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RESTRICTED POISSON ALGEBRAS

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We reformulate Bezrukavnikov and Kaledin's definition of a restricted Poisson algebra, provide some natural and interesting examples, and discuss connections with other research topics.

Introduction

The Poisson bracket was introduced by Poisson [1809] as a tool for classical dynamics. Poisson geometry has become an active research field during the past 50 years. The study of Poisson algebras over $\mathbb{R}$ or a field of characteristic zero [Laurent-Gengoux et al. 2013] also has a long history, and is closely related to noncommutative algebra, differential geometry, deformation quantization, number theory, and other areas. The notion of a *restricted Poisson algebra* was introduced about ten years ago in an important paper of Bezrukavnikov and Kaledin [2008] in the study of deformation quantization in positive characteristic. The project in that paper is a natural extension of the classical deformation quantization of symplectic (or Poisson) manifolds.

Our first goal is to better understand Bezrukavnikov and Kaledin's definition via a Lie-algebraic approach. We reinterpret their definition in the following way.

Throughout the paper let $\Bbbk$ be a base field of characteristic $p \geq 3$. All vector spaces and algebras are over $\Bbbk$.

Definition 0.1. Let $(A, \{-,-\})$ be a Poisson algebra over $\Bbbk$.

(1) We call A a *weakly restricted Poisson algebra* if there is a p-map operation $x \mapsto x^{\{p\}}$ such that $(A, \{-,-\}, (-)^{\{p\}})$ is a restricted Lie algebra.

(2) We call A a *restricted Poisson algebra* if A is a weakly restricted Poisson algebra and the p-map $(-)^{\{p\}}$ satisfies

(E0.1.1) $$(x^2)^{\{p\}} = 2x^p x^{\{p\}}$$

for all $x \in A$.

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The formulation in (E0.1.1) is slightly simpler than the original definition. We will show that Definition 0.1(2) is equivalent to [Bezrukavnikov and Kaledin 2008, Definition 1.8] in Lemma 3.7. Generally it is not easy to prove basic properties for restricted Poisson algebras. For example, it is not straightforward to show that the tensor product preserves the restricted Poisson structure. Different formulations are helpful in understanding and proving some elementary properties.

Since there are several structures on a restricted Poisson algebra, it is delicate to verify all of the compatibility conditions. There are not many examples given in the literature. Our second goal is to provide several canonical examples from different research subjects. Restricted Poisson algebras can be viewed as a Poisson version of restricted Lie algebras, so the first few examples come from restricted (or modular) Lie theory. Let L be a restricted Lie algebra over $\Bbbk$. Then the trivial extension algebra $\Bbbk \oplus L$ (with $L^2 = 0$) is a restricted Poisson algebra. More naturally we have the following.

Theorem 0.2 (Theorem 6.5). *Let L be a restricted Lie algebra over $\Bbbk$ and let $s(L)$ be the p-truncated symmetric algebra. Then $s(L)$ admits a natural restricted Poisson structure induced by the restricted Lie structure of L.*

To use ideas from Poisson geometry, it is a good idea to extend the restricted Poisson structure to the symmetric algebra of a restricted Lie algebra (Example 6.2). The following result is slightly more general and useful in other settings.

Theorem 0.3 (Theorem 6.1). *Let T be an index set and $A = \Bbbk[x_i \mid i \in T]$ be a polynomial Poisson algebra. If, for each $i \in T$, there exists $\gamma(x_i) \in A$ such that $\mathrm{ad}_{x_i}^p = \mathrm{ad}_{\gamma(x_i)}$, then A admits a restricted Poisson structure $(-)^{\{p\}}\colon A \to A$ such that $x_i^{\{p\}} = \gamma(x_i)$ for all $i \in T$.*

The next example comes from deformation theory, which is also considered in [Bezrukavnikov and Kaledin 2008]. See (E7.0.1) for the definition of $M_n^p(f)$.

Proposition 0.4 (Proposition 7.1). *Let $(A, \cdot, \{-,-\})$ be a Poisson algebra over $\Bbbk$ and let $(A[\![t]\!], *)$ be a deformation quantization of A. If $M_n^p(f) = 0$ for $1 \le n \le p-2$ and f^p is central in $A[\![t]\!]$ for all $f \in A$, then A admits a restricted Poisson structure.*

A Lie–Rinehart algebra is an algebraic counterpart of a Lie algebroid, and appears naturally in the study of Gerstenhaber algebras, Batalin–Vilkovisky algebras and Maurer–Cartan algebras [Huebschmann 1990; 2005]. In this paper, we also study the relationship between restricted Poisson algebras and restricted Lie–Rinehart algebras. To save space we refer to [Dokas 2012] for the definition and some properties of restricted Lie–Rinehart algebras.

Theorem 0.5 (Theorem 8.2). *Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a (weakly) restricted Poisson algebra. If the module of Kähler differentials $\Omega_{A/\Bbbk}$ is free over A, then*

$(A, \Omega_{A/\Bbbk}, (-)^{[p]})$ *is a restricted Lie–Rinehart algebra, where the* p*-map of* $\Omega_{A/\Bbbk}$ *is determined by*

$$(x\,\mathrm{d}u)^{[p]} = x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u$$

for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$.

The category of restricted Poisson algebras is a symmetric monoidal category. In particular, the tensor product of two restricted Poisson algebras is again a restricted Poisson algebra (Proposition 9.2). Advances in algebra benefit tremendously from a geometric viewpoint and methods and vice versa. Restricted Poisson algebras are, to some extent, the algebraic counterpart of symplectic differential geometry in positive characteristic. Following this idea, restricted Poisson–Lie groups should correspond to restricted Poisson Hopf algebras which connect both Poisson geometry in positive characteristic and quantum groups at the root of unity. Hence, it is meaningful to introduce the notion of a restricted Poisson Hopf algebra; see Definition 9.3. One natural example of such an algebra is given in Example 9.4.

The paper is organized as follows. Sections 1 and 2 contain basic definitions about restricted Lie algebras and Poisson algebras. In Section 3, we reintroduce the notion of a restricted Poisson algebra. In Sections 4 to 7, we give several natural examples. In Section 8, we prove Theorem 0.5. The notion of a restricted Poisson Hopf algebra is introduced in Section 9.

1. Restricted Lie algebras

We give a short review of restricted Lie algebras.

Lie algebras over a field of positive characteristic often admit an additional structure involving a so-called p-map. The Lie algebra together with a p-map is called a *restricted Lie algebra*, which was first introduced and systematically studied by Jacobson [1941; 1962]. Let $L := (L, [-,-])$ be a Lie algebra over $\Bbbk$. For convenience, for each $x \in L$ we denote by $\mathrm{ad}_x\colon L \to L$ the adjoint representation given by $\mathrm{ad}_x(y) = [x, y]$ for all $y \in L$. We recall the definition of a restricted Lie algebra from [Jacobson 1941, Section 1]. As always, we assume that $\Bbbk$ is of positive characteristic $p \geq 3$.

Definition 1.1 [Jacobson 1941]. A *restricted Lie algebra* $(L, (-)^{[p]})$ over $\Bbbk$ is a Lie algebra L over $\Bbbk$ together with a *p-map* $(-)^{[p]} : x \mapsto x^{[p]}$ such that

(1) $\mathrm{ad}_x^p = \mathrm{ad}_{x^{[p]}}$ for all $x \in L$;
(2) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for all $\lambda \in \Bbbk$ and $x \in L$;
(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \Lambda_p(x, y)$, where $\Lambda_p(x, y) = \sum_{i=1}^{p-1} s_i(x, y)/i$ for all $x, y \in L$ and $s_i(x, y)$ is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}_{tx+y}^{p-1}(x)$.

For simplicity, we write all multiple Lie brackets with the notation

$$[x_1,[x_2,\ldots,[x_{n-1},x_n]\ldots]] =: [x_1,x_2,\ldots,x_{n-1},x_n] \tag{E1.1.1}$$

for $x_1,\ldots,x_n \in L$. Clearly,

$$\mathrm{ad}_x^i(y) = [\underbrace{x,\ldots,x}_{i \text{ copies}}, y]$$

for every i. With this notation, we have

$$s_i(x,y) = \sum_{\substack{x_k = x \text{ or } y \\ \#\{k|x_k=x\}=i-1}} [x_1,\ldots,x_{p-2},y,x], \tag{E1.1.2}$$

and hence

$$\Lambda_p(x,y) = \sum_{\substack{x_k = x \text{ or } y \\ x_{p-1}=y,\, x_p=x}} \frac{1}{\#(x)}[x_1,\ldots,x_{p-1},x_p]. \tag{E1.1.3}$$

Note that $\Lambda_p(x,y)$ is denoted by $L(x,y)$ in [Bezrukavnikov and Kaledin 2008] and denoted by $\sigma(x,y)$ in [Hochschild 1954]. Another way of understanding $\Lambda_p(x,y)$ is to use the universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebra L. By [Hochschild 1954, Condition (3) on p. 559],

$$\Lambda_p(x,y) = (x+y)^p - x^p - y^p \tag{E1.1.4}$$

for all $x,y \in L \subset \mathcal{U}(L)$, where $(-)^p$ is the multiplicative p-th power in $\mathcal{U}(L)$.

We give a well-known example which will be used later.

Example 1.2. Let A be an associative algebra over $\Bbbk$. We denote by A_L the induced Lie algebra with the bracket given by $[x,y] := xy - yx$ for all $x,y \in A$. Then $(A_L,(-)^p)$ is a restricted Lie algebra, where $(-)^p$ is the Frobenius map given by $x \mapsto x^p$.

Jacobson gave a necessary and sufficient condition in which an ordinary Lie algebra over $\Bbbk$ is restricted:

Lemma 1.3 [Jacobson 1962, Theorem 11]. *Let L be a Lie algebra with a $\Bbbk$-basis $\{x_i\}_{i\in I}$ for some index set I. Suppose that there exists an element $\gamma(x_i) \in L$ for each $i \in I$ such that*

$$\mathrm{ad}_{x_i}^p = \mathrm{ad}_{\gamma(x_i)}.$$

Then there exists a unique restricted structure on L such that $x_i^{[p]} = \gamma(x_i)$ for all $i \in I$.

2. Poisson algebras and their enveloping algebras

In this section we recall some definitions. We begin with some basics concerning Poisson algebras.

Definition 2.1 [Laurent-Gengoux et al. 2013, Definition 1.1]. Let A be a commutative algebra over $\Bbbk$. A *Poisson structure* on A is a Lie bracket $\{-,-\}\colon A \otimes A \to A$ such that the following Leibniz rule holds:

(E2.1.1) $$\{xy, z\} = x\{y, z\} + y\{x, z\} \quad \forall\, x, y, z \in A.$$

The algebra A together with a Poisson structure is called a *Poisson algebra*.

The Lie bracket $\{-,-\}$ (which replaces $[-,-]$ in the previous section) is called the *Poisson bracket*, and the associative multiplication of A is sometimes denoted by $\cdot$.

Recall that the module of Kähler differentials, denoted by $\Omega_{A/\Bbbk}$, of a commutative algebra A over $\Bbbk$ is an A-module generated by elements (or symbols) $\mathrm{d}x$ for all $x \in A$, and subject to the relations

$$\mathrm{d}(x+y) = \mathrm{d}x + \mathrm{d}y, \quad \mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x, \quad \mathrm{d}\lambda = 0,$$

where $x, y \in A$ and $\lambda \in \Bbbk \subseteq A$. When $(A, \{-,-\})$ is a Poisson algebra, the module of Kähler differentials $\Omega_{A/\Bbbk}$ admits a Lie algebra structure with Lie bracket given by

$$[x\,\mathrm{d}u, y\,\mathrm{d}v] = x\{u, y\}\,\mathrm{d}v + y\{x, v\}\,\mathrm{d}u + xy\,\mathrm{d}\{u, v\}$$

for all $x\,\mathrm{d}u, y\,\mathrm{d}v \in \Omega_{A/\Bbbk}$. Moreover, A is also a Lie module over $\Omega_{A/\Bbbk}$ with the action given by $(x\,\mathrm{d}u).a = x\{u, a\}$ for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$ and $a \in A$. In fact, the pair $(A, \Omega_{A/\Bbbk})$ is a Lie–Rinehart algebra in the following sense.

Definition 2.2 [Dokas 2012, Definition 1.5]. A *Lie–Rinehart algebra* over A is a pair (A, L), where A is a commutative associative algebra over $\Bbbk$ and L is a Lie algebra equipped with the structure of an A-module together with an *anchor map*

$$\alpha\colon L \to \mathrm{Der}_{\Bbbk}(A)$$

which is both an A-module and a Lie algebra homomorphism such that

(E2.2.1) $$[X, aY] = a[X, Y] + \alpha(X)(a)Y$$

for all $a \in A$ and $X, Y \in L$.

Note that, in the case of a Poisson algebra, the anchor map $\alpha : \Omega_{A/\Bbbk} \to \mathrm{Der}(A)$ is given by

(E2.2.2) $$\alpha(x\,\mathrm{d}u)(z) = x\{u, z\}$$

for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$ and $z \in A$.

Let (A, L) be a Lie–Rinehart algebra. Rinehart [1963] introduced the notion of the universal enveloping algebra $\mathcal{U}(A, L)$ of (A, L), which is an associative $\Bbbk$-algebra satisfying the appropriate universal property; see [Huebschmann 1990] for more details. We recall the definition next.

Denote by $A \rtimes L$ the semidirect product of the Lie algebra L and the L-module A. More precisely, $A \rtimes L$ is the direct sum of A and L as a vector space, and the Lie bracket is given by

$$[(a, X), (b, Y)] = (X(b) - Y(a), [X, Y])$$

for all $(a, X), (b, Y) \in A \rtimes L$. Let $(\mathcal{U}(A \rtimes L), \iota)$ be the universal enveloping algebra of the Lie algebra $A \rtimes L$, where $\iota\colon A \rtimes L \to \mathcal{U}(A \rtimes L)$ is the canonical embedding. We consider the subalgebra $\mathcal{U}^+(A \rtimes L)$ (without unit) generated by $A \rtimes L$. Moreover, $A \rtimes L$ has the structure of an A-module via $a(a', X) = (aa', aX)$ for all $a, a' \in A$ and $X \in L$. The (universal) enveloping algebra $\mathcal{U}(A, L)$ associated to the Lie–Rinehart algebra (A, L) is defined to be the quotient

$$\mathcal{U}(A, L) = \frac{\mathcal{U}^+(A \rtimes L)}{\big(\iota((a, 0))\iota((a', X)) - \iota(a(a', X))\big)}.$$

Note that $(1_A, 0)$ becomes the algebra identity of $\mathcal{U}(A, L)$. There are two canonical maps

$$\iota_1\colon A \to \mathcal{U}(A, L),\ \ a \mapsto (a, 0), \qquad \text{and} \qquad \iota_2\colon L \to \mathcal{U}(A, L),\ \ X \mapsto (0, X).$$

Observe that ι_1 is an algebra homomorphism and ι_2 is a Lie algebra homomorphism. Moreover, we have the relations

$$\iota_1(a)\iota_2(X) = \iota_2(aX) \quad \text{and} \quad [\iota_2(X), \iota_1(a)] = \iota_1(X(a))$$

for all $a \in A$ and $X \in L$.

As a consequence of [Rinehart 1963, Theorem 3.1], we have the following.

Lemma 2.3. *Let (A, L) be a Lie–Rinehart algebra and $\mathcal{U}(A, L)$ the enveloping algebra of (A, L). If L is a projective A-module, then the Lie algebra homomorphism $\iota_2\colon L \to \mathcal{U}(A, L)$ is injective.*

It is worth restating the above construction for Poisson algebras since it is needed later. Denote by $A \rtimes \Omega_{A/\Bbbk}$ the semidirect product of A and $\Omega_{A/\Bbbk}$ with the Lie bracket given by

$$[(a, x\,\mathrm{d}u), (b, y\,\mathrm{d}v)] = \big(x\{u, b\} - y\{v, a\}, x\{u, y\}\,\mathrm{d}v + y\{x, v\}\,\mathrm{d}u + xy\,\mathrm{d}\{u, v\}\big)$$

for $(a, x\,\mathrm{d}u), (b, y\,\mathrm{d}v) \in A \rtimes \Omega_{A/\Bbbk}$. The Poisson enveloping algebra of A, denoted by $\mathcal{P}(A)$ (which is a new notation), is defined to be the enveloping algebra of the

Lie–Rinehart algebra $(A, \Omega_{A/\Bbbk})$, which can be realized as an associated algebra

$$\mathcal{P}(A) := \mathcal{U}(A, \Omega_{A/\Bbbk}) = \mathcal{U}^{+}(A \rtimes \Omega_{A/\Bbbk})/J,$$

where $\mathcal{U}(A \rtimes \Omega_{A/\Bbbk})$ is the universal enveloping algebra of the Lie algebra $A \rtimes \Omega_{A/\Bbbk}$, and J is the ideal generated by

$$(a,0)(b, x\,\mathrm{d}u) - (ab, ax\,\mathrm{d}u) \tag{E2.3.1}$$

for all $a, b \in A$ and $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$ [Moerdijk and Mrčun 2010; Rinehart 1963]. Here we have two maps

$$\iota_1 \colon A \to A \rtimes \Omega_{A/\Bbbk} \to \mathcal{P}(A), \quad \iota_1(a) = (a, 0),$$

and

$$\iota_2 \colon \Omega_{A/\Bbbk} \to A \rtimes \Omega_{A/\Bbbk} \to \mathcal{P}(A), \quad \iota_2(x\,\mathrm{d}u) = (0, x\,\mathrm{d}u).$$

Then ι_1 and ι_2 are homomorphisms of associative algebras and Lie algebras, respectively. Moreover, we have

$$\iota_1(\{x, y\}) = [\iota_2(\mathrm{d}x), \iota_1(y)], \tag{E2.3.2}$$

$$\iota_2(\mathrm{d}(xy)) = \iota_1(x)\iota_2(\mathrm{d}y) + \iota_1(y)\iota_2(\mathrm{d}x) \tag{E2.3.3}$$

for all $x, y \in A$.

If $\Omega_{A/\Bbbk}$ is a projective A-module, then the canonical map $\iota_2 \colon \Omega_{A/\Bbbk} \to \mathcal{P}(A)$ is injective (Lemma 2.3). It follows that $\Omega_{A/\Bbbk}$ can be seen as a Lie subalgebra of $\mathcal{P}(A)$.

We now recall the definition of a free Poisson algebra; see [Shestakov 1993, Section 3]. Let V be $\Bbbk$-vector space. Let $\mathrm{Lie}(V)$ be the free Lie algebra generated by V. The *free Poisson algebra generated by* V, denoted by $\mathrm{FP}(V)$, is the symmetric algebra over $\mathrm{Lie}(V)$, namely

$$\mathrm{FP}(V) = \Bbbk[\mathrm{Lie}(V)]. \tag{E2.3.4}$$

The following universal property is well known [Shestakov 1993, Lemma 1, p. 312].

Lemma 2.4. *Let A be a Poisson algebra and V be a vector space. Every $\Bbbk$-linear map $g : V \to A$ extends uniquely to a Poisson algebra morphism $G : \mathrm{FP}(V) \to A$ such that g factors through G.*

Shestakov [1993, Section 3] defined the notion of a free Poisson algebra by the universal property stated in Lemma 2.4, and then proved that the free Poisson algebra can be constructed by using (E2.3.4) [Shestakov 1993, Lemma 1, p. 312]. In the same paper, Shestakov also considered the super (or $\mathbb{Z}_2$-graded) version of Poisson algebras.

For each associative commutative algebra A over a base field $\Bbbk$ of characteristic p, let A^p denote the subalgebra generated by $\{f^p \mid f \in A\}$. The free Poisson algebras have the following special property.

Lemma 2.5. *Let A be a free Poisson algebra* $\mathrm{FP}(V)$.

(1) *$\Omega_{A/\Bbbk}$ is a free module over A. As a consequence, the Lie algebra map $\iota_2\colon \Omega_{A/\Bbbk} \to \mathcal{P}(A)$ is injective.*
(2) *The kernel of* $\mathrm{d} : A \to \Omega_{A/\Bbbk}$ *is A^p.*

Proof. (1) Since A is a commutative polynomial ring, $\Omega_{A/\Bbbk}$ is free over A. (The proof is omitted). The consequence follows from Lemma 2.3.

(2) Check directly. □

Let V be a $\Bbbk$-vector space. There are two gradings that can naturally be assigned to $\mathrm{FP}(V)$. The first one is determined by

$$\deg_1(x) = 1 \quad \forall\, 0 \neq x \in \mathrm{Lie}(V).$$

Since $\mathrm{FP}(V)$ is the symmetric algebra associated to $\mathrm{Lie}(V)$, the above extends to an $\mathbb{N}$-grading on $\mathrm{FP}(V)$. Since the Lie bracket $\{-,-\}$ has degree -1, the Poisson bracket on $\mathrm{FP}(V)$ has degree -1. Note that the multiplication on $\mathrm{FP}(V)$ is homogeneous with respect to $\deg_1$.

For the second grading, we assume that

$$\deg_2(x) = 1 \quad \forall\, 0 \neq x \in V$$

and make the free Lie algebra $\mathrm{Lie}(V)$ $\mathbb{N}$-graded (namely, $[-,-]$ is homogeneous of degree zero). Then we extend the $\mathbb{N}$-grading to $\mathrm{FP}(V)$ so that both the Poisson bracket and the multiplication are homogeneous of degree zero.

Let $\{v_i\}_{i\in I}$ be a $\Bbbk$-basis of V and $\{x_j\}_{j\in J}$ a $\Bbbk$-basis of $\mathrm{Lie}(V)$. Let A be the free Poisson algebra $\mathrm{FP}(V)$ and let A^c be the A^p-submodule of A generated by monomials $x_1^{i_1}\cdots x_n^{i_n}$, for $x_1,\dots,x_n \in \mathrm{Lie}(V)$, which are not in A^p.

Recall that

$$\{f_1, f_2, \dots, f_n\} := \{f_1, \{f_2, \dots, \{f_{n-1}, f_n\}\dots\}\} \tag{E2.5.1}$$

for all $f_i \in A$.

Lemma 2.6. *Let A be a free Poisson algebra* $\mathrm{FP}(V)$.

(1) *Let $f_1,\dots,f_n$ be polynomials in v_i (not x_i). If p does not divide $n-1$, then $\{f_1, f_2, \dots, f_n\} \in A^c$.*
(2) *Let f, g be polynomials in v_i. Then $\Lambda_p(f,g) \in A^c$.*
(3) *The following elements are in A^c for any polynomials in f, g, h in v_i:*
   (a) $\Lambda_p(f,g)$, $\Lambda_p(f^2,g^2)$, $\Lambda_p(f^2+g^2, 2fg)$.

(b) $\Lambda_p(fg,h)$, $\Lambda_p((fg)^2,h^2)$, $\Lambda_p((fg)^2+h^2,2fgh)$.

(c) $\Lambda_p(fg,fh)$.

Proof. (1) By linearity, we may assume that all f_s are monomials in $\{v_i\} \subseteq V$. Then $\deg_1 f_s = \deg_2 f_s$ for $s=1,\dots,n$. Let $F := \{f_1, f_2, \dots, f_n\}$. Then

$$\deg_1 F = -n + 1 + \deg_2 F.$$

Since p does not divide $n-1$, p cannot divide both $\deg_1 F$ and $\deg_2 F$. This implies that $F \in A^c$.

(2) Note that $\Lambda_p(f,g)$ is a linear combination of terms of the form (E2.5.1) when $n=p$ and $f_i = f$ or g. By part (1), $\Lambda_p(f,g) \in A^c$.

(3) This is a special case of part (2) for different choices of f,g. □

3. Restricted Poisson algebras, definition

In this section we present a formulation of a restricted Poisson algebra that is equivalent to [Bezrukavnikov and Kaledin 2008, Definition 1.8].

Inspired by the notion of a restricted Lie algebra, we first introduce the definition of a weakly restricted Poisson structure over a field $\Bbbk$ of characteristic $p \geq 3$.

Definition 3.1. Let $(A,\cdot,\{-,-\})$ be a Poisson algebra. If there exists a p-map $(-)^{\{p\}}\colon A \to A$ such that $(A,\{-,-\},(-)^{\{p\}})$ is a restricted Lie algebra, then A is called a *weakly restricted Poisson algebra*.

This definition requires no compatibility condition between the p-map $(-)^{\{p\}}$ and the multiplication $\cdot$. We will see that an additional requirement is very natural from a Lie-algebraic point of view.

Lemma 3.2. *Let $(A,\cdot,\{-,-\})$ be a Poisson algebra and let $x,y \in A$.*

(1) *If there exist $\widetilde{x}$ and $\widetilde{y}$ in A such that $\mathrm{ad}_x^p = \mathrm{ad}_{\widetilde{x}}$ and $\mathrm{ad}_y^p = \mathrm{ad}_{\widetilde{y}}$, then*

$$\mathrm{ad}_{xy}^p = \mathrm{ad}_{x^p\widetilde{y}+y^p\widetilde{x}+\Phi_p(x,y)},$$

where

(E3.2.1) $$\Phi_p(x,y) = (x^p+y^p)\Lambda_p(x,y) - \tfrac{1}{2}\big(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\big).$$

In particular, $\mathrm{ad}_{x^2}^p = \mathrm{ad}_{2x^p\widetilde{x}}$.

(2) *If $(A,\cdot,\{-,-\})$ is a weakly restricted Poisson algebra, then*

(E3.2.2) $$\mathrm{ad}_{(xy)^{\{p\}}} = \mathrm{ad}_{x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x,y)}.$$

In particular,

(E3.2.3) $$\mathrm{ad}_{(x^2)^{\{p\}}} = \mathrm{ad}_{2x^p x^{\{p\}}}.$$

Proof. (1) We first prove the assertion when $x = y$. By the Leibniz rule, we have $\mathrm{ad}_{(fg)} = f\,\mathrm{ad}_g + g\,\mathrm{ad}_f$ for any $f, g \in A$. Clearly,

$$\mathrm{ad}^p_{x^2} = (2x\,\mathrm{ad}_x)^p = (2x)^p(\mathrm{ad}_x)^p = 2x^p\,\mathrm{ad}^p_x = 2x^p\,\mathrm{ad}_{\tilde{x}} = \mathrm{ad}_{2x^p\tilde{x}}.$$

In the general case, considering the universal enveloping algebra of the Lie algebra $(A, \{-,-\})$ and using (E1.1.4), we get $\mathrm{ad}_{\Lambda_p(f,g)} = \mathrm{ad}^p_{f+g} - \mathrm{ad}^p_f - \mathrm{ad}^p_g$ for any $f, g \in A$. Therefore,

$$\begin{aligned}
\mathrm{ad}_{x^p\tilde{y}+y^p\tilde{x}+\Phi(x,y)} &= \mathrm{ad}_{x^p\tilde{y}+y^p\tilde{x}+(x^p+y^p)\Lambda_p(x,y)-\frac{1}{2}\left(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\right)}\\
&= x^p\,\mathrm{ad}^p_y + y^p\,\mathrm{ad}^p_x + (x^p+y^p)(\mathrm{ad}^p_{x+y} - \mathrm{ad}^p_x - \mathrm{ad}^p_y)\\
&\qquad + \tfrac{1}{2}(\mathrm{ad}^p_{x^2} + \mathrm{ad}^p_{y^2} + \mathrm{ad}^p_{2xy} - \mathrm{ad}^p_{(x+y)^2})\\
&= x^p\,\mathrm{ad}^p_y + y^p\,\mathrm{ad}^p_x + (x^p+y^p)(\mathrm{ad}^p_{x+y} - \mathrm{ad}^p_x - \mathrm{ad}^p_y)\\
&\qquad + x^p\,\mathrm{ad}^p_x + y^p\,\mathrm{ad}^p_y + \mathrm{ad}^p_{xy} - (x+y)^p\,\mathrm{ad}^p_{x+y}\\
&= \mathrm{ad}^p_{xy},
\end{aligned}$$

which completes the proof.

(2) This is an immediate consequence of (1). □

Concerning the notation Φ_p in (E3.2.1), we also have the following characterization by considering the Poisson enveloping algebra.

Proposition 3.3. *Let A be a Poisson algebra and $\mathcal{P}(A)$ the Poisson enveloping algebra of A. Then, for all $x, y \in A$, we have*

(E3.3.1) $\iota_2(\mathrm{d}\Phi_p(x,y)) = (\iota_2(\mathrm{d}(xy)))^p - \iota_1(x^p)(\iota_2(\mathrm{d}y))^p - \iota_1(y^p)(\iota_2(\mathrm{d}x))^p.$

Proof. By the definition of $\mathcal{P}(A)$, we have

$$(0, \mathrm{d}x^2)^p = (0, 2x\,\mathrm{d}x)^p = ((2x,0)(0,\mathrm{d}x))^p = (2x,0)^p(0,\mathrm{d}x)^p = 2(x^p,0)(0,\mathrm{d}x)^p$$

and hence

(E3.3.2) $$(\iota_2(\mathrm{d}x^2))^p = 2\iota_1(x^p)(\iota_2(\mathrm{d}x))^p$$

for any $x \in A$. It follows that (E3.3.1) holds when $x = y$.

Considering the Frobenius map of $\mathcal{P}(A)$, we have

$$\begin{aligned}
(\iota_2(\mathrm{d}(x+y)))^p &= (0, \mathrm{d}(x+y))^p = ((0,\mathrm{d}x) + (0,\mathrm{d}y))^p\\
&= (0,\mathrm{d}x)^p + (0,\mathrm{d}y)^p + \Lambda_p((0,\mathrm{d}x),(0,\mathrm{d}y))\\
&= (\iota_2(\mathrm{d}x))^p + (\iota_2(\mathrm{d}y))^p + \iota_2(\mathrm{d}\Lambda_p(x,y))
\end{aligned}$$

since ι_2 is a homomorphism of Lie algebras. By the above computation and (E3.3.2),

$$\begin{aligned}(\iota_2(\mathrm{d}(x+y)^2))^p &= 2\iota_1((x+y)^p)(\iota_2(\mathrm{d}(x+y)))^p\\ &= 2\iota_1(x^p+y^p)\big((\iota_2(\mathrm{d}x))^p+(\iota_2(\mathrm{d}y))^p+\iota_2(\mathrm{d}\Lambda_p(x,y))\big).\end{aligned}$$

By a direct calculation and (E3.3.2),

$$\begin{aligned}(\iota_2(\mathrm{d}(x+y)^2))^p &= (\iota_2(\mathrm{d}x^2+\mathrm{d}y^2+2\,\mathrm{d}(xy)))^p\\ &= (\iota_2(\mathrm{d}x^2+\mathrm{d}y^2))^p+(\iota_2(2\,\mathrm{d}(xy)))^p+\iota_2(\mathrm{d}\Lambda_p(x^2+y^2,2xy))\\ &= (\iota_2(\mathrm{d}x^2))^p+(\iota_2(\mathrm{d}y^2))^p+\iota_2(\mathrm{d}\Lambda_p(x^2,y^2))\\ &\qquad +2(\iota_2(\mathrm{d}(xy)))^p+\iota_2(\mathrm{d}\Lambda_p(x^2+y^2,2xy))\\ &= 2\iota_1(x^p)(\iota_2(\mathrm{d}x))^p+2\iota_1(y^p)(\iota_2(\mathrm{d}y))^p+\iota_2(\mathrm{d}\Lambda_p(x^2,y^2))\\ &\qquad +2(\iota_2(\mathrm{d}(xy)))^p+\iota_2(\mathrm{d}\Lambda_p(x^2+y^2,2xy)).\end{aligned}$$

Comparing the above two equations, we get

$$\begin{aligned}&(\iota_2(\mathrm{d}(xy)))^p+\tfrac{1}{2}\big(\iota_2\big(\mathrm{d}\big(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\big)\big)\big)\\ &\quad= \iota_1(x^p)(\iota_2(\mathrm{d}y))^p+\iota_1(y^p)(\iota_2(\mathrm{d}x))^p+\iota_1(x^p+y^p)\iota_2(\mathrm{d}\Lambda_p(x,y))\\ &\quad= \iota_1(x^p)(\iota_2(\mathrm{d}y))^p+\iota_1(y^p)(\iota_2(\mathrm{d}x))^p+\iota_2\big(\mathrm{d}((x^p+y^p)\Lambda_p(x,y))\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\iota_2(\mathrm{d}\Phi_p(x,y))\\ &\quad= \iota_2\big(\mathrm{d}\big((x^p+y^p)\Lambda_p(x,y)-\tfrac{1}{2}\big(\Lambda_p(x^2,y^2)+\Lambda_p(x^2+y^2,2xy)\big)\big)\big)\\ &\quad= (\iota_2(\mathrm{d}(xy)))^p-\iota_1(x^p)(\iota_2(\mathrm{d}y))^p-\iota_1(y^p)(\iota_2(\mathrm{d}x))^p. \qquad \square\end{aligned}$$

For a weakly restricted Poisson algebra, it is desirable to consider the compatibility between the p-map and the associative multiplication. By removing ad from (E3.2.3) (which can be done in some cases), we obtain (E3.4.1) below. Similarly, if we remove ad from (E3.2.2), we obtain (E3.5.1) below. Both Lemma 3.2 and Proposition 3.3 suggest the following definition. Following Lemma 3.2(2), condition (E3.4.1) is forced.

Definition 3.4. Let $(A,\cdot,\{-,-\},(-)^{\{p\}})$ be a weakly restricted Poisson algebra over $\Bbbk$. We call A a *restricted Poisson algebra* if, for every $x\in A$,

$$(x^2)^{\{p\}} = 2x^p x^{\{p\}}. \tag{E3.4.1}$$

In this case, the p-map $(-)^{\{p\}}$ is a *restricted Poisson structure* on A.

Next we give another description of condition (E3.4.1) which is convenient for some computations.

Proposition 3.5. *Let A be a weakly restricted Poisson algebra.*

(1) *Suppose* (E3.4.1) *holds. Then $(\lambda 1_A)^{\{p\}} = 0$ for all $\lambda \in \Bbbk$.*

(2) *Equation* (E3.4.1) *holds for all $x \in A$ if and only if every pair of elements (x, y) in A satisfies*

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x, y). \tag{E3.5.1}$$

Consequently, A is a restricted Poisson algebra if and only if (E3.5.1) *holds.*

(3) *Suppose* (E3.5.1) *holds. Then*

$$(x^n)^{\{p\}} = n x^{(n-1)p} x^{\{p\}} \tag{E3.5.2}$$

for all n. As a consequence, $(x^p)^{\{p\}} = 0$ for all $x \in A$.

(4) *If $(1_A)^{\{p\}} = 0$, then* (E3.5.1) *holds for pairs $(x, \lambda 1_A)$ and $(\lambda 1_A, x)$ for all $x \in A$ and all $\lambda \in \Bbbk$.*

Proof. (1) Clearly, $1_A^{\{p\}} = 2 \cdot 1_A^p 1_A^{\{p\}}$ and hence $1_A^{\{p\}} = 0$. For every $\lambda \in \Bbbk$, $(\lambda 1_A)^{\{p\}} = \lambda^p 1_A^{\{p\}} = 0$.

(2) The "if" part is trivial since $\Phi_p(x, x) = 0$ for any $x \in A$. Next, we show the "only if" part. By (E3.4.1) and Definition 1.1(3), we have

$$((x + y)^2)^{\{p\}} = 2(x + y)^p (x + y)^{\{p\}} = 2(x^p + y^p)(x^{\{p\}} + y^{\{p\}} + \Lambda_p(x, y)).$$

Since $(A, \{-,-\}, (-)^{\{p\}})$ is a restricted Lie algebra, by Definition 1.1(2,3) we have

$$\begin{aligned}
&((x + y)^2)^{\{p\}} \\
&\quad = (x^2 + y^2 + 2xy)^{\{p\}} \\
&\quad = (x^2 + y^2)^{\{p\}} + 2^p (xy)^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy) \\
&\quad = (x^2)^{\{p\}} + (y^2)^{\{p\}} + \Lambda_p(x^2, y^2) + 2^p (xy)^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy) \\
&\quad = 2x^p x^{\{p\}} + 2y^p y^{\{p\}} + \Lambda_p(x^2, y^2) + 2(xy)^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy).
\end{aligned}$$

Comparing the above two equations and using $2 \neq 0$, we obtain (E3.5.1).

(3) This follows by induction.

(4) First of all, $(\lambda 1_A)^{\{p\}} = \lambda^p 1_A^{\{p\}} = 0$ for all $\lambda \in \Bbbk$. The assertion follows by the fact $\Phi_p(\lambda 1_A, x) = \Phi_p(x, \lambda 1_A) = 0$. □

Remark 3.6. Several remarks are collected below.

(1) As in [Bezrukavnikov and Kaledin 2008], we assume that $p \geq 3$. So the polynomial $\Phi_p(x, y)$ in (E3.2.1) is well defined. When $p = 3$, we have

$$\Phi_3(x, y) = x^2 y\{y, y, x\} + xy^2\{x, x, y\} + xy\{x, y\}^2.$$

For $p > 3$, it is too long to write out all the terms as above.

(2) Considering $\Phi_p(x, y)$ as an element in $\mathrm{FP}(V)$, where $V = \Bbbk x \oplus \Bbbk y$, it is homogeneous of degree $p+1$ with respect to $\deg_2$ and homogeneous of degree $2p$ with respect to $\deg_1$.

(3) Bezrukavnikov and Kaledin [2008, Definition 1.8] defined a *restricted Poisson algebra* as a weakly restricted Poisson algebra $(A, \{-,-\}, (-)^{\{p\}})$ such that the p-map satisfies

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + P(x, y) \tag{E3.6.1}$$

for all $x, y \in A$. Here $P(x, y)$ is a canonical quantized polynomial determined by [Bezrukavnikov and Kaledin 2008, (1.3)]. We will show that (E3.6.1) is equivalent to (E3.5.1).

(4) $P(x, y)$ is defined implicitly, but it follows from [Bezrukavnikov and Kaledin 2008, (1.3)] that $P(x, x) = 0$. Therefore a restricted Poisson algebra in the sense of [Bezrukavnikov and Kaledin 2008, Definition 1.8] is a restricted Poisson algebra in the sense of Definition 3.4.

(5) There are other interpretations of $\Phi_p(x, y)$. Using

$$xy = \tfrac{1}{4}[(x+y)^2 - (x-y)^2],$$

we obtain that

$$(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi'_p(x, y), \tag{E3.6.2}$$

where

$$\begin{aligned}\Phi'_p(x, y) = {} & \tfrac{1}{4}\Lambda_p\big((x+y)^2, -(x-y)^2\big) \\ & + \tfrac{1}{2}\big((x^p + y^p)\Lambda_p(x, y) - (x^p - y^p)\Lambda_p(x, -y)\big).\end{aligned} \tag{E3.6.3}$$

One can show that $\Phi_p(x, y) = \Phi'_p(x, y)$ in the free Poisson algebra generated by x and y.

(6) The following are clear by definition.

(a) $\Lambda_p(x, y) = \Lambda_p(y, x)$ for all $x, y \in A$.

(b) If $\{x, y\} = 0$, then $\Lambda_p(x, y) = 0$.

(c) $\Phi_p(x, y) = \Phi_p(y, x)$ for all $x, y \in A$.

(d) If $\{x, y\} = 0$, then $\Phi_p(x, y) = 0$.

Lemma 3.7. *The definitions of restricted Poisson algebras in Definition 3.4 and* [Bezrukavnikov and Kaledin 2008, Definition 1.8] *are equivalent.*

Proof. Let $P(x, y)$ be the polynomial defined in [Bezrukavnikov and Kaledin 2008, (1.3)]. By Proposition 3.5(2), it remains to show that $P(x, y) = \Phi_p(x, y)$. Let $\mathrm{Lie}(V)$ be the free Lie algebra over a vector space V and consider the tensor (free) algebra $T(V)$ as a universal enveloping algebra over $\mathrm{Lie}(V)$. Then we have a Poincaré–Birkhoff–Witt filtration on $T(V)$. The free quantized algebra $Q^\bullet(V)$ is the Rees algebra associated to this filtration. By definition, for each n,

$$\mathcal{F}_n := F_nT(V) = \Bbbk \oplus L^\bullet(V) \oplus (L^\bullet(V))^2 \oplus \cdots \oplus (L^\bullet(V))^n.$$

We are omitting the symbol h which represents the natural embedding $h : \mathcal{F}_\bullet \to \mathcal{F}_{\bullet+1}$ in the Rees ring. Taking $V = \Bbbk x \oplus \Bbbk y$, inside the Rees ring we have

$$\begin{aligned}(x^p + y^p)^2 &+ (\Lambda_p(x,y))^2 + \Lambda_p(x,y)(x^p+y^p) + (x^p+y^p)\Lambda_p(x,y)\\ &= (x^p + y^p + \Lambda_p(x,y))^2\\ &= (x+y)^{2p}\\ &= (x^2+y^2+xy+yx)^p\\ &= (x^2+y^2)^p + (xy+yx)^p + \Lambda_p(x^2+y^2, xy+yx)\\ &= x^{2p} + y^{2p} + \Lambda_p(x^2,y^2) + (xy)^p + (yx)^p + \Lambda_p(xy,yx)\\ &\qquad + \Lambda_p(x^2+y^2, xy+yx),\end{aligned}$$

and hence

$$\begin{aligned}(xy)^p &+ (yx)^p - x^py^p - y^px^p\\ &= \Lambda_p(x,y)(x^p+y^p) + (x^p+y^p)\Lambda_p(x,y) + (\Lambda_p(x,y))^2\\ &\qquad - \Lambda_p(x^2,y^2) - \Lambda_p(xy,yx) - \Lambda_p(x^2+y^2, xy+yx).\end{aligned}$$

On the other hand,

$$[x,y]^p = (xy-yx)^p = (xy)^p - (yx)^p + \Lambda_p(xy, -yx).$$

So we have

$$\begin{aligned}2P(x,y) &= 2((xy)^p - x^py^p)\\ &= \Lambda_p(x,y)(x^p+y^p) + (x^p+y^p)\Lambda_p(x,y) - \Lambda_p(x^2,y^2)\\ &\qquad - \Lambda_p(x^2+y^2, xy+yx) + (\Lambda_p(x,y))^2 - \Lambda_p(xy,yx)\\ &\qquad - \Lambda_p(xy,-yx) + [x,y]^p - [x^p,y^p].\end{aligned}$$

In fact, it is easily seen that $(\Lambda_p(x,y))^2 \in \mathcal{F}_2$, $[x,y]^p \in \mathcal{F}_p$. On the other hand,

$$[x^p, y^p] = \mathrm{ad}_x^p(y^p) = -\,\mathrm{ad}_x^{p-1}(\mathrm{ad}_y^p(x)) \in \mathcal{F}_1,$$

where $\mathrm{ad}_x(y) = [x,y]$. By (E1.1.3), we have

$$\Lambda_p(xy,yx) = \sum_{x_k = xy \text{ or } yx} \frac{1}{\#(xy)}\, \mathrm{ad}_{x_1}\cdots \mathrm{ad}_{x_{p-2}}([yx,xy]),$$

where $\#(xy)$ is the number of xy in the collection of possibly repeated elements $\{x_1, x_2, \dots, x_{p-2}, yx, yx\}$. Since

$$[yx, xy] = [yx, yx + [x, y]] = [yx, [x, y]] \in \mathcal{F}_2,$$

we have $\Lambda_p(xy, yx) \in \mathcal{F}_p$. Similarly, $\Lambda_p(xy, -yx) \in \mathcal{F}_p$. By definition (see [Bezrukavnikov and Kaledin 2008, (1.3)]), $P(x, y)$ is homogeneous of degree $p+1$. Therefore, after removing lower-degree components,

$$2P(x, y) = \Lambda_p(x, y)(x^p + y^p) + (x^p + y^p)\Lambda_p(x, y) \\ - \Lambda_p(x^2, y^2) - \Lambda_p(x^2 + y^2, xy + yx).$$

Since the multiplication is commutative in a Poisson algebra, we have

$$P(x, y) = (x^p + y^p)\Lambda_p(x, y) - \tfrac{1}{2}\big(\Lambda_p(x^2, y^2) + \Lambda_p(x^2 + y^2, 2xy)\big) = \Phi_p(x, y). \quad \square$$

4. Elementary properties and examples

We start with something obvious.

Definition 4.1. Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a restricted Poisson algebra. A Poisson ideal I of A is said to be *restricted* if $x^{\{p\}} \in I$ for any $x \in I$.

The proofs of the following three assertions are easy and omitted.

Lemma 4.2. *Let A be a restricted Poisson algebra. Suppose that I is a Poisson ideal of A that is generated by $\{x_i \mid i \in S\}$ as an ideal of the commutative ring A. If $x_i^{\{p\}} \in I$ for any $i \in S$, then I is a restricted Poisson ideal.*

Proposition 4.3. *Let A be a restricted Poisson algebra and I a restricted Poisson ideal of A. Then the quotient Poisson algebra A/I is a restricted Poisson algebra.*

Clearly, we have the following fact.

Proposition 4.4. *Let $f : A \to A'$ be a homomorphism of restricted Poisson algebras. Then* Ker f *is a restricted Poisson ideal of A.*

Let A^p be the subalgebra of A generated by $\{f^p \mid f \in A\}$ — the image of the Frobenius map.

Lemma 4.5. *Let A be a Poisson algebra and $f, g, h \in A$. Then the following hold*:

(1) $f^p \Phi_p(g, h) - \Phi_p(fg, h) + \Phi_p(f, gh) - h^p \Phi_p(f, g) = 0$.

(2) *If f is in the Poisson center of A, then* $f^p \Phi_p(g, h) = \Phi_p(fg, h) = \Phi_p(g, fh)$.

(3) $\Phi_p(f, g + h) - \Phi_p(f, g) - \Phi_p(f, h) = \Lambda_p(fg, fh) - f^p \Lambda_p(g, h)$.

Proof. It is clear that (2) is a consequence of (1). It suffices to show assertions (1) and (3) for the free Poisson algebra $\mathrm{FP}(A)$ since there is a surjective Poisson algebra map $\mathrm{FP}(A) \to A$ (Lemma 2.4). So the hypothesis becomes that f, g, h are in a $\Bbbk$-space V sitting inside a free Poisson algebra $\mathrm{FP}(V)$.

When A is a free Poisson algebra $\mathrm{FP}(V)$, by Lemma 2.5(1), ι_2 is injective. It follows from Lemma 2.5(2) that

(a) the kernel of the map

$$A \xrightarrow{\mathrm{d}} \Omega_{A/\Bbbk} \xrightarrow{\iota_2} \mathcal{P}(A)$$

is A^p.

Let $\{v_i\}_{i\in S}$ be a basis of the V. Let A^c be the A^p-submodule of $A = \mathrm{FP}(V)$ defined before Lemma 2.6. Then

(b) $A^c \cap A^p = \{0\}$ and $\Lambda_p(x, y) \in A^c$ for all $x, y \in \Bbbk[V]$ by Lemma 2.6(2).

Now we prove (1) and (3) under conditions (a) and (b).

(1) For all $f, u \in A$, we have $\mathrm{d}(f^p u) = f^p\,\mathrm{d}u$ and $\iota_2(\mathrm{d}(f^p u)) = (f^p, 0)(0, \mathrm{d}u) \in \mathcal{P}(A)$. By Proposition 3.3,

$$\begin{aligned}
\iota_2(\mathrm{d}(f^p\Phi_p(g,h))) &= (f^p,0)(0,\mathrm{d}(gh))^p - (f^p g^p,0)(0,\mathrm{d}h)^p - (f^p h^p,0)(0,\mathrm{d}g)^p,\\
\iota_2(\mathrm{d}\Phi_p(fg,h)) &= (0,\mathrm{d}(fgh))^p - ((fg)^p,0)(0,\mathrm{d}h)^p - (h^p,0)(0,\mathrm{d}(fg))^p,\\
\iota_2(\mathrm{d}\Phi_p(f,gh)) &= (0,\mathrm{d}(fgh))^p - (f^p,0)(0,\mathrm{d}(gh))^p - ((gh)^p,0)(0,\mathrm{d}f)^p,\\
\iota_2(\mathrm{d}(h^p\Phi_p(f,g))) &= (h^p,0)(0,\mathrm{d}(fg))^p - (h^p f^p,0)(0,\mathrm{d}g)^p - (h^p g^p,0)(0,\mathrm{d}f)^p
\end{aligned}$$

for all $f, g, h \in V$. It follows that

$$\iota_2(\mathrm{d}(f^p\Phi_p(g,h) - \Phi_p(fg,h) + \Phi_p(f,gh) - \Phi_p(f,g)h^p)) = 0.$$

By condition (a), we get

$$X := f^p\Phi_p(g,h) - \Phi_p(fg,h) + \Phi_p(f,gh) - h^p\Phi_p(f,g) \in A^p.$$

By definition, X is in the A^p-submodule generated by $\Lambda_p(x, y)$ for all $x, y \in A$, or in A^c as given in condition (b). But since $A^p \cap A^c = \{0\}$ by condition (b), we obtain that $X = 0$ and that the desired identity holds.

(3) The proof is similar to that of (1) and is omitted. □

Proposition 4.6. *Let A be a weakly restricted Poisson algebra.*

(1) *If (x, y) satisfies (E3.5.1), then so do $(x, \lambda y)$ and $(\lambda x, y)$ for all $\lambda \in \Bbbk$.*

(2) *Let $f, g, h \in A$. Suppose that (f, g) and (g, h) satisfy (E3.5.1). Then (fg, h) satisfies (E3.5.1) if and only if (f, gh) does.*

(3) *If (f, g) and (f, h) satisfy (E3.5.1), then so does $(f, g + h)$.*

(3′) *If* (g, f) *and* (h, f) *satisfy* (E3.5.1), *then so does* $(g+h, f)$.

(4) *Fix an* $x \in A$ *and let* R_x *be the set of* $y \in A$ *such that* (x, y) *satisfies* (E3.5.1). *Then* R_x *is a* $\Bbbk$*-subspace of* A.

(4′) *Fix an* $x \in A$ *and let* L_x *be the set of* $y \in A$ *such that* (y, x) *satisfies* (E3.5.1). *Then* L_x *is a* $\Bbbk$*-subspace of* A.

Proof. (1) Assuming (E3.5.1) for (x, y), we have

$$\begin{aligned}(x\lambda y)^{\{p\}} &= (\lambda(xy))^{\{p\}} = \lambda^p (xy)^{\{p\}}\\ &= \lambda^p(x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x, y))\\ &= x^p\big((\lambda y)^{\{p\}} + (\lambda y)^p x^{\{p\}} + \lambda^p \Phi_p(x, y)\big)\\ &= x^p\big((\lambda y)^{\{p\}} + (\lambda y)^p x^{\{p\}} + \Phi_p(x, \lambda y)\big),\end{aligned}$$

where the last equation is Lemma 4.5(2). So $(x, \lambda y)$ satisfies (E3.5.1). Similarly for $(\lambda x, y)$.

(2) By symmetry, we only prove one implication and assume that (fg, h) satisfies (E3.5.1). We show next that (f, gh) satisfies (E3.5.1):

$$\begin{aligned}(f(gh))^{\{p\}} &= ((fg)h)^{\{p\}} = (fg)^p h^{\{p\}} + h^p (fg)^{\{p\}} + \Phi_p(fg, h)\\ &= (fg)^p h^{\{p\}} + h^p\big(f^p g^{\{p\}} + g^p f^{\{p\}} + \Phi_p(f, g)\big) + \Phi_p(fg, h)\\ &= f^p g^p h^{\{p\}} + f^p h^p g^{\{p\}} + g^p h^p f^{\{p\}} + \Phi_p(fg, h) + h^p \Phi_p(f, g)\\ &= f^p g^p h^{\{p\}} + f^p h^p g^{\{p\}} + g^p h^p f^{\{p\}}\\ &\qquad\qquad + f^p \Phi_p(g, h) + \Phi_p(f, gh) \qquad \text{by Lemma 4.5(1)}\\ &= f^p\big(g^p h^{\{p\}} + h^p g^{\{p\}} + \Phi_p(g, h)\big) + (gh)^p f^{\{p\}} + \Phi_p(f, gh)\\ &= f^p (gh)^{\{p\}} + (gh)^p f^{\{p\}} + \Phi_p(f, gh).\end{aligned}$$

(3) Assume (f, g) and (f, h) satisfy (E3.5.1). Then

$$\begin{aligned}(f(g+h))^{\{p\}} &= (fg + fh)^{\{p\}}\\ &= (fg)^{\{p\}} + (fh)^{\{p\}} + \Lambda_p(fg, fh)\\ &= f^p g^{\{p\}} + g^p x^{\{p\}} + \Phi_p(f, g) + x^p h^{\{p\}} + h^p f^{\{p\}}\\ &\qquad\qquad + \Phi_p(f, h) + \Lambda_p(fg, fh)\\ &= f^p\big(g^{\{p\}} + h^{\{p\}} + \Lambda_p(g, h)\big) + (g+h)^p f^{\{p\}} + \Phi_p(f, g+h)\\ &= f^p (g+h)^{\{p\}} + (g+h)^p f^{\{p\}} + \Phi_p(f, g+h),\end{aligned}$$

where the second-to-last equality is deduced from Lemma 4.5(3). So $(f, g+h)$ satisfies (E3.5.1).

(3′) is equivalent to (3).

(4) Let

$$R_x = \{y \in A \mid \text{(E3.5.1) holds for the pair } (x, y)\}.$$

By Proposition 4.6(1), we have

(i) if $y \in R_x$, then so is λy for all $\lambda \in \Bbbk$.

By Proposition 4.6(3),

(ii) if $g, h \in R_x$, then so is $g + h$.

By (i) and (ii) above, R_x is a $\Bbbk$-subspace of A.

(4′) This is true because $L_x = R_x$. □

The following result will be used several times.

Theorem 4.7. *Let A be a weakly restricted Poisson algebra. Let $\boldsymbol{b} := \{b_i\}_{i \in S}$ be a $\Bbbk$-basis of A. If (E3.5.1) holds for every pair $(x, y) \subseteq \boldsymbol{b}$, then A is a restricted Poisson algebra.*

Proof. We need to show that (E3.5.1) holds for all $x, y \in A$. First we fix any $x \in \boldsymbol{b}$ and let

$$R_x = \{y \in A \mid \text{(E3.5.1) holds for the pair } (x, y)\}.$$

By Proposition 4.6(4), R_x is a $\Bbbk$-subspace of A. By hypothesis, we see that $\boldsymbol{b} \subseteq R_x$. Since $\boldsymbol{b}$ is a basis of A, $R_x = A$.

Next we fix $y \in A$ and consider

$$L_y = \{x \in A \mid \text{(E3.5.1) holds for the pairs } (x, y)\}.$$

Similarly, by Proposition 4.6(4′), L_y is a $\Bbbk$-subspace. It contains $\boldsymbol{b}$ because $R_x = A$ for all $x \in \boldsymbol{b}$ (see the first paragraph). Hence, $L_y = A$. This means that (E3.5.1) holds for all pairs (x, y) in A. Therefore A is a restricted Poisson algebra. □

One of the main goals of this paper is to provide some interesting examples of restricted Poisson algebras. In the rest of this section we give some elementary (but nontrivial) examples. We would like to give a gentle warning before the examples. We have checked that all p-maps given below satisfy (E3.5.1); however, our proofs are tedious computations and therefore omitted. On the other hand, since the p-maps are explicitly expressed by partial derivatives, one can verify the assertions with enough patience. More-sophisticated examples are given in later sections.

Example 4.8. Let $A = \Bbbk[x, y]$ be a polynomial algebra in two variables x, y, where the (classical) Poisson bracket is given by

$$\{f, g\} = f_x g_y - f_y g_x \tag{E4.8.1}$$

for all $f, g \in A$, and f_x and f_y are the partial derivatives of f with respect to the variables x and y, respectively. (The bracket defined in (E4.8.1) was the original Poisson bracket studied by many people including Poisson [1809] when $\Bbbk = \mathbb{R}$.)

(1) Let $\Bbbk$ be a base field of characteristic 3. For every $f \in A$, we define

(E4.8.2) $$f^{\{3\}} = f_x^2 f_{yy} + f_y^2 f_{xx} + f_x f_y f_{xy},$$

where f_{xx}, f_{yy} and f_{xy} are the second order partial derivatives of f. Then $(A, \cdot, \{-,-\}, (-)^{\{3\}})$ is a restricted Poisson algebra.

(2) Let $\Bbbk$ be a base field of characteristic 5. For every $f \in A$, define

(E4.8.3) $$\begin{aligned} f^{\{5\}} = {} & f_1^4 f_{2222} + f_1^3 f_2 f_{1222} + f_1^2 f_2^2 f_{1122} + f_1 f_2^3 f_{1112} + f_2^4 f_{1111} \\ & + f_{12}(f_1^3 f_{222} - f_1^2 f_2 f_{122} - f_1 f_2^2 f_{112} + f_2^3 f_{111}) \\ & - f_1 f_{22}(f_1^2 f_{122} - 2 f_1 f_2 f_{112} + f_2^2 f_{111}) \\ & - f_2 f_{11}(f_2^2 f_{112} - 2 f_2 f_1 f_{122} + f_1^2 f_{222}) \\ & + 2(f_{12}^2 - f_{11} f_{22})(f_1^2 f_{22} - 2 f_1 f_2 f_{12} + f_2^2 f_{11}), \end{aligned}$$

where $f_{i_1 i_2 \cdots i_k}$ denotes the k-th order partial derivative of f with respect to the variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$. Then $(A, \cdot, \{-,-\}, (-)^{\{5\}})$ is a restricted Poisson algebra.

See Example 7.3 for general p. It would be interesting to understand the meaning of (E4.8.2) and (E4.8.3) and to find its connection with other subjects.

The next two are slight generalizations of the previous example.

Example 4.9. Suppose $\operatorname{char} \Bbbk = 3$ and let $A = \Bbbk[x, y]$ be a polynomial Poisson algebra in two variables x, y, where the Poisson bracket is given by

$$\{f, g\} = \varphi(f_x g_y - f_y g_x),$$

and $\varphi = \lambda x + \mu y + \nu$, $\lambda, \mu, \nu \in \Bbbk$. For every $f \in A$, we define

(E4.9.1) $$\begin{aligned} f^{\{3\}} = {} & \lambda \varphi f_x f_y^2 + \mu \varphi f_x^2 f_y \\ & + \varphi^2 (f_x^2 f_{yy} + f_y^2 f_{xx} + f_x f_y f_{xy}) + \lambda^2 y f_y^3 + \mu^2 x f_x^3. \end{aligned}$$

Then $(A, \cdot, \{-,-\}, (-)^{\{3\}})$ is a restricted Poisson algebra.

Example 4.10. Suppose $\operatorname{char} \Bbbk = 3$ and let $A = \Bbbk[x_1, x_2, \ldots, x_n]$ be a Poisson algebra, where the Lie bracket is given by $\{x_i, x_j\} = 2c_{ij} \in \Bbbk$ with $c_{ij} + c_{ji} = 0$ for $1 \le i, j \le n$. Clearly, $\{f, g\} = \sum_{1 \le i, j \le n} c_{ij}(f_i g_j - f_j g_i)$ for $f, g \in A$, where f_i denotes the partial derivative of f with respect to the variable x_i for $i = 1, 2, \ldots, n$.

Then A is a restricted Poisson algebra with the p-map given by

$$f^{\{3\}} = \sum_{1 \le i,j,k,l \le n} c_{ij} c_{kl} f_i f_k f_{jl}$$

for any $f \in A$, where f_{jl} is the second partial derivative of f with respect to the variables x_j and x_l.

5. Existence and uniqueness of restricted structures

By Lemma 3.2(2), a weakly restricted Poisson structure on a Poisson algebra is very close to a restricted Poisson structure (up to a factor in the Poisson center). In this section, we study the existence and uniqueness of (weakly) restricted Poisson structures. First we consider the trivial extension.

Lemma 5.1. *Let A be a Poisson algebra, and let $A = \Bbbk 1_A \oplus \mathfrak{m}$ be its decomposition as a Lie algebra.*

(1) *If $x \mapsto x^{\{p\}}$ is a restriction p-map of the Lie algebra $\mathfrak{m}$, it can naturally be extended to A by defining $1_A^{\{p\}} = 0$. As a consequence, A is a weakly restricted Poisson algebra.*

(2) *If, further, the p-map on $\mathfrak{m}$ satisfies (E3.4.1), then so does the extended p-map on A. In this case, A is a restricted Poisson algebra.*

Proof. (1) This follows from Lemma 1.3. For all $\lambda \in \Bbbk$ and $x \in \mathfrak{m}$, the p-map is defined by $(\lambda 1_A + x)^{\{p\}} = x^{\{p\}}$.

(2) We check (E3.4.1) for elements in A as follows:

$$\begin{aligned}
((\lambda 1_A + x)^2)^{\{p\}} &= (\lambda^2 1_A + 2\lambda x + x^2)^{\{p\}} \\
&= (2\lambda x + x^2)^{\{p\}} = (2\lambda x)^{\{p\}} + (x^2)^{\{p\}} \\
&= 2\lambda^p x^{\{p\}} + 2x^p x^{\{p\}} = 2(\lambda 1_A + x)^p x^{\{p\}} \\
&= 2(\lambda 1_A + x)^p (\lambda 1_A + x)^{\{p\}}.
\end{aligned}$$

Therefore A is a restricted Poisson algebra. □

The following example is immediate.

Example 5.2. (1) Let L be a restricted Lie algebra and let $A = \Bbbk 1_A \oplus L$, where the associate product on L is 0. Then A is a Poisson algebra in the obvious way. Both sides of (E3.4.1) are zero for elements in L (since $L \cdot L = 0$). By Lemma 5.1(2), A is a restricted Poisson algebra.

(2) Consider the special case when $L = \Bbbk x + \Bbbk y$ is a solvable Lie algebra with $[x, y] = x$. For $f = \lambda_1 x + \lambda_2 y \in L$, we define the p-map by

$$f^{\{p\}} = \lambda_2^{p-1}(\lambda_1 x + \lambda_2 y).$$

It is straightforward to check that $(L,(-)^{\{p\}})$ is a restricted Lie algebra. Let $A=\Bbbk 1_A\oplus L$. Then, by part (1), A is a restricted Poisson algebra. As a commutative algebra, $A=\Bbbk[x,y]/(x^2,xy,y^2)$ with $\Bbbk$-linear basis $\{1,x,y\}$. The Poisson bracket is given by $\{x,y\}=x$.

Let L be a restricted Lie algebra. It is well known that the p-map of L is unique up to a semilinear map from L to $Z(L)$, where $Z(L)$ is the center of L. Recall that a map $\gamma\colon L\to Z(L)$ being semilinear means that for any $x,y\in A$ and $\lambda\in\Bbbk$,

$$\gamma(x+y)=\gamma(x)+\gamma(y),$$
$$\gamma(\lambda x)=\lambda^p\gamma(x).$$

The following lemma is well known and easy to prove.

Lemma 5.3. *Let $(L,(-)^{[p]})$ be a restricted Lie algebra.*

(1) *Let $(-)^{\{p\}}$ be another restricted Lie structure on L. Then there is a map $\gamma: L\to Z(L)$ such that $(-)^{\{p\}}=(-)^{[p]}+\gamma$.*
(2) *Let γ be a map from L to $Z(L)$. Then $(-)^{[p]}+\gamma$ is a restricted Lie structure on L if and only if γ is a semilinear map from L to $Z(L)$.*

Let A be a Poisson algebra over $\Bbbk$ and $Z(A)$ the center of A. Observe that $Z(A)$ is a left A-module with the action given by

$$A\times Z(A)\to Z(A),\quad (a,z)\mapsto a^pz.$$

A semilinear map $\psi\colon A\to Z(A)$ is called a *Frobenius derivation* of A with the values in $Z(A)$ provided that $\psi(ab)=a^p\psi(b)+b^p\psi(a)$ for any $a,b\in A$. For example, if $\psi_0\colon A\to A$ is a derivation, then $\psi\colon A\to Z(A)$, defined by $\psi(a)=(\psi_0(a))^p$ for all $a\in A$, is a Frobenius derivation of A with the values in $Z(A)$.

By Lemma 5.3(1), any two restricted Poisson structures on A differ by a semilinear map γ which appears in the next proposition, which was mentioned in [Bezrukavnikov and Kaledin 2008, p. 414].

Proposition 5.4. *Let $(A,\cdot,\{-,-\},(-)^{\{p\}})$ be a restricted Poisson algebra and γ a map from A to itself. Then the map $(-)^{\{p\}}+\gamma$ is a restricted Poisson structure if and only if γ is a Frobenius derivation of A with values in $Z(A)$.*

Proof. Let $(-)^{\{p\}_1}\colon A\to A$ be another p-map such that $(A,\cdot,\{-,-\},(-)^{\{p\}_1})$ is also a restricted Poisson algebra. Since $(-)^{\{p\}_1}$ and $(-)^{\{p\}}$ are restricted structures on Lie algebra $(A,\{-,-\})$, $\gamma=(-)^{\{p\}_1}-(-)^{\{p\}}$ is a semilinear map from A to $Z(A)$ by Lemma 5.3. Moreover, for any $x,y\in A$, $(xy)^{\{p\}_1}=x^py^{\{p\}_1}+y^px^{\{p\}_1}+\Phi_p(x,y)$, and

$$\begin{aligned}\gamma(xy)&=(xy)^{\{p\}_1}-(xy)^{\{p\}}\\&=x^p(y^{\{p\}_1}-y^{\{p\}})+y^p(x^{\{p\}_1}-x^{\{p\}})\\&=x^p\gamma(y)+y^p\gamma(x).\end{aligned}$$

It follows that γ is a Frobenius derivation of A with values in $Z(A)$.

Conversely, it follows from Lemma 5.3 that the map $(-)^{\{p\}}+\gamma$ is also a restricted Lie structure on $(A,\{-,-\})$, since γ is a semilinear map from A to $Z(A)$ and $(-)^{\{p\}}$ is a p-map of Lie algebra $(A,\{-,-\})$. Moreover, for any $x,y\in A$,

$$(xy)^{\{p\}}+\gamma(xy)=x^p(y^{\{p\}}+\gamma(y))+y^p(x^{\{p\}}+\gamma(x))+\Phi_p(x,y).$$

It follows that the Poisson algebra A together with the map $(-)^{\{p\}}+\gamma$ is a restricted structure. □

By Proposition 5.4, the p-map of a restricted Poisson algebra is unique up to Frobenius derivations.

Remark 5.5. Let $(A,\cdot,\{-,-\},(-)^{\{p\}})$ be a restricted Poisson algebra and let $\gamma : A\to Z(A)$ be a semilinear map. Suppose that γ is not a Frobenius derivation (which is possible for many A) and defines a new p-map $(-)'^{\{p\}}=(-)^{\{p\}}+\gamma$. Then by Proposition 5.4, $(A,\cdot,\{-,-\},(-)'^{\{p\}})$ is not a restricted Poisson algebra, but it is still a weakly restricted Poisson algebra by Lemma 5.3(2).

6. Restricted Poisson algebras from restricted Lie algebras

We start with a general result.

Theorem 6.1. *Let $A=\Bbbk[x_i\mid i\in T]$ be a polynomial Poisson algebra with an index set T. If for each $i\in T$, there exists $\gamma(x_i)\in A$ such that $\mathrm{ad}^p_{x_i}=\mathrm{ad}_{\gamma(x_i)}$, then A admits a restricted Poisson structure $(-)^{\{p\}}$ such that $x_i^{\{p\}}=\gamma(x_i)$ for all $i\in T$.*

Proof. First we show that A has a weakly restricted Poisson structure, and then verify that the weakly restricted Poisson structure satisfies (E3.5.1).

For the sake of simplicity, we assume that $T=\{1,2,\dots,n\}$. To apply Lemma 1.3, we choose a canonical monomial $\Bbbk$-basis of A, which is

$$\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\mid i_1,i_2,\dots,i_n\geq 0\}.$$

We define $(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})^{\{p\}}$ inductively on the degree $i_1+i_2+\cdots+i_n$ such that

$$\mathrm{ad}^p_{(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})}=\mathrm{ad}_{(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})^{\{p\}}},$$

and therefore get the restricted Lie structure on $(A,\{-,-\})$ by Lemma 1.3. For convenience, we write $x^I=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ and $|I|=i_1+\cdots+i_n$ for $I=(i_1,\dots,i_n)$.

If $|I|=0$, then $x^I=1$ and we define $1^{\{p\}}=0$, and if $|I|=1$, then $x^I=x_i$ for some $1\leq i\leq n$. We define $x_i^{\{p\}}=\gamma(x_i)$ for each $1\leq i\leq n$. By hypothesis, $\mathrm{ad}^p_{x^I}=\mathrm{ad}_{(x^I)^{\{p\}}}$ for any I with $|I|=0,1$.

We proceed by induction and assume that $(x^I)^{\{p\}}$ has been defined such that $\mathrm{ad}^p_{x^I}=\mathrm{ad}_{(x^I)^{\{p\}}}$ for any x^I with $|I|\leq m$. For each monomial x^I of degree

$m+1$, we assume that k is the smallest subscript such that $i_k \geq 1$ in I, i.e., $I=(0,\ldots,0,i_k,\ldots,i_n)$, and define

$$(x^I)^{\{p\}} = x_k^p(x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})^{\{p\}} + (x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})^p x_k^{\{p\}} + \Phi_p(x_k, x_k^{i_1-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n}). \tag{E6.1.1}$$

By Lemma 3.2(1) for $(x,y)=(x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})$ and the above definition, we have $\mathrm{ad}^p_{x^I} = \mathrm{ad}_{(x^I)^{\{p\}}}$ for any $|I|=m+1$, which completes the induction. By Lemma 1.3, A has a weakly restricted Poisson structure.

Now let $\boldsymbol{b}$ be the set of all monomials, which is a $\Bbbk$-basis of A. We prove that (E3.5.1) holds for any pair of elements (x,y) in $\boldsymbol{b}$ by induction on $\deg x + \deg y$. If x or y is 1, then (E3.5.1) holds trivially, which also takes care of the case when $m := \deg x + \deg y \leq 1$. Suppose that the assertion holds for m and now assume that $\deg x + \deg y = m+1$. Let

$$xy = x_k^{i_k}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n}, \quad \text{where } i_k > 0.$$

By (E6.1.1), the pair $(x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}}\cdots x_n^{i_n})$ satisfies (E3.5.1). By symmetry, we may assume that $x = x_k g$. Then the above says that the pair (x_k, gy) satisfies (E3.5.1). By the induction hypothesis, the pairs (x_k, g) and (g,y) satisfy (E3.5.1). By Proposition 4.6(2), $(x,y)=(x_k g, y)$ satisfies (E3.5.1). By induction, (E3.5.1) holds for any two elements in $\boldsymbol{b}$. Finally the main statement follows from Theorem 4.7. □

As a consequence, we have the following.

Example 6.2. Let L be a restricted Lie algebra. We claim that the polynomial Poisson algebra $A := \Bbbk[L]$ (also denoted by $S(L)$) is a restricted Poisson algebra. Let $\{x_i\}_{i\in I}$ be a basis of L. Then, for each i, there is an $\gamma(x_i) := x_i^{[p]} \in L$ such that $\mathrm{ad}^p_{x_i} = \mathrm{ad}_{\gamma(x_i)}$ when restricted to L. Since A is a polynomial ring over L, both $\mathrm{ad}^p_{x_i}$ and $\mathrm{ad}_{\gamma(x_i)}$ extend uniquely to derivations of A. Thus $\mathrm{ad}^p_{x_i} = \mathrm{ad}_{\gamma(x_i)}$ holds when applied to A. The claim follows from Theorem 6.1 and there is a unique restricted structure $(-)^{\{p\}}$ on A such that

$$x^{\{p\}} = x^{[p]} \quad \forall\, x \in L.$$

Let V be a vector space. Then the free restricted Lie algebra $\mathrm{RLie}(V)$ can be defined by using the universal property or by taking the restricted Lie subalgebra of the free associative algebra generated by V with the p-map being the p-powering map. Now we can define the free restricted Poisson algebra generated by V.

Definition 6.3. Let V be a $\Bbbk$-space. The free restricted Poisson algebra generated by V is defined to be

$$\mathrm{FRP}(V) = \Bbbk[\mathrm{RLie}(V)].$$

The following universal property is standard [Shestakov 1993, Lemma 1, p. 312].

Lemma 6.4. *Let A be a restricted Poisson algebra and V be a vector space. Every $\Bbbk$-linear map $g : V \to A$ extends uniquely to a restricted Poisson algebra morphism $G : \mathrm{FRP}(V) \to A$ such that g factors through G.*

Continuing Example 6.2, when L is a restricted Lie algebra over $\Bbbk$ and $S(L) := \Bbbk[L]$ the symmetric algebra on L, then $S(L)$ admits an induced restricted Poisson structure. One natural setting in positive characteristic is to replace the symmetric algebra $S(L)$ by the truncated (or small) symmetric algebra $s(L)$. By definition, when L has a $\Bbbk$-basis $\{x_i\}_{i\in I}$,

$$s(L) = \Bbbk[x_i \mid i \in I]/(x_i^p,\ \forall\, i \in I). \tag{E6.4.1}$$

It is easily seen that $s(L)$ admits a Poisson structure with the bracket

$$\{f, g\} = \sum_{i,j} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) \{x_i, x_j\}$$

for any $f, g \in s(L)$. Next we show that $s(L)$ has a natural restricted Poisson structure.

Theorem 6.5. *Let L be a restricted Lie algebra over $\Bbbk$ of characteristic p and let $s(L)$ be the Poisson algebra with the bracket induced by L. Then $s(L)$ admits a natural restricted Poisson structure induced by the p-map of L.*

Proof. By Example 6.2, $S(L)$ has an induced restricted Poisson algebra structure. By (E6.4.1),

$$s(L) = S(L)/J,$$

where J is the Poisson ideal generated by x_i^p for all $i \in I$. By Proposition 3.5(3), $(x_i^p)^{\{p\}} = 0$. By Lemma 4.2, J is a restricted Poisson ideal as desired. □

7. Restricted Poisson algebras from deformation quantization

Bezrukavnikov and Kaledin [2008, Section 1.2] showed that a Frobenius-constant quantization automatically gives a restricted Poisson algebra. In this section, we consider a special deformation quantization of a Poisson algebra to produce more examples under a weaker condition.

Let A be a commutative (associative) algebra. Let $\Bbbk[\![t]\!]$ be the formal power series ring in one variable t. A *formal deformation* of A means an associative algebra $A[\![t]\!]$ over $\Bbbk[\![t]\!]$ with multiplication, denoted by m_t, satisfying

$$m_t(a \otimes b) = a * b = ab + m_1(a, b)t + \cdots + m_n(a, b)t^n + \cdots$$

for all $a, b \in A \subset A[\![t]\!]$. We should view $A[\![t]\!]$ as the power series ring in one variable t with coefficients in A where the associative multiplication m_t (or the star product $*$) is induced by a family of $\Bbbk$-bilinear maps $\{m_i\colon A\otimes A\to A\}_{i\geq 0}$ with $m_0(a,b)=ab$.

