathrm{PGL}_3(\mathbb{A})} E(h, s)\theta(z \cdot f)_{N_H, \chi_0}(h)\, dh \\
&= \int_{T(\mathbb{A})\backslash \mathrm{PGL}_3(\mathbb{A})} \int_{[T]} E(th, s)\theta(z \cdot f)_{N_H, \chi_0}(th)\, dt\, dh \\
&= \int_{T(\mathbb{A})\backslash \mathrm{PGL}_3(\mathbb{A})} \theta(z \cdot f)_{N_H, \chi_0}(h) \int_{[T]} E(th, s)\, dt \ dh,
\end{aligned}$$

where we have used the fact that $\theta(z\cdot f)_{N_H,\chi_0}(th) = \theta(z\cdot f)_{N_H,\chi_0}(h)$ for all $t \in T(\mathbb{A})$, which can be shown as follows: it is true for $t \in T(F)$, and it is true for $t \in T(F_v)$ for all finite places v by the local formulae; by weak approximation ([Platonov and Rapinchuk 1994, Theorem 7.7, p. 415]), $T(F)$ is dense in $T(F_\infty)$, where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, so it is true for all $t \in T(\mathbb{A})$.

Lemma 3.3. *Keep the notation above. Then*

$$\int_{[T]} E(th, s)\, dt = \int_{T(\mathbb{A})} \varphi^0_s(th)\, dt,$$

where φ^0_s is the spherical vector in $I_Q(s)$.

Proof. We have

$$\int_{[T]} E(th, s)\, dt = \int_{T(F)\backslash T(\mathbb{A})} \sum_{\gamma \in Q(F)\backslash \mathrm{PGL}_3(F)} \varphi^0_s(\gamma th)\, dt.$$

Note that $Q(F)\backslash \mathrm{PGL}_3(F)$ is the Grassmannian $\mathbb{P}^2(F)$, which is just $E^\times/F^\times = T(F)$, where E is the cubic field extension of F generated by one of the roots of the polynomial $x^3 + bx - c$. The lemma then follows. □

In summary, we have shown the following result:

Proposition 3.4. *We have*

$$I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) = \int_{\mathrm{PGL}_3(\mathbb{A})} \theta(z \cdot f)_{N_H, \chi_0}(h)\varphi^0_s(h)\, dh.$$

Next we analyze $\theta(z\cdot f)_{N_H,\chi_0}(h)=\int_{[N_H]}\theta(z\cdot f)(nh)\overline{\chi_0(n)}dn$. We follow the arguments in [Gan 2008, §3].

First note that the mapping $L:\phi\mapsto\theta(\phi)_{N_H,\chi_0}(1)$ gives a nonzero element in the space $\mathrm{Hom}_{N_H(\mathbb{A})}(\Pi,\chi_0)$. Fix a finite place v of F. Since $\chi_{0,v}\in\Omega_v$, it follows from [Magaard and Savin 1997, Lemma 6.2] that

$$\dim\mathrm{Hom}_{N_H(F_v)}(\Pi_v,\chi_0)=1.$$

Note that in this case, a nonzero element of the space $\mathrm{Hom}_{N_H(F_v)}(\Pi_v,\chi_0)$ is given by

$$L_v(\phi)=i_v(\bar{\phi})(\chi_{0,v}),$$

where $\bar{\phi}$ is the image of ϕ in $(\Pi_v)_{Z_H}$, and i_v is the mapping on page 239.

Next we consider the archimedean places. Let the archimedean part of Π be $\Pi_\infty=\bigotimes_{v|\infty}\Pi_v$. Then the global functional $L\in\mathrm{Hom}_{N_H(\mathbb{A})}(\Pi,\chi_0)$ can be decomposed as $L=L_\infty\otimes\bigotimes_{v\nmid\infty}L_v$, where $L_\infty\in\mathrm{Hom}_{N_H(F_\infty)}(\Pi_\infty,\chi_0)$ is nontrivial.