Define a bilinear map $\{-,-\}\colon A\otimes A\to A$ by setting $\{a,b\}=m_1(a,b)-m_1(b,a)$. It is easy to check that A together with the bracket $\{-,-\}$ is a Poisson algebra. Then $(A,\{-,-\})$ is called the *classical limit* of $(A[\![t]\!],m_t)$, and $(A[\![t]\!],m_t)$ is called a *deformation quantization* of the Poisson algebra $(A,\{-,-\})$.

For every $f\in A$, we write the p-power of f as

$$f^{*p}=\sum_{n=0}^{\infty} M_n^p(f)t^n = f^p + M_1^p(f)t + M_2^p(f)t^2+\cdots\in A[\![t]\!], \tag{E7.0.1}$$

where $M_i^p(f)\in A$ for all $i=0,1,2,\dots$.

Proposition 7.1. *Let $(A,\cdot,\{-,-\})$ be a Poisson algebra over $\Bbbk$ and let $(A[\![t]\!],*)$ be a deformation quantization of A. If $M_n^p(f)=0$ for $1\leq n\leq p-2$ and f^p is central in $A[\![t]\!]$ for all $f\in A$, then A admits a restricted Poisson structure.*

Proof. Recall that $f*g=\sum_{n=0}^{\infty}m_n(f,g)t^n\in A[\![t]\!]$ for all $f,g\in A$, where $m_n(f,g)\in A$ for all n. By the definition of the deformation quantization,

$$\{f,g\}=m_1(f,g)-m_1(g,f)$$

for all $f,g\in A$. Considering the Frobenius map $f\mapsto f^{*p}$ in $A[\![t]\!]$, we get

$$[f^{*p},g]_* = [\underbrace{f,\dots,f}_{p\text{ copies}},g]_* \tag{E7.1.1}$$

for all $f,g\in A$.

Since $[f,g]_*=\{f,g\}t \pmod{t^2}$ and $[-,-]_*$ is $\Bbbk[\![t]\!]$-bilinear, we have

$$[\underbrace{f,\dots,f}_{p\text{ copies}},g]_*\equiv\{\underbrace{f,\dots,f}_{p\text{ copies}},g\}t^p \pmod{t^{p+1}}.$$

By assumption, $M_n^p(f)=0$ for $1\leq n\leq p-2$ and f^p is central in $A[\![t]\!]$. Using the fact that (E7.1.1) or $\mathrm{ad}_{f^{*p}}(g)=(\mathrm{ad}_f)^p(g)$, it follows that

$$\{M_{p-1}^p(f),g\}t^p=\{\underbrace{f,\dots,f}_{p\text{ copies}},g\}t^p \pmod{t^{p+1}}$$

or

$$\{M_{p-1}^p(f),g\}=m_1(M_{p-1}^p(f),g)-m_1(g,M_{p-1}^p(f))=\{\underbrace{f,\dots,f}_{p\text{ copies}},g\} \tag{E7.1.2}$$

for all $g\in A$. We define $f^{\{p\}}=M_{p-1}^p(f)$ for any $f\in A$, and prove that the map $f\mapsto M_{p-1}^p(f)$ gives rise to a restricted Poisson structure on A.

Note that Definition 1.1(1) follows from (E7.1.2). Definition 1.1(2) follows from the fact that $(\lambda f)^{*p} = \lambda^p f^{*p}$. For the condition in Definition 1.1(3), we consider the Frobenius map of $A[\![t]\!]$, and get a restricted Lie structure of $(A[\![t]\!], [-,-]_*)$. It follows from Example 1.2 that

$$(f+g)^{*p} - f^{*p} - g^{*p} = \Lambda_p^*(f, g).$$

Computing the coefficients of t^{p-1} in the above equation, we get

$$(f+g)^{\{p\}} - f^{\{p\}} - g^{\{p\}} = \Lambda_p(f, g) \tag{E7.1.3}$$

as desired.

Finally it remains to show (E3.4.1). By assumption, $M_n^p(f) = 0$ for all $1 \leq n \leq p-2$. We compute the coefficient of t^{p-1} in the expression of $f^{*,2p}$ as follows:

$$\begin{aligned} f^{*,2p} &= f^{*p} * f^{*p} \\ &= (f^p + t^{p-1} M_{p-1}^p(f) + \cdots) * (f^p + t^{p-1} M_{p-1}^p(f) + \cdots) \\ &\equiv f^{2p} + 2 f^p M_{p-1}^p(f) t^{p-1} \pmod{t^p}. \end{aligned}$$

Assume that $f * f = f^2 + tW$, where

$$W = m_1(f, f) + m_2(f, f)t + \cdots .$$

It follows that

$$\begin{aligned} f^{*,2p} &= (f^{*2})^{*p} = (f^2 + tW)^{*p} \\ &= (f^2)^{*p} + (tW)^{*p} + \Lambda_p^*(f^2, tW) \\ &\equiv f^{2p} + M_{p-1}^p(f^2) t^{p-1} \pmod{t^p}. \end{aligned}$$

Therefore, for all $f \in A$, $f^{2^{\{p\}}} = 2 f^p f^{\{p\}}$, which is (E3.4.1). □

Before giving some explicit examples, we recall a result.

Lemma 7.2 [Bezrukavnikov and Kaledin 2008, Lemma 1.3]. *Let B be an associative algebra over a base field k of characteristic $p > 0$, and let $B_{(k)}$, $B_{(1)} = B$, $B_{(k)} = [B, B_{(k-1)}]$ be its central series with respect to the commutator. If $B_{(p)} = 0$ and $B_{(2)}^p = 0$, then the Frobenius map $x \mapsto x^p$ preserves the addition and the multiplication.*

Example 7.3. Let $A := \Bbbk[x, y]$ be a polynomial Poisson algebra over a field $\Bbbk$ of characteristic $p \geq 3$ with the bracket given by $\{x, y\} = 1$.

By a direct calculation, the Poisson algebra A admits a deformation quantization $(A[\![t]\!], *)$ with the star product given by

$$f * g = \sum_{0 \leq n \leq p-1} m_n(f, g) t^n = \sum_{0 \leq n \leq p-1} \frac{t^n}{n!} (\partial_1^n f)(\partial_2^n g) \tag{E7.3.1}$$

for all $f, g \in A$, where ∂_1 and ∂_2 are the partial derivatives of f with respect to the variables x and y, respectively.

Clearly, $f^p * g = f^p g = g * f^p$ for any $f, g \in A$, whence f^p is central in $A[\![t]\!]$. Moreover, for every $f \in A$, we claim that

$$M_n^p(f) = 0 \quad \text{for } 1 \le n \le p-2.$$

In fact, for any $f, g \in A \subset A[\![t]\!]$, we have

$$[f, g]_* = f * g - g * f \in (t).$$

It follows that

$$[f, g]_*^{*,p} \in (t^p),$$

and

$$[f_1, \dots, f_p]_* \in (t^{p-1})$$

for all $f_1, \dots, f_p \in A[\![t]\!]$. Since t is central, we can define the quotient algebra $B := A[\![t]\!]/(t^{p-1})$. The above computation shows that $B_{(p)} = 0$ and $B_{(2)}^p = 0$. By Lemma 7.2, it follows that the Frobenius map $b \mapsto b^{*p}$ of B is additive and multiplicative. By (E7.3.1), an easy computation shows that $x^{*p} = x^p$ and $y^{*p} = y^p$ in $A[\![t]\!]$. For each $f \in A$, let $\bar{f}$ be the corresponding element in B. Then

$$\bar{f}^{*p} = \overline{f^p} \in B$$

since the map $\bar{f} \mapsto \bar{f}^{*p}$ preserves the addition and the multiplication in B. It follows that $f^{*p} - f^p \in (t^{p-1})$ and therefore $M_n^p = 0$ for any $1 \le n \le p-2$.

By Proposition 7.1, A admits a restricted Poisson structure with the p-map $f^{\{p\}} = M_{p-1}^p(f)$ for any $f \in A$. The p-map agrees with (E4.8.2) when $p = 3$ and (E4.8.3) when $p = 5$.

The next example is a generalization of the previous one.

Example 7.4. Let $A := \Bbbk[x_1, \dots, x_m]$ be a polynomial Poisson algebra with Poisson bracket determined by $\{x_i, x_j\} = c_{ij} \in \Bbbk$ for $1 \le i < j \le m$. Let μ denote the associative product of A which is extended to the power series ring of A. Let $\partial_i := \partial/\partial x_i$ for all $1 \le i \le m$. For each scalar $c \in \Bbbk$, let $\exp(tc\partial_i \otimes \partial_j)$ be the operator

$$\sum_{0 \le n \le p-1} \frac{(ct)^n}{n!} \partial_i^n \otimes \partial_j^n \,:\, A[\![t]\!] \otimes A[\![t]\!] \to A[\![t]\!] \otimes A[\![t]\!].$$

By a direct calculation, a deformation quantization $(A[\![t]\!], *)$ of the Poisson algebra A is given by

$$f * g = \mu\Big(\prod_{1 \le i < j \le m} \big(\exp(c_{ij} t\, \partial_i \otimes \partial_j)\big)(f \otimes g) \Big)$$

for all $f, g \in A$. Clearly, $f^p \in A \subset A[\![t]\!]$ is central for any $f \in A$. Being similar to the proof of Example 7.3, we have $M_n^p(f) = 0$ for $1 \le n \le p-2$ and all $f \in A$. By Proposition 7.1, A admits a restricted Poisson structure with the p-map $f^{\{p\}} = M_{p-1}^p(f)$ for any $f \in A$. When $p = 3$, the p-map is given in Example 4.10.

Example 7.5. Let $B_{2n} = \Bbbk[x_1, \dots, x_{2n}]/I$ be the p-truncated polynomial Poisson algebra in $2n$ variables over $\Bbbk$, where the Poisson bracket is defined by

$$\{f, g\} = \sum_{i=1}^{n} \bigl(\partial_i(f)\partial_{n+i}(g) - \partial_{n+i}(f)\partial_i(g)\bigr)$$

for all $f, g \in B_{2n}$, and I is generated by x_i^p, $i = 1, \dots, 2n$. Skryabin [2002] introduced the notion of the normalized p-map on $(B_{2n}, \{-,-\})$, say, $1^{\{p\}} = 0$ and $f^{\{p\}} \in \mathfrak{m}^2$ for all $f \in \mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of B_{2n} as an associative algebra.

We consider the Poisson algebra $A = \Bbbk[x_1, \dots, x_{2n}]$ in Example 7.4 with the bracket given by $c_{ij} = \delta_{i+n,j}$ for all $1 \le i < j \le 2n$. Clearly, x_i^p is central and I is a Poisson ideal of A. By Proposition 3.5(3), $(x_i^p)^{\{p\}} = 0$ for all $i \in I$, and by Lemma 4.2, I is a restricted Poisson ideal of A. Therefore, it follows from Proposition 4.3 that the Poisson algebra B_{2n} admits a restricted Poisson structure. Clearly, this p-map is normalized.

8. Connection with restricted Lie–Rinehart algebras

Some definitions concerning Lie–Rinehart algebras were given in Section 2. Let A be a Poisson algebra and $\Omega_{A/\Bbbk}$ its Kähler differentials module. Then the pair $(A, \Omega_{A/\Bbbk})$ is a Lie–Rinehart algebra over $\Bbbk$, where the anchor map $\alpha : \Omega_{A/\Bbbk} \to \mathrm{Der}(A)$ is given in (E2.2.2). Dokas [2012] introduced the notion of a restricted Lie–Rinehart algebra and studied its cohomology theory. The goal of this section is to show that the Lie–Rinehart algebra $(A, \Omega_{A/\Bbbk})$ admits a natural restricted structure if the Poisson algebra A is weakly restricted and $\Omega_{A/\Bbbk}$ is a free module over A.

Let $(L, (-)^{[p]}$ and $(L', (-)^{[p]})$ be restricted Lie algebras. A map $f : (L, (-)^{[p]}) \to (L', (-)^{[p]})$ is called a restricted Lie homomorphism if f is a Lie algebra homomorphism and satisfies $f(x^{[p]}) = f(x)^{[p]}$ for all $x \in L$.

Definition 8.1 [Dokas 2012, Definition 1.7]. A *restricted Lie–Rinehart algebra* $(A, L, (-)^{[p]})$ over a commutative $\Bbbk$-algebra A is a Lie–Rinehart algebra over A such that

(a) $(L, (-)^{[p]})$ is a restricted Lie algebra over $\Bbbk$,

(b) the anchor map is a restricted Lie homomorphism, and

(c) we have
$$(aX)^{[p]} = a^p X^{[p]} + (aX)^{p-1}(a)X$$
for all $a \in A$ and $X \in L$.

We now prove Theorem 0.5.

Theorem 8.2. *Let $(A, \cdot, \{-,-\}, (-)^{\{p\}})$ be a weakly restricted Poisson algebra. If the module of Kähler differentials $\Omega_{A/\Bbbk}$ is free, then the Lie–Rinehart algebra $(A, \Omega_{A/\Bbbk}, (-)^{[p]})$ is restricted, where the p-map of $\Omega_{A/\Bbbk}$ is defined by*
$$(x\,\mathrm{d}u)^{[p]} = x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u$$
for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$.

Proof. Since $\Omega_{A/\Bbbk}$ is a free A-module, $\Omega_{A/\Bbbk}$ can be embedded into the universal enveloping algebra $\mathcal{U}(A, \Omega_{A/\Bbbk})$ (Lemma 2.3). By the proof of [Dokas 2012, Proposition 2.2], it suffices to show that
$$\mathrm{ad}^p_{x\,\mathrm{d}u}(y\,\mathrm{d}v) = [x^p\,\mathrm{d}u^{\{p\}} + (x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u, y\,\mathrm{d}v]$$
for all $x\,\mathrm{d}u$ and $y\,\mathrm{d}v \in \Omega_{A/\Bbbk}$.

By Hochschild's relation [1955, Lemma 1], we get in $\mathcal{U}(A, L)$ the relation
$$(\iota_2(x\,\mathrm{d}u))^p = \iota_1(x^p)(\iota_2(\mathrm{d}u))^p + \iota_2((x\,\mathrm{d}u)^{p-1}(x)\,\mathrm{d}u)$$
for all $x\,\mathrm{d}u \in \Omega_{A/\Bbbk}$. Considering the Frobenius map of $\mathcal{U}(A, L)$, we have
$$[(\iota_2(\mathrm{d}u))^p, \iota_1(y)] = [\iota_2(\mathrm{d}u), \dots, \iota_2(\mathrm{d}u), \iota_1(y)] = \iota_1((\mathrm{ad}_u)^p(y)),$$
and hence
$$\iota_2(\mathrm{d}u)^p\iota_1(y) = \iota_1(y)\iota_2(\mathrm{d}u)^p + \iota_1((\mathrm{ad}_u)^p(y))$$
for all $\mathrm{d}u \in \Omega_{A/\Bbbk}$ and $y \in A$. Moreover, for $x\,\mathrm{d}u, y\,\mathrm{d}v \in \Omega_{A/\Bbbk} \subset \mathcal{U}(A, L)$,
$$\begin{aligned}
&[\iota_1(x^p)(\iota_2(\mathrm{d}u))^p, \iota_2(y\,\mathrm{d}v)]\\
&\quad= \iota_1(x^p)(\iota_2(\mathrm{d}u))^p\iota_1(y)\iota_2(\mathrm{d}v) - \iota_1(y)\iota_2(\mathrm{d}v)\iota_1(x^p)(\iota_2(\mathrm{d}u))^p\\
&\quad= \iota_1(x^p)\big(\iota_1(y)(\iota_2(\mathrm{d}u))^p + \iota_1((\mathrm{ad}_u)^p(y))\big)\iota_2(\mathrm{d}v)\\
&\qquad\qquad - \iota_1(y)\big(\iota_1(x^p)\iota_2(\mathrm{d}v) + \iota_1(\{v, x^p\})\big)(\iota_2(\mathrm{d}u))^p\\
&\quad= \iota_1(x^p y)[(\iota_2(\mathrm{d}u))^p, \iota_2(\mathrm{d}v)] + \iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v)\\
&\quad= \iota_1(x^p y)\iota_2(\mathrm{ad}^p_{\mathrm{d}u}(\mathrm{d}v)) + \iota_2(x^p(\mathrm{d}u)^p(y)\,\mathrm{d}v)\\
&\quad= \iota_1(x^p y)\iota_2(\mathrm{d}(\mathrm{ad}^p_u(v))) + \iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v),
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\iota_2(\mathrm{ad}^p_{x\,\mathrm{d}u}(y\,\mathrm{d}v))\\
&\quad=[(\iota_2(x\,\mathrm{d}u))^p,\iota_2(y\,\mathrm{d}v)]\\
&\quad=[\iota_1(x^p)(\iota_2(\mathrm{d}u))^p+\iota_2((x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u),\iota_2(y\,\mathrm{d}v)]\\
&\quad=\iota_1(x^py)\iota_2(\mathrm{d}(\mathrm{ad}^p_u(v)))+\iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v)+\iota_2([(x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u,y\,\mathrm{d}v])\\
&\quad=\iota_2(x^py\,\mathrm{d}(\mathrm{ad}^p_u(v)))+\iota_2(x^p(\mathrm{ad}_u)^p(y)\,\mathrm{d}v)+\iota_2([(x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u,y\,\mathrm{d}v])\\
&\quad=\iota_2([x^p\,\mathrm{d}u^{\{p\}}+(x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u,y\,\mathrm{d}v]),
\end{aligned}$$

and hence

$$\mathrm{ad}^p_{x\,\mathrm{d}u}(y\,\mathrm{d}v)=[x^p\,\mathrm{d}u^{\{p\}}+(x\,\mathrm{ad}_u)^{p-1}(x)\,\mathrm{d}u,\,y\,\mathrm{d}v]$$

as desired. □

For Poisson algebras A in Examples 4.8–4.10, Example 6.2, Theorem 6.5, and Examples 7.3–7.5, it is automatic that $\Omega_{A/\Bbbk}$ is free over A.

9. Restricted Poisson Hopf algebras

We first recall the definition of Poisson Hopf algebras. The notion of a Poisson Hopf algebra was probably first introduced by Drinfel'd [1985; 1987]; see also [Doebner et al. 1990].

Definition 9.1. Let A be a Poisson algebra. We say that A is a Poisson Hopf algebra if

(1) A is a Hopf algebra with the usual operations Δ,ϵ,S;

(2) $\Delta: A\to A\otimes A$ and $\epsilon: A\to\Bbbk$ are Poisson algebra morphisms and $S: A\to A$ is a Poisson algebra antiautomorphism.

To define restricted Poisson Hopf algebras, we first need to show that the tensor product of two restricted Poisson algebras is again a restricted Poisson algebra.

Proposition 9.2. *Let A and B be two restricted Poisson algebras. Then there is a unique restricted Poisson structure on $A\otimes B$ such that*

(E9.2.1) $$(a\otimes b)^{\{p\}}=a^{\{p\}}\otimes b^p+a^p\otimes b^{\{p\}}$$

for all $a\in A$ and $b\in B$.

Proof. First of all, it is well known that $A\otimes B$ is a Poisson algebra with bracket defined by

$$\{a_1\otimes b_1,a_2\otimes b_2\}=\{a_1,a_2\}\otimes b_1b_2+a_1a_2\otimes\{b_1,b_2\}$$

for all $a_1,a_2\in A$ and $b_1,b_2\in B$.

Let $\{a_i\}_{i\in I}$ (respectively, $\{b_j\}_{j\in J}$) be a $\Bbbk$-basis of A (respectively, B) and assume that $1_A \in \{a_i\}_{i\in I}$ and $1_B \in \{b_j\}_{i\in J}$. Then $\{a_i \otimes b_j\}_{i\in I, j\in J}$ is a $\Bbbk$-basis of $A \otimes B$.

For any $a \in A$ and $b \in B$, $\mathrm{ad}^p_{a\otimes b}$ is a derivation. For any $c \otimes d \in A \otimes B$, we have

$$\begin{aligned}
\mathrm{ad}^p_{a\otimes b}(c\otimes d) &= (1\otimes d)\,\mathrm{ad}^p_{a\otimes b}(c\otimes 1) + (c\otimes 1)\,\mathrm{ad}^p_{a\otimes b}(1\otimes d)\\
&= (1\otimes d)(\mathrm{ad}^p_a(c)\otimes b^p) + (c\otimes 1)(a^p \otimes \mathrm{ad}^p_b(d))\\
&= (1\otimes d)(\mathrm{ad}_{a^{\{p\}}}(c)\otimes b^p) + (c\otimes 1)(a^p \otimes \mathrm{ad}_{b^{\{p\}}}(d))\\
&= (1\otimes d)(\mathrm{ad}_{a^{\{p\}}\otimes b^p}(c\otimes 1)) + (c\otimes 1)(\mathrm{ad}_{a^p\otimes b^{\{p\}}}(1\otimes d))\\
&= (1\otimes d)(\mathrm{ad}_{a^{\{p\}}\otimes b^p}(c\otimes 1)) + (c\otimes 1)(\mathrm{ad}_{a^{\{p\}}\otimes b^p}(1\otimes d))\\
&\qquad + (1\otimes d)(\mathrm{ad}_{a^p\otimes b^{\{p\}}}(c\otimes 1)) + (c\otimes 1)(\mathrm{ad}_{a^p\otimes b^{\{p\}}}(1\otimes d))\\
&= \mathrm{ad}_{a^{\{p\}}\otimes b^p}(c\otimes d) + \mathrm{ad}_{a^p\otimes b^{\{p\}}}(c\otimes d)\\
&= \mathrm{ad}_{a^{\{p\}}\otimes b^p + a^p\otimes b^{\{p\}}}(c\otimes d).
\end{aligned}$$

In particular,

$$\mathrm{ad}^p_{a_i\otimes b_j} = \mathrm{ad}_{(a_i^{\{p\}}\otimes b_j^p + a_i^p\otimes b_j^{\{p\}})}$$

for all i and j. Since $\{a_i \otimes b_j\}_{i\in I, j\in J}$ is a $\Bbbk$-basis of $A\otimes B$, by Lemma 1.3, there is a unique weak restricted Poisson structure on $A\otimes B$ such that

(E9.2.2) $$(a_i\otimes b_j)^{\{p\}} = a_i^{\{p\}}\otimes b_j^p + a_i^p \otimes b_j^{\{p\}}$$

for all i, j, which agrees with (E9.2.1). It remains to show that this weak restricted Poisson structure on $A\otimes B$ is indeed a restricted Poisson structure and (E9.2.1) holds.

We first prove (E9.2.1). By (E9.2.2), $(a_i\otimes 1)^{\{p\}} = a_i^{\{p\}}\otimes 1$. It follows from Definition 1.1 that

(E9.2.3) $$(a\otimes 1)^{\{p\}} = a^{\{p\}}\otimes 1$$

for all $a \in A$. By symmetry, $(1\otimes b)^{\{p\}} = 1\otimes b^{\{p\}}$ for all $b\in B$. Since $\{a_i\otimes 1, 1\otimes b_j\} = 0$, (E9.2.2) implies that the pair $(a_i\otimes 1, 1\otimes b_j)$ satisfies (E3.5.1). By Proposition 4.6(4), $R_{a_i\otimes 1}$ is a $\Bbbk$-vector space, and by assumption, $\{b_j\}$ is a $\Bbbk$-basis of B, so we have that $R_{a_i\otimes 1}\supseteq B$. Or, for any $b\in B$, the pair $(a_i\otimes 1, 1\otimes b)$ satisfies (E3.5.1). By switching a and b and applying the same argument, one sees that any pair $(a\otimes 1, 1\otimes b)$ satisfies (E3.5.1). This means that

$$\begin{aligned}
(a\otimes b)^{\{p\}} &= (a\otimes 1)^{\{p\}}(1\otimes b)^p + (a\otimes 1)^p(1\otimes b)^{\{p\}} + \Phi_p(a\otimes 1, 1\otimes b)\\
&= (a\otimes 1)^{\{p\}}(1\otimes b)^p + (a\otimes 1)^p(1\otimes b)^{\{p\}}\\
&= a^{\{p\}}\otimes b^p + a^p\otimes b^{\{p\}}.
\end{aligned}$$

So we proved (E9.2.1).

For the rest, we claim that for any pair of elements $(a_i \otimes b_j, a_k \otimes b_l)$, (E3.5.1) holds. By using (E9.2.3), (E3.5.1) holds for all pairs of the form $(a \otimes 1, a' \otimes 1)$. By symmetry, (E3.5.1) holds for all pairs of the form $(1 \otimes b, 1 \otimes b')$. By (E9.2.1), (E3.5.1) holds for pairs of the form $(a \otimes 1, 1 \otimes b)$. Set $f = a \otimes 1$, $g = a' \otimes 1$ and $h = 1 \otimes b$ for any $a, a' \in A$ and $b \in B$. Then (f, g), (g, h) and (fg, h) satisfy (E3.5.1). By Proposition 4.6(2), (f, gh) satisfies (E3.5.1). Or equivalently, $(a \otimes 1, a' \otimes b)$ satisfies (E3.5.1). By symmetry, $(1 \otimes b, a \otimes b')$, $(a \otimes b, a' \otimes 1)$ and $(a \otimes b, 1 \otimes b')$ satisfy (E3.5.1). Recycle the letters and let $f = a \otimes b$, $g = a' \otimes 1$ and $h = 1 \otimes b'$. We have that (f, g), (g, h) and (fg, h) all satisfy (E3.5.1). By Proposition 4.6(2), (f, gh) satisfies (E3.5.1). By choosing special a, a', b, b' we have that $(a_i \otimes b_j, a_k \otimes b_l)$ satisfies (E3.5.1) as desired. This says that every pair of elements from the $\Bbbk$-basis $\{a_i \otimes b_j\}_{i \in I, j \in J}$ satisfies (E3.5.1). By Theorem 4.7, the weak restricted Poisson structure on $A \otimes B$ is actually a restricted Poisson structure.

The above proof shows that there is a unique restricted Poisson structure on $A \otimes B$ satisfying (E9.2.2). Since (E9.2.1) is a consequence of (E3.5.1), the assertion follows. □

Now it is reasonable to define a restricted Poisson Hopf algebra.

Definition 9.3. A restricted Poisson algebra H is called a *restricted Poisson Hopf algebra* if there are restricted Poisson algebra maps $\Delta : H \to H \otimes H$ and $\epsilon : H \to \Bbbk$ and a restricted Poisson algebra antiautomorphism $S : H \to H$ such that H together with (Δ, ϵ, S) becomes a Hopf algebra.

One canonical example is the following.

Example 9.4. Let L be a restricted Lie algebra. Then $s(L)$ (given in Theorem 6.5) is a restricted Poisson Hopf algebra with the structure maps determined by

$$\Delta : x \to x \otimes 1 + 1 \otimes x,$$
$$\epsilon : x \to 0,$$
$$S : x \to -x$$

for all $x \in L$. It is straightforward to check that $s(L)$ is a restricted Poisson Hopf algebra. Similarly, $S(L)$ (given in Example 6.2) is a restricted Poisson Hopf algebra with structure maps determined as above.

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REMARKS ON GJMS OPERATOR OF ORDER SIX

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We study analysis aspects of the sixth-order GJMS operator P_g^6. Under conformal normal coordinates around a point, we present the expansions of Green's function of P_g^6 with pole at this point. As a starting point of the study of P_g^6, we manage to give some existence results of the prescribed Q-curvature problem on Einstein manifolds. One among them is that for $n \geq 10$, let (M^n, g) be a closed Einstein manifold of positive scalar curvature and f a smooth positive function in M. If the Weyl tensor is nonzero at a maximum point of f and f satisfies a vanishing order condition at this maximum point, then there exists a conformal metric $\tilde{g}$ of g such that its Q-curvature $Q_{\tilde{g}}^6$ equals f.

1. Introduction

Recently, some remarkable developments have been achieved in the existence theory of the positive constant Q-curvature problem associated to the Paneitz–Branson operator. One key ingredient in such works is that a strong maximum principle for the fourth-order Paneitz–Branson operator is discovered under a hypothesis on the positivity of some conformal invariants or Q-curvature of the background metric. The readers are referred to [Gursky et al. 2016; Gursky and Malchiodi 2015; Hang and Yang 2016; Li and Xiong 2015] and the references therein. This naturally stimulates us to study the GJMS operator of order six and its associated Q-curvature problem, the analogue to the Yamabe problem and Q-curvature problem for the Paneitz–Branson operator. Except for the aforementioned cases, due to the lack of a maximum principle for higher order elliptic equations in general, the existence theory of such problems needs to be developed. Until an analogue of Aubin's result [1976] for the Yamabe problem is verified in Proposition 3.2 below, by adapting some ideas for the Paneitz–Branson operator from [Esposito and Robert 2002; Djadli et al. 2000], we establish some existence results of the prescribed Q-curvature problem on Einstein manifolds, in which case the sixth-order GJMS operator has constant coefficients.

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The conformally covariant GJMS operators with principle part $(-\Delta_g)^k$, $k\in\mathbb{N}$ were discovered by Graham, Jenne, Mason and Sparling [Graham et al. 1992]. In particular, the GJMS operator of order six and the associated Q-curvature are given as follows (see [Juhl 2013; Wünsch 1986]): on manifolds (M^n, g) of dimension $n\geq 3$ and $n\neq 4$, denote by $\sigma_k(A_g)$ the k-th elementary symmetric function of the Schouten tensor

$$A_{ij}=\frac{1}{n-2}\Big(R_{ij}-\frac{R_g}{2(n-1)}g_{ij}\Big).$$

Denote by

$$C_{ijk}=\nabla_k A_{ij}-\nabla_j A_{ik},\quad B_{ij}=\Delta_g A_{ij}-\nabla^k\nabla_j A_{ik}-A^{kl}W_{kijl}=\nabla^k C_{ijk}-A^{kl}W_{kijl}$$

the Cotton tensor and Bach tensor, respectively. Let

$$\begin{aligned}T_2=&(n-2)\sigma_1(A_g)g-8A_g=-\frac{8}{n-2}\operatorname{Ric}_g+\frac{n^2-4n+12}{2(n-1)(n-2)}R_g g;\\ T_4=&-\frac{3n^2-12n-4}{4}\sigma_1(A_g)^2g+4(n-4)|A|_g^2g+8(n-2)\sigma_1(A_g)A_g\\ &+(n-6)\Delta_g\sigma_1(A_g)g-48A_g^2-\frac{16}{n-4}B_g;\\ v_6=&-\frac{1}{8}\sigma_3(A_g)-\frac{1}{24(n-4)}\langle B, A\rangle_g.\end{aligned}$$

Then, the Q-curvature Q_g^6 is defined by

$$\begin{aligned}Q_g^6=&-3!\,2^6v_6-\frac{n+2}{2}\Delta_g(\sigma_1(A_g)^2)+4\Delta_g|A|_g^2\\ &-8\delta(A_g d\sigma_1(A_g))+\Delta_g^2\sigma_1(A_g)-\frac{n-6}{2}\sigma_1(A_g)\Delta_g\sigma_1(A_g)\\ &-4(n-6)\sigma_1(A_g)|A|_g^2+\frac{(n-6)(n+6)}{4}\sigma_1(A_g)^3,\end{aligned}\tag{1-1}$$

and the GJMS operator of sixth-order P_g^6 is given by[1]

$$-P_g^6=\Delta_g^3+\Delta_g\delta T_2 d+\delta T_2 d\Delta_g+\frac{n-2}{2}\Delta_g(\sigma_1(A_g)\Delta_g)+\delta T_4 d-\frac{n-6}{2}Q_g^6,\tag{1-2}$$

where $-\delta d=\Delta_g$. The operator P_g^6 is conformally covariant in the sense that if $\tilde g=u^{4/(n-6)}g,\ 0<u\in C^\infty(M)$ with $n\geq 3$ and $n\neq 4, 6$,

$$u^{\frac{n+6}{n-6}}P_{\tilde g}^6\varphi=P_g^6(u\varphi),\tag{1-3}$$

and in dimension 6,

$$P^6_{e^{2u}g}\varphi=e^{-6u}P_g^6\varphi$$

[1] The definition of P_g^6 differs from the formula (10.15) in [Juhl 2013] by a minus sign.

for all $\varphi \in C^\infty(M)$. When (M, g) is Einstein, P_g^6 has constant coefficients; explicitly,

$$Q_g^6 = \frac{n^4-20n^2+64}{32n^2(n-1)^3} R_g^3,$$

$$-P_g^6 = \Delta_g^3 + \frac{-3n^2+6n+32}{4n(n-1)} R_g \Delta_g^2 + \frac{3n^4-12n^3-52n^2+128n+192}{16n^2(n-1)^2} R_g^2 \Delta_g - \frac{n-6}{2} Q_g^6.$$

Obviously, when $n \geq 7$, Q_g^6 is a positive constant whenever the scalar curvature R_g is positive. Through a direct computation, the GJMS operator P_g^6 has the following factorization:

$$P_g^6 = \Big(-\Delta_g + \frac{(n-6)(n+4)}{4n(n-1)} R_g\Big)\Big(-\Delta_g + \frac{(n-4)(n+2)}{4n(n-1)} R_g\Big)\Big(-\Delta_g + \frac{n-2}{4(n-1)} R_g\Big). \tag{1-4}$$

In general, as shown in [Fefferman and Graham 2012] and [Gover 2006], on Einstein manifolds the GJMS operator of order $2k$ for all positive integers k satisfies the above property as

$$P_g^{2k} = \prod_{i=1}^{k} \Big(-\Delta_g + \frac{R_g}{4n(n-1)}(n+2i-2)(n-2i)\Big).$$

In particular, choose $M^n = S^n$, $g = g_{S^n}$, then

$$Q_{S^n}^6 = \frac{n(n^4-20n^2+64)}{32},$$

$$\begin{aligned} P_{S^n}^6 &= -\Delta_{S^n}^3 - \frac{-3n^2+6n+32}{4}\Delta_{S^n}^2 - \frac{3n^4-12n^3-52n^2+128n+192}{16}\Delta_{S^n} + \frac{n-6}{2} Q_{S^n}^6 \\ &= \Big(-\Delta_{S^n} + \frac{(n-6)(n+4)}{4}\Big)\Big(-\Delta_{S^n} + \frac{(n-4)(n+2)}{4}\Big)\Big(-\Delta_{S^n} + \frac{n(n-2)}{4}\Big). \end{aligned}$$

From now on, we set $P_g = P_g^6$ and $Q_g = Q_g^6$ unless stated otherwise. Then, for any $\varphi \in H^3(M, g)$, we get

$$\begin{aligned} &\int_M \varphi P_g \varphi \, d\mu_g \\ &= \int_M \Big(|\nabla\Delta\varphi|_g^2 - 2T_2(\nabla\Delta\varphi, \nabla\varphi) - \frac{n-2}{2}\sigma_1(A)(\Delta_g\varphi)^2 - T_4(\nabla\varphi,\nabla\varphi) + \frac{n-6}{2} Q_g \varphi^2\Big)\, d\mu_g. \end{aligned}$$

As a starting point of the study on the sixth-order GJMS operator, we obtain some existence results of conformal metrics with positive Q-curvature candidates on closed Einstein manifolds under some additional natural assumptions.

Theorem 1.1. *Suppose (M^n, g) is a closed Einstein manifold of dimension $n \geq 10$ and has positive scalar curvature. Let f be a smooth positive function on M.*

Assume the Weyl tensor W_g is nonzero at a maximum point p of f and f satisfies the vanishing order condition at p:

$$\begin{cases} \Delta_g f(p) = 0 & \textit{if } n = 10, \\ \nabla^k f(p) = 0,\ k = 2, 3, 4 & \textit{if } n \geq 11. \end{cases} \tag{1-5}$$

Then there exists a smooth solution to the Q-curvature equation

$$P_g u = f u^{\frac{n+6}{n-6}}, \quad u > 0 \quad \textit{in } M.$$

We remark that the condition (1-5) imposed on the Q-curvature candidates f is conformally invariant. The condition that (M, g) is Einstein is only used to seek a *positive* solution. Theorem 1.1 is a special case of a generalized Theorem 3.3.

This paper is organized as follows. In Section 2, the expansions of Green's function for P_g when $n \geq 7$ are presented under conformal normal coordinates around a point. The technique used here is basically inspired by Lee and Parker [1987]; see also [Hang and Yang 2016]. The complicated computations of the term $P_g(r^{6-n})$ are left to the Appendix, where r is the geodesic distance from this point. In Section 3, we prove an analogue (cf., Proposition 3.2) of Aubin's result for any closed manifold of dimension $n \geq 10$, which is not locally conformally flat. Based on this result, using the mountain pass lemma we state in Theorem 3.3 some results of the prescribed Q-curvature problem associated to the sixth-order GJMS operator on Einstein manifolds. Then our main Theorem 1.1 directly follows from Theorem 3.3.

2. Expansion of Green's function of $\mathbf{P_g}$

Based on the survey paper by Lee and Parker [1987] on the Yamabe problem, the method of deriving expansions of Green's function of P_g is more or less standard except for careful computations on some lower-order terms involved in P_g. One may also refer to [Hang and Yang 2016] for the Paneitz–Branson operator case. Green's functions of conformally covariant operators play an important role in the solvability of the constant curvature problems, for instance, the Yamabe problem (see [Lee and Parker 1987] etc.) and the constant Q-curvature problem for the Paneitz–Branson operator (see [Djadli et al. 2000; Esposito and Robert 2002; Gursky et al. 2016; Hang and Yang 2016], etc.). In particular, F. Hang and P. Yang [2016] set up a dual variational method of the minimization for the Paneitz–Branson functional to seek a positive maximizer of the dual functional; such a scheme heavily relies on the positivity and expansion of its Green's function. We expect that the expansion of Green's function for P_g^6 will be useful to some possible future applications.

Throughout, we use the following notation: $2^\sharp = 2n/(n-6)$, $\omega_n = \mathrm{vol}(S^n, g_{S^n})$ and when $n > 6$, $c_n = 1/(8(n-2)(n-4)(n-6)\omega_{n-1})$. For $m \in \mathbb{Z}_+$, let

$$\mathcal{P}_m := \{\text{homogeneous polynomials in } \mathbb{R}^n \text{ of degree } m\}$$

and

$$\mathcal{H}_m := \{\text{harmonic polynomials in } \mathbb{R}^n \text{ of degree } m\}.$$

Then $\mathcal{P}_m$ has the following decomposition (see [Stein 1970], p. 68–70):

$$\mathcal{P}_m = \bigoplus_{k=0}^{[m/2]} (r^{2k}\mathcal{H}_{m-2k}).$$

Proposition 2.1. *Assume $n > 6$ and $\ker P_g = 0$. Let $G_p(x)$ be the Green's function of the sixth-order GJMS operator at the pole $p \in M^n$ with the property that $P_g G_p = c_n\delta_p$ in the sense of distributions. Then, under the conformal normal coordinates around p with conformal metric g, $G_p(x)$ has the following expansions*:

(a) *If n is odd, then*

$$G_p(x) = r^{6-n}\Big(1 + \sum_{k=1}^{n} \psi_k\Big) + A + O(r),$$

where A is a constant and $\psi_k \in \mathcal{P}_k$.

(b) *If n is even, then*

$$G_p(x) = r^{6-n}\Big(1 + \sum_{k=1}^{n} \psi_k\Big) + r^{6-n}\Big(\sum_{k=n-4}^{n} \varphi_k\Big)\log r + r^{6-n}\Big(\sum_{k=n-4}^{n} \varphi'_k\Big)\log^2 r + r^{6-n}\Big(\sum_{k=n-2}^{n} \varphi''_k\Big)\log^3 r + \varphi'''_n \log^4 r + A + O(r),$$

where A is a constant and $\psi_k, \varphi_k, \varphi'_k, \varphi''_k, \varphi'''_k \in \mathcal{P}_k$.

Moreover, we may restate some of the above results in another way.

(c) *If $n = 7, 8, 9$ or M is conformally flat near p, then*

$$G_p(x) = c_n r^{6-n} + A + O(r),$$

where A is a constant.

(d) *If $n = 10$, then*

$$G_p(x) = c_n r^{-4} + \frac{1}{17280}|W(p)|^2 \log r + O(1).$$

(e) *If $n \geq 11$, then*

$$G_p(x) = c_n r^{6-n} + \psi_4 r^{6-n} + O(r^{11-n}),$$

where $\psi_4 \in \mathcal{P}_4$ and

$$\psi_4(x) = \frac{1}{135(n-2)}\Big[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{r^2}{n+4}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 + \frac{3}{2(n+4)(n+2)}|W(p)|^2r^4\Big] + \frac{3n-20}{270(n+4)(n-4)(n-8)}r^2\Big[\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n}|W(p)|^2r^2\Big] - \frac{5n^2-66n+224}{120(n-8)(n-4)}r^2\Big[\sigma_1(A)_{,ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2\Big] + \frac{3n^4-16n^3-164n^2+400n+2432}{576(n+4)(n+2)n(n-1)}|W(p)|^2r^4.$$

Before starting to derive the expansion of Green's function of P_g, we first need to introduce some notation. For $\alpha \in \mathbb{R}$, set

$$A_\alpha = r^2\Delta_0 + 2\alpha r\partial_r + \alpha(\alpha+n-2), \qquad A_{\alpha,g} = r^2\Delta_g + 2\alpha r\partial_r + \alpha(\alpha+n-2),$$

where Δ_0 denotes the Euclidean Laplacian, and

$$B_\alpha = \frac{\partial}{\partial\alpha}A_\alpha = 2r\partial_r + 2\alpha + n - 2.$$

For $k \in \mathbb{Z}_+$, a straightforward computation yields (also see [Hang and Yang 2016, Lemma 2.4])

$$A_\alpha(\varphi\log^k r) = A_\alpha\varphi\log^k r + kB_\alpha\varphi\log^{k-1} r + k(k-1)\varphi\log^{k-2} r.$$

From this, for $\alpha, \beta, \gamma \in \mathbb{R}$ we get

$$\begin{aligned}
\text{(2-1)}\quad & A_\gamma A_\beta A_\alpha(\varphi\log^k r)\\
&= A_\gamma A_\beta A_\alpha\varphi\log^k r + k(B_\gamma A_\beta A_\alpha + A_\gamma B_\beta A_\alpha + A_\gamma A_\beta B_\alpha)\varphi\log^{k-1} r\\
&\quad + k(k-1)(A_\beta A_\alpha + B_\gamma B_\beta A_\alpha + B_\gamma A_\beta B_\alpha + A_\gamma B_\beta B_\alpha + A_\gamma A_\alpha + A_\gamma A_\beta)\varphi\log^{k-2} r\\
&\quad + k(k-1)(k-2)\\
&\qquad (B_\beta A_\alpha + A_\beta B_\alpha + B_\gamma A_\alpha + B_\gamma B_\beta B_\alpha + B_\gamma A_\beta + A_\gamma B_\alpha + A_\gamma B_\beta)\varphi\log^{k-3} r\\
&\quad + k(k-1)(k-2)(k-3)(A_\alpha + A_\beta + A_\gamma + B_\gamma B_\beta + B_\gamma B_\alpha + B_\beta B_\alpha)\varphi\log^{k-4} r\\
&\quad + k(k-1)(k-2)(k-3)(k-4)(B_\alpha + B_\beta + B_\gamma)\varphi\log^{k-5} r\\
&\quad + k(k-1)(k-2)(k-3)(k-4)(k-5)\varphi\log^{k-6} r.
\end{aligned}$$

A direct computation yields

$$\Delta_0(r^\alpha\varphi) = r^{\alpha-2}A_\alpha\varphi, \qquad \Delta_0^2(r^\alpha\varphi) = \Delta_0(r^{\alpha-2}A_\alpha\varphi) = r^{\alpha-4}A_{\alpha-2}A_\alpha\varphi,$$
$$\Delta_0^3(r^\alpha\varphi) = r^{\alpha-6}A_{\alpha-4}A_{\alpha-2}A_\alpha\varphi.$$

In particular,

$$\Delta_0^3(r^{6-n}\varphi) = r^{-n}A_{2-n}A_{4-n}A_{6-n}\varphi.$$

Define

$$M_g := \Delta_g \delta T_2 d + \delta T_2 d \Delta_g + \frac{n-2}{2}\Delta_g(\sigma_1(A)\Delta_g) + \delta T_4 d,$$

then rewrite (1-2) as $-P_g = (\Delta_g)^3 + M_g - (n-6)/2Q_g$. Notice that

$$\begin{aligned} A_{\alpha,g} =& A_\alpha + r^2(\Delta_g - \Delta_0) = A_\alpha + r^2\partial_i((g^{ij} - \delta^{ij})\partial_j), \\ -P_g(r^\alpha\varphi) =& r^{\alpha-6}(A_{\alpha-4}A_{\alpha-2}A_\alpha\varphi + K_\alpha\varphi), \end{aligned}$$

where

$$\begin{aligned} K_\alpha\varphi = {}& r^2(\Delta_g - \Delta_0)A_{\alpha-2}A_\alpha\varphi + A_{\alpha-4}(r^2(\Delta_g - \Delta_0))A_\alpha\varphi \\ & + A_{\alpha-4}A_{\alpha-2}(r^2(\Delta_g - \Delta_0))\varphi + r^{6-\alpha}M_g(r^\alpha\varphi) - \frac{n-6}{2}r^6Q_g\varphi. \end{aligned} \tag{2-2}$$

We first state the expression of $P_g(r^{6-n})$ and leave the complicated computations to the Appendix.

Lemma 2.2. *Under conformal normal coordinates around p with metric g, we have*

$$\begin{aligned} & -P_g(r^{6-n}) \\ & = -c_n\delta_p + (n-6)r^{-n}\Bigg\{\frac{64(n-4)}{9} \\ & \Bigg[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{r^2}{n+4}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 + \frac{3}{2(n+4)(n+2)}|W(p)|^2r^4\Bigg] \\ & \qquad + \frac{16(3n-20)}{9(n+4)}r^2\Bigg[\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n}|W(p)|^2r^2\Bigg] \\ & \qquad - 4(5n^2-66n+224)r^2\Bigg[\sigma_1(A)_{,ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2\Bigg] \\ & \qquad + \frac{3n^4-16n^3-164n^2+400n+2432}{3(n+4)(n+2)n(n-1)}|W(p)|^2r^4\Bigg\} + O(r^{5-n}), \end{aligned}$$

where W_{ijkl} is the Weyl tensor of metric g and each term in square brackets on the right-hand side of the identity is a harmonic polynomial.

Consequently, we rewrite the above equation in Lemma 2.2 as

$$P_g(r^{6-n}) = c_n\delta_p + r^{-n}f,$$

with $f = O(r^4)$.

Observe that for $i = 0, 1, \dots, [m/2]$,

$$A_\alpha|_{r^{2i}\mathcal{H}_{m-2i}} = (\alpha + 2i)(2m - 2i + \alpha + n - 2)$$

and

$$B_\alpha|_{r^{2i}\mathcal{H}_{m-2i}} = 2m + 2\alpha + n - 2.$$

Then

$$\begin{aligned} &A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}\mathcal{H}_{m-2i}} \\ &= (6-n+2i)(4-n+2i)(2-n+2i)(2m+4-2i)(2m+2-2i)(2m-2i). \end{aligned} \tag{2-3}$$

We start to find a formal asymptotic solution like $G_p(x) = r^{6-n}\big(1+\sum_{k=1}^n \psi_k\big)+\varphi$ with $\psi_k \in \mathcal{P}_k$. If we can find $\bar{\psi} = \sum_{k=1}^n \psi_k$ such that

$$A_{2-n}A_{4-n}A_{6-n}\bar{\psi} + K_{6-n}\bar{\psi} + f = O(r^{n+1}), \tag{2-4}$$

the regularity theory for elliptic equations gives that there exists a solution $\varphi \in C^{6,\alpha}_{\mathrm{loc}}$ for any $0 < \alpha < 1$ to

$$P_g(\varphi) = -r^{-n}(A_{2-n}A_{4-n}A_{6-n}\bar{\psi} + K_{6-n}\bar{\psi} + f) \in C^{\alpha}_{\mathrm{loc}}.$$

Thus it only remains to seek $\bar{\psi}$ satisfying (2-4) via induction. For any nonnegative integer k, it is not hard to see from the definition (2-2) of K_{6-n} that $K_{6-n}\varphi \in \mathcal{P}_{k+2}$ when $\varphi \in \mathcal{P}_k$. We first set $\psi_1 = \psi_2 = \psi_3 = 0$ by (2-4) and define

$$f_3 = f = O(r^4).$$

Case 1. n is odd.

If we have found $\psi_1, \dots, \psi_k$ for $3 \le k \le n-1$ with $\psi_k \in \mathcal{P}_k$ and

$$f_k = A_{2-n}A_{4-n}A_{6-n}\left(\sum_{i=1}^k \psi_i\right) + K_{6-n}\left(\sum_{i=1}^k \psi_i\right) + f := b_{k+1} + O(r^{k+2}),$$

then it follows from (2-3) that $A_{2-n}A_{4-n}A_{6-n}$ is invertible on $\mathcal{P}_{k+1}$ for $0 \le k \le n-1$. Thus there exists a unique $\psi_{k+1} \in \mathcal{P}_{k+1}$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi_{k+1} + b_{k+1} = 0.$$

This implies that

$$\begin{aligned} f_{k+1} =& A_{2-n}A_{4-n}A_{6-n}\left(\sum_{i=1}^{k+1} \psi_i\right) + K_{6-n}\left(\sum_{i=1}^{k+1} \psi_i\right) + f \\ =& f_k + A_{2-n}A_{4-n}A_{6-n}\psi_{k+1} + K_{6-n}\psi_{k+1} \\ =& O(r^{k+2}). \end{aligned}$$

This finishes the induction and assertion (a) follows.

Case 2. n is even and not less than 10.

Since $A_{2-n}A_{4-n}A_{6-n}$ is invertible on $\mathcal{P}_k$ for $0 \le k \le n-7$, by the same induction in Case 1, we may find $\psi_1, \ldots, \psi_{n-7}$ such that

$$f_{n-7} = A_{2-n}A_{4-n}A_{6-n}\left(\sum_{k=1}^{n-7}\psi_k\right) + K_{6-n}\left(\sum_{k=1}^{n-7}\psi_k\right) + f = O(r^{n-6}) := b_{n-6} + O(r^{n-5}).$$

Let $\psi^{(0)}_{n-6} = \alpha^{(0)}_{n-6}(x) + \beta^{(0)}_{n-6}(x)\log r$, where $\alpha^{(0)}_{n-6}(x) \in \mathcal{P}_{n-6}\backslash r^{n-6}\mathcal{H}_0$ and $\beta^{(0)}_{n-6}(x) \in r^{n-6}\mathcal{H}_0$, then it follows from (2-1) that

$$\begin{aligned}
&A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-6} \\
&= A_{2-n}A_{4-n}A_{6-n}\alpha^{(0)}_{n-6} + (B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})\beta^{(0)}_{n-6}.
\end{aligned}$$

Notice that for $0 \le i \le (n-8)/2$, we have

$$A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}\mathcal{H}_{m-2i}} \neq 0$$

by (2-3) and

$$\begin{aligned}
(B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})|_{r^{n-6}\mathcal{H}_0} &= 8(n-2)(n-4)(n-6) \\
&\neq 0.
\end{aligned}$$

Hence there exists a unique $\psi^{(0)}_{n-6} \in \mathcal{P}_{n-6} + \mathcal{P}_{n-6}\log r$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-6} + b_{n-6} = 0.$$

This indicates that

$$\begin{aligned}
f_{n-6} &= f_{n-7} + A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-6} + K_{6-n}\psi^{(0)}_{n-6} \\
&= O(r^{n-5}) + (K_{6-n}\beta^{(0)}_{n-6})\log r \\
&:= b_{n-5} + O(r^{n-4})\log r + O(r^{n-4}).
\end{aligned}$$

Let $\psi^{(0)}_{n-5} = \alpha^{(0)}_{n-5} + \beta^{(0)}_{n-5}\log r$, where $\alpha^{(0)}_{n-5} \in \mathcal{P}_{n-5} \setminus r^{n-6}\mathcal{H}_1$ and $\beta^{(0)}_{n-5} \in r^{n-6}\mathcal{H}_1$. Then we have

$$\begin{aligned}
&A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-5} \\
&= A_{2-n}A_{4-n}A_{6-n}\alpha^{(0)}_{n-5} + (B_{2-n}A_{4-n}A_{6-n} + A_{2-n}B_{4-n}A_{6-n} + A_{2-n}A_{4-n}B_{6-n})\beta^{(0)}_{n-5}.
\end{aligned}$$

By similar arguments, there exists a unique $\psi^{(0)}_{n-5} \in \mathcal{P}_{n-5} + r^{n-6}\mathcal{H}_1\log r$ such that

$$A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-5} + b_{n-5} = 0.$$

This implies that

$$\begin{aligned} f_{n-5} =& f_{n-6} + A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-5} + K_{6-n}\psi^{(0)}_{n-5} \\ =& O(r^{n-4})\log r + O(r^{n-4}) \\ :=& b^{(1)}_{n-4}\log r + O(r^{n-4}) + O(r^{n-3})\log r. \end{aligned}$$

Choose $\psi^{(1)}_{n-4} = \alpha^{(1)}_{n-4}\log r + \beta^{(1)}_{n-4}\log^2 r \in \mathcal{P}_{n-4}\log r + (r^{n-6}\mathcal{H}_2 + r^{n-4}\mathcal{H}_0)\log^2 r$. Then (2-1) gives

$$\begin{aligned} & A_{2-n}A_{4-n}A_{6-n}\psi^{(1)}_{n-4} \\ &= [A_{2-n}A_{4-n}A_{6-n}\alpha^{(1)}_{n-4} \\ &\qquad + 2(B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})\beta^{(1)}_{n-4}]\log r \\ &\qquad\qquad + A_{2-n}A_{4-n}A_{6-n}\beta^{(1)}_{n-4}\log^2 r + O(r^{n-4}). \end{aligned}$$

Since

$$\begin{aligned} (B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})|_{r^{n-6}\mathcal{H}_2} &= 8(n+2)n(n-2) \\ &\neq 0; \\ (B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})|_{r^{n-4}\mathcal{H}_0} &= -4n(n-2)(n-4) \\ &\neq 0 \end{aligned}$$

and $A_{2-n}A_{4-n}A_{6-n}|_{r^{2i}\mathcal{H}_{n-4-2i}} \neq 0$ for $0 \le i \le (n-8)/2$, there exists a unique $\psi^{(1)}_{n-4}$ such that

$$\begin{aligned} & A_{2-n}A_{4-n}A_{6-n}\alpha^{(1)}_{n-4} \\ &\qquad + 2(B_{6-n}A_{4-n}A_{2-n} + A_{6-n}B_{4-n}A_{2-n} + A_{6-n}A_{4-n}B_{2-n})\beta^{(1)}_{n-4} + b^{(1)}_{n-4} = 0 \end{aligned}$$

and

$$\begin{aligned} f^{(1)}_{n-4} =& f_{n-5} + A_{2-n}A_{4-n}A_{6-n}\psi^{(1)}_{n-4} + K_{6-n}\psi^{(1)}_{n-4} \\ =& O(r^{n-4}) + O(r^{n-3})\log r + O(r^{n-2})\log^2 r \\ :=& b^{(0)}_{n-4} + O(r^{n-3})\log r + O(r^{n-3}) + O(r^{n-2})\log^2 r. \end{aligned}$$

Choose $\psi^{(0)}_{n-4} \in \mathcal{P}_{n-4} + (r^{n-6}\mathcal{H}_2 + r^{n-4}\mathcal{H}_0)\log r$ to remove the term $b^{(0)}_{n-4}$ and set

$$\begin{aligned} f^{(0)}_{n-4} &= f^{(1)}_{n-4} + A_{2-n}A_{4-n}A_{6-n}\psi^{(0)}_{n-4} + K_{6-n}\psi^{(0)}_{n-4} \\ &= O(r^{n-3})\log r + O(r^{n-3}) + O(r^{n-2})\log^2 r. \end{aligned}$$

By similar arguments and (2-1), we get

$$\begin{aligned}
&\psi_{n-3}^{(1)} \in \mathcal{P}_{n-3} \log r + (r^{n-6}\mathcal{H}_3 + r^{n-4}\mathcal{H}_1) \log^2 r;\\
&\psi_{n-3}^{(0)} \in \mathcal{P}_{n-3} + (r^{n-6}\mathcal{H}_3 + r^{n-4}\mathcal{H}_1) \log r;\\
&\psi_{n-2}^{(i)} \in \mathcal{P}_{n-2} \log^i r + (r^{n-6}\mathcal{H}_4 + r^{n-4}\mathcal{H}_2 + r^{n-2}\mathcal{H}_0) \log^{i+1} r, \qquad \text{for } i = 0, 1, 2;\\
&\psi_{n-1}^{(i)} \in \mathcal{P}_{n-1} \log^i r + (r^{n-6}\mathcal{H}_5 + r^{n-4}\mathcal{H}_3 + r^{n-2}\mathcal{H}_1) \log^{i+1} r, \qquad \text{for } i = 0, 1, 2;\\
&\psi_{n}^{(i)} \in \mathcal{P}_{n} \log^i r + (r^{n-6}\mathcal{H}_6 + r^{n-4}\mathcal{H}_4 + r^{n-2}\mathcal{H}_2) \log^{i+1} r, \qquad \text{for } i = 0, 1, 2, 3.
\end{aligned}$$

Now we set

$$\psi_{n-6} = \psi_{n-6}^{(0)},\ \psi_{n-5} = \psi_{n-5}^{(0)},\ \psi_{n-4} = \psi_{n-4}^{(0)} + \psi_{n-4}^{(1)},\ \psi_{n-3} = \psi_{n-3}^{(0)} + \psi_{n-3}^{(1)}$$

and

$$\psi_{n-2} = \sum_{i=0}^{2} \psi_{n-2}^{(i)}, \quad \psi_{n-1} = \sum_{i=0}^{2} \psi_{n-1}^{(i)}, \quad \psi_{n} = \sum_{i=0}^{3} \psi_{n}^{(i)}.$$

Eventually, we obtain

$$\begin{aligned}
f_n &= A_{2-n}A_{4-n}A_{6-n}\Big(\sum_{k=1}^{n} \psi_k\Big) + K_{6-n}\Big(\sum_{k=1}^{n} \psi_k\Big) + f\\
&= O(r^{n+1})(\log^3 r + \log^2 r + \log r + 1) + O(r^{n+2}) \log^4 r.
\end{aligned}$$

Hence, $r^{-n} f_n \in C^\alpha$ for any $0 < \alpha < 1$. This finishes the induction and we obtain assertion (b) as desired.

Case 3. $n = 8$.

Notice that

$$P_g(G_p(x) - c_n r^{-2}) = O(r^{-4}) \in L^p,$$

for some $\frac{8}{5} < p < 2$. Then it follows from the regularity theory of elliptic equations that $G_p(x) - c_n r^{-2} \in C^{6-8/p}_{\mathrm{loc}}$. From this, we have $G_p(x) = c_n r^{-2} + A + O(r)$.

Case 4. M is locally conformally flat.

One may choose g flat near p and $P_g = -\Delta_0^3$. Hence, $P_g(G(x) - c_n r^{6-n}) = 0$ and then $G_p(x) - c_n r^{6-n}$ is smooth near p.

Therefore, the assertion (c) follows from cases 1,3,4. In some special cases, the leading term ψ_4 can be computed with the help of Lemma 2.2. The proof of Proposition 2.1 is complete.

3. $n \geq 10$ and not locally conformally flat

Similar to the Yamabe constant, for $n \geq 3$ and $n \neq 4, 6$, we define

$$Y_6^+(M, g) = \inf_{0<u\in H^3(M,g)} \frac{\int_M u P_g u \, d\mu_g}{\left(\int_M u^{\frac{2n}{n-6}} \, d\mu_g\right)^{\frac{n-6}{n}}}.$$

It follows from (1-3) that $Y_6^+(M, g)$ is a conformal invariant. However, due to the lack of a maximum principle for higher order elliptic equations in general, we first study another conformally invariant quantity,

$$Y_6(M, g) = \inf_{u\in H^3(M,g)\setminus\{0\}} \frac{\int_M u P_g u \, d\mu_g}{\left(\int_M |u|^{\frac{2n}{n-6}} \, d\mu_g\right)^{\frac{n-6}{n}}}.$$

In particular, we have $Y_6(S^n) = Y_6^+(S^n) = (n-6)/2 Q_{S^n} \omega_n^{6/n}$. For $w \in C_c^\infty(\mathbb{R}^n)$, let

$$\|w\|_{\mathcal{D}^{3,2}} := \sum_{|\beta|=3} \|D^\beta w\|_{L^2(\mathbb{R}^n)} \approx \|\nabla\Delta w\|_{L^2(\mathbb{R}^n)},$$

and let $\mathcal{D}^{3,2}(\mathbb{R}^n)$ denote the completion of $C_c^\infty(\mathbb{R}^n)$ under this norm. The equivalence of the above last two norms can be easily deduced by the formula (3-4) below. We first recall an optimal Euclidean Sobolev inequality (see [Lions 1985, p.154–165], [Lieb 1983]).

Lemma 3.1. *For $n \geq 7$, the following sharp Sobolev embedding inequality holds*:

$$Y_6(S^n)\left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-6}} dy\right)^{\frac{n-6}{n}} \leq \int_{\mathbb{R}^n} |\nabla\Delta w|^2 \, dy \quad \text{for all} \quad w \in \mathcal{D}^{3,2}(\mathbb{R}^n).$$

The equality holds if and only if $w(y) = (2/(1+|y|^2))^{(n-6)/2}$ up to any nonzero constant multiple, as well as all translations and dilations.

Proposition 3.2. *On a closed Riemannian manifold (M^n, g) of dimension $n \geq 10$, if there exists $p \in M^n$ such that the Weyl tensor $W_g(p) \neq 0$, then $Y_6(M^n) < Y_6(S^n)$.*

Proof. Recall the definition of P_g:

$$-P_g = \Delta_g^3 + \Delta_g \delta T_2 d + \delta T_2 d \Delta_g + \frac{n-2}{2} \Delta_g(\sigma_1(A)\Delta_g) + \delta T_4 d - \frac{n-6}{2} Q_g.$$

Then for all $\varphi \in H^3(M, g)$,

$$\int_M \varphi P_g \varphi \, d\mu_g = \int_M |\nabla\Delta\varphi|_g^2 d\mu_g - 2\int_M T_2(\nabla\varphi, \nabla\Delta\varphi) d\mu_g - \frac{n-2}{2}\int_M \sigma_1(A)(\Delta\varphi)^2 d\mu_g$$
$$- \int_M T_4(\nabla\varphi, \nabla\varphi) d\mu_g + \frac{n-6}{2}\int_M Q_g \varphi^2 d\mu_g.$$

Fix $\rho > 0$ small and choose test functions

$$\varphi(x) = \eta_\rho(x) u_\epsilon(x), \quad u_\epsilon(x) = \Big(\frac{2\epsilon}{\epsilon^2 + |x|^2}\Big)^{\frac{n-6}{2}}, \quad \epsilon > 0,$$

where $r = |x| = d_g(x, p)$ and

$$\eta_\rho \in C_c^\infty, \quad 0 \le \eta_\rho \le 1, \quad \eta_\rho \equiv 1 \quad \text{in} \quad B_\rho \quad \text{and} \quad \eta_\rho \equiv 0 \quad \text{in} \quad B_{2\rho}^c.$$

It is known from Lee and Parker [1987] that up to a conformal factor, under conformal normal coordinates around p with metric g, for all $N \ge 5$, we have

$$\sigma_1(A_g)(p) = 0, \quad \sigma_1(A_g)_{,i}(p) = 0, \quad \Delta_g \sigma_1(A_g)(p) = -\frac{|W(p)|_g^2}{12(n-1)}$$

and $\sqrt{\det g} = 1 + O(r^N)$.

Our purpose is to estimate $\int_M \varphi P_g \varphi \, d\mu_g$ and $\int_M \varphi^{2n/(n-6)} \, d\mu_g$. A direct computation shows

$$u_\epsilon' = -(n-6) u_\epsilon \frac{r}{\epsilon^2 + r^2}, \quad u_\epsilon'' = -(n-6) u_\epsilon \frac{\epsilon^2 - (n-5) r^2}{(\epsilon^2 + r^2)^2}$$

and

$$\Delta_0 u_\epsilon = -(n-6) \frac{u_\epsilon}{(\epsilon^2 + r^2)^2} (n\epsilon^2 + 4r^2),$$
$$(\Delta_0 u_\epsilon)' = (n-6)(n-4) \frac{u_\epsilon r}{(\epsilon^2 + r^2)^3} [(n+2)\epsilon^2 + 4r^2].$$

We start with $\int_M |\nabla \Delta \varphi|_g^2 \, d\mu_g$ and divide its integral into two parts: $\int_M = \int_{B_\rho} + \int_{M \setminus \overline{B_\rho}}$. Compute

$$\begin{aligned}
\int_{B_\rho} &|\nabla \Delta \varphi|_g^2 \, d\mu_g \\
&= \int_{B_\rho} g^{ij} (\Delta\varphi)_{,i} (\Delta\varphi)_{,j} \, d\mu_g \\
&= \int_{B_\rho} (\delta^{ij} + O(r^2)) (\Delta_0 \varphi + O(r^{N-1}) \varphi')_{,i} (\Delta_0 \varphi + O(r^{N-1}) \varphi')_{,j} (1 + O(r^N)) \, dx \\
&= \int_{B_\rho} |(\nabla\Delta)_0 \varphi|^2 \, dx + \int_{B_\rho} (\Delta_0 \varphi)' (O(r^{N-2}) \varphi' + O(r^{N-1}) \varphi'') \, dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus \overline{B_\rho}} |(\nabla\Delta)_0 \varphi|^2 \, dx &= (n-6)^2 (n-4)^2 \int_{\mathbb{R}^n \setminus \overline{B_\rho}} \frac{u_\epsilon^2 r^2}{(\epsilon^2 + r^2)^6} [(n+2)\epsilon^2 + 4r^2]^2 \, dx \\
&\le C \int_{\rho/\epsilon}^\infty \sigma^{5-n} d\sigma = O(\epsilon^{n-6}).
\end{aligned}$$

Similarly, we estimate $\int_{M\setminus\overline{B_\rho}}|\nabla\Delta\varphi|_g^2\,d\mu_g=O(\epsilon^{n-6})$. Thus, we obtain

$$\int_M|\nabla\Delta\varphi|_g^2\,d\mu_g=\int_{\mathbb{R}^n}|\nabla\Delta_0 u_\epsilon|^2dx+O(\epsilon^{n-6}).$$

Secondly, we compute

$$\begin{aligned}
&\int_{B_\rho}\sigma_1(A)(\Delta\varphi)^2\,d\mu_g\\
&\quad=\int_{B_\rho}\left(\frac{1}{2}\sigma_1(A)_{,ij}(p)x^ix^j+O(r^3)\right)(\Delta_0\varphi+O(r^{N-1})\varphi')^2(1+O(r^N))\,dx\\
&\quad=\int_{B_\rho}\frac{1}{2n}\Delta\sigma_1(A)(p)|x|^2(\Delta_0\varphi)^2\,dx+\int_{B_\rho}O(r^3)\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}(n\epsilon^2+4r^2)^2\,dx\\
&\quad=-\frac{(n-6)^2|W(p)|^2}{24n(n-1)}\omega_{n-1}\int_0^\rho\frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2r^{n+1}dr+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx,
\end{aligned}$$

and for some large enough N

$$\begin{aligned}
\int_{B_{2\rho}\setminus\overline{B_\rho}}\sigma_1(A)(\Delta\varphi)^2\,d\mu_g&\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}|\Delta_0\varphi+O(r^{N-1})\varphi'|^2(1+O(r^N))\,dx\\
&\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}[(\Delta_0\varphi)^2+O(r^{2(N-1)})|\varphi'|^2]\,dx\\
&\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}(u_\epsilon\Delta_0\eta_\rho+2\nabla u_\epsilon\cdot\nabla\eta_\rho+\eta_\rho\Delta_0u_\epsilon)^2\,dx+O(\epsilon^{n-6})\\
&\le C\int_\rho^{2\rho}\frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2r^{n-1}\,dr+O(\epsilon^{n-6})\\
&\overset{\sigma=r/\epsilon}{\le}C\epsilon^2\int_{\rho/\epsilon}^{2\rho/\epsilon}\frac{(n+4\sigma^2)^2\sigma^{n-1}}{(1+\sigma^2)^{n-2}}\,d\sigma+O(\epsilon^{n-6})\\
&\le C\epsilon^2\left(\frac{\rho}{\epsilon}\right)^{8-n}+O(\epsilon^{n-6})=O(\epsilon^{n-6}).
\end{aligned}$$

Observe that

$$\int_{B_\rho}\frac{r^3u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx=\begin{cases}O(\epsilon^4)&\text{if } n=10,\\ O(\epsilon^5|\log\epsilon|)&\text{if } n=11,\\ O(\epsilon^5)&\text{if } n\ge 12.\end{cases}\tag{3-1}$$

Hence,

$$-\frac{n-2}{2}\int_M \sigma_1(A)(\Delta\varphi)^2\,d\mu_g$$
$$=\frac{(n-6)^2(n-2)|W(p)|^2}{48n(n-1)}\omega_{n-1}\int_0^\rho \frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2 r^{n+1}\,dr$$
$$+\begin{cases} O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\geq 12.\end{cases}$$

Thirdly, we compute $\int_M T_2(\nabla\varphi,\nabla\Delta\varphi)\,d\mu_g$.

$$\int_{B_\rho} T_2(\nabla\varphi,\nabla\Delta\varphi)\,d\mu_g=\int_{B_\rho}[(n-2)\sigma_1(A)\langle\nabla\varphi,\nabla\Delta\varphi\rangle-8A_{ij}\varphi_{,i}(\Delta\varphi)_{,j}]\,d\mu_g.$$

Observe that $u_{\epsilon,i}=(x^i/r)u_\epsilon'$ and $(\Delta_0 u_\epsilon)_{,i}=(x^i/r)(\Delta_0 u_\epsilon)'$. Then we get

$$(n-2)\int_{B_\rho}\sigma_1(A)\langle\nabla\varphi,\nabla\Delta\varphi\rangle\,d\mu_g$$
$$=(n-2)\int_{B_\rho}\left(\frac{1}{2}\sigma_1(A)_{,ij}(p)x^ix^j+O(r^3)\right)g^{kl}\varphi_{,k}(\Delta\varphi)_{,l}\,d\mu_g$$
$$=(n-2)\int_{B_\rho}\left(\frac{1}{2}\sigma_1(A)_{,ij}(p)x^ix^j+O(r^3)\right)(\delta^{kl}+O(r^2))\varphi_{,k}(\Delta_0\varphi+O(r^{N-1})\varphi')_{,l}\,d\mu_g$$
$$=\frac{n-2}{2}\int_{B_\rho}\frac{1}{n}\Delta\sigma_1(A)(p)|x|^2\varphi_{,i}(\Delta_0\varphi)_{,i}\,dx+\int_{B_\rho}O(r^3)|\varphi'||(\Delta_0\varphi)'|\,dx$$
$$=-\frac{(n-2)|W(p)|^2}{24n(n-1)}\int_{B_\rho}\left\{-(n-6)^2(n-4)\frac{u_\epsilon^2 r^4}{(\epsilon^2+r^2)^4}\left[(n+2)\epsilon^2+4r^2\right]\right\}dx$$
$$+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx$$
$$=\frac{(n-2)(n-4)(n-6)^2}{24n(n-1)}|W(p)|^2\int_{B_\rho}\frac{r^4}{(\epsilon^2+r^2)^4}u_\epsilon^2[(n+2)\epsilon^2+4r^2]\,dx$$
$$+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx,$$

and

$$
\begin{aligned}
&-8\int_{B_\rho} A_{ij}\varphi_{,i}(\Delta\varphi)_{,j}\,d\mu_g\\
&=-8\int_{B_\rho}\Big(A_{ij,k}(p)x^k+\frac{1}{2}A_{ij,kl}(p)x^kx^l+O(r^3)\Big)\varphi_{,i}(\Delta_0\varphi+O(r^{N-1})\varphi')_{,j}\,d\mu_g\\
&=-4\int_{B_\rho} A_{ij,kl}(p)x^kx^lx^ix^j\Big[-(n-4)(n-6)^2\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\Big]\,dx\\
&\qquad\qquad+\int_{B_\rho} O(r^3)|\varphi'||(\Delta_0\varphi)'|\,dx\\
&=4(n-4)(n-6)^2\int_{B_\rho}\Big[-\frac{2}{9}\frac{1}{n-2}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2-\frac{\sigma_1(A)_{,ij}(p)x^ix^jr^2}{n-2}\Big]\\
&\qquad\qquad\times\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dx+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx\\
&=-\frac{8(n-4)(n-6)^2}{9(n-2)}\int_{B_\rho}\sum_{k,l}(W_{iklj}(p)W_{sklt}(p)x^ix^jx^sx^t)\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dx\\
&\qquad-\frac{4(n-4)(n-6)^2}{n(n-2)}\int_{B_\rho}\frac{\Delta\sigma_1(A)(p)r^4}{(\epsilon^2+r^2)^4}u_\epsilon^2[(n+2)\epsilon^2+4r^2]\,dx+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx\\
&=-\frac{(n-4)(n-6)^2}{(n-1)n(n+2)}\omega_{n-1}|W(p)|^2\int_0^\rho\frac{r^{n+3}u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx,
\end{aligned}
$$

where the last identity follows from

$$
\begin{aligned}
&\sum_{k,l}W_{iklj}(p)W_{sklt}(p)\int_{B_\rho}x^ix^jx^sx^t\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dx\\
&=\sum_{k,l}W_{iklj}(p)W_{sklt}(p)\int_{\mathbb{S}^{n-1}}\xi^i\xi^j\xi^s\xi^t\,d\mu_{\mathbb{S}^{n-1}}\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\\
&=\frac{\omega_{n-1}}{n(n+2)}\sum_{k,l}W_{iklj}(p)W_{sklt}(p)[\delta_{ij}\delta_{st}+\delta_{is}\delta_{jt}+\delta_{it}\delta_{js}]\int_0^\rho\frac{r^{n+3}u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\\
&=\frac{\omega_{n-1}}{n(n+2)}[|W(p)|^2+W_{iklj}(p)W_{jkli}(p)]\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\\
&=\frac{3}{2}\frac{\omega_{n-1}}{n(n+2)}|W(p)|^2\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&-2\int_{B_\rho} T_2(\nabla\varphi, \nabla\Delta\varphi)\, d\mu_g \\
&= -\frac{(n^2-28)(n-4)(n-6)^2}{12n(n-1)(n+2)} |W(p)|^2 \omega_{n-1} \int_0^\rho r^{n+3} \frac{u_\epsilon^2}{(\epsilon^2+r^2)^4} [(n+2)\epsilon^2 + 4r^2]\, dr \\
&\qquad + \int_{B_\rho} \frac{O(r^3) u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx.
\end{aligned}
$$

By a similar argument, one has

$$
\begin{aligned}
\left| \int_{B_{2\rho}\setminus \overline{B_\rho}} T_2(\nabla\varphi, \nabla\Delta\varphi) \right| &\le C \int_{B_{2\rho}\setminus \overline{B_\rho}} |\nabla\varphi| |\nabla\Delta\varphi|\, d\mu_g \\
&\le C \int_{B_{2\rho}\setminus \overline{B_\rho}} |u_\epsilon'| |(\Delta u_\epsilon)'|\, dx + O(\epsilon^{n-6}) = O(\epsilon^{n-6}).
\end{aligned}
$$