So we have the following result:

Lemma 3.5. *Keep the notation above. For a pure tensor $f=\bigotimes f_v\in\Pi$, we have*

$$\theta(f)_{N_H,\chi_0}(h)=\theta(h\cdot f)_{N_H,\chi_0}(1)=L_\infty(h_\infty\cdot f_\infty)\cdot\prod_{v\nmid\infty}L_v(h_v\cdot f_v),$$

where h_∞ (resp. f_∞) is the archimedean part of h (resp. f).

Combining Proposition 3.4 and Lemma 3.5, we obtain the following result:

Proposition 3.6. *Suppose $f=\bigotimes f_v\in\Pi$ is a pure tensor. Then*

$$I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1)=I(f_\infty,s)\cdot I(z\cdot f_{v_0},s)\cdot\prod_{v\neq v_0,v\nmid\infty}I(f_v,s),$$

where

$$I(f_\infty,s)=\int_{\mathrm{PGL}_3(F_\infty)}L_\infty(h_\infty\cdot f_\infty)\varphi^0_{s,\infty}(h_\infty)\,dh_\infty,$$

and for a finite place v and $\phi_v\in\Pi_v$,

$$I(\phi_v,s)=\int_{\mathrm{PGL}_3(F_v)}L_v(h_v\cdot\phi_v)\varphi^0_{s,v}(h_v)\,dh_v,$$

where

$$L_v(\phi_v)=i_v(\bar{\phi}_v)(\chi_0).$$

Recall v_0 is a finite place of F such that z comes from the Bernstein center of $\mathrm{PGL}_3(F_{v_0})$.

In the next section, we will compute the local integrals $I(f_v,s)$ in the finite unramified case, and we will see that they are quotients of local zeta factors. See Proposition 4.3 in the next section for the precise result.

4. Unramified computation

This section is devoted to the computation of the local unramified factors $I(f_v, s)$ of the Fourier coefficient $I^{\text{reg}}_{\psi_\sigma}(f, s)(1)$. So v is a finite place of F such that v is unramified in the cubic field extension E of F, b and c are units in $\mathcal{O}_v$, the character ψ_v is unramified, f_v and $\varphi_{s,v}$ are spherical vectors.

For simplicity, we omit the subscript v from notation. So F is a p-adic local field with $p \neq 2$, $\mathcal{O}$ its ring of integers, ϖ a uniformizer of $\mathcal{O}$, $k_F = \mathcal{O}/\varpi\mathcal{O}$ the residue field of F, q the cardinality of k_F, $\nu : F^\times \to \mathbb{Z}$ the valuation given by $\nu(\varpi^n \mathcal{O}^\times) = n$, and $|\cdot| : F^\times \to \mathbb{R}_+^\times$ the absolute value given by $|x| = q^{-\nu(x)}$. The Haar measure dx on F satisfies $\text{Vol}(\mathcal{O}) = 1$, and take the Haar measure $d^\times x$ on $F^\times$ such that $\text{Vol}(\mathcal{O}^\times) = 1$.

Reduction to a volume computation. Let $G = \text{PGL}_3(F)$, $K = \text{PGL}_3(\mathcal{O})$, $Q = UL$ the parabolic subgroup of G with Levi subgroup $L \cong \text{GL}_2(F)$ and unipotent radical $U \cong F^2$. Then there is an Iwasawa decomposition $G = QK$.

We want to compute the integral

$$I(s) = \int_G i(\overline{hf})(\chi)\varphi_s(h)\, dh,$$

where f is the spherical vector in Π, φ_s is the spherical vector in $I_Q(s)$, i is the mapping as on page 239, and $\chi = (1, A_0, A_0^\sharp, \det(A_0)) \in F \oplus M_3(F) \oplus M_3(F) \oplus F$, where

$$A_0 = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & -b \\ 0 & 1 & 0 \end{pmatrix}$$

with $b, c \in \mathcal{O}^\times$.

We have

$$\begin{aligned} I(s) &= \int_K \int_Q i(\overline{qkf})(\chi)\varphi_s(qk) dq\, dk \\ &= \int_Q i(\overline{qf})(\chi)\varphi_s(q) dq \\ &= \int_L \int_U i(\overline{ulf})(\chi)\varphi_s(ul)\delta_Q(ul)^{-1} du\, dl \\ &= \int_L \int_U i(\overline{ulf})(\chi)|\det(l)|^{s-1/2} du\, dl, \end{aligned}$$

since $\varphi_s(ul) = |\det(l)|^{s+1/2}$ and $\delta_Q(ul) = |\det(l)|$.

Note that $i(\bar{f}) \in C^\infty(\Omega)$, where $\Omega = F \oplus M_3(F) \oplus M_3(F) \oplus F$, and it is constant on $\Omega(n)$, where $\Omega(n) = \Omega \cap (\varpi^n \Lambda - \varpi^{n+1}\Lambda)$, with

$$\Lambda = \mathcal{O} \oplus M_3(\mathcal{O}) \oplus M_3(\mathcal{O}) \oplus \mathcal{O}.$$

Now we have

$$i(\overline{ulf})(\chi) = i(\bar{f})(l^{-1}u^{-1}\chi).$$

But

$$\chi = (1, A_0, A_0^\sharp, \det(A_0)) \in \Omega(0).$$

So

$$l^{-1}u^{-1}\chi = (1, l^{-1}u^{-1}A_0ul, \dots),$$

and

$$i(\bar{f})(l^{-1}u^{-1}\chi) = \begin{cases} 1 & \text{if } l^{-1}u^{-1}A_0ul \in M_3(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} I(s) &= \int_L \int_U 1_{M_3(\mathcal{O})}(l^{-1}u^{-1}A_0ul)|\det(l)|^{s-1/2} du\, dl \\ &= \int_{\mathrm{GL}_2(F)} \int_{F^2} 1_{M_3(\mathcal{O})}(l(g)^{-1}u(x,y)^{-1}A_0u(x,y)l(g))|\det(g)|^{s-1/2}\, dx\, dy\, dg. \end{aligned}$$

Let B be the group of upper triangular matrices in $\mathrm{GL}_2(F)$. Then $\mathrm{GL}_2(F) = \mathrm{GL}_2(\mathcal{O})B$, and

$$\begin{aligned} I(s) &= \int_{\mathrm{GL}_2(\mathcal{O})} \int_B \int_{F^2} 1_{M_3(\mathcal{O})}(l(k)^{-1}l(p)^{-1}u(x,y)^{-1} \\ &\qquad\qquad A_0u(x,y)l(p)l(k))|\det(p)|^{s-1/2}\, dx\, dy\, dp\, dk \\ &= \int_B \int_{F^2} 1_{M_3(\mathcal{O})}(l(p)^{-1}u(x,y)^{-1}A_0u(x,y)l(p))|\det(p)|^{s-1/2}\, dx\, dy\, dp. \end{aligned}$$

Now $B = N_B M_B$, where $N_B \cong F$ is the unipotent radical and $M_B \cong F^\times \times F^\times$ is the Levi subgroup. So

$$\begin{aligned} I(s) &= \int_{M_B} \int_{N_B} \int_{F^2} 1_{M_3(\mathcal{O})}(l(nm)^{-1}u(x,y)^{-1}A_0u(x,y)l(nm)) \\ &\qquad\qquad \delta_B(m)^{-1}|\det(nm)|^{s-1/2}\, dx\, dy\, dn\, dm \\ &= \int_{F^{\times 2}} \int_{F^3} 1_{M_3(\mathcal{O})}(l(n(\beta)m(\alpha,\delta))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\alpha,\delta))) \\ &\qquad\qquad \cdot |\alpha/\delta|^{-1}|\alpha\delta|^{s-1/2}\, dx\, dy\, d\beta\, d^\times\alpha\, d^\times\delta, \end{aligned}$$

where

$$n(\beta) = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}, \quad m(\alpha,\delta) = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}.$$