Fourthly, we compute $\int_M T_4(\nabla\varphi, \nabla\varphi)\, d\mu_g$.

$$
\begin{aligned}
(n-6)\int_{B_\rho} \Delta\sigma_1(A) |\nabla\varphi|_g^2\, d\mu_g &= (n-6)\int_{B_\rho} (\Delta\sigma_1(A)(p) + O(r))(|\varphi'|^2 + O(r^2)|\varphi|^2)\, dx \\
&= -(n-6)^3 \frac{|W(p)|^2}{12(n-1)} \int_{B_\rho} \frac{u_\epsilon^2 r^2}{(\epsilon^2+r^2)^2}\, dx + \int_{B_\rho} \frac{O(r^3) u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx.
\end{aligned}
$$

Using (A-5), we get

$$
\begin{aligned}
&-\frac{16}{n-4} \int_{B_\rho} B_{ij} \varphi_{,i} \varphi_{,j}\, d\mu_g \\
&= -\frac{16}{n-4} \int_{B_\rho} (n-6)^2 u_\epsilon^2 \frac{B_{ij} x^i x^j}{(\epsilon^2+r^2)^2}\, dx \\
&= -\frac{16(n-6)^2}{n-4} \int_{B_\rho} \Big[-\frac{2}{9} \frac{1}{n-2} \sum_{k,l,s} [(W_{ikls}(p) + W_{ilks}(p)) x^i]^2 \\
&\quad + \frac{1}{12(n-2)(n-1)} |W(p)|^2 r^2 - \frac{7n-8}{n-2} \sigma_1(A)_{,ij}(p) x^i x^j + O(r^3) \Big] \frac{u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx \\
&= -\frac{16(n-6)^2}{n-4} \Big[-\frac{2}{3n(n-2)} + \frac{1}{12(n-2)(n-1)} + \frac{7n-8}{12(n-2)(n-1)n} \Big] \\
&\qquad |W(p)|^2 \int_{B_\rho} \frac{r^2 u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx + \int_{B_\rho} \frac{O(r^3) u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx \\
&= \int_{B_\rho} \frac{O(r^3) u_\epsilon^2}{(\epsilon^2+r^2)^2}\, dx,
\end{aligned}
$$

where the second identity follows from

$$\sum_{i,k,l,s}(W_{ikls}(p)+W_{ilks}(p))^2=2|W(p)|^2+2\sum_{i,k,l,s}W_{ikls}(p)W_{ilks}(p)=3|W(p)|^2,$$

in view of

$$0=W_{ikls}(W_{ilks}+W_{iksl}+W_{islk})=W_{ikls}W_{ilks}+W_{ikls}W_{iksl}+W_{ikls}W_{islk}=2W_{ikls}W_{ilks}-|W|^2$$

at p. Also we have

$$\int_{B_{2\rho}\setminus\overline{B_\rho}}T_4(\nabla\varphi,\nabla\varphi)\,d\mu_g\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}|\nabla\varphi|_g^2\,d\mu_g=O(\epsilon^{n-6}).$$

Hence, collecting the above terms together with (3-1), we obtain

$$\begin{aligned}
-\int_M & T_4(\nabla\varphi,\nabla\varphi)\,d\mu_g\\
&=-(n-6)\int_{B_\rho}\Delta\sigma_1(A)|\nabla\varphi|_g^2\,d\mu_g+\frac{16}{n-4}\int_{B_\rho}B_{ij}\varphi_{,i}\varphi_{,j}\,d\mu_g+O(\epsilon^{n-6})\\
&=(n-6)^3\frac{|W(p)|^2}{12(n-1)}\int_{B_\rho}\frac{u_\epsilon^2r^2}{(\epsilon^2+r^2)^2}\,dx+\begin{cases}O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\ge 12.\end{cases}
\end{aligned}$$

Finally, we compute $((n-6)/2)\int_M Q_g\varphi^2\,d\mu_g$. By the definition (1-1) of Q_g, integration by parts gives

$$\begin{aligned}
\frac{n-6}{2}\int_M Q_g\varphi^2\,d\mu_g&=\frac{n-6}{2}\int_M\Delta^2\sigma_1(A)\varphi^2\,d\mu_g+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx+O(\epsilon^{n-6})\\
&=\frac{n-6}{2}\int_M\Delta\sigma_1(A)\Delta\varphi^2\,d\mu_g+\int_{B_\rho}\frac{O(r^3)u_\epsilon^2}{(\epsilon^2+r^2)^2}\,dx+O(\epsilon^{n-6})\\
&=-\frac{(n-6)^2|W(p)|^2}{12(n-1)}\omega_{n-1}\int_0^\rho\frac{u_\epsilon^2r^{n-1}}{(\epsilon^2+r^2)^2}[(n-10)r^2-n\epsilon^2]\,dr\\
&\qquad+\begin{cases}O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\ge 12,\end{cases}
\end{aligned}$$

by (3-1), where the last identity follows from

$$\begin{aligned}
&\frac{n-6}{2}\int_{B_\rho}\Delta\sigma_1(A)\Delta\varphi^2\,d\mu_g\\
&=\frac{n-6}{2}\int_{B_\rho}(\Delta\sigma_1(A)(p)+O(r))(\Delta_0\varphi^2+O(r^{N-1})(\varphi^2)')\,dx\\
&=\frac{n-6}{2}\Delta\sigma_1(A)(p)\int_{B_\rho}2(\varphi\Delta_0\varphi+|\nabla\varphi|_0^2)\,dx+\int_{B_\rho}\frac{O(r)u_\epsilon^2}{\epsilon^2+r^2}\,dx\\
&=-\frac{(n-6)^2|W(p)|^2}{12(n-1)}\omega_{n-1}\int_0^\rho\frac{u_\epsilon^2r^{n-1}}{(\epsilon^2+r^2)^2}[(n-10)r^2-n\epsilon^2]dr+\int_{B_\rho}\frac{O(r)u_\epsilon^2}{\epsilon^2+r^2}\,dx
\end{aligned}$$

and the first identity follows from

$$\left|\int_{B_{2\rho}\setminus\overline{B_\rho}}Q_g\varphi^2\,d\mu_g\right|\le C\int_{B_{2\rho}\setminus\overline{B_\rho}}u_\epsilon^2\,dx=O(\epsilon^{n-6}).$$

Therefore collecting all the above terms together, we obtain

$$\int_M\varphi P_g\varphi\,d\mu_g=\int_{\mathbb{R}^n}|\nabla\Delta_0u_\epsilon|^2\,dx+A_{n,\rho,\epsilon}|W(p)|^2\omega_{n-1}+O(\epsilon^{\min\{n-6,5\}}),$$

where $A_{n,\rho,\epsilon}$ is a constant given by

$$\begin{aligned}
&(n-6)^2\biggl(\frac{n-2}{48n(n-1)}\int_0^\rho\frac{(n\epsilon^2+4r^2)^2}{(\epsilon^2+r^2)^4}u_\epsilon^2r^{n+1}\,dr+\frac{n-6}{12(n-1)}\int_0^\rho\frac{u_\epsilon^2r^{n+1}}{(\epsilon^2+r^2)^2}\,dr\\
&\qquad-\frac{1}{12(n-1)}\int_0^\rho\frac{u_\epsilon^2r^{n-1}}{(\epsilon^2+r^2)^2}[(n-10)r^2-n\epsilon^2]\,dr\\
&\qquad-\frac{(n^2-28)(n-4)}{12n(n-1)(n+2)}\int_0^\rho r^{n+3}\frac{u_\epsilon^2}{(\epsilon^2+r^2)^4}[(n+2)\epsilon^2+4r^2]\,dr\biggr)\\
&=2^{n-6}\frac{(n-6)^2}{12(n-1)}\epsilon^4\biggl(\frac{n-2}{4n}\int_0^{\rho/\epsilon}\frac{(n+4\sigma^2)^2}{(1+\sigma^2)^4}(1+\sigma^2)^{-(n-6)}\sigma^{n+1}\,d\sigma\\
&\qquad+(n-6)\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^2}(1+\sigma^2)^{-(n-6)}\sigma^{n+1}\,d\sigma\\
&\qquad-\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^2}(1+\sigma^2)^{-(n-6)}\sigma^{n-1}[(n-10)\sigma^2-n]\,d\sigma\\
&\qquad-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^{\rho/\epsilon}\frac{\sigma^{n+3}}{(1+\sigma^2)^4}(1+\sigma^2)^{-(n-6)}[(n+2)+4\sigma^2]\,d\sigma\biggr),
\end{aligned}$$

where $r=\epsilon\sigma$. When $n=10$, we claim that the leading term of the constant in the parentheses on the right-hand side of the above identity:

$$\frac{1}{5}\int_0^{\rho/\epsilon}\frac{(4\sigma^2+10)^2}{(1+\sigma^2)^4}(1+\sigma^2)^{-4}\sigma^{11}\,d\sigma+\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^2}(1+\sigma^2)^{-4}(4\sigma^2+10)\sigma^9\,d\sigma \\ -\frac{18}{5}\int_0^{\rho/\epsilon}\frac{1}{(1+\sigma^2)^4}(1+\sigma^2)^{-4}(4\sigma^2+12)\sigma^{13}\,d\sigma$$

is a negative constant multiple of $|\log\epsilon|$. To see this, notice it is obviously true for the third term, and the first two terms equal

$$\frac{1}{5}\int_0^{\rho/\epsilon}\{\sigma^2[(4\sigma^2+10)^2-18\sigma^2(4\sigma^2+12)]+5(4\sigma^2+10)(1+\sigma^2)^2\}(1+\sigma^2)^{-8}\sigma^9\,d\sigma \\ =\frac{1}{5}\int_0^{\rho/\epsilon}(-36\sigma^6-46\sigma^4+220\sigma^2+50)(1+\sigma^2)^{-8}\sigma^9\,d\sigma,$$

whose leading term is also a negative constant multiple of $|\log\epsilon|$. For $n\geq 11$, let $t=\sigma^2$. The limit of the coefficient of $|W(p)|^2\omega_{n-1}$ as $\epsilon\to 0$ is

$$2^{n-7}\frac{(n-6)^2}{12(n-1)}\epsilon^4\Bigg\{\frac{n-2}{4n}\int_0^\infty\frac{(n+4t)^2}{(1+t)^{n-2}}t^{\frac{n}{2}}\,dt \\ +(n-6)\int_0^\infty\frac{1}{(1+t)^{n-4}}t^{\frac{n}{2}}\,dt-\int_0^\infty\frac{(n-10)t-n}{(1+t)^{n-4}}t^{\frac{n}{2}-1}\,dt \\ -\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^\infty\frac{(n+2)+4t}{(1+t)^{n-2}}t^{\frac{n}{2}+1}\,dt\Bigg\}.$$

With the help of the Beta function:

$$\int_0^\infty\frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,dx=B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

for $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, we have

$$\begin{aligned}&\frac{n-2}{4n}\int_0^\infty\frac{(n+4t)^2}{(1+t)^{n-2}}t^{\frac{n}{2}}\,dt\\ =&\frac{n-2}{4n}\int_0^\infty\frac{(n-4)^2+8(n-4)(1+t)+16(1+t)^2}{(1+t)^{n-2}}t^{\frac{n}{2}}\,dt\\ =&\frac{n-2}{4n}\Big[(n-4)^2B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big)+8(n-4)B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)+16B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)\Big],\\ &(n-6)\int_0^\infty\frac{1}{(1+t)^{n-4}}t^{\frac{n}{2}}\,dt=(n-6)B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big),\\ &-\int_0^\infty\frac{(n-10)t-n}{(1+t)^{n-4}}t^{\frac{n}{2}-1}\,dt=-(n-10)B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)+nB\Big(\frac{n}{2},\frac{n}{2}-4\Big),\end{aligned}$$

and

$$
\begin{aligned}
&-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^\infty \frac{(n+2)+4t}{(1+t)^{n-2}}t^{\frac{n}{2}+1}\,dt\\
&=-\frac{(n^2-28)(n-4)}{n(n+2)}\int_0^\infty \frac{4(1+t)^2+(n-6)(1+t)-(n-2)}{(1+t)^{n-2}}t^{\frac{n}{2}}\,dt\\
&=-\frac{4(n^2-28)(n-4)}{n(n+2)}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)-\frac{(n^2-28)(n-4)(n-6)}{n(n+2)}B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)\\
&\quad+\frac{(n^2-28)(n-4)(n-2)}{n(n+2)}B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big).
\end{aligned}
$$

Hence, the above limit of the coefficient of $|W(p)|^2\omega_{n-1}$ is rewritten as

$$
\begin{aligned}
\text{(3-2)}\quad & 2^{n-7}\frac{(n-6)^2}{12(n-1)}\epsilon^4\Big\{nB\Big(\frac{n}{2},\frac{n}{2}-4\Big)\\
&+B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big)\Big[\frac{n-2}{4n}(n-4)^2+\frac{(n^2-28)(n-4)(n-2)}{n(n+2)}\Big]\\
&+B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)\Big[\frac{2(n-2)(n-4)}{n}-\frac{(n^2-28)(n-4)(n-6)}{n(n+2)}\Big]\\
&+B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)\Big[\frac{4(n-2)}{n}-n+10+n-6-\frac{4(n^2-28)(n-4)}{n(n+2)}\Big]\Big\}\\
&=2^{n-7}\frac{(n-6)^2}{12(n-1)}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big)\\
&\qquad\epsilon^4\Big\{(n-10)+\frac{(n-2)(\frac{n}{2}-4)(\frac{n}{2}-5)}{4n(n+2)(n-3)}(5n^2-2n-120)\\
&\qquad\qquad+\frac{\frac{n}{2}-5}{n(n+2)}(-n^3+8n^2+28n-176)\\
&\qquad\qquad\qquad+\frac{4}{n(n+2)}(-n^3+6n^2+30n-116)\Big\},
\end{aligned}
$$

where we have used some elementary identities

$$
\begin{aligned}
B\Big(\frac{n}{2}+1,\frac{n}{2}-3\Big)&=\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2}-3)}{\Gamma(n-2)}&&=\frac{(\frac{n}{2}-4)(\frac{n}{2}-5)}{(n-3)(n-4)}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big),\\
B\Big(\frac{n}{2}+1,\frac{n}{2}-4\Big)&=\frac{\frac{n}{2}-5}{n-4}B(\tfrac{n}{2}+1,\tfrac{n}{2}-5),\\
B\Big(\frac{n}{2},\frac{n}{2}-4\Big)&=\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}-4)}{\Gamma(n-4)}&&=\frac{n-10}{n}B\Big(\frac{n}{2}+1,\frac{n}{2}-5\Big).
\end{aligned}
$$

The constant in the last brace of (3-2) when $n \geq 11$ is

$$
\begin{aligned}
&n-10+\frac{1}{16n(n+2)(n-3)}\{(n-2)(n-8)(n-10)(5n^2-2n-120)\\
&+8(n-3)[(n-10)(-n^3+8n^2+28n-176)+8(-n^3+6n^2+30n-116)]\}\\
&\quad = n-10+\frac{1}{16n(n+2)(n-3)}[-3n^5+2n^4+228n^3-264n^2-1760n-768]\\
&\quad = \frac{-3n^5+18n^4+52n^3-200n^2-800n-768}{16n(n+2)(n-3)} < 0.
\end{aligned}
$$

On the other hand, we have

$$
\int_M \varphi^{\frac{2n}{n-6}}\,d\mu_g = \int_{B_\rho} u_\epsilon^{\frac{2n}{n-6}}\,d\mu_g + \int_{B_{2\rho}\setminus \overline{B_\rho}} \varphi^{\frac{2n}{n-6}}\,d\mu_g = \int_{\mathbb{R}^n} u_\epsilon^{\frac{2n}{n-6}}\,dx + O(\epsilon^n).
$$

Therefore, putting these facts together, we conclude by Lemma 3.1 that

$$
\begin{aligned}
\frac{\int_M \varphi P_g \varphi\,d\mu_g}{\left(\int_M \varphi^{\frac{2n}{n-6}}\,d\mu_g\right)^{\frac{n-6}{n}}} &= Y_6(S^n) + A_{n,\rho,\epsilon}|W(p)|^2\omega_{n-1} + \begin{cases} O(\epsilon^4) & \text{if } n=10,\\ O(\epsilon^5|\log\epsilon|) & \text{if } n=11,\\ O(\epsilon^5) & \text{if } n\geq 12,\end{cases}\\
&= \begin{cases} Y_6(S^n) - C_n|W(p)|^2\epsilon^4|\log\epsilon| + O(\epsilon^4) & \text{if } n=10,\\ Y_6(S^n) - C_n|W(p)|^2\epsilon^4 + o(\epsilon^4) & \text{if } n\geq 11,\end{cases}
\end{aligned}
$$

for some positive constant $C_n > 0$. Consequently, choosing ϵ sufficiently small, we obtain $Y_6(M^n) < Y_6(S^n)$. This finishes the proof. □

Given a smooth positive function f on M^n, we define a "free" energy functional by

$$
E_f[u] = \frac{1}{2}\int_M u P_g u\,d\mu_g - \frac{1}{2^\sharp}\int_M f|u|^{2^\sharp}\,d\mu_g.
$$

Let $u_{,i}$ or $\nabla_i u$ denote the covariant derivatives of u with respect to the metric g and R^l_{ijk} be the Riemannian curvature tensor of metric g. Notice that

$$
\nabla_j\nabla_i\nabla^i u = \nabla_i\nabla_j\nabla^i u + R^k_{iij}\nabla_k u = \nabla_i\nabla^i\nabla_j u - R^k_j\nabla_k u.
$$

We have

$$
\int_M |\nabla\Delta u|_g^2\,d\mu_g = \int_M |\Delta\nabla_j u - R^k_j\nabla_k u|_g^2\,d\mu_g. \tag{3-3}
$$

Under g-normal coordinates around a point, one gets

$$
\begin{aligned}
&\tfrac{1}{2}\Delta_g|\nabla^2 u|_g^2\\
&\qquad = |\nabla^3 u|_g^2 + \langle \nabla\Delta\nabla_i u, \nabla\nabla^i u\rangle_g + u_{,ij}(R^l_{ijk}u_{,lk} + R^l_j u_{,il} + R^l_{ijk,k}u_{,l} + R^l_{ijk}u_{,lk}).
\end{aligned}
$$

Integrating the above identity over M gives

$$\begin{aligned}(3\text{-}4)\quad &\int_M |\Delta\nabla u|_g^2\, d\mu_g \\ &= \int_M |\nabla^3 u|_g^2\, d\mu_g + \int_M O\big(|\mathrm{Rm}||\nabla^2 u|_g + |\nabla\,\mathrm{Rm}||\nabla u|_g\big)|\nabla^2 u|_g\, d\mu_g.\end{aligned}$$

From (3-3) and (3-4), it yields that the following two norms are equivalent:

$$\begin{aligned}\|u\|_{H^3} :=& \left(\int_M (|\nabla\Delta u|_g^2\, d\mu_g + |\nabla^2 u|_g^2 + |\nabla u|_g^2 + u^2)\, d\mu_g\right)^{1/2} \\ \approx& \left(\int_M (|\nabla^3 u|_g^2\, d\mu_g + |\nabla^2 u|_g^2 + |\nabla u|_g^2 + u^2)\, d\mu_g\right)^{1/2}, \quad u \in H^3(M, g).\end{aligned}$$

Let $\|\cdot\|_p$ denote the norm of $L^p(M, g)$ for $1 \le p \le \infty$.

A sequence $\{u_k\}$ in $H^3(M, g)$ is called a Palais–Smale (P–S)$_\beta$ sequence for E_f if $E_f[u_k] \to \beta \in \mathbb{R}$ and $DE_f[u_k] \to 0$ as $k \to \infty$. The energy E_f satisfies the (P–S)$_\beta$ condition if any Palais–Smale sequence of E_f has a strongly convergent subsequence. We call P_g is coercive if there exists a constant $\mu(g) > 0$ such that

$$\int_M \psi P_g \psi\, d\mu_g \ge \mu(g) \int_M \psi^2\, d\mu_g, \quad \text{for all} \quad \psi \in H^3(M, g).$$

Remark. If (M, g) is Einstein and of positive constant scalar curvature, from the factorization (1-4) of P_g, the coercivity of P_g is automatically satisfied.

As an application, we adapt some arguments in Esposito and Robert [2002] to show some existence results of the prescribed Q-curvature equation, whose solution may change signs due to the lack of maximum principles (in general).

Theorem 3.3. *Let (M^n, g) be a smooth closed manifold of dimension $n \ge 10$ and f be a smooth positive function in M^n. Suppose the Weyl tensor W_g is nonzero at a maximum point of f and f satisfies the vanishing order condition (1-5) at this maximum point. If P_g is coercive, then there exists a nontrivial $C^{6,\mu}$ $(0 < \mu < 1)$ solution to*

$$P_g u = f|u|^{2^\sharp - 2} u \quad \text{in } M. \tag{3-5}$$

In addition, if (M, g) is Einstein and of positive scalar curvature, then there exists a smooth solution to the Q-curvature equation

$$P_g u = f u^{\frac{n+6}{n-6}}, u > 0 \quad \text{in } M. \tag{3-6}$$

Proof. By the assumptions, there exists $p \in M$ such that $f(p) = \max_{x\in M^n} f(x)$, $W_g(p) \neq 0$ and the vanishing order condition (1-5) of f is true at p. Let

$$\gamma_\epsilon(t) = t\frac{\varphi}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}},$$

where $\varphi = \eta_\rho u_\epsilon$ is the test function chosen in Proposition 3.2. By choosing t_0 large enough, we get $E[\gamma_\epsilon(t_0)] < 0$. Let

$$\Gamma = \left\{\gamma(t) \in C([0,t_0], H^3(M,g)); \gamma(0) = 0, \gamma(t_0) = t_0\frac{\varphi}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}}\right\}.$$

From the coercivity of P_g and the Sobolev embedding theorem, we have

$$E_f\left[\frac{\varphi}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}}\right] = \frac{1}{2}\frac{\int_M \varphi P_g\varphi\, d\mu_g}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}^2} - \frac{1}{2^\sharp} \geq \frac{1}{2}C - \frac{1}{2^\sharp}.$$

It suffices to only estimate the term:

$$\begin{aligned}\int_M f\varphi^{\frac{2n}{n-6}}\, d\mu_g =& \int_{B_\rho}\left[f(p) + \sum_{k=2}^{4}\frac{1}{k!}\partial_{i_1\cdots i_k}f(p)x^{i_1}\cdots x^{i_k} + O(|x|^5)\right]u_\epsilon^{2^\sharp}\, dx + O(\epsilon^n)\\ =& f(p)\int_{\mathbb{R}^n} u_\epsilon^{\frac{2n}{n-6}}\, dx + \begin{cases} O(\epsilon^4) & \text{if } n = 10,\\ o(\epsilon^4) & \text{if } n \geq 11,\end{cases}\end{aligned}$$

where the second equality follows from the vanishing order condition (1-5) of f at p. From this and some existing estimates in the proof of Proposition 3.2, we conclude that there exist some sufficiently small $\epsilon > 0$ and a constant $C_n' > 0$ such that

$$\begin{aligned}\sup_{t\geq 0} E_f[\gamma_\epsilon(t)] =& E_f[\gamma_\epsilon(t^*)]\\ =& \frac{3}{n}\left(\frac{\int_M \varphi P_g\varphi d\mu_g}{\|f^{1/2^\sharp}\varphi\|_{2^\sharp}^2}\right)^{2^\sharp/(2^\sharp-2)}\\ \leq& \begin{cases} \frac{3}{n}(\max_M f)^{\frac{6-n}{6}}Y_6(S^n)^{\frac{n}{6}} - C_n'|W(p)|^2\epsilon^4|\log\epsilon| + O(\epsilon^4) & \text{if } n = 10,\\ \frac{3}{n}(\max_M f)^{\frac{6-n}{6}}Y_6(S^n)^{\frac{n}{6}} - C_n'|W(p)|^2\epsilon^4 + o(\epsilon^4) & \text{if } n \geq 11,\end{cases}\end{aligned}$$

where $t^* = \left(\int_M \varphi P_g\varphi\, d\mu_g / \|f^{1/2^\sharp}\varphi\|_{2^\sharp}^2\right)^{1/(2^\sharp-2)}$. Then it follows from the mountain pass lemma (see [Ambrosetti and Rabinowitz 1973] or [Esposito and Robert 2002, Proposition 1]) that

$$\beta = \inf_{\gamma\in\Gamma}\sup_{0\leq t\leq t_0} E_f[\gamma(t)] \leq \sup_{t\geq 0} E_f[\gamma_\epsilon(t)] < \frac{3}{n}Y_6(S^n)^{\frac{n}{6}}(\max_M f)^{\frac{6-n}{6}}$$

is a critical value of E_f and there exists a $(\text{P–S})_\beta$ sequence $\{u_k\}$ of E_f in $H^3(M,g)$.

Next we claim that E_f satisfies the (P–S)$_\beta$ condition. For the above (P–S)$_\beta$ sequence $\{u_k\}$ satisfying $E_f[u_k] \to \beta$ and $DE_f[u_k] \to 0$ as $k \to \infty$, we have

$$2\beta + o(\|u_k\|_{H^3}) = 2E_f[u_k] - \langle DE_f[u_k], u_k\rangle = \frac{6}{n}\int_M f|u_k|^{2^\sharp}\, d\mu_g.$$

Together with the coercivity of P_g, one has

$$\mu(g)\|u_k\|_{H^3} \le 2E_f[u_k] + \frac{2}{2^\sharp}\int_M f|u_k|^{2^\sharp}\, d\mu_g \le C + o(\|u_k\|_{H^3}).$$

From this, we get $\{u_k\}$ is bounded in $H^3(M, g)$. Then up to a subsequence, as $k \to \infty$, $u_k \rightharpoonup u$ in $H^3(M, g)$ and $u_k \to u$ in $L^p(M, g)$ for $1 \le p < 2^\sharp$. It is easy to verify that u is a weak solution to (3-5), that is, for all $\psi \in H^3(M, g)$,

$$\int_M \psi P_g u\, d\mu_g = \int_M f|u|^{2^\sharp-2}u\psi\, d\mu_g.$$

Choosing $\psi = u$, one has

$$\int_M u P_g u\, d\mu_g = \int_M f|u|^{2^\sharp}\, d\mu_g,$$

whence

$$E_f[u] = \frac{3}{n}\int_M f|u|^{2^\sharp}\, d\mu_g \ge 0.$$

Applying the Brezis–Lieb lemma to

$$\int_M |\nabla\Delta u_k|_g^2\, d\mu_g = \int_M |\nabla\Delta u|_g^2\, d\mu_g + \int_M |\nabla\Delta(u-u_k)|_g^2\, d\mu_g + o(1),$$
$$\int_M f|u_k|^{2^\sharp}\, d\mu_g = \int_M f|u|^{2^\sharp}\, d\mu_g + \int_M f|u-u_k|^{2^\sharp}\, d\mu_g + o(1),$$

we have

$$\begin{aligned} E_f[u_k] - E_f[u] &= \frac{1}{2}\int_M |\nabla\Delta(u-u_k)|_g^2 - \frac{1}{2^\sharp}\int_M f|u-u_k|^{2^\sharp}\, d\mu_g + o(1) \\ &= E_f[u-u_k] + o(1). \end{aligned}$$

Since $DE_f[u_k] \to 0$ in $(H^3(M, g))'$, we have

$$\begin{aligned} o(1) &= \langle u_k - u, DE_f[u_k]\rangle \\ &= \langle u_k - u, DE_f[u_k] - DE_f[u]\rangle \\ &= \int_M |\nabla\Delta(u-u_k)|_g^2\, d\mu_g - \int_M f|u-u_k|^{2^\sharp}\, d\mu_g + o(1). \end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{3}{n}\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g + o(1) &= E_f[u_k-u]\\ &= E_f[u_k]-E_f[u]+o(1)\le E_f[u_k]+o(1)\to\beta,\end{aligned}$$

as $k\to\infty$, which yields

$$\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g \le \frac{n}{3}\beta + o(1). \tag{3-7}$$

Mimicking a cut-and-paste argument as in [Djadli et al. 2000], we obtain that given $\epsilon>0$, there exists a constant $B_\epsilon>0$ such that

$$\left(\int_M |\psi|^{2^\sharp}\,d\mu_g\right)^{2/2^\sharp} \le (1+\epsilon)Y_6(S^n)^{-1}\int_M (|\nabla\Delta\psi|_g^2+|\nabla^2\psi|_g^2+|\nabla\psi|_g^2)\,d\mu_g + B_\epsilon\int_M \psi^2\,d\mu_g,$$

for all $\psi\in H^3(M,g)$. Choosing $\psi=u_k-u$ and k sufficiently large, we get

$$\left(\int_M |u-u_k|^{2^\sharp}\,d\mu_g\right)^{2/2^\sharp} \le (1+\epsilon)Y_6(S^n)^{-1}\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g + o(1).$$

Hence we have

$$\begin{aligned}o(1) &= \int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g - \int_M f|u-u_k|^{2^\sharp}\,d\mu_g\\ &\ge \int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g\\ &\qquad\left[1-\big(\max_M f\big)(1+\epsilon)^{\frac{2^\sharp}{2}}Y_6(S^n)^{-\frac{2^\sharp}{2}}\left(\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g\right)^{\frac{6}{n-6}}\right].\end{aligned}$$

From (3-7) and $\beta<(3/n)Y_6(S^n)^{n/6}(\max_M f)^{(6-n)/6}$, choosing ϵ sufficiently small, we get

$$o(1)\ge C\int_M |\nabla\Delta(u-u_k)|_g^2\,d\mu_g.$$

Combining the above inequality and the coercivity of P_g to show that $u_k\to u$ in $H^3(M,g)$. Using the regularity result in Lemma 3.4 below, we know that $u\in C^{6,\mu}(M)$ for any $0<\mu<1$.

In addition, assume (M,g) is Einstein and has positive constant scalar curvature. We define the modified energy in $H^3(M,g)$ by

$$E_f^+[u]=\frac{1}{2}\int_M uP_gu\,d\mu_g - \frac{1}{2^\sharp}\int_M f u_+^{2^\sharp}\,d\mu_g,$$

where $u_+=\max\{u,0\}$. Using the above similar arguments associated with the mountain pass lemma and mimicking what we did in Lemma 3.4 below for E_f^+, we get that there exists a nontrivial C^6-solution u to

$$P_g u = f u_+^{\frac{n+6}{n-6}} \quad \text{in} \quad M. \tag{3-8}$$

Since P_g is coercive by the remark on page 57, testing equation (3-8) with $u_- = \min\{u, 0\}$ we conclude that $u \geq 0$ in M. Together with R_g being a positive constant and the factorization (1-4) of GJMS operator:

$$\left(-\Delta_g + \frac{(n-6)(n+4)}{4n(n-1)} R_g\right)\left(-\Delta_g + \frac{(n-4)(n+2)}{4n(n-1)} R_g\right)\left(-\Delta_g + \frac{n-2}{4(n-1)} R_g\right) u \geq 0$$

and $u \not\equiv 0$ in M, we employ the maximum principle twice and strong maximum principle once for elliptic equations of second-order to show that u is a positive solution to the equation (3-6). From this and Schauder estimates for elliptic equations, we conclude that $u \in C^\infty(M)$. This completes the proof. □

We are now concerned with the regularity of mountain pass critical points for E.

Lemma 3.4. *Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 7$. Assume $u \in H^3(M, g)$ is a weak solution of equation (3-5). Then $u \in C^{6,\mu}(M)$ for any $0 < \mu < 1$.*

Proof. Rewrite $P_g = (-\Delta_g)^3 - M_g + (n-6)/2 Q_g$ by (1-2). Let $u \in H^3(M, g)$ be a weak solution of equation (3-5) and rewrite this equation as

$$\begin{aligned}(-\Delta_g + 1)^3 u =& M_g u + 3\Delta_g^2 u - 3\Delta_g u + (1 - \tfrac{n-6}{2} Q_g) u + f |u|^{2^\sharp - 2} u \\ :=& b + f |u|^{2^\sharp - 2} u,\end{aligned} \tag{3-9}$$

where $b \in H^{-1}(M, g)$. By the Sobolev embedding theorem we have $u \in L^{2^\sharp}(M, g)$ and $|u|^{2^\sharp - 2} \in L^{n/6}(M, g)$. Given $\epsilon > 0$, there exist a $K_\epsilon > 0$ and a decomposition of $f |u|^{2^\sharp - 2} = h_\epsilon + \eta_\epsilon$ with $\|h_\epsilon\|_{n/6} \leq \epsilon$, $\|\eta_\epsilon\|_\infty \leq K_\epsilon$. Inspired by the arguments in [Esposito and Robert 2002, Proposition 3], for $s > 1$ we define an operator

$$H_\epsilon : v \in L^s(M, g) \to (-\Delta_g + 1)^{-3}(h_\epsilon v) \in L^s(M, g).$$

Indeed, from the Sobolev embedding theorem, the standard $W^{2,p}$-regularity theory of the elliptic operator $-\Delta_g + 1$ and Hölder's inequality, we have

$$\begin{aligned}\|H_\epsilon v\|_s \leq C \|(-\Delta_g + 1)^{-3}(h_\epsilon v)\|_{W^{6, \frac{ns}{n+6s}}} \leq& C \|h_\epsilon v\|_{\frac{ns}{n+6s}} \\ \leq& C \|h_\epsilon\|_{\frac{n}{6}} \|v\|_s \leq C\epsilon \|v\|_s,\end{aligned}$$

where the constant C is independent of u. If we choose $\epsilon > 0$ small enough, then the norm of H_ϵ on the space $L^s(M, g)$ satisfies

$$\|H_\epsilon\|_{L^s \to L^s} \leq C\epsilon \leq \tfrac{1}{2}.$$

With the help of the operator H_ϵ, we rewrite equation (3-9) as

$$(\mathrm{Id}-H_\epsilon)u=(-\Delta_g+1)^{-3}(b+\eta_\epsilon u),$$

then it is easy to show $\mathrm{Id}-H_\epsilon : L^s \to L^s$ is bounded and invertible. We intend to prove $u \in H^6(M,g)$. To see this, notice that $(-\Delta_g+1)^{-3}(b+\eta_\epsilon u) \in H^5(M,g)$ since $b+\eta_\epsilon u \in H^{-1}(M,g)$. In the following, we first show $u \in H^4(M,g)$. Apply the Sobolev embedding theorem and the L^s-boundedness of the operator $(\mathrm{Id}-H_\epsilon)^{-1}$ to show that if $n \le 10$, $u \in L^p(M,g)$ for all $p>1$, and if $n>10$, $u \in L^{2n/(n-10)}(M,g)$. In the latter case we have $|u|^{2^\sharp-2}u \in L^{2n(n-6)/((n+6)(n-10))}(M,g)$. From equation (3-9), we get

$$(-\Delta_g+1)^2u=(-\Delta_g+1)^{-1}b+(-\Delta_g+1)^{-1}(f|u|^{2^\sharp-2}u).$$

From $(-\Delta_g+1)^{-1}(|u|^{2^\sharp-2}u) \in W^{2,2n(n-6)/((n+6)(n-10))}(M,g) \hookrightarrow L^2(M,g)$ and $(-\Delta_g+1)^{-1}b \in L^2(M,g)$, we have $u \in H^4(M,g)$ in both cases. Repeat the above step with $u \in H^4(M,g)$ and $b \in L^2(M,g)$ in this situation. Notice that $(-\Delta_g+1)^{-3}(b+\eta_\epsilon u) \in H^6(M,g)$, similar arguments in the above step show that if $n \le 12$, $u \in L^p(M,g)$ for all $p>1$ and if $n>12$, $u \in L^{2n/(n-12)}(M,g)$. In the latter case, we get $|u|^{2^\sharp-2}u \in L^2(M,g)$ due to $2n(n-6)/((n+6)(n-12))>2$. Hence we obtain $u \in H^6(M,g)$.

Finally we start with the classical bootstrap. We now construct a nondecreasing sequence $s_k \in \mathbb{R}\cup\{+\infty\}$ such that $u \in W^{6,s_k}(M,g)$ for all $k \in \mathbb{N}$. Set $s_0=2$, and find $k \ge 0$ such that $u \in W^{6,s_k}(M,g)$. Next we will define s_{k+1} by induction. The Sobolev embedding theorem yields

$$b \in L^{\frac{ns_k}{n-2s_k}}(M,g),$$

with the convention that $ns_k/(n-2s_k)=+\infty$ if $s_k \ge n/2$, and

$$|u|^{2^\sharp-2}u \in L^{\frac{ns_k(n-6)}{(n-6s_k)(n+6)}}(M,g),$$

with the convention that $ns_k/(n-6s_k)=+\infty$ if $s_k \ge n/6$. In view of equation (3-9), we have

$$u \in W^{6,s_{k+1}}(M,g) \quad \text{with } s_{k+1}=\min\left\{\frac{ns_k}{n-2s_k},\ \frac{ns_k(n-6)}{(n-6s_k)(n+6)}\right\}.$$

If $s_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, it must hold that $s_k \to +\infty$. Then we have $u \in W^{6,p}(M,g)$ for all $1 \le p < +\infty$. If $s_k=+\infty$ for all $k \ge k_0+1$, then $s_{k_0} \ge n/6$, whence $b \in L^{n/4}(M,g)$ and $|u|^{2^\sharp-2}u \in L^q(M,g)$ for all $1 \le q < +\infty$. The equation (3-9) leads to $u \in W^{6,n/4}(M)$. Repeating the argument twice, we obtain $u \in W^{6,p}(M,g)$ for all $1 \le p < +\infty$. From this and the Sobolev embedding theorem, we have $u \in C^{5,\nu}(M)$ for all $0<\nu<1$. By the regularity theory for the classical solution of the elliptic operator $-\Delta_g + 1$, we get $u \in C^{6,\mu}(M)$ for some $0 < \mu < 1$. This completes the proof. $\square$

Appendix: proof of Lemma 2.2

As in Proposition 3.2, one may employ all computations under conformal normal coordinates of the metric g around a point in M. From Lee and Parker [1987] that up to a conformal factor, under g-conformal normal coordinates around this point, for all $N \geq 5$ we have

$$\sigma_1(A_g) = 0, \quad \sigma_1(A_g)_{,i} = 0, \quad \Delta_g \sigma_1(A_g) = -\frac{|W|_g^2}{12(n-1)}$$

at this point and $\sqrt{\det g} = 1 + O(r^N)$ near this point.

To simplify the notation, we will omit the subscript g. Notice that

$$\begin{aligned} -P_g(r^{6-n}) =& \Big[\Delta^3 + \Delta\delta T_2 d + \delta T_2 d\Delta + \frac{n-2}{2}\Delta(\sigma_1(A)\Delta) + \delta T_4 d - \frac{n-6}{2} Q_g\Big](r^{6-n}) \\ :=& \sum_{k=1}^{6} I_k. \end{aligned}$$

Next, we begin to estimate all terms I_1–I_6.

For I_1, let $u = u(r)$ be a radial function. We have

$$\begin{aligned} \Delta u(r) =& \Delta_0 u(r) + O(r^{N-1})u'; \\ \Delta^2 u(r) =& \Delta_0(\Delta_0 u(r) + O(r^{N-1})u') + O(r^{N-1})(\Delta_0 u(r) + O(r^{N-1})u')' \\ =& \Delta_0^2 u(r) + O(r^{N-1})u''' + O(r^{N-2})u'' + O(r^{N-3})u'; \\ \Delta^3 u(r) =& \Delta_0^3 u(r) + O(r^{N-1})u^{(5)} + O(r^{N-2})u^{(4)} + O(r^{N-3})u''' \\ &+ O(r^{N-4})u'' + O(r^{N-5})u'. \end{aligned}$$

Hence we obtain

$$I_1 = \Delta^3(r^{6-n}) = -c_n\delta_p + O(r^{N-n}).$$

To estimate I_2, notice that

$$I_2 = \Delta\delta T_2 d(r^{6-n}) = -\Delta[(T_2)_{ij}(r^{6-n})_{,j}]_{,i} = -\Delta[(T_2)_{ij,i}(r^{6-n})_{,j} + (T_2)_{ij}(r^{6-n})_{,ji}].$$

Using

$$(r^{6-n})_{,j} = (6-n)r^{4-n}x^j,$$

$$(r^{6-n})_{,ji} = (4-n)(6-n)r^{2-n}x^i x^j + (6-n)r^{4-n}\delta_{ij} + O(r^{6-n}), \tag{A-1}$$

one has

$$(T_2)_{ij,i}(r^{6-n})_{,j} = (n-10)\sigma_1(A)_{,j}(6-n)r^{4-n}x^j = (n-10)(6-n)\sigma_1(A)_{,j}x^j r^{4-n}$$

and

$$
\begin{aligned}
(T_2)_{ij}(r^{6-n})_{,ji} &= [(n-2)\sigma_1(A)g_{ij}-8A_{ij}](6-n)[(4-n)r^{2-n}x^ix^j+r^{4-n}\delta_{ij}+O(r^{6-n})]\\
&= (6-n)[4(n-4)\sigma_1(A)r^{4-n}-8(4-n)A_{ij}x^ix^jr^{2-n}]+O(r^{7-n}).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
I_2 &= -(6-n)\Delta[(n-10)\sigma_1(A)_{,j}x^jr^{4-n}+4(n-4)\sigma_1(A)r^{4-n}-8(4-n)A_{ij}x^ix^jr^{2-n}]\\
&= (n-6)\{(n-10)[4(4-n)\sigma_1(A)_{,j}x^jr^{2-n}+2(4-n)\sigma_1(A)_{,jk}x^jx^kr^{2-n}\\
&\qquad +\sigma_1(A)_{,jkk}x^jr^{2-n}+2\Delta\sigma_1(A)r^{4-n}]+O(r^{5-n})\\
&\qquad +4(n-4)[\Delta\sigma_1(A)r^{4-n}+2(4-n)\sigma_1(A)_{,k}x^kr^{2-n}+2(4-n)\sigma_1(A)r^{2-n}]\\
&\qquad +8(n-4)[4(2-n)A_{ij}x^ix^jr^{-n}+\Delta A_{ij}x^ix^jr^{2-n}\\
&\qquad +4\sigma_1(A)_{,i}x^ir^{2-n}+2\sigma_1(A)r^{2-n}]\}\\
&= (n-6)\Big\{-4(n-4)(3n-26)\sigma_1(A)_{,j}x^jr^{2-n}+6(n-6)\Delta\sigma_1(A)r^{4-n}\\
&\qquad -2(n-10)(n-4)\sigma_1(A)_{,jk}x^jx^kr^{2-n}\\
&\qquad +(n-10)\sigma_1(A)_{,jkk}x^jr^{4-n}+O(r^{5-n})-8(n-6)(n-4)\sigma_1(A)r^{2-n}\\
&\qquad -32(n-4)(n-2)A_{ij}x^ix^jr^{-n}+8(n-4)\Delta A_{ij}x^ix^jr^{2-n}\Big\}\\
&= (n-6)\Big\{-4(n-4)(3n-26)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}-\frac{n-6}{2(n-1)}|W(p)|^2r^{4-n}\\
&\qquad -2(n-10)(n-4)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\\
&\qquad -4(n-6)(n-4)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\\
&\qquad -16(n-4)(n-2)A_{ij,kl}(p)x^ix^jx^kx^lr^{-n}\\
&\qquad +8(n-4)\Delta A_{ij}x^ix^jr^{2-n}\Big\}+O(r^{5-n})\\
&= (n-6)\Big\{-2(n-4)(9n-74)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}-\frac{n-6}{2(n-1)}|W(p)|^2r^{4-n}\\
&\qquad -16(n-4)(n-2)A_{ij,kl}(p)x^ix^jx^kx^lr^{-n}\\
&\qquad +8(n-4)\Delta A_{ij}x^ix^jr^{2-n}\Big\}+O(r^{5-n}).
\end{aligned}
$$

To estimate

$$
I_3=\delta T_2 d\Delta(r^{6-n})=-[(T_2)_{ij}(\Delta r^{6-n})_{,j}]_{,i}=-(T_2)_{ij,i}(\Delta r^{6-n})_{,j}-(T_2)_{ij}(\Delta r^{6-n})_{,ji}.
$$

Recall that $T_2=(n-2)\sigma_1(A)g-8A$. Then

$$
(T_2)_{ij,i}=(n-10)\sigma_1(A)_{,j}.
$$

Observe that

$$
\begin{aligned}
\Delta r^{6-n} &= 4(6-n)r^{4-n}+O(r^{N+4-n}),\\
(\Delta r^{6-n})_{,j} &= 4(6-n)(4-n)x^jr^{2-n}+O(r^{N+3-n}),
\end{aligned}
$$

and

$$(\Delta r^{6-n})_{,ji} = 4(6-n)(4-n)[(2-n)x^i x^j r^{-n} + r^{2-n}\delta_{ij}] + O(r^{4-n}).$$

Then we have

$$\begin{aligned}
&(T_2)_{ij}(\Delta r^{6-n})_{,ji}\\
&= 4(n-6)(n-4)[(n-2)\sigma_1(A)g_{ij} - 8A_{ij}][(2-n)x^i x^j r^{-n} + r^{2-n}\delta_{ij} + O(r^{4-n})]\\
&= 4(n-6)(n-4)[-(n-2)^2\sigma_1(A)r^{2-n} + n(n-2)\sigma_1(A)r^{2-n}\\
&\quad + 8(n-2)r^{-n}A_{ij}x^i x^j - 8\sigma_1(A)r^{2-n}] + O(r^{5-n})\\
&= 4(n-6)(n-4)[2(n-6)\sigma_1(A)r^{2-n} + 8(n-2)r^{-n}A_{ij}x^i x^j] + O(r^{5-n})\\
&= 4(n-6)(n-4)[(n-6)\sigma_1(A)_{,ij}(p)x^i x^j r^{2-n} + 4(n-2)r^{-n}(A_{ij,kl}(p)x^i x^j x^k x^l)]\\
&\quad + O(r^{5-n}).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
I_3 =& -4(n-6)(n-4)[(n-6)\sigma_1(A)_{,ij}(p)x^i x^j r^{2-n} + 4(n-2)r^{-n}(A_{ij,kl}(p)x^i x^j x^k x^l)]\\
&\quad - 4(n-6)(n-4)(n-10)r^{2-n}\sigma_1(A)_{,i}x^i + O(r^{5-n})\\
=& -8(n-8)(n-6)(n-4)\sigma_1(A)_{,ij}(p)x^i x^j r^{2-n}\\
&\quad - 16(n-6)(n-4)(n-2)r^{-n}(A_{ij,kl}(p)x^i x^j x^k x^l) + O(r^{5-n}).
\end{aligned}$$

We now compute

$$\begin{aligned}
I_4 &= \frac{n-2}{2}\Delta(\sigma_1(A)\Delta(r^{6-n}))\\
&= 2(n-2)(6-n)\Delta(\sigma_1(A)r^{4-n}) + O(r^{N+4-n})\\
&= 2(n-2)(6-n)r^{2-n}[\Delta\sigma_1(A)r^2 + 2(4-n)\sigma_1(A)_{,i}x^i + 2(4-n)\sigma_1(A)]\\
&\quad + O(r^{N+2-n})\\
&= 2(n-2)(n-6)r^{2-n}\Big[\frac{1}{12(n-1)}|W(p)|^2 r^2 + 3(n-4)\sigma_1(A)_{,ij}(p)x^i x^j\Big]\\
&\quad + O(r^{5-n}).
\end{aligned}$$

For I_5, from (A-1) we have

$$\begin{aligned}
I_5 =& \delta T_4 d(r^{6-n})\\
=& -((T_4)_{ij}r^{6-n}{}_{,j})_{,i}\\
=& -(T_4)_{ij,i}(r^{6-n})_{,j} - (T_4)_{ij}(r^{6-n})_{,ji}\\
=& (n-6)[r^{4-n}(T_4)_{ij,i}x^j - (n-4)r^{2-n}(T_4)_{ij}x^i x^j + r^{4-n}\operatorname{tr}(T_4)]\\
:=& (n-6)[I_1^{(5)} + I_2^{(5)} + I_3^{(5)}].
\end{aligned}$$

Also from [Lee and Parker 1987], we have

$$\mathrm{Sym}\big(R_{kl,ij}+\tfrac{2}{9}R_{nklm}R_{nijm}\big)(p)=0 \quad \text{and} \quad R_{ij}(p)=0,$$

then

$$R_{kl,ij}(p)x^ix^jx^kx^l=-\tfrac{2}{9}W_{nklm}(p)W_{nijm}(p)x^ix^jx^kx^l.$$

Thus we have

$$A_{kl,ij}(p)x^ix^jx^kx^l=-\frac{2}{9}\frac{1}{n-2}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2-\frac{\sigma_1(A)_{,ij}(p)x^ix^jr^2}{n-2}. \tag{A-2}$$

To estimate $I_3^{(5)}$. From the definition of T_4, one gets

$$\begin{aligned}\mathrm{tr}(T_4)&=-\frac{3n^3-12n^2-36n+64}{4}\sigma_1(A)^2+4(n^2-4n-12)|A|^2+n(n-6)\Delta\sigma_1(A)\\&=-\frac{n(n-6)}{12(n-1)}|W(p)|^2+O(r).\end{aligned}$$

Thus one obtains

$$I_3^{(5)}=-\frac{n(n-6)}{12(n-1)}|W(p)|^2r^{4-n}+O(r^{5-n}).$$

For the term $I_1^{(5)}$, it is easy to see

$$I_1^{(5)}=r^{4-n}(T_4)_{ij,i}x^j=O(r^{5-n}).$$

It remains to estimate the term $I_2^{(5)}$. One has

$$(T_4)_{ij}x^ix^j=(n-6)\Delta\sigma_1(A)r^2-\frac{16}{n-4}B_{ij}x^ix^j+O(r^4). \tag{A-3}$$

Notice that

$$\begin{aligned}B_{ij}x^ix^j&=[C_{ijk,k}-A_{kl}W_{kijl}]x^ix^j=[(A_{ij,k}-A_{ik,j})_{,k}-A_{kl}W_{kijl}]x^ix^j\\&=[\Delta A_{ij}-A_{ik,jk}+O(r)]x^ix^j\end{aligned}$$

and

$$\begin{aligned}\Delta(A_{ij}x^ix^j)&=(A_{ij,k}x^ix^j+A_{ij}(x^i\delta_{jk}+x^j\delta_{ik}))_{,k}\\&=(\Delta A_{ij})x^ix^j+2A_{ij,k}(x^i\delta_{jk}+x^j\delta_{ik})+2\sigma_1(A)+O(r^3)\\&=(\Delta A_{ij})x^ix^j+4\sigma_1(A)_{,i}x^i+2\sigma_1(A)+O(r^3).\end{aligned}$$

By (A-2), one gets

$$
\begin{aligned}
(\Delta A_{ij})x^ix^j &= \Delta(A_{ij}x^ix^j) - 4\sigma_1(A)_{,i}x^i - 2\sigma_1(A) + O(r^3)\\
&= \Delta\left[\frac{1}{2}A_{ij,kl}(p)x^ix^jx^kx^l + O(r^5)\right] - 4[\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)]\\
&\quad - \sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\\
&= \Delta\left[-\frac{1}{9}\frac{1}{n-2}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 - \frac{\sigma_1(A)_{,ij}(p)x^ix^jr^2}{2(n-2)}\right]\\
&\quad - 5\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\\
&= -\frac{2}{9}\frac{1}{n-2}\sum_{k,l,s}[(W_{ikls}(p) + W_{ilks}(p))x^i]^2\\
&\quad + \frac{1}{12(n-2)(n-1)}|W(p)|^2r^2 - 6\frac{n-1}{n-2}\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3),
\end{aligned}
\tag{A-4}
$$

where the last identity follows from the following two estimates:

$$
\begin{aligned}
&\Delta(\sigma_1(A)_{,ij}(p)x^ix^jr^2)\\
&\quad = \Delta(\sigma_1(A)_{,ij}(p)x^ix^j)r^2 + 2\nabla_s(\sigma_1(A)_{,ij}(p)x^ix^j)\nabla_s r^2 + (\sigma_1(A)_{,ij}(p)x^ix^j)\Delta r^2\\
&\quad = 2\Delta\sigma_1(A)(p)r^2 + 8\sigma_1(A)_{,ij}(p)x^ix^j + 2n\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)\\
&\quad = -\frac{1}{6(n-1)}|W(p)|^2r^2 + 2(n+4)\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3)
\end{aligned}
$$

and

$$
\Delta\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 = 2\sum_{k,l,s}[W_{iklj}(p)(x^i\delta_{js} + x^j\delta_{is})]^2 = 2\sum_{k,l,s}[(W_{ikls}(p) + W_{ilks}(p))x^i]^2,
$$

which follows from

$$
\Delta\left[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2\right] = 2\sum_{k,l}[(W_{iklj}(p)x^ix^j)\Delta(W_{sklt}(p)x^sx^t) + |\nabla(W_{iklj}(p)x^ix^j)|^2]
$$

and $\Delta(W_{sklt}(p)x^sx^t) = (W_{sklt}(p)(x^s\delta_{it} + x^t\delta_{is}))_{,i} = 2W_{sklt}(p)\delta_{st} = 0$. Using $A_{ik,jk} = A_{ik,kj} + R^l_{ijk}A_{lk} + R^l_{kjk}A_{il} = \sigma_1(A)_{,ij} + R_{lijk}A_{lk} + R_{lj}A_{il}$, one has

$$
\begin{aligned}
A_{ik,jk}x^ix^j &= \sigma_1(A)_{,ij}x^ix^j + R_{lijk}A_{lk}x^ix^j + R_{lj}A_{il}x^ix^j\\
&= (\sigma_1(A)_{,ij}(p) + O(r))x^ix^j\\
&\qquad + (W_{lijk}(p) + O(r))(A_{lk,m}(p)x^m + O(r^2))x^ix^j + O(r^4)\\
&= \sigma_1(A)_{,ij}(p)x^ix^j + O(r^3).
\end{aligned}
$$

Thus, one obtains

$$B_{ij}x^ix^j = -\frac{2}{9}\frac{1}{n-2}\sum_{k,l,s}[(W_{ikls}(p)+W_{ilks}(p))x^i]^2+\frac{1}{12(n-2)(n-1)}|W(p)|^2r^2 - \frac{7n-8}{n-2}\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3). \tag{A-5}$$

Inserting the above equations into (A-3), one gets

$$\begin{aligned}(T_4)_{ij}x^ix^j = &-\frac{n-6}{12(n-1)}|W(p)|^2r^2+\frac{32}{9(n-4)(n-2)}\sum_{k,l,s}[(W_{ikls}(p)+W_{ilks}(p))x^i]^2\\ &-\frac{4}{3(n-4)(n-2)(n-1)}|W(p)|^2r^2\\ &+\frac{16(7n-8)}{(n-4)(n-2)}\sigma_1(A)_{,ij}(p)x^ix^j + O(r^3),\end{aligned}$$

whence

$$\begin{aligned}I_2^{(5)} = r^{2-n}\bigg[&\frac{(n-6)(n-4)}{12(n-1)}|W(p)|^2r^2 - \frac{32}{9(n-2)}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2\\ &+\frac{4}{3(n-2)(n-1)}|W(p)|^2r^2 - \frac{16(7n-8)}{n-2}\sigma_1(A)_{,ij}(p)x^ix^j\bigg] + O(r^{5-n}).\end{aligned}$$

Combining all the terms together, one has

$$\begin{aligned}I_5 = \bigg[&-\frac{n^2-8n+8}{3(n-1)(n-2)}|W(p)|^2r^{4-n}\\ &-\frac{32}{9(n-2)}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2r^{2-n}\\ &-\frac{16(7n-8)}{n-2}\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\bigg](n-6) + O(r^{5-n}).\end{aligned}$$

Finally, from the definition of Q_g in (1-1), it is not difficult to show that $I_6 = -(n-6)/2Q_gr^{6-n} = O(r^{6-n})$.

Therefore, collecting all the terms I_1–I_6 together with (A-2) and (A-4), we conclude that

$$\begin{aligned}-P_g(r^{6-n}) = -c_n\delta_p + (n-6)\bigg[&-\frac{16}{9}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2r^{2-n}\\ &-\frac{2(n-8)}{3(n-1)}|W(p)|^2r^{4-n}+\frac{64(n-4)}{9}\sum_{k,l}(W_{iklj}(p)x^ix^j)^2r^{-n}\\ &-4(5n^2-66n+224)\sigma_1(A)_{,ij}(p)x^ix^jr^{2-n}\bigg] + O(r^{5-n})\end{aligned}$$

$$
\begin{aligned}
&= -c_n\delta_p + (n-6)r^{-n}\Bigg\{\frac{64(n-4)}{9}\Bigg[\sum_{k,l}(W_{iklj}(p)x^ix^j)^2 \\
&\qquad\qquad - \frac{r^2}{n+4}\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 \\
&\qquad\qquad + \frac{3}{2(n+4)(n+2)}|W(p)|^2r^4\Bigg] \\
&\quad + \frac{16(3n-20)}{9(n+4)}r^2\Bigg[\sum_{k,l,s}((W_{ikls}(p)+W_{ilks}(p))x^i)^2 - \frac{3}{n}|W(p)|^2r^2\Bigg] \\
&\quad - 4(5n^2-66n+224)r^2\Bigg[\sigma_1(A)_{,ij}(p)x^ix^j + \frac{|W(p)|^2}{12n(n-1)}r^2\Bigg] \\
&\quad + \frac{3n^4-16n^3-164n^2+400n+2432}{3(n+4)(n+2)n(n-1)}|W(p)|^2r^4\Bigg\} + O(r^{5-n}),
\end{aligned}
$$

where each term in square brackets on the right-hand side of the last identity is a harmonic polynomial. This finishes the proof of Lemma 2.2.

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ON THE ASYMPTOTIC BEHAVIOR OF BERGMAN KERNELS FOR POSITIVE LINE BUNDLES

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Let L be a positive line bundle on a projective complex manifold. We study the asymptotic behavior of Bergman kernels associated with the tensor powers L^p of L as p tends to infinity. The emphasis is the dependence of the uniform estimates on the positivity of the Chern form of the metric on L. This situation appears naturally when we approximate a semipositive singular metric by smooth positively curved metrics.

1. Introduction

Let L be an ample holomorphic line bundle over a projective manifold X of dimension n. Fix a (reference) smooth Hermitian metric h_0 on L whose first Chern form ω_0 is a Kähler form. Recall that $\omega_0 = (\sqrt{-1}/2\pi)R_0^L$, where R_0^L is the curvature of the Chern connection on (L, h_0).

Let h^L be a semipositive singular metric on L. For various applications, one needs to understand the asymptotic behavior of the Bergman kernel associated with L^p and h^L when p tends to infinity. A natural approach is to approximate the considered metric by smooth positively curved metrics, and therefore, it is necessary to understand the dependence of the Bergman kernels in terms of the positivity of the curvature of the metric. See [Błocki and Kołodziej 2007; Demailly 1992; Dinh et al. 2015] for the regularization of metrics. This method was already used in our previous work on the speed of convergence of Fekete points, see [Berman et al. 2011; Dinh et al. 2015]. In §2.3 of the latter, inspired by [Berndtsson 2003], an L^1-estimate for Bergman kernels was obtained. Here, we investigate the uniform estimate which can be useful for applications in geometry.

Fix a smooth Kähler form θ on X (one can take $\theta = \omega_0$). Consider a metric $h = e^{-2\phi}h_0$ on L with weight ϕ of class $\mathcal{C}^{n+6}$ whose first Chern form $\omega := dd^c\phi + \omega_0$ (here $d^c := (\sqrt{-1}/2\pi)(\bar\partial - \partial)$) satisfies

$$\omega \geq \zeta\theta \quad \text{for some constant} \quad 0 < \zeta \leq 1. \tag{1-1}$$

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Consider the natural metric on the space of smooth sections of L^p, induced by the metric h on L and the volume form $\theta^n/n!$ on X, which is defined by

$$\|s\|^2_{L^2(p\phi)} := \int_X |s(x)|^2_{p\phi}\theta^n/n!\,. \tag{1-2}$$

Here, $|s(x)|_{p\phi}$ stands for the norm of $s(x)$ with respect to the metric $h^{\otimes p}$ on L^p. Let $\langle\cdot,\cdot\rangle_{p\phi}$ be the associated Hermitian product on $\mathcal{C}^\infty(X, L^p)$, the space of smooth sections of L^p. Let P_p be the orthogonal projection from $(\mathcal{C}^\infty(X, L^p), \langle\cdot,\cdot\rangle_{p\phi})$ onto the subspace of holomorphic sections $H^0(X, L^p)$. The Bergman kernel associated with the above data is the kernel associated with the last projection where we use the volume form $\theta^n/n!$ to integrate functions on X. This kernel is denoted by $P_p(x, x')$, with $x, x' \in X$. It is a section of the line bundle over $X \times X$ which is the tensor product of two line bundles: the first one is the pullback to $X \times X$ of the line bundle L^p over the first factor, and the second one is the pullback of the dual line bundle $(L^*)^p$ of L^p over the second factor. In particular, its restriction to the diagonal of $X \times X$, i.e., $P_p(x, x)$, can be identified to a positive-valued function on X. See [Ma and Marinescu 2007] for details. In fact, if $\{s_j\}_j$ is an orthonormal basis of $(H^0(X, L^p), \langle\cdot,\cdot\rangle)$, then

$$P_p(x,x)=\sum_j|s_j(x)|^2_{p\phi}=\sup\{|s(x)|^2_{p\phi}, s\in H^0(X,L^p) \quad \text{with } \|s\|_{L^2(p\phi)}=1\}. \tag{1-3}$$

Here is the main result in this paper which gives us a uniform estimate of the Bergman kernel in terms of ϕ, ω, p and ζ. This is a version of Tian's theorem [1990]. See [Berndtsson 2003; Boutet de Monvel and Sjöstrand 1976; Catlin 1999; Coman and Marinescu 2016; Dai et al. 2006; Hsiao and Marinescu 2014; Ma and Marinescu 2015; Xu 2012; Zelditch 1998] for various generalizations. We also refer to [Ma and Marinescu 2007] for a comprehensive study of several analytic and geometric aspects of Bergman kernels. The last reference is inspired by the analytic localization technique in [Bismut and Lebeau 1991].

Theorem 1.1. *Under the above assumptions, there exist $\delta > 0$, $c > 0$ satisfying the following condition: for any $l \in \mathbb{N}^*$, there is a constant $c_l > 0$ such that for $p \in \mathbb{N}^*$, $p\zeta > \delta$, and $x \in X$, we have*

$$\Big|p^{-n}P_p(x,x) - \frac{\omega(x)^n}{\theta(x)^n}\Big| \leqslant c|d\phi|^{2n+8}_{n+5}|\omega|^{4n+20}_0|d\phi|^{2n+2}_{n+2}\zeta^{-2n-10}p^{-1} \\ +c_l|\omega|^{2n+2}_n(|d\phi|_2\zeta^{-1})^{6n+6+3l}p^{-l}. \tag{1-4}$$

Note that $|\cdot|_k$ stands for $1 + \|\cdot\|_{\mathcal{C}^k}$. As a direct consequence, we infer the following result by taking $l = 1$.

Corollary 1.2. *There exist $\delta > 0,\ c > 0$ such that for any $0 < \zeta \leq 1$, any weight ϕ of class $\mathcal{C}^{n+6}$ with $dd^c\phi + \omega_0 \geq \zeta\theta$, and any $p \in \mathbb{N}^*$ with $\zeta p > \delta$, we have*

$$\left| p^{-n} P_p(x,x) - \frac{\omega(x)^n}{\theta(x)^n} \right| \leq c\zeta^{-6n-9} |d\phi|_{n+5}^{8n+30} p^{-1}. \tag{1-5}$$

If $\phi \in \mathcal{C}^{n+2k+6}$, we can adapt easily the proof of Theorem 1.1 to get the estimate for $\mathcal{C}^k$-norm of the left-hand side of (1-4). Cf., Remark 3.9.

The article is organized as follows. In Section 2, we reduce the problem to the local setting. In Section 3, we establish Theorem 1.1. We need an approach different from previous ones which use the normal coordinates and the extension of connections on L; see [Dai et al. 2006, §4.2] and [Ma and Marinescu 2007, §4.1.3]. Note that throughout the paper, the constants $c, c', c_l, \dots$ may change from line to line.

2. Localization of the problem

Recall that the complex structure on X is given by a smooth section J of the vector bundle $\mathrm{End}(TX)$ such that $-J^2$ is the identity section. Here, TX denotes the real tangent bundle of X. Denote also by $T^{(1,0)}X$ and $T^{(0,1)}X$ the holomorphic and antiholomorphic tangent bundles of X. They are complex vector subbundles of $TX \otimes_{\mathbb{R}} \mathbb{C}$. The Kähler form θ induces a Riemannian metric g^{TX} on X defined by $g^{TX} := \theta(\cdot, J\cdot)$.

Let $\bar\partial^{L^p}$ be the $\bar\partial$-operator acting on L^p and $\bar\partial^{L^p,*}$ its dual operator with respect to the metric $h = e^{-2\phi}h_0$ on L and the Kähler form θ. Consider the Dirac and Laplacian-type operators

$$D_p := \sqrt{2}\big(\bar\partial^{L^p} + \bar\partial^{L^p,*}\big) \quad \text{and} \quad \Box_p := \tfrac{1}{2}D_p^2 = \bar\partial^{L^p}\bar\partial^{L^p,*} + \bar\partial^{L^p,*}\bar\partial^{L^p}. \tag{2-1}$$

They act on $\Omega^{0,\bullet}(X, L^p)$, the space of the forms of bidegree $(0,\cdot)$ with values in L^p.

Let ∇^L be the Chern connection on $(L, h = e^{-2\phi}h_0)$ and $R^L = (\nabla^L)^2$ its curvature which is related to the first Chern form ω by

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{2-2}$$

Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}). It preserves $T^{(1,0)}X$, $T^{(0,1)}X$, and its restriction to $T^{(1,0)}X$ is the Chern connection $\nabla^{T^{(1,0)}X}$. Let $\nabla^{\Lambda^{0,\bullet}}$ be the connection on $\Lambda(T^{*(0,1)}X)$ induced by $\nabla^{T^{(1,0)}X}$, and $\nabla^{\Lambda^{0,\bullet}\otimes L^p}$ the connection on $\Lambda(T^{*(0,1)}X) \otimes L^p$ induced by $\nabla^{\Lambda^{0,\bullet}}$ and ∇^L. For $u \in T^{(1,0)}X$ and $v \in T^{(0,1)}X$, let $u^* \in T^{*(0,1)}X$ be the metric dual of u with respect to g^{TX}, define the operator $c(\cdot)$ depending linearly on a vector in $T^{(1,0)}X \oplus T^{(0,1)}X$ by setting

$$c(u) := \sqrt{2}u^* \wedge \quad \text{and} \quad c(v) := \sqrt{2}i_v, \tag{2-3}$$

where i denotes, as usual, the contraction operator. Then by [Ma and Marinescu 2007, p.31], for $\{e_j\}$ an orthonormal frame of (TX, g^{TX}), we have

$$D_p = \sum_j c(e_j)\nabla_{e_j}^{\Lambda^{0,\bullet}\otimes L^p}. \tag{2-4}$$

Denote by K_X^* the anticanonical bundle of X. The curvature of K_X^* with respect to the above Riemannian metric is denoted by $R^{K_X^*}$. Then $\sqrt{-1}R^{K_X^*}$ is the Ricci curvature of (X, g^{TX}). Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$ with dual frame $\{w^j\}_{j=1}^n$. Set

$$\omega_d := -\sum_{l,m} R^L(w_l, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l}. \tag{2-5}$$

Recall that $(\sqrt{-1}/2\pi)R^L = \omega \geq \zeta\theta$. Then ω_d is a section of $\mathrm{End}(\Lambda(T^{*(0,1)}X))$ and R^L acts as the derivative ω_d on $\Lambda(T^{*(0,1)}X)$. By [Ma and Marinescu 2007, (1.4.63)] and using that $\langle\Delta^{0,\bullet}s, s\rangle_{p\phi} \geq 0$, where $\Delta^{0,\bullet}$ is a holomorphic Kodaira type Laplacian, we obtain for $s \in \Omega^{0,\bullet}(X, L^p)$ that

$$\begin{aligned} \|D_p s\|^2_{L^2(p\phi)} &= 2\langle\Box_p s, s\rangle_{p\phi} \\ &\geq -2p\langle\omega_d s, s\rangle_{p\phi} + 2\sum_{l,m}\big\langle R^{K_X^*}(w_l, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_l}s, s\big\rangle_{p\phi}. \end{aligned} \tag{2-6}$$

Now by (1-1), (2-2), (2-5) and some standard arguments (see the proof of [Ma and Marinescu 2007, Theorem 1.5.5]) there exists $\delta > 0$, depending only on the Ricci curvature $R^{K_X^*}$, such that if $\zeta p > \delta$, then the spectrum of D_p^2 satisfies

$$\mathrm{Spec}(D_p^2) \subset \{0\} \cup [2\pi\zeta p, +\infty[. \tag{2-7}$$

Let a_X denote the injectivity radius of (X, θ). For $0 < \epsilon_0 < a_X/4$, let $f_{\epsilon_0} : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$f_{\epsilon_0}(v) = \begin{cases} 1 & \text{for} \quad |v| \leqslant \epsilon_0/2, \\ 0 & \text{for} \quad |v| \geqslant \epsilon_0. \end{cases} \tag{2-8}$$

Set

$$\begin{aligned} F_{\epsilon_0}(a) :&= \left(\int_{-\infty}^{+\infty} f_{\epsilon_0}(v)\,dv\right)^{-1} \int_{-\infty}^{+\infty} e^{iv\zeta a} f_{\epsilon_0}(v)\,dv \\ &= \left(\int_{-\infty}^{+\infty} f_{\epsilon_0}(\zeta^{-1}v)\,dv\right)^{-1} \int_{-\infty}^{+\infty} e^{iva} f_{\epsilon_0}(\zeta^{-1}v)\,dv. \end{aligned} \tag{2-9}$$

Then $F_{\epsilon_0}(a)$ lies in Schwartz space $\mathcal{S}(\mathbb{R})$ and $F_{\epsilon_0}(0) = 1$.

Proposition 2.1. *Let $\delta > 0$ satisfy (2-7). Then, for all $l \in \mathbb{N}$, $0 < \epsilon_0 < a_X/4$ and F_{ϵ_0} as above, there exists $c > 0$ such that for $p \geq 1$, $\delta/p < \zeta \leq 1$, $x, x' \in X$*

$$\|F_{\epsilon_0}(D_p)(x,x') - P_p(x,x')\|_{L^\infty(p\phi)} \leq c|\omega|_n^{2n+2}\zeta^{-6n-3l-6}p^{-l}. \tag{2-10}$$

Proof. For $a \in \mathbb{R}$, set

$$\phi_p(a) := 1_{[\sqrt{\zeta p},+\infty[}(|a|)F_{\epsilon_0}(a). \tag{2-11}$$

By (2-7) and (2-11), for $\zeta p > \delta$, we get

$$F_{\epsilon_0}(D_p) - P_p = \phi_p(D_p). \tag{2-12}$$

By (2-9), for any $m \in \mathbb{N}$ there exists $c > 0$ such that for all $\zeta \in]0,1[$,

$$\sup_{a\in\mathbb{R}} |a|^m |F_{\epsilon_0}(a)| \leq c\zeta^{-m}. \tag{2-13}$$

Thus, for any $m \in \mathbb{N}$ and $\zeta p > \delta$, we have

$$\|(D_p)^m F_{\epsilon_0}(D_p)\|^{0,0} := \sup_{s\in\Omega^{0,\bullet}(X,L^p)\setminus\{0\}} \frac{\|(D_p)^m F_{\epsilon_0}(D_p)s\|_{L^2(p\phi)}}{\|s\|_{L^2(p\phi)}} \leq c\zeta^{-m}. \tag{2-14}$$

As X is compact, there exists a finite set of points a_i, $1 \leq i \leq r$, such that the family of balls $U_i := B^X(a_i, \epsilon_0)$ of center a_i and radius ϵ_0, is a covering of X. We identify the ball $B^{T_{a_i}X}(0,\epsilon_0)$ in the tangent space of X at a_i with the ball $B^X(a_i,\epsilon_0)$ using the exponential map. We then identify $(TX)_Z$, $\Lambda(T^{*(0,1)}X)_Z$, L^p_Z for $Z \in B^{T_{a_i}X}(0,\epsilon_0)$ with $T_{a_i}X$, $\Lambda(T^{*(0,1)}X)_{a_i}$, $L^p_{a_i}$ by parallel transport with respect to the connections ∇^{TX}, $\nabla^{\Lambda^{0,\bullet}}$, ∇^{L^p} along the curve $\gamma_Z : [0,1] \ni u \mapsto \exp^X_{a_i}(uZ)$. Then $(L,h)|_{U_i}$ is identified as the trivial bundle (L_{a_i}, h_{a_i}).

Let $\{e_j\}_j$ be an orthonormal basis of $T_{a_i}X \simeq \mathbb{R}^{2n}$. Let $\tilde{e}_j(Z)$ be the parallel transport of e_j with respect to ∇^{TX} along the above curve. Let Γ^L, $\Gamma^{\Lambda^{0,\bullet}}$ be the corresponding connection forms of ∇^L and $\nabla^{\Lambda^{0,\bullet}}$ with respect to any fixed frame for L and $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve γ_Z under the trivialization on U_i. Denote by ∇_v the ordinary differentiation operator on $T_{a_i}X$ in the direction v. As we are working in the Kähler case, by [Ma and Marinescu 2007, Proposition 1.2.6, Theorem 1.4.5, Remark 1.4.8], we can write on U_i

$$D_p = \sum_j c(\tilde{e}_j)\big(\nabla_{\tilde{e}_j} + p\Gamma^L(\tilde{e}_j) + \Gamma^{\Lambda^{0,\bullet}}(\tilde{e}_j)\big)\big). \tag{2-15}$$

In fact, the last identity is a consequence of (2-4). Consider the radial vector field $\mathcal{R} = \sum_j Z_j e_j$. By [Ma and Marinescu 2007, (1.2.32)], the Lie derivative $L_{\mathcal{R}}\Gamma^L$ is equal to $i_{\mathcal{R}}R^L$. Therefore, we get the identity

$$\Gamma^L_Z = \int_0^1 (i_{\mathcal{R}}R^L)_{tZ}\,dt, \tag{2-16}$$

which allows us to bound Γ^L.

Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$. For $m \in \mathbb{N}$, we define a Sobolev norm on the m-th Sobolev space $H^m(X, \Lambda(T^{*(0,1)}X) \otimes L^p)$ by

$$\|s\|^2_{H^m} = \sum_{i=1}^{r} \sum_{k=0}^{m} \sum_{j_1,\dots,j_k=1}^{2n} \|\nabla_{e_{j_1}} \cdots \nabla_{e_{j_k}} (\varphi_i s)\|^2_{L^2}. \tag{2-17}$$

Note that here we trivialize the line bundle L using a unitary section; so the section s above is identified with a function. Therefore, we drop the subscript $p\phi$ since this weight is already taken into account.