Since $F^\times = \bigcup_{n\in\mathbb{Z}} \varpi^n \mathcal{O}^\times$, we have

$$\begin{aligned}
&I(s)\\
&= \sum \int_{F^3}\int_{\varpi^n\mathcal{O}^\times}\int_{\varpi^m\mathcal{O}^\times} 1_{M_3(\mathcal{O})}(l(n(\beta)m(\alpha,\delta))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\alpha,\delta)))\\
&\qquad\qquad \cdot |\alpha/\delta|^{-1}|\alpha\delta|^{s-1/2}d^\times\alpha\, d^\times\delta\, dx\, dy\, d\beta\\
&= \sum \int_{F^3} 1_{M_3(\mathcal{O})} l(n(\beta)m(\varpi^m,\varpi^n))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\varpi^m,\varpi^n)))\\
&\qquad \cdot |\varpi|^{(m+n)(s-1/2)}|\varpi^{m-n}|^{-1}\mathrm{Vol}(\varpi^m\mathcal{O}^\times, d^\times\alpha)\mathrm{Vol}(\varpi^n\mathcal{O}^\times, d^\times\delta)\, dx\, dy\, d\beta\\
&= \sum \int_{F^3} 1_{M_3(\mathcal{O})}(l(n(\beta)m(\varpi^m,\varpi^n))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\varpi^m,\varpi^n)))\\
&\qquad\qquad \cdot q^{-(m+n)(s-1/2)+m-n}\, dx\, dy\, d\beta,
\end{aligned}$$

where the sum is taken over all $m, n \in \mathbb{Z}$, and we have used $\mathrm{Vol}(\varpi^m\mathcal{O}^\times, d^\times\alpha) = \mathrm{Vol}(\varpi^n\mathcal{O}^\times, d^\times\delta) = \mathrm{Vol}(\mathcal{O}^\times) = 1$, and $|\varpi|^{-1} = q$.

Note that

$$\begin{aligned}
&l(n(\beta)m(\varpi^m, \varpi^n))^{-1}u(x, y)^{-1}A_0u(x, y)l(n(\beta)m(\varpi^m, \varpi^n))\\
&\qquad = \begin{pmatrix} -\beta & -\varpi^{n-m}(x+\beta^2-y\beta) & \varpi^{-m}(c-xy-\beta(x-y^2-b))\\ \varpi^{m-n} & \beta-y & \varpi^{-n}(x-y^2-b)\\ 0 & \varpi^n & y\end{pmatrix}.
\end{aligned}$$

So

$$\begin{aligned}
I(s) &= \sum_{m\ge n\ge 0} S(m, n)q^{-(m+n)(s-1/2)+m-n}\\
&= \sum_{k\ge 0} q^{-k(s+1/2)} \sum_{\substack{m+n=k\\ m\ge n\ge 0}} S(m, n)q^{2m}\\
&= 1 + \sum_{k\ge 1} q^{-k(s+1/2)} \sum_{\substack{m+n=k\\ m\ge n\ge 0}} S(m, n)q^{2m},
\end{aligned}$$

where $S(m, n)$ is the volume of the set of $(x, y, \beta) \in \mathcal{O}^3$ such that $\nu(x+\beta^2-y\beta) \ge m-n$, $\nu(x-y^2-b) \ge n$ and $\nu(c-xy-\beta(x-y^2-b)) \ge m$ (with respect to the Haar measure on F such that $\mathrm{Vol}(\mathcal{O}) = 1$).

Let r be the number of solutions of x^3+bx-c over F. Then $r = 0$, 1 or 3.

Lemma 4.1. (1) *If* $r = 0$, *then* $S(m, n) = 0$ *for* $m \ge n \ge 0$ *with* $m+n \ge 1$.

(2) *If* $r = 1$, *then*

$$S(m, n) = \begin{cases} 0 & \text{if } m > n > 0,\\ q^{-2m} & \text{if } m = n > 0, /\\ q^{-2m} & \text{if } m > n = 0.\end{cases}$$

(3) *If* $r = 3$, *then*

$$S(m,n) = \begin{cases} 6q^{-2m} & \text{if } m > n > 0, \\ 3q^{-2m} & \text{if } m = n > 0, \\ 3q^{-2m} & \text{if } m > n = 0. \end{cases}$$