By (2-15), (2-16) and [Ma and Marinescu 2007, (1.6.9)], for P a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_i, we get

$$\begin{aligned} \|Ps\|_{H^1} &\leqslant c(\|D_p Ps\|_{L^2} + p|\omega|_0 \|Ps\|_{L^2}) \\ &\leqslant c'\Big(\|PD_p s\|_{L^2} + p\sum_{k=0}^{m} |\omega|_k \|s\|_{H^{m-k}}\Big), \end{aligned} \tag{2-18}$$

for some constants $c, c' > 0$. From (2-18), we get by induction for (other) suitable constants $c, c' > 0$

$$\begin{aligned} \|s\|_{H^{m+1}} &\le c \sum_{k=0}^{m+1} p^{m+1-k} \|D_p^k s\|_{L^2} \prod_{\sum(k_r+1)=m-k+1} |\omega|_{k_r} \\ &\le c' \sum_{k=0}^{m+1} p^{m+1-k} \|D_p^k s\|_{L^2} |\omega|^{m-k+1}_{m-k}. \end{aligned} \tag{2-19}$$

Note that for $k = m+1$ we set $|\omega|^{m-k+1}_{m-k} = 1$.

Let Q be a differential operator of order $m' \in \mathbb{N}$ with scalar principal symbol and with compact support in U_j. We deduce from (2-19) with suitable sections instead of s that

$$\begin{aligned} \|Q^*\phi_p(D_p)D^k s\|_{L^2} &\le c \sum_{k'=0}^{m'} p^{m'-k'} \|D_p^{k'} \phi_p(D_p) D_p^k s\|_{L^2} |\omega|^{m'-k'}_{m'-k'-1} \\ &= c \sum_{k'=0}^{m'} p^{m'-k'} \|D_p^{k+k'} \phi_p(D_p) s\|_{L^2} |\omega|^{m'-k'}_{m'-k'-1}. \end{aligned} \tag{2-20}$$

Note that the operators, considered in the last two lines, commute. Thanks to (2-7), (2-11), (2-12) and then (2-14), if $0 < \zeta \le 1$ and $\zeta p \ge \delta$, for any $q \in \mathbb{N}$, the main factor in the last line can be bounded using

$$\begin{aligned}\|D_p^{k+k'}\phi_p(D_p)s\|_{L^2} &\le (\zeta p)^{-q/2}\|D_p^{k+k'+q}\phi_p(D_p)s\|_{L^2}\\ &\le c(\zeta p)^{-q/2}\zeta^{-k-k'-q}\|s\|_{L^2}.\end{aligned}\tag{2-21}$$

Take any $l>0$ and choose $q:=2(m+m'-k-k'+l)$ in (2-21). Using the identity

$$\langle D_p^k\phi_p(D_p)Qs,\,s'\rangle=\langle s,\,Q^*\phi_p(D_p)D_p^ks'\rangle,$$

then by (2-19)-(2-21), there exists $c_l>0$ such that for $0<\zeta\le 1,\ \zeta p\ge\delta$, we have

$$\begin{aligned}\|P\phi_p(D_p)Qs\|_{L^2}&\\ &\le c\sum_{k=0}^{m}\sum_{k'=0}^{m'}p^{m+m'-k-k'}(\zeta p)^{-q/2}\zeta^{-k-q-k'}|\omega|_{m-k-1}^{m-k}|\omega|_{m'-k'-1}^{m'-k'}\|s\|_{L^2}\\ &\le c_l\zeta^{-3m-3m'-3l}p^{-l}|\omega|_{m-1}^{m}|\omega|_{m'-1}^{m'}\|s\|_{L^2}.\end{aligned}\tag{2-22}$$

Finally, on $U_i\times U_j$, by using the standard Sobolev's inequality and (2-12), we get (2-10). Proposition 2.1 follows. □

Remark 2.2. By (2-9) and the finite propagation speed of solutions of hyperbolic equations [Ma and Marinescu 2007, Theorem D.2.1], $F_{\epsilon_0}(D_p)(x,x')$ only depends on the restriction of D_p to $B^X(x,\epsilon_0\zeta)$, and

$$F_{\epsilon_0}(D_p)(x,x')=0\quad\text{when}\quad \operatorname{dist}(x,x')\geqslant\epsilon_0\zeta.\tag{2-23}$$

To get the uniform estimate of the Bergman kernels in terms of ζ, p, we need an approach different from the use of the normal coordinates and the extension of connections on L in [Dai et al. 2006, §4.2] and [Ma and Marinescu 2007, §4.1.3]. Let $\psi: X\supset U\to V\subset\mathbb{C}^n$ be a holomorphic local chart such that $0\in V$ and V is convex (by abuse of notation, we sometimes identify U with V and x with $\psi(x)$). Then, for any $x\in\frac12 V:=\{y\in\mathbb{C}^n:2y\in V\}$, we will use the holomorphic coordinates induced by ψ and let $0<\epsilon_0\le 1$ be such that $B(x,4\epsilon_0)\subset V$ for any $x\in\frac12 V$. We choose ϵ_0 smaller than $a_X/4$ in order to use the estimates given in the proof of Proposition 2.1. Consider the holomorphic family of holomorphic local coordinates $\psi_x:\psi^{-1}(B(x,4\epsilon_0))\to B(0,4\epsilon_0)$ for $x\in\frac12 V$ given by $\psi_x(y):=\psi(y)-x$.

Let σ be a holomorphic frame of L on U and define the function $\varphi(Z)$ on U by $|\sigma|_\phi^2(Z)=:e^{-2\varphi(Z)}$. Consider the holomorphic family of holomorphic trivializations of L associated with the coordinates ψ_x and the frame σ. These trivializations are given by $\Psi_x: L|_{\psi^{-1}(B(x,4\epsilon_0))}\to B(0,4\epsilon_0)\times\mathbb{C}$ with $\Psi_x(y,v):=(\psi_x(y),v/\sigma(y))$ for v a vector in the fiber of L over the point y.

Consider a point $x_0\in\frac12 V$. Denote by $\varphi_{x_0}:=\varphi\circ\psi_{x_0}^{-1}$ the function φ in local coordinates ψ_{x_0}. Denote also by $\varphi_{x_0}^{[1]}$ and $\varphi_{x_0}^{[2]}$ the first and second order Taylor expansions of φ_{x_0}, i.e.,

$$\begin{aligned}\varphi_{x_0}^{[1]}(Z) &:= \sum_{j=1}^{n}\Big(\frac{\partial\varphi}{\partial z_j}(x_0)z_j + \frac{\partial\varphi}{\partial \bar z_j}(x_0)\bar z_j\Big),\\ \varphi_{x_0}^{[2]}(Z) &:= \operatorname{Re}\sum_{j,k=1}^{n}\Big(\frac{\partial^2\varphi}{\partial z_j\partial z_k}(x_0)z_jz_k + \frac{\partial^2\varphi}{\partial z_j\partial \bar z_k}(x_0)z_j\bar z_k\Big),\end{aligned}\tag{2-24}$$

where we write $z = (z_1, \dots, z_n)$ the complex coordinates of Z.

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\rho(t) = 1 \quad \text{if} \quad |t| < 2; \qquad \rho(t) = 0 \quad \text{if} \quad |t| > 4.\tag{2-25}$$

We denote in the sequel $X_0 = \mathbb{R}^{2n} \simeq T_{x_0}X$ and equip X_0 with the metric $g^{TX_0}(Z) := g^{TX}(\rho(\epsilon_0^{-1}|Z|)Z)$. Now let $0 < \epsilon < \epsilon_0$ and define

$$\varphi_\epsilon(Z) := \rho(\epsilon^{-1}|Z|)\varphi_{x_0}(Z) + \big(1-\rho(\epsilon^{-1}|Z|)\big)\big(\varphi(x_0) + \varphi_{x_0}^{[1]}(Z) + \varphi_{x_0}^{[2]}(Z)\big).\tag{2-26}$$

Let $h_\epsilon^{L_0}$ be the metric on $L_0 = X_0 \times \mathbb{C}$ defined by

$$|1|^2_{h_\epsilon^{L_0}}(Z) := e^{-2\varphi_\epsilon(Z)}.\tag{2-27}$$

Here, as above, subscript ϵ implies the use of the weight φ_ϵ. Let $\nabla_\epsilon^{L_0}$ be the Chern connection on $(L_0, h_\epsilon^{L_0})$ and $R_\epsilon^{L_0}$ be the curvature of $\nabla_\epsilon^{L_0}$.

Then there exists a constant A with $c|d\phi|_2^{-1} < A < 1$ for $c > 0$ such that when $\epsilon \le A\zeta$, the following estimate holds for every $x_0 \in U$:

$$\inf\Big\{\sqrt{-1}R^{L_0}_{\epsilon,Z}(u, Ju)/|u|^2_{g^{TX_0}} : u \in T_ZX_0 \quad \text{and} \quad Z \in X_0\Big\} \geqslant \tfrac{4}{5}\zeta;\tag{2-28}$$

because there exists $C > 0$ such that for $|Z| \le 4\epsilon,\ 0 \le j \le 2$, we have

$$\Big|\varphi_{x_0}(Z) - \big(\varphi(x_0) + \varphi_{x_0}^{[1]}(Z) + \varphi_{x_0}^{[2]}(Z)\big)\Big|_{C^j} \le C|d\phi|_2|Z|^{3-j}.\tag{2-29}$$

From now on, we take

$$\epsilon := \epsilon_0 A\zeta.\tag{2-30}$$

Let S_{x_0} be the unitary section of $(L_0, h_\epsilon^{L_0})$ which is parallel with respect to $\nabla_\epsilon^{L_0}$ along the curve $[0, 1] \ni u \to uZ$ for any $Z \in X_0$. We can write it as $S_{x_0} = e^{-\tau}1$ with $\tau(x_0) = \varphi(x_0)$, then

$$\nabla_Z^{L_0} S_{x_0} = i_Z(-d\tau - 2\partial\varphi_\epsilon)S_{x_0} = 0,\tag{2-31}$$

and hence the function τ is given by

$$\tau(Z) = \varphi(x_0) - 2\int_0^1 (i_Z\partial\varphi_\epsilon)_{tZ}\,dt.\tag{2-32}$$

Let

$$D_p^{X_0} = \sqrt{2}(\bar\partial^{L_0^p} + \bar\partial^{L_0^p *}_{p\varphi_\epsilon}) \tag{2-33}$$

be the Dolbeault operator on X_0 associated with the above data, i.e., $\bar\partial^{L_0^p *}_{p\varphi_\epsilon}$ is the adjoint of $\bar\partial^{L_0^p}$ with respect to the metrics g^{TX_0} and $h_\epsilon^{L_0}$. Over the ball $B(x_0, 2\epsilon)$, D_p is just the restriction of $D_p^{X_0}$. Now by [Ma and Marinescu 2007, Theorem 1.4.7], and the observation that the tensors associated with g^{TX_0} do not depend on ζ and ϵ, as in (2-7), we get from (2-28) the existence of a constant $\delta > 0$ such that for $\zeta p > \delta$,

$$\operatorname{Spec}(D_p^{X_0})^2 \subset \{0\} \cup [\zeta p, +\infty[. \tag{2-34}$$

Using S_{x_0}, we get an isometry $L_0^p \simeq \mathbb{C}$. Let P_p^0 be the orthogonal projection from $\mathcal{C}^\infty(X_0, L_0^p) \simeq \mathcal{C}^\infty(X_0, \mathbb{C})$ on $\operatorname{Ker}(D_p^{X_0})$. Let $P_p^0(x, x')$ be the smooth kernel of P_p^0 with respect to the volume form $dv_{X_0}(x')$ induced by the metric g^{TX_0}. We have the following result:

Proposition 2.3. *For all $l \in \mathbb{N}$, there exists $c > 0$ such that for $\zeta p > \delta$, $x, x' \in B(x_0, \epsilon)$,*

$$\|(P_p^0 - P_p)(x, x')\|_{\mathcal{C}^0} \le c(|d\phi|_2^{-1}\zeta)^{-6n-3l-6} p^{-l} |\omega|_n^{2n+2}. \tag{2-35}$$

Proof. First, we replace $f_{\epsilon_0}(v)$ in (2-8) by $f_{\epsilon_0}(v/A)$. By Remark 2.2 and (2-30), for $x, x' \in B(x_0, \epsilon)$, we have $F_\epsilon(D_p)(x, x') = F_\epsilon(D_p^0)(x, x')$. Now we have a version of Proposition 2.1 for P_p^0 with $A\zeta$ instead of ζ. Estimate (2-35) follows. □

3. Uniform estimate of the Bergman kernels

We continue to use the notations introduced at the end of the last section. By Proposition 2.3, in order to study the kernel P_p, it suffices to study the kernel P_p^0. For this purpose, we will rescale the operator $(D_p^{X_0})^2$. Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z), \tag{3-1}$$

with $\kappa(0) = 1$.

Let $\{e_j\}_{j=1}^{2n}$ be an oriented orthonormal basis of $T_{x_0}X$, and let $\{e^j\}_{j=1}^{2n}$ be its dual basis. They allow us to identify $X_0 = \mathbb{C}^n$ with $\mathbb{R}^{2n}$ and we write $Z = (Z_1, \dots, Z_{2n})$. If $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is a multi-index, set $Z^\alpha := Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U, and set $\partial_j := \nabla_{e_j}$. Set

$t := p^{-1/2}$. For $s \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})$ and $Z \in \mathbb{R}^{2n}$, define

$$\begin{aligned} (S_t s)(Z) &:= s(Z/t), \qquad \nabla_t := t S_t^{-1} \kappa^{1/2} \nabla^{L_0^p} \kappa^{-1/2} S_t, \\ \mathscr{L}_t &:= S_t^{-1} t^2 \kappa^{1/2} (D_p^{X_0})^2 \kappa^{-1/2} S_t. \end{aligned} \tag{3-2}$$

Once we have done the trivialization of L_0 on X_0, (3-2) is well defined for any $p \in \mathbb{R},\ p \geq 1$.

The notations $\langle \cdot, \cdot \rangle_0$ and $\| \cdot \|_0$ mean respectively the inner product and the L^2-norm on $\mathcal{C}^\infty(X_0, \mathbb{C})$ induced by g^{TX_0}. For $s \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$, set

$$\begin{aligned} \|s\|_{t,0}^2 &:= \|s\|_0^2 = \int_{\mathbb{R}^{2n}} |s(Z)|^2 \, dv_{TX}(Z), \\ \|s\|_{t,m}^2 &:= \sum_{l=0}^{m} \sum_{j_1, \cdots, j_l = 1}^{2n} \|\nabla_{t, e_{j_1}} \cdots \nabla_{t, e_{j_l}} s\|_{t,0}^2. \end{aligned} \tag{3-3}$$

We then, for convenience, denote by $\langle s, s' \rangle_{t,0}$ the inner product on $\mathcal{C}^\infty(X_0, L_{x_0}^{\otimes p})$ corresponding to the norm $\| \cdot \|_{t,0}$. Let H_t^m be the Sobolev space of order m with norm $\| \cdot \|_{t,m}$. Let H_t^{-1} be the Sobolev space of order -1 and let $\| \cdot \|_{t,-1}$ be the norm on H_t^{-1} defined by $\|s\|_{t,-1} := \sup_{0 \neq s' \in H_t^1} |\langle s, s' \rangle_{t,0}| / \|s'\|_{t,1}$. If $B : H_t^m \to H_t^{m'}$ is a bounded linear operator for $m, m' \in \mathbb{Z}$, denote by $\|B\|_t^{m,m'}$ the norm of B with respect to the norms $\| \cdot \|_{t,m}$ and $\| \cdot \|_{t,m'}$.

Theorems 3.1, 3.2, 3.4 and Proposition 3.3 below are the analogues of [Ma and Marinescu 2007, Theorem 4.1.9–4.1.14] (cf., also [Dai et al. 2006, Theorem 4.7–4.10]). The emphasis here is the precise dependence of the involved constants on the curvature form ω.

Theorem 3.1. *There exist $c_1, c_2, c_3 > 0$ such that for $t \in]0, 1],\ \zeta \in]0, 1]$, and $s, s' \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbb{C})$,*

$$\begin{aligned} \langle \mathscr{L}_t s, s \rangle_{t,0} &\geqslant c_1 \|s\|_{t,1}^2 - c_2 |\omega|_0 \|s\|_{t,0}^2, \\ |\langle \mathscr{L}_t s, s' \rangle_{t,0}| &\leqslant c_3 |\omega|_0 \|s\|_{t,1} \|s'\|_{t,1}. \end{aligned} \tag{3-4}$$

Proof. By using the Lichnerowicz formula [Ma and Marinescu 2007, (4.1.33)], the same arguments as in (4.1.38)–(4.1.39) of the same work give the result. □

Let δ_ζ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\zeta/2$.

Theorem 3.2. *There exists $\delta > 0$ such that the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for all $\lambda \in \delta_\zeta$ and $t \in]0, \sqrt{\zeta/\delta}]$. There exists $c > 0$ such that for all $t \in]0, \sqrt{\zeta/\delta}],\ \lambda \in \delta_\zeta$, we have*

$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{0,0} \leqslant 2\zeta^{-1}, \quad \|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant c|\omega|_0^2 \zeta^{-1}. \tag{3-5}$$

Proof. By (2-34) and (3-2), we have

$$\operatorname{Spec}(\mathscr{L}_t) \subset \{0\} \cup [\zeta, +\infty[. \tag{3-6}$$

Thus, the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \delta_\zeta$ and $t \in]0, \sqrt{\zeta/\delta}]$, and we get the first inequality of (3-5).

By (3-4), $(\lambda_0 - \mathscr{L}_t)^{-1}$ exists for $\lambda_0 \in \mathbb{R},\ \lambda_0 \leqslant -2c_2|\omega|_0$. Moreover, as

$$c_1\|s\|^2_{t,1} \le -\langle(\lambda_0 - \mathscr{L}_t)s, s\rangle_{t,0} \le \|(\lambda_0 - \mathscr{L}_t)s\|_{t,-1}\|s\|_{t,1},$$

we have

$$\|(\lambda_0 - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{c_1}. \tag{3-7}$$

On the other hand, we have

$$(\lambda - \mathscr{L}_t)^{-1} = (\lambda_0 - \mathscr{L}_t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_t)^{-1}(\lambda_0 - \mathscr{L}_t)^{-1}. \tag{3-8}$$

Therefore, for $\lambda \in \delta_\zeta$, from the first estimate in (3-5) and (3-8), we get

$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,0} \le \frac{1}{c_1}(1 + 2|\lambda - \lambda_0|\zeta^{-1}). \tag{3-9}$$

In (3-8), we can interchange the last two factors. Then, applying (3-7) and (3-9) gives

$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{c_1} + \frac{|\lambda - \lambda_0|}{c_1{}^2}(1 + 2|\lambda - \lambda_0|\zeta^{-1}) \leqslant c|\omega|_0^2\zeta^{-1}. \tag{3-10}$$

The theorem follows. □

Proposition 3.3. *Take $m \in \mathbb{N}^*$. There is a $c > 0$ such that for $t \in]0, 1]$, $Q_1, \dots, Q_m \in \{\nabla_{t,e_j}, Z_j\}_{j=1}^{2n}$ and $s, s' \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$,*

$$\left|\left\langle [Q_1, [Q_2, \dots [Q_m, \mathscr{L}_t] \dots]]s, s'\right\rangle_{t,0}\right| \leqslant c|d\phi|_{m+1}^{\min(2,m)}\|s\|_{t,1}\|s'\|_{t,1}. \tag{3-11}$$

Proof. By [Ma and Marinescu 2007, (1.6.31)] and as in the proof of Proposition 1.6.9 of the same work, we know that $[Q_1, [Q_2, \dots [Q_m, \mathscr{L}_t] \dots]]$ has the same structure as $\mathscr{L}_t$ for $t \in [0, 1]$. More precisely, it has the form

$$\sum_{i,j} a_{ij}(t, tZ)\nabla_{t,e_i}\nabla_{t,e_j} + \sum_j d_j(t, tZ)\nabla_{t,e_j} + c(t, tZ), \tag{3-12}$$

where $a_{ij}(t, Z)$ and its derivatives in Z are uniformly bounded, $d_j(t, Z)$, $c(t, Z)$ and their first derivatives in Z are bounded by $c|d\phi|_{m+1}^{\min(2,m)}$ for $Z \in \mathbb{R}^{2n}$ and $t \in [0, 1]$ and a constant $c > 0$. We then get estimate (3-11). □

Theorem 3.4. *For $Q_1, \dots, Q_m \in \{\nabla_{t,e_j}, Z_j\}_{j=1}^{2n}$, there exists $c > 0$ such that we have for $t \in]0, \sqrt{\zeta/\delta}]$, $\lambda \in \delta_\zeta$ and $s \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$,*

$$\|Q_1 \cdots Q_m(\lambda - \mathscr{L}_t)^{-1} s\|_{t,1} \leqslant c \sum_{k=0}^{m} \sum_{1 \le j_1 < \cdots < j_k \le m} |d\phi|_{m-k+1}^{m-k} (|\omega|_0^2 \zeta^{-1})^{m-k+1} \|Q_{j_1} \cdots Q_{j_k} s\|_{t,0}. \tag{3-13}$$

Proof. For $Q_1, \dots, Q_m \in \{\nabla_{t,e_j}, Z_j\}_{j=1}^{2n}$, we can express $Q_1 \cdots Q_m(\lambda - \mathscr{L}_t)^{-1}$ as the sum of $(\lambda - \mathscr{L}_t)^{-1} Q_1 \cdots Q_m$ with a linear combination of operators of the type

$$[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_{m_1}}, (\lambda - \mathscr{L}_t)^{-1}] \dots]] Q_{j_{m_1+1}} \cdots Q_{j_m}, \tag{3-14}$$

with $j_1 < j_2 \cdots < j_{m_1}$, $j_{m_1+1} < \cdots < j_m$. The coefficients of this combination are bounded when m is bounded. Let $\mathcal{S}_t$ be the family of operators

$$[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \mathscr{L}_t] \dots]] = -[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \lambda - \mathscr{L}_t] \dots]].$$

Note that

$$[Q, (\lambda - \mathscr{L}_t)^{-1}] = -(\lambda - \mathscr{L}_t)^{-1} [Q, \lambda - \mathscr{L}_t] (\lambda - \mathscr{L}_t)^{-1} = (\lambda - \mathscr{L}_t)^{-1} [Q, \mathscr{L}_t] (\lambda - \mathscr{L}_t)^{-1},$$

thus by the recurrence on m_1 we know that every commutator

$$[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_{m_1}}, (\lambda - \mathscr{L}_t)^{-1}] \dots]]$$

is a linear combination of operators of the form

$$(\lambda - \mathscr{L}_t)^{-1} S_1 (\lambda - \mathscr{L}_t)^{-1} S_2 \cdots S_{m_2} (\lambda - \mathscr{L}_t)^{-1} \tag{3-15}$$

with $S_1, \dots, S_{m_2} \in \mathcal{S}_t$ and $m_2 \le m_1$. The coefficients of this combination are bounded when m_1 is bounded.

From Proposition 3.3 we deduce that the $\|\cdot\|_t^{1,-1}$ norms of the operators $[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \mathscr{L}_t] \dots]]$ are uniformly bounded from above by a constant times $|d\phi|_{l+1}^{l}$. Hence, by Theorem 3.2, the $\|\cdot\|_t^{0,1}$ norm of the operator (3-15) is bounded by a constant times

$$\zeta^{-m_2-1} |\omega|_0^{2m_2+2} \sum_{\substack{l_1 + \cdots + l_{m_2} = m_1 \\ l_1, \dots, l_{m_2} \ge 1}} \prod_{j=1}^{m_2} |d\phi|_{l_j+1}^{l_j}.$$

The theorem follows. □

Let $\mathcal{P}_t : (\mathcal{C}^\infty(X_0, \mathbb{C}), \|\cdot\|_0) \to \operatorname{Ker}(\mathscr{L}_t)$ be the orthogonal projection corresponding to the norm $\|\cdot\|_{t,0}$ given in (3-3). Let $\mathcal{P}_t(Z, Z')$, (with $Z, Z' \in X_0$) be the smooth kernel of $\mathcal{P}_t$ with respect to $dv_{TX}(Z')$. Note that $\mathscr{L}_t$ is a family of differential operators on $T_{x_0}X$ with coefficients in $\mathbb{C}$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX with itself on X. We can view $\mathcal{P}_t(Z, Z')$ as smooth functions over $TX \times_X TX$ by identifying a section $F \in \mathcal{C}^\infty(TX \times_X TX, \mathbb{C})$ with the family $(F_{x_0})_{x_0 \in X}$, where $F_{x_0} := F|_{\pi^{-1}(x_0)}$. In the following result we adapt [Ma and Marinescu 2007, Theorem 4.1.24] to the present situation.

Theorem 3.5. *For any $r \in \mathbb{N}$, $\sigma > 0$, there exists $c > 0$, such that for $t \in]0, \sqrt{\zeta/\delta}]$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leqslant \sigma$,*

$$\left\| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z, Z') \right\|_{\mathcal{C}^0(X)} \leqslant c\zeta^{-2n-4r-2} |d\phi|_{2r+n+1}^{4r+2n} |\omega|_0^{8r+4n+4} |d\phi|_{n+2}^{2n+2}. \tag{3-16}$$

Proof. By (3-6), for every $k \in \mathbb{N}^*$,

$$\mathcal{P}_t = \frac{1}{2\pi\sqrt{-1}} \int_{\delta_\zeta} \lambda^{k-1} (\lambda - \mathscr{L}_t)^{-k} d\lambda. \tag{3-17}$$

For $m \in \mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\nabla_{t,e_{i_1}} \cdots \nabla_{t,e_{i_j}}$ with $j \leqslant m$. We apply Theorem 3.4 to $m-1$ operators $Q_2, \dots, Q_m$ instead of m operators. We deduce that for $l, m \in \mathbb{N}^*$ with $l \geq m$, and $Q = Q_1 \cdots Q_m \in \mathcal{Q}^m$, there are $c, c' > 0$ such that for $t \in]0, \sqrt{\zeta/\delta}]$, $\zeta \in [0, 1]$, $s \in \mathcal{C}_0^\infty(X_0, \mathbb{C})$ and $\lambda \in \delta_\zeta$

$$\begin{aligned} \|Q_1 \cdots Q_m (\lambda - \mathscr{L}_t)^{-l} s\|_{t,0} &\leqslant c \|Q_2 \cdots Q_m (\lambda - \mathscr{L}_t)^{-l} s\|_{t,1} \\ &\leqslant c' \sum_{k=0}^{m-1} \sum_{1 < i_1 < \cdots < i_k \leq m} |d\phi|_{m-k}^{m-k-1} (|\omega|_0^2 \zeta^{-1})^{m-k} \|Q_{i_1} \cdots Q_{i_k} (\lambda - \mathscr{L}_t)^{-l+1} s\|_{t,0}. \end{aligned} \tag{3-18}$$

Then, by induction and using (3-5), we get

$$\|Q_1 \cdots Q_m (\lambda - \mathscr{L}_t)^{-l} s\|_{t,0} \leqslant c\zeta^{-m-l+1} |d\phi|_m^{m-1} |\omega|_0^{2m} \|s\|_{t,0}. \tag{3-19}$$

As $\mathscr{L}_t$ is symmetric, we can consider the adjoint of the operator in (3-19) and get for $Q' = Q'_1 \cdots Q'_{m'} \in \mathcal{Q}^{m'}$,

$$\|(\lambda - \mathscr{L}_t)^{-l} Q'_1 \cdots Q'_{m'} s\|_{t,0} \leqslant c\zeta^{-m'-l+1} |d\phi|_{m'}^{m'-1} |\omega|_0^{2m'} \|s\|_{t,0}. \tag{3-20}$$

Note that for $m = 0$ and $l \in \mathbb{N}$ we also have $\|(\lambda - \mathscr{L}_t)^{-l} s\|_{t,0} \leq c\zeta^{-l} \|s\|_{t,0}$. Thus, for $Q \in \mathcal{Q}^m$, $Q' \in \mathcal{Q}^{m'}$ with $m, m' > 0$, by taking $k = m + m'$, we get

$$\begin{aligned} \|Q \mathcal{P}_t Q'\|_t^{0,0} &\leqslant \frac{1}{2\pi} \int_{\delta_\zeta} |\lambda|^{m+m'-1} \|Q(\lambda - \mathscr{L}_t)^{-m-m'} Q'\|_t^{0,0} |d\lambda| \\ &\leqslant c |d\phi|_m^{m-1} |\omega|_0^{2m} |d\phi|_{m'}^{m'-1} |\omega|_0^{2m'} \zeta^{-2m-2m'+2} \zeta^{m+m'} \\ &= c |d\phi|_m^{m-1} |\omega|_0^{2m} |d\phi|_{m'}^{m'-1} |\omega|_0^{2m'} \zeta^{-m-m'+2}. \end{aligned} \tag{3-21}$$

By [Ma and Marinescu 2007, Lemma 1.2.4], (2-31), (2-32) and (3-2), on $B^{T_{x_0}X}(0, \epsilon/t)$,

$$\nabla_{t,e_i}|_Z = \nabla_{e_i} + \tfrac{1}{2} R^L_{x_0}(Z, e_i) + O(t|Z|^2)|d\phi|_2. \tag{3-22}$$

Let $|\cdot|_{(\sigma),m}$ denote the usual Sobolev norm on $\mathcal{C}^\infty(B^{T_{x_0}X}(0, \sigma+1), \mathbb{C})$ induced by the volume form $dv_{TX}(Z)$ as in (3-3). Observe that by (3-3), (3-22), for $m > 0$, there exists $c > 0$ such that for $s \in \mathcal{C}^\infty(X_0, \mathbb{C})$ with $\operatorname{supp}(s) \subset B(0, \sigma+1)$,

$$\frac{1}{c|d\phi|^m_{m+1}} \|s\|_{t,m} \leqslant |s|_{(\sigma),m} \leqslant c|d\phi|^m_{m+1} \|s\|_{t,m}. \tag{3-23}$$

Now, we want to estimate $Q_Z Q'_{Z'} \mathcal{P}_t(Z, Z')$ using the standard Sobolev's inequality for $Q \in \mathcal{Q}^m$ and $Q' \in \mathcal{Q}^{m'}$. If we define $S := Q\mathcal{P}_t Q'$ then we have for $|Z|, |Z'| \leq \sigma$

$$|Q_Z Q'_{Z'} \mathcal{P}_t(Z, Z')| \leq c \sup\Big\{ \Big\| \frac{\partial^{|\alpha|}}{\partial Z^\alpha} S \frac{\partial^{|\alpha'|} s}{\partial Z'^{\alpha'}} \Big\|_{(\sigma),n+1}, \ \|s\|_{L^2} = 1, \\ \operatorname{supp}(s) \subset B(0, \sigma+1), |\alpha|, |\alpha'| \leq n+1 \Big\}. \tag{3-24}$$

Hence, by (3-23), applied twice to $n+1$ instead of m, and also (3-21), applied to $m+n+1$, $m'+n+1$ instead of m, m', we get

$$\sup_{|Z|,|Z'|\leqslant\sigma} |Q_Z Q'_{Z'} \mathcal{P}_t(Z, Z')| \\ \leqslant c' |d\phi|^{m+n}_{m+n+1} |\omega|_0^{2m+2n+2} |d\phi|^{m'+n}_{m'+n+1} |\omega|_0^{2m'+2n+2} |d\phi|^{2n+2}_{n+2} \zeta^{-m-m'-2n}. \tag{3-25}$$

By (3-22) and (3-25) for $m = m' = 0$, estimate (3-16) holds for $r = 0$.

Consider now $r \geq 1$. Set

$$I_{k,r} := \Big\{ (\boldsymbol{k}, \boldsymbol{r}) = \{(k_i, r_i)\}_{i=0}^j : \sum_{i=0}^j k_i = k + j, \quad \sum_{i=1}^j r_i = r, \quad k_i, r_i \in \mathbb{N}^* \Big\}. \tag{3-26}$$

Then there exist $a^{\boldsymbol{k}}_{\boldsymbol{r}} \in \mathbb{R}$ such that

$$A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t) = (\lambda - \mathscr{L}_t)^{-k_0} \frac{\partial^{r_1} \mathscr{L}_t}{\partial t^{r_1}} (\lambda - \mathscr{L}_t)^{-k_1} \cdots \frac{\partial^{r_j} \mathscr{L}_t}{\partial t^{r_j}} (\lambda - \mathscr{L}_t)^{-k_j}, \\ \frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_t)^{-k} = \sum_{(\boldsymbol{k},\boldsymbol{r}) \in I_{k,r}} a^{\boldsymbol{k}}_{\boldsymbol{r}} A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t). \tag{3-27}$$

Set $g_{ij}(Z) := \langle \partial/\partial Z_i, \partial/\partial Z_j \rangle_Z$, and (g^{ij}) the inverse matrix of (g_{ij}). Note that $(\partial^u/\partial t^u)(g^{ij}(tZ))$, $(\partial^u/\partial t^u)(\nabla_{t,e_i} - (1/t)\Gamma^{L_0}_\epsilon(tZ))$ are functions which do not depend on ζ, and $(\partial^u/\partial t^u) R^{L_0}_\epsilon(tZ)$, $(\partial^u/\partial t^u)\Gamma^{L_0}_\epsilon(tZ)$ are functions of type $d'(tZ)Z^\beta$, and $\nabla_{e_{j_1}} \cdots \nabla_{e_{j_l}} d'(tZ)$ is uniformly controlled by $|d\phi|_{l+u+1}$.

We handle now the operator $A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)Q'$. We will move first all the terms Z^β in $d'(tZ)Z^\beta$ (defined above) to the right-hand side of this operator. To do so, we always use the commutator trick as in the proof of [Ma and Marinescu 2007, Theorem 1.6.10], i.e., each time, we perform only the commutation with Z_i (not directly with Z^β with $|\beta| > 1$). Then $A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)Q'$ is as the form $\sum_{|\beta|\leqslant 2r} \mathscr{L}_{\beta,t} Q''_\beta Z^\beta$, and Q''_β is obtained from Q' and its commutation with Z^β. Observe that $[Z_i, \mathscr{L}_t]$ is a first order differential operator and $[Z_{j_1}, [Z_{j_2}, \mathscr{L}_t]] = g^{j_1 j_2}(tZ)$ is a bounded function. Therefore, $\mathscr{L}_{\beta,t}$ is a linear combination of operators of the form

$$(\lambda - \mathscr{L}_t)^{-k'_0} S_1 (\lambda - \mathscr{L}_t)^{-k'_1} S_2 \cdots S_{l'} (\lambda - \mathscr{L}_t)^{-k'_{l'}}, \tag{3-28}$$

with $S_i \in \{a(tZ)\nabla_{t,e_{j_1}} \nabla_{t,e_{j_2}}, d_{j_1}(tZ)\nabla_{t,e_{j_1}}, d'(tZ)\}$ and the number of $\nabla_{t,e_{j_1}}$ in all $\{S_i\}_i$ is less than $\sum_i r_i + 2j = r + 2j$. As $k > 2(r+1) + m + m'$, we can split the above operator into two parts as in [Ma and Marinescu 2007, (4.1.51)] and use the fact that the term $\nabla_{t,e_j}(\lambda - \mathscr{L}_t)^{-l_1}$ will contribute ζ^{-l_1}. Similarly to (3-18), we get that $A^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)$ is well defined and for $m, m' \in \mathbb{N}$, $k > 2(r+1) + m + m'$, $Q \in \mathcal{Q}^m$, $Q' \in \mathcal{Q}^{m'}$, there exists $c > 0$ such that for $\lambda \in \delta_\zeta$ and $t \in]0, \sqrt{\zeta/\delta}]$,

$$\begin{aligned} &\|QA^{\boldsymbol{k}}_{\boldsymbol{r}}(\lambda, t)Q's\|_{t,0} \\ &\qquad \leqslant c|d\phi|^{m+2r-1}_{m+2r} |\omega|^{2m+4r}_0 |d\phi|^{m'+2r-1}_{m'+2r} |\omega|^{2m'+4r}_0 \zeta^{-\sum_{i=0}^{j} k_i - m - m' - 3r} \sum_{|\beta|\leqslant 2r} \|Z^\beta s\|_{t,0} \\ &\qquad \leq c|d\phi|^{m+2r-1}_{m+2r} |\omega|^{2m+4r}_0 |d\phi|^{m'+2r-1}_{m'+2r} |\omega|^{2m'+4r}_0 \zeta^{-k-m-m'-4r} \sum_{|\beta|\leqslant 2r} \|Z^\beta s\|_{t,0}. \end{aligned} \tag{3-29}$$

By (3-17), (3-27) and (3-29), as in (3-21), for $m, r \in \mathbb{N}$, $Q \in \mathcal{Q}^m$ and $Q' \in \mathcal{Q}^{m'}$, there exists $c > 0$ such that for $t \in]0, \sqrt{\zeta/\delta}]$ and $s \in \mathcal{C}^\infty_0(X_0, \mathbb{C})$,

$$\begin{aligned} &\left\| Q \frac{\partial^r}{\partial t^r} \mathcal{P}_t Q' s \right\|_{t,0} \\ &\qquad \leqslant c|d\phi|^{m+2r-1}_{m+2r} |d\phi|^{m'+2r-1}_{m'+2r} |\omega|^{2m+2m'+8r}_0 \zeta^{-m-m'-4r} \sum_{|\beta|\leqslant 2r} \|Z^\beta s\|_{t,0}. \end{aligned} \tag{3-30}$$

Finally, equations (3-23) and (3-30) together with Sobolev's inequalities imply for $|Z|, |Z'| \leqslant \sigma$,

$$\sup_{|Z|,|Z'|\leqslant\sigma} \left| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z, Z') \right| \leqslant c|d\phi|^{2n+4r}_{2r+n+1} |\omega|^{2(2n+2+4r)}_0 |d\phi|^{2n+2}_{n+2} \zeta^{-2n-4r-2}. \tag{3-31}$$

This ends the proof of the theorem. ∎

Note that by (3-22), the operator $\mathscr{L}_t$ has a limit when $t \to 0$ which we denote by $\mathscr{L}_0$. For k big enough, set

$$F_r := \frac{1}{2\pi\sqrt{-1}\, r!} \int_{\delta_\zeta} \lambda^{k-1} \sum_{(\boldsymbol{k},\boldsymbol{r})\in I_{k,r}} a_{\boldsymbol{r}}^{\boldsymbol{k}} A_{\boldsymbol{r}}^{\boldsymbol{k}}(\lambda, 0)\, d\lambda. \tag{3-32}$$

Let $F_r(Z, Z') \in \mathcal{C}^\infty(TX \times_X TX, \mathbb{C})$ be the smooth kernel of F_r with respect to $dv_{TX}(Z')$.

Theorem 3.6. *For all $j \in \mathbb{N}$, $\sigma > 0$, there exists $c > 0$ such that for $t \in \,]0, \sqrt{\zeta/\delta}]$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$, we have*

$$\begin{aligned} \Big\| \Big(\mathcal{P}_t - \sum_{r=0}^{j} F_r t^r \Big)(Z, Z') \Big\|_{\mathcal{C}^0(X)} \qquad\qquad\qquad\qquad \\ \leqslant c\, |d\phi|_{2j+n+3}^{2(2j+n+2)} |\omega|_0^{2(4j+2n+6)} |d\phi|_{n+2}^{2n+2} \zeta^{-4j-2n-6} t^{j+1}. \end{aligned} \tag{3-33}$$

Proof. By [Ma and Marinescu 2007, (4.1.69)], we have

$$\frac{1}{r!} \frac{\partial^r}{\partial t^r} \mathcal{P}_t \Big|_{t=0} = F_r. \tag{3-34}$$

Recall that the Taylor expansion with integral rest of a function $G \in \mathcal{C}^{j+1}([0, 1])$ is

$$G(t) - \sum_{r=0}^{j} \frac{1}{r!} \frac{\partial^r G}{\partial t^r}(0) t^r = \frac{1}{j!} \int_0^t (t - t_0)^j \frac{\partial^{j+1} G}{\partial t^{j+1}}(t_0)\, dt_0, \qquad t \in [0, 1]. \tag{3-35}$$

Theorem 3.5 and (3-34) show estimate (3-16) holds if we replace $(1/r!)(\partial^r/\partial t^r)\mathcal{P}_t$ with F_r. Using this new estimate together with (3-35) and (3-16), we get (3-33). □

Let $\mathcal{P}$ be the orthogonal projection from $L^2(X_0, \mathbb{C})$ onto $\mathrm{Ker}(\mathscr{L}_0)$, and let $\mathcal{P}(Z, Z')$ be the smooth kernel of $\mathcal{P}$ with respect to $dv_{TX}(Z')$. Then $\mathcal{P}(Z, Z')$ is the Bergman kernel of $\mathscr{L}_0$. By [Ma and Marinescu 2007, (4.1.84)], if we choose $\{w_j\}$ to be an orthonormal basis of $T_{x_0}^{(1,0)}X$ such that $\dot{R}_{x_0}^L = \mathrm{diag}(a_1, \dots, a_n) \in \mathrm{End}(T_{x_0}^{(1,0)}X)$ with $\langle \dot{R}_{x_0}^L W, \overline{Y} \rangle = R^L(W, \overline{Y})$ for $W, Y \in T_{x_0}^{(1,0)}X$, then

$$\mathcal{P}(Z, Z') = \prod_{i=1}^{n} \frac{a_i}{2\pi} \exp\Big(-\frac{1}{4} \sum_i a_i \big(|z_i|^2 + |z_i'|^2 - 2 z_i \bar{z}_i' \big) \Big). \tag{3-36}$$

The following result was established in [Ma and Marinescu 2007, Theorem 4.1.21]:

Theorem 3.7. *There exist polynomials $J_r(Z, Z')$ in Z, Z' with the same parity as r and* $\deg J_r(Z, Z') \leqslant 3r$, *whose coefficients are polynomials in R^{TX} (resp. R^L) and their derivatives of order $\leqslant r-2$ (resp. $\leqslant r$), and reciprocals of linear combinations of eigenvalues of R^L at x_0, such that*

$$F_r(Z, Z') = J_r(Z, Z') \mathcal{P}(Z, Z'). \tag{3-37}$$

Moreover, we have

$$J_0 = 1 \quad \textit{and} \quad F_0 = \mathcal{P}. \tag{3-38}$$

Owing to (3-1), (3-2), as in [Ma and Marinescu 2007, (4.1.96)], we have

$$P_p^0(Z, Z') = t^{-2n}\kappa^{-1/2}(Z)\mathcal{P}_t(Z/t, Z'/t)\kappa^{-1/2}(Z'), \quad \text{for all } Z, Z' \in \mathbb{R}^{2n}. \tag{3-39}$$

From Theorems 3.6 and 3.7 and (3-39), we get the following near-diagonal expansion of the Bergman kernels. Recall that we are working with $t = p^{-1/2}$.

Theorem 3.8. *For every $j \in \mathbb{N}$, there exists $c > 0$ such that the estimate*

$$\begin{aligned} \left|\left(\frac{1}{p^n} P_p^0(Z, Z') - \sum_{r=0}^{j} F_r(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-1/2}(Z)\kappa^{-1/2}(Z')p^{-r/2}\right)\right| \\ \leqslant c|d\phi|_{2j+n+3}^{2(2j+n+2)}|\omega|_0^{2(2n+4j+6)}|d\phi|_{n+2}^{2n+2}p^{-(j+1)/2}\zeta^{-2n-4j-6} \end{aligned} \tag{3-40}$$

holds for all $0 < \zeta \leq 1$, $\zeta p > \delta$, and all $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leq \sigma/\sqrt{p}$.

End of the proof of Theorem 1.1. We apply Theorem 3.8 to $Z = Z' = 0$ and $j = 1$. Note that $F_1(0, 0) = 0$ because the function F_1 is odd. By equation (3-36), $\mathcal{P}(0, 0) = \omega(x_0)^n/\theta(x_0)^n$. So from (3-40), we get

$$\left\| \frac{1}{p^n} P_p^0(0, 0) - \frac{\omega(x_0)^n}{\theta(x_0)^n} \right\|_{\mathcal{C}^0(X)} \leqslant c|d\phi|_{n+5}^{2n+8}|\omega|_0^{4n+20}|d\phi|_{n+2}^{2n+2}\zeta^{-2n-10}p^{-1}. \tag{3-41}$$

We then deduce the result form Propositions 2.1, 2.3 and (3-41). □

Remark 3.9. Assume now $\phi \in \mathcal{C}^{n+2k+6}$. Then by the usual $\mathcal{C}^k$-norm on each U_j and Sobolev embedding theorem, from (2-22), we get

$$\| F_{\epsilon_0}(D_p)(x, x') - P_p(x, x')\|_{\mathcal{C}^k} \leq c|\omega|_{n+k}^{2n+2+2k}\zeta^{-6n-3l-6-3k}p^{-l}. \tag{3-42}$$

Note that $\nabla^{L^p} = d + p\Gamma^L$ (cf., (2-15)), thus if we use the $\mathcal{C}^k$-norm induced by ∇^{L^p}, then we get

$$\begin{aligned} \| F_{\epsilon_0}(D_p)(x, x') - P_p(x, x')\|_{\mathcal{C}^k(X\times X)} \\ \leq c\sum_{r=0}^{k}|\omega|_{n+r}^{2n+2+2r}\zeta^{-6n-3(k-r+1)-6-3k}p^{-k+r-1}|\omega|_{k-r}^{k-r}p^{k-r} \\ \leq c|\omega|_{n+k}^{2n+2+2k}\zeta^{-6n-9-3k}p^{-1}. \end{aligned} \tag{3-43}$$

In the same way as (2-35) and above, we get

$$\|(P_p^0 - P_p)(x, x')\|_{\mathcal{C}^k(X\times X)} \leq c(|d\phi|_2^{-1}\zeta)^{-6n-3k-9}p^{-1}|\omega|_{n+k}^{2n+2+2k}. \tag{3-44}$$

Combining [Ma and Marinescu 2007, (4.1.64)] and the argument for (3-16), we get

$$\left\| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z, Z') \right\|_{\mathcal{C}^{m'}} \leqslant c\zeta^{-2n-4(r+m')-2} |d\phi|^{4(r+m')+2n}_{2(r+m')+n+1} |\omega|_0^{8(r+m')+4n+4} |d\phi|^{2n+2}_{n+2}; \tag{3-45}$$

here $\mathcal{C}^{m'}$ is the usual $\mathcal{C}^{m'}$-norm for the parameter x_0.

Thus we get an extension of (1-4):

$$\begin{aligned} \left\| p^{-n} P_p(x,x) - \frac{\omega(x)^n}{\theta(x)^n} \right\|_{\mathcal{C}^k} \leqslant c |d\phi|^{2n+4k+8}_{n+2k+5} |\omega|_0^{4n+8k+20} |d\phi|^{2n+2}_{n+2} \zeta^{-2n-4k-10} p^{-1} \\ + c|\omega|^{2n+2k+2}_{n+k} (|d\phi|_2 \zeta^{-1})^{6n+9+3k} p^{-1}. \end{aligned} \tag{3-46}$$

Remark 3.10. Let ϕ be a function of class $\mathcal{C}^\alpha$, with $0 < \alpha \leq 1$, which is ω_0-plurisubharmonic, i.e., $dd^c\phi + \omega_0 \geq 0$. For each $0 < \zeta \leq 1$, we can find a smooth ω_0-plurisubharmonic function ϕ_ζ such that $\|\phi_\zeta\|_{\mathcal{C}^k} \leq c\zeta^{-k+\alpha}$ and $dd^c\phi_\zeta + \omega_0 \geq \zeta\omega_0$, see [Dinh et al. 2015]. As mentioned in Section 1, we can study ϕ by applying our results to ϕ_ζ. Some steps in the proof of our estimates can be strengthened using $\|\phi_\zeta\|_{\mathcal{C}^k} \leq c\zeta^{-k+\alpha}$ for each $0 \leq k \leq n+6$ instead of using only the $\mathcal{C}^{n+6}$-norm.

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MOLINO THEORY FOR MATCHBOX MANIFOLDS

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A matchbox manifold is a foliated space with totally disconnected transversals, and an equicontinuous matchbox manifold is the generalization of Riemannian foliations for smooth manifolds in this context. We develop the Molino theory for all equicontinuous matchbox manifolds. Our work extends the Molino theory developed by Álvarez López and Moreira Galicia, which required the hypothesis that the holonomy actions for these spaces satisfy the strong quasianalyticity condition. The methods of this paper are based on the authors' previous work on the structure of weak solenoids, and provide many new properties of the Molino theory for the case of totally disconnected transversals, and examples to illustrate these properties. In particular, we show that the Molino space need not be uniquely well defined, unless the global holonomy dynamical system is stable, a notion defined in this work. We show that examples in the literature for the theory of weak solenoids provide examples for which the strong quasianalytic condition fails. Of particular interest is a new class of examples of equicontinuous minimal Cantor actions by finitely generated groups, whose construction relies on a result of Lubotzky. These examples have nontrivial Molino sequences, and other interesting properties.

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1. Introduction

A smooth foliation $\mathcal{F}$ of a connected compact manifold is a smooth decomposition of M into leaves, which are connected submanifolds of M with constant leaf dimension n and codimension q, where $m = n + q$ is the dimension of M. This structure is defined by a finite covering of M by coordinate charts whose image is the product space

$$(-1, 1)^n \times (-1, 1)^q \subset \mathbb{R}^m,$$

such that the leaves are mapped into linear planes of dimension n, and the transition functions between charts preserve these planes. The space $(-1, 1)^q$ is called the local transverse model for $\mathcal{F}$. A smooth foliation $\mathcal{F}$ is said to be *Riemannian*, or *bundle-like*, if there exists a Riemannian metric on the normal bundle $Q \to M$ which is invariant under the transverse holonomy transport along the leaves of $\mathcal{F}$. This condition was introduced by Reinhart [1959], and is a very strong assumption to impose on a foliation. The Molino theory for Riemannian foliations gives a complete structure theory for the geometry and dynamics of this class of foliations on compact smooth manifolds [Haefliger 1989; Moerdijk and Mrčun 2003; Molino 1982; 1988].

An *n-dimensional foliated space* $\mathfrak{M}$, as introduced by Moore and Schochet [2006], is a continuum — a compact connected metrizable space — with a continuous decomposition of $\mathfrak{M}$ into leaves, which are connected manifolds with constant leaf dimension n. Moreover, the decomposition has a local product structure analogous to that for smooth foliations [Candel and Conlon 2000; Moore and Schochet 2006]; that is, every point of $\mathfrak{M}$ has an open neighborhood homeomorphic to the open subset $(-1, 1)^n \subset \mathbb{R}^n$ times an open subset of a Polish space $\mathfrak{X}$, which is said to be the *local transverse model*. Thus, $\mathfrak{M}$ has a foliation denoted by $\mathcal{F}_{\mathfrak{M}}$ whose leaves are the maximal path-connected components, with respect to the fine topology on $\mathfrak{M}$ induced by the plaques of the local product structure.

An *equicontinuous foliated space* is the topological analog of a Riemannian foliation. In this case, the transverse holonomy pseudogroup associated to the foliation is assumed to act via an equicontinuous collection of local homeomorphisms on the transverse model spaces. The transverse holonomy maps are not assumed to be differentiable, so there is no natural normal bundle associated to a foliated space, and the standard methods for showing an analog of the Molino theory do not apply. In a series of papers, Álvarez López and Candel [2009; 2010] and Álvarez López and Moreira Galicia [2016] formulated a *topological Molino theory* for equicontinuous foliated spaces, which is a partial generalization of the Molino theory for smooth Riemannian foliations. They formulated the notion of *strongly quasianalytic* "regularity" for a foliated space, which is a condition on the pseudogroup associated to the foliation, as discussed in Section 9. The topological

Molino theory in [Álvarez López and Moreira Galicia 2016] applies to foliated spaces which satisfy the strongly quasianalytic condition.

The topological Molino theory for an equicontinuous foliated space $\mathfrak{M}$ with *connected* transversals essentially reduces to the smooth theory, by [Álvarez López and Candel 2010; Álvarez López and Moreira Galicia 2016; Álvarez López and Barral Lijó 2016]. In contrast, when the transversals to $\mathcal{F}_{\mathfrak{M}}$ are *totally disconnected*, and we then say that $\mathfrak{M}$ is a matchbox manifold, the development of a Molino theory in [Álvarez López and Moreira Galicia 2016] does not address several key issues, which can be seen as the result of using techniques developed for the smooth theory in the context of totally disconnected spaces. In this work, we apply a completely different approach to developing a topological Molino theory for the case of totally disconnected transversals. The techniques we use were developed in the authors' works [Dyer 2015; Dyer et al. 2016; 2017]. They are used here to develop a topological Molino theory for matchbox manifolds in full generality, and to reveal the far greater complexity of the theory in this case. In particular, we show by our results and examples that the classification of equicontinuous matchbox manifolds via Molino theory is far from complete.

We recall in Section 2 the definitions of a foliated space $\mathfrak{M}$, and of a *matchbox manifold*, which is a foliated space whose local transverse models for the foliation $\mathcal{F}_{\mathfrak{M}}$ are totally disconnected. The terminology "matchbox manifold" follows the usage introduced in continua theory [Aarts and Oversteegen 1991; 1995; Aarts and Martens 1988]. A matchbox manifold with 2-dimensional leaves is a lamination by surfaces, as defined in [Ghys 1999; Lyubich and Minsky 1997]. If all leaves of $\mathfrak{M}$ are dense, then it is called a *minimal matchbox manifold*. A compact minimal set $\mathfrak{M} \subset M$ for a foliation $\mathcal{F}$ on a manifold M yields a foliated space with foliation $\mathcal{F}_{\mathfrak{M}} = \mathcal{F}|\mathfrak{M}$. If the minimal set is exceptional, then $\mathfrak{M}$ is a minimal matchbox manifold. It is an open problem to determine which minimal matchbox manifolds are homeomorphic to exceptional minimal sets of C^r-foliations of compact smooth manifolds, for $r \geq 1$. For example, the issues associated with this problem are discussed in [Cass 1985; Clark and Hurder 2011; Hurder 2013].

It was shown in [Clark and Hurder 2013, Theorem 4.12] that an equicontinuous matchbox manifold $\mathfrak{M}$ is minimal; that is, every leaf is dense in $\mathfrak{M}$. This result generalized a result of Joe Auslander [1988] for equicontinuous group actions. Examples of equicontinuous matchbox manifolds are given by *weak solenoids*, which are discussed in Section 3. Briefly, a weak solenoid $\mathcal{S}_{\mathcal{P}}$ is the inverse limit of a sequence of covering maps

$$\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\},$$

called a *presentation* for $\mathcal{S}_{\mathcal{P}}$, where M_ℓ is a compact connected manifold without boundary and $p_{\ell+1}$ is a finite-to-one covering space. The results of [Clark and

Hurder 2013] reduce the study of equicontinuous matchbox manifolds to the study of weak solenoids:

Theorem 1.1 [Clark and Hurder 2013, Theorem 1.4]. *An equicontinuous matchbox manifold $\mathfrak{M}$ is homeomorphic to a weak solenoid.*

The idea of the proof of this result is to choose a clopen transversal $V_0 \subset \mathfrak{M}$; then associated to the induced holonomy action of $\mathcal{F}_{\mathfrak{M}}$ on V_0, one defines (see Proposition 3.4) a chain of subgroups of finite index, $\mathcal{G} = \{G_0 \supset G_1 \supset \cdots\}$, where G_0 is the fundamental group of the first shape approximation M_0 to $\mathfrak{M}$, where M_0 is a compact manifold without boundary. Then $\mathfrak{M}$ is shown to be homeomorphic to the inverse limit of the infinite chain of coverings of M_0 associated to the subgroup chain $\mathcal{G}$.

The theory of inverse limits for covering spaces, as developed for example in [Fokkink and Oversteegen 2002; McCord 1965; Rogers 1970; Rogers and Tollefson 1971a; 1971b; Schori 1966], reduces many questions about the classification of weak solenoids to questions about properties of the group chain $\mathcal{G}$ associated with the presentation $\mathcal{P}$. Thus, every equicontinuous matchbox manifold $\mathfrak{M}$ admits a presentation which determines its homeomorphism type. In Section 3A, the notion of a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with presentation $\mathcal{P}$ is recalled, and the notion of a dynamical partition of the transversal space V_0 is introduced in Section 3B. As discussed in Section 3C, the homeomorphism constructed in the proof of Theorem 1.1 is well defined up to return equivalence for the action of the respective holonomy pseudogroups [Clark et al. 2013a, Section 4]. Thus, we are interested in invariants for group chains that are independent of the choice of the chain, up to the corresponding notion of return equivalence for group chains. This is the approach we use in this work to formulate and study "Molino theory" for weak solenoids.

Section 4 introduces the group chain model for the holonomy action of weak solenoids, following the approach in [Dyer 2015; Dyer et al. 2016; 2017]. Section 5 then recalls results in the literature about homogeneous matchbox manifolds and the associated group chain models for their holonomy actions, which are fundamental for developing the notion of a "Molino space". Section 6 introduces the notion of the Ellis group associated to the holonomy action of a weak solenoid. Ellis semigroups were developed in [Auslander 1988; Ellis and Gottschalk 1960; Ellis 1960; 1969; Ellis and Ellis 2014], and also appeared in [Álvarez López and Candel 2010]. A key point of our approach is to use this concept as the foundation of our development of a topological Molino theory.

A key aspect of the Molino space for a foliation is that it is *foliated homogeneous*. A continuum $\mathfrak{M}$ is said to be *homogeneous* if given any pair of points $x, y \in \mathfrak{M}$, then there exists a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$. A homeomorphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ preserves the path-connected components, hence a homeomorphism of a matchbox manifold preserves the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. It follows that if $\mathfrak{M}$ is homogeneous, then it is also foliated homogeneous. Our first result is that every equicontinuous matchbox manifold admits a foliated homogeneous Molino space.

Theorem 1.2. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and let $\mathcal{P}$ be a presentation of $\mathfrak{M}$, such that $\mathfrak{M}$ is homeomorphic to a solenoid $\mathcal{S}_{\mathcal{P}}$. Then there exists a homogeneous matchbox manifold $\widehat{\mathfrak{M}}$ with foliation $\widehat{\mathcal{F}}$, called a Molino space of $\mathfrak{M}$, and a compact totally disconnected group $\mathcal{D}$ (the discriminant group for $\mathcal{P}$ as defined in Section 6C) such that there exists a fibration*

$$\mathcal{D} \longrightarrow \widehat{\mathfrak{M}} \xrightarrow{\hat{q}} \mathfrak{M}, \tag{1}$$

*where the restriction of $\hat{q}$ to each leaf in $\widehat{\mathfrak{M}}$ is a covering map of some leaf in $\mathfrak{M}$. We say that (1) is a **Molino sequence** for $\mathfrak{M}$.*

The construction of the spaces in (1) is given in Section 7. The homeomorphism type of the fibration (1) depends on the choice of a homeomorphism of $\mathfrak{M}$ with a weak solenoid $\mathcal{S}_{\mathcal{P}}$, and this in turn depends on the choice of the presentation $\mathcal{P}$ associated to $\mathfrak{M}$ and a section $V_0 \subset \mathfrak{M}$, as discussed in Section 3C. Examples show that the topological isomorphism type of $\mathcal{D}$ may depend on the choice of the section V_0, and the sequence (1) need not be an invariant of the homeomorphism type of $\mathfrak{M}$. This motivates the introduction of the following definition.

Definition 1.3. A matchbox manifold $\mathfrak{M}$ is said to be *stable* if the topological type of the sequence (1) is well defined by choosing a sufficiently small transversal V_0 to the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. A matchbox manifold $\mathfrak{M}$ is said to be *wild* if it is not stable.

In Section 7D we discuss the relation between the above definition and the notion of a stable group chain as given in Definition 7.5. Our next result concerns the existence of stable matchbox manifolds.

Proposition 1.4. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and suppose $\mathfrak{M}$ admits a transverse section V_0 with presentation $\mathcal{P}$, such that the group $\mathcal{D}$ in the Molino sequence (1) is finite. Then $\mathfrak{M}$ is stable.*

Proposition 1.4 is proved in Section 7. Theorem 10.8 shows that every separable Cantor group $\mathcal{D}$ can be realized as the discriminant of a stable weak solenoid, but we do not know of a general criterion for when a weak solenoid whose discriminant is a Cantor group must be stable.

The Molino space $\widehat{\mathfrak{M}}$ is always a homogeneous matchbox manifold. By the results in [Dyer et al. 2016], $\mathfrak{M}$ is homogeneous if and only if for some section V_0, the fibration (1) has trivial fiber $\mathcal{D}$. Each leaf of a homogeneous foliated space has trivial germinal holonomy, and thus the properties of holonomy for a matchbox manifold $\mathfrak{M}$ are closely related to its nonhomogeneity. Section 8 considers the germinal holonomy groups associated to the global holonomy action for a matchbox manifold.

Of special importance is the notion of *locally trivial germinal holonomy*, introduced by Sacksteder and Schwartz [1965], and used by Inaba [1977; 1983] in his study of Reeb stability for noncompact leaves in smooth foliations. A leaf L_x in a matchbox manifold $\mathfrak{M}$, which intersects a transversal section V_0 at a point x, has locally trivial germinal holonomy if there is an open neighborhood $U \subset V_0$ of x such that the holonomy pseudogroup acts trivially on U. A leaf with locally trivial germinal holonomy has trivial germinal holonomy, but the converse need not be true. In particular, we prove the following result in Section 8. We say that a leaf L_x has finite π_1-type if its fundamental group is finitely generated. A matchbox manifold $\mathfrak{M}$ has finite π_1-type if all leaves in the foliation $\mathcal{F}_{\mathfrak{M}}$ have finite π_1-type.

Lemma 1.5. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold with finite π_1-type. Let L_x be a leaf with trivial germinal holonomy. Then L_x has locally trivial germinal holonomy.*

The statement of Lemma 1.5 is implicit in the authors' work [Dyer et al. 2017]. The notion of locally trivial germinal holonomy and the germinal holonomy properties of equicontinuous matchbox manifolds turn out to be important in the study of topological Molino theory. Since a weak solenoid is a foliated space, by a fundamental result of Epstein, Millett and Tischler [Epstein et al. 1977] it contains leaves with trivial germinal holonomy. A *Schori solenoid* is an example of a weak solenoid, and was first constructed in [Schori 1966]. Each leaf in the foliation of a Schori solenoid is a surface of infinite genus.

Proposition 1.6. *The Schori solenoid contains leaves which have trivial germinal holonomy*, *but do not have locally trivial germinal holonomy.*

Proposition 1.6 is proved in Section 9. Proposition 1.6 shows that the condition of finite generation of the fundamental group is essential for the conclusion of Lemma 1.5. Another result, proved in Section 8, relates the existence of leaves with nontrivial holonomy with nontriviality of the fiber $\mathcal{D}$ in the Molino sequence (1).

Theorem 1.7. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. If $\mathfrak{M}$ has a leaf with nontrivial holonomy*, *then the Molino sequence* (1) *is nontrivial for any choice of section $V_0 \subset \mathfrak{M}$.*

The example in [Fokkink and Oversteegen 2002] and new examples in Section 10 show that nontrivial holonomy is not a necessary condition for (1) to be nontrivial, as one can construct nonhomogeneous equicontinuous matchbox manifolds with simply connected leaves.

Álvarez López and Moreira Galicia [2016] investigated Molino theory in the case when the closure of the pseudogroup of an equicontinuous foliated space (in the compact-open topology) satisfies the condition of strong quasianalyticity (SQA). Geometrically, this means that the pseudogroup action is *locally determined*; that is, if a holonomy map acts trivially on an open subset of its domain, then it is trivial everywhere on its domain. A natural problem is to determine which classes of equicontinuous matchbox manifolds are SQA. This question is studied in Section 9.

Note that for equicontinuous actions on Cantor sets the compact-open topology, the uniform topology and the topology of pointwise convergence coincide. The following result is proved in Section 9. The set V_n in the statement below is a partition set of $V_0 \subset \mathfrak{T}$ as defined in Proposition 3.4.

Theorem 1.8. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold which has finite π_1-type. Then there exists a transverse section V_0 such that the action of the holonomy pseudogroup on this section is SQA. In addition, if V_0 can be chosen so that the fiber $\mathcal{D}$ in the Molino sequence (1) is finite, then there exists a section $V_n \subset V_0$ such that the **closure** of the pseudogroup action on V_n is SQA as well.*

On the other hand, there are equicontinuous matchbox manifolds which do not satisfy SQA condition.

Theorem 1.9. *For every transverse section V_0 in the Schori solenoid, the holonomy pseudogroup associated to the section is not SQA.*

Theorem 1.2 proves that the Molino space exists for any matchbox manifold $\mathfrak{M}$, including those that do not admit a section with the SQA holonomy pseudogroup. Thus, for equicontinuous matchbox manifolds, our results are more general than those in [Álvarez López and Moreira Galicia 2016].

Analyzing the results of Lemma 1.5 and Theorem 1.8, one concludes that the condition of finite π_1-type, imposed on a matchbox manifold $\mathfrak{M}$, and the condition of finiteness of the fiber $\mathcal{D}$ in the Molino sequence (1), are quite strong and force the holonomy pseudogroup to possess various nice properties, such as locally trivial germinal holonomy and the SQA condition.

It is natural to ask, how diverse is the class of examples with finite fiber $\mathcal{D}$ in the Molino sequence? The authors' work [Dyer et al. 2016] constructed new examples of equicontinuous matchbox manifolds with finite fiber $\mathcal{D}$, which are weakly normal, that is, restricting to a smaller transverse section one can arrange that the Molino sequence (1) has a trivial fiber. One of these examples is also described in Example 8.6 in this paper. Rogers and Tollefson [1971c] constructed an example of a weak solenoid which turns out to be stable and have finite fiber $\mathcal{D}$, where the nontriviality of $\mathcal{D}$ is due to the presence of a leaf with nontrivial holonomy. This example illustrates Proposition 1.4 and Theorem 1.7.

The concluding section (Section 10) gives the construction of a variety of new classes of examples which illustrate the concepts and results of this work. We first give in Section 10A a reformulation of the constructions of the discriminant groups in Section 6, in terms of closed subgroups of inverse limit groups, which is analogous to a construction attributed to Lenstra in [Fokkink and Oversteegen 2002]. This alternate formulation is of strong interest in itself, as it gives a deeper understanding of the Molino spaces introduced in this work. This construction can be applied to the examples constructed by Lubotzky [1993] showing the existence of various products of torsion groups in the profinite completion of torsion-free groups, as recalled in Section 10D. We then give three applications of these results, which are included in Section 10E. The first construction is based on the conclusions of Theorem 10.4.

Theorem 1.10. *Fix an integer $n \geq 3$. Then there exists a finite index, torsion-free subgroup $G \subset \mathrm{SL}_n(\mathbb{Z})$ of the $n \times n$ integer matrices such that given any finite group F of cardinality $|F|$ which satisfies $4(|F|+2) \leq n$, there exists an irregular group chain $\mathcal{G}_F$ in G with the properties that*

(1) *the discriminant group of $\mathcal{G}_F$ is isomorphic to F;*

(2) *the group chain $\mathcal{G}_F$ is stable, with constant discriminant group isomorphic to F;*

(3) *the kernel $K(\mathcal{G}_F^{\hat{g}})$ of each conjugate $\mathcal{G}_F^{\hat{g}}$ of this group chain is trivial.*

The terminology used in Theorem 1.10 will be explained in later sections, where we will show that given such a group chain, one can construct matchbox manifolds with the following properties:

Corollary 1.11. *Let F be a finite group. Then there exists a nonhomogeneous matchbox manifold $\mathfrak{M}$ such that every leaf of $\mathcal{F}_{\mathfrak{M}}$ has trivial germinal holonomy, and for any sufficiently small transverse section in $\mathfrak{M}$, its Molino sequence is nontrivial with fiber group $\mathcal{D} \cong F$.*

Note that it follows by Theorem 1.8 that for the examples constructed in the proof of Corollary 1.11, there is a section $V \subset \mathfrak{M}$ such that the closure of the pseudogroup action on V satisfies the SQA condition of Álvarez López and Moreira Galicia [2016].

The next two constructions are based on the conclusions of Theorem 10.5, due to Lubotzky. Again, the terminology used in the statements will be explained in later sections.

Theorem 1.12. *There exists a finite index, torsion-free finitely generated group G such that given any separable profinite group K, there exists an irregular group chain $\mathcal{G}_K$ in G such that*

(1) *the discriminant group of $\mathcal{G}_K$ is isomorphic to K;*

(2) *the group chain $\mathcal{G}_F$ is stable, with constant discriminant group isomorphic to K.*

Corollary 1.13. *Let K be a Cantor group. Then there exists a nonhomogeneous matchbox manifold $\mathfrak{M}$ such that, for any sufficiently small transverse section in $\mathfrak{M}$, its Molino sequence is nontrivial with fiber group $\mathcal{D} \cong K$.*

Finally, Theorem 10.10 gives the first examples of equicontinuous matchbox manifolds which are not virtually regular. The *virtually regular* condition was introduced in [Dyer et al. 2017], and is defined in Definition 10.9. As the terminology suggests, this notion is related to the homogeneity properties of finite-to-one coverings of a matchbox manifold $\mathfrak{M}$.

The concluding section (Section 10F) lists some open problems.

2. Equicontinuous Cantor foliated spaces

In this section, we recall background concepts about foliated spaces, and introduce the group chains associated to their equicontinuous Cantor holonomy actions.

2A. ***Equicontinuous Cantor foliated spaces.*** Recall that an *n-dimensional matchbox manifold* $\mathfrak{M}$ is a compact connected metrizable topological space such that every point $x \in \mathfrak{M}$ has an open neighborhood $U \subset \mathfrak{M}$ such that there is a homeomorphism

$$\varphi_x : \overline{U}_x \to [-1, 1]^n \times \mathfrak{T}_x, \tag{2}$$

where $\mathfrak{T}_x$ is a totally disconnected space. The homeomorphism φ_x is called a *local foliation chart*, and the space $\mathfrak{T}_x$ is called a *local transverse model*. As usual in foliation theory, one can choose a finite atlas $\mathcal{U} = \{(\varphi_i, U_i)\}_{1 \le i \le \nu}$ of local charts such that the intersections of the path-connected components in $U_x \cap U_y$ are connected and simply connected, and the images $\mathcal{T}_i = \varphi_i^{-1}(\{0\} \times \mathfrak{T}_i)$ are disjoint. The leaves of the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$ are defined to be the path-connected components of $\mathfrak{M}$, which are then a union of the path-connected components (the plaques) in the open sets U_i. A matchbox manifold is (*topologically*) *minimal* if each leaf $L \subset \mathfrak{M}$ is dense in $\mathfrak{M}$.

We require the matchbox manifold $\mathfrak{M}$ to be *smooth*; that is, the transition maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i^{-1}(\overline{U}_i \cap \overline{U}_j) \to \varphi_j(\overline{U}_i \cap \overline{U}_j)$$

are C^∞-maps in the first coordinate $x \in [-1, 1]^n$, and the restrictions to plaques depend continuously on $y \in \mathfrak{T}_i$, in the C^∞-topology on leaves, for $1 \le i, j \le \nu$.

Let $\mathrm{pr}_2 : [-1, 1]^n \times \mathfrak{T}_i \to \mathfrak{T}_i$ be the projection onto the second factor. Then $\pi_i = \mathrm{pr}_2 \circ \varphi_i : \overline{U}_i \to \mathfrak{T}_i$ for $1 \le i \le \nu$ are the local defining maps for the foliation $\mathcal{F}_{\mathfrak{M}}$. Set $\mathfrak{T}_{i,j} = \pi_i(U_i \cap U_j)$ for $1 \le i, j \le \nu$. Since the path-connected components of the charts are either disjoint or have a connected intersection, there is a well-defined change-of-coordinates homeomorphism

$$h_{i,j} = \pi_j \circ \pi_i^{-1} : \mathfrak{T}_{i,j} \to \mathfrak{T}_{j,i} \tag{3}$$

with domain $\mathfrak{T}_{i,j}$ and range $\mathfrak{T}_{j,i}$. Let $\mathcal{G}^1_{\mathcal{F}} = \{(h_{i,j}, \mathfrak{T}_{i,j}) \mid 1 \le i, j \le \nu\}$. Set $\mathfrak{T} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_\nu$. Then the collection of maps $\mathcal{G}^1_{\mathcal{F}}$ generates the *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ acting on the transverse space $\mathfrak{T}$. The construction and properties of $\mathcal{G}_{\mathcal{F}}$ are described in full detail in [Clark and Hurder 2013, Section 3].

For the study of the dynamical properties of $\mathcal{F}_{\mathfrak{M}}$, it is useful to introduce also the collection of maps $\mathcal{G}^*_{\mathcal{F}} \subset \mathcal{G}_{\mathcal{F}}$, defined as follows. Let $\mathcal{G}_0 \subset \mathcal{G}_{\mathcal{F}}$ denote the collection consisting of all possible compositions of homeomorphisms in $\mathcal{G}^1_{\mathcal{F}}$. Then $\mathcal{G}^*_{\mathcal{F}}$ consists of all possible restrictions of homeomorphisms in $\mathcal{G}_0$ to open subsets of their domains. The collection of maps $\mathcal{G}^*_{\mathcal{F}}$ is closed under the operations of compositions, taking inverses, and restrictions to open sets, and is called a *pseudo⋆group* in [Álvarez López and Moreira Galicia 2016; Matsumoto 2010], while $\mathcal{G}^*_{\mathcal{F}}$ is called a *localization* of $\mathcal{G}_0$ in [Álvarez López and Moreira Galicia 2016].

Remark 2.1. The standard definition of a pseudogroup [Candel and Conlon 2000] requires the pseudogroup to be closed under the operations of composition, taking inverses, restriction to open subsets, and combination of maps. A combination of two local homeomorphisms h_1 and h_2, with possibly disjoint domains $D(h_1)$ and $D(h_2)$ and with disjoint ranges, is a homeomorphism h defined on $D(h_1) \cup D(h_2)$ where $h|D(h_1) = h_1$ and $h|D(h_2) = h_2$. However, allowing such arbitrary gluings of maps is unnatural. For example, a composition $h_{j,k} \circ h_{i,j}$ can be associated with the existence of a leafwise path $\gamma_x : [0, 1] \to L_x \in \mathfrak{M}$ with $\gamma_x(0) \in U_i$ and $\gamma_y(1) \in U_k$, where L_x is a leaf such that $\pi_i(x) \in D(h_{j,k} \circ h_{i,j})$. If $\pi_i(y) \in D(h_{j,k} \circ h_{i,j})$, then the path γ_x can be lifted to a nearby leaf L_y to a "parallel" path γ_y with $\gamma_y(0) \in U_i$ and $\gamma_y(1) \in U_k$. Thus a holonomy transformation $h_{j,k} \circ h_{i,j}$ has a geometric meaning as the transverse transport in leaves along a leafwise path. Therefore, in the definitions of $\mathcal{G}_0$ and $\mathcal{G}^*_{\mathcal{F}}$ (and of a pseudo⋆group in [Matsumoto 2010]), one does not allow combinations of local homeomorphisms, unless such homeomorphisms can be obtained by restrictions to open subsets of maximal domains of elements in $\mathcal{G}_0$.