Proof. The results follow from Lemmas 4.7, 4.8 and 4.9. □

Corollary 4.2. (1) *If* $r = 0$, *then for* $k \geq 1$,

$$\sum_{\substack{m+n=k \\ m \geq n \geq 0}} S(m,n) q^{2m} = 0.$$

(2) *If* $r = 1$, *then for* $k \geq 1$,

$$\sum_{\substack{m+n=k \\ m \geq n \geq 0}} S(m,n) q^{2m} = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ 2 & \text{if } k \text{ is even}. \end{cases}$$

(3) *If* $r = 3$, *then for* $k \geq 1$,

$$\sum_{\substack{m+n=k \\ m \geq n \geq 0}} S(m,n) q^{2m} = 3k.$$

Recall that r is the number of solutions of $x^3 + bx - c$ over F. Let E be the cubic algebra over F determined by the polynomial $x^3 + bx - c$ as follows. If $r = 0$, then E is a field generated by one of the roots of $x^3 + bx - c$ over F, which is an unramified cubic field extension of F; if $r = 1$, say

$$x^3 + bx - c = (x - \alpha) \cdot g(x),$$

then $E = F \oplus F'$, where F' is the splitting field of $g(x)$ over F, which is an unramified quadratic field extension of F; if $r = 3$, then $E = F \oplus F \oplus F$.

Proposition 4.3. *Let E be the cubic algebra determined by $x^3 + bx - c$ as above. Then*

$$I(s) = \frac{\zeta_E\left(s + \frac{1}{2}\right)}{\zeta_F\left(3s + \frac{3}{2}\right)}.$$

More precisely,

$$I(s) = \begin{cases} 1 & \text{if } r = 0, \\ \dfrac{(1 - q^{-(s+1/2)})^{-1}(1 - q^{-2(s+1/2)})^{-1}}{(1 - q^{-3(s+1/2)})^{-1}} & \text{if } r = 1, \\ \dfrac{(1 - q^{-(s+1/2)})^{-3}}{(1 - q^{-3(s+1/2)})^{-1}} & \text{if } r = 3. \end{cases}$$

Computation of the volume. In this section, we will compute the volume $S(m,n)$ in the last section by using the higher-dimensional Hensel's lemma.

Let $f=(f_1,\dots,f_n):\mathcal{O}^n\to\mathcal{O}^n$ be a polynomial map, where each f_i is a polynomial over $\mathcal{O}$. We say $f:\mathcal{O}^n\to\mathcal{O}^n$ is *strongly regular* at $x_0\in\mathcal{O}^n$ if $\det D_f(x_0)\in\mathcal{O}^\times$, where $D_f(x_0)$ is the Jacobian matrix of f at x_0.

Theorem 4.4 (higher-dimensional Hensel's lemma). *Suppose $f:\mathcal{O}^n\to\mathcal{O}^n$ is a polynomial map which is strongly regular at $x_0\in\mathcal{O}^n$, and $y\in\mathcal{O}^n$ satisfies $y\equiv f(x_0)\ (\mathrm{mod}\ \varpi)$. Then there exists a unique $x\in\mathcal{O}^n$ such that $f(x)=y$ and $x\equiv x_0\ (\mathrm{mod}\ \varpi)$.*

Proof. See [Green et al. 1995, Theorem 8.1, p. 71] or [Kuhlmann 2011, §4.4]. □

Theorem 4.5. *Let $f:\mathcal{O}^n\to\mathcal{O}^n$ be a polynomial map which is strongly regular at $x_0\in\mathcal{O}^n$. Then $f:x_0+(\varpi\mathcal{O})^n\to f(x_0)+(\varpi\mathcal{O})^n$ is a measure-invariant bijection.*

Corollary 4.6. *Suppose $y_0\in\mathcal{O}^n$ and $f:\mathcal{O}^n\to\mathcal{O}^n$ is strongly regular at all solutions of $f(x)=y_0$ over $\mathcal{O}$. Then for an open subset $U\subset y_0+(\varpi\mathcal{O})^n$,*

$$\mathrm{Vol}(f^{-1}(U))=N\cdot\mathrm{Vol}(U),$$

where $\mathrm{Vol}(X)$ is the volume of the subset X of $\mathcal{O}^n$, and N is the number of $x\in k_F^n$ with $f(x)\equiv y_0\ (\mathrm{mod}\ \varpi)$.