Let $d_{\mathfrak{M}}$ be a metric on $\mathfrak{M}$, and denote by $d_{\mathcal{T}_i}$ the restriction of $d_{\mathfrak{M}}$ to the embedded image $\mathcal{T}_i$ of the transversal $\mathfrak{T}_i$, $1 \le i \le \nu$. For each $1 \le i \le \nu$, consider the pullback $d_{\mathfrak{T}_i}$ of $d_{\mathcal{T}_i}$ along the embedding. Then define a metric $d_{\mathfrak{T}}$ on $\mathfrak{T}$ by the formula

$$d_{\mathfrak{T}}(x, y) = \begin{cases} d_{\mathfrak{T}_i}(x, y) & \text{if } x, y \in \mathfrak{T}_i \text{ for some } i, \\ \infty & \text{otherwise.} \end{cases}$$

For a homeomorphism $\gamma \in \mathcal{G}^*_{\mathcal{F}}$, denote by $D(\gamma)$ and $R(\gamma)$ the domain and the range of γ, respectively.

Definition 2.2. The action of the pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on the transversal $\mathfrak{T}$ is *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\gamma \in \mathcal{G}^*_{\mathcal{F}}$, if $x, x' \in D(\gamma)$ and $d_{\mathfrak{T}}(x, x') < \delta$, then $d_{\mathfrak{T}}(\gamma(x), \gamma(x')) < \varepsilon$.

The following notion is used in the statement of various results in this work.

Definition 2.3. A path-connected topological space X is said to have finite π_1-type if the fundamental group $\pi_1(X, x)$ is a finitely generated group for the choice of some basepoint $x \in X$. A matchbox manifold $\mathfrak{M}$ is said to have finite π_1-type if each leaf $L \subset \mathfrak{M}$ is a space of finite π_1-type.

2B. ***Suspensions.*** There is a well-known construction which yields a foliated space from a group action, called the *suspension construction*, as discussed in [Candel and Conlon 2000, Chapter 3], for example. We state this construction in the restricted context which we use in this work.

Let X be a Cantor space and H a finitely generated group, and assume there is given an action $\varphi : H \to \mathrm{Homeo}(X)$. Suppose that H admits a generating set $\{g_1, \dots, g_k\}$; then there is a homomorphism $\alpha_k : \mathbb{Z} * \cdots * \mathbb{Z} \twoheadrightarrow H$ of the free group on k generators onto H, given by mapping generators to generators. Of course, the map α_k will have nontrivial kernel, unless H happens to be a free group. Next, let Σ_k be a compact surface without boundary of genus k. Then for a choice of basepoint $x_0 \in \Sigma_k$ set $G = \pi_1(\Sigma_k, x_0)$. Then there is a homomorphism $\beta_k : G \to \mathbb{Z} * \cdots * \mathbb{Z}$ onto the free group of k generators. Denote the composition of these maps by $\Phi = \varphi \circ \alpha_k \circ \beta_k$ to obtain the homomorphism $\Phi : G = \pi_1(\Sigma_k, x_0) \to \mathbb{Z} * \cdots * \mathbb{Z} \to H \to \mathrm{Homeo}(X)$.

Now, let $\widetilde{\Sigma}_k$ denote the universal covering space of Σ_k, equipped with the right action of G by covering transformations. Form the product space $\widetilde{\Sigma}_k \times X$ which has a foliation $\widetilde{\mathcal{F}}$ whose leaves are the slices $\widetilde{\Sigma}_k \times \{x\}$ for each $x \in X$. Define a right action of G on $\widetilde{\Sigma}_k \times X$, which for $g \in G$ is given by $(y, x) \cdot g = (y \cdot g, \Phi(g^{-1})(x))$. For each g, this action preserves the foliation $\widetilde{\mathcal{F}}$, so we obtain a foliation $\mathcal{F}_{\mathfrak{M}}$ on the quotient space $\mathfrak{M} = (\widetilde{\Sigma}_k \times X)/G$. Note that all leaves of $\mathcal{F}_{\mathfrak{M}}$ are surfaces, which are in general noncompact.

Note that $\mathfrak{M}$ is a foliated Cantor bundle over Σ_k, and the holonomy of this bundle $\pi : \mathfrak{M} \to \Sigma_k$ acting on the fiber $V_0 = \pi^{-1}(x_0)$ is canonically identified with the action $\Phi : G \to \mathrm{Homeo}(X)$. Consequently, if the action Φ is minimal in the sense of topological dynamics [Auslander 1988], then the foliation $\mathcal{F}_{\mathfrak{M}}$ is minimal. If the action Φ is equicontinuous in the sense of topological dynamics [Auslander 1988], then $\mathcal{F}_{\mathfrak{M}}$ is an equicontinuous foliation in the sense of Definition 2.2.

There is a variation of the above construction, where we assume that G is a *finitely presented* group, and there is given a homomorphism $\Phi : G \to \mathrm{Homeo}(X)$. In this case, it is a well-known folklore result (for example, see [Massey 1991]) that there exists a closed connected 4-manifold B such that for a choice of basepoint $b_0 \in B$, $\pi_1(B, b_0)$ is homeomorphic to G. Then the suspension construction can be applied to the homomorphism $\Phi : \pi_1(B, b_0) \to \mathrm{Homeo}(X)$, where we replace Σ_k above with B, and the space $\widetilde{\Sigma}_k$ with the universal covering $\widetilde{B}$ of B. The resulting foliated space $\mathfrak{M}$ will have holonomy given by the map Φ.

In summary, the suspension construction translates results about equicontinuous minimal Cantor actions to results about equicontinuous matchbox manifolds.

3. Weak solenoids

In this section, we first recall the construction procedure for (*weak*) *solenoids*, and describe some of their properties. In Section 3B, we discuss the construction from [Clark and Hurder 2013] which associates a group chain to an equicontinuous matchbox manifold, which leads to a more precise statement of Theorem 1.1. Then in Section 3C, we make some observations about the conclusion of Theorem 1.1 which are important when considering the definition of the Molino space for matchbox manifolds.

3A. *Weak solenoids.* Let $n \geq 1$. Then for each $\ell \geq 0$, let M_ℓ be a compact connected simplicial complex of dimension n. A *presentation* is a collection $\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$, where each map $p_{\ell+1}$ is a proper surjective map of simplicial complexes with discrete fibers, which is called a *bonding* map. For $\ell \geq 0$ and $x \in M_\ell$, the preimage $\{p_{\ell+1}^{-1}(x)\} \subset M_{\ell+1}$ is compact and discrete, so the cardinality $\#\{p_{\ell+1}^{-1}(x)\}$ is finite. For a presentation $\mathcal{P}$ defined in this generality, the cardinality of the fibers of the maps $p_{\ell+1}$ need not be constant in either ℓ or x.

Associated to a presentation $\mathcal{P}$ is an inverse limit space,

$$
\begin{aligned}
(4) \qquad \mathcal{S}_{\mathcal{P}} &\equiv \varprojlim \{p_{\ell+1} : M_{\ell+1} \to M_\ell\} \\
&= \{(x_0, x_1, \dots) \in \mathcal{S}_{\mathcal{P}} \mid p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0\} \subset \prod_{\ell \geq 0} M_\ell.
\end{aligned}
$$

The set $\mathcal{S}_{\mathcal{P}}$ is given the relative topology, induced from the product (Tychonoff) topology, so that $\mathcal{S}_{\mathcal{P}}$ is itself compact and connected.

Definition 3.1. The inverse limit space $\mathcal{S}_{\mathcal{P}}$ in (4) is called a (*weak*) *solenoid* if for each $\ell \geq 0$ the space M_ℓ is a compact connected manifold without boundary, and $p_{\ell+1}$ is a proper covering map of degree $m_{\ell+1} > 1$.

Weak solenoids are a generalization of 1-dimensional (Vietoris) solenoids, described in Example 3.2 below. Weak solenoids were originally considered by McCord [1965], Rogers and Tollefson [1971a; 1971c] and Schori [1966], and later by Fokkink and Oversteegen [2002].

Example 3.2. Let $M_\ell = \mathbb{S}^1$ for each $\ell \geq 0$, and let the map $p_{\ell+1}$ be a proper covering map of degree $m_{\ell+1} > 1$ for $\ell \geq 0$. Then $\mathcal{S}_{\mathcal{P}}$ is an example of a classic 1-dimensional solenoid, discovered independently by van Dantzig [1930] and Vietoris [1927]. If $m_{\ell+1} = 2$ for $\ell \geq 0$, then $\mathcal{S}_{\mathcal{P}}$ is called the *dyadic* solenoid.

Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid as in Definition 3.1. For each $\ell \geq 1$, the composition

$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell : M_\ell \to M_0 \tag{5}$$

is a finite-to-one covering map of the base manifold M_0. For each $\ell \geq 0$, projection onto the ℓ-th factor in the product $\prod_{\ell \geq 0} M_\ell$ in (4) yields a fibration map denoted by $\Pi_\ell : \mathcal{S}_{\mathcal{P}} \to M_\ell$. For $\ell = 0$ this yields the fibration $\Pi_0 : \mathcal{S}_{\mathcal{P}} \to M_0$, and for $\ell \geq 1$ we have

$$\Pi_0 = q_\ell \circ \Pi_\ell : \mathcal{S}_{\mathcal{P}} \to M_0. \tag{6}$$

A choice of basepoint $x_0 \in M_0$ fixes a fiber $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$, which is a Cantor set by the assumption that the fibers of each map $p_{\ell+1}$ have cardinality at least 2. McCord [1965] showed that (6) is a fiber bundle over M_0 with a Cantor set fiber, and the solenoid $\mathcal{S}_{\mathcal{P}}$ has a local product structure as in (2). The path-connected components of $\mathcal{S}_{\mathcal{P}}$ thus define a foliation denoted by $\mathcal{F}_{\mathcal{P}}$. We then have:

Proposition 3.3. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid whose base space M_0 is a compact manifold of dimension $n \geq 1$. Then $\mathcal{S}_{\mathcal{P}}$ is a minimal matchbox manifold of dimension n with foliation $\mathcal{F}_{\mathcal{P}}$.*

Denote by $G_0 = \pi_1(M_0, x_0)$ the fundamental group of M_0 with basepoint x_0, and choose a point $x \in \mathfrak{X}_0$ in the fiber over x_0. This defines basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$ for $\ell \geq 1$.

Let $y \in \mathfrak{X}_0$ be another point, set $y_\ell = \Pi_\ell(y) \in M_\ell$, and note that $y_0 = x_0$ by construction. We will interchangeably write $y = (y_\ell)$ to denote a point in $\mathfrak{X}_0$ or $\mathcal{S}_{\mathcal{P}}$. Let L_y denote the leaf of $\mathcal{F}_{\mathcal{P}}$ containing y. Then the restriction $\Pi_0 | L_y : L_y \to M_0$ of the bundle projection to each path-connected component L_y is a covering map. For $g = [\gamma_0] \in G_0$, let $\gamma_\ell : [0, 1] \to M_\ell$ be a lift of γ_0 with the starting point $\gamma_\ell(0) = y_\ell$. Define a homeomorphism $h_g : \mathfrak{X}_0 \to \mathfrak{X}_0$ by $h_g(y_\ell) = (\gamma_\ell(1))$. Thus there is a representation

$$\Phi_0 : G_0 \to \mathrm{Homeo}(\mathfrak{X}_0) : \gamma \to h_g, \tag{7}$$

called the *global holonomy map* of the solenoid $\mathcal{S}_{\mathcal{P}}$.

3B. ***Dynamical partitions.*** It was shown in [Clark and Hurder 2013, Theorem 4.12] that an equicontinuous matchbox manifold $\mathfrak{M}$ is minimal, that is, every leaf is dense in $\mathfrak{M}$. This result generalizes to pseudogroups by a corresponding result of Auslander [1988] for equicontinuous group actions. It follows that for any clopen subset $V_0 \subset \mathfrak{T}$, the restricted pseudo⋆group $\mathcal{G}^*_{V_0} = \mathcal{G}^*_{\mathcal{F}} | V_0$ is *return equivalent* to the pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on $\mathfrak{T}$, where return equivalence is defined and studied in [Clark et al. 2013a, Section 4]. Thus, for the study of the dynamical properties of $\mathcal{F}_{\mathfrak{M}}$ one can restrict to the study of $\mathcal{G}^*_{V_0}$. The following result is based on the constructions in [Clark and Hurder 2013].

Proposition 3.4. *Let $\mathfrak{M}$ be a matchbox manifold with totally disconnected transversal $\mathfrak{T}$ and equicontinuous holonomy pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ on $\mathfrak{T}$, let $x \in \mathfrak{T}$ be a point, and let $W \subset \mathfrak{T}$ be a clopen (closed and open) neighborhood of x. Then there exists a clopen subset $x \in V_0 \subset W$ and a descending chain of clopen sets $V_0 \supset V_1 \supset \cdots$ of $\mathfrak{T}$ with $\{x\} = \bigcap_\ell V_\ell$ such that:*

(1) *The restriction $\mathcal{G}^*_{\mathcal{F}}|V_0$ is generated by a group G_0 of transformations of V_0.*

(2) *For each $\ell \geq 1$ the collection $\mathcal{Q}_\ell = \{g \cdot V_\ell\}_{g \in G_0}$ is a finite partition of V_0 into clopen sets.*

(3) *We have $\operatorname{diam}(g \cdot V_\ell) < 2^{-\ell}$ for all $g \in G_0$ and all $\ell \geq 0$.*

(4) *The collection of elements which fix V_ℓ, that is,*

$$G^x_\ell = \{g \in G_0 \mid g \cdot V_\ell = V_\ell\},$$

is a subgroup of finite index in G_0. More precisely, $|G_0 : G^x_\ell| = \operatorname{card} \mathcal{Q}_\ell$.

There are many choices involved in the construction of the partitions $\mathcal{Q}_\ell$ and consequently the stabilizer groups G^x_ℓ:

(1) The choice of a transverse section $V_0 \subset \mathfrak{T}$, which results in the choice of the group G_0.

(2) The choice of a basepoint $x \in V_0$.

(3) Given V_0, x and G_0, there is freedom to choose clopen sets $V_1 \supset V_2 \supset \cdots$, which results in the choice of the sequence of groups $G_0 = G^x_0 \supset G^x_1 \supset G^x_2 \supset \cdots$.

Thus, the algebraic and geometric data encoded by these choices must be considered up to suitable notions of equivalence, which will be introduced in Section 4A.

3C. *Homeomorphisms.* Let $\mathfrak{M}$ be a matchbox manifold with totally disconnected transversal $\mathfrak{T}$ and equicontinuous holonomy pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$ acting on $\mathfrak{T}$, let $x \in \mathfrak{T}$ be a point, and let $\{V_{\ell+1} \subset V_\ell \mid \ell \geq 0\}$ be a descending chain of clopen subsets of $\mathfrak{T}$ with $x \in V_\ell$ for all $\ell \geq 0$, as introduced in Proposition 3.4, where G_0 is a group of transformations of V_0, and G_ℓ denotes the stabilizer subgroup of G_0 of the set V_ℓ.

The basic idea of the proof of Theorem 1.1 is that if we choose the section $V_0 \subset \mathfrak{M}$ appropriately and it is sufficiently small, then there is a compact manifold M_0 and a fibration $\Pi'_0 : \mathfrak{M} \to M_0$ for which the inverse image $(\Pi'_0)^{-1}(x_0)$ equals V_0, where $x_0 = \Pi'_0(x)$. Moreover, the restrictions of the map Π'_0 to the leaves of $\mathcal{F}_{\mathfrak{M}}$ are coverings of M_0. The definition of the map Π'_0 requires the highly technical results of [Clark et al. 2013b] to define a transverse Cantor foliation $\mathcal{H}_0$ to $\mathcal{F}_{\mathfrak{M}}$, so that the quotient space $M_0 = \mathfrak{M}/\mathcal{H}_0$ is a compact manifold, and then Π'_0 is the projection along the leaves of the transverse foliation $\mathcal{H}_0$, or better said the equivalence classes defined by the leaves of $\mathcal{H}_0$. Then V_0 is the $\mathcal{H}_0$-equivalence class of the point $x \in V_0 \subset \mathfrak{M}$.

Let $V_\ell \subset V_0$ be the clopen set in Proposition 3.4 and $G_\ell^x = \{g \in G_0 \mid g \cdot V_\ell = V_\ell\}$ the isotropy subgroup of V_ℓ. Then there is a Cantor subfoliation $\mathcal{H}_\ell$ of $\mathcal{H}_0$ such that V_ℓ is the $\mathcal{H}_\ell$-equivalence class of x. Moreover, there is a quotient map $\Pi'_\ell : \mathfrak{M} \to \mathfrak{M}/\mathcal{H}_\ell \equiv M_\ell$, where is M_ℓ is identified with the covering of M_0 associated to the subgroup $G_\ell \subset G_0 = \pi_1(M_0, x_0)$. Note that the fiber $(\Pi'_\ell)^{-1}(x_0)$ equals V_ℓ and the monodromy action of G_0 on V_0 partitions V_0 into the translates of V_ℓ. There is then a quotient covering map $q_\ell : M_\ell \to M_0$, and as in (6), we have

$$\Pi'_0 = q_\ell \circ \Pi'_\ell : \mathfrak{M} \to M_0. \tag{8}$$

For each $\ell \geq 0$ let $p_{\ell+1} : M_{\ell+1} \to M_\ell$ be the quotient map defined by expanding the equivalence classes of $\mathfrak{M}$ defined by $\mathcal{H}_{\ell+1}$ to the equivalence classes defined by $\mathcal{H}_\ell$. Then we obtain a collection of covering maps $\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$ which defines a weak solenoid $\mathcal{S}_\mathcal{P}$. As the diameters of the clopen partition sets V_ℓ tend to 0 as ℓ increases, it is then standard that the collection of maps $\{\Pi'_\ell : \mathfrak{M} \to M_\ell \mid \ell \geq 0\}$ induces a foliated homeomorphism $\Pi_0^* : \mathfrak{M} \to \mathcal{S}_\mathcal{P}$.

In later sections, we will also consider the presentations $\mathcal{P}_n$ obtained by truncating the initial n terms in the presentation $\mathcal{P}$. That is, for $n \geq 0$ we have

$$\mathcal{P}_n = \{p'_{\ell+1} : M'_{\ell+1} \to M'_\ell \mid \ell \geq 0\}, \quad \text{where } M'_\ell = M_{\ell+n} \text{ and } p'_{\ell+1} = p_{\ell+n+1}. \tag{9}$$

It is a basic property of inverse limit spaces [McCord 1965; Rogers 1970] that for $n \geq 1$ and $m \geq 0$, there is a homeomorphism $\sigma_n : \mathcal{S}_{\mathcal{P}_{m+n}} \cong \mathcal{S}_{\mathcal{P}_n}$, where the homeomorphism is given by the "shift in coordinates" map σ_n in the inverse sequences defining these spaces. Also, by the same reasoning as above, there is a foliated homeomorphism $\Pi_n^* : \mathfrak{M} \to \mathcal{S}_{\mathcal{P}_n}$ and we have a commutative diagram of fibrations:

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "="] \arrow[d, "\Pi^*_{n+m}"'] & \mathfrak{M} \arrow[d, "\Pi^*_m"] \\ \mathcal{S}_{\mathcal{P}_{n+m}} \arrow[r, "\sigma_n"] & \mathcal{S}_{\mathcal{P}_m} \end{tikzcd} \tag{10}$$

Note that if the presentation $\mathcal{P}$ is constructed using the holonomy of $\mathcal{F}_\mathfrak{M}$ acting on the transversal $V_0 \subset \mathfrak{M}$, then for $n > 0$ and $m \geq 0$, the map $\sigma_n : \mathcal{S}_{\mathcal{P}_{n+m}} \to \mathcal{S}_{\mathcal{P}_m}$ satisfies $\sigma_n(V_{m+n}) \subset V_n$. That is, the induced map on $\mathfrak{M}$ sends the transversal $(\Pi^*_{m+n})^{-1}(V_{m+n}) \subset \mathfrak{M}$ into the transversal $(\Pi^*_n)^{-1}(V_n) \subset \mathfrak{M}$. On the other hand, given a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ there is no reason it should map the transversal V_0 into itself. In particular, the induced map

$$(\Pi^*_n) \circ h \circ (\Pi^*_{m+n})^{-1} : \mathcal{S}_{\mathcal{P}_{m+n}} \to \mathcal{S}_{\mathcal{P}_n} \tag{11}$$

on weak solenoids need not be fiber preserving. On the other hand, as discussed in [Fokkink and Oversteegen 2002], there is always a map $h' : \mathfrak{M} \to \mathfrak{M}$ which is homotopic to h such that the induced map as in (11) maps a clopen subset of V_{m+n} into a clopen subset of V_n. Thus, by allowing sufficiently large values of n and m and choice of basepoints in the range and domain, we can always ensure that a given homeomorphism of $\mathfrak{M}$ induces a fiber-preserving map between the weak solenoids $\mathcal{S}_{\mathcal{P}_{m+n}}$ and $\mathcal{S}_{\mathcal{P}_n}$.

4. Group chain models

Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid defined by a presentation $\mathcal{P}$, with basepoint $x \in \mathfrak{X}_0 \equiv \Pi_0^{-1}(x_0) \subset \mathcal{S}_{\mathcal{P}}$. For $G_0 = \pi_1(M_0, x_0)$, let $\Phi_0 : G_0 \to \mathrm{Homeo}(\mathfrak{X}_0)$ be the holonomy action in (7).

The following "combinatorial model" for the action (7) allows for a deeper analysis of the relation between the action Φ_0 and the algebraic structure of G_0. For each $\ell \geq 1$, recall that

$$G_\ell^x = \mathrm{image}\{(q_\ell)_\# : \pi_1(M_\ell, x_\ell) \to G_0\} \tag{12}$$

denotes the image of the induced map $(q_\ell)_\#$ on fundamental groups. In this way, associated to the presentation $\mathcal{P}$ and basepoint $x \in \mathfrak{X}_0$, we obtain a descending chain of subgroups of finite index

$$\mathcal{G}^x : G_0 \supset G_1^x \supset G_2^x \supset \cdots \supset G_\ell^x \supset \cdots . \tag{13}$$

Each quotient $X_\ell^x = G_0/G_\ell^x$ is a finite set equipped with a left G_0-action, and there are surjections $X_{\ell+1}^x \to X_\ell^x$ which commute with the action of G_0. The inverse limit

$$X_\infty^x = \varprojlim\{p_{\ell+1} : X_{\ell+1}^x \to X_\ell^x\} = \{(eG_0, g_1G_1^x, \dots) \mid g_\ell G_\ell^x = g_{\ell+1}G_\ell^x\} \subset \prod_{\ell \geq 0} X_\ell^x \tag{14}$$

is then a totally disconnected compact perfect set, so is a Cantor set. The fundamental group G_0 acts on the left on X_∞^x via the coordinatewise multiplication on the product in (14). We denote this Cantor action by $(X_\infty^x, G_0, \Phi_x)$.

Lemma 4.1. *There is a homeomorphism $\tau_x : \mathfrak{X}_0 \to X_\infty^x$ equivariant with respect to the action (7) of G_0 on $\mathfrak{X}_0$ and Φ_x on X_∞; that is, $\tau_x \circ h_g(y) = \Phi_x(g) \circ \tau_x(y)$ for all $y \in \mathfrak{X}_0$.*

In particular, this allows us to conclude that the action Φ_0 of G_0 on the fiber of the solenoid $\mathcal{S}_{\mathcal{P}}$ is minimal. Indeed, the left action of G_0 on each quotient space X_ℓ^x is transitive, so the orbits are dense in the product topology on X_∞^x.

Remark 4.2. The group chain (14) and the homeomorphism in Lemma 4.1 depend on the choice of a point $x \in \mathfrak{X}_0$. For a different basepoint $y \in \mathfrak{X}_0$ in the fiber over x_0, let $\tau_x(y) = (g_i G_\ell^x) \in X_\infty^x$; then the group chain $\mathcal{G}^y$ associated to y is given by a chain of conjugate subgroups in G_0, where $G^y_\ell = g_\ell G^x_\ell g_\ell^{-1}$ for $\ell \geq 0$. The group chains $\mathcal{G}^y$ and $\mathcal{G}^x$ are said to be *conjugate chains*. The composition $\tau_y \circ \tau_x^{-1} : X^x_\infty \to X^y_\infty$ gives a topological conjugacy between the minimal Cantor actions $(X^x_\infty, G_0, \Phi_x)$ and $(X^y_\infty, G_0, \Phi_y)$. The map $\tau_x : \mathfrak{X}_0 \to X^x_\infty$ can be viewed as "coordinates" on the inverse limit space $\mathfrak{X}_0$, and the composition $\tau_y \circ \tau_x^{-1}$ as a "change of coordinates". Properties of the minimal Cantor action $(X^x_\infty, G_0, \Phi_x)$ which are independent of the choice of these coordinates are thus properties of the topological type of $\mathcal{S}_{\mathcal{P}}$.

4A. ***Equivalence of group chains.*** Fokkink and Oversteegen [2002] and the authors [Dyer et al. 2016] studied equivalences of group chains associated to a given equicontinuous minimal Cantor system (V_0, G_0, Φ). We now briefly recall the key results.

Denote by $\mathfrak{G}$ the collection of all possible subgroup chains in G_0. Then there are two equivalence relations on $\mathfrak{G}$. The first was introduced by Rogers and Tollefson [1971b]:

Definition 4.3. In a finitely generated group G_0, two group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ with $G_0 = H_0$ are *equivalent* if and only if there is a group chain $\{K_\ell\}_{\ell\geq 0}$ and infinite subsequences $\{G_{\ell_k}\}_{k\geq 0}$ and $\{H_{j_k}\}_{k\geq 0}$ such that $K_{2k} = G_{\ell_k}$ and $K_{2k+1} = H_{j_k}$ for $k \geq 0$.

The next definition was introduced by Fokkink and Oversteegen [2002].

Definition 4.4. Two group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ in $\mathfrak{G}$ are *conjugate equivalent* if and only if there exists a sequence $(g_\ell) \subset G_0$ for which the compatibility condition $g_\ell G_\ell = g_{\ell+1} G_\ell$ for all $\ell \geq 0$ is satisfied, and such that the group chains $\{g_\ell G_\ell g_\ell^{-1}\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ are equivalent.

The dynamical meaning of the equivalences in Definitions 4.3 and 4.4 is given by the following theorem, which follows from results in [Fokkink and Oversteegen 2002]; see also [Dyer et al. 2016].

Theorem 4.5. *Let $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ be group chains in G_0, with $H_0 = G_0$, and let*

$$G_\infty = \varprojlim \{G_0/G_{\ell+1} \to G_0/G_\ell\},$$
$$H_\infty = \varprojlim \{G_0/H_{\ell+1} \to G_0/H_\ell\}.$$

Then

(1) *the group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ are **equivalent** if and only if there exists a homeomorphism $\tau : G_\infty \to H_\infty$ equivariant with respect to the G_0-actions on G_∞ and H_∞, and such that $\varphi(eG_\ell) = (eH_\ell)$;*

(2) *the group chains $\{G_\ell\}_{\ell\geq 0}$ and $\{H_\ell\}_{\ell\geq 0}$ are **conjugate equivalent** if and only if there exists a homeomorphism $\tau : G_\infty \to H_\infty$ equivariant with respect to the G_0-actions on G_∞ and H_∞.*

That is, an equivalence of two group chains corresponds to the existence of a *basepoint-preserving* equivariant homeomorphism between their inverse limit systems, while a conjugate equivalence of two group chains corresponds to the existence of a equivariant conjugacy between their inverse limit systems, which need not preserve the basepoint.

Let $\mathfrak{G}(\Phi_0)$ denote the class of group chains in G_0 which are *conjugate equivalent* to the group chain $\{G^x_\ell\}_{\ell\geq 0}$ with basepoint x. The following result gives a geometric interpretation of the conjugate equivalence class $\mathfrak{G}(\Phi_0)$ of a group chain $\{G^x_\ell\}_{\ell\geq 0}$.

Proposition 4.6. *Given an equicontinuous minimal Cantor action* (V_0, G_0, Φ_0), *let* $\{G^x_\ell\}_{\ell\geq 0}$ *be a group chain with partitions* $\{\mathcal{Q}_\ell\}_{\ell\geq 0}$ *and basepoint* x, *as in Proposition 3.4. Then a group chain* $\{H_\ell\}_{\ell\geq 0}$ *is in* $\mathfrak{G}(\Phi_0)$ *if and only if there exists a collection of* G_0*-invariant partitions* $\mathcal{S}_\ell = \{g\cdot U_\ell\}_{g\in G_0}$ *of* V_0, *where* $U_\ell \subset V_0$ *is a clopen set, and* $\bigcap_\ell U_\ell = \{y\} \subset V_0$, *such that* $H_\ell = H^y_\ell$ *is the isotropy group at* U_ℓ *of the action of* G_0 *on the partition* $\mathcal{S}_\ell$, *for all* $\ell \geq 0$.

4B. ***Kernels of group chains.*** The following notion is important for the study of group chains.

Definition 4.7. The *kernel* of a group chain $\mathcal{G} = \{G_\ell\}_{\ell\geq 0}$ is the subgroup of G_0 given by

$$K(\mathcal{G}) = \bigcap_{\ell\geq 0} G_\ell. \tag{15}$$

The following property is immediate from the definitions.

Lemma 4.8. *Suppose that the group chains* $\mathcal{G} = \{G_\ell\}_{\ell\geq 0}$ *and* $\mathcal{H} = \{H_\ell\}_{\ell\geq 0}$ *with* $G_0 = H_0$ *are equivalent. Then* $K(\mathcal{G}) = K(\mathcal{H}) \subset G_0$.

If the chains $\mathcal{G}$ and $\mathcal{H}$ are only conjugate equivalent, then the kernels need not be equal.

An infinite group G_0 which admits a group chain $\mathcal{C} = \{C_\ell\}_{\ell\geq 0}$ where each C_ℓ is a *normal* subgroup of G_0, and such that $\bigcap C_\ell = \{e\}$, where e denotes the identity element in G_0, is said to be *residually finite*. It is an elementary fact that given any group chain $\mathcal{G} = \{G_\ell\}_{\ell\geq 0}$ in G_0, there is an associated core group chain $\mathcal{G}$ for which $C_\ell \subset G_\ell$ with C_ℓ normal in G_0 for all $\ell > 0$, as will be discussed in Section 6B below. Thus, if the group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell\geq 0}$ introduced above has $K(\mathcal{G}^x)$ the trivial group, then G_0 must be a residually finite group. On the other hand, there are many classes of groups which are not residually finite, and thus any group chain for these groups must have nontrivial kernels. For example, many of the types of Baumslag–Solitar groups are not residually finite [Levitt 2015a; 2015b; Meskin 1972], so every equicontinuous minimal Cantor system defined by an action of one of these groups will have nontrivial kernels.

The kernel $K(\mathcal{G}^x)$ has an interpretation in terms of the topology of the leaves of the foliation $\mathcal{F}_{\mathcal{P}}$ of a weak solenoid. Let (V_0, G_0, Φ_0) be the holonomy action for a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with presentation $\mathcal{P}$ and basepoint $x \in V_0$, and let $\mathcal{G}^x = \{G_i^x\}_{i \geq 0}$ be the group chain at x. Recall that the restriction of the bundle projection $\Pi_0 | L_x : L_x \to M_0$ to the leaf L_x containing x is a covering map. Let $\widetilde{M}_0$ be the universal cover of M_0. Then by standard arguments of covering space theory (see [McCord 1965]) there is a homeomorphism

$$\widetilde{M}_0 / K(\mathcal{G}^x) \to L_x. \tag{16}$$

Now let $y \in \mathfrak{X}_0$ be another point. Then by Remark 4.2, the group chain associated to y is given by $\mathcal{G}^y = \{g_i G_i^x g_i^{-1}\}_{i \geq 0}$ where $\tau_x(y) = (g_i G_i^x)$. If y is in the orbit of x under the G_0-action, then we can take $g_i = g$ for some $g \in G_0$, and thus $K(\mathcal{G}^y) = g K(\mathcal{G}^x) g^{-1}$; that is, the kernels of $\mathcal{G}^x$ and $\mathcal{G}^y$ are conjugate, reflecting the fact that the fundamental group of the leaf L_x is replaced by a conjugate as x changes. If y is not in the orbit of x, then the relationship between $K(\mathcal{G}^x)$ and $K(\mathcal{G}^y)$ depends on the dynamical properties of the solenoid.

In particular, in Section 8 we relate the algebraic properties of the kernels $K(\mathcal{G}^y)$ with the germinal holonomy groups of the foliation $\mathcal{F}_{\mathcal{P}}$. Recall from Section 1 that a manifold L has π_1-finite type if its fundamental group is finitely generated. A matchbox manifold $\mathfrak{M}$ has finite π_1-type if all leaves in $\mathcal{F}_{\mathfrak{M}}$ have finite π_1-type. The following statement is immediate from the above discussion.

Lemma 4.9. *An equicontinuous matchbox manifold $\mathfrak{M}$ has finite π_1-type if and only if, for the associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$, for all $\mathcal{G}^y \in \mathfrak{G}(\Phi)$, the kernel $K(\mathcal{G}^y)$ is a finitely generated subgroup of G_0.*

We next give two examples to illustrate the above concepts.

Example 4.10. Let $\mathcal{S}_{\mathcal{P}}$ be a Vietoris solenoid, as in Example 3.2, where $m_\ell > 1$ is the degree of p_ℓ. Choose $x \in \mathcal{S}_{\mathcal{P}}$ so that $\Pi_\ell(x) = 0$ for $\ell \geq 0$. Then $G_0 = \mathbb{Z}$, and $G_\ell^x = \widetilde{m}_\ell \mathbb{Z}$, where $\widetilde{m}_\ell = m_1 m_2 \cdots m_\ell$ is the product of the degrees of the coverings. Then the kernel $K(\mathcal{G}^x)$ is $\{0\}$, and the path-connected component L_x is homeomorphic to the real line. Let $y \in \mathfrak{X}_0$ be any other point in the fiber. Since $\mathbb{Z}$ is abelian, any subgroup conjugate to $G_\ell^x = \widetilde{m}_\ell \mathbb{Z}$ is equal to it. It follows that $K(\mathcal{G}^y) = \{0\}$, and L_y is homeomorphic to the real line for any $y \in \mathfrak{X}_0$.

More generally, suppose $\mathcal{S}_{\mathcal{P}}$ is an n-dimensional solenoid and G_ℓ^x is a normal subgroup of G_0 for all $\ell \geq 1$. Then for any $y \in \mathfrak{X}_0$ we have $\mathcal{G}^y = \mathcal{G}^x$, and so $K(\mathcal{G}^y) = K(\mathcal{G}^x)$. It follows that all leaves in $\mathcal{S}_{\mathcal{P}}$ are homeomorphic. The Vietoris solenoid $\mathcal{S}_{\mathcal{P}}$ is of finite π_1-type.

Example 4.11. This example is due to Rogers and Tollefson [1971c]. Consider a map of the plane given by a translation by $\frac{1}{2}$ in the first component, and by reflection in the second component, i.e.,

$$r \times i : \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{where } (x, y) \mapsto \left(x + \tfrac{1}{2}, -y\right).$$

This map commutes with translations by the elements in the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, and so induces the map $r \times i : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$ of the torus. This map is an involution, and the quotient space $K = \mathbb{T}^2/(x, y) \sim r \times i(x, y)$ is homeomorphic to the Klein bottle.

Consider the double covering map $L : \mathbb{T}^2 \to \mathbb{T}^2$ given by $L(x, y) = (x, 2y)$. The inverse limit $\mathbb{T}_\infty = \varprojlim\{L : \mathbb{T}^2 \to \mathbb{T}^2\}$ is a solenoid with 2-dimensional leaves. Let $x_0 = (0, 0) \in M_0 = \mathbb{T}^2$. The fundamental group $G_0 = \mathbb{Z}^2$ is abelian, so for any $x, y \in \mathfrak{X}_0$ the kernels $K(\mathcal{G}^x) = K(\mathcal{G}^y)$ are isomorphic to $\mathbb{Z}$, and every leaf is homeomorphic to an open two-ended cylinder.

The involution $r \times i$ is compatible with the covering maps L, and so it induces an involution $(r \times i)_\infty : \mathbb{T}_\infty \to \mathbb{T}_\infty$, which is seen to have a single fixed point $(0, 0, \dots) \in \mathbb{T}_\infty$ and permute other path-connected components. Let $p : K \to K$ be the double covering of the Klein bottle by itself, given by $p(x, y) = (x, 2y)$, and consider the inverse limit space $K_\infty = \varprojlim\{p : K \to K\}$. Note that taking the quotient by the involution $r \times i$ is compatible with the covering maps L and p; that is, $p \circ (r \times i) = L$, and so induces the map $i_\infty : \mathbb{T}_\infty \to K_\infty$ of the inverse limit spaces. Under this map, the path-connected component of the fixed point $(0, 0, \dots)$ is identified so as to become a nonorientable one-ended cylinder. The image of any other path-connected component is an orientable two-ended cylinder.

Let $x = (x_\ell) \in K_\infty$ for $x_\ell \in K$. Then $G_0 = \pi_1(K, x_0) = \langle a, b \mid bab^{-1} = a^{-1}\rangle$. Fokkink and Oversteegen [2002] computed the kernel $K(\mathcal{G}^x) = \langle b\rangle$ of the group chain $\mathcal{G}^x$. They also computed kernels for group chains at any other basepoint $y \in \mathfrak{X}_0$ and found that either $K(\mathcal{G}^y)$ is conjugate to $\langle b\rangle$, or $K(\mathcal{G}^y)$ is equal to $\langle b^2\rangle$. This example has finite π_1-type.

5. Homogeneous solenoids and actions

In this section, we review the results from various works about the criteria for homogeneity of matchbox manifolds. These data will be of use later, when we give the proof Theorem 1.2.

A continuum $\mathfrak{M}$ is said to be *homogeneous* if given any pair of points $x, y \in \mathfrak{M}$, there exists a homeomorphism $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$. A homeomorphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ preserves the path-connected components, hence preserves the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. It follows that if $\mathfrak{M}$ is homogeneous, then it is also foliated homogeneous.

By [Clark and Hurder 2013, Theorem 5.2] a homogeneous matchbox manifold $\mathfrak{M}$ is equicontinuous. Hence by Theorem 1.1 above, which is proved in [Clark and Hurder 2013, Theorem 1.4], the foliated space $\mathfrak{M}$ is homeomorphic to a weak solenoid $\mathcal{S}_{\mathcal{P}}$. We restrict our attention to equicontinuous foliated spaces, so consider the problem of giving conditions for when a weak solenoid $\mathcal{S}_{\mathcal{P}}$ is homogeneous, which is thus equivalent to asking when an equicontinuous matchbox manifold is homogeneous. This is one of the original motivating problems in the study of solenoids, to obtain necessary and sufficient conditions for when the solenoid $\mathcal{S}_{\mathcal{P}}$ is homogeneous [Fokkink and Oversteegen 2002; Rogers 1970; Rogers and Tollefson 1971a; Schori 1966]. In this section, we recall the relevant results of these previous works, and of [Dyer 2015; Dyer et al. 2016; 2017].

5A. ***Regular actions.*** An *automorphism* of (V_0, G_0, Φ_0) is a homeomorphism $h : V_0 \to V_0$ which commutes with the G_0-action on V_0. Denote by $\mathrm{Aut}(V_0, G_0, \Phi_0)$ the group of automorphisms of the action (V_0, G_0, Φ_0). Note that $\mathrm{Aut}(V_0, G_0, \Phi_0)$ is a topological group for the compact-open topology on maps, and is a closed subgroup of $\mathrm{Homeo}(V_0)$.

Definition 5.1. The equicontinuous minimal Cantor action (V_0, G_0, Φ_0) is

(1) *regular* if the action of $\mathrm{Aut}(V_0, G_0, \Phi_0)$ on V_0 has a single orbit;

(2) *weakly normal* if the action of $\mathrm{Aut}(V_0, G_0, \Phi_0)$ decomposes V_0 into a finite collection of orbits;

(3) *irregular* if the action of $\mathrm{Aut}(V_0, G_0, \Phi_0)$ decomposes V_0 into an infinite collection of orbits.

The terminology in Definition 5.1 is chosen to be consistent with the terminology in [Dyer et al. 2016; Fokkink and Oversteegen 2002].

Recall that $\mathfrak{G}$ denotes the collection of all possible subgroup chains in G_0, and let $\mathfrak{G}(\Phi_0) \subset \mathfrak{G}$ denote the collection of all group chains in $\mathfrak{G}$ which are conjugate equivalent to a given group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$. Theorem 4.5 states that a group chain $\{G^x_\ell\}_{\ell \geq 0}$ is equivalent to the group chain $\{H^y_\ell\}_{\ell \geq 0}$ if and only if there exists a conjugacy $h : V_0 \to V_0$ of the G_0-action on V_0 such that $h(x) = y$. Such an h is an automorphism of (V_0, G_0, Φ_0), which gives the following result.

Theorem 5.2. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor action, and $\{G^x_\ell\}_{\ell \geq 0} \in \mathfrak{G}$ be a group chain associated to the action. Then (V_0, G_0, Φ_0) is*

(1) ***regular*** *if all group chains in $\mathfrak{G}(\Phi_0)$ are equivalent*;

(2) ***weakly normal*** *if $\mathfrak{G}(\Phi_0)$ contains a finite number of classes of equivalent group chains*;

(3) ***irregular*** *if $\mathfrak{G}(\Phi_0)$ contains an infinite number of classes of equivalent group chains.*

McCord [1965] studied the case when the chain $\{G^x_\ell\}_{\ell \geq 0}$ consists of normal subgroups of G_0. In this case, every quotient $X^x_\ell = G_0/G^x_\ell$ is a finite group, and the inverse limit X^x_∞, defined by (14), is then a profinite group. The group X^x_∞ is identified with V_0 as a topological space, and it acts transitively on V_0 on the right. The right action of X^x_∞ commutes with the left action of G_0 on X^x_∞, and thus $X^x_\infty \subset \mathrm{Aut}(V_0, G_0, \Phi_0)$, and so the automorphism group acts transitively on H_∞. McCord [1965] used this observation to show that the group $\mathrm{Homeo}(\mathcal{S}_\mathcal{P})$ acts transitively on $\mathcal{S}_\mathcal{P}$, proving the following theorem.

Theorem 5.3. *Let $\mathcal{S}_\mathcal{P}$ be a solenoid with a group chain $\{G^x_\ell\}_{\ell\geq 0}$ such that G^x_ℓ is a normal subgroup of G_0 for all $\ell \geq 0$. Then $\mathcal{S}_\mathcal{P}$ is homogeneous.*

For example, if G_0 is abelian, then every group chain $\{G^x_\ell\}_{\ell\geq 0}$ consists of normal subgroups, and the solenoid $\mathcal{S}_\mathcal{P}$ is homogeneous.

5B. ***Weakly normal actions.*** We next consider the problem of giving necessary and sufficient conditions for when a solenoid $\mathcal{S}_\mathcal{P}$ is homogeneous.

The converse to Theorem 5.3 is not true. Indeed, Rogers and Tollefson [1971b] gave an example of a weak solenoid for which the presentation yields a chain of subgroups which are not normal in G_0, yet the inverse limit is a profinite group, and so the solenoid is homogeneous. This example was the motivation for the work of Fokkink and Oversteegen [2002], where they gave a necessary and sufficient condition on the chain $\{G^x_\ell\}_{\ell\geq 0}$ for the weak solenoid to be homogeneous. In particular, they proved the following result. Let $N_{G_0}(G_\ell)$ denote the normalizer of the subgroup G_ℓ in G_0; that is, $N_{G_0}(G_\ell) = \{g \in G_0 \mid g\, G_\ell\, g^{-1} = G_\ell\}$.

Theorem 5.4 [Fokkink and Oversteegen 2002]. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor action, $x \in V_0$ be a point, and $\{G^x_\ell\}_{\ell\geq 0}$ be an associated group chain with conjugate equivalence class $\mathfrak{G}(\Phi_0)$. Then*

(1) *(V_0, G_0, Φ_0) is regular if and only if there exists a group chain $\{N_\ell\}_{\ell\geq 0} \in \mathfrak{G}(\Phi_0)$ such that N_ℓ is a normal subgroup of G_0 for each $\ell \geq 0$;*

(2) *(V_0, G_0, Φ_0) is weakly normal if and only if there exists $\{G^{x\prime}_\ell\}_{i\geq 0} \in \mathfrak{G}(\Phi_0)$ and an $n > 0$ such that $G^{x\prime}_\ell \subset G^x_n \subseteq N_{G_0}(G^{x\prime}_\ell)$ for all $\ell \geq n$.*

In Theorem 5.4, the set $\mathfrak{G}(\Phi_0)$ contains group chains which are conjugate equivalent to the given chain $\{G^x_\ell\}_{\ell\geq 0}$. The condition that the group chain $\{N_\ell\}_{\ell\geq 0}$ consists of normal subgroups implies that every chain in $\mathfrak{G}(\Phi_0)$ is equivalent to $\{N_\ell\}_{\ell\geq 0}$, and so $\{G^x_\ell\}_{\ell\geq 0}$ is equivalent to $\{N_\ell\}_{\ell\geq 0}$. In statement (2), the condition $G^{x\prime}_\ell \subset G^x_n \subseteq N_{G_0}(G^{x\prime}_\ell)$ implies that the group chain $\{G^{x\prime}_\ell\}_{\ell\geq 0}$ is equivalent to $\{G^x_\ell\}_{\ell\geq 0}$. Indeed, suppose that $G^{x\prime}_\ell \subset G^{x}_m{}' \subseteq N_{G_0}(G^{x\prime}_\ell)$ for some m. Then for $n \leq m$ and $\ell \leq n$ we have $G^{x\prime}_\ell \subset G^{x\prime}_n \subseteq N_{G_0}(G^{x\prime}_\ell)$. If $\{G^{x\prime}_\ell\}_{i\geq 0}$ is equivalent to $\{G^x_\ell\}_{i\geq 0}$, then for some $n \leq m$ we have $G^{x\prime}_n \subset G^x_n \subset G^{x}_m{}'$, which yields the statement.

Recall that Proposition 3.4 introduced the descending chain of clopen sets $\{V_{\ell+1} \subset V_\ell \mid \ell \geq 0\}$ of V_0 such that V_ℓ is stabilized by the action of G_ℓ. Thus, the weak normality condition in Theorem 5.4 implies that if we restrict the G_0 action to the clopen set $V_n \subset V_0$, then the restricted action (V_n, G_n, Φ_n) with associated group chain $\mathcal{G}_n^x = \{G_\ell^x\}_{\ell \geq n}$ is regular. In the case where the group chain $\{G_\ell\}_{\ell \geq 0}$ is associated to a weak solenoid $\mathcal{S}_{\mathcal{P}}$, restricting to the action (V_n, G_n, Φ_n) amounts to discarding the initial manifolds $\{M_0, \dots, M_{n-1}\}$ in the presentation $\mathcal{P}$, to obtain the presentation $\mathcal{P}_n$ defined in (9). Then as discussed in Section 3C, there is a homeomorphism $\mathcal{S}_{\mathcal{P}_n} \cong \mathcal{S}_{\mathcal{P}}$, where the homeomorphism is given by the "shift" map σ_n. Thus, $\mathcal{S}_{\mathcal{P}}$ is homogeneous if and only if $\mathcal{S}_{\mathcal{P}_n}$ is homogeneous, and so by Theorem 5.3 a weak solenoid whose associated group chain is weakly normal is homogeneous. We thus obtain the following result of Fokkink and Oversteegen [2002], giving a criterion for when a weak solenoid is homogeneous.

Proposition 5.5. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid, defined by a presentation $\mathcal{P}$ with associated group chain $\{G_\ell^x\}_{\ell \geq 0}$. Then $\mathcal{S}_{\mathcal{P}}$ is homogeneous if and only if $\{G_\ell^x\}_{\ell \geq 0}$ is weakly normal.*

We also have the following property of presentations of homogeneous solenoids.

Proposition 5.6 [Fokkink and Oversteegen 2002]. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid, defined by a presentation $\mathcal{P}$ with associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$. If $\mathcal{S}_{\mathcal{P}}$ is homogeneous, then the kernel $K(\mathcal{G}^x) \subset G_0$ has a finite number of conjugacy classes in G_0.*

Proof. Suppose that $\mathcal{S}_{\mathcal{P}}$ is homogeneous. Then by Theorem 5.4, there exists $\mathcal{G}^{x\prime} = \{G_\ell^{x\prime}\}_{\ell \geq 0} \in \mathfrak{G}(\Phi_0)$ and an $n > 0$ such that $G_\ell^{x\prime} \subset G_n^x \subseteq N_{G_0}(G_\ell^{x\prime})$ for all $\ell \geq n$. Then $G_n^{x\prime} \subseteq N_{G_0}(G_\ell^{x\prime})$ for all $\ell \geq n$, which implies that $G_n^{x\prime} \subset N_{G_0}(K(\mathcal{G}_x'))$. Indeed, the chain $\{G_\ell^{x\prime}\}_{\ell \geq n}$ contains subgroups normal in $G_n^{x\prime}$, and its intersection is then again normal in $G_n^{x\prime}$. Then for any $h \in G_n^{x\prime}$ we have

$$h \cdot K(\mathcal{G}^{x\prime}) \cdot h^{-1} = K(\mathcal{G}^{x\prime}), \tag{17}$$

and $K(\mathcal{G}^{x\prime})$ has only a finite number of conjugacy classes, at most $[G_0 : G_n^{x\prime}]$. Since $\mathcal{G}^x$ is equivalent to $\mathcal{G}^{x\prime}$, we have that $G_0^x = G_0^{x\prime} \supset G_1^x \supset G_1^{x\prime} \supset G_2^x \supset G_2^{x\prime} \supset \cdots$, and so $K(\mathcal{G}^x) = K(\mathcal{G}^{x\prime})$, which yields the statement. □

6. Ellis group of equicontinuous minimal systems

In [Ellis and Gottschalk 1960; Ellis 1960], the *Ellis* (*enveloping*) *semigroup* associated to a continuous group action $\Phi : G \times X \to X$ was introduced, and it is treated in [Auslander 1988; Ellis 1969; Ellis and Ellis 2014]. The construction of $\widehat{E}(X, G, \Phi)$ is abstract, and it can be difficult to calculate this group exactly. A key problem is to understand the relation between the algebraic properties of $\widehat{E}(X, G, \Phi)$ and the dynamics of the action. In this section, we briefly recall some basic properties of $\widehat{E}(X, G, \Phi)$, then consider the results for the special case of equicontinuous minimal systems.

6A. ***Ellis (enveloping) group.*** Let X be a compact Hausdorff topological space and G be a finitely generated group. Consider the space $X^X = \mathrm{Maps}(X, X)$ with the topology of *pointwise convergence on maps*. With this topology, X^X is a compact Hausdorff space. Each $g \in G$ defines an element $\hat{g} \in \mathrm{Homeo}(X) \subset X^X = \mathrm{Maps}(X, X)$. Denote by $\widehat{G}$ the set of all such elements. Ellis [1960] showed that the closure $\overline{\widehat{G}} \subset X^X$ has the structure of a right topological semigroup. Moreover, if the action (X, G, Φ) is equicontinuous, then the semigroup $\overline{\widehat{G}}$ is a group naturally identified with the closure $\overline{\Phi(G)}$ of $\Phi(G) \subset \mathrm{Homeo}(X)$ in the *uniform topology on maps*. Each element of $\overline{\Phi(G)}$ is the limit of a sequence of points in $\widehat{G}$, and we use the notation (g_i) to denote a sequence $\{g_i \mid i \geq 1\} \subset G$ such that the sequence $\{\hat{g}_i = \Phi(g_i) \mid i \geq 1\} \subset \mathrm{Homeo}(X)$ converges in the uniform topology.

Assume the action of G on X is minimal, that is, the orbit $\Phi(G)(x)$ is dense in X for any $x \in X$. It then follows that the orbit of the Ellis group $\overline{\Phi(G)}(x)$ equals X for any $x \in X$. That is, the group $\overline{\Phi(G)}$ acts transitively on X. Then for the isotropy group of the action at x,

$$\overline{\Phi(G)}_x = \{(g_i) \in \overline{\Phi(G)} \mid (g_i) \cdot x = x\}, \tag{18}$$

we have the natural identification $X \cong \overline{\Phi(G)}/\overline{\Phi(G)}_x$ of left G-spaces.

Given an equicontinuous minimal Cantor system (X, G, Φ), the Ellis group $\overline{\Phi(G)}$ depends only on the image $\Phi(G) \subset \mathrm{Homeo}(X)$. On the other hand, the isotropy group $\overline{\Phi(G)}_x$ may depend on the point $x \in X$. Since the action of $\overline{\Phi(G)}$ is transitive on X, given any $y \in X$, there exists $(g_i) \in \overline{\Phi(G)}$ such that $(g_i) \cdot x = y$. It follows that

$$\overline{\Phi(G)}_y = (g_i) \cdot \overline{\Phi(G)}_x \cdot (g_i)^{-1}. \tag{19}$$

Thus, the *cardinality* of the isotropy group $\overline{\Phi(G)}_x$ is independent of the point $x \in X$, and so the Ellis group $\overline{\Phi(G)}$ and the cardinality of $\overline{\Phi(G)}_x$ are invariants of (X, G, Φ).

6B. ***Ellis group for group chains.*** We consider the Ellis group for an equicontinuous minimal Cantor action (V_0, G_0, Φ), in terms of an associated group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ for $x \in V_0$. For each subgroup G^x_ℓ consider the maximal normal subgroup of G^x_ℓ which is given by

$$C_\ell \equiv \mathrm{core}_{G_0}\, G^x_\ell \equiv \bigcap_{g \in G_0} g G^x_\ell g^{-1} \subseteq G^x_\ell. \tag{20}$$

The group C_ℓ is called the *core* of G_ℓ in G_0. Since C_ℓ is normal in G_0, the quotient G_0/C_ℓ is a finite group, and the collection $\mathcal{C} = \{C_\ell\}_{\ell \geq 0}$ forms a descending chain of normal subgroups of G_0. The inclusions of coset spaces define bonding maps

$\delta_\ell^{\ell+1}$ for the inverse sequence of quotients G_0/C_ℓ, and the inverse limit space

$$C_\infty = \{(eG_0, g_1C_1, \dots) \mid g_\ell C_\ell = g_{\ell+1}C_\ell\} \subset \prod_{\ell \geq 0} G_0/C_\ell \tag{21}$$

$$\cong \varprojlim \{\delta_\ell^{\ell+1} : G_0/C_{\ell+1} \to G_0/C_\ell\} \tag{22}$$

is a profinite group. Let $\hat{\iota} : G_0 \to C_\infty$ be the homomorphism defined by $\hat{\iota}(g) = (gC_\ell)$ for $g \in G_0$. Then the induced left action of G_0 on C_∞ yields a minimal Cantor system, denoted by $(C_\infty, G_0, \widehat{\Phi}_0)$.

Also, introduce the descending chain of clopen neighborhoods of the identity $(eC_\ell) \in C_\infty$, which for $n \geq 0$ defines a neighborhood system for C_∞:

$$C_{n,\infty} = \{(g_\ell C_\ell) \in C_\infty \mid g_n \in C_n\}, \tag{23}$$

$$\cong \varprojlim \{\delta_\ell^{\ell+1} : C_n/C_{\ell+1} \to C_n/C_\ell \mid \ell \geq n\}. \tag{24}$$

6C. ***The discriminant.*** Observe that for each $\ell \geq 0$, the quotient group $D_\ell^x = G_\ell^x/C_\ell \subset G_0/C_\ell$. It follows that the inverse limit space

$$\mathcal{D}_x = \varprojlim \{\delta_\ell^{\ell+1} : D_{\ell+1}^x \to D_\ell^x\} \tag{25}$$

is a closed subgroup of C_∞. The group $\mathcal{D}_x$ is called the *discriminant group* of the action (V_0, G_0, Φ_0).

The relationship between C_∞ and the Ellis group of (V_0, G_0, Φ_0) is given by the following result.

Theorem 6.1 [Dyer et al. 2016, Theorem 4.4]. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor action, let $x \in V_0$, and let $\mathcal{G}^x \equiv \{G_\ell^x\}_{i \geq 0}$ be the associated group chain at x. Then there is a natural isomorphism of topological groups $\widehat{\Theta} : \overline{\Phi(G_0)} \cong C_\infty$ such that the restriction $\widehat{\Theta} : \overline{\Phi(G_0)}_x \cong \mathcal{D}_x$.*

Moreover, the discriminant subgroup is simple by the next result.

Proposition 6.2 [Dyer et al. 2016, Proposition 5.3]. *Let (V_0, G_0, Φ_0) be an equicontinuous minimal Cantor system, $x \in V_0$ a basepoint, and $\overline{\Phi_0(G_0)}_x$ the isotropy group of x. Then*

$$\mathrm{core}_{G_0} \overline{\Phi_0(G_0)}_x = \bigcap_{k \in G_0} k \, \overline{\Phi_0(G_0)}_x \, k^{-1} \tag{26}$$

is the trivial group. Thus, the maximal normal subgroup of $\overline{\Phi_0(G_0)}_x$ in $\overline{\Phi_0(G_0)}$ is also trivial.

We next consider the homogeneity properties of a solenoid $\mathcal{S}_\mathcal{P}$ in terms of $\mathcal{D}_x$ (see [Dyer et al. 2016]). It follows from Proposition 6.2 that if $\mathcal{D}_x$ is nontrivial, then it is not normal in C_∞, and therefore the quotient $X_\infty^x = C_\infty/\mathcal{D}_x$ is not a group. We thus conclude:

Proposition 6.3 [Dyer et al. 2016]. *The action* (V_0, G_0, Φ_0) *is regular if and only if* $\mathcal{D}_x$ *is trivial.*

Note that Proposition 6.3 does not take into account the possibility that the action of a subgroup G^x_ℓ on a smaller section V_ℓ is regular. The general formulation is then as follows.

Corollary 6.4. *An equicontinuous matchbox manifold* $\mathfrak{M}$ *is homogeneous if and only if it admits a transverse section* V_0 *and a presentation* $\mathcal{P}$ *with associated group chain* $\{G^x_\ell\}_{\ell \geq 0}$ *such that the discriminant group* $\mathcal{D}_x$ *is trivial.*

7. Molino theory for weak solenoids

In this section, we obtain a Molino theory for weak solenoids, and hence for all equicontinuous matchbox manifolds, including those for which the hypotheses of [Álvarez López and Moreira Galicia 2016] are not satisfied. There are often subtle, and not so subtle, differences between the theory for matchbox manifolds and for smooth Riemannian foliations, as will be discussed further in the following sections.

7A. ***Molino overview.*** Molino theory for Riemannian foliations gives a structure theory for the geometry and dynamics of this class of foliations on compact smooth manifolds. The Séminaire Bourbaki article by Haefliger [1989] gives a concise overview of the theory and its applications, and Molino's book [1988] and its multiple appendices give a more detailed treatment of this theory and its applications. The book [Moerdijk and Mrčun 2003] is also an excellent reference about the essentials of Molino theory. We give a very brief summary below of some key properties of the Molino space $\widehat{M}$ associated to a smooth Riemannian foliation $\mathcal{F}$ of a compact connected manifold M.

Given a Riemannian foliation $\mathcal{F}$ of a compact connected manifold M, the associated *Molino space* $\widehat{M}$ is a compact connected manifold with a Riemannian foliation $\widehat{\mathcal{F}}$ whose leaves have the same dimension as those of $\mathcal{F}$. In the case where $\mathcal{F}$ is a minimal foliation, in the sense that each leaf of $\mathcal{F}$ is dense in M, then we can assume that the foliation $\widehat{\mathcal{F}}$ is also minimal.

Associated to a minimal Riemannian foliation $\mathcal{F}$ is the *structural Lie algebra* $\mathfrak{h}$, given by the algebra of holonomy-invariant vector fields normal to $\mathcal{F}$, and which is well defined up to isomorphism.

There is a fibration $\hat{\pi} : \widehat{M} \to M$ equipped with a fiber-preserving right action of a connected Lie group H whose Lie algebra is $\mathfrak{h}$, and for which the foliation $\widehat{\mathcal{F}}$ is invariant under the action of H. Moreover, for each leaf $\widehat{L} \subset \widehat{M}$, there is a leaf $L \subset M$ such that the restriction $\hat{\pi} : \widehat{L} \to L$ is the holonomy covering of L. We say

$$H \longrightarrow \widehat{M} \xrightarrow{\hat{\pi}} M \tag{27}$$

is a *Molino sequence* for M, and H is the structural Lie group for $\widehat{\mathcal{F}}$.

A key property of the Molino space $\widehat{M}$ of $\mathcal{F}$ is that it is *transversally parallelizable*, or TP. This condition states that there are nonvanishing vector fields $\{\vec{v}_1, \dots, \vec{v}_q\}$ on M which span the normal bundle to $\mathcal{F}$ at each $x \in M$, and the vector fields are locally projectable. As a consequence, given any pair of points $x, y \in \widehat{M}$ there exists a diffeomorphism $h : \widehat{M} \to \widehat{M}$ which maps leaves of $\widehat{\mathcal{F}}$ to leaves of $\widehat{\mathcal{F}}$, and satisfies $h(x) = y$. A foliation $\widehat{\mathcal{F}}$ satisfying this condition is said to be *foliated homogeneous*.

7B. ***Molino sequences for weak solenoids.*** For a matchbox manifold, the TP condition cannot be defined, as the transversal space to the foliation is totally disconnected. Thus, we need an alternative approach to defining the Molino fibration (27) in the case where the transversal space to the foliation is a Cantor set. The basic observation is that the foliated homogeneous condition for $\widehat{M}$ admits a natural generalization to all foliated spaces, as discussed for weak solenoids in Section 5. For weak solenoids, we will see below that the structural Lie group H is replaced by the discriminant subgroup $\mathcal{D}_x \subset C_\infty$ of Section 6C, and the foliated homogeneous condition is a consequence of the Ellis group construction. We now restate and prove Theorem 1.2.

Theorem 7.1. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and let $\mathcal{P}$ be a presentation of $\mathfrak{M}$, such that $\mathfrak{M}$ is homeomorphic to a solenoid $\mathcal{S}_{\mathcal{P}}$. Then there exists a homogeneous matchbox manifold $\widehat{\mathfrak{M}}$ with foliation $\widehat{\mathcal{F}}$, called a Molino space of $\mathfrak{M}$, a compact totally disconnected group $\mathcal{D}$, and a fibration*

$$\mathcal{D} \longrightarrow \widehat{\mathfrak{M}} \xrightarrow{\hat{q}} \mathfrak{M}, \tag{28}$$

*where the restriction of $\hat{q}$ to each leaf in $\widehat{\mathfrak{M}}$ is a covering map of some leaf in $\mathfrak{M}$. We say that (28) is a **Molino sequence** for $\mathfrak{M}$.*

Proof. Let $V_0 \subset \mathfrak{M}$ be a transverse section to the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$, as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let G_0 be the restricted holonomy group acting on V_0. Let $\mathcal{P} = \{p_{\ell+1} : M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$ be a presentation at x such that there is a homeomorphism $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$, and for $x \in V_0$ let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$. Let $\Pi_0 : \mathcal{S}_{\mathcal{P}} \to M_0$ and set $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$. Let $\tau : V_0 \to \mathfrak{X}_0$ with $\tau_x(x) = (eG^x_\ell)$ be the homeomorphism defined in Lemma 4.1.

Recall that the covering map $q_\ell : M_\ell \to M_0$ defined in (5) is associated to the subgroup $G^x_\ell \subset G_0 = \pi_1(M_0, x_0)$. Recall that the core subgroup $C_\ell \subset G^x_\ell$ is the maximal normal subgroup of G_0 contained in G^x_ℓ, and has finite index in G^x_ℓ. For each $\ell > 0$, let $\hat{q}_\ell : \widehat{M}_\ell \to M_0$ be the proper covering space associated to the normal subgroup C_ℓ. Each inclusion $C_{\ell+1} \subset C_\ell$ induces a normal covering map $\hat{p}_{\ell+1} : \widehat{M}_{\ell+1} \to \widehat{M}_\ell$, and so yields a presentation $\widehat{\mathcal{P}} = \{\hat{p}_{\ell+1} : \widehat{M}_{\ell+1} \to \widehat{M}_\ell \mid \ell \geq 0\}$.

Definition 7.2. The *Molino space* associated to a weak solenoid $\mathcal{S}_{\mathcal{P}}$ defined by a presentation $\mathcal{P}$ is the inverse limit space associated to the presentation $\widehat{\mathcal{P}}$,

$$\widehat{\mathcal{S}}_{\mathcal{P}} \equiv \varprojlim \{\hat{p}_{\ell+1} : \widehat{M}_{\ell+1} \to \widehat{M}_{\ell}\}. \tag{29}$$

Let $\widehat{\Pi}_0 : \widehat{\mathcal{S}}_{\mathcal{P}} \to M_0$ be the projection map, with fiber $\widehat{\mathfrak{X}}_0 = \widehat{\Pi}_0^{-1}(x_0)$.

We state some of the basic properties of the space $\widehat{\mathcal{S}}_{\mathcal{P}}$. The proofs of the following statements are omitted, as they follow by arguments analogous to the corresponding statements for $\mathcal{S}_{\mathcal{P}}$.

Proposition 7.3. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid defined by a presentation $\mathcal{P}$, and let $\widehat{\mathcal{S}}_{\mathcal{P}}$ be the solenoid defined by (29). Then*

(1) *there is a natural isomorphism $\widehat{\mathfrak{X}}_0 \cong C_\infty$, where C_∞ is the profinite group defined by (22);*

(2) *there is a natural map of fibrations $\hat{q} : \widehat{\mathcal{S}}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, whose fiber over $x \in \mathfrak{X}_0$ is the discriminant group $\mathcal{D}_x$;*

(3) *the global holonomy of the fibration $\widehat{\Pi}_0 : \widehat{\mathcal{S}}_{\mathcal{P}} \to M_0$ is naturally conjugate as G_0-actions with the minimal Cantor system $(C_\infty, G_0, \widehat{\Phi}_0)$.*

Definition 7.4. The *Molino sequence* for the weak solenoid $\mathcal{S}_{\mathcal{P}}$ is the principal fibration

$$\mathcal{D}_x \longrightarrow \widehat{\mathcal{S}}_{\mathcal{P}} \xrightarrow{\hat{q}} \mathcal{S}_{\mathcal{P}}. \tag{30}$$

Proposition 7.3(3) implies that the foliation $\widehat{\mathcal{F}}_{\mathcal{P}}$ on $\widehat{\mathcal{S}}_{\mathcal{P}}$ is minimal, and the restrictions of $\hat{q}$ to the leaves of $\widehat{\mathcal{F}}_{\mathcal{P}}$ are covering maps by construction, as there is a covering map $\widehat{M}_\ell \to M_\ell$ for each $\ell \geq 1$ which induces $\hat{q}$. Finally, the space $\widehat{\mathcal{S}}_{\mathcal{P}}$ is homogeneous by Proposition 5.5, as it is defined using the normal group chain $\{C_\ell\}_{\ell \geq 0}$.

Set $\widehat{\mathfrak{M}} = \widehat{\mathcal{S}}_{\mathcal{P}}$ and $\mathcal{D} = \mathcal{D}_x$. Then we have established Theorem 7.1. □

The construction of the sequence in (30) may depend on the various choices made, and this is a fundamental aspect of the Molino theory for weak solenoids. We consider in Section 7C the dependence of the discriminant group on the partition sets $V_n \subset V_0$. Then in Section 8, we consider the dependence of the sequence (30) on the choice of the basepoint $x \in V_0$ and the role of the holonomy of the leaf L_x in the properties of $\mathcal{D}_x$.

7C. *Stability of the Molino sequence.* We next consider the stability of the discriminant group for an equicontinuous Cantor minimal system (V_0, G_0, Φ_0) when one restricts to a section $V_n \subset V_0$.

We start with an example that highlights the importance of the "asymptotic algebraic structure" of the group chain $\mathcal{G}^x$ for the definition of the Molino space. Consider a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ defined by the holonomy action (V_0, G_0, Φ_0) for a clopen subset $V_0 \subset \mathfrak{X}_0$, and suppose that $\mathcal{G}^x$ is not regular. Then by Proposition 6.3, the discriminant group $\mathcal{D}_x$ is nontrivial, and thus the sequence (30) has nontrivial fiber. Now suppose that, in addition, the group chain $\mathcal{G}^x$ is weakly normal. Then by Theorem 5.4, there exists some $n > 0$ such that the restricted action (V_n, G_n, Φ_n) is regular; hence the discriminant group $\mathcal{D}_x^n$ for the truncated chain $\mathcal{G}_n^x = \{G_\ell^x\}_{\ell \geq n}$ associated to the restricted action is trivial. For the truncated presentation $\mathcal{P}_n$ defined by (9), we have $\widehat{\mathcal{S}}_{\mathcal{P}_n} = \mathcal{S}_{\mathcal{P}_n}$ as $\mathcal{D}_x^n$ is the trivial group, and $\mathcal{S}_{\mathcal{P}_n} \cong \mathcal{S}_{\mathcal{P}}$ as remarked in Section 3C; hence we can consider $\widehat{\mathcal{S}}_{\mathcal{P}_n}$ as a Molino space for $\mathcal{S}_{\mathcal{P}}$ as well. That is, for this choice of V_n as a section, the Molino sequence (30) has trivial fiber.

We next develop a comparison, for $n \geq 0$, of the discriminant groups $\mathcal{D}_x^n$ for the group chain $\mathcal{G}_n^x$ associated to the truncated presentation $\mathcal{P}_n$ defined by (9). We work with the group chain model $(X_\infty^x, G_0, \Phi_x)$ of Lemma 4.1 for the holonomy action $\Phi_0 : G_0 \to \mathrm{Homeo}(\mathfrak{X}_0)$. By definition (25) of the discriminant group, it suffices to consider this invariant in sufficiently small clopen neighborhoods of the identity in the core group associated with the group chains. For $n \geq 0$, we have the clopen neighborhoods of $\{e\} \in X_\infty$:

$$U_n = \{(g_\ell G_\ell) \in X_\infty \mid g_n \in G_n^x\} \subset X_\infty \tag{31}$$

$$\cong \varprojlim \{\delta_\ell^{\ell+1} : G_n^x / G_{\ell+1}^n \to G_n^x / G_\ell \mid \ell \geq n\}. \tag{32}$$

Note that U_n is just the inverse limit group defined by the truncated group chain $\mathcal{G}_n^x$. Next, we introduce the core groups of $\mathcal{G}_n^x$ for arbitrarily small neighborhoods of $\{e\} \in U_n$. For $\ell \geq n \geq 0$, set

$$E_{n,\ell} \equiv \mathrm{core}_{G_n^x}\, G_\ell^x \equiv \bigcap_{g \in G_n^x} g\, G_\ell^x g^{-1}. \tag{33}$$

Note that $E_{0,\ell} = C_\ell$, and that for all $m \geq n \geq 0$ and $\ell > m$, we have $E_{n,\ell} \subset E_{m,\ell} \subset G_\ell^x$.

For $k \geq n \geq 0$, define the clopen neighborhood $V_{n,k}$ of $\{e\}$ for the core group of $\mathcal{G}_n^x$ by

$$V_{n,k} = \{(g_\ell E_{n,\ell}) \mid \ell \geq k,\ g_k \in G_k^x,\ g_{\ell+1} E_{n,\ell} = g_\ell E_{n,\ell}\} \tag{34}$$

$$\cong \varprojlim \{\delta_\ell^{\ell+1} : G_k^x / E_{n,\ell} \to G_k^x / E_{n,\ell+1} \mid \ell \geq k\}. \tag{35}$$

Then $V_{n,n}$ is the core limit group, or the Ellis group, for the truncated group chain $\mathcal{G}_n^x$, and $\{e\} \in V_{n,k} \subset V_{n,n}$ for all $k \geq n$. Note also that $V_{0,0} = C_\infty$ is the Ellis group for $\mathcal{G}^x$.

For each $\ell \geq k \geq m \geq n$, the inclusions $E_{n,\ell} \subset E_{m,\ell}$ induce group surjections

$$G_k^x / E_{n,\ell} \xrightarrow{\varphi_{k,n,m}^\ell} G_k^x / E_{m,\ell}, \tag{36}$$

so we obtain surjective homomorphisms of profinite groups $\varphi_{n,m} : V_{n,k} \to V_{m,k}$ for each $m > n \geq 0$. In particular, for $k = m$, this states that the clopen neighborhood $V_{n,m}$ of $\{e\}$ in the limit core group for $\mathcal{G}^x_n$ maps onto the limit core group $V_{m,m}$ of $\mathcal{G}^x_m$.

We consider next the discriminant groups associated to the group chains $\mathcal{G}^x_n$ for $n \geq 0$, $\mathcal{D}^n_x \subset V_{n,n}$:

$$\mathcal{D}^n_x = \varprojlim\{\delta^{\ell+1}_\ell : G^x_{\ell+1}/E_{n,\ell+1} \to G^x_\ell/E_{n,\ell} \mid \ell \geq n\} \tag{37}$$

$$\cong \varprojlim\{\delta^{\ell+1}_\ell : G^x_{\ell+1}/E_{n,\ell+1} \to G^x_\ell/E_{n,\ell} \mid \ell \geq m\} \quad \text{for } m \geq n. \tag{38}$$

It follows from (36) and (38) that for $m > n$, there are surjective homomorphisms:

$$\mathcal{D}_x \xrightarrow{\psi_{0,n}} \mathcal{D}^n_x \xrightarrow{\psi_{n,m}} \mathcal{D}^m_x. \tag{39}$$

Definition 7.5. A group chain $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ is said to be *stable* if there exists $n_0 \geq 0$ such that the maps $\psi_{n,m} : \mathcal{D}^n_x \to \mathcal{D}^m_x$ defined in (39) are isomorphisms for all $m \geq n \geq n_0$. Otherwise, the group chain is said to be *wild*.

Theorem 5.4 implies that if the group chain $\{G^x_\ell\}_{\ell \geq 0}$ is weakly normal, then it is stable, as there exists some $n_0 \geq 0$ such that $\mathcal{D}^n_x$ is the trivial group for all $n \geq n_0$. This discussion and Lemma 7.6 yield Proposition 1.4 of the Introduction.

Lemma 7.6. *If the discriminant group $\mathcal{D}_x$ for $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ is finite, then $\mathcal{G}^x$ is stable.*

Proof. The map $\psi_{0,n} : \mathcal{D}_x \to \mathcal{D}^n_x$ is surjective for all $n \geq 0$, so the assumption that the cardinality $\#\mathcal{D}_x$ is finite implies that the cardinality $\#\mathcal{D}^n_x$ of the group $\mathcal{D}^n_x$ is decreasing with n, and thus there exists $n_0 \geq 0$ such that the cardinality of its image must stabilize for $n \geq n_0$. Then for $n \geq n_0$, the homomorphism $\psi_{n,m} : \mathcal{D}^{n_0}_x \to \mathcal{D}^n_x$ is an isomorphism. □

7D. ***Stable matchbox manifolds.*** We next consider the relationship between the notion of stable for a matchbox manifold as given in Definition 1.3, and stable for a group chain as given in Definition 7.5.