Now we compute the volume $S(m,n)$.

For $m\ge n\ge 0$, let $\Omega_{m,n}$ be the set of $(x,y,\beta)\in\mathcal{O}^3$ such that

$$\begin{cases} \nu(x+\beta^2-y\beta)\ge m-n,\\ \nu(x-y^2-b)\ge n,\\ \nu(c-xy-\beta(x-y^2-b))\ge m.\end{cases}$$

Then $S(m,n)=\mathrm{Vol}(\Omega_{m,n})$.

Let r be the number of solutions of $x^3+bx-c=0$ over F. Then $r=0$, 1 or 3.

Lemma 4.7. *Suppose $m>n>0$. Then*

$$\mathrm{Vol}(\Omega_{m,n})=N\cdot q^{-2m},$$

where

$$N=\begin{cases}0 & \text{if } r=0,1,\\ 6 & \text{if } r=3.\end{cases}$$

Proof. Define a polynomial map $f:\mathcal{O}^3\to\mathcal{O}^3$ by

$$(x,y,\beta)\mapsto(w_1,w_2,w_3),$$

where

$$\begin{cases} w_1=x+\beta^2-y\beta,\\ w_2=x-y^2-b,\\ w_3=c-xy-\beta(x-y^2-b).\end{cases}$$

It is routine to check that f is strongly regular at all solutions of $f(x, y, \beta) = 0$.

Let

$$E_{m,n} = \{(w_1, w_2, w_3) \in \mathcal{O}^3 \mid \nu(w_1) \geq m - n,\, \nu(w_2) \geq n,\, \nu(w_3) \geq m\}.$$

Then $\Omega_{m,n} = f^{-1}(E_{m,n})$. By Corollary 4.6,

$$\mathrm{Vol}(\Omega_{m,n}) = N \cdot \mathrm{Vol}(E_{m,n}) = N \cdot q^{-2m},$$

where N is the number of $(x, y, \beta) \in k_F^3$ such that $f(x, y, \beta) = 0$.

Suppose $N \neq 0$. Take $(x_0, y_0, \beta_0) \in k_F^3$ satisfying $f(x_0, y_0, \beta_0) = 0$. It is easy to see that y_0 and $-\beta_0$ are two distinct roots of $x^3 + bx - c$ over k_F, so $r = 3$. Then $N = 6$.

If $r \neq 3$ (i.e., $r = 0$ or 1), then $N = 0$ (since $N \neq 0$ implies that $r = 3$).

The desired result follows. □

Lemma 4.8. *Suppose $m = n > 0$. Then*

$$\mathrm{Vol}(\Omega_{m,m}) = r \cdot q^{-2m}.$$

Proof. In this case, for $(x, y, \beta) \in \mathcal{O}^3$, the condition $\nu(x + \beta^2 - y\beta) \geq 0$ holds automatically. So

$$\begin{aligned}\Omega_{m,m} &= \{(x, y, \beta) \in \mathcal{O}^3 \mid \nu(x - y^2 - b) \geq m,\, \nu(c - xy - \beta(x - y^2 - b)) \geq m\} \\ &= \{(x, y, \beta) \in \mathcal{O}^3 \mid \nu(x - y^2 - b) \geq m,\, \nu(c - xy) \geq m\}.\end{aligned}$$

For $\beta \in \mathcal{O}$, let

$$\Omega_m(\beta) = \{(x, y) \in \mathcal{O}^2 \mid (x, y, \beta) \in \Omega_{m,m}\}.$$

Then $\Omega_m(\beta) = \Omega_m(0)$ for any $\beta \in \mathcal{O}$, and

$$\begin{aligned}\mathrm{Vol}(\Omega_{m,m}) &= \iiint_{(x,y,\beta)\in\Omega_{m,m}} dx\, dy\, d\beta \\ &= \int_{\beta\in\mathcal{O}} \iint_{(x,y)\in\Omega_m(\beta)} dx\, dy\, d\beta \\ &= \int_{\mathcal{O}} \mathrm{Vol}(\Omega_m(\beta))\, d\beta \\ &= \mathrm{Vol}(\Omega_m(0)) \cdot \mathrm{Vol}(\mathcal{O}) \\ &= \mathrm{Vol}(\Omega_m(0)).\end{aligned}$$

Define a polynomial map $f_\beta : \mathcal{O}^2 \to \mathcal{O}^2$ by

$$(x, y) \mapsto (w_2 = x - y^2 - b,\, w_3 = c - xy).$$

Then it is routine to check that f is strongly regular at all solutions of $f(x, y) = 0$.

Let

$$E_m = \{(w_2, w_3) \in \mathcal{O}^2 \mid \nu(w_2) \geq m,\ \nu(w_3) \geq m\}.$$

Then $\Omega_m(0) = f^{-1}(E_m)$.

By Corollary 4.6, $\mathrm{Vol}(f^{-1}(E_m)) = N \cdot \mathrm{Vol}(E_m) = N \cdot q^{-2m}$, where N is the number of $(x, y) \in k_F^2$ such that $f(x, y) = 0$. It is easy to check that $N = r$. So

$$\mathrm{Vol}(\Omega_{m,m}) = \mathrm{Vol}(\Omega_m(0)) = \mathrm{Vol}(f^{-1}(E_m)) = r \cdot q^{-2m}. \qquad \square$$

Lemma 4.9. *Suppose $m > n = 0$. Then*

$$\mathrm{Vol}(\Omega_{m,0}) = r \cdot q^{-2m}.$$

Proof. In this case, for $(x, y, \beta) \in \mathcal{O}^3$, the condition $\nu(x - y^2 - b) \geq 0$ holds automatically. So

$$\Omega_{m,0} = \{(x, y, \beta) \in \mathcal{O}^3 \mid \nu(x + \beta^2 - y\beta) \geq m,\ \nu(c - xy - \beta(x - y^2 - b)) \geq m\}.$$

For $y \in \mathcal{O}$, let

$$\Omega_m(y) = \{(x, \beta) \in \mathcal{O}^2 \mid (x, y, \beta) \in \Omega_{m,0}\}.$$

Then

$$\begin{aligned}\mathrm{Vol}(\Omega_{m,0}) &= \iiint_{(x,y,\beta)\in\Omega_{m,0}} dx\, d\beta\, dy \\ &= \int_{y\in\mathcal{O}} \iint_{(x,\beta)\in\Omega_m(y)} dx\, d\beta\, dy \\ &= \int_{\mathcal{O}} \mathrm{Vol}(\Omega_m(y))\, dy.\end{aligned}$$

For $y \in \mathcal{O}$, define a polynomial map $f_y : \mathcal{O}^2 \to \mathcal{O}^2$ by

$$(x, \beta) \mapsto (w_1 = x + \beta^2 - y\beta,\ w_3 = c - xy - \beta(x - y^2 - b)).$$

Then it is routine to check that f_y is strongly regular at all solutions of $f_y(x, \beta) = 0$.

Let

$$E_m = \{(w_1, w_3) \in \mathcal{O}^2 \mid \nu(w_1) \geq m,\ \nu(w_3) \geq m\}.$$

Then $\Omega_m(y) = f_y^{-1}(E_m)$. By Corollary 4.6,

$$\mathrm{Vol}(f_y^{-1}(E_m)) = N_y \cdot \mathrm{Vol}(E_m) = N_y \cdot q^{-2m},$$

where N_y is the number of $(x, \beta) \in k_F^2$ such that $f_y(x, \beta) = 0$. It is easy to check that $N_y = r$. So $\mathrm{Vol}(\Omega_m(y)) = r \cdot q^{-2m}$, and

$$\mathrm{Vol}(\Omega_{m,0}) = \int_{\mathcal{O}} \mathrm{Vol}(\Omega_m(y))\, dy = r \cdot q^{-2m} \cdot \mathrm{Vol}(\mathcal{O}) = r \cdot q^{-2m}. \qquad \square$$

5. Proof of the main result

In this section, we prove Theorem 1.1.