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Let G_0 be the group of transformations of V_0 which induces the restricted holonomy group acting on V_0, and let $G^x_\ell \subset G_0$ be the stabilizer group of the set V_ℓ. Let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$, let $\mathcal{P}_n$ be the presentation (9) associated to the truncated group chain $\mathcal{G}^x_n = \{G^x_\ell\}_{\ell \geq n}$, and let $\mathcal{S}_{\mathcal{P}_n}$ be the inverse limit solenoid. For each $n \geq 0$, let $\widehat{\mathcal{S}}_{\mathcal{P}_n}$ be the homogeneous solenoid associated to the normal group chain $\{E_{n,\ell}\}_{\ell \geq n}$ defined by (33).

Assume that the group chain $\mathcal{G}^x$ is stable in the sense of Definition 7.5. That is, there exists an index n_0 such that for any $m > n \geq n_0$ restricting to the smaller sections $V_m \subset V_n \subset V_0$ with induced presentations $\mathcal{P}_m$ and $\mathcal{P}_n$, then the induced map $\psi_{n,m} : \mathcal{D}^n_x \to \mathcal{D}^m_x$ in (39) is a topological isomorphism. Then we have a commutative diagram of fibrations:

$$\begin{tikzcd} \mathcal{D}^n_x \arrow[r, "\psi_{n,m}"] \arrow[d] & \mathcal{D}^m_x \arrow[d] \\ \widehat{\mathcal{S}}_{\mathcal{P}_n} \arrow[r, "\hat{\sigma}_{m-n}"] \arrow[d] & \widehat{\mathcal{S}}_{\mathcal{P}_m} \arrow[d] \\ \mathcal{S}_{\mathcal{P}_n} \arrow[r, "\sigma_{m-n}"] & \mathcal{S}_{\mathcal{P}_m} \end{tikzcd} \tag{40}$$

By the discussion in Section 3C, the shift map σ_{m-n} is a homeomorphism, and by assumption, the map $\psi_{n,m} : \mathcal{D}_n \cong \mathcal{D}_m$ is a topological isomorphism. Thus the map $\hat{\sigma}_{m-n} : \widehat{\mathcal{S}}_{\mathcal{P}_n} \to \widehat{\mathcal{S}}_{\mathcal{P}_m}$ is a homeomorphism. Hence, the Molino sequences for the presentations $\mathcal{P}_n$ and $\mathcal{P}_m$ yield isomorphic topological fibrations. Conversely, if the topological type of the Molino sequence

$$\mathcal{D}^n_x \longrightarrow \widehat{\mathcal{S}}_{\mathcal{P}_n} \longrightarrow \mathcal{S}_{\mathcal{P}_n} \tag{41}$$

is well defined up to homeomorphism of fibrations, for given V_0 and $n \geq 0$ sufficiently large, then there exists $n_0 \geq 0$ such that $m > n \geq n_0$ implies that $\mathcal{D}^n_x \xrightarrow{\psi_{n,m}} \mathcal{D}^m_x$ is a topological isomorphism. Thus, the map of fibers $\psi_{n,m} : \mathcal{D}^n_x \to \mathcal{D}^m_x$ is a topological isomorphism, and hence $\mathcal{G}^x$ is stable.

The following statement summarizes these conclusions.

Theorem 7.7. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Let G_0 be the restricted holonomy group acting on V_0, and let $G^x_\ell \subset G_0$ be the stabilizer group of the set V_ℓ. Let $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$, let $\mathcal{P}_n$ be the presentation (9) associated to the truncated group chain $\mathcal{G}^x_n = \{G^x_\ell\}_{\ell \geq n}$, and let $\mathcal{S}_{\mathcal{P}_n}$ be the inverse limit solenoid. For each $n \geq 0$, let $\widehat{\mathcal{S}}_{\mathcal{P}_n}$ be the homogeneous solenoid associated to the normal group chain $\{E^n_\ell\}_{\ell \geq n}$ defined by (33).*

1. *If $\mathcal{G}^x$ is stable, then there exists $n_0 \geq 0$ such that for all $n \geq n_0$ the fibration (41) is a Molino sequence for $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}_n}$, and the fiber group $\mathcal{D}^n_x$ is well defined up to topological isomorphism.*
2. *If $\mathcal{G}^x$ is wild, then the topological isomorphism type of the fiber in the sequence (41) does not stabilize as n tends to infinity.*

Theorem 7.7 implies that the Molino sequence of a matchbox manifold $\mathfrak{M}$ need not be well defined, though if the associated group chain $\mathcal{G}^x$ is stable, then $\mathfrak{M}$ does have a well-defined Molino sequence.

8. Germinal holonomy in solenoids

In this section, we investigate the relationship between the germinal holonomy groups of leaves in a solenoid, the kernels of the associated group chains, and the discriminant group of the action.

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold with transverse section V_0, let $x \in V_0$ be a point, and let $\mathcal{P} = \{f_i^{\ell+1} : M_{\ell+1} \to M_\ell\}$ be a presentation with associated group chain $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ in $G_0 = \pi_1(M_0, x_0)$. Then by Theorem 1.1, there is a foliated homeomorphism $\mathfrak{M} \cong \mathcal{S}_{\mathcal{P}}$.

Let $C_\infty = \varprojlim\{G_0/C_{\ell+1} \to G_0/C_\ell\}$, where C_ℓ is the maximal normal subgroup of G_ℓ^x, $\ell \geq 0$, and let $\mathcal{D}_x$ be the discriminant group at x. Denote by $L_x \subset \mathcal{S}_{\mathcal{P}}$ the leaf of $\mathcal{F}_{\mathcal{P}}$ through x. Recall that the kernel of $\mathcal{G}^x$ is the subgroup $K(\mathcal{G}^x) \subset G_0$ as defined in Definition 4.7, and is the isotropy subgroup of the action (V_0, G_0, Φ_0) at x.

8A. ***Locally trivial germinal holonomy.*** The following properties of pseudogroup actions are basic for understanding their dynamical properties.

Definition 8.1. Given $g_1, g_2 \in K(\mathcal{G}^x)$, we say g_1 and g_2 have the same *germinal holonomy* at x if there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that the restrictions $\Phi_0(g_1)|U_x$ and $\Phi_0(g_2)|U_x$ agree on U_x. In particular, we say that $g \in K(\mathcal{G}^x)$ has *trivial germinal holonomy* at x if there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that the restriction $\Phi_0(g)|U_x$ is the trivial map.

By straightforward checking of definitions, one can see that the notion "germinal holonomy at x" defines an equivalence relation on the image of the isotropy subgroup $K(\mathcal{G}^x)$ under the global holonomy map $\Phi_0 : G_0 \to \mathrm{Homeo}(V_0)$. Denote by $\mathrm{Germ}(\Phi_0, x)$ the quotient of $\Phi_0(K(\mathcal{G}^x))$ by this equivalence relation. Thus the composition of $\Phi_0 : K(\mathcal{G}^x) \to \mathrm{Homeo}(V_0)$ with the quotient map gives us a surjective map $K(\mathcal{G}^x) \to \mathrm{Germ}(\Phi_0, x)$. A standard argument shows that if $\mathrm{Germ}(\Phi_0, x)$ is trivial, and y is in the same G_0-orbit of x, then $\mathrm{Germ}(\Phi_0, y)$ is trivial. This leads to the following definition.

Definition 8.2. We say that a leaf L_x is *without holonomy*, or that L_x has *trivial holonomy*, if $\mathrm{Germ}(\Phi_0, x)$ is trivial. We say that $\mathrm{Germ}(\Phi_0, x)$ is *locally trivial* if there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that for *every* $g \in K(\mathcal{G}_x)$ the restriction $\Phi_0(g)|U_x$ is the trivial map.

The distinction between the holonomy group $\mathrm{Germ}(\Phi_0, x)$ being trivial and it being locally trivial may seem technical, but this distinction is related to fundamental dynamical properties of the foliation $\mathcal{F}_{\mathcal{P}}$ of $\mathcal{S}_{\mathcal{P}}$. For example, it is a key concept in the generalizations of the Reeb stability theorem from compact leaves to the noncompact case for codimension-one foliations, as discussed in [Sacksteder and Schwartz 1965; Inaba 1977; 1983]. The nomenclature "locally trivial" was introduced by Inaba [1977; 1983]. As we see below, this distinction is also important for the study of the dynamics of weak solenoids. First, we make an elementary observation, which implies Lemma 1.5 of the Introduction.

Lemma 8.3. *Suppose that $K(\mathcal{G}^x)$ is finitely generated. If* $\mathrm{Germ}(\Phi_0, x)$ *is trivial, then* $\mathrm{Germ}(\Phi_0, x)$ *is locally trivial.*

Proof. Let $\{g_1, \ldots, g_k\} \subset K(\mathcal{G}^x)$ be a set of generators. Then $\mathrm{Germ}(\Phi_0, x)$ being trivial implies that for each $1 \leq i \leq k$ there exists an open $U_i \subset V_0$ with $x \in U_i$ such that the restriction $\Phi_0(g_i)|U_i$ is the trivial map. Then let $U_x = U_1 \cap \cdots \cap U_k$, which is an open neighborhood of x, and the restriction $\Phi_0(g)|U_x$ is then trivial for all $g \in K(\mathcal{G}^x)$. □

We also recall a basic result, which is a version of the fundamental result of Epstein, Millett and Tischler [Epstein et al. 1977] in the language of group actions on Cantor sets.

Theorem 8.4. *Let (V_0, G_0, Φ_0) be a given action, and suppose that V_0 is a Baire space. Then the union of all $x \in V_0$ such that* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group forms a G_δ subset of V_0. In particular, there exists at least one $x \in V_0$ such that* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group.*

The following is an immediate consequence of this result and Definition 5.1.

Corollary 8.5. *Let (V_0, G_0, Φ_0) be a* ***regular*** *equicontinuous minimal Cantor system. Then* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group for all $x \in V_0$. Consequently, if $\mathfrak{M}$ is a homogeneous matchbox manifold, then all leaves of $\mathcal{F}_{\mathfrak{M}}$ are without germinal holonomy.*

8B. ***Algebraic conditions.*** Next, we explore the relation between the structure of a group chain $\mathcal{G}^x$ and the germinal holonomy group at x. First, note that for a given section V_0 and the holonomy action (V_0, G_0, Φ_0), the assumption that the germinal holonomy group $\mathrm{Germ}(\Phi_0, x)$ is trivial need not imply that $K(\mathcal{G}^x)$ is trivial, or even that it is a normal subgroup of G_0, as the following example shows.

Example 8.6. Let Γ be a finitely presented group and $\{\Gamma_\ell\}_{\ell \geq 0}$ be a chain of normal subgroups in Γ with kernel $\Gamma_x = \bigcap_\ell \Gamma_\ell$. Let H be a finite simple group, and let $K \subset H$ be a nontrivial subgroup. Since H is simple, K is not normal in H.

Let $G_0 = H \times \Gamma$ and $G_\ell = K \times \Gamma_\ell$, $\ell \geq 0$. Note that G_ℓ is a normal subgroup of $G_1 = K \times \Gamma_1$ for all $\ell \geq 1$, but G_ℓ is not normal in G_0. Thus, the group chain $\{G_\ell\}_{\ell \geq 0}$ is weakly normal. Let M_0 be a compact connected manifold without boundary such that $\pi_1(M_0, x_0) = G_0$, where $x_0 \in M_0$ is some basepoint. Then the group chain $\mathcal{G}^x = \{G_\ell\}_{\ell \geq 0}$ yields a presentation $\mathcal{P} = \{f_\ell^{\ell+1} : M_{\ell+1} \to M_\ell\}$, and the corresponding solenoid $\mathcal{S}_\mathcal{P}$ is homogeneous by Proposition 5.5.

By Theorem 8.4, $\mathcal{S}_\mathcal{P}$ has a leaf L_y without holonomy. By Remark 4.2, a group chain with basepoint y is given by $\mathcal{G}^y = \{g_i G_i g_i^{-1}\}_{i \geq 0}$, where $g_i = (c_i, \gamma_i)$. Since the projection $G_0/G_{\ell+1} \to G_0/G_\ell$ restricts to the identity map on the factor H/K, for all $\ell \geq 0$, one can write $g_i = (c, \gamma_i)$ for some $c \in H$. Since each Γ_ℓ is a normal subgroup, we have that $g_i G_i g_i^{-1} = cKc^{-1} \times \Gamma_i$. Thus, $K(\mathcal{G}^y) = cKc^{-1} \times \Gamma_x$ is not a normal subgroup of G_0, since H is simple.

Next, we consider the holonomy action of the elements in $K(\mathcal{G}^x)$ on V_0 in more detail, using the inverse limit model $\tau_x : V_0 \cong X_\infty^x = \{G_0/G_{\ell+1}^x \to G_0/G_\ell^x\}$. For each $n \geq 0$, set

$$U(x, n) = \{(g_\ell G_\ell^x) \in X_\infty^x \mid g_\ell = e \text{ if } \ell \leq n;\ g_\ell G_\ell^x = g_{\ell+1} G_\ell^x \text{ for all } \ell \geq n\}, \tag{42}$$

which is a “cylinder neighborhood” of $(eG_\ell^x) \in V_0$. Note that $\tau_x(V_n) = U(x, n)$ for $n \geq 0$, where V_n is a generating set in the partition introduced in Proposition 3.4.

Since $K(\mathcal{G}^x)$ is a subgroup of G_0, for each $n \geq 1$ one can consider its left action on the cosets in G_0/G_n^x. Such an action fixes the coset eG_n^x; thus the action of $g \in K(\mathcal{G}_n^x)$ fixes the neighborhood of the identity as a set, $\Phi_0(g) : U(x, n) \to U(x, n)$ for $g \in G_n^x$, and permutes the points in $U(x, n)$.

Now observe that the action of g has trivial germinal holonomy at x if for some $n_g > 0$, g acts trivially on the clopen neighborhood $U(x, n_g)$ of x; that is, $\Phi_0(g)|U(x, n_g)$ is the trivial map. The following algebraic characterization of elements without holonomy was obtained in [Dyer et al. 2017, Lemma 5.3].

Lemma 8.7. *The action of $g \in K(\mathcal{G}^x)$ has **trivial germinal holonomy** at x if and only if there exists some index $i_g \geq 0$ such that multiplication by g satisfies $g \cdot hK(\mathcal{G}^x) = hK(\mathcal{G}^x)$ for all $h \in G_{i_g}$. That is, $h^{-1}gh \in K(\mathcal{G}^x)$ for all $h \in G_{i_g}$.*

In the case where the kernel $K(\mathcal{G}^x)$ is finitely generated, we have the following consequence of Lemma 8.7, whose proof can be compared with that of Lemma 8.3.

Proposition 8.8. *Let $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ be a group chain, and suppose the kernel $K(\mathcal{G}^x)$ is finitely generated. Suppose that* $\mathrm{Germ}(\Phi_0, x)$ *is the trivial group. Then there is an index $\ell_x \geq 0$ such that $K(\mathcal{G}^x)$ is a normal subgroup of $G_{\ell_x}^x$.*

Proof. Let $\{g_1, \ldots, g_k\} \subset K(\mathcal{G}^x)$ be a set of generators. Then for each $1 \leq \ell \leq k$, there exists $i_\ell \geq 0$ such that $h^{-1}gh \in K(\mathcal{G}^x)$ for all $h \in G_{i_\ell}$. Let $\ell_x = \max\{i_1, \ldots, i_k\}$. Then this implies that $h^{-1}gh \in K(\mathcal{G}^x)$ for all $g \in K(\mathcal{G}^x)$ and $h \in G_{\ell_x}$; that is, $K(\mathcal{G}^x)$ is a normal subgroup of $G_{\ell_x}^x$. □

Remark 8.9. The condition that the kernel $K(\mathcal{G}^x)$ of the group chain $\mathcal{G}^x$ is finitely generated is essential. Example 9.7 gives a group chain whose kernel at x is infinitely generated, and the germinal holonomy group $\mathrm{Germ}(\Phi_0, x)$ is not locally trivial.

Proposition 8.8 implies the following result, which is an algebraic analog of Reeb stability.

Proposition 8.10. *Let (V_0, G_0, Φ_0) be a minimal equicontinuous Cantor group action. Let $x, y \in V_0$ be such that both germinal holonomy groups $\mathrm{Germ}(\Phi_0, x)$ and $\mathrm{Germ}(\Phi_0, y)$ are **locally trivial**. Then for associated group chains $\mathcal{G}^x$ and $\mathcal{G}^y$, the kernels $K(\mathcal{G}^x)$ and $K(\mathcal{G}^y)$ are conjugate subgroups of G_0.*

Proof. Let $\mathcal{G}^x$ and $\mathcal{G}^y$ be group chains at x and y, respectively, for the action (V_0, G_0, Φ_0). Let $\tau_x : \mathfrak{X}_0 \to X^x_\infty$ and $\tau_y : \mathfrak{X}_0 \to X^y_\infty$ be the corresponding homeomorphisms defined in Lemma 4.1, each of which is equivariant with respect to the action (7) of G_0.

By the assumption that $\mathrm{Germ}(\Phi_0, x)$ is locally trivial, there exists an open set $U_x \subset V_0$ with $x \in U_x$ such that for every $g \in K(\mathcal{G}^x)$ the restriction $\Phi_0(g)|U_x$ is the trivial map. As the image $\tau_x(U_x) \subset X^x_\infty$ is open and contains $(eG^x_i) = \tau_x(x)$, there exists an index $\ell_x > 0$ such that $U((eG^x_i), \ell_x) \subset \tau_x(U_x)$, where $U((eG^x_i), \ell_x)$ is defined in (42). Note that $G^x_{\ell_x}$ is the stabilizer of $U((eG^x_i), \ell_x)$ for the action of G_0. Then $K(\mathcal{G}^x)$ acts trivially on $U((eG^x_i), \ell_x)$, so $K(\mathcal{G}^x)$ is a normal subgroup of $G^x_{\ell_x}$ by Lemma 8.7.

Set $V_1 = \tau_x^{-1}(U((eG^x_\ell), \ell_x)) \subset U_x$ and let $z \in V_1$ with $z \neq x$. Then the image $\tau_x(z)$ is $(h_i G^x_i)$, where $h_i \in G^x_{\ell_x}$ for $i \geq \ell_x$ and $h_i = e$ for $i \leq \ell_x$. As usual, the sequence (h_i) also satisfies the compatibility condition $h_i G^x_i = h_j G^x_i$ for all $i \geq 0$ and $j > i$. By Remark 4.2, we have that $\mathcal{G}^z = \{h_i G^x_i h_i^{-1}\}_{i \geq 0}$.

Note that $h_i K(\mathcal{G}^x) h_i^{-1} = K(\mathcal{G}^x)$ for $i \geq 0$, since $K(\mathcal{G}^x)$ is normal in $G^x_{\ell_x}$, so we have

$$K(\mathcal{G}^x) = \bigcap_{i \geq 0} G^x_i = \bigcap_{i \geq 0} h_i K(\mathcal{G}^x) h_i^{-1} \subseteq \bigcap_{i \geq 0} h_i G^x_i h_i^{-1} = K(\mathcal{G}^z). \tag{43}$$

In general, this inclusion may be proper, as illustrated in Example 9.6.

Now assume that $\mathrm{Germ}(\Phi_0, z)$ is locally trivial. We show that $K(\mathcal{G}^z) \subseteq K(\mathcal{G}^x)$. First, note that there exists an open set $U_z \subset V_0$ with $z \in U_z$ such that for every $g \in K(\mathcal{G}^z)$ the restriction $\Phi_0(g)|U_z$ is the trivial map. Recall that $\tau_x(z) = (h_i G^x_i) \in U((eG^x_\ell), \ell_x)$. Then there exists $\ell_z \geq \ell_x$ such that

$$U((h_i G^x_i), \ell_z) = \{(g_i G^x_i) \in X^x_\infty \mid g_i = h_i \text{ for } i \leq \ell\} \subset \tau_x(U_z). \tag{44}$$

That is, $g \in K(\mathcal{G}^z)$ acts trivially on the cylinder set $U((h_i G^x_i), \ell_z)$ in X^x_∞. Let $h = h_{\ell_z} \in G^x_{\ell_x}$, so we obtain an element $(hG^x_i) \in X^x_\infty$. By choice of h and (44) we have $(hG^x_i) \in U((h_i G^x_i), \ell_z)$. Now let $g \in K(\mathcal{G}^z)$. Then the restricted map $\Phi_0(g)|U_z$ is the identity, so we have $g \cdot (hG^x_i) = (hG^x_i)$. But this means that $h^{-1} g h G^x_i = G^x_i$

for all $i \geq 0$, and thus $h^{-1}gh \in K(\mathcal{G}^x)$, or $g \in hK(\mathcal{G}^x)h^{-1}$. Since $h \in G^x_{\ell_z} \subset G^x_{\ell_x}$, and $K(\mathcal{G}^x)$ is a normal subgroup of $G^x_{\ell_x}$, this implies that $K(\mathcal{G}^z) \subseteq K(\mathcal{G}^x)$.

Now suppose that $y \in V_0$ is such that $\mathrm{Germ}(\Phi_0, y)$ is locally trivial. The action of G_0 on V_0 is assumed to be minimal, so there exists $g \in G_0$ such that $z = \Phi_0(g)(y) \in V_1$. Then the holonomy at z is also locally trivial, so $K(\mathcal{G}^z) = K(\mathcal{G}^x)$ by the argument above. On the other hand, we have $K(\mathcal{G}^y) = g^{-1}K(\mathcal{G}^z)g$ as $K(\mathcal{G}^y)$ is the isotropy subgroup of y. The claim of the proposition then follows. □

8C. ***Kernels and discriminants.*** We give two results concerning the relation between the kernel of a group chain and its discriminant.

Proposition 8.11. *Let* (V_0, G_0, Φ_0) *be an equicontinuous minimal Cantor system*, $x \in V_0$ *be a choice of basepoint*, *and* $\mathcal{G}^x = \{G^x_\ell\}_{\ell \geq 0}$ *be a group chain associated to* (V_0, G_0, Φ_0) *at* x. *Let* $\mathcal{L}_0 = \mathrm{Ker}(\Phi_0)$ *denote the kernel of* $\Phi_0 : G_0 \to \mathrm{Homeo}(V_0)$. *Then* $K(\mathcal{G}^x) \subset \mathcal{L}_0$ *if and only if the intersection* $\Phi_0(G_0) \cap \overline{\Phi_0(G_0)}_x$ *is the trivial group.*

Proof. By Theorem 6.1, we can identify $\overline{\Phi_0(G_0)} \cong C_\infty$ and $\overline{\Phi_0(G_0)}_x \cong \mathcal{D}_x$, where the image $\Phi_0(G_0)$ is identified with the elements $(g_\ell C_\ell) \in C_\infty$ such that $g_\ell C_\ell = gC_\ell$ for all $\ell \geq 0$, for some $g \in G_0$.

First, suppose that $g \in G_0$ satisfies $\Phi(g) \in \overline{\Phi_0(G_0)}_x$ and $\Phi(g)$ is not the trivial element. Then $\hat{g} = (gC_\ell) \in \mathcal{D}_x$ and $(gC_\ell) \neq (eC_\ell)$, so there exists $\ell_0 > 0$ such that $g \notin C_{\ell_0}$. By the definition of $\mathcal{D}_x$ in (25), we have that $\hat{g}$ is in the image of the map $\delta^{\ell+1}_\ell : D^x_{\ell+1} \to D^x_\ell$ for all $\ell > 0$ where $D^x_\ell = G^x_\ell / C_\ell$. This implies that $gC_\ell \subset G^x_\ell$, and hence $g \in G^x_\ell$ for all $\ell \geq 0$, and so $g \in K(\mathcal{G}^x)$. We claim that $\Phi_0(g)$ is not the trivial action, so that $g \notin \mathcal{L}_0$. It is given that $g \notin C_{\ell_0}$; hence $gC_{\ell_0} \neq C_{\ell_0}$. Then for all $\ell \geq \ell_0$, we have $gC_\ell \neq C_\ell$, so $g \cdot (eC_\ell) \neq (eC_\ell)$, which implies $g \notin \mathcal{L}_0$. It follows that $K(\mathcal{G}^x) \not\subset \mathcal{L}_0$, as was to be shown.

Conversely, let $g \in K(\mathcal{G}^x)$ and suppose that $g \notin \mathcal{L}_0$. First note that $g \in G^x_\ell$ for all $\ell \geq 0$, and so we have $\hat{g} = (gC_\ell) \in \mathcal{D}_x$. The assumption that $g \notin \mathcal{L}_0$ implies there exists some $(h_\ell C_\ell) \in C_\infty$ such that $g \cdot (h_\ell C_\ell) \neq (h_\ell C_\ell)$. Thus, there exists $\ell_0 > 0$ such that for all $\ell \geq \ell_0$ we have $gh_\ell C_\ell \neq h_\ell C_\ell$, which implies that $h_\ell^{-1} g h_\ell \notin C_\ell$ and so $g \notin C_\ell$ as C_ℓ is a normal subgroup of G_0. Thus, $(eC_\ell) \neq (gC_\ell)$ for all $\ell \geq \ell_0$, and so $(gC_\ell) \in \mathcal{D}_x$ is nontrivial. That is, $\Phi(g) \in \overline{\Phi_0(G_0)}_x$ is a nontrivial element, as was to be shown. □

Compare the following application of Proposition 8.11 with the conclusions of Theorem 7.7.

Proposition 8.12. *Let* $\mathfrak{M}$ *be an equicontinuous matchbox manifold*, *let* V_0 *be a transverse section in* $\mathfrak{M}$ *as given in Proposition 3.4*, *and let* $x \in V_0$ *be a choice of basepoint. Let* V_ℓ *be defined as in Proposition 3.4*, *so that* $x \in V_\ell$ *for all* $\ell \geq 0$. *Let* G_0 *be the restricted holonomy group acting on* V_0, *and let* $G^x_\ell \subset G_0$ *be the* *stabilizer group of the set V_ℓ. Let $\mathcal{G}^x = \{G^x_\ell\}_{\ell\geq 0}$ be the associated group chain in $G_0 = \pi_1(M_0, x_0)$, and let $\mathcal{G}^x_n = \{G^x_\ell\}_{\ell\geq n}$ be the associated truncated group chain. Assume that the leaf L_x containing x has nontrivial germinal holonomy. Then the discriminant $\mathcal{D}^x_n$ for the chain $\mathcal{G}^x_n$ is nontrivial for all $n \geq 0$.*

Proof. Let $n \geq 0$, and let $\mathcal{L}_n \subset G^x_n$ be the kernel of the restricted action $\Phi_n : G^x_n \to \mathrm{Homeo}(V_n)$. Then by Lemma 8.7, the kernel $K(\mathcal{G}^x_n) \subset G^x_n$ is not a normal subgroup, so $\mathcal{L}_n \subset K(\mathcal{G}^x_n)$ is a proper inclusion. Then by Proposition 8.11, the discriminant group $\mathcal{G}^x_n$ is nontrivial also. □

This yields the proof of Theorem 1.7 of the Introduction, which we restate now.

Theorem 8.13. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. If there exists a leaf with nontrivial holonomy for $\mathcal{F}_{\mathfrak{M}}$, then for any choice of transversal $V_0 \subset \mathfrak{M}$, the resulting Molino sequence (28) has nontrivial fiber $\mathcal{D}$.*

The converse to Theorem 8.13 is not true. Fokkink and Oversteegen [2002, Theorem 35] constructed an example of a solenoid with simply connected leaves which is nonhomogeneous. Since the leaves are simply connected, they have trivial holonomy. In Section 10 we construct further examples of actions with nontrivial Molino fiber and simply connected leaves.

9. Strongly quasianalytic actions

In this section, we study the condition of *strong quasianalyticity*, abbreviated as the SQA condition, for equicontinuous matchbox manifolds, as defined in Definition 9.2 below. We identify classes of matchbox manifolds for which this condition holds, and also give examples for which it does not. The generalization of Molino theory in [Álvarez López and Moreira Galicia 2016] applies to equicontinuous foliated spaces such that the closure of their holonomy pseudo⋆groups satisfies the SQA condition. Thus, it is important to characterize the weak solenoids with this property.

9A. ***The strong quasianalyticity condition.*** The precise notion of the SQA condition has evolved in the literature, motivated by the search for a condition equivalent to the quasianalyticity condition for the pseudo⋆groups of smooth foliations as introduced by Haefliger [1985]. Álvarez López and Candel [2009] introduced the notion of a *quasieffective* pseudo⋆group as part of their study of equicontinuous foliated spaces. This terminology was replaced by the notion of a *strongly quasianalytic* pseudo⋆group in [Álvarez López and Moreira Galicia 2016].

Definition 9.1. [Haefliger 1985] A pseudo⋆group $\mathcal{G}^*$ acting on a locally compact locally connected space $\mathfrak{T}$ is *quasianalytic* if for every $h \in \mathcal{G}^*$ the following holds: Let $U \subset \mathrm{Dom}(h) \subset \mathfrak{T}$ be an open set, and suppose $x \in \mathfrak{T}$ is in the closure of U. Suppose the restriction $h|U$ is the identity map. Then there is an open neighborhood V of x such that the restriction $h|V$ is the identity map.

Definition 9.1 describes the properties of pseudo$\star$groups, which were discussed in Remark 2.1, where the action of an element is *locally determined*; that is, if h is the identity on an open set, then it is the identity on a larger set. For the case where the space $\mathfrak{T}$ is not locally connected, Álvarez López and Candel [2009] introduced the following modification of this notion.

Definition 9.2. A pseudo$\star$group $\mathcal{G}^*$ acting on a locally compact space $\mathfrak{T}$ is *strongly quasianalytic*, or SQA, if for every $h \in \mathcal{G}^*$ the following holds: Let $U \subset \mathrm{Dom}(h)$ be a nonempty open set, and suppose the restriction $h|U$ is the identity map. Then h is the identity map on its domain $\mathrm{Dom}(h)$. A matchbox manifold $\mathfrak{M}$ satisfies the SQA condition if there exists a traversal $V_0 \subset \mathfrak{M}$ such that the induced pseudo$\star$group $\mathcal{G}^*_{\mathcal{F}}$ on V_0 satisfies the SQA condition.

Definition 9.2 says that the action of an equicontinuous strongly quasianalytic pseudo$\star$group $\mathcal{G}$ is locally determined. That is, if h is the identity on a nonempty open subset of its domain, then it is the identity on $\mathrm{Dom}(h)$. In the case where the transversal $\mathfrak{T}$ is locally compact and locally connected, this condition is equivalent to quasianalyticity by [Álvarez López and Candel 2009, Lemma 9.8]. However, when $\mathfrak{T}$ is totally disconnected, the SQA condition becomes a statement about the algebraic properties of the group chain associated to the action, as we next discuss.

Recall from Proposition 3.4(1) that if $\mathfrak{M}$ is an equicontinuous matchbox manifold, then we can assume that the pseudo$\star$group action on the transversal is given by an equicontinuous minimal Cantor action (V_0, G_0, Φ_0). Thus, for each $h \in G_0$ we have $\mathrm{Dom}(h) = V_0$. Moreover, the assumption that the restriction $h|U$ is the identity in the statement of Definition 9.2 means that the SQA condition need only be checked for $h \in G_0$ such that there exists $x \in V_0$ for which $\Phi_0(h)(x) = x$, that is, those elements whose action fixes at least a point.

Recall from Section 6A that the closure $\overline{\Phi_0(G_0)} \subset \mathrm{Homeo}(V_0)$ in the uniform topology of the image $\Phi_0(G_0) \subset \mathrm{Homeo}(V_0)$ is called the Ellis group of the Cantor system (V_0, G_0, Φ_0), which yields a Cantor system $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$, where $\widehat{\Phi}_0 : \overline{\Phi_0(G_0)} \to \mathrm{Homeo}(V_0)$. Given $x \in V_0$ then $\overline{\Phi_0(G_0)}_x \subset \overline{\Phi_0(G_0)}$ denotes the isotropy subgroup at x for the action, and then the SQA condition must be checked for all elements of $\overline{\Phi_0(G_0)}_x$. We set $\widehat{\Phi}_0(G_0) = \{\Phi_0(g) \mid g \in G_0\}$, which is a dense subgroup of $\overline{\Phi_0(G_0)}$. The following result follows from the definitions.

Lemma 9.3. *If $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ satisfies the SQA condition, then (V_0, G_0, Φ_0) also satisfies the SQA condition. Conversely, suppose that $\overline{\Phi_0(G_0)}_x \subset \widehat{\Phi}_0(G_0)$. Then (V_0, G_0, Φ_0) satisfying the SQA condition implies that $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ satisfies the SQA condition.*

Proof. Let $g \in G_0$ and set $\hat{g} = \Phi_0(g) \in \mathrm{Homeo}(V_0)$. Then $\hat{g} \in \overline{\Phi_0(G_0)}$, so that if $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ satisfies the SQA condition then so must the action of $\hat{g}$. Conversely, suppose (V_0, G_0, Φ_0) satisfies the SQA condition. As noted above, the

SQA property need only be checked for $h \in \overline{\Phi_0(G_0)}_x$. By assumption, such an h belongs to $\Phi_0(G_0)$ and so satisfies the SQA condition. □

Note that the assumption that $\overline{\Phi_0(G_0)}_x \subset \widehat{\Phi}_0(G_0)$ implies that the compact set $\overline{\Phi_0(G_0)}_x$ is contained in a countable set, hence it must be finite. Thus, by Theorem 6.1, this implies that the discriminant group $\mathcal{D}_x$ of the action is finite. The converse need not be true. That is, if the discriminant $\overline{\Phi_0(G_0)}_x$ is finite, then it may be possible to choose a point $y \in V_0$ such that $\overline{\Phi_0(G_0)}_y$ has trivial intersection with $\widehat{\Phi}_0(G_0)$; for instance, this is the case for Example 9.6. Examples in Section 10 show that it is possible to construct actions (V_0, G_0, Φ_0) such that $\overline{\Phi_0(G_0)}_x$ has trivial intersection with $\widehat{\Phi}_0(G_0)$ for any choice of $x \in V_0$.

We next consider the SQA property for an equicontinuous minimal Cantor system (V_0, G_0, Φ_0) and its associated Ellis system $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$. This condition for the system (V_0, G_0, Φ_0) can be formulated in terms of the group chain model developed in Sections 4 and 6B, in which case Lemma 8.7 and Proposition 8.8 imply that the condition is a statement about the holonomy action of the kernel $K(\mathcal{G}^x)$ of the chain $\mathcal{G}^x$ for each $x \in V_0$. Examples 9.6 and 9.7 below and the discussion in Section 10 illustrate the possibilities.

The SQA property for the system $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ can be much more subtle to check, as now it is a condition on the action of the isotropy group $\overline{\Phi_0(G_0)}_x \cong \mathcal{D}_x$ which depends on the algebraic properties of the closed subgroup $\mathcal{D}_x \subset C_\infty$. Note that in this case, for any $x, y \in V_0$ the isotropy groups $\mathcal{D}_x$ and $\mathcal{D}_y$ are conjugate in C_∞, so it suffices to consider the condition for a fixed choice of basepoint $x \in V_0$.

9B. ***Sufficient conditions for the SQA property.*** We next indicate a few classes of solenoids which satisfy the quasianalyticity condition.

Lemma 9.4. *If a matchbox manifold $\mathfrak{M}$ is homogeneous, then there exists a section V_0 with associated presentation $\mathcal{P}$ such that the actions (V_0, G_0, Φ_0) and $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ are SQA.*

Proof. By Corollary 6.4 one can assume that V_0 and $\mathcal{P}$ are chosen so that the associated group chain $\{G^x_\ell\}_{\ell \geq 0}$ consists of normal subgroups. Then $K(\mathcal{G}^x)$ is a normal subgroup of G_0, so by Lemma 8.7, each $g \in K(\mathcal{G}^x)$ defines a trivial holonomy action on V_0. Hence the action of G_0 on V_0 is SQA.

Since $\{G^x_\ell\}_{\ell \geq 0}$ is a chain of normal subgroups, the isotropy group $\overline{\Phi_0(G_0)}_x$ is trivial by Proposition 6.3, and so the condition $\overline{\Phi_0(G_0)}_x \subset \widehat{\Phi}_0(G_0)$ is trivially satisfied. Then by Lemma 9.3 the action $(V_0, \overline{\Phi_0(G_0)}, \widehat{\Phi}_0)$ is SQA. □

Note that the holonomy pseudogroups associated to homogeneous solenoids, as in Lemma 9.4, satisfy a stronger condition than SQA. Recall from [Álvarez López and Moreira Galicia 2016, Definition 2.22] that the action of G_0 on V_0 is *strongly locally free* if for all $h \in G_0$, if $h(x) = x$, then $h(y) = y$ for all $y \in V_0$. If $\mathfrak{M}$ is homogeneous, then the action on a local section V_0, as given by Lemma 9.4, is strongly locally free. The actions in Lemma 9.4 are the actions in [Álvarez López and Moreira Galicia 2016, Example 2.35].

The following result gives a class of equicontinuous matchbox manifolds which satisfy the SQA condition. This is Theorem 1.8 of the Introduction.

Theorem 9.5. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold of finite π_1-type. Then there exists a section V_0 with a presentation $\mathcal{P}$ such that the action (V_0, G_0, Φ_0) is SQA. Further, if V_0 can be chosen so that the discriminant group $\mathcal{D}_x = \overline{\Phi_0(G_0)}_x$ is finite, then there exists an $n \geq 0$ such that the restricted action (V_n, G_n^x, Φ_n) and the action $(V_n, \overline{\Phi_n(G_n^x)}, \widehat{\Phi}_n)$ are both SQA.*

Proof. Let V_0 be a transverse section in $\mathfrak{M}$ as given in Proposition 3.4, and let $x \in V_0$ be a choice of basepoint. By Theorem 8.4 we can assume that x is chosen so that L_x is a leaf without holonomy. As the leaves of $\mathcal{F}_{\mathfrak{M}}$ are assumed to have finite π_1-type, by Lemma 8.7 and Proposition 8.8, and restricting to a smaller section is necessary, we can assume that V_0 and $\{\mathcal{G}_\ell^x\}_{\ell \geq 0}$ are chosen so that $K(\mathcal{G}^x)$ is a normal subgroup of G_0. Then by Proposition 8.10, $K(\mathcal{G}^x) \subseteq K(\mathcal{G}^y)$ for all $y \in V_0$, and, if $\mathrm{Germ}(y, \Phi_0)$ is trivial, then $K(\mathcal{G}^x) = K(\mathcal{G}^y)$.

Since the G_0-orbit of x is dense in V_0, any $g \in G_0$ which is the identity on a nonempty open set in V_0 must be contained in $K(\mathcal{G}^x)$, and so it is the identity on V_0. Thus, (V_0, G_0, Φ_0) is SQA.

Now let C_∞ be the Ellis group, associated to (V_0, G_0, Φ_0), and suppose the discriminant group $\mathcal{D}_x \cong \overline{\Phi_0(G_0)}_x$ is finite. Suppose there exists a nontrivial element $\hat{g} \in \overline{\Phi_0(G_0)}_x$ which fixes an open subset U of V_0 around x.

Let V_ℓ be defined as in Proposition 3.4, so that $x \in V_\ell$ for all $\ell \geq 0$. Choose an index $n \geq 0$ large enough so that $V_n \subset U$. Let $y \in V_n$. Then

$$\hat{g} = (g_i C_i) \in \overline{\Phi_0(G_0)}_y,$$

and it follows that

$$\hat{g} \in \bigcap_{y \in V_n} \overline{\Phi_0(G_0)}_y,$$

that is, the intersection $\bigcap_{y \in V_n} \mathcal{D}_y$ is nontrivial.

Consider the truncated chain $\{G_\ell^x\}_{\ell \geq n}$ and the corresponding action (V_n, G_n^x, Φ_n). Recall from Section 7C that $E_\ell^n = \mathrm{core}_{G_n^x} G_\ell^x$ is a maximal normal subgroup of G_ℓ^x in G_n^x, and there is an inclusion

$$C_\ell \subset E_\ell^n \subset G_\ell^x, \tag{45}$$

where C_ℓ is the maximal normal subgroup of G_ℓ^x in G_0. The Ellis group E_∞^n of the restricted action (V_n, G_n^x, Φ_n) is defined by (34) as the inverse limit of coset spaces

G^x_n/E^n_ℓ. The inclusions (45) yield a commutative diagram

$$\begin{tikzcd} G^x_n/C_\ell \arrow[rr, "\varphi_{n,\ell}"] \arrow[dr] & & G^x_n/E^n_\ell \arrow[dl] \\ & G^x_n/G^x_\ell & \end{tikzcd} \tag{46}$$

which is equivariant with respect to the natural action of G^x_n on its coset spaces. Taking the inverse limits, we obtain the commutative diagram

$$\begin{tikzcd} C^n_\infty \arrow[rr, "\varphi_{n,\infty}"] \arrow[dr] & & E^n_\infty \arrow[dl] \\ & G^n_\infty \cong V_n & \end{tikzcd} \tag{47}$$

where C^n_∞ is the profinite subgroup of C_∞, defined by (31), which is again equivariant with respect to the action of G^x_n on the inverse limits, and $\varphi_{n,\infty}$ is a surjective group homomorphism.

Let $\hat{g}_n = \varphi_{n,\infty}(\hat{g})$. We will show that $\hat{g}_n$ acts trivially on V_n. Indeed, let $\hat{g} = (g_\ell C_\ell)$, where $g_\ell \in G^x_n$. Then $\hat{g}_n = (g_\ell E^n_\ell)$ for $\ell \geq n$. Since C_ℓ and E^n_ℓ are normal subgroups of G^x_n, the actions of $g_\ell C_\ell$ and $g_\ell E^n_\ell$ on G^x_n/G_ℓ are well defined; for example, for any $h \in G^x_n$ we have

$$g_\ell C_\ell h G^x_\ell = g_\ell h C_\ell h^{-1} h G^x_\ell = g h G^x_\ell,$$

and similarly for $g_\ell E^n_\ell$. Since diagram (46) is a commutative diagram of equivariant maps, we obtain that

$$g_\ell C_\ell h G^x_\ell = h G^x_\ell \implies g_\ell E^n_\ell h G^x_\ell = h G^x_\ell,$$

and it follows that if $\hat{g}$ acts trivially on $y = (h_i G^x_\ell) \in V_n$, then $\hat{g}_n$ acts trivially on y as well.

Then by an argument similar to the one at the beginning of this proof, we obtain that $\hat{g}_n \in \bigcap_{y\in V_n} \mathcal{D}^n_y$, where $\mathcal{D}^n_y$ is the discriminant group of the truncated action (V_0, G^x_n, Φ_n) at $y \in V_n$. We note that $\bigcap_{y\in V_n} \mathcal{D}^n_y$ is the maximal normal subgroup of $\mathcal{D}^n_x$, and so by Proposition 6.2 it must be trivial. Therefore, $\hat{g}_n = \varphi_{n,\infty}(\hat{g})$ is the identity in E^n_∞.

We note that the restricted group action (V_n, G^x_n, Φ_n) is SQA since (V_0, G_0, Φ_0) is SQA. By restricting to a smaller section and applying the above argument a finite number of times we may assume that no element of the discriminant group $\mathcal{D}^n_x$ fixes an open subset of V_n. It follows that the action $(V_n, \overline{\Phi_n(G^x_n)}, \widehat{\Phi}_n)$ of the closure is SQA. $\square$

9C. ***SQA counterexamples.*** We give two classes of examples to illustrate the above results.

Example 9.6. We first give an example of a group action, corresponding to the holonomy of a solenoid with leaves of finite π_1-type, that is not strongly locally free.

Let K be the Klein bottle, with fundamental group $G_0 = \langle a, b \mid bab^{-1} = a^{-1} \rangle$, and let $K_\infty = \varprojlim \{p : K \to K\}$ be the inverse limit space, as described in Example 4.11. The solenoid K_∞ contains one nonorientable leaf with one end, and every other leaf is an open two-ended cylinder. Thus, each leaf is homotopic to a circle, and thus has finite π_1-type.

The group chain $\mathcal{G}^x$, associated to the choice of basepoint as in Example 4.11, consists of subgroups $G^x_\ell = \langle a^{2^\ell}, b \rangle$, $\ell \geq 0$, and $K(\mathcal{G}^x) = \langle b \rangle$. This leaf has nontrivial holonomy, with $\mathrm{Germ}(x, \Phi) \cong \mathbb{Z}_2$. Fokkink and Oversteegen [2002] computed that the kernel of a group chain based at any point which is not in the orbit of x is $K(\mathcal{G}^y) = \langle b^2 \rangle$, which is easily seen to be a normal subgroup of G_0. Thus for the chosen section V_0, for every point y with trivial $\mathrm{Germ}(y, \Phi)$ the kernel $K(\mathcal{G}^y)$ is a normal subgroup of G^y_ℓ, $\ell \geq 0$, and the section satisfies Proposition 8.10. So the action (V_0, G_0, Φ) satisfies the SQA condition.

This action is not strongly locally free. Indeed, the action of the element b fixes x, but it does not fix any y with trivial $\mathrm{Germ}(y, \Phi)$. The nontrivial element in $\overline{\Phi(G_0)}_x$ acts nontrivially on any open subset of V_0, and so the action $(V_0, \overline{\Phi(G_0)}_x, \widehat{\Phi})$ satisfies the SQA condition.

Example 9.7. We next give an example of a solenoid for which the action of the holonomy group on the fiber is not SQA for any choice of a transverse section V_0. This example is the Schori solenoid [1966]. We now recall its construction, as described in [Clark et al. 2014].

Let X_0 be a genus-2 surface. Recall that a 1-handle is a 2-torus without an open disc, and note that the genus-2 surface X_0 can be seen as the union of two 1-handles H_0 and F_0 intersecting along the boundaries of the open discs taken out. Let x_0 be a point in the intersection of the handles. Recall that the fundamental group of the genus-2 surface can be presented as

$$\pi_1(X_0, x_0) = \langle a, b, c, d \mid aca^{-1}c^{-1}bdb^{-1}d^{-1} = 1 \rangle,$$

where a and b are longitudinal loops in X_0.

Cut the handle H_0 (resp. F_0) along a closed curve C_0 (resp. D_0), as shown in Figure 1, top left. Pull the cut handles apart to obtain the surface with boundary $\overline{X}_0$ (see Figure 1, top center). Take three copies of $\overline{X}_0$, denoted by $\overline{X}^1_0$, $\overline{X}^2_0$, $\overline{X}^3_0$, and identify their boundaries as shown in Figure 1, top right. The resulting surface X_1 (see Figure 1, bottom) has genus 4, and there is an obvious 3-to-1 covering map $f^1_0 : X_1 \to X_0$. Let x_1 be the preimage of x_0 in the second copy of the handle. We

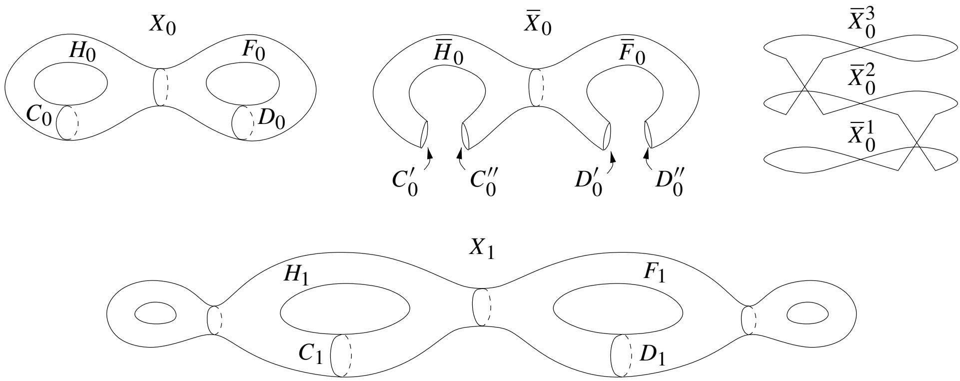

Figure 1. Construction of the Schori example. Top left: choice of the handles H_0 and F_0 and closed curves C_0 and D_0 in X_0. Top center: the cut surface $\overline{X}_0$. Top right: identifications between $\overline{X}_0^\ell$, $\ell = 1, 2, 3$. Each $\overline{X}_0^i$ is represented by a cut copy of a figure 8, and identifications are depicted with straight lines. Bottom: the surface X_1 and the choice of the handles H_1 and F_1 and closed curves C_1 and D_1.

note that the covering f_0^1 is not regular; that is, the image $(f_0^1)_*\pi_1(X_1, x_1)$ of the fundamental group of X_1 is not a normal subgroup of $\pi_1(X_0, x_0)$. Geometrically, we can see that f_0^1 is irregular as follows: Take a longitudinal loop γ in X_0, which represents an equivalence class of loops in $\pi_1(X_0, x_0)$. The fiber of f_0^1 consists of three points, and we see from Figure 1, top right, that depending on the initial point of the lift, γ may lift to a loop or to a nonclosed curve [Schori 1966].

Proceed inductively to obtain a collection of 3-to-1 coverings $f_\ell^{\ell+1} : X_{\ell+1} \to X_\ell$. That is, we can see X_ℓ as the union of two handles H_ℓ and F_ℓ, intersecting along their boundaries (see Figure 1, bottom) for $\ell = 1$. We cut the handle H_ℓ (resp. F_ℓ) along a closed curve C_ℓ (resp. D_ℓ), pull the handles apart to obtain the surface with boundary $\overline{X}_\ell$, take three copies of $\overline{X}_\ell$, denoted by $\overline{X}_\ell^1$, $\overline{X}_\ell^2$, $\overline{X}_\ell^3$, and identify their boundaries in a way similar to Figure 1, top right. The resulting surface $X_{\ell+1}$ is a 3-to-1 nonregular cover of X_ℓ. This defines a presentation $\mathcal{P} = \{f_\ell^{\ell+1} : X_{\ell+1} \to X_\ell, \ell \geq 0\}$ of the Schori solenoid $\mathcal{S}_\mathcal{P}$. Let $\mathfrak{X}_0$ be the fiber of $\mathcal{S}_\mathcal{P}$ at x_0.

For each $\ell \geq 0$, we choose $x_{\ell+1}$ to be a preimage of x_ℓ under the covering map $f_\ell^{\ell+1}$ in the second copy of X_ℓ. Denote by $\mathcal{G}^x = \{G_\ell^x\}_{\ell \geq 0}$ the corresponding group chain, and recall that there is a conjugacy

$$\varphi : \mathfrak{X}_0 \to X_\infty^x = \varprojlim \{G_0/G_{\ell+1}^x \to G_0/G_\ell^x, \ell \geq 0\}.$$

As before, we denote by $U(x, \ell)$ the cylinder set in X_∞^x containing (eG_ℓ^x). If $y \in \mathfrak{X}_0$ is a point with $\varphi(y) = (g_\ell G_\ell^x)$, then $g_\ell \cdot U(x, \ell) = \Phi(g_\ell)(U(x, \ell))$ is a cylinder set containing $\varphi(y)$. Set $U_\ell^y = \varphi^{-1}(g_\ell \cdot U(x, \ell))$. The group chain $\mathcal{G}^y = \{G_\ell^y = g_\ell G_\ell^x g_\ell^{-1}\}_{\ell \geq 0}$ corresponds to a presentation $\mathcal{P}'$ of the Schori solenoid with basepoint y.

The following result is Theorem 1.9 of the Introduction.

Theorem 9.8. *In the Schori solenoid, for any choice of basepoint $y \in \mathfrak{X}_0$, and any choice of section U_n^y, $n \geq 0$, the holonomy action (U_n^y, G_n^y, Φ_n) is not SQA.*

Proof. Let $y \in \mathfrak{X}_0$, and let (U_n^y, G_n^y, Φ_n) be the holonomy action. At the end of Section 2 we described the procedure of restricting to a smaller section, which gives us a presentation $\mathcal{P}_n' = \{f_\ell^{\ell+1} : X_{\ell+1} \to X_\ell, \ \ell \geq n\}$. By a slight abuse of notation, we now set $G_n^y = \pi_1(X_n, y_n)$ and $G_\ell^y = (f_n^\ell)_* \pi_1(X_\ell, y_\ell)$ (these groups are isomorphic to the groups $(f_0^\ell)_* \pi_1(X_\ell, y_\ell)$, which we denoted by G_ℓ^y earlier). Thus we have a homeomorphism

$$\varphi_n' : U_n^y \to X_{\infty,n}^y = \varprojlim\{G_n^y / G_{\ell+1}^y \to G_n^y / G_\ell^y\},$$

which commutes with the action of G_n^y on U_y^n and $X_{\infty,n}^y$. Denote by $U(y, \ell)$ the cylinder neighborhoods of (eG_ℓ^y) in $X_{\infty,n}^y$. In particular, $U(y, n) = X_{\infty,n}$.

The surface X_ℓ in the presentation $\mathcal{P}'$ has genus $m_\ell = 3^\ell + 1$ (see [Clark et al. 2014]), so G_ℓ^y has m_ℓ generators, represented by longitudinal loops. In particular, there are loops γ_n and δ_n which wind around the handles H_n and F_n in X_n, respectively. Denote by g_γ and g_δ the elements represented by γ_n and δ_n in G_n^y, respectively.

Now consider the construction of the surface X_{n+1}. It is obtained by the identification of three copies $\overline{X}_n^{1,2,3}$ of X_n similar to the identification in Figure 1, bottom left. There is a point y_{n+1} in one of the copies which satisfies $f_n^{n+1}(y_{n+1}) = y_n$, and which corresponds to our choice of the basepoint y. Denote by z_{n+1} and v_{n+1} the other two points such that

$$f_n^{n+1}(z_{n+1}) = f_n^{n+1}(v_{n+1}) = y_n.$$

Denote by $\gamma_{y_{n+1}}$, $\gamma_{z_{n+1}}$, and $\gamma_{v_{n+1}}$ the copies of γ_n in $\overline{X}_n^{1,2,3}$ with respective basepoints y_{n+1}, z_{n+1}, and v_{n+1}. Note that these loops are cut when constructing $\overline{X}_n^{1,2,3}$. We now proceed to identify the boundaries of $\overline{X}_n^{1,2,3}$ according to the construction, which would close one of the loops back, and would intertwine the boundaries of the other two loops, so as to create a single loop of twice the length of γ_n.

We have the following alternatives: first, suppose $\gamma_{z_{n+1}}$ is identified into a loop, and $\gamma_{y_{n+1}}$ and $\gamma_{v_{n+1}}$ are identified to make a single loop of twice the length. Then the lift of γ_n with the starting point y_{n+1} is the curve $\gamma_{y_{n+1}}$ which is not closed and has v_{n+1} as its ending point. This means that the action of g_γ on the coset space G_n^y / G_{n+1}^y maps eG_{n+1}^y onto $g_\gamma G_n^y$, and so maps the cylinder neighborhood $U(y, n+1)$ onto the clopen set $g_\gamma(U(y, n+1))$. At the same time, the lift of γ_n

with the starting point z_{n+1} is a closed loop. So the action of g_γ fixes the coset $\gamma_\delta G^y_{n+1}$, and the clopen set $g_\delta(U(y, n+1))$. We note that on the subsequent steps of the construction, when creating X_{n+i}, the lifts of the loop $\gamma_{z_{n+1}}$ are never cut and identified, which means that the action of g_γ is the identity on $g_\delta(U(y, n+1))$.

Another alternative is that $\gamma_{y_{n+1}}$ is identified into a loop, and $\gamma_{z_{n+1}}$ and $\gamma_{v_{n+1}}$ are identified to make a single loop. Arguing similarly, in this case we obtain that the action of g_γ is the identity on $U(y, n+1)$, and it permutes the sets $g_\delta(U(y, n+1))$ and $g_\gamma \circ g_\delta(U(y, n+1))$. Thus in both cases we obtain an element which is the identity on a clopen subset of the section U^y_n, which permutes two other subsets of U^n_y, which means that (U^y_n, G^y_n, Φ) is not SQA. Since the choice of y and n was arbitrary, we conclude that the holonomy pseudogroup for the Schori solenoid is not SQA. □

From the proof of Theorem 9.8 we obtain the following corollary, which shows that the hypotheses of Proposition 8.8 are necessary.

Corollary 9.9. *In the Schori solenoid, for any choice of a transverse section V_0, and any choice of a point x,* $\mathrm{Germ}(\Phi_0, x)$ *is not locally trivial.*

Proof. From the proof of Theorem 9.8 we conclude that, for any choice of basepoint $y \in \mathfrak{X}_0$, and any choice of group chain $\mathcal{G}^y_n = \{G^y_n\}_{i\geq 0}$, the kernel $K(G^y_n)$ is not a normal subgroup of G^x_n. It follows that even if $\mathrm{Germ}(\Phi_0, x)$ is trivial, it is not locally trivial. □

10. A universal construction

In this section, we give a general method of constructing examples of group chains with prescribed discriminant groups. This construction is inspired by the proof of Lemma 37 in Section 8 of [Fokkink and Oversteegen 2002], which they attribute to Hendrik Lenstra. The construction of Lenstra is given in Section 10A, and Section 10B discusses some properties of this construction. Then in Section 10C we give criteria for when the resulting group chains are stable.

Section 10D recalls two basic results of Lubotzky [1993]. The first, given here as Theorem 10.4, realizes any given finite group F embedded into the profinite completion of a finitely generated, torsion-free group G. A second result of Lubotzky, given here as Theorem 10.5, embeds the infinite product $\boldsymbol{H}$ of a collection of finite groups as a subgroup of the profinite completion of a finitely generated, torsion-free group G. Then in Section 10E, these constructions of Lubotzky are used to construct the examples used in the proofs of Theorems 1.10 and 1.12 of the Introduction.

There is an extensive literature on embedding groups into the profinite completion of a given torsion-free, finitely generated group (see [Ribes and Zalesskii 2000] for a discussion of this topic and further references). The methods of this section apply in this generality to yield an enormous range of equicontinuous minimal Cantor actions with infinite, hence Cantor discriminant, groups.

10A. ***A profinite construction.*** We first give a reformulation of the constructions in Sections 6B and 6C, in analogy with the construction of Lenstra in [Fokkink and Oversteegen 2002]. This alternate formulation is of strong interest in itself, as it gives a deeper understanding of the Molino spaces introduced in this work.

Let G_0 be a finitely generated group, $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ be a group chain in G_0, and $\mathcal{C} = \{C_\ell\}_{\ell \geq 0}$ be the core group chain associated to $\mathcal{G}$, with C_∞ the core group associated with $\mathcal{C}$. Assume the kernel $K(\mathcal{G})$ is trivial, so the map $\widehat{\Phi} : G_0 \to C_\infty$ is an injective homomorphism with dense image $\widehat{G}_0 = \widehat{\Phi}(G_0) \subset C_\infty$. Then the discriminant of $\mathcal{G}$ is a compact subgroup $\mathcal{D} \subset C_\infty$, whose rational core as defined in (26) is trivial by Proposition 6.2.

Let $C_{n,\infty} \subset C_\infty$ be the clopen normal subgroup neighborhood of the identity $\{e\}$ defined in (23). As $\bigcap_{n \geq 1} C_{n,\infty} = \{e\}$, the collection $\{C_{n,\infty} \mid n \geq 1\}$ is a clopen neighborhood system about the identity in C_∞. Observe that from the definition (21), we have that $C_\infty / C_{n,\infty} \cong G_0 / C_n$ and $\widehat{G}_0 \cap C_{n,\infty} \cong G_n$. As each subgroup $C_{n,\infty}$ is normal and $\mathcal{D}$ is compact, the product $V_n = \mathcal{D} \cdot C_{n,\infty} \subset C_\infty$ is a clopen subgroup of C_∞ containing $\mathcal{D}$, and we have $\mathcal{D} = \bigcap_{n \geq 1} V_n$. Thus, $\mathcal{D}$ is realized as the countable intersection of clopen subgroups of C_∞. It is an exercise to show that this formulation of $\mathcal{D}$ agrees the definition of $\mathcal{D}$ as an inverse limit in (25).

We now turn the order of the above remarks around to obtain a construction of a group chain with prescribed discriminant group.

Proposition 10.1. *Let C_∞ be a profinite group, and let $G \subset C_\infty$ be a finitely generated dense subgroup. Let $\mathcal{D} \subset C_\infty$ be a compact subgroup of infinite index which has trivial rational core,*

$$\mathrm{core}_G\, \mathcal{D} = \bigcap_{k \in G} k \mathcal{D} k^{-1} = \{e\}. \tag{48}$$

Then there exists a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ with $G_0 = G$, with discriminant group $\mathcal{D}$.

Proof. By the assumption that C_∞ is a profinite group, there exists a group chain $\{U_\ell \mid \ell \geq 1\}$ that is a clopen neighborhood system about the identity in C_∞, such that

(1) each U_ℓ is normal in C_∞,

(2) for each $\ell \geq 0$ there is a proper inclusion $U_{\ell+1} \subset U_\ell$,

(3) $\bigcap_{\ell \geq 1} U_\ell = \{e\}$.

In particular, each quotient $H^\ell \equiv C_\infty / U_\ell$ is a finite group. Let $\iota_\ell^{\ell+1} : H^{\ell+1} \to H^\ell$ be the map induced by inclusion of cosets. Then there is a natural identification

$$C_\infty \cong \varprojlim \{\iota_\ell^{\ell+1} : H^{\ell+1} \to H^\ell\}. \tag{49}$$

Next, for each $\ell \geq 1$, set $W_\ell = \mathcal{D} \cdot U_\ell$, which is a subgroup of C_∞, as U_ℓ is normal. Moreover, the assumption that $\mathcal{D}$ is compact implies that each W_ℓ is a clopen subset of C_∞. Then set $G_\ell = G \cap W_\ell$, which is a subgroup of finite index in G, and so $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is a subgroup chain in G. Note that

$$K(\mathcal{G}) = \bigcap_{\ell \geq 0} G_\ell = \bigcap_{\ell \geq 0} G \cap W_\ell = G \cap \bigcap_{\ell \geq 0} W_\ell = G \cap \mathcal{D}. \tag{50}$$

We next calculate the discriminant of the chain $\mathcal{G}$. Let $\pi^\ell : C_\infty \to H^\ell$ be the quotient map. As each H^ℓ is finite, the image $\mathcal{D}^\ell \equiv \pi^\ell(\mathcal{D})$ is a finite set. The group G is dense in C_∞ so has nontrivial intersection with each clopen set gU_ℓ. Thus,

$$\mathcal{D}^\ell = \pi^\ell(\mathcal{D}) = \pi^\ell(W_\ell) = \pi^\ell(G \cap W_\ell) = \pi^\ell(G_\ell) \subset H^\ell. \tag{51}$$

The core of the group G_ℓ is the group $C_\ell \equiv \mathrm{core}_G\, G_\ell = \bigcap_{g \in G} gG_\ell g^{-1}$. We have

$$\begin{aligned} \pi^\ell(C_\ell) &= \pi^\ell\Big(\bigcap_{g \in G} g\, G_\ell\, g^{-1}\Big) \\ &= \bigcap_{g \in G} \pi^\ell(g)\pi^\ell(G_\ell)\pi^\ell(g)^{-1} \\ &= \bigcap_{g \in G} \pi^\ell(g)\pi^\ell(\mathcal{D})\pi^\ell(g)^{-1} \\ &= \{e^\ell\}, \end{aligned} \tag{52}$$

where $e^\ell \in H^\ell$ is the identity, and the last equality follows since G is dense in C_∞ and the core of $\mathcal{D}$ is trivial. It follows that $C_\ell = G \cap U_\ell$, and thus we obtain induced maps on the quotients, $\bar{\pi}^\ell : G/C_\ell \to H_\ell$. Then note that $\pi^\ell(G_\ell/C_\ell) = \pi^\ell(\mathcal{D}) = \mathcal{D}^\ell$ for all $\ell \geq 0$.

The map $\iota_\ell^{\ell+1} : H^{\ell+1} \to H^\ell$ induces a map (denoted the same), $\iota_\ell^{\ell+1} : \mathcal{D}^{\ell+1} \to \mathcal{D}^\ell$. Then for the inverse limits we have

$$\varprojlim\{\delta_\ell^{\ell+1} : G_{\ell+1}/C_{\ell+1} \to G_\ell/C_\ell\} = \varprojlim\{\iota_\ell^{\ell+1} : \mathcal{D}^{\ell+1} \to \mathcal{D}^\ell\}. \tag{53}$$

The term on the left-hand side of (53) is by definition the discriminant of the chain $\mathcal{G}$, while the term on the right-hand side of (53) is homeomorphic to the subgroup $\mathcal{D}$, as $\{U_\ell \mid \ell \geq 1\}$ is a clopen neighborhood system about the identity in C_∞. □

10B. ***Properties of the Lenstra construction.*** We make some remarks about the construction in Proposition 10.1. First, note that the proof of [Fokkink and Oversteegen 2002, Lemma 37] defined the chain $\mathcal{G}_n$ using a collection of clopen neighborhoods of $e \in C_\infty$. However, the proof in that paper that the chain $\mathcal{G}_n$ is not weakly regular used Proposition 5.6, that is, the fact that if the number of conjugacy classes of the kernel $K(\mathcal{G}_n)$ is infinite, then $\mathcal{G}_n$ cannot be weakly regular. Our approach is to calculate the discriminant group for the chain directly.

Assume there is given a profinite group C_∞, a compact subgroup $\mathcal{D} \subset C_\infty$, and a dense subgroup $G \subset C_\infty$ satisfying the hypotheses of Proposition 10.1. Set $X = C_\infty/\mathcal{D}$, which is a Cantor space. The left action of G on X defines a map $\Phi : G \to \mathrm{Homeo}(X)$, which is a minimal action as G is dense in C_∞. Thus, the construction yields an equicontinuous minimal Cantor system (X, G, Φ).

Next, given a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in C_∞ which satisfies the conditions in the proof of Proposition 10.1, let $\mathcal{G} \equiv \{G_\ell\}_{\ell \geq 0}$ be the group chain in G constructed with respect to this clopen neighborhood system. Then it is an exercise, using the techniques of the proof of Proposition 10.1, to show that there is a G-equivariant homeomorphism of spaces

$$\tau : X \cong \varprojlim \{\iota_{\ell+1} : G/C_{\ell+1} \to G/G_\ell\} \equiv X_\infty.$$

Now suppose that $\{V_\ell \mid \ell \geq 1\}$ is another clopen neighborhood system about the identity in C_∞ which also satisfies the conditions in the proof of Proposition 10.1, and let $\mathcal{H} \equiv \{H_\ell\}_{\ell \geq 0}$ be the group chain in G constructed with respect to this second clopen neighborhood system. A basic property of neighborhood systems is that given any $\ell \geq 0$ there exists $\ell' \geq 0$ such that $V_{\ell'} \subset U_\ell$, and $\ell'' \geq 0$ such that $U_{\ell''} \subset V_\ell$. It follows from their definitions that the group chains $\mathcal{G}$ and $\mathcal{H}$ are equivalent in the sense of Definition 4.3.

Suppose that $G \cap \mathcal{D} = \{e\}$. Then the calculation (50) shows that the kernel $K(\mathcal{G}) = \{e\}$ is trivial. Moreover, suppose the choice of $\mathcal{D}$ is made so that $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} = \{e\}$ for all $\hat{g} \in C_\infty$. Given $y \in X$ let $\tau(y) = (g_\ell G_\ell) \in X_\infty$, and let $\mathcal{G}^y = \{g_\ell G_\ell g_\ell^{-1}\}_{\ell \geq 0}$ be the conjugate group chain. Choose $\hat{g} \in C_\infty$ such that $\tau(\hat{g}\mathcal{D}) = (g_\ell G_\ell)$. Then

$$K(\mathcal{G}^y) = G \cap (\hat{g}\mathcal{D}\hat{g}^{-1}) = \{e\} \tag{54}$$

so that $\mathcal{G}^y$ also has trivial kernel. Thus, if we choose $\mathcal{D}$ so that $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} = \{e\}$ for all $\hat{g} \in C_\infty$ is satisfied, then the Cantor system (X, G, Φ) has trivial kernel for the group chain $\mathcal{G}^y$ at y for all points $y \in X$. For example, suppose that G is a torsion-free group and $\mathcal{D}$ is a torsion group. Then the condition $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} = \{e\}$ for all $\hat{g} \in C_\infty$ is automatically satisfied, as each nontrivial element of $\mathcal{D}$, and hence $\hat{g}\mathcal{D}\hat{g}^{-1}$, has finite order. We use this observation in Theorem 10.7 below.

On the other hand, given $G \subset C_\infty$ as in Proposition 10.1, suppose that the compact subgroup $\mathcal{D} \subset C_\infty$ is chosen so that $G \cap \hat{g}\mathcal{D}\hat{g}^{-1} \neq \{e\}$ for some $\hat{g} \in C_\infty$. Then by Proposition 8.11 there exists $y \in X$ such that the Cantor system (X, G, Φ) has nontrivial kernel $K(\mathcal{G}^y)$ for the group chain $\mathcal{G}^y$ about y. It then follows that the germinal holonomy group $\mathrm{Germ}(\Phi, y)$ is nontrivial, so this method can also be used to construct examples with nontrivial germinal holonomy groups.

10C. ***Stable actions.*** Recall from Definition 7.5 that a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is said to be *stable* if there exists $n_0 \geq 0$ such that the maps $\psi_{n,m} : \mathcal{D}^n \to \mathcal{D}^m$

defined in (39) are isomorphisms for all $m \geq n \geq n_0$. We consider the problem of when a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ constructed using the method in the proof of Proposition 10.1 is stable.

We assume the hypotheses of Proposition 10.1, and the constructions of its proof. Fix $n > 0$, and consider the truncated group chain $\mathcal{G}_n = \{G_\ell\}_{\ell \geq n}$. Then the calculation of the kernel $K(\mathcal{G}_n) = G \cap \mathcal{D}$ is the same as (50). Also, note that $\mathcal{D} \subset W_\ell$ for all $\ell \geq 0$, so the calculations in (51) also proceed analogously. However, the last equality in (52) requires the additional assumption

$$\operatorname{core}_U \mathcal{D} = \bigcap_{k \in U} k\mathcal{D}k^{-1} = \{e\} \tag{55}$$

for the clopen neighborhoods $U = U_\ell$ of the identity in order to conclude that $\mathcal{D}$ is the discriminant group for $\mathcal{G}_n$. In other words, we require that the subgroup $\mathcal{D}$ is "totally not-normal" for every neighborhood of the identity in $\widehat{G}$. The above remarks yield:

Proposition 10.2. *Let C_∞ be a profinite group, let $G \subset C_\infty$ be a finitely generated dense subgroup, and let $\mathcal{D} \subset C_\infty$ be a compact subgroup of infinite index, such that (55) holds for every clopen neighborhood $\{e\} \in U \subset C_\infty$. Choose a group chain $\{U_\ell \mid \ell \geq 1\}$ which is a clopen neighborhood system about the identity in C_∞. Then the associated group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ with $G_0 = G$ has discriminant group $\mathcal{D}$ and is stable.*

Finally, in the case where $\mathcal{D} \subset C_\infty$ is a compact subgroup of infinite index, but need not satisfy the condition that its core is trivial, then noting that the core is a normal subgroup, we can modify the construction above as follows to obtain a minimal Cantor action.

Corollary 10.3. *Let C'_∞ be a profinite group, $G' \subset C'_\infty$ be a finitely generated dense subgroup, and $\mathcal{D}' \subset C'_\infty$ be a nontrivial compact subgroup of infinite index, and let $\operatorname{core}_{G'} \mathcal{D}'$ denote the rational core of $\mathcal{D}'$ as in (55), which is a normal subgroup of C'_∞ as G' is dense. Set*

$$C_\infty = C'_\infty/(\operatorname{core}_{G'} \mathcal{D}'), \quad G = G'/(G' \cap \operatorname{core}_{G'} \mathcal{D}'), \quad \mathcal{D} = \mathcal{D}'/\operatorname{core}_{G'} \mathcal{D}'.$$

Then there exists a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ with $G_0 = G$ and discriminant group $\mathcal{D}$.

10D. ***Constructing embedded groups.*** We next recall the remarkable constructions of Lubotzky, which when combined with the techniques of Proposition 10.1, make possible the construction of a wide class of equicontinuous minimal Cantor actions by a finitely generated, torsion-free, residually finite group G, with prescribed discriminant group $\mathcal{D}$. There are two cases of the construction.

Theorem 10.4 [Lubotzky 1993, Theorem 2(b)]. *Let F be a nontrivial finite group, and set $\boldsymbol{F}_i = F$ for all integers $i \geq 1$. Let $\boldsymbol{F} = \prod \boldsymbol{F}_i$ denote the infinite cartesian product of F. Then there exists a finitely generated, residually finite, torsion-free group $G \subset \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ sufficiently large whose profinite completion $\widehat{G}$ contains $\boldsymbol{F}$.*

Proof. We give just an outline of the construction used in the proof of Theorem 2(b) in [Lubotzky 1993], with details as required for the constructions of our examples. First recall some basic facts. For $n \geq 3$, let $\Gamma_n = \mathrm{SL}_n(\mathbb{Z})$ denote the $n \times n$ integer matrices. The group Γ_n is finitely generated and residually finite, and hence so are all finite index subgroups of Γ_n. Let $\Gamma_n(m)$ denote the congruence subgroup

$$\Gamma_n(m) \equiv \mathrm{Ker}\{\varphi_m : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})\}.$$

For $m \geq 3$, $\Gamma_n(m)$ is torsion-free. Moreover, by the congruence subgroup property, every finite index subgroup of Γ_n contains $\Gamma_n(m)$ for some nonzero m. Then this implies

$$\widehat{\mathrm{SL}_n(\mathbb{Z})} \cong \varprojlim \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z}) \cong \mathrm{SL}_n(\widehat{\mathbb{Z}}) \cong \prod_p \mathrm{SL}_n(\mathbb{Z}_p), \tag{56}$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and we use that $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers, and the product is taken over all primes. Note that the factors in the cartesian product on the right-hand side of (56) commute with each other.

Let $G \subset \Gamma_n$ be a finite index, torsion-free subgroup, which is then finitely generated, and its profinite completion $\widehat{G}$ is an open subgroup of $\widehat{\mathrm{SL}_n(\mathbb{Z})}$. Then there exists a cofinite subgroup $\mathcal{P}(G)$ of the primes such that

$$\prod_{p \in \mathcal{P}(G)} \mathrm{SL}_n(\mathbb{Z}_p) \subset \widehat{G}. \tag{57}$$

Let $d_F = |F|$ denote the cardinality of F, and let $n \geq |F| + 2$. Then F embeds in the alternating group $\mathrm{Alt}(n)$ on n symbols. Then F being nontrivial implies that $n \geq 4 > 3$. For each $p \in \mathcal{P}(F)$, the group $\mathrm{Alt}(n)$ embeds into $\mathrm{SL}_n(\mathbb{Z}_p)$, and thus we obtain an embedding

$$\iota_\infty : \boldsymbol{F} \cong \prod_{p \in \mathcal{P}(G)} F_p \subset \prod_{p \in \mathcal{P}(G)} \mathrm{Alt}(n) \subset \prod_{p \in \mathcal{P}(G)} \mathrm{SL}_n(\mathbb{Z}_p) \subset \widehat{G}, \tag{58}$$

where $F_p = F$ for each $p \in \mathcal{P}(G)$. This completes the construction. □

Lubotzky [1993, Theorem 1] extended the above construction to obtain an embedding for a group $\mathcal{D}$ which is an infinite product of possibly distinct finite groups $\{\boldsymbol{H}_i \mid i = 1, 2, \ldots\}$. The extension is highly nontrivial, as if all of the groups $\boldsymbol{H}_i$ are distinct, then the degrees $|\boldsymbol{H}_i|$ must tend to infinity, and so the above straightforward strategy for embedding no longer works.

Theorem 10.5 [Lubotzky 1993, Theorem 1]. *Let* $\{\boldsymbol{H}_i \mid i = 1, 2, \ldots\}$ *be an infinite collection of nontrivial finite groups, and let* $\boldsymbol{H} = \prod \boldsymbol{H}_i$ *denote their cartesian product. Then there exists a finitely generated, residually finite, torsion-free group* G *whose profinite completion* $\widehat{G}$ *contains* $\boldsymbol{H}$.

Proof. Again, we only sketch some key aspects of the proof from [Lubotzky 1993]. Let $G \subset \Gamma_n$ be the finitely generated, torsion-free, residually finite group constructed on page 330 of [Lubotzky 1993], and let $\widehat{G}$ be its profinite completion. Lubotzky constructs by induction an increasing sequence of primes $\{p_n \mid n \geq 3\}$ such that

$$\prod_{n=3}^{\infty} \mathrm{SL}_n(\mathbb{Z}_{p_n}) \subset \widehat{G}. \tag{59}$$

For $i \geq 1$, let $d_i = |\boldsymbol{H}_i|$ denote the cardinality of $\boldsymbol{H}_i$. Then each $\boldsymbol{H}_i$ embeds in the alternating group $\mathrm{Alt}(d_i + 2)$ on $d_i + 2$ symbols. Now choose an increasing sequence of integers $\{n_i \mid i \geq 1\}$ such that $n_i \geq d_i + 2$. Then for each $i \geq 1$, the group $\mathrm{Alt}(d_i + 2)$ embeds into the alternating group $\mathrm{Alt}(n_i)$ by taking only the permutations on the first $d_i + 2$ symbols. For each $i \geq 1$ the group $\mathrm{Alt}(n_i)$ embeds into $\mathrm{SL}_{n_i}(\mathbb{Z}_{p_{n_i}})$. Thus, we have embeddings $\boldsymbol{H}_i \subset \mathrm{Alt}(n_i) \subset \mathrm{SL}_{n_i}(\mathbb{Z}_{p_{n_i}})$.

The product in (59) is over all $n \geq 3$, while the group $\boldsymbol{H}_n = \boldsymbol{H}_{n_i}$ if $n = n_i$ for some n_i as chosen above. For $n \neq n_i$ for some i, let $\boldsymbol{H}_n$ be the trivial group. Set $\boldsymbol{A}_n = \mathrm{Alt}(n_i)$ if $n = n_i$ for some n_i and let $\boldsymbol{A}_n$ be the trivial group otherwise. Then we obtain an embedding of the infinite product $\boldsymbol{H}$,

$$\iota_\infty : \mathcal{D} \cong \prod_{n \geq 3} \boldsymbol{H}_n \subset \prod_{n \geq 3} \boldsymbol{A}_n \subset \prod_{i \geq 1} \mathrm{SL}_{n_i}(\mathbb{Z}_{p_{n_i}}) \subset \widehat{G}. \tag{60}$$

This completes the construction. □

10E. *Constructing stable actions.* We next use Theorems 10.4 and 10.5, and observations from their proofs in [Lubotzky 1993], to construct examples of stable equicontinuous minimal Cantor group actions.

We first require a simple observation. For $n \geq 2$, the alternating group $\mathrm{Alt}(n)$ on n symbols embeds into the alternating group $\mathrm{Alt}(4n)$ on $4n$ symbols, by considering $\mathrm{Alt}(n)$ as acting on the first n symbols and fixing the remaining $3n$ symbols. We thus consider $\mathrm{Alt}(n) \subset \mathrm{Alt}(4n)$ as a subgroup.

Lemma 10.6. *The core of* $\mathrm{Alt}(n)$ *in* $\mathrm{Alt}(4n)$ *is the trivial group.*

Proof. There exists an element $\sigma \in \mathrm{Alt}(4n)$ which swaps the first $2n$ symbols for the last $2n$ symbols. Then $\sigma^{-1}\mathrm{Alt}(n)\sigma$ is contained in the alternating group which permutes the last $2n$ symbols, and hence is disjoint from the subgroup $\mathrm{Alt}(n)$. □

Lemma 10.6 is used to ensure that the chains constructed below satisfy the conditions of Section 10C.

Theorem 10.7. *Let F be a finite group. Then there exists a finite index, torsion-free group $G \subset \mathrm{SL}_n(\mathbb{Z})$ and an embedding of F into the profinite completion $\widehat{G}$, so that the resulting group chain $\mathcal{G}_K = \{G_\ell\}_{\ell \geq 0}$ constructed as in Section 10A yields an equicontinuous minimal Cantor system (X_∞, G, Φ) whose discriminant group for the truncated group chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to F for all $k \geq 0$. Hence the action is stable and irregular. Moreover, the germinal holonomy group for each $x \in X$ is trivial.*

Proof. As noted in the proof of Theorem 10.4, if F is a nontrivial finite group of order $d_F = |F|$, then F embeds in the alternating group $\mathrm{Alt}(d_F + 2)$. We then embed $\mathrm{Alt}(d_F + 2)$ in the alternating group $\mathrm{Alt}(n)$ for $n \geq 4(d_F + 2)$, by considering $\mathrm{Alt}(n)$ as acting on the first n symbols, as in the proof of Lemma 10.6. We identify F with its image, and then note that the core of F in $\mathrm{Alt}(n)$ is the trivial group. Note that $d_F \geq 2$, so we have that $n \geq 16$. Also note that if F' is any other finite group of order at most d_F, then it also embeds into $\mathrm{Alt}(d_F + 2)$, and hence the following construction is universal for all finite groups F' with $|F'| \leq |F|$.

For $n \geq 4(d_F + 2)$, let $G \subset \Gamma_n = \mathrm{SL}_n(\mathbb{Z})$ be the finite index, torsion-free subgroup constructed in the proof of Theorem 10.4. Set $\boldsymbol{H}_\ell = \mathrm{Alt}(n)$ for all integers $\ell \geq 1$, and let $\boldsymbol{H} = \prod \boldsymbol{H}_\ell$ denote their cartesian product. Then the embedding (58) becomes

$$\iota_\infty : \boldsymbol{H} \cong \prod_{p \in \mathcal{P}(G)} \mathrm{Alt}_p(n) \subset \prod_{p \in \mathcal{P}(G)} \mathrm{SL}_n(\mathbb{Z}_p) \subset \widehat{G}, \tag{61}$$

where $\mathrm{Alt}_p(n) = \mathrm{Alt}(n)$ for each prime p.

For each $i \geq 1$, we have the embedding

$$F \subset \mathrm{Alt}(d_F + 2) \subset \mathrm{Alt}(n) = \boldsymbol{H}_\ell.$$

Let $F \to \boldsymbol{H}$ be the diagonal embedding into the infinite product, which then yields an embedding $\iota_F : F \to \widehat{G}$ into the profinite completion of G, with image denoted by $\mathcal{D} = \iota_F(F)$.

Next, use the method of Section 10A to construct a group chain in G. The group G is residually finite, so there exists a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in $\widehat{G}$, where each U_ℓ is normal in $\widehat{G}$. Note that G is dense in $\widehat{G}$ and each U_ℓ is closed, so the closure of $G \cap U_\ell$ in $\widehat{G}$ is equal to U_ℓ. Set $W_\ell = \mathcal{D} \cdot U_\ell$ for $\ell \geq 1$, and $G_\ell = G \cap W_\ell$. Let $\mathcal{G}_F = \{G_\ell\}_{\ell \geq 0}$ denote the resulting group chain.

Let $\{e\} \in U \subset \widehat{G}$ be a normal clopen neighborhood of the identity, so that $\widehat{G}/U$ is a finite group with cardinality $|\widehat{G}/U|$. We claim $\mathrm{core}_U\, \mathcal{D} = \{e\}$. The normal subgroup U has finite index; hence, as argued in the proof of [Lubotzky 1993, Theorem 2], there exists a cofinite subset of primes $\mathcal{P}(G, U) \subset \mathcal{P}(G)$ of the list in the product in (61) such that

$$\prod_{p \in \mathcal{P}(G,U)} \mathrm{Alt}_p(n) \subset \prod_{p \in \mathcal{P}(G,U)} \mathrm{SL}_n(\mathbb{Z}_p) \subset U \subset \widehat{G}.$$

For $p \in \mathcal{P}(G, U)$, note that for the diagonal embedding of F into $\boldsymbol{H}$, the projection to each factor of $\boldsymbol{H}$ is an isomorphism. For the image of F in the p-th factor, we have

$$F \subset \mathrm{Alt}(d_F + 2) \subset \mathrm{Alt}(n) = \mathrm{Alt}_p(n) \subset \mathrm{SL}_n(\mathbb{Z}_p).$$

The image group has trivial core by Lemma 10.6. The projection of $\mathcal{D}$ to $F \subset \mathrm{Alt}_p(n)$ is an isomorphism, so this implies that $\mathcal{D}$ has trivial core in U as well. Then by Proposition 10.2, for all $k \geq 0$, the discriminant group for the truncated group chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to F. In particular, $\mathcal{G}_F$ is a stable group chain.

Next, observe that $\mathcal{D}$ being compact implies that the closure $\overline{G_\ell}$ of G_ℓ in $\widehat{G}$ equals W_ℓ, and $\mathcal{D} = \bigcap \overline{G_\ell}$. For the kernel of $\mathcal{G}_F$ as defined in Section 4B, we then have

$$K(\mathcal{G}_F) = \bigcap_{\ell \geq 0} G_\ell \subset \bigcap_{\ell \geq 0} \overline{G_\ell} = \mathcal{D}. \tag{62}$$

The group $\mathcal{D}$ is finite, hence every element of $\mathcal{D}$ has finite order, while $K(\mathcal{G}_F)$ is a torsion-free subset of G. Thus, $K(\mathcal{G}_F) \subset \mathcal{D} \cap G = \{e\}$; hence $K(\mathcal{G}_F)$ is the trivial group. Moreover, for each $\hat{g} \in \widehat{G}$ let

$$\mathcal{G}_F^{\hat{g}} = \{\hat{g}\, G_\ell\, \hat{g}^{-1}\}_{\ell \geq 0}$$

denote the conjugate group chain. Then by the same reasoning, we also have $K(\mathcal{G}_F^{\hat{g}}) = \{e\}$, as $\hat{g}^{-1}\mathcal{D}\hat{g} \subset \widehat{G}$ is again a finite subgroup, hence has trivial intersection with G.

Let (X, G, Φ) be the equicontinuous minimal Cantor system with $X = \widehat{G}/\mathcal{D}$ with the associated group chain $\mathcal{G}_F$, as discussed in Section 10B. The discriminant group of $\mathcal{G}_F$ is $\mathcal{D}$, and each nontrivial element $h \in \mathcal{D}$ is torsion, hence its image in $\widehat{G}$ is torsion, and thus any conjugate of it is not contained in the torsion-free subgroup G. Thus, for each $y \in X$, the action Φ has trivial germinal holonomy at y.

The discriminant group of the truncated chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to F for all $k \geq 0$. Thus, $\mathcal{G}_F$ cannot be a weakly regular group chain. This establishes all of the claims of Theorem 10.7. □

Note that the action (X, G, Φ) satisfies the SQA condition by default, as all germinal holonomy groups are trivial. The action of $\widehat{G}$ on $X = \widehat{G}/\mathcal{D}$ satisfies the SQA condition by Theorem 9.5. Corollary 1.11 now follows by using the construction in Section 2B to obtain a matchbox manifold with section $V_0 \cong X$ and induced holonomy action (X, G, Φ).

We remark that it is tempting to use the fact that $G \subset \mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{SL}_n(\mathbb{R})$ is a torsion-free subgroup, and then use the quotient space $M_0 = \mathrm{SL}_n(\mathbb{R})/G$ as the base of a presentation for a weak solenoid $\mathcal{S}_{\mathcal{P}}$. However, this quotient space is not compact, and we do not have a "theory of weak solenoids" over noncompact manifolds.

We next use Theorem 10.5 to construct two types of embeddings of Cantor groups into profinite groups. Theorem 10.8 embeds a profinite group such that the resulting action is stable. Theorem 10.10 embeds a Cantor group such that the resulting action is not virtually regular.

Theorem 10.8. *Let K be a separable profinite group. There exists a finitely generated, residually finite, torsion-free group G, and an embedding of K into its profinite completion $\widehat{G}$, such that the resulting group chain $\mathcal{G}_K = \{G_\ell\}_{\ell\geq 0}$ constructed as in Section 10A yields an equicontinuous minimal Cantor system (X, G, Φ) whose discriminant group for the truncated group chain $\{G_\ell\}_{\ell\geq k}$ is isomorphic to K for all $k \geq 0$. Hence the action is stable and irregular.*

Proof. Let $G \subset \Gamma_n$ be the finitely generated, torsion-free, residually finite group used in the proof of Theorem 10.5, as constructed on page 330 of [Lubotzky 1993], and let $\widehat{G}$ be its profinite completion.

The assumption that K is a separable profinite group implies that K is isomorphic to an inverse system of finite groups

$$K \cong \varprojlim\{\varphi_\ell^{\ell+1} : \boldsymbol{K}_{\ell+1} \to \boldsymbol{K}_\ell \mid \ell \geq 0\} \subset \boldsymbol{K} \cong \prod \boldsymbol{K}_\ell, \tag{63}$$

where each $\boldsymbol{K}_\ell$ is a finite group, and the bonding maps $\varphi_\ell^{\ell+1}$ are epimorphisms for all $\ell \geq 0$, but not isomorphisms. Thus, their cardinalities $\{|\boldsymbol{K}_\ell| \mid \ell \geq 0\}$ form an increasing sequence of integers. Note that we have isomorphisms for all $k > 0$, induced by the shift map σ_i on indices,

$$\sigma_i : K \cong \varprojlim\{\varphi_\ell^{\ell+1} : \boldsymbol{K}_{\ell+1} \to \boldsymbol{K}_\ell \mid \ell \geq k\}. \tag{64}$$

For each $\ell \geq 0$, set $d_\ell = 4(|\boldsymbol{K}_\ell| + 2)$. Then as in the construction in Theorem 10.7, there is an embedding of $\boldsymbol{K}_\ell$ into the alternating group, $\boldsymbol{K}_\ell \subset \mathrm{Alt}(|\boldsymbol{K}_\ell| + 2) \subset \mathrm{Alt}(d_\ell)$. Choose an increasing sequence of integers $\{n_\ell \mid \ell \geq 1\}$ so that $n_\ell \geq d_\ell$ for all $\ell \geq 1$.

Then as in the proof of Theorem 10.5, we set $\boldsymbol{H}_n = \mathrm{Alt}(d_\ell)$ if $n = n_\ell$ for some n_ℓ as chosen above. If $n \neq n_\ell$ for all ℓ, let $\boldsymbol{H}_n$ be the trivial group. Set $\boldsymbol{A}_n = \mathrm{Alt}(n_\ell)$ if $n = n_\ell$ for some n_ℓ, and let $\boldsymbol{A}_n$ be the trivial group otherwise. Then we obtain an embedding of the infinite product,

$$\boldsymbol{H} \equiv \prod_{n\geq 3} \boldsymbol{H}_n \subset \boldsymbol{A} \equiv \prod_{n\geq 3} \boldsymbol{A}_n \subset \prod_{\ell\geq 1} \mathrm{SL}_{n_\ell}(\mathbb{Z}_{p_{n_\ell}}) \subset \widehat{G}. \tag{65}$$

Now observe that the inverse limit presentation in (63), along with the above embedding (65), gives an embedding

$$\Delta_K : K \subset \prod \boldsymbol{K}_\ell \subset \prod_{n\geq 3} \boldsymbol{H}_n \subset \prod_{n\geq 3} \boldsymbol{A}_n \subset \widehat{G}. \tag{66}$$

Set $\mathcal{D} = \Delta_K(K) \subset \widehat{G}$. Then as in the proof of Theorem 10.7, use the method of Section 10A to construct a group chain in G. The group G is residually finite, so there exists a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in $\widehat{G}$, where each U_ℓ is normal in $\widehat{G}$. Set $W_\ell = \mathcal{D} \cdot U_\ell$ for $\ell \geq 1$, and $G_\ell = G \cap W_\ell$. Let $\mathcal{G}_K = \{G_\ell\}_{\ell \geq 0}$ denote the resulting group chain.

Let $\{e\} \in U \subset \widehat{G}$ be a normal clopen neighborhood of the identity, so that $\widehat{G}/U$ is a finite group with cardinality $|\widehat{G}/U|$. We claim $\mathrm{core}_U \mathcal{D} = \{e\}$. Note that for $m \geq 5$, the alternating group $\mathrm{Alt}(m)$ is simple, and its cardinality $|\mathrm{Alt}(m)| = \frac{1}{2}m!$ tends to infinity as m increases. As the sequence $\{n_\ell\}$ is increasing, for some $\ell_0 > 0$, we have $\ell \geq \ell_0$, and $\boldsymbol{A}_m = \mathrm{Alt}(n_\ell)$ being nontrivial implies that $\boldsymbol{H}_m$ has order $|\boldsymbol{H}_m| = \frac{1}{2}(n_\ell)! > |\widehat{G}/U|$. Thus, the projection $\boldsymbol{A}_m \subset \widehat{G} \to \widehat{G}/U$ cannot be an injection, and as $\boldsymbol{A}_m$ is a simple group, it must be contained in the kernel, so $\boldsymbol{A}_m \subset U$. Let $\pi_m : \boldsymbol{A} \to \boldsymbol{A}_m$ be the projection onto the m-th factor. We have that $\mathcal{D} \subset \boldsymbol{H} \subset \boldsymbol{A}$. Let $\mathcal{D}_m \subset \boldsymbol{A}_m$ denote its image. By the choice of m, and because $n_\ell \geq d_\ell = 4(|\boldsymbol{K}_\ell| + 2)$, Lemma 10.6 implies the subgroup $\mathcal{D}_m$ has trivial core in $\boldsymbol{A}_m$. It follows that $\mathcal{D}$ has trivial core in U.

Then by Proposition 10.2, for all $k \geq 0$, the discriminant group for the truncated group chain $\{G_\ell\}_{\ell \geq k}$ is isomorphic to K. In particular, $\mathcal{G}_K$ is a stable group chain and is not weakly normal.

The rest of the proof proceeds as for that of Theorem 10.7. □

Note that in the above proof, we cannot assert that all leaves of the suspended foliation $\mathcal{F}_{\mathfrak{M}}$ have trivial holonomy, as examples show that some conjugate of $\mathcal{D}$ in $\widehat{G}$ may intersect G nontrivially.

Our final example, which is again based on the application of Theorem 10.5, answers a question posed in [Dyer et al. 2016]. In that work, the notion of a virtually regular action (X, G, Φ) with group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ was introduced:

Definition 10.9. [Dyer et al. 2016, Definition 1.12] A group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is said to be *virtually regular* if there exists a normal subgroup $G'_0 \subset G_0$ of finite index such that the restricted chain $\mathcal{G}' = \{G'_\ell\}_{\ell \geq 0}$, where $G'_\ell = G_\ell \cap G'_0$, is weakly normal in G'_0.

There is an alternate definition of this concept, which was shown in [Dyer et al. 2016] to be equivalent: a matchbox manifold $\mathfrak{M}$ is *virtually regular* if there exists a homogeneous matchbox manifold $\mathfrak{M}'$ and a finite-to-one normal covering map $h : \mathfrak{M}' \to \mathfrak{M}$. Thus, the notion of virtually regular is a natural property of a matchbox manifold $\mathfrak{M}$, and can be checked by considering a group chain model for the holonomy action of the foliation $\mathcal{F}_{\mathfrak{M}}$.

The following example is the first known to the authors which is not virtually regular, and gives a natural paradigm for the construction of group chains which are not virtually regular.

Theorem 10.10. *There exists a finitely generated, residually finite, torsion-free group G with profinite completion $\widehat{G}$ such that for any infinite collection $\{F_\ell\}_{\ell\geq 1}$ of nontrivial finite simple groups, their cartesian product $\boldsymbol{F}=\prod F_\ell$ can be embedded into $\widehat{G}$, so that the resulting group chain $\mathcal{G}_{\boldsymbol{F}}=\{G_\ell\}_{\ell\geq 0}$ constructed as in Section 10A yields an equicontinuous minimal Cantor system (X, G, Φ) whose discriminant group for the group chain $\mathcal{G}_{\boldsymbol{F}}$ is isomorphic to $\boldsymbol{F}$. Moreover, $\mathcal{G}_{\boldsymbol{F}}$ is not virtually regular.*

Proof. The proof follows the same approach as that used in the proof of Theorem 10.8. Let $G\subset\Gamma_n$ be the finitely generated, torsion-free, residually finite group used in the proof of Theorem 10.5, as constructed on page 330 of [Lubotzky 1993], and let $\widehat{G}$ be its profinite completion.

For each $\ell\geq 0$, set $d_\ell=4(|F_\ell|+2)$. Then there is an embedding of F_ℓ into the alternating groups, $F_\ell\subset \mathrm{Alt}(|F_\ell|+2)\subset\mathrm{Alt}(d_\ell)$, as in the proof of Theorem 10.8. Choose an increasing sequence of integers $\{n_\ell\mid \ell\geq 1\}$ so that $n_\ell\geq d_\ell$ for all $\ell\geq 1$. Let $\mathrm{Alt}(d_\ell)\subset\mathrm{Alt}(n_\ell)$ be the embedding as the permutations on the first d_ℓ symbols. Then we obtain an embedding $\iota_{\boldsymbol{F}}:\boldsymbol{F}\to\widehat{G}$, of the infinite product $\boldsymbol{F}$ into $\widehat{G}$, given by the composition

$$\iota_{\boldsymbol{F}}:\boldsymbol{F}\cong\prod_{\ell\geq 1}F_\ell\subset\prod_{\ell\geq 1}\mathrm{Alt}(d_\ell)\subset\prod_{\ell\geq 1}\mathrm{Alt}(n_\ell)\subset\prod_{\ell\geq 1}\mathrm{SL}_{n_\ell}(\mathbb{Z}_{p_{n_\ell}})\subset\widehat{G}. \tag{67}$$

Set $\mathcal{D}=\iota_{\boldsymbol{F}}(\boldsymbol{F})\subset\widehat{G}$. Use the method of Section 10A to construct a group chain in G. The group G is residually finite, so there exists a clopen neighborhood system $\{U_\ell\mid\ell\geq 1\}$ about the identity in $\widehat{G}$, where each U_ℓ is normal in $\widehat{G}$. Set $W_\ell=\mathcal{D}\cdot U_\ell$ for $\ell\geq 1$, and $G_\ell=G\cap W_\ell$. Let $\mathcal{G}_{\boldsymbol{F}}=\{G_\ell\}_{\ell\geq 0}$ denote the resulting group chain.

Let $U\subset\widehat{G}$ be a normal clopen neighborhood of the identity. For example, given a normal subgroup $G'\subset G$ with finite index, we can take U to be the profinite completion of G' in $\widehat{G}$. Let $\mathcal{G}^U_{\boldsymbol{F}}=\{G'_\ell\}_{\ell\geq 0}$ be the group chain defined by $G'_\ell=G_\ell\cap U$ for $\ell\geq 0$. Then $\mathcal{D}\cap U=\bigcap(U_\ell\cap U)$.

We next show that the normal core, $\mathrm{core}_U\,\mathcal{D}\subset\mathcal{D}$, of $\mathcal{D}\cap U$ in U is a finite subgroup, and then apply Corollary 10.3 to conclude that the discriminant of the action defined by the group chain $\mathcal{G}^U_{\boldsymbol{F}}$ is a nontrivial Cantor group. The following argument is similar to that used in the proof of Theorem 10.8, and uses that the alternating group $\mathrm{Alt}(m)$ is simple for $m\geq 5$ and has order $|\mathrm{Alt}(m)|=\frac{1}{2}m!$. Let $d_U=|\widehat{G}/U|$ be the order of the finite group.

Choose $\ell_U\geq 1$ such that $n_{\ell_U}\geq 5$ and $|\mathrm{Alt}(n_{\ell_U})|=\frac{1}{2}(n_{\ell_U})!>d_U$. Then for all $\ell\geq\ell_U$, the factor $\mathrm{Alt}(n_\ell)$ in the product in (67) is contained in the kernel of the projection $\widehat{G}\to\widehat{G}/U$, and thus, $F_\ell\subset\mathrm{Alt}(n_\ell)\subset U$. Consequently, we have that

$$\mathcal{D}_{\ell_U}\equiv\prod_{\ell\geq\ell_U}F_\ell\subset\boldsymbol{A}_{\ell_U}\equiv\prod_{\ell\geq\ell_U}\mathrm{Alt}(n_\ell)\subset\mathcal{D}\cap U. \tag{68}$$

In particular, this shows that $\mathcal{D} \cap U$ contains a nontrivial Cantor group. Moreover, by applying Lemma 10.6 to each factor of the product in $\mathcal{D}_{\ell_U}$, we see that $\mathcal{D}_{\ell_U}$ has trivial core in U_ℓ as well. Thus, we have

$$\operatorname{core}_U \mathcal{D} \subset \prod_{1 \le \ell < \ell_U} F_\ell, \tag{69}$$

and so $\operatorname{core}_U \mathcal{D}$ is a finite normal subgroup of U.

By Corollary 10.3, the quotient group chain $\{(G_\ell \cap U)/(\operatorname{core}_U \mathcal{D})\}_{\ell \ge 0}$ has a nontrivial discriminant group $\mathcal{D}/(\operatorname{core}_U \mathcal{D})$ which contains a subgroup isomorphic to the nontrivial Cantor group $\mathcal{D}_{\ell_U}$. For $\ell > 0$, apply this to the case $U = U_\ell$ to obtain that the quotient chain $\{(G_\ell \cap U_\ell)/(\operatorname{core}_{U_\ell} \mathcal{D})\}_{\ell \ge 0}$ is not equivalent to a normal chain. Now suppose that the restricted group chain $\mathcal{G}_{\boldsymbol{F}}^{U_\ell}$ is equivalent to a normal chain. Then as $\operatorname{core}_{U_\ell} \mathcal{D}$ is a normal subgroup of U_ℓ, this implies that the quotient group chain $\{(G_\ell \cap U_\ell)/(\operatorname{core}_{U_\ell} \mathcal{D})\}_{\ell \ge 0}$ is equivalent to a normal chain, hence has trivial discriminant, which is a contradiction. Thus, the group chain $\mathcal{G}_{\boldsymbol{F}}$ is not virtually regular. □

10F. ***Open problems.*** There are many variations of the above method that can be considered, and open questions about the resulting minimal Cantor actions. First, it is interesting to understand the answer to the following.

Problem 10.11. Given a separable profinite group $\widehat{H}$ and an embedding into a profinite group $\widehat{G}$ with trivial rational core, constructed using the methods of [Lubotzky 1993], give criteria for when the resulting equicontinuous minimal Cantor system (X, G, Φ) is weakly normal, and whether the action is stable or wild. Furthermore, when do the resulting actions satisfy the SQA condition of Section 9A?

There is also an extensive literature for the construction of embeddings of groups H into the profinite completions of torsion-free, finitely generated nilpotent and solvable groups. For example, [Crawley-Boevey et al. 1988] showed that if G is a finitely generated, torsion-free nilpotent group, then the profinite completion $\widehat{G}$ is torsion-free, so if $D \subset \widehat{G}$ is a closed subgroup, then it must be a Cantor group.

On the other hand, [Evans 1990; Kropholler and Wilson 1993] showed that there exists a countable, torsion-free, residually finite, metabelian group G such that its profinite completion contains a nontrivial torsion subgroup. Quick [2001] studied the profinite topology of nilpotent groups of class two and finitely generated center-by-metabelian groups, and used this to construct embeddings of finite groups into the profinite completions of these classes of groups. However, the embedding obtained in [Quick 2001] is contained in the center of G, so does not satisfy the trivial core condition. We conclude with an open question, suggested by the examples and results of [Dyer 2015; Dyer et al. 2016; 2017; Fokkink and Oversteegen 2002; Rogers and Tollefson 1971b; Schori 1966].

Problem 10.12. Determine which groups H can be embedded as a closed subgroup of $\widehat{G}$ with trivial rational core, where G is a finitely generated, torsion-free amenable group.

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CYCLIC PURSUIT ON COMPACT MANIFOLDS

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We study a form of cyclic pursuit on Riemannian manifolds with positive injectivity radius. We conjecture that on a compact manifold, the piecewise geodesic loop formed by connecting consecutive pursuit agents either collapses to a point in finite time or converges to a closed geodesic. The main result is that this conjecture is valid for nonpositively curved compact manifolds.

1. Introduction

Our starting point is the classical three bug problem, first posed by Edouard Lucas [1877]: Three bugs start on the corners of an equilateral triangle, and each chases the next at unit speed. What happens? Answer: The bugs wind around the center of the triangle infinitely many times as they head inward along logarithmic spirals. They collide at the center of the triangle in finite time. We get similar behavior in general for a system of n bugs starting at the vertices of a regular n-gon, each chasing its clockwise neighbor at unit speed; see, e.g., [Behroozi and Gagnon 1979]. For illustrations of cyclic pursuit with initial conditions on a regular n-gon, see clips 1 through 4 at http://tinyurl.com/Gekhtman-bugs. For details on the history of various versions of the n bug problem, see the introduction of [Richardson 2001b].

Next, consider n bugs starting at arbitrary positions in $\mathbb{R}^d$, with bug i chasing bug $i+1 \bmod n$ at unit speed. Clips 5 through 7 at the address below demonstrate cyclic pursuit with randomly chosen initial conditions in the unit cube of $\mathbb{R}^3$. The typical observed behavior is as follows: Starting from the initial random configuration, chains of closely spaced bugs form, the chains come together to form a close approximation of a smooth knot, the knot unknots into an approximately circular loop, and the loop collapses to a point in finite time. The evolution of the piecewise linear loop connecting the bugs qualitatively resembles the curve-shortening flow on the space of smooth loops in $\mathbb{R}^3$. Richardson [2001b; 2001a] analyzed aspects of cyclic pursuit in $\mathbb{R}^d$. He showed that for $n \geq 7$, the only stable configuration of n bugs in cyclic pursuit is a planar regular n-gon [2001a]. Based on simulations, Richardson conjectured that, if the initial positions of n bugs are chosen uniformly

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at random in $[0, 1]^d$, the probability of converging asymptotically to the regular n-gon configuration approaches 1 as $n \to \infty$.

In this paper, we study cyclic pursuit on Riemannian manifolds with positive injectivity radius. To define pursuit in this case, we choose initial positions such that each bug is within the injectivity radius of the next, and we have each bug chase the next with velocity equal to the unit vector tangent to the shortest geodesic connecting it to the next bug. Unlike in the Euclidean case, the bugs do not necessarily all collide in finite time. Certainly, they cannot if the piecewise geodesic loop connecting consecutive bugs is not nullhomotopic. This leads to the conjecture that on a compact manifold, if the bugs do not collide in finite time, the loop connecting them converges to a closed geodesic. The main result of this paper is that the conjecture is valid for pursuit on manifolds of nonpositive curvature. Clips 8 and 9 demonstrate convergence on a flat Möbius band and a flat torus, respectively. Numerical simulations suggest the conjecture is valid in general. Clip 10 demonstrates cyclic pursuit on S^2 and clips 11 and 12 show pursuit on $\mathbb{RP}^2$.

The organization of the paper is as follows: In Section 3, we study basic properties of cyclic pursuit in Euclidean space. In Section 4, we introduce cyclic pursuit on Riemannian manifolds. In Section 5, we prove a result which states roughly that, if the bugs enter a convex subset of a manifold, they stay in that subset. We derive as a consequence a condition for the pursuit to end in finite time. In Section 6, we prove subsequential convergence of the loop of bugs to a closed geodesic, and we obtain another criterion for pursuit to end in finite time. In Section 7, we give a condition for the loop of bugs to converge to a closed geodesic. In Section 8, we discuss convergence to closed geodesics which are locally length-minimizing, in the sense that any other loop uniformly close to the geodesic is longer. Then, we prove our main result: *for pursuit on a nonpositively curved compact manifold, the loop of bugs either collapses to a point in finite time or converges to a closed geodesic.*

2. Notation

Unless otherwise stated, geodesics are parametrized at constant speed.

If $(M, \langle\,\cdot\,,\cdot\,\rangle)$ is a Riemannian manifold and $p, q \in M$ are connected by a unique shortest geodesic, $[p, q]$ denotes the shortest geodesic from p to q, parametrized as a map from the unit interval $[0, 1]$. Unless otherwise stated, $\|\cdot\| : T_pM \to \mathbb{R}$ denotes the norm associated to $\langle\,\cdot\,,\cdot\,\rangle_p$, and $d : M \times M \to \mathbb{R}$ denotes the distance function associated to the metric. We identify S^1 with $\mathbb{R}/\mathbb{Z}$. For each $x \in S^1$, we define $T_x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ as the translation $T_x(s) = s + x$. If $\alpha, \gamma : S^1 \to M$ are two loops, $d(\alpha, \gamma) = \sup_{s\in S^1} d(\alpha(s), \gamma(s))$ denotes the supremum distance. If $p \in M$ and $r > 0$, $B_r(p)$ denotes the metric ball of radius r centered at p. For $p \in M$ and $K \subset M$, $d(p, K)$ denotes $\inf_{q\in K} d(p, q)$. If $\alpha : S^1 \to M$ is a closed geodesic and $\delta > 0$, $\overline{N}_\delta(\alpha) = \{p \in M \mid d(p, \alpha(S^1)) \le \delta\}$ denotes the closed δ-neighborhood of $\alpha(S^1)$.

3. Cyclic pursuit in Euclidean space

We define cyclic pursuit of n bugs in $\mathbb{R}^d$ with initial positions $\{b_i(0)\}_{i\in\mathbb{Z}/n}$ as the unique collection of piecewise smooth functions $\{b_i : [0,\infty) \to \mathbb{R}^n\}_{i\in\mathbb{Z}/n}$ with the given initial conditions satisfying

(1) If $b_i(t) \neq b_{i+1}(t)$, then

$$\dot{b}_i(t) = \frac{b_{i+1}(t) - b_i(t)}{\|b_{i+1}(t) - b_i(t)\|}. \tag{1}$$

(2) If $b_i(t_0) = b_{i+1}(t_0)$, then $b_i(t) = b_{i+1}(t)$ for all $t > t_0$.

(3) If $b_i(t_0) = b_0(t_0)$ for all i, then $b_i(t) = b_0(t_0)$ for all $t > t_0$.

The following result is well known. We include a proof, as it will be useful later.

Proposition 3.1. *For any set of initial conditions $\{b_i(0)\}_{i\in\mathbb{Z}/n}$, cyclic pursuit in $\mathbb{R}^d$ ends in finite time, i.e., there is a $t_0 > 0$ so that $b_i(t) = b_0(t_0)$ for all i and all $t \geq t_0$.*

Proof. Let $l_i(t) = d(b_i(t), b_{i+1}(t))$, and let $l(t) = \sum_{i=1}^n l_i(t)$ be the length of the piecewise linear loop connecting the $b_i(t)$. We recall the following fact, which can be verified by direct computation: Fix $p \in \mathbb{R}^d$ and consider the function $d_p : \mathbb{R}^d \setminus \{p\} \to \mathbb{R}$, $d_p(q) = d(p,q)$. Then the gradient of d_p at q is the unit vector $(q-p)/\|q-p\|$.

Let $u_i = (b_{i+1}(t) - b_i(t))/\|b_{i+1}(t) - b_i(t)\|$. Assuming for now that $l_i(t) > 0$ for all i, we get

$$\begin{aligned}\frac{d}{dt} l_i(t) &= \langle u_i(t), \dot{b}_{i+1}(t)\rangle + \langle -u_i(t), \dot{b}_i(t)\rangle \\ &= \langle u_i(t), u_{i+1}(t)\rangle + \langle -u_i(t), u_i(t)\rangle = \cos\theta_i(t) - 1,\end{aligned}$$

where $\theta_i(t) \in [0,\pi]$ is the angle between $u_i(t)$ and $u_{i+1}(t)$. By a theorem of Borsuk [1947], the sum of the exterior angles of a piecewise linear loop in $\mathbb{R}^d$ is at least 2π. So some θ_i is at least $\frac{2\pi}{n}$, and we find that $\frac{d}{dt} l(t) \leq \cos\frac{2\pi}{n} - 1$. In other words, $\frac{d}{dt} l(t)$ is negative, with absolute value bounded below by $1 - \cos\frac{2\pi}{n}$. If some $l_i(t)$ is 0, this effectively reduces n, so we still have the same bound on $\frac{d}{dt} l(t)$. Thus, pursuit ends by time $l(0)[1 - \cos\frac{2\pi}{n}]^{-1}$. □

Remark. If the θ_i are all $\frac{\pi}{2}$ or less, then Jensen's inequality applied to $1 - \cos\theta$ on $[0, \frac{\pi}{2}]$ yields $\left|\frac{d}{dt} l(t)\right| \geq n\left(1 - \cos\frac{2\pi}{n}\right)$. On the other hand, if at least one of the θ_i is greater than $\frac{\pi}{2}$, then $\left|\frac{d}{dt} l(t)\right| \geq 1 - \cos\frac{\pi}{2} = 1$. Thus, assuming that $l_i(t) > 0$ for all i, we have $\left|\frac{d}{dt} l(t)\right| \geq \min[1, n(1 - \cos\frac{2\pi}{n})]$. Since $\min\left[1, n\left(1 - \cos\frac{2\pi}{n}\right)\right]$ is a nonincreasing function of $n \geq 2$, we still have $\left|\frac{d}{dt} l(t)\right| \geq \min\left[1, n\left(1 - \cos\frac{2\pi}{n}\right)\right]$ if some (but not all) of the $l_i(t)$ are 0. Hence, the time from the start of the pursuit process to its end is bounded above by $l(0) \cdot \left(\min\left[1, n(1 - \cos\frac{2\pi}{n})\right]\right)^{-1}$, which grows linearly in n. (Compare this to the $O(n^2)$ bound on the time obtained from the estimate $\left|\frac{d}{dt}l(t)\right| \geq 1-\cos\frac{2\pi}{n}$ in the last paragraph.) Note also that, in the case that the $b_i(0)$ are vertices of a regular planar n-gon, we get that the time to mutual capture is precisely $l(0)\cdot\left[n(1-\cos\frac{2\pi}{n})\right]^{-1}$.

4. Pursuit on Riemannian manifolds

Cyclic pursuit on a Riemannian manifold is defined just as in the Euclidean case: each bug's velocity is the unit vector pointing towards the next bug along the shortest geodesic connecting the two. To ensure that there is a unique shortest geodesic connnecting each pair of bugs, we consider only manifolds with positive injectivity radius, and we choose initial positions so that the distance between each bug and its prey is less than the injectivity radius.

Let (M, g) be a manifold with positive injectivity radius denoted by $\mathrm{inj}(M)$, and let $\{b_i(0)\}_{i\in\mathbb{Z}/n}$ be initial positions in M satisfying $d(b_i(0), b_{i+1}(0)) < \mathrm{inj}(M)$. Then we define $\{b_i : [0,\infty) \to M\}_{i\in\mathbb{Z}/n}$ as the unique collection of piecewise smooth functions with the given initial conditions satisfying

(1) If $b_i(t) \neq b_{i+1}(t)$, then

$$\dot{b}_i(t) = \frac{\exp^{-1}_{b_i(t)}(b_{i+1}(t))}{\left\|\exp^{-1}_{b_i(t)}(b_{i+1}(t))\right\|}. \tag{2}$$

(2) If $b_i(t_0) = b_{i+1}(t_0)$, then $b_i(t) = b_{i+1}(t)$ for all $t > t_0$.

(3) If $b_i(t_0) = b_0(t_0)$ for all i, then $b_i(t) = b_0(t_0)$ for all $t > t_0$.

Let $l_i(t) = d(b_i(t), b_{i+1}(t))$. To see that the pursuit process is well defined for all $t \geq 0$, we need to check that each $l_i(t)$ is nonincreasing and thus stays less than $\mathrm{inj}(M)$.

To compute $\frac{d}{dt}l_i(t)$, we recall the following fact, which follows from the Gauss lemma of Riemannian geometry: If $p \in M$ and U is a normal neighborhood of p, consider the function $d_p : U\setminus\{p\} \to \mathbb{R}$ given by $d_p(q) = d(p,q)$. The gradient of d_p at q is the tangent at q of the shortest unit speed geodesic going from p to q.

Now, if $l_i(t) > 0$, we use the above fact and the law of motion (2) to compute, just as in the last section, that

$$\frac{d}{dt}l_i(t) = \cos\theta_i(t) - 1,$$

where θ_i is the angle at $b_{i+1}(t)$ between $[b_i(t), b_{i+1}(t)]$ and $[b_{i+1}(t), b_{i+2}(t)]$. So if $l_i(t) > 0$, then $\frac{d}{dt}l_i(t) \leq 0$ and thus l_i is locally nonincreasing at t. On the other hand, if $l_i(t) = 0$, then $l_i(t') = 0$ for all $t' > t$. So each l_i is indeed nonincreasing, and the pursuit process is well defined.

For each $t \geq 0$, $i \in \mathbb{Z}/n$, let $\beta_i^t = [b_i(t), b_{i+1}(t)]$ be the shortest geodesic connecting $b_i(t)$ to $b_{i+1}(t)$. Let $\beta^t : \mathbb{R}/\mathbb{Z} \to M$ be the constant-speed piecewise geodesic loop formed by concatenating the β_i^t, with $\beta^t(0) = \beta^t(1) = b_0(t)$. Then $t \mapsto \beta^t$ is a homotopy of loops, so if β^0 is not nullhomotopic, the pursuit process will not end in finite time. So Proposition 3.1 does not generalize to pursuit on Riemannian manifolds. This leads to the following conjecture for compact manifolds:

Conjecture 4.1. If M is a compact Riemannian manifold, and $\{b_i(0)\}_{i\in\mathbb{Z}/n}$ are initial conditions for cyclic pursuit, then the associated family of loops β^t either collapses to a point in finite time or converges to a closed geodesic as $t \to \infty$.

By convergence above, we mean convergence in the quotient of $C^0(S^1, M)$ by rotations in the domain. In other words, a sequence of loops $\{\gamma_j\}_{j=1}^{\infty}$ converges to $\gamma : \mathbb{R}/\mathbb{Z} \to M$ if

$$\lim_{j\to\infty} \inf_{c\in\mathbb{R}} \sup_{s\in\mathbb{R}/\mathbb{Z}} d(\gamma_j(s), \gamma(s+c)) = 0. \tag{3}$$

We prove Conjecture 4.1 in the case of pursuit on nonpositively curved compact manifolds in Section 8.

Remark. As observed above, if pursuit ends in finite time, then β^0 is nullhomotopic. The converse is not true. For instance, suppose α is a nullhomotopic closed geodesic along which all sectional curvatures are negative, e.g., the neck of a dumbbell. We will see in Section 8 that if β^0 is sufficiently close to α, then β^t will converge to α.

5. Convex submanifolds

We will need the following result, which states roughly that, if at some time the b_i all belong to a convex set $K \subset M$, then they stay in K.

Proposition 5.1. *Let M^d be a Riemannian manifold with* $\operatorname{inj}(M) > 0$, $\{b_i(t)\}_{i\in\mathbb{Z}/n}$ *cyclic pursuit curves on M, $l_i(t) = d(b_i(t), b_{i+1}(t))$. Let $K^d \subset M$ be a smoothly embedded submanifold with boundary, topologically closed in M. Suppose there is an $R \in (0, \operatorname{inj}(M))$ so that for any two points $p_1, p_2 \in K$ with $d(p_1, p_2) < R$, the geodesic segment $[p_1, p_2]$ is contained in K. If for some $t_0 \in [0, \infty)$, all of the $b_i(t_0)$ are in K and all of the $l_i(t_0)$ are less than R, then $b_i(t) \in K$ for all $i \in \mathbb{Z}/n$ and for all $t \geq t_0$.*

Proof idea. If one of the bugs reaches ∂K, then, by the convexity assumption, the bug's velocity will not point out of K. So the bug will stay in K.

Proof. Since K is closed, embedded, and of the same dimension as M, its topological boundary in M is the boundary manifold ∂K. Suppose for the sake of contradiction that there is a $t_1 > t_0$ and $j \in \mathbb{Z}/n$ so that $b_j(t_1)$ is not in K. Set

$$t' = \sup\{t \in [t_0, t_1] \mid b_i(t) \in K \text{ for all } i\} \quad \text{and} \quad \varepsilon = t_1 - t'.$$

Then since K is closed and the b_i are continuous, all of the $b_i(t')$ are in K, and at least one of the $b_i(t')$ is in ∂K. Furthermore, for all $t \in (t', t' + \varepsilon]$, at least one of the $b_i(t)$ is in $M \setminus K$.

For each $i \in \mathbb{Z}/n$, let $(V_i, x_i^1, \dots, x_i^d)$ be a coordinate neighborhood of $b_i(t')$ with the property that $V_i \cap K = \{p \in V_i \mid x_i^d(p) \le 0\}$, thus that $V_i \cap \partial K = \{p \in V_i \mid x_i^d(p) = 0\}$. (If $b_i(t')$ is in the interior of K, it may be that $V_i \cap \partial K = \varnothing$ and $x_i^d < 0$ on all of V_i.) Shrinking ε if necessary, we may assume $b_i([t', t' + \varepsilon]) \subset V_i$ for all i. Let $b_i^d = x_i^d \circ b_i$ denote the d-th component of b_i. Let $h(t) = \max_i b_i^d(t)$. Since all of the $b_i(t')$ are in K, and at least one of $b_i(t')$ is in ∂K, $h(t') = 0$. For each $t \in (t', t' + \varepsilon]$, at least one of the $b_i(t)$ is in the complement of K, so $h(t) > 0$. Assume without loss of generality that $b_i(t') \neq b_{i+1}(t')$, for all i. Then take ε small enough that $b_i(t) \neq b_{i+1}(t)$, for all $t \in [t', t' + \varepsilon]$ and all $i \in \mathbb{Z}/n$. Since each b_i^d is smooth on $[t', t' + \varepsilon]$, h is absolutely continuous on $[t', t' + \varepsilon]$. So h is almost everywhere differentiable, and

$$h(t) = \int_{t'}^{t} \frac{d}{ds} h(s)\, ds$$

for each $t \in [t', t' + \varepsilon]$. Thus, for some $c_1 \in [0, \varepsilon]$, $\frac{d}{dt} h(t' + c_1)$ is defined and

$$h(t' + \varepsilon) \le \varepsilon \frac{d}{dt} h(t' + c_1) = \varepsilon \frac{d}{dt} b_j^d(t' + c_1),$$

for some j for which $b_j^d(t' + c_1) = h(t' + c_1)$. Take ε small enough that if $b_i^d(t) = h(t)$ for some $t \in [t', t' + \varepsilon]$, $b_i^d(t') = 0$. Then in particular, $b_j^d(t') = 0$ for the j in the last displayed formula.

For each $i \in \mathbb{Z}/n$, let $v_i^d(p, q)$ be the d-th component of the initial unit tangent to $[p, q]$, for $(p, q) \in V_i \times V_{i+1}$ with $0 < d(p, q) < \mathrm{inj}(M)$. By the law of motion (2),

$$\frac{d}{dt} b_j^d(t' + c_1) = v_j^d(b_j(t' + c_1), b_{j+1}(t' + c_1)).$$

For each i, let $B_i \subset V_i$ be an open coordinate ball centered at $b_i(t')$ with $\bar{B}_i \subset V_i$. Shrinking the B_i if necessary, assume there is a $\delta > 0$ so that $\delta < d(p, q) < R$ for all $(p, q) \in B_i \times B_{i+1}$. Then v^d is C^1 on $\bar{B}_i \times \bar{B}_{i+1}$. Since K contains $[p, q]$ whenever $p, q \in K$ and $d(p, q) < R$, we have for all $(p, q) \in B_i \times B_{i+1}$ with $x_i^d(p) = 0$ and $x_{i+1}^d(q) \le 0$, that $v_i^d(p, q) \le 0$. Let μ_i and ν_i be the maximum on $\bar{B}_i \times \bar{B}_{i+1}$ of the absolute value of the derivative of $v_i^d(p, q)$ with respect to the d-th component of p and q, respectively. Let $\mu = \max_i \mu_i$ and $\nu = \max_i = \nu_i$. Taking ε small enough that $b_i([t', t' + \varepsilon]) \subset B_i$ for all i, we have

$$v_j^d(b_j(t' + c_1), b_{j+1}(t' + c_1)) \le (\mu + \nu) h(t' + c_1).$$

From the last three displayed formulas, we get

$$h(t' + \varepsilon) \le (\mu + \nu) \varepsilon h(t' + c_1).$$

Similarly, $h(t'+c_1) \leq (\mu+\nu)\varepsilon h(t'+c_2)$ for some $c_2 \in [0, c_1]$, so we obtain $h(t'+\varepsilon) \leq ((\mu+\nu)\varepsilon)^2 h(t'+c_2)$. Inductively, we get for each positive integer k,

$$h(t'+\varepsilon) \leq ((\mu+\nu)\varepsilon)^k h(t'+c_k)$$

for some $c_k \in [0, \varepsilon]$. Let $C = \max_{i \in \mathbb{Z}/n} \sup_{p \in B_i} |x_i^d(p)|$. Taking $\varepsilon < \frac{1}{2}(\mu+\nu)^{-1}$, we get $h(t'+\varepsilon) < 2^{-k}C$. Letting $k \to \infty$ yields $h(t'+\varepsilon) = 0$, a contradiction. $\square$

We say a subset K of a Riemannian manifold M is *convex* if for each pair $p, q \in K$, there is a unique shortest geodesic in M connecting p, q and this geodesic is contained in K. Proposition 5.1 is the key ingredient in the proof of the following result:

Proposition 5.2. *Suppose M is compact and $\{b_i(t)\}_{i \in \mathbb{Z}/n}$ are cyclic pursuit curves in M. Let $l_i(t) = d(b_i(t), b_{i+1}(t))$ for each i. If $l_i(t) \to 0$ for all i, then pursuit ends in finite time.*

Proof idea. Reduce to the Euclidean case by noting that the bugs will eventually lie in a small, convex, approximately Euclidean ball.

Proof. Assume for the sake of contradiction that pursuit does not end in finite time. Without loss of generality, assume $l_i(t) > 0$ for all i. Since M is compact, there is a $p \in M$ and a sequence of times $(t_j)_{j=1}^{\infty}$ with $t_j \to \infty$ so that $b_0(t_j) \to p$ as $j \to \infty$. Let $r \in (0, \operatorname{inj}(M))$ be small enough that $\overline{B}_r(p)$, the closed r-ball centered at p, is convex. Since $d(b_i(t_j), b_{i+1}(t_j)) \to 0$, $b_0(t_j) \to p$, and $b_i(t_j) \to p$ for all i, there is a J for which all of the $b_i(t_J)$ belong to $\overline{B}_r(p)$. By Proposition 5.1, the $b_i(t)$ remain in $\overline{B}_r(p)$ for all $t > t_J$.

Let (U, x^i) be a normal coordinate neighborhood centered at p. By Corollary A.2, we have that for small enough r, $\overline{B}_r(p)$ is a convex subset of U and has the following property: for any two geodesics $\gamma_1 : [0, a_1] \to \overline{B}_r(p)$ and $\gamma_2 : [0, a_1] \to \overline{B}_r(p)$ with $\gamma_1(0) = \gamma_2(0)$, the metric angle between $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ is within $\frac{\pi}{n}$ of the Euclidean angle, computed in the coordinates (U, x^i), between $\gamma_1(a_1) - \gamma_1(0)$ and $\gamma_2(a_2) - \gamma_2(0)$.

Now, choose r as above and find t_0 so that all of the $b_i(t)$ are in $\overline{B}_r(t)$ for $t \geq t_0$. As we showed in the proof of Proposition 3.1, at least one of the Euclidean angles of the piecewise linear loop connecting the $b_i(t)$ is at least $\frac{2\pi}{n}$. So by the result quoted in the last paragraph, at least one of the angles of the piecewise geodesic loop connecting the $b_i(t)$ is at least $\frac{\pi}{n}$. Thus, $\frac{d}{dt} l(t) \leq \cos\frac{\pi}{n} - 1$ for $t \geq t_0$ and so pursuit ends by time $t_0 + l(t_0)(1 - \cos\frac{\pi}{n})^{-1}$. This is a contradiction. $\square$

6. Subsequential convergence

In this section, M is a compact Riemannian manifold.

Proposition 6.1. *Let $\{b_i(t)\}_{i\in\mathbb{Z}/n}$ be pursuit curves on M, let β^t be the associated family of piecewise geodesic loops, and let $l(t)$ the length of β^t. If the pursuit does not end in finite time, then there is a sequence of times $(t_j)_{j=1}^{\infty}$, $t_j \to \infty$ so that β^{t_j} converges uniformly to a closed geodesic of length $L = \lim_{t\to\infty} l(t)$ as $j \to \infty$.*

Proof sketch. Take a sequence t_j so that the $b_i(t_j)$ converge and so that $\frac{d}{dt} l(t_j) \to 0$. Then β^{t_j} converges to a piecewise geodesic loop. The condition $\frac{d}{dt} l(t_j) \to 0$ implies that the angles between segments of the limiting loop are 0.

Proof. Let $\beta_i^t = [b_i(t), b_{i+1}(t)]$. Recall that β^t is the constant-speed piecewise geodesic loop formed from the β_i^t, with $\beta^t(0) = b_0(t)$. We have $L > 0$, else by Proposition 5.2, pursuit ends in finite time. Assume without loss of generality that $d(b_i(t), b_{i+1}(t)) > 0$ for all $i \in \mathbb{Z}/n$ and all $t > 0$. Then we have for all t that

$$\frac{d}{dt} l(t) = \sum_i (\cos\theta_i(t) - 1),$$

where $\theta_i(t)$ is the angle at $b_{i+1}(t)$ between β_i^t and β_{i+1}^t. Since l is differentiable, nonincreasing, and bounded from below, there is a sequence $(t_j)_{j=1}^{\infty}$, $t_j \to \infty$, so that $\frac{d}{dt} l(t_j) \to 0$. This implies that for each i, $\theta_i(t_j) \to 0$ as $j \to \infty$. Since M is compact, we may pass to a subsequence and assume that for each i, $b_i(t_j)$ converges to some point $a_i \in M$. Then $[b_i(t_j), b_{i+1}(t_j)]$ converges uniformly to $[a_i, a_{i+1}]$. Let α be the constant speed piecewise geodesic loop formed from the geodesic segments $[a_i, a_{i+1}]$, with $\alpha(0) = a_0$. Then β^{t_j} converges to α uniformly. By continuity, α has length L.

We need to show that α is a closed geodesic. To do this, it suffices to show that the angles between successive geodesic segments comprising α are 0. We need to include the case that $a_i = a_{i+1}$ for some i. To this end, suppose $a_{i-1} \neq a_i$, and let k be the largest integer so that $a_i = a_{i+1} = \cdots = a_{i+k}$. We need to show that the angle at a_i between $[a_{i-1}, a_i]$ and $[a_i, a_{i+k+1}]$ is 0. Let (U, x^i) be a normal coordinate neighborhood centered at a_i, and $\|\cdot\|_U$ be the Euclidean norm on TU coming from the coordinates. Fix $\varepsilon > 0$. Then for large enough j, $b_i(t_j), \dots, b_{i+k}(t_j)$ are in U and

$$\left\| \frac{\dot\beta_m^{t_j}(1)}{\|\dot\beta_m^{t_j}(1)\|} - \frac{\dot\beta_m^{t_j}(0)}{\|\dot\beta_m^{t_j}(0)\|} \right\|_U < \varepsilon,$$

for $m = i, \dots, i+k-1$. (See (9) in the Appendix.) Since $\theta_i(t_j) \to 0$ for all i, for large enough j,

$$\left\| \frac{\dot\beta_m^{t_j}(0)}{\|\dot\beta_m^{t_j}(0)\|} - \frac{\dot\beta_{m-1}^{t_j}(1)}{\|\dot\beta_{m-1}^{t_j}(1)\|} \right\|_U < \varepsilon,$$

for $m = i, \ldots, i+k$. From the last two displayed expressions, we obtain

$$\left\| \frac{\dot{\beta}^{t_j}_{i+k}(0)}{\|\dot{\beta}^{t_j}_{i+k}(0)\|} - \frac{\dot{\beta}^{t_j}_{i-1}(1)}{\|\dot{\beta}^{t_j}_{i-1}(1)\|} \right\|_U < (2k-1)\varepsilon.$$

Thus, the expression on the left of the last inequality converges to 0 as $j \to \infty$. But

$$\frac{\dot{\beta}^{t_j}_{i+k}(0)}{\|\dot{\beta}^{t_j}_{i+k}(0)\|}$$

converges to the unit tangent to $[a_i, a_{i+k+1}]$ at a_i, and

$$\frac{\dot{\beta}^{t_j}_{i-1}(1)}{\|\dot{\beta}^{t_j}_{i-1}(1)\|}$$

converges to the unit tangent to $[a_{i-1}, a_i]$ at a_i. Hence, these two unit tangent directions are the same, and the angle between $[a_i, a_{i+k+1}]$ and $[a_{i-1}, a_i]$ at a_i is 0, as claimed. $\square$

As a consequence of the last proposition, we have the following corollaries.

Corollary 6.2. *If for some $t_0 \geq 0$, the length of β^{t_0} is less than the length $\lambda_{\min}$ of the shortest closed geodesic of M, pursuit ends in finite time. In particular, if the length of β^0 is less than $\lambda_{\min}$, pursuit ends in finite time.*

Corollary 6.3. *If for some $t_0 \geq 0$, the $b_i(t_0)$ all lie in a convex, smoothly embedded, closed metric ball $\bar{B} \subset M$, then pursuit ends in finite time.*

Proof. By Proposition 5.1, the $b_i(t)$ stay in $\bar{B}$ for $t > t_0$. If pursuit does not end in finite time, then arguing as in Proposition 6.1, there is a sequence $t_j \to \infty$ so that β^{t_j} converges to a closed geodesic in $\bar{B}$. But there are no closed geodesics contained in $\bar{B}$. $\square$

It follows, for example, that if the $b_i(t_0)$ all lie in an open hemisphere of S^2 with its standard metric, pursuit ends in finite time.

7. A criterion for convergence

The next result gives a criterion for convergence of β^t to a closed geodesic α.

Proposition 7.1. *Let M be a Riemannian manifold with $\operatorname{inj}(M) > 0$. Let $\{b_i(t)\}_{i \in \mathbb{Z}/n}$ be cyclic pursuit curves on M, and let β^t the associated family of loops. Suppose there is a sequence $t_j \to \infty$ and a closed geodesic α so that $\beta^{t_j} \to \alpha$ uniformly. If $\sup_{s \in S^1} d(\beta^t(s), \alpha(S^1)) \to 0$ as $t \to \infty$, then β^t converges to α in the sense of (3), as $t \to \infty$.*

Proof idea. For large t, the curves β^t and α have approximately the same length. In addition, β^t fits into a small tubular neighborhood of α. These two facts force β^t to be uniformly close to α.

Proof. Let U be an open neighborhood of $\alpha(S^1)$ such that each $p \in U$ has a unique closest point $\pi(p)$ on $\alpha(S^1)$. Shrinking U if necessary, we may construct a smooth unit vector field X on U extending the unit tangent field $\dot{\alpha}/\|\dot{\alpha}\|$ of α.

Fix $\varepsilon > 0$. Let $l_i(t) = d(b_i(t), b_{i+1}(t))$, and let λ be the minimum of $\lim_{t\to\infty} l_i(t)$ over i for which $\lim_{t\to\infty} l_i(t) > 0$. The continuous dependence of the initial unit tangent of a geodesic $[p, q]$ on the endpoints p and q implies: There is a δ such that $\overline{N}_\delta(\alpha) \subset U$ and if $\gamma : [0, 1] \to \overline{N}_\delta(\alpha)$ is a geodesic of length at least λ, then the component of $\dot{\gamma}(0)/\|\dot{\gamma}(0)\|$ normal to $X(\gamma(0))$ has length less than ε.

Consider i such that $\lim_{t\to\infty} l_i(t) > 0$. Let $a_i^t = \pi(\beta_i^t(0))$. By hypothesis,

$$d(\beta_i^t(0), a_i^t) \to 0 \tag{4}$$

as $t \to \infty$. By the observation in the previous paragraph, the component of $\dot{\beta}_i^t(0)$ orthogonal to $X(\beta_i^t(0))$ goes to 0 as well. Since $\beta^{t_j} \to \alpha$ uniformly, we have by continuity that

$$\|\dot{\beta}_i^t(0) - l_i(t) \cdot X(\beta_i^t(0))\| \to 0. \tag{5}$$

Now, let α_i^t be the segment of α starting at a_i^t with initial velocity $l_i(t) \cdot X(a_i(t))$. Since a geodesic depends continuously on its initial parameters, (4) and (5) give

$$\lim_{t\to\infty} \sup_{s\in[0,1]} d(\beta_i^t(s), \alpha_i^t(s)) = 0. \tag{6}$$

Let $i_1, \ldots, i_m$ be the values of i for which $\lim_{t\to\infty} l_i(t) > 0$, listed in order. Now, let $\gamma^t : \mathbb{R}/\mathbb{Z} \to M$ be the piecewise continuous loop formed by concatenating the segments $\alpha_{i_j}^t([0, 1))$. We parametrize γ^t so that each $\alpha_{i_j}^t$ is traversed at the same constant speed, and $\gamma^t(0) = \alpha_{i_1}^t(0)$. Then as a consequence of (6),

$$\lim_{t\to\infty} \sup_{s\in S^1} d(\beta^t(s), \gamma^t(s)) = 0. \tag{7}$$

Let c^t be such that $\alpha(c^t) = \gamma^t(0)$. Then by applying the triangle inequality, $\lim_{t\to\infty} d(\alpha_{i_j}^t(1), \alpha_{i_{j+1}}^t(0)) = 0$ for $j = 1, \ldots, m-1$. Also, $\sum_{j=1}^m l_{i_j}(t)$ converges to the length of α as $t \to \infty$. It follows that

$$\lim_{t\to\infty} \sup_{s\in S^1} d(\gamma^t(s), \alpha(s + c^t)) = 0. \tag{8}$$

From (7) and (8), we get

$$\lim_{t\to\infty} \sup_{s\in S^1} d(\beta^t(s), \alpha(s + c^t)) = 0,$$

which completes the proof. $\square$

8. Nonpositive curvature

In the next proposition, we show that if a subsequence β^{t_j} converges to a closed geodesic α which is a local minimizer of length, then β^t converges to α.

We recall the following notation: For each $x \in S^1$, let $T_x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the translation $T_x(s) = s + x$. If $\alpha, \gamma : S^1 \to M$ are two loops, $d(\alpha, \gamma)$ denotes the supremum distance $\sup_{s \in S^1} d(\alpha(s), \gamma(s))$.

Proposition 8.1. *Let M be a Riemannian manifold with* $\operatorname{inj}(M) > 0$, $\{b_i(t)\}_{i \in \mathbb{Z}/n}$ *cyclic pursuit curves on M, β^t the associated family of loops. Let α be a closed geodesic of length L, and suppose there is $\varepsilon > 0$ so that any rectifiable, constant speed loop γ of length L with $d(\alpha, \gamma) < \varepsilon$ is a reparametrization of α, i.e., $\gamma = \alpha \circ T_x$ for some $x \in S^1$. If there is a sequence of times $t_j \to \infty$ so that $\beta^{t_j} \to \alpha$ as $j \to \infty$, then $\beta^t \to \alpha$ as $t \to \infty$.*

Proof sketch. If β^t does not converge to α, then β^t has another length L subsequential limit γ. It follows that for any k, there is a homotopy from α to γ through curves of length between L and $L + 1/k$. The homotopy passes through a curve η_k so that $\inf_{x \in S^1} d(\eta_k, \alpha \circ T_x) = \frac{\varepsilon}{2}$. Taking a subsequential limit of the η_k yields a contradiction.

Proof. Suppose for the sake of contradiction that β^t does not converge to α. Then by Proposition 7.1, there is $\delta > 0$ and a sequence of times $t'_j \to \infty$ so that $\sup_{s \in S^1} d(\beta^{t'_j}(s), \alpha(S^1)) > \delta$. Passing to a subsequence, we may assume $\beta^{t'_j}$ converges uniformly to a constant speed piecewise geodesic loop γ as $j \to \infty$.

Since $\sup_{s \in S^1} d(\beta^{t'_j}(s), \alpha(S^1)) > \delta$ for all j, $\sup_{s \in S^1} d(\gamma(s), \alpha(S^1)) \geq \delta$, so γ is not a reparametrization of α. But since $\beta^{t_j} \to \alpha$, $\lim_{t \to \infty} l(t) = L$, and thus γ has length L. So by hypothesis, $d(\gamma \circ T_x, \alpha) \geq \varepsilon$ for all $x \in S^1$, which is to say that $\inf_{x \in S^1} d(\gamma \circ T_x, \alpha) \geq \varepsilon$. We may assume that $t_j < t'_j$, $d(\beta^{t_j}, \alpha) < \frac{\varepsilon}{2}$, and $d(\beta^{t'_j}, \gamma) < \frac{\varepsilon}{2}$ for all j. Then

$$\inf_{x \in S^1} d(\beta^{t_j} \circ T_x, \alpha) \leq d(\beta^{t_j}, \alpha) < \frac{\varepsilon}{2},$$

and

$$\inf_{x \in S^1} d(\beta^{t'_j} \circ T_x, \alpha) \geq \Big(\inf_{x \in S^1} d(\gamma \circ T_x, \alpha)\Big) - d(\beta^{t'_j}, \gamma) > \frac{\varepsilon}{2}.$$

Since the function $(t, x) \mapsto d(\beta^t \circ T_x, \alpha)$ is continuous and S^1 is compact, the function $t \mapsto \inf_{x \in S^1} d(\beta^t \circ T_x, \alpha)$ is continuous. Thus, for each j, there is $t''_j \in (t_j, t'_j)$, so that $\inf_{x \in S^1} d(\beta^{t''_j} \circ T_x, \alpha) = \frac{\varepsilon}{2}$. A subsequence of $\beta^{t''_j}$ converges to a constant speed loop η of length L with

$$\inf_{x \in S^1} d(\eta \circ T_x, \alpha) = \frac{\varepsilon}{2}.$$

Hence, η is not a reparametrization of α, yet there is an x so that $d(\eta \circ T_x, \alpha) = \frac{\varepsilon}{2} < \varepsilon$. This is a contradiction. □

With the notation as above we have the following:

Corollary 8.2. *Let α be a closed geodesic of length L such that all sectional curvatures are negative at each point of the image of α. If $\beta^{t_j} \to \alpha$ for some sequence $t_j \to \infty$, then $\beta^t \to \alpha$.*

Proof. From the formula for second variation of arclength, we know that α is isolated in the space of loops of length L, i.e., that α satisfies the hypotheses of Proposition 8.1. □

Remark. Suppose a subsequence β^{t_j} converges to a closed geodesic α. Then Proposition 8.1 shows that if α is isolated in the space of *rectifiable loops* of its length, then $\beta^t \to \alpha$. Suppose instead α is merely isolated in the space of *closed geodesics* of its length. Suppose in addition that $\theta_i(t)$ converges to 0 for all i. Then all subsequential limits of β^t are geodesics, so arguing as in the proof of Proposition 8.1, we can show $\beta^t \to \alpha$. In particular, if the following conjecture holds, then β^t converges to a closed geodesic (or a point) for pursuit on any compact manifold whose space of closed geodesics is discrete.

Conjecture 8.3. Let M be a compact manifold, $\{b_i(t)\}_{i \in \mathbb{Z}/n}$ pursuit curves on M, $l_i(t)$ the associated lengths, $\theta_i(t)$ the associated angles. If $l_i(t) > 0$ for all $i \in \mathbb{Z}/n$ and $t \geq 0$, then $\theta_i(t) \to 0$.

Corollary 8.2 and Proposition 6.1 imply Conjecture 4.1 for compact manifolds of negative curvature. Next, we prove Conjecture 4.1 for manifolds of nonpositive curvature. First, we need a lemma:

Lemma 8.4. *Let α be a closed geodesic on a Riemannian manifold M. Suppose $\alpha(S^1)$ is contained in a nonpositively curved open submanifold $U \subset M$. Suppose further that there is an $r > 0$ so that* $\operatorname{inj}(p) \geq r$ *for all $p \in U$. Fix $\varepsilon > 0$. For sufficiently small δ, the following property holds: for any $p_1, p_2 \in \overline{N}_\delta(\alpha)$ with $d(p_1, p_2) < r - \varepsilon$, we have $[p_1, p_2] \subset \overline{N}_\delta(\alpha)$.*

Proof. We will show that it suffices to take δ small enough that

(i) $\delta < \frac{\varepsilon}{4}$,

(ii) for any $p, q \in \overline{N}_\delta(\alpha)$ with $d(p, q) < r - \frac{\varepsilon}{4}$, we have $[p, q] \subset U$.

(For (ii), we use the continuous dependence of $[p, q]$ on p, q.)

Now, take $p_1, p_2 \in \overline{N}_\delta(\alpha)$ with $d(p_1, p_2) < r - \varepsilon$. Let $\alpha : [0, 1] \to M$ be the shortest geodesic connecting p_1 to p_2. Choose q_i in the image of α with $d(p_i, q_i) \leq \delta$ for $i = 1, 2$. By condition (i) and the triangle inequality, $d(q_1, q_2) < r - \frac{\varepsilon}{2}$, so there is a unique shortest geodesic $\gamma : [0, 1] \to M$ connecting q_1 to q_2.

Observe that $d(\alpha(t), \gamma(t)) < r - \frac{\varepsilon}{4}$ for all $t \in [0, 1]$. Indeed, for $t \in [0, 1/2]$, the path consisting of segments $[\alpha(t), p_1]$, $[p_1, q_1]$, $[q_1, \gamma(t)]$ has length less than $r - \frac{\varepsilon}{4}$. Similarly, for $t \in [1/2, 1]$, the path consisting of $[\alpha(t), p_2]$, $[p_2, q_2]$, $[q_2, \gamma(t)]$ has length less than $r - \frac{\varepsilon}{4}$.

By (ii), it follows that $[\alpha(t), \gamma(t)] \subset U$ for $t \in [0, 1]$. The fact that $d(\alpha(t), \gamma(t)) < r - \frac{\varepsilon}{4}$ for all $t \in [0, 1]$ also implies that the geodesic $[\alpha(t), \gamma(t)]$ varies smoothly in t. Since U is nonpositively curved, we may apply the formula for the second variation of energy to the family of geodesics $[\alpha(t), \gamma(t)]$ to conclude that $d^2(\alpha(t), \gamma(t))$ is convex as a function of t. Therefore, for $t \in [0, 1]$,

$$\begin{aligned} d(\alpha(t), \gamma(t)) &\leq \max[d(\alpha(0), \gamma(0)), d(\alpha(1), \gamma(1))] \\ &= \max[d(p_1, q_1), d(p_2, q_2)] \leq \delta. \end{aligned}$$

Thus, $[p_1, p_2] \subset \overline{N}_\delta(\alpha)$. □

Proposition 8.5. *If pursuit on a nonpositively curved compact manifold M does not end in finite time, there is a closed geodesic α so that* $\sup_{s \in S^1} d(\beta^t(s), \alpha(S^1)) \to 0$ *as $t \to \infty$.*

Proof. By Proposition 6.1, there is a closed geodesic α and a sequence $t_j \to \infty$ so that $\beta^{t_j} \to \alpha$ uniformly.

Let $\varepsilon > 0$ be such that $d(b_i(0), b_{i+1}(0)) < \mathrm{inj}(M) - \varepsilon$ for all i. Take $\delta < \frac{\varepsilon}{4}$. Then by the proof of Lemma 8.4, if $p_1, p_2 \in \overline{N}_\delta(\alpha)$ and $d(p_1, p_2) < \mathrm{inj}(M) - \varepsilon$, we have $[p_1, p_2] \subset \overline{N}_\delta(\alpha)$. Also take δ small enough that $\overline{N}_\delta(\alpha)$ is a closed manifold with boundary, smoothly embedded in M. Since $\beta^{t_j} \to \alpha$, we have that the $b_i(t_J)$ are all in $\overline{N}_\delta(\alpha)$ for some sufficiently large J. Now, by Proposition 5.1, we have $b_i(t) \in \overline{N}_\delta(\alpha)$ for all $i \in \mathbb{Z}/n$, $t \geq t_J$. By Lemma 8.4, $\beta^t \subset \overline{N}_\delta(\alpha)$ for $t \geq t_J$. Thus, $\sup_{s \in S^1} d(\beta^t(s), \alpha) \to 0$ as $t \to \infty$. □

As a consequence of Proposition 7.1 and Proposition 8.5, we have Conjecture 4.1 for manifolds of nonpositive curvature:

Theorem 8.6. *Let M be a compact manifold of nonpositive sectional curvature. Suppose pursuit on M with initial positions $\{b_i(0)\}_{i \in \mathbb{Z}/n}$ does not end in finite time, and let β^t be the associated family of piecewise geodesic loops. Then there is a closed geodesic α so that $\beta^t \to \alpha$.*

As an improvement of Corollary 8.2, we have the following result, which states that if β^t gets close enough to a geodesic along which all sectional curvatures are negative, then β^t converges to that geodesic:

Proposition 8.7. *Let M be a Riemannian manifold with* $\mathrm{inj}(M) > 0$, *and let α be a closed geodesic such that all sectional curvatures are negative at each point of the image of α. Fix $\varepsilon > 0$. Then there is a $\delta > 0$ so that, if* $d(b_i(0), b_{i+1}(0)) < \mathrm{inj}(M) - \varepsilon$

for all i and β^{t_0} is uniformly δ-close to $\alpha \circ T_x$ for some $t_0 > 0$ and some $x \in \mathbb{R}/\mathbb{Z}$, then $\beta^t \to \alpha$ as $t \to \infty$.

Proof. Take δ small enough so that:

(i) if $p_1, p_2 \in \overline{N}_\delta(\alpha)$ and $d(p_1, p_2) < \mathrm{inj}(M) - \varepsilon$, then $[p_1, p_2] \subset \overline{N}_\delta(\alpha)$;

(ii) $\overline{N}_\delta(\alpha)$ is a closed manifold with boundary, smoothly embedded in M;

(iii) any loop uniformly δ-close to $\alpha \circ T_x$ for some $x \in \mathbb{R}/\mathbb{Z}$ is homotopic to α through a family of loops in $\overline{N}_\delta(\alpha)$;

(iv) any closed geodesic γ in $\overline{N}_\delta(\alpha)$ homotopic to α through a family of loops in $\overline{N}_\delta(\alpha)$ differs from α by a rotation in the domain.