Take $f = f^0 = \bigotimes f_v^0 \in \Pi$ to be the spherical vector. Consider the regularized theta integral $I^{\mathrm{reg}}(f,s)(1)$. Then for $\mathrm{Re}(s) \gg 0$, $I^{\mathrm{reg}}(f,s)(1) = P_z(s) E(\Phi(f,s),1)$.

By Proposition 3.6 and Proposition 4.3, the Fourier coefficient of $I^{\mathrm{reg}}(f,s)(1)$ with respect to ψ_σ is given by

$$I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) = I(f_\infty,s)\cdot I(z\cdot f_{v_0},s)\cdot \prod_{\substack{v\neq v_0\\ v\nmid\infty}} I(f_v,s) = \frac{\zeta_E\bigl(s+\frac{1}{2}\bigr)}{\zeta_F\bigl(3s+\frac{3}{2}\bigr)}\cdot\alpha_\infty(s)\cdot\prod_v \alpha_v(s),$$

where

$$\alpha_\infty(s) = I(f_\infty,s)\cdot\frac{\zeta_\infty\bigl(3s+\frac{3}{2}\bigr)}{\zeta_{E_\infty}\bigl(s+\frac{1}{2}\bigr)},$$

and for a finite place v of F,

$$\alpha_v(s) = \begin{cases} I(f_v,s)\cdot\dfrac{\zeta_v\bigl(3s+\frac{3}{2}\bigr)}{\zeta_{E_v}\bigl(s+\frac{1}{2}\bigr)} & \text{if } v\neq v_0,\\[2ex] I(z\cdot f_v,s)\cdot\dfrac{\zeta_v\bigl(3s+\frac{3}{2}\bigr)}{\zeta_{E_v}\bigl(s+\frac{1}{2}\bigr)} & \text{if } v = v_0,\end{cases}$$

and $\alpha_v(s) = 1$ for almost all finite v.

On the other hand, by Proposition 3.1, for a pure tensor $\Phi_s = \bigotimes \Phi_{s,v} \in I_P(s)$,

$$\frac{E_{\psi_\sigma}(\Phi(f,s),1)}{E_{\psi_\sigma}(\Phi_s,1)} = \prod_v \frac{E_{\psi_\sigma,v}(\Phi(f,s),1)}{E_{\psi_\sigma,v}(\Phi_s,1)} = \zeta_F\bigl(s+\tfrac{1}{2}\bigr)\zeta_F\bigl(s+\tfrac{3}{2}\bigr)\zeta_F(2s+1)\prod_v \beta_v(\Phi_s)$$

where

$$\beta_v(\Phi_s) = \frac{1}{\zeta_v\bigl(s+\frac{1}{2}\bigr)\zeta_v\bigl(s+\frac{3}{2}\bigr)\zeta_v(2s+1)}\frac{E_{\psi_\sigma,v}(\Phi(f,s),1)}{E_{\psi_\sigma,v}(\Phi_s,1)},$$

and $\beta_v(\Phi_s) = 1$ for almost all finite v.

Since $I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) = P_z(s)E_{\psi_\sigma}(\Phi(f,s),1)$, we get

$$E_{\psi_\sigma}(\Phi_s,1) = \frac{\zeta_E\bigl(s+\frac{1}{2}\bigr)}{\zeta_F\bigl(s+\frac{1}{2}\bigr)\zeta_F\bigl(s+\frac{3}{2}\bigr)\zeta_F(2s+1)\zeta_F\bigl(3s+\frac{3}{2}\bigr)}\cdot J_\infty(\Phi_s)\cdot\prod_v J_v(\Phi_s),$$

where

$$J_\infty(\Phi_s) = \alpha_\infty(s)/\beta_\infty(\Phi_s),$$

and for a finite place v of F,

$$J_v(\Phi_s) = \begin{cases}\alpha_v(s)/\beta_v(\Phi_s) & \text{if } v\neq v_0,\\ P_z(s)\alpha_v(s)/\beta_v(\Phi_s) & \text{if } v = v_0.\end{cases}$$

Note that $J_v(\Phi_s)$ is equal to 1 for almost all finite v. This is the desired result.

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