For condition (iv), we argue on general grounds that taking δ small forces γ to be uniformly close to $\alpha \circ T_x$ for some $x \in S^1$, and then we use the fact that a closed geodesic on M along which all sectional curvatures are negative is isolated in the space of closed geodesics on M; the argument is straightforward, and we omit the details. For (i), we use Lemma 8.4. For (ii) and (iii), we take the image under exp of a neighborhood of the zero section in the normal bundle of $\alpha(S^1)$.

Suppose we have initial conditions for pursuit $\{b_i(0)\}_{i \in \mathbb{Z}/n}$ with relative distances $d(b_i(0), b_{i+1}(0)) < \mathrm{inj}(M) - \varepsilon$ for all i, and the associated piecewise geodesic loop β^{t_0} is uniformly δ-close to $\alpha \circ T_x$ for some $t_0 > 0$, $x \in \mathbb{R}/\mathbb{Z}$. By (i), (ii), and Proposition 5.1, we have $\beta^t \subset \overline{N}_\delta(\alpha)$ for all $t \geq t_0$. Using Proposition 6.1, we get a sequence $t_j \to \infty$ and a geodesic γ contained in $\overline{N}_\delta(\alpha)$ so that $\beta^{t_j} \to \gamma$ uniformly. So β^{t_0} is homotopic through a family of loops in $\overline{N}_\delta(\alpha)$ to γ. But by (iii), β^{t_0} is also homotopic through a family of loops in $\overline{N}_\delta(\alpha)$ to α. Now by (iv), γ differs from α by a rotation in the domain. So by Corollary 8.2, $\beta^t \to \alpha$ as $t \to \infty$. □

Appendix

We prove a result (Corollary A.2 below) needed for the proof of Proposition 5.2.

Proposition A.1. *Let (M^n, g) be a Riemannian manifold, p a point in M, (U, x^i) a normal coordinate neighborhood centered at p. Let $\|\cdot\|$ be the Euclidean norm on (U, x^i). Then for every $\varepsilon > 0$, there is an r such that $B_r(p) \subset U$ and for every geodesic $\gamma : [0, a] \to B_r(p)$,*

$$\left\| \frac{\dot\gamma(0)}{\|\dot\gamma(0)\|} - \frac{\gamma(t) - \gamma(0)}{\|\gamma(t) - \gamma(0)\|} \right\| < \varepsilon,$$

for all $t \in (0, a]$.

Proof. Fix $\varepsilon > 0$. Let V be an open neighborhood of p with closure contained in U. Then the Christoffel symbols Γ^k_{ij} associated to (U, x^i) are bounded on V. Find μ so that $|\Gamma^k_{ij}| < \mu$ on V for all i, j, k. Take r small enough that $B_r(p) \subset V$. Since $g_{ij}(p) = \delta_{ij}$, we may take r small enough that any vector in $TB_r(p)$ of unit length with respect to g has length less than 2 with respect to the Euclidean norm. Now, if $\gamma : [0, a] \to B_r(p)$ is a unit speed geodesic, we have

$$\frac{d^2\gamma^k}{dt^2} = -\Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt},$$

so $\left|\frac{d^2\gamma^k}{dt^2}\right| < 4\mu n^2$; thus $\left\|\frac{d^2\gamma}{dt^2}\right\| < 4\mu n^{\frac{5}{2}}$. Integrating, we find for each $t \in [0, a]$,

$$\|\dot\gamma(t) - \dot\gamma(0)\| < 4t\mu n^{\frac{5}{2}} \le 8r\mu n^{\frac{5}{2}},$$

so taking r smaller than $[8\mu n^{\frac{5}{2}}]^{-1}\varepsilon$,

$$\|\dot\gamma(t) - \dot\gamma(0)\| < \varepsilon. \tag{9}$$

Integrating again, we get $\|\gamma(t) - \gamma(0) - t\dot\gamma(0)\| < t\varepsilon$, so

$$\left\|\frac{\gamma(t) - \gamma(0)}{t} - \dot\gamma(0)\right\| < \varepsilon$$

for $t \in (0, a]$. Assuming r is chosen small enough so that any $v \in TB_r(p)$ with $g(v, v) = 1$ satisfies $\big\|v - v/\|v\|\big\| < \varepsilon$, we have

$$\left\|\frac{\gamma(t) - \gamma(0)}{t} - \frac{\dot\gamma(0)}{\|\dot\gamma(0)\|}\right\| < 2\varepsilon.$$

Hence, the Euclidean distance from the unit vector $\dot\gamma(0)/\|\dot\gamma(0)\|$ to the line spanned by $\gamma(t) - \gamma(0)$ is less than 2ε. For ε small enough, this implies

$$\left\|\frac{\gamma(t) - \gamma(0)}{\|\gamma(t) - \gamma(0)\|} - \frac{\dot\gamma(0)}{\|\dot\gamma(0)\|}\right\| < 3\varepsilon. \qquad \square$$

With notation as in the last proposition, we have the following:

Corollary A.2. *For every* $\varepsilon > 0$, *there is an* r *so that* $B_r(p) \subset U$ *and for any two geodesics* $\gamma_1 : [0, a_1] \to B_r(p)$ *and* $\gamma_2 : [0, a_2] \to B_r(p)$ *with* $\gamma_1(0) = \gamma_2(0)$, *the metric angle between* $\dot\gamma_1(0)$ *and* $\dot\gamma_2(0)$ *is within* ε *of the Euclidean angle between* $\gamma_1(a_1) - \gamma_1(0)$ *and* $\gamma_2(a_2) - \gamma_2(0)$.

Proof. By the last part and uniform continuity of the spherical distance function $S^{n-1} \times S^{n-1} \to \mathbb{R}$ on the unit sphere, we can choose r small enough that the Euclidean angle between the vectors $\gamma_1(a_1) - \gamma_1(0)$ and $\gamma_2(a_2) - \gamma_2(0)$ is within $\frac{\varepsilon}{2}$ of the Euclidean angle between $\dot\gamma_1(0)$ and $\dot\gamma_2(0)$, for any two unit speed geodesics $\gamma_1 : [0, a_1] \to B_r(p)$ and $\gamma_2 : [0, a_2] \to B_r(p)$ with $\gamma_1(0) = \gamma_2(0)$. Then, if necessary, we choose r smaller so that for any two vectors $u, v \in TB_r(p)$ based at the same point, the Euclidean angle between u and v is within $\frac{\varepsilon}{2}$ of the metric angle. $\square$

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CRITICALITY OF THE AXIALLY SYMMETRIC NAVIER–STOKES EQUATIONS

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Smooth solutions to the axisymmetric Navier–Stokes equations obey the following maximum principle:

$$\sup_{t\geq 0} \|rv^\theta(t,\cdot)\|_{L^\infty} \leq \|rv^\theta(0,\cdot)\|_{L^\infty}.$$

We prove that all solutions with initial data in $H^{1/2}$ are smooth globally in time if rv^θ satisfies a kind of form boundedness condition (FBC) which is invariant under the natural scaling of the Navier–Stokes equations. In particular, if rv^θ satisfies

$$\sup_{t\geq 0}|rv^\theta(t,r,z)| \leq C_*|\ln r|^{-2}, \quad \text{where} \quad r \leq \delta_0 \in \left(0,\tfrac{1}{2}\right), \quad C_* < \infty,$$

then our FBC is satisfied. Here δ_0 and C_* are independent of neither the profile nor the norm of the initial data. So the gap from regularity is logarithmic in nature. We also prove the global regularity of solutions if $\|rv^\theta(0,\cdot)\|_{L^\infty}$ or $\sup_{t\geq 0}\|rv^\theta(t,\cdot)\|_{L^\infty(r\leq r_0)}$ is small but the smallness depends on a certain dimensionless quantity of the initial data.

1. Introduction

The global regularity problem of three-dimensional incompressible Navier–Stokes equations is commonly considered as supercritical because the a priori estimates based on energy equality become worse when looking into finer and finer scales; see, for instance, [Tao 2007]. Such a "supercriticality" barrier is one of the main reasons why this is such a hard problem.

Recently, the axisymmetric Navier–Stokes equations have attracted tremendous interest from experts. See, for instance, [Burke Loftus and Zhang 2010; Chae and Lee 2002; Chen et al. 2008; 2009; 2015; Hou and Li 2008; Hou et al. 2008; Jiu and Xin 2003; Koch et al. 2009; Lei et al. 2013; Lei and Zhang 2011b; 2011a; Leonardi et al. 1999; Neustupa and Pokorný 2000; 2001; Pan 2016; Seregin and Šverák 2009; Tian and Xin 1998; Zhang and Zhang 2014]. These results heavily depend on the maximum principle of the dimensionless quantity $\Gamma = rv^\theta$, which

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makes the axisymmetric Navier–Stokes equations partially critical (only the swirl component v^θ of the velocity field satisfies a dimensionless a priori estimate). Although the axially symmetric Navier–Stokes equation is a special case of the full three-dimensional one, our level of understanding had been roughly the same, with the previously mentioned difficulty unresolved because the effective a priori bound available is still the energy estimate, which has positive dimension $\frac{1}{2}$.

The aim of this article is to show that the axisymmetric Navier–Stokes equation is, in fact, fully critical. More precisely, we prove all solutions with initial data in $H^{1/2}$ are smooth globally in time if rv^θ satisfies a kind of FBC which is invariant under the natural scaling of the Navier–Stokes equations. In particular, if rv^θ satisfies $\sup_{t\geq 0}|rv^\theta(t,r,z)| \leq C_*|\ln r|^{-2}$, where $r \leq \delta_0 \in \left(0, \frac{1}{2}\right)$, $C_* < \infty$, then our FBC is satisfied. Here δ_0 and C_* are independent of neither the profile nor the norm of the initial data. The proof is based on the observation that the vorticity equations can be transformed into a system such that the vortex-stretching terms are critical. This means that the potentials in front of unknown functions scale as $1/|x|^2$. For example, in (1-8) below, the function J is regarded as unknown and the potential in front of it is $-2v^\theta/r$ which scales as $1/|x|^2$.

We also prove the global regularity of solutions if $\sup_{t\geq 0}\|rv^\theta(t,\cdot)\|_{L^\infty(r\leq r_0)}$ or $\|rv^\theta(0,\cdot)\|_{L^\infty}$ is small but the smallness depends on a certain dimensionless quantity of the initial data. Our work is inspired by the recent interesting result of Chen, Fang and Zhang in [Chen et al. 2015] where, among other things, global regularity is obtained if $rv^\theta(t,\cdot,z)$ is Hölder continuous in the r variable.

To state our result more precisely, let us recall that in cylindrical coordinates r, θ, z with $(x_1,x_2,x_3) = (r\cos\theta, r\sin\theta, z)$, axially symmetric solutions of the Navier–Stokes equations are of the following form:

$$\begin{cases} v(t,x) = v^r(t,r,z)e_r + v^\theta(t,r,z)e_\theta + v^z(t,r,z)e_z, \\ p(t,x) = p(t,r,z). \end{cases}$$

The components v^r, v^θ, v^z are all independent of the angle of rotation θ. Here e_r, e_θ, e_z are the basis vectors for $\mathbb{R}^3$ given by

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^\top, \quad e_\theta = \left(\frac{-x_2}{r}, \frac{x_1}{r}, 0\right)^\top, \quad e_z = (0,0,1)^\top.$$

In terms of (v^r, v^θ, v^z, p), the axisymmetric Navier–Stokes equations are

$$\begin{cases} \partial_t v^r + (v^r e_r + v^z e_z)\cdot\nabla v^r - \dfrac{(v^\theta)^2}{r} + \partial_r p = \left(\Delta - \dfrac{1}{r^2}\right)v^r, \\ \partial_t v^\theta + (v^r e_r + v^z e_z)\cdot\nabla v^\theta + \dfrac{v^r v^\theta}{r} = \left(\Delta - \dfrac{1}{r^2}\right)v^\theta, \\ \partial_t v^z + (v^r e_r + v^z e_z)\cdot\nabla v^z + \partial_z p = \Delta v^z, \\ \partial_r v^r + \dfrac{v^r}{r} + \partial_z v^z = 0. \end{cases} \tag{1-1}$$

It is well known that finite energy smooth solutions of the Navier–Stokes equations satisfy the following energy identity

$$\|v(t,\cdot)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s,\cdot)\|_{L^2}^2\,ds = \|v_0\|_{L^2}^2, \quad \forall\, t \geq 0. \tag{1-2}$$

Set

$$\Gamma = r v^\theta.$$

One can easily check that

$$\partial_t \Gamma + (v^r e_r + v^z e_z)\cdot\nabla\Gamma = \Big(\Delta - \frac{2}{r}\partial_r\Big)\Gamma. \tag{1-3}$$

A significant consequence of (1-3) is that smooth solutions of the axisymmetric Navier–Stokes equations satisfy the following maximum principle; see, for instance, [Chae and Lee 2002; Hou and Li 2008; Chen et al. 2008; Neustupa and Pokorný 2000; 2001]:

$$\sup_t \|\Gamma(t,\cdot)\|_{L^\infty} \leq \|\Gamma_0\|_{L^\infty}. \tag{1-4}$$

We emphasize that $\|\Gamma(t,\cdot)\|_{L^\infty}$ is a dimensionless quantity with respect to the natural scaling of the Navier–Stokes equations. From this point of view, the axisymmetric Navier–Stokes equations can be seen as partially critical, while the general Navier–Stokes equations are known to be supercritical; see [Tao 2007].

Now let us introduce the function class where v^θ lives. It is defined in an integral way which is usually called the form boundedness condition (FBC), which is similar to a condition that a certain Hardy-type inequality holds.

Definition 1.1. We say that the angular velocity $v^\theta(t,r,z)$ is in a (δ_*, C_*)-critical class if

$$\int \frac{|v^\theta|}{r}|f|^2\,dx \leq C_* \int |\partial_r f|^2\,dx + C_0\int_{r\geq r_0} |f|^2\,dx, \tag{1-5}$$

$$\int |v^\theta|^2|f|^2\,dx \leq \delta_* \int |\partial_r f|^2\,dx + C_0\int_{r\geq r_0} |f|^2\,dx, \tag{1-6}$$

hold for some $r_0 > 0$, some $C_0 > 0$ and for all $t \geq 0$ and all axisymmetric scalar and vector functions $f \in H^1$.

Clearly, under the natural scaling of the Navier–Stokes equations, namely,

$$v^\lambda(t,x) = \lambda v(\lambda^2 t, \lambda x), \qquad p^\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x),$$

the above definition of FBC is invariant: $(v^\lambda)^\theta$ satisfies (1-5)–(1-6) if v^θ also does.

We now state the first result of this article.

Theorem 1.2. *For any $C_* > 1$, there exists a constant $\delta_* > 0$ depending on C_* such that the following conclusion holds for all local strong solutions to the axially symmetric Navier–Stokes equations with initial data $\|v_0\|_{H^{1/2}} < \infty$ and $\|\Gamma_0\|_{L^\infty} < \infty$. If the angular velocity field v^θ is in the (δ_*, C_*)-critical class, i.e., v^θ satisfies the critical form boundedness condition in (1-5)–(1-6), then v is regular globally in time.*

An important corollary of Theorem 1.2 is this:

Corollary 1.3. *Let $\delta_0 \in \left(0, \frac{1}{2}\right)$ and $C_1 > 1$. Let v be the local strong solution of the axially symmetric Navier–Stokes equations with initial data $v_0 \in H^{1/2}$ and $\|\Gamma_0\|_{L^\infty} < \infty$. If*

$$\sup_{0\le t<T} |\Gamma(t, r, z)| \le C_1 |\ln r|^{-2}, \quad r \le \delta_0, \tag{1-7}$$

then v is regular globally in time.

We emphasize that C_* in Theorem 1.2 and C_1 in Corollary 1.3 are independent of neither the profile nor the norm of the given initial data. The proof of this corollary will be given at the end of Section 2. The point is that if (1-7) is satisfied, then the FBC of (1-5)–(1-6) is true. Then one can apply Theorem 1.2 to get the desired conclusion.

Our work is inspired by a recent very interesting work by Chen, Fang and Zhang [2015] where, among other things, the authors proved that v is regular if Γ is Hölder continuous. Let

$$\Omega = \frac{\omega^\theta}{r}, \qquad J = -\frac{\partial_z v^\theta}{r}.$$

We emphasize that J was introduced in [Chen et al. 2015], while Ω appeared much earlier and can be at least tracked back to the book of Majda and Bertozzi [2002]. Both of the two new variables are of great importance in our work. Following [Majda and Bertozzi 2002; Hou and Li 2008; Chen et al. 2015], we also study the equations for J and Ω:

$$\begin{cases} \partial_t J + (b\cdot\nabla)J = \left(\Delta + \frac{2}{r}\partial_r\right)J + (\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r}, \\ \partial_t\Omega + (b\cdot\nabla)\Omega = \left(\Delta + \frac{2}{r}\partial_r\right)\Omega - 2\frac{v^\theta}{r}J. \end{cases} \tag{1-8}$$

Here ω^θ is the angular component of the vorticity $\omega = \nabla\times v$, which reads

$$\omega(t, x) = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z,$$

with

$$\omega^r = -\partial_z v^\theta, \quad \omega^\theta = \partial_z v^r - \partial_r v^z, \quad \omega^z = \partial_r v_\theta + \frac{v^\theta}{r}.$$

Our new observation is that the axisymmetric Navier–Stokes equations exhibit certain critical nature when being formulated in terms of a new set of unknowns,

J and Ω. Our second observation is that, with the FBC assumptions (1-5)–(1-6), the stretching term $(\omega^r \partial_r + \omega^z \partial_z) v^r / r$ in the equation for J could be arbitrarily small, by using the relation of v^r, v^z and Ω in Lemma 2.1 (which was originally proved by Hou, Lei and Li [Hou et al. 2008] in the periodic case and later on extended to the general case by Lei [2015]. Alternatively, one may also use the magic formula given by Miao and Zheng [2013] to prove it). Then we can derive a closed a priori estimate for J and Ω using the first two observations and the structure of the stretching term in the equation for Ω.

Our second goal is to prove that the smallness of $\sup_{t\geq 0} \|\Gamma(t,\cdot)\|_{L^\infty(r\leq r_0)}$ or $\|\Gamma_0\|_{L^\infty}$ implies the global regularity of the solutions. Recently, Chen, Fang and Zhang [2015] proved that, among many other interesting results, if $\Gamma(t,\cdot,z)$ is Hölder continuous in the r variable, then the solution of the axisymmetric Navier–Stokes equations is smooth. Both of the results depend on given initial data. More precisely, the smallness of $\sup_{t\geq 0} \|\Gamma(t,\cdot)\|_{L^\infty(r\leq r_0)}$ or $\|\Gamma_0\|_{L^\infty}$ in our Theorem 1.4 depends on other dimensionless norms of the initial data. From this point of view, our result improves the one in [Chen et al. 2015].

We define
$$V = \frac{v^\theta}{\sqrt{r}}.$$
Here is the second main result:

Theorem 1.4. *Let $r_0 > 0$. Suppose that $v_0 \in H^{1/2}$ such that $\Omega_0 \in L^2$, $V_0^2 \in L^2$ and $\Gamma_0 \in L^2 \cap L^\infty$. Denote*
$$\big(\|\Omega_0\|_{L^2} + \|V_0^2\|_{L^2}\big)\|\Gamma_0\|_{L^2} = M_0$$
and
$$\big(\|V_0^2\|_{L^2} + \|\Omega_0\|_{L^2} + r_0^{-2}\|v_0\|_{L^2}\|\Gamma_0\|_{L^\infty}^{3/2}\big)\|\Gamma_0\|_{L^2} = M_1.$$
There exists an absolute positive (small) constant $\delta > 0$ such that if either
$$\|\Gamma_0\|_{L^\infty} \leq \delta M_0^{-1},$$
or
$$\sup_{t\geq 0} \|\Gamma(t,\cdot)\|_{L^\infty(r\leq r_0)} \leq \delta M_1^{-1},$$
then the axially symmetric Navier–Stokes equations are globally well-posed.

The proof of Theorem 1.4 is based on a new formulation of the axisymmetric Navier–Stokes equations (1-1) in terms of V and $\Omega = \omega^\theta / r$, and also on the estimate of v^r / r in terms of Ω and its derivative (see Lemma 2.1).

Now let us recall some highlights on the study of the axisymmetric Navier–Stokes equations. It has been known since the late 1960s (see [Ladyzhenskaya 1968; Ukhovskii and Iudovich 1968]) that if the swirl $v_\theta = 0$, then finite energy solutions to (1-1) are smooth for all time. See also [Leonardi et al. 1999], by

Leonardi, Málek, Nečas and Pokorný. In the presence of swirl, it is not known in general if finite energy solutions blow up or not in finite time. Hou and Li [2008] constructed a family of large solutions based on some deep insights on a one-dimensional model. See also some extended results in [Hou et al. 2008] by Hou, Lei and Li. We also mention various a priori estimates of smooth solutions by Chae and Lee [2002] and Burke Loftus and Zhang [2010]. To the best of our knowledge, the best a priori bound of the velocity field is given in [Lei et al. 2013]:

$$|v(t,x)| \le C_* r^{-2} |\ln r|^{1/2}.$$

In [Chen et al. 2008], Chen, Strain, Tsai and Yau obtained a lower bound for the possible blow up rate of singularities: if

$$|v(t,x)| \le \frac{C_*}{r},$$

then v is regular. This seems to be the first time that people have been able to exclude possible singularities in the presence of assumptions on $|x|^{-1}$-type nonsmallness quantities. Soon afterward, Chen, Strain, Yau and Tsai [Chen et al. 2009] and Koch, Nadirashvili, Seregin and Šverák [Koch et al. 2009] extended the result of [Chen et al. 2008] and in particular, excluded the possibility of type I singularities of v. See also a local version by Seregin and Šverák [2009] and various extensions by Pan [2016]. We also mention that Lei and Zhang [2011b] excluded the possibility of singularities under

$$v^r e_r + v^z e_z \in L^\infty([0,T], \mathrm{BMO}^{-1})$$

based on an observation in [Lei and Zhang 2011a]. This solves the regularity problem of $L^\infty\big([0,T], \mathrm{BMO}^{-1}\big)$ solutions of Navier–Stokes equations in the axisymmetric case. Moreover, it extends the result of [Chen et al. 2008] and [Koch et al. 2009] since the assumptions on the axial component of velocity $|v^z| \le C_* r^{-1}$ itself imply $v^r e_r + v^z e_z \in L^\infty([0,T], \mathrm{BMO}^{-1})$ (see [Lei et al. 2013] for details).

Let us also mention that Neustupa and Pokorný [2000] proved that the regularity of one component (either v^r or v^θ) implies regularity of the other components of the solution. The work of Jiu and Xin [2003] also proves regularity under an assumption of sufficiently small zero-dimension scaled norms. See more refined results in [Neustupa and Pokorný 2001] and the work of Ping Zhang and Ting Zhang [Zhang and Zhang 2014]. Chae and Lee [2002] also proved regularity results assuming finiteness of another certain zero-dimensional integral. Tian and Xin [1998] constructed a family of singular axially symmetric solutions with singular initial data.

The remainder of the paper is simply organized as follows. In Section 2 we recall two basic lemmas and prove Corollary 1.3 by assuming the validity of Theorem 1.2. In Section 3 we prove Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.4.

Remark. This paper was posted on the arXiv in May 2015. In August 2015, Dongyi Wei [2016] improved one of our regularity conditions by a factor of $\sqrt{|\ln r|}$.

2. Notations and lemmas

For abbreviation, we denote

$$b(t,x) = v^r e_r + v^z e_z.$$

The last equation in (1-1) shows that b is divergence-free. The Laplacian operator Δ and the gradient operator ∇ in the cylindrical coordinate are

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2, \qquad \nabla = e_r\partial_r + \frac{e_\theta}{r}\partial_\theta + e_z\partial_z.$$

For the scalar axisymmetric function $f(t,r,z)$, we often use the commutation property:

$$\nabla\partial_r f(r,z) = \partial_r \nabla f(r,z).$$

Throughout the proof, we will denote

$$\|f\|_{L^2}^2 = \int |f|^2 r\,dr\,dz, \qquad dx = r\,dr\,dz.$$

The estimate in Lemma 2.1 will be used often. It was originally proved in [Hou et al. 2008] in the periodic case and then extended to the general case in [Lei 2015] (see (4.5)–(4.6) there, noting the relation $v^r = -\partial_z\psi^\theta$). Alternatively, one may also use the magic formula given by Miao and Zheng [2013] to prove it.

Lemma 2.1. *Let v^r be the radial component of the velocity field and $\Omega = \omega^\theta/r$. Then there exists an absolute positive constant $K_0 > 0$ such that*

$$\left\|\nabla\frac{v^r}{r}\right\|_{L^2} \le K_0\|\Omega\|_{L^2}, \qquad \left\|\nabla^2\frac{v^r}{r}\right\|_{L^2} \le K_0\|\partial_z\Omega\|_{L^2}.$$

Lemma 2.2 gives the uniform decay estimate for the angular component of vorticity in the r direction for large r. We point out that a weaker estimate for ω^θ has appeared in [Chae and Lee 2002]. Even though we don't need to use the estimate for ω^r and ω^z in this paper, we will include them below for possible future use.

Lemma 2.2. *Suppose that $v_0 \in L^2$ is an axially symmetric divergence-free vector and $(r\omega_0^r, r^2\omega_0^\theta, r\omega_0^z) \in L^2$. Then the smooth solution of the Navier–Stokes equation with initial data v_0 satisfies the following a priori estimates*:

$$\text{(2-1)} \quad \sup_{0\le t<T}\big(\|r\omega^r(t,\cdot)\|_{L^2}^2, \|r\omega^z(t,\cdot)\|_{L^2}^2\big) + \int_0^T \big(\|\nabla[r\omega^r(t,\cdot)]\|_{L^2}^2, \|\nabla[r\omega^z(t,\cdot)]\|_{L^2}^2\big)\,dt \le \|r\omega_0^r\|_{L^2}^2 + \|r\omega_0^z\|_{L^2}^2 + 4(\|\Gamma_0\|_{L^\infty}^2+1)\|v_0\|_{L^2}^2,$$

and

$$\sup_{0\le t<T}\|r^2\omega^\theta(t,\cdot)\|_{L^2}^2+\int_0^T\|\nabla(r^2\omega^\theta)\|_{L^2}^2\,dt \le C_0\Big(\|r^2\omega_0^\theta\|_{L^2}^2+\big(\|v_0\|_{L^2}^4+\|\Gamma_0\|_{L^3}^2\big)\|v_0\|_{L^2}^2\Big)\exp\Big\{\frac{T}{\|\Gamma_0\|_{L^3}^2+\|v_0\|_{L^2}^4}\Big\}, \tag{2-2}$$

where C_0 is a generic positive constant.

Proof. First of all, let us recall that

$$\begin{cases}\partial_t\omega^r+b\cdot\nabla\omega^r-\partial_r v^r\omega^r=\Big(\Delta-\dfrac{1}{r^2}\Big)\omega^r+\partial_z v^r\omega^z,\\ \partial_t\omega^\theta+b\cdot\nabla\omega^\theta-\dfrac{v^r}{r}\omega^\theta=\Big(\Delta-\dfrac{1}{r^2}\Big)\omega^\theta+\partial_z\dfrac{(v^\theta)^2}{r},\\ \partial_t\omega^z+b\cdot\nabla\omega^z-\partial_z v^z\omega^z=\Delta\omega^z+\partial_r v^z\omega^r.\end{cases} \tag{2-3}$$

Let us first prove (2-1). Taking the L^2 inner product of the first equation of (2-3) with $r^2\omega^r$, and of the third equation with $r^2\omega^z$, we have

$$\frac12\frac{d}{dt}\int(\omega^r)^2r^2\,dx-\int r^2\omega^r\Big(\Delta-\frac{1}{r^2}\Big)\omega^r\,dx = -\int r^2\omega^r b\cdot\nabla\omega^r\,dx+\int r^2\partial_r v^r(\omega^r)^2\,dx+\int r^2\omega^r\partial_z v^r\omega^z\,dx$$

and

$$\frac12\frac{d}{dt}\int(\omega^z)^2r^2\,dx-\int r^2\omega^z\Delta\omega^z\,dx = -\int r^2\omega^z b\cdot\nabla\omega^z\,dx+\int r^2\partial_z v^z(\omega^z)^2\,dx+\int r^2\omega^z\partial_r v^z\omega^r\,dx.$$

Using integration by parts, we have

$$-\int r^2\omega^r\Big(\Delta-\frac{1}{r^2}\Big)\omega^r\,dx=\int|\nabla(r\omega^r)|^2\,dx$$

and

$$-\int r^2\omega^z\Delta\omega^z\,dx=\int|\nabla(r\omega^z)|^2\,dx-2\int|\omega^z|^2\,dx.$$

Using integration by parts and the incompressibility constraint, one has

$$\begin{aligned}-\int r^2\omega^r b\cdot\nabla\omega^r\,dx-\int r^2\omega^z b\cdot\nabla\omega^z\,dx+\int r^2\partial_r v^r(\omega^r)^2dx+\int r^2\partial_z v^z(\omega^z)^2dx\\ =\int(rv^r+r^2\partial_r v^r)(\omega^r)^2+(rv^r+r^2\partial_z v^z)(\omega^z)^2\,dx\\ =-\int\big[\partial_z v^z(r\omega^r)^2+\partial_r v^r(r\omega^z)^2\big]\,dx.\end{aligned}$$

Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\int\big[(\omega^r)^2+(\omega^z)^2\big]r^2\,dx+\int\big(|\nabla(r\omega^r)|^2+|\nabla(r\omega^z)|^2\big)\,dx$$
$$=\int|\omega^z|^2\,dx-\int\big[\partial_z v^z(r\omega^r)^2+\partial_r v^r(r\omega^z)^2\big]\,dx+\int\big(r^2\omega^r\partial_z v^r\omega^z+r^2\omega^z\partial_r v^z\omega^r\big)\,dx$$
$$\le 2\int|\omega^z|^2\,dx+\|\nabla b\|_{L^2}\big(\|r\omega^r\|_{L^4}^2+\|r\omega^z\|_{L^4}^2\big).$$

Note that by Gagliardo–Nirenberg's inequality and the maximum principle $\|\Gamma\|_{L^\infty}\le\|\Gamma_0\|_{L^\infty}$, one has

$$\begin{aligned}\|r\omega^r\|_{L^4}^2+\|r\omega^z\|_{L^4}^2&=\|\nabla\Gamma\|_{L^4}^2\\&=\Big(\int-\Gamma\nabla\cdot(\nabla\Gamma|\nabla\Gamma|^2)\,dx\Big)^{1/2}\le 3\|\Gamma\|_{L^\infty}\|\Delta\Gamma\|_{L^2}\\&\le 3\|\Gamma_0\|_{L^\infty}\big(\|\partial_r(r\omega^z)\|_{L^2}+\|\partial_z(r\omega^r)\|_{L^2}+\|\omega^z\|_{L^2}\big).\end{aligned}$$

Hence, by Hölder's inequality, we have

$$\frac{d}{dt}\int\big[(\omega^r)^2+(\omega^z)^2\big]r^2\,dx+\int\big(|\nabla(r\omega^r)|^2+|\nabla(r\omega^z)|^2\big)\,dx\le 4\big(\|\Gamma_0\|_{L^\infty}^2+1\big)\|\nabla b\|_{L^2}^2.$$

Integrating the above differential inequality with respect to time and recalling the basic energy estimate, one gets (2-1).

Next, let us prove (2-2). Let us first write the second equation of (2-3) as:

$$\partial_t(r^2\omega^\theta)+b\cdot\nabla(r^2\omega^\theta)-3rv^r\omega^\theta=\frac{\partial_z\Gamma^2}{r}+\Delta(r^2\omega^\theta)-\frac{4}{r}\partial_r(r^2\omega^\theta)+3\omega^\theta.$$

The standard energy estimate gives that

$$\frac{1}{2}\frac{d}{dt}\|r^2\omega^\theta\|_{L^2}^2+\|\nabla(r^2\omega^\theta)\|_{L^2}^2=3\int rv^r\omega^\theta r^2\omega^\theta\,dx+\int\partial_z\Gamma^2 r\omega^\theta\,dx+3\int\omega^\theta r^2\omega^\theta\,dx.$$

It is easy to estimate that

$$\begin{aligned}\int\partial_z\Gamma^2 r\omega^\theta\,dx&\le 2\|\Gamma\|_{L^3}\,\|\nabla v^\theta\|_{L^2}\,\|r^2\omega^\theta\|_{L^6}\\&\le 4\|\Gamma_0\|_{L^3}^2\,\|\nabla v^\theta\|_{L^2}^2+\tfrac{1}{4}\|\nabla(r^2\omega^\theta)\|_{L^2}^2.\end{aligned}$$

Next, one also has

$$\begin{aligned}\int\omega^\theta r^2\omega^\theta\,dx&\le\|\omega^\theta\|_{L^2(r\le R(t))}^2R^2(t)+\|r^2\omega^\theta\|_{L^2(r>R(t))}^2R^{-2}(t)\\&\le\|\omega^\theta\|_{L^2}^2R^2(t)+\|r^2\omega^\theta\|_{L^2}^2R^{-2}(t).\end{aligned}$$

Finally, we estimate that

$$\begin{aligned}\int r v^r \omega^\theta r^2 \omega^\theta \, dx &\le \|v^r\|_{L^2} \, \|(r^2\omega^\theta)^{3/2}\|_{L^4} \, \|(\omega^\theta)^{1/2}\|_{L^4} \\ &\le \|v_0\|_{L^2}^4 \, \|\omega^\theta\|_{L^2}^2 + \tfrac{1}{4}\|\nabla(r^2\omega^\theta)\|_{L^2}^2.\end{aligned}$$

By taking $R(t) = \|\Gamma_0\|_{L^3} + \|u_0\|_{L^2}^2$, we arrive at

$$\begin{aligned}&\frac{d}{dt}\|r^2\omega^\theta\|_{L^2}^2 + \|\nabla(r^2\omega^\theta)\|_{L^2}^2 \\ &\qquad \lesssim \big(\|\Gamma_0\|_{L^3}^2 + \|v_0\|_{L^2}^4\big)\|\nabla v\|_{L^2}^2 + \big(\|\Gamma_0\|_{L^3}^2 + \|u_0\|_{L^2}^4\big)^{-1}\|r^2\omega^\theta\|_{L^2}^2.\end{aligned}$$

Clearly, (2-2) follows by the basic energy estimate and applying Gronwall's inequality to the above differential inequality. $\square$

Finally, let us prove Corollary 1.3 by using Theorem 1.2.

Proof. It suffices to check the validity of FBC in (1-5)–(1-6) under the assumptions in Corollary 1.3. Let $\delta_0 \in \big(0, \frac{1}{2}\big)$ and $C_1 > 1$ be arbitrarily large. Noting $\|\Gamma_0\|_{L^\infty} < \infty$ and using the maximum principle, we have

$$\|\Gamma(t,\cdot)\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty}.$$

Take a smooth cut-off function of r such that

$$\phi \equiv 1 \text{ if } 0 \le r \le 1, \quad \phi \equiv 0 \text{ if } r \ge 2.$$

For all $\delta < \delta_0/2$, using (1-7), one has

$$\int \frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \le \int \frac{C_1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz.$$

Using integration by parts, one has

$$\begin{aligned}&\int \frac{1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \\ &\qquad = \int \Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 d|\ln r|^{-1}\,dz = \int |\ln r|^{-1}\phi\Big(\frac{r}{\delta}\Big)f\,\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\,dr\,dz \\ &\qquad \le \frac{1}{2}\int \frac{1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz + \frac{1}{2}\int \Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz.\end{aligned}$$

Hence, we have

$$\int \frac{1}{r^2|\ln r|^2}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \le \int \Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz,$$

which further gives that

$$\int \frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz \le C_1\int \Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz.$$

On the other hand, it is easy to see that

$$\int \frac{|v^\theta|}{r}\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz \le \|\Gamma_0\|_{L^\infty}\delta^{-2}\int\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz.$$

Consequently, we have

$$\begin{aligned}(2\text{-}4)\quad &\int \frac{|v^\theta|}{r}|f|^2 r\,dr\,dz\\ &\le 2C_1\int\Big|\partial_r\Big[\phi\Big(\frac{r}{\delta}\Big)f\Big]\Big|^2 r\,dr\,dz+2\|\Gamma_0\|_{L^\infty}\delta^{-2}\int\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz\\ &\le 4C_1\int|\partial_r f|^2 r\,dr\,dz+C\delta^{-2}\int_{r\ge\delta}|f|^2 r\,dr\,dz.\end{aligned}$$

Here and in the next inequality we use C to denote some generic positive constant whose value may change from line to line and which may depend on $\|\Gamma_0\|_{L^\infty}$ and C_1.

Next, using (2-4), we have

$$\begin{aligned}(2\text{-}5)\quad &\int |v^\theta|^2|f|^2 r\,dr\,dz\\ &\le 2\int|rv^\theta|\frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz+2\int|rv^\theta|^2r^{-2}\Big|\Big[1-\phi\Big(\frac{r}{\delta}\Big)\Big]f\Big|^2 r\,dr\,dz\\ &\le 2C_1|\ln\delta|^{-2}\int\frac{|v^\theta|}{r}\Big|\phi\Big(\frac{r}{\delta}\Big)f\Big|^2 r\,dr\,dz+2\|\Gamma_0\|^2_{L^\infty}\delta^{-2}\int_{r\ge\delta}|f|^2 r\,dr\,dz\\ &\le C\delta^{-2}\int_{r\ge\delta}|f|^2 r\,dr\,dz+8C_1^2|\ln\delta|^{-2}\int|\partial_r f|^2 r\,dr\,dz.\end{aligned}$$

Hence, one may choose δ small enough so that $16C_1^2|\ln\delta|^{-2}\le\delta_*$ and choose $C_*=4C_1$. Then it is clear from (2-4) and (2-5) that the assumptions in equations (1-5)–(1-6) are satisfied. Using Theorem 1.2, one concludes that v is smooth for all $t>0$. □

3. Criticality of axisymmetric Navier–Stokes equations

Proof of Theorem 1.2. First of all, for initial data $v_0\in H^{1/2}$, by the classical results of Leray [1934] and Fujita and Kato [1964], there exists a unique local strong solution v to the Navier–Stokes equations (1-1). Moreover, $v(t,\cdot)\in H^s$ for any $s\ge 0$, at least on a short time interval $[\epsilon, 2\epsilon]$. In particular, $\nabla\omega(t,\cdot)\in L^2$, at least on a short time interval $[\epsilon, 2\epsilon]$. A consequence is that $\nabla\omega^r$, $\nabla\omega^\theta$, $\nabla\omega^z$, ω^r/r, ω^θ/r are all L^2-functions. In particular, recalling that

$$J=\frac{\omega^r}{r}\quad\text{and}\quad\Omega=\frac{\omega^\theta}{r},$$

one has $J(t,\cdot) \in L^2$ and $\Omega(t,\cdot) \in L^2$ for $t \in [\epsilon, 2\epsilon]$. Inductively, one also has $J(t,\cdot) \in H^2$ and $\Omega(t,\cdot) \in H^2$. Without loss of generality, we may assume that

$$J_0 \in H^2 \quad \text{and} \quad \Omega_0 \in H^2.$$

Otherwise we may start from $t = \epsilon$. As long as the solution is still smooth, one has

$$\|J(t,\cdot)\|_{L^2} + \|\Omega(t,\cdot)\|_{L^2} < \infty, \qquad \|\nabla J(t,\cdot)\|_{L^2}^2 + \|\nabla \Omega(t,\cdot)\|_{L^2}^2 < \infty$$

and

$$\int_{-\infty}^{\infty} \big(|J(t,0,z)|^2 + |\Omega(t,0,z)|^2\big) dz \lesssim \|J(t,\cdot)\|_{H^2}^2 + \|\Omega(t,\cdot)\|_{H^2}^2 < \infty.$$

So all calculations below are legal as long as the solution is still smooth. Our task is to derive a certain sufficiently strong a priori estimate.

By applying the standard energy estimate to the first equation in (1-8), we have

$$\frac{1}{2}\frac{d}{dt}\|J\|_{L^2}^2 = -\int J(b\cdot\nabla)J r\,dr\,dz + \int J\Big(\Delta + \frac{2}{r}\partial_r\Big)J + \int J(\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r} r\,dr\,dz.$$

Using the incompressibility constraint, one has

$$-\int J(b\cdot\nabla)J r\,dr\,dz = \frac{1}{2}\int J^2 \nabla\cdot b r\,dr\,dz = 0.$$

On the other hand, by direct calculations, one has

$$\int J\Big(\Delta + \frac{2}{r}\partial_r\Big)J = -\|\nabla J\|_{L^2}^2 - \int |J(t,0,z)|^2\,dz.$$

Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\|J\|_{L^2}^2 + \|\nabla J\|_{L^2}^2 + \int_{-\infty}^{\infty} |J(t,0,z)|^2\,dz = \int J(\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r} r\,dr\,dz. \tag{3-1}$$

Similarly, by applying the energy estimate to the second equation in (1-8), one obtains that

$$\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 + \int_{-\infty}^{\infty} |\Omega(t,0,z)|^2\,dz = -2\int \frac{v^\theta}{r} J\Omega r\,dr\,dz. \tag{3-2}$$

In the remaining part of the proof of Theorem 1.2, we will use C to denote a generic positive constant whose value may change from line to line and which may depend on $\|\Gamma_0\|_{L^\infty}$, C_0, C_* and r_0. Using $\|\Gamma\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty}$ and the form boundedness condition in (1-5), one has

$$\begin{aligned}(3\text{-}3)\quad &\left|\int \frac{v^\theta}{r} J\Omega r\,dr\,dz\right| \\ &\le \frac{1}{4C_*}\int\left|\frac{v^\theta}{r}\right|\Omega^2 r\,dr\,dz + C_*\int\left|\frac{v^\theta}{r}\right| J^2 r\,dr\,dz \\ &\le \frac{1}{4}\int|\partial_r\Omega|^2 r\,dr\,dz + C_*^2\int|\partial_r J|^2 r\,dr\,dz + C\int_{r\ge r_0}\big(|J|^2+|\Omega|^2\big) r\,dr\,dz.\end{aligned}$$

Inserting (3-3) into (3-2), one has

$$(3\text{-}4)\qquad \frac{d}{dt}\|\Omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 + 2\int_{-\infty}^{\infty}|\Omega(t,0,z)|^2\,dz \le 2C_*^2\|\nabla J\|_{L^2}^2 + C\|\omega\|_{L^2}^2.$$

Next, we estimate that

$$\begin{aligned}\left|\int J(\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r}\,dx\right| &= \left|\int[\nabla\times(v^\theta e_\theta)]\cdot\Big(J\nabla\frac{v^r}{r}\Big)\,dx\right| \\ &\le \|\nabla J\|_{L^2}\left\|v^\theta\nabla\frac{v^r}{r}\right\|_{L^2}.\end{aligned}$$

Again, using the form boundedness condition in (1-6), one has

$$\left\|v^\theta\nabla\frac{v^r}{r}\right\|_{L^2}^2 \le \delta_*\left\|\partial_r\nabla\frac{v^r}{r}\right\|_{L^2}^2 + C_0\int_{r\ge r_0}\left|\nabla\frac{v^r}{r}\right|^2 dr\,dr\,dz.$$

Using Lemma 2.1 and the identity

$$\nabla\frac{v^r}{r} = \frac{\nabla v^r}{r} - e_r\frac{v^r}{r^2} = \frac{\nabla v^r}{r} + e_r\frac{\partial_r v^r + \partial_z v^z}{r},$$

we have

$$\begin{aligned}(3\text{-}5)\quad &\left|\int J(\omega^r\partial_r + \omega^z\partial_z)\frac{v^r}{r} r\,dr\,dz\right| \\ &\le \tfrac{1}{2}\|\nabla J\|_{L^2}^2 + \frac{\delta_*}{2}\left\|\partial_r\nabla\frac{v^r}{r}\right\|_{L^2}^2 + \frac{C_0}{2}\int_{r\ge r_0}\left|\nabla\frac{v^r}{r}\right|^2 dr\,dr\,dz \\ &\le \tfrac{1}{2}\|\nabla J\|_{L^2}^2 + \frac{K_0\delta_*}{2}\|\partial_z\Omega\|_{L^2}^2 + C\int|\nabla v|^2\,dr\,dr\,dz.\end{aligned}$$

Inserting (3-5) into (3-1), we have

$$\begin{aligned}(3\text{-}6)\quad &\frac{d}{dt}\|J\|_{L^2}^2 + \|\nabla J\|_{L^2}^2 + 2\int_{-\infty}^{\infty}|J(t,0,z)|^2\,dz \\ &\le K_0\delta_*\|\partial_z\Omega\|_{L^2}^2 + C\int|\nabla v|^2\,dr\,dr\,dz.\end{aligned}$$

Multiplying (3-6) by $3C_*^2$ and then adding it to (3-4), we have

$$\frac{d}{dt}\big(3C_*^2\|J\|_{L^2}^2+\|\Omega\|_{L^2}^2\big)+\big(C_*^2\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\big) + \int_{-\infty}^{\infty}\big(6C_*^2|J(t,0,z)|^2+2|\Omega(t,0,z)|^2\big)\,dz \le 3C_*^2K_0\delta_*\|\nabla\Omega\|_{L^2}^2+C\int|\nabla v|^2r\,drdz.$$

Integrating the above inequality with respect to time, we have

$$3C_*^2\|J(t,\cdot)\|_{L^2}^2+\|\Omega(t,\cdot)\|_{L^2}^2+\int_0^t\big(C_*^2\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\big)\,ds + \int_0^t\int_{-\infty}^{\infty}\big(6C_*^2|J(t,0,z)|^2+2|\Omega(t,0,z)|^2\big)\,dzds \le CC_*^2\big(\|J_0\|_{L^2}^2+\|\Omega_0\|_{L^2}^2\big)+3C_*^2K_0\delta_*\int_0^t\|\nabla\Omega\|_{L^2}^2\,ds+C\|v_0\|_{L^2}.$$

Here we used the basic energy identity (1-2). Recall that K_0 is an absolute positive constant determined in Lemma 2.1. Hence, we may take δ_* so that $3C_*^2K_0\delta_*<\frac{1}{2}$. Consequently, we have

$$\sup_{0\le t<T}\big(\|J(t)\|_{L^2}^2+\|\Omega(t)\|_{L^2}^2\big)+\int_0^T\big(\|\nabla J\|_{L^2}^2+\|\nabla\Omega\|_{L^2}^2\big)dt + \int_0^T\int_{-\infty}^{\infty}\big(|J(t,0,z)|^2+|\Omega(t,0,z)|^2\big)dzdt<\infty \tag{3-7}$$

for all $T<\infty$.

Clearly, the a priori estimate (3-7) and Sobolev imbedding theorem imply that

$$\sup\nolimits_{0\le t<T}\|b(t,\cdot)\|_{L^p(r\le 1)}<\infty,\qquad 3\le p\le 6,\quad T<\infty. \tag{3-8}$$

Remark 3.1. There are several ways to prove the regularity of v from here. For instance, the easiest way is just to use the result in [Neustupa and Pokorný 2000]. If one would like to assume further decay properties on the initial data so that the conditions in Lemma 2.2 are satisfied, then one can easily derive an $L^\infty\big([0,T],L^p\big)$ estimate for b when $r\ge 1$ and $3\le p\le 6$, by using the basic energy estimate and Sobolev imbedding theorem. This, combined with the local L^p estimate of v in (3-8), immediately gives that b is in $L^\infty\big([0,T],L^3\big)$ for any $T<\infty$. Using the Sobolev imbedding theorem once more, one has $b\in L^\infty\big([0,T],\mathrm{BMO}^{-1}\big)$. Then the result in [Lei and Zhang 2011b] implies that v is regular up to time T.

Let us give an alternative and self-contained proof. We first derive an L^4 a priori estimate for v^θ without using the result of [Lei and Zhang 2011b] and Lemma 2.2. Using the equation of v^θ in (1-1) and the standard energy estimate, one has

$$\frac{d}{dt}\|v^\theta\|_{L^4}^4 + \|\nabla(v^\theta)^2\|_{L^2}^2 + \|r^{-1}(v^\theta)^2\|_{L^2}^2 \le C\left|\int \frac{v^r(v^\theta)^4}{r} r\,dr\,dz\right| \le C\|r^{-1}v^r\|_{L^\infty}\|v^\theta\|_{L^4}^4.$$

Hence, by using Lemma 2.1 and the three-dimensional interpolation inequality $\|f\|_{L^\infty}^2 \lesssim \|\nabla f\|_{L^2}\|\nabla^2 f\|_{L^2}$, one has

$$\left\|\frac{v^r}{r}\right\|_{L^\infty} \lesssim \left\|\nabla\partial_z\frac{\psi^\theta}{r}\right\|_{L^2}^{1/2}\left\|\nabla^2\partial_z\frac{\psi^\theta}{r}\right\|_{L^2}^{1/2} = \left\|\nabla\frac{v^r}{r}\right\|_{L^2}^{1/2}\left\|\nabla^2\frac{v^r}{r}\right\|_{L^2}^{1/2} \lesssim \|\Omega\|_{L^2}^{1/2}\|\partial_z\Omega\|_{L^2}^{1/2},$$

and using (3-7), one concludes from Gronwall's inequality that

$$\|v^\theta\|_{L^4} < \infty, \quad \text{and} \quad \int_0^T \|r^{-1}(v^\theta)^2\|_{L^2}^2\,dt < \infty, \quad 0 \le t \le T.$$

Then we use the second equation of (2-3) to derive that

$$\begin{aligned}\frac{d}{dt}\|\omega^\theta\|_{L^2}^2 + 2\|\nabla\omega^\theta\|_{L^2}^2 + 2\|r^{-1}\omega^\theta\|_{L^2}^2 &= -\int \frac{v^r}{r}(\omega^\theta)^2 r\,dr\,dz + \int \omega^\theta\partial_z\frac{(v^\theta)^2}{r} r\,dr\,dz \\ &\le C\|r^{-1}v^r\|_{L^\infty}\|\omega^\theta\|_{L^2}^2 + \|\partial_z\omega^\theta\|_{L^2}^2 + \tfrac{1}{4}\|r^{-1}(v^\theta)^2\|_{L^2}^2.\end{aligned}$$

Hence, Gronwall's inequality similarly gives that

$$\omega^\theta \in L^2, \qquad T < \infty.$$

By the basic energy identity (1-2) and Sobolev imbedding, one has

$$b \in L^p, \qquad 2 \le p \le 6.$$

Hence, $v \in L_T^\infty(L_x^4)$. So the Serrin-type criterion implies v is regular up to time T. □

4. Small $\|\Gamma_0\|_{L^\infty}$ or $\|\Gamma(t,\cdot)\|_{L^\infty(r\le r_0)}$ global regularity

This section is devoted to proving Theorem 1.4.

Proof of Theorem 1.4. Recall that

$$V = \frac{v^\theta}{\sqrt{r}}, \qquad \Omega = \frac{\omega^\theta}{r}.$$

Let us first formulate the axisymmetric Navier–Stokes equations (1-1) in terms of V and Ω as follows:

$$\begin{cases}\partial_t V + b\cdot\nabla V + \dfrac{3v^r}{2r}V = \Big(\Delta + \dfrac{1}{r}\partial_r - \dfrac{3}{4r^2}\Big)V, \\ \partial_t\Omega + b\cdot\nabla\Omega = \Big(\Delta + \dfrac{2}{r}\partial_r\Big)\Omega + \dfrac{2\partial_z V^2}{r}.\end{cases} \tag{4-1}$$

By the energy estimate, one has

$$\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 \le \tfrac{1}{2}\|\partial_z\Omega\|_{L^2}^2 + \tfrac{1}{2}\big\|r^{-1}|V|^2\big\|_{L^2}^2. \tag{4-2}$$

Let us first prove the global regularity under

$$\|\Gamma_0\|_{L^\infty} \le \delta M_0^{-1}.$$

Using Lemma 2.1, one has

$$\Big\|\frac{v^r}{r}\Big\|_{L^\infty} \lesssim \|\Omega\|_{L^2}^{1/2}\,\|\partial_z\Omega\|_{L^2}^{1/2}.$$

Noting that

$$\|V\|_{L^4}^4 \lesssim \big\|r^{-1}|V|^2\big\|_{L^2}^{3/2}\,\|\Gamma\|_{L^4},$$

one can apply the L^4 energy estimate for V to get

$$\begin{aligned}\frac{d}{dt}\big\||V|^2\big\|_{L^2}^2 + \big\|\nabla|V|^2\big\|_{L^2}^2 + \big\|r^{-1}|V|^2\big\|_{L^2}^2 \\ &\lesssim \Big\|\frac{v^r}{r}\Big\|_{L^\infty}\|V\|_{L^4}^4 \lesssim \|\Omega\|_{L^2}^{1/2}\,\|\partial_z\Omega\|_{L^2}^{1/2}\,\big\|r^{-1}|V|^2\big\|_{L^2}^{3/2}\,\|\Gamma\|_{L^4} \\ &\lesssim \|\Omega\|_{L^2}^{1/2}\,\|\Gamma\|_{L^2}^{1/2}\,\|\Gamma\|_{L^\infty}^{1/2}\,\big(\|\partial_z\Omega\|_{L^2}^2 + \big\|r^{-1}|V|^2\big\|_{L^2}^2\big).\end{aligned} \tag{4-3}$$

Combining (4-2) and (4-3), we arrive at

$$\begin{aligned}\frac{d}{dt}\Big(\big\||V|^2\big\|_{L^2}^2 + \|\Omega\|_{L^2}^2\Big) + \Big(\big\|\nabla|V|^2\big\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2\Big) + \big\|r^{-1}|V|^2\big\|_{L^2}^2 \\ &\lesssim \|\Omega\|_{L^2}^{1/2}\,\|\Gamma\|_{L^2}^{1/2}\,\|\Gamma\|_{L^\infty}^{1/2}\Big(\|\partial_z\Omega\|_{L^2}^2 + \big\|r^{-1}|V|^2\big\|_{L^2}^2\Big).\end{aligned} \tag{4-4}$$

Recall that we have the following a priori estimate:

$$\|\Gamma\|_{L^2} \le \|\Gamma_0\|_{L^2}, \qquad \|\Gamma\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty}.$$

Hence, under the condition of the theorem, there exists $T > 0$ such that

$$\big\||V|^2\big\|_{L^2}^2 + \|\Omega\|_{L^2}^2 < 2\big\||V_0|^2\big\|_{L^2}^2 + 2\|\Omega_0\|_{L^2}^2, \quad \forall\, 0 \le t < T.$$

If δ is a suitably small positive constant and

$$\|\Gamma_0\|_{L^\infty} \le \delta M_0^{-1}$$

is satisfied, then we have

$$\|\Omega\|_{L^2}^{1/2}\,\|\Gamma\|_{L^2}^{1/2}\,\|\Gamma\|_{L^\infty}^{1/2} \lesssim M_0^{1/2}(\delta M_0^{-1})^{1/2} \lesssim \delta^{1/2}, \quad \forall\, 0 \le t < T.$$

Hence, by (4-4), we derive that

$$\frac{d}{dt}\Big(\big\||V|^2\big\|_{L^2}^2 + \|\Omega\|_{L^2}^2\Big) \le 0, \quad \forall\, 0 \le t < T,$$

which implies that

$$\big\| |V|^2 \big\|_{L^2}^2 + \|\Omega\|_{L^2}^2 \le \big\| |V_0|^2 \big\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2, \quad 0 \le t \le T.$$

The above argument implies, by the standard continuation method that

$$\big\| |V|^2 \big\|_{L^2}^2 + \|\Omega\|_{L^2}^2 \le \big\| |V_0|^2 \big\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2, \quad \forall\, t \ge 0.$$

Hence the proof for first part of the theorem is finished.

On the other hand, if

$$\|\Gamma(t,\cdot)\|_{L^\infty(r\le r_0)} \le \delta M_1^{-1}$$

is satisfied, then one may treat (4-3) as follows:

$$\begin{aligned}
\frac{d}{dt}\big\| |V|^2 \big\|_{L^2}^2 &+ \big\| \nabla |V|^2 \big\|_{L^2}^2 + \big\| r^{-1} |V|^2 \big\|_{L^2}^2 \\
&\lesssim \Big\| \frac{v^r}{r} \Big\|_{L^\infty} \|V\|_{L^4(r\le r_0)}^4 + \int_{r\ge r_0} \Big| \frac{v^r}{r} \frac{(v^\theta)^4}{r^2} \Big| r\, dr\, dz \\
&\lesssim \|\Omega\|_{L^2}^{1/2} \|\Gamma\|_{L^2}^{1/2} \|\Gamma\|_{L^\infty(r\le r_0)}^{1/2} \big(\|\partial_z \Omega\|_{L^2}^2 + \big\| r^{-1}|V|^2 \big\|_{L^2}^2 \big) \\
&\qquad + r_0^{-4} \Big\| \frac{v^r}{r} \Big\|_{L^2} \Big\| \frac{v^\theta}{r} \Big\|_{L^2} \|\Gamma\|_{L^\infty(r\ge r_0)}^3,
\end{aligned}$$

which, combined with (4-2), gives that

$$\begin{aligned}
\big\| |V(t,\cdot)|^2 \big\|_{L^2}^2 &+ \|\Omega(t,\cdot)\|_{L^2}^2 + \int_0^t \Big(\big\| \nabla |V|^2 \big\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 \Big)\, ds \\
&\lesssim \big\| |V_0|^2 \big\|_{L^2}^2 + \|\Omega_0\|_{L^2}^2 + r_0^{-4} \|v_0\|_{L^2}^2 \|\Gamma_0\|_{L^\infty}^3 \\
&\quad + \big(\|\Gamma\|_{L^\infty(r\le r_0)} \|\Gamma_0\|_{L^2} \sup_{0\le s\le t} \|\Omega(s,\cdot)\|_{L^2} \big)^{1/2} \big(\|\partial_z \Omega\|_{L^2}^2 + \big\| r^{-1}|V|^2 \big\|_{L^2}^2 \big).
\end{aligned}$$

Here $r_0 > 0$ is arbitrary. Then similar continuation arguments as in the proof used in the first part imply that if δ is a suitable small absolute positive constant, then the solution v is regular. Here M_1 is given in the statement of Theorem 1.4. This shows that the smallness of Γ locally in r implies the regularity of the solutions. $\square$

Remark. Since $\big\| |V_0|^2 \big\|_{L^2}$ and $\|\Omega_0\|_{L^2}$ have dimension $-\frac{3}{2}$, and $\|\Gamma_0\|_{L^2}$ has dimension $\frac{3}{2}$, the constant M_0 in Theorem 1.4 has dimension 0. Similarly, one can check that M_1 is also dimensionless if one assigns r_0 dimension 1.

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CONVEXITY OF THE ENTROPY OF POSITIVE SOLUTIONS TO THE HEAT EQUATION ON QUATERNIONIC CONTACT AND CR MANIFOLDS

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A proof of the monotonicity of an entropy like energy for the heat equation on a quaternionic contact and CR manifolds is given.

1. Introduction

The purpose of this note is to show the monotonicity of the entropy type energy associated to the (subelliptic) heat equation in a sub-Riemannian setting. The result is inspired by the corresponding Riemannian fact related to Perelman's entropy formula for the heat equation on a static Riemannian manifold; see [Ni 2004]. More recently a similar quantity was considered in the CR case [Chang and Wu 2010]. Our goal is to prove the convexity of the entropy of a positive solution to the (subelliptic) heat equation on a quaternionic contact manifold, henceforth abbreviated to QC, and give an alternative proof of the result in that work, more in line with the Riemannian case. We resolve directly the difficulties arising in the sub-Riemannian setting of both quaternionic contact and CR manifolds. Section 3 contains the alternative proof of the result of [Chang and Wu 2010] in the CR case while the remaining parts of the paper focus on the QC case.

To state the problem, let M be a quaternionic contact or a CR manifold of real dimensions $4n+3$ and $2n+1$, respectively, and u be a smooth *positive* solution to the (QC or CR) heat equation

$$\frac{\partial}{\partial t}u = \Delta u. \tag{1-1}$$

Hereafter, $\triangle u = \mathrm{tr}^g(\nabla^2 u)$ denotes the negative sub-Laplacian with the trace taken with respect to an orthonormal basis of the respective horizontal $4n$ or $2n$ dimensional spaces in the QC and CR cases. Associated to such a solution are the

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(Boltzmann–Nash like) entropy

$$\mathcal{N}(t) = \int_M u \ln u \,\mathrm{Vol}_\eta \tag{1-2}$$

and entropy energy functional

$$\mathcal{E}(t) = \int_M |\nabla f|^2 u \,\mathrm{Vol}_\eta, \tag{1-3}$$

where, as usual, $f = -\ln u$ and Vol_η is the naturally associated volume form on M, see (2-4) and (3-3). Exactly as in the Riemannian case, we have that the entropy is decreasing (i.e., nonincreasing) because of the formula

$$\frac{d}{dt}\mathcal{N} = -\mathcal{E}(t).$$

Our goal is the computation of the second derivative of the entropy. In order to state the result in the QC case we consider the Ricci type tensor

$$\mathcal{L}(X, X) \overset{def}{=} 2Sg(X, X) + \alpha_n T^0(X, X) + \beta_n U(X, X), \tag{1-4}$$

where X is any vector from the horizontal distribution, $\alpha_n = 2(2n+3)/(2n+1)$, $\beta_n = 4(2n-1)(n+2)/((2n+1)(n-1))$, and T^0 and U are certain invariant components of the torsion; see Section 2. The tensor (1-4) appeared earlier as a natural assumption in the QC Lichnerowicz–Obata type results; see [Ivanov and Vassilev 2015, Section 8.1] and references therein. In addition, following [Ivanov et al. 2013], we define the P-form of a fixed smooth function f on M by the following equation:

$$P_f(X) = \sum_{b=1}^{4n} \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3}\sum_{b=1}^{4n} \nabla^3 f(I_t X, e_b, I_t e_b) \\ -4nS\,df(X) + 4nT^0(X, \nabla f) - \frac{8n(n-2)}{n-1}U(X, \nabla f), \tag{1-5}$$

which in the case $n = 1$ is defined by formally dropping the last term. The P-function of f is the function $P_f(\nabla f)$. The C-operator of M is the 4th order differential operator

$$f \mapsto Cf = -\nabla^* P_f = \sum_{a=1}^{4n} (\nabla_{e_a} P_f)(e_a).$$

In many respects the C-operator plays a role similar to the Paneitz operator in CR geometry. We say that the P-function of f is nonnegative if

$$\int_M f \cdot Cf \,\mathrm{Vol}_\eta = -\int_M P_f(\nabla f)\,\mathrm{Vol}_\eta \geq 0.$$

If the above holds for any $f \in \mathcal{C}_o^\infty(M)$ we say that the C-operator is nonnegative, $\mathcal{C} \geq 0$.

We are ready to state our first result.

Proposition 1.1. *Let M be a compact QC manifold of dimension $4n+3$. If $u = e^{-f}$ is a positive solution to heat equation (1-1), then we have*

$$\frac{2n+1}{4n}\mathcal{E}'(t) = -\int_M \Big[|(\nabla^2 f)_0|^2 + \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f) + \frac{1}{16n}|\nabla f|^4 \Big] u \,\mathrm{Vol}_\eta + \frac{3}{n}\int_M P_F(\nabla F)\mathrm{Vol}_\eta,$$

where $u = F^2$ ($f = -2\ln F$) and $(\nabla^2 f)_0$ is the traceless part of horizontal Hessian of f.

Several important properties of the C-operator were found in [Ivanov et al. 2013], most notable of which is the fact that the C-operator is nonnegative for $n > 1$. In dimension seven, $n = 1$, the condition of nonnegativity of the C-operator is nontrivial. However, [Ivanov et al. 2013] showed that on a 7-dimensional compact QC-Einstein manifold with positive QC-scalar curvature the P-function of an eigenfunction of the sub-Laplacian is nonnegative. In particular, this property holds on any 3-Sasakian manifold, see [Ivanov et al. 2014a, Corollary 4.13]. Clearly, these facts together with Proposition 1.1 imply the following theorem:

Theorem 1.2. *Let M be a compact QC manifold of dimension $4n+3$ of nonnegative Ricci type tensor $\mathcal{L}(X, X) \geq 0$. In the case $n = 1$ assume, in addition, that the C-operator is nonnegative. If $u = e^{-f}$ is a positive solution of the heat equation (1-1), then the energy $\mathcal{E}(t)$ is monotone decreasing (i.e., nonincreasing).*

The proof of Proposition 1.1 follows one of L. Ni's arguments [2004] in the Riemannian case, and thus it relies on Bochner's formula. More precisely, after Ni's initial step, in order to handle the extra terms in Bochner's formula, we will follow the presentation of [Ivanov and Vassilev 2015] where this was done for the QC Lichnerowicz-type lower eigenvalue bound under positive Ricci-type tensor; see [Ivanov et al. 2013; Ivanov et al. 2014b] for the original result. In the QC case, similar to the CR case, the Bochner formula has additional hard to control terms, which include the P-function of f. In our case, since the integrals are with respect to the measure $u\,\mathrm{Vol}_\eta$, rather than Vol_η as in the Lichnerowicz-type estimate, some new estimates are needed. The key is the following proposition which can be considered as an estimates from above of the integral of the P-function of f with respect to the measure $u\,\mathrm{Vol}_\eta$ when the C-operator is nonnegative.

Proposition 1.3. *Let (M, η) be a compact QC manifold of dimension $4n+3$. If $u = e^{-f}$ is a positive solution to heat equation (1-1), then with $f = -2\ln F$ we have the identity*

$$\int_M P_f(\nabla f) u \,\mathrm{Vol}_\eta = \frac{1}{4}\int_M |\nabla f|^4 u \,\mathrm{Vol}_\eta + 4\int_M P_F(\nabla F)\,\mathrm{Vol}_\eta\,. \tag{1-6}$$

In the last section of the paper we apply the same method in the case of a strictly pseudoconvex pseudo-Hermitian manifold and prove the following proposition:

Proposition 1.4. *Let M be a compact strictly pseudoconvex pseudo-Hermitian CR manifold of dimension $2n+1$. If $u = e^{-f}$ is a positive solution to the heat equation (1-1), then we have*

$$\frac{n+1}{2n}\mathcal{E}'(t) = -\int_M \Big[|(\nabla^2 f)_0|^2 + \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f) + \frac{1}{8n}|\nabla f|^4\Big] u \,\mathrm{Vol}_\eta - \frac{6}{n}\int_M F\mathcal{C}(F)\,\mathrm{Vol}_\eta,$$

where $u = F^2$, $(\nabla^2 f)_0$ is the traceless part of horizontal Hessian of f and $\mathcal{C}$ is the CR-Paneitz operator of M.

We refer to Section 3 for the relevant notation and definitions. As a consequence of Proposition 1.4 we recover the monotonicity of the entropy energy shown previously in [Chang and Wu 2010]. We note that one of the motivations to consider the problem was the application of the CR version of the monotonicity of the entropy-like energy (see their Lemma 3.3) in obtaining (nonoptimal) estimate on the bottom of the L^2 spectrum of the CR sub-Laplacian. However, the proof of their Corollary 1.9 and Section 6 is not fully justified since their Lemma 3.3 is proved for a compact manifold. It should be noted that a proof of S.-Y. Cheng's type (even nonoptimal) estimate in a sub-Riemannian setting, such as CR or QC manifold, is an interesting problem in particular because of the lack of general comparison theorems.

We conclude by mentioning another proof of the monotonicity of the energy in the recent preprint [Ivanov and Petkov 2016], which was the result of a past collaborative work with Ivanov and Petkov. Remarkably, [Chang and Wu 2010] is also not acknowledged in [Ivanov and Petkov 2016] despite the line for line substantial overlap of their Section 3 with Chang and Wu's [2010, Lemma 3.3] proof. In this paper we give an independent direct approach to the problem.

2. Proofs of the propositions

Some preliminaries. Throughout this section M will be a QC manifold of dimension $4n+3$, [Biquard 1999], with horizontal space H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, and Biquard connection ∇ with torsion T. Below we record some of the properties needed for this paper, see also [Biquard 2000] and [Ivanov and Vassilev 2011] for a more expanded exposition.

The $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ structure on H is fixed by a positive definite symmetric tensor g and a rank-three bundle $\mathbb{Q}$ of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions and also the conditions

$$g(I_s\cdot, I_s\cdot) = g(\cdot, \cdot) \quad \text{and} \quad 2g(I_sX, Y) = d\eta_s(X, Y).$$

Associated with the Biquard connection is the vertical space V, which is complementary to H in TM. In the case $n = 1$ we shall make the usual assumption of existence of Reeb vector fields ξ_1, ξ_2, ξ_3, so that the connection is defined following D. Duchemin [2006]. The fundamental 2-forms ω_s of the fixed QC structure will be denoted by ω_s,

$$2\omega_{s|H} = d\eta_{s|H}, \quad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

In order to give some idea of the involved quantities we list a few more essential for us details. Recall that ∇ preserves the decomposition $H \oplus V$ and the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ structure on H,

$$\nabla g = 0, \quad \nabla \Gamma(\mathbb{Q}) \subset \Gamma(\mathbb{Q})$$

and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$. Furthermore, for a vertical field $\xi \in V$, the endomorphism $T_\xi \equiv T(\xi, \cdot)_{|H}$ of H belongs to the space $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$ hence $T(\xi, X, Y) = g(T_\xi X, Y)$ is a well defined tensor field. The two $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$-invariant trace-free symmetric 2-tensors

$$T^0(X, Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y) \quad \text{and} \quad U(X, Y) = g(uX, Y)$$

on H, introduced in [Ivanov et al. 2014a], satisfy

$$\begin{aligned} T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) &= 0, \\ U(X,Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) &= U(I_3X, I_3Y). \end{aligned} \tag{2-1}$$

Note that when $n = 1$, the tensor U vanishes. The tensors T^0 and U determine completely the torsion endomorphism due to the identity [Ivanov and Vassilev 2010, Proposition 2.3]

$$4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y),$$

which in view of (2-1) implies

$$\sum_{s=1}^{3} T(\xi_s, I_s X, Y) = T^0(X, Y) - 3U(X, Y). \tag{2-2}$$

The curvature of the Biquard connection is $R = [\nabla, \nabla] - \nabla_{[\,,\,]}$ with QC-Ricci tensor and *normalized* QC-scalar curvature, defined by respectively by

$$\mathrm{Ric}(X, Y) = \sum_{a=1}^{4n} g(R(e_a, X)Y, e_a), \qquad 8n(n+2)S = \sum_{a=1}^{4n} \mathrm{Ric}(e_a, e_a).$$

According to [Biquard 2000] the Ricci tensor restricted to H is a symmetric tensor. Remarkably, the torsion tensor determines the QC-Ricci tensor of the Biquard connection on M in view of the formula, [Ivanov et al. 2014a],

$$\mathrm{Ric}(X,Y) = (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + \frac{S}{4n}g(X,Y). \tag{2-3}$$

Finally, Vol_η will denote the volume form, see [Ivanov et al. 2014a, Chapter 8],

$$\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \Omega^n, \tag{2-4}$$

where $\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$ is the fundamental 4-form. We note the integration by parts formula

$$\int_M (\nabla^* \sigma)\,\mathrm{Vol}_\eta = 0, \tag{2-5}$$

where the (horizontal) divergence of a horizontal vector field $\sigma \in \Lambda^1(H)$ is given by $\nabla^*\sigma = -\,\mathrm{tr}\,|_H \nabla\sigma = -\nabla\sigma(e_a, e_a)$ for an orthonormal frame $\{e_a\}_{a=1}^{4n}$ of the horizontal space.

Proof of Proposition 1.3. We start with a formula for the change of the dependent function in the P-function of f. To this effect, with $f = f(F)$, a short calculation shows the next identity

$$\begin{aligned}\nabla^3 f(Z,X,Y) = f'\nabla^3 F(Z,X,Y) + f'''dF(Z)dF(X)dF(Y) + f''\nabla^2 F(Z,X)dF(Y)\\ + f''\nabla^2 F(Z,Y)dF(X) + f''\nabla^2 F(X,Y)dF(Z).\end{aligned}$$

Recalling definition (1-5) we obtain

$$\begin{aligned}P_f(Z) = f'P_F(Z) + f'''|\nabla F|^2 dF(Z) + 2f''^2 F(Z,\nabla F)\\ + f''(\Delta F)dF(Z) + f''\sum_{s=1}^{3} g(\nabla^2 F, \omega_s)dF(I_s Z),\end{aligned} \tag{2-6}$$

which implies the identity

$$P_f(\nabla f) = f'^2 P_F(\nabla F) + f'f'''|\nabla F|^4 + 2f'f''\nabla^2 F(\nabla F, \nabla F) + f'f''|\nabla F|^2\Delta F. \tag{2-7}$$

In our case, since we are interested in expressing the integral of $uP_f(\nabla f) = e^{-f}P_f(\nabla f)$ in terms of the integral of a P -function of some function, equation (2-7) leads to the ordinary differential equation $u(-u'/u)^2 = constant$. Therefore, we let $u = F^2$ and find

$$uP_f(\nabla f) = 4P_F(\nabla F) + 8F^{-2}|\nabla F|^4 - 8F^{-1}\nabla^2 F(\nabla F, \nabla F) - 4F^{-1}|\nabla F|^2\Delta F. \tag{2-8}$$

Now, the last three terms will be expressed back in the variable f which gives

$$uP_f(\nabla f) = 4P_F(\nabla F) + \left[-\tfrac{1}{4}|\nabla f|^4 + \tfrac{1}{2}|\nabla f|^2\Delta f + \nabla^2 f(\nabla f, \nabla f)\right]u. \tag{2-9}$$

At this point, we integrate the above identity and then apply the (integration by parts) divergence formula (2-5) in order to show

$$\int_M \nabla^2 f(\nabla f, \nabla f) u \,\mathrm{Vol}_\eta = \frac{1}{2}\int_M \big[|\nabla f|^4 - |\nabla f|^2 \Delta f\big] u \,\mathrm{Vol}_\eta,$$

which leads to (1-6). The proof of Proposition 1.3 is complete. □

Proof of Proposition 1.1. The initial steps are identical to the Riemannian case [Ni 2004] thus we skip the detailed computations and only sketch the common steps. Let $w = 2\Delta f - |\nabla f|^2$. Using the heat equation, exactly as in the Riemannian case, we have the identities

$$\partial_t f = \Delta f - |\nabla f|^2, \quad u\Delta f = u|\nabla f|^2 - \Delta u, \quad \text{and} \quad \Delta u = (|\nabla f|^2 - \Delta f)u, \tag{2-10}$$

which imply

$$\mathcal{E}'(t) = \int_M (\partial_t - \Delta)(uw) \,\mathrm{Vol}_\eta \tag{2-11}$$

and also

$$(\partial_t - \Delta)(uw) = [2g(\nabla(\Delta f), \nabla f) - \Delta|\nabla f|^2]u. \tag{2-12}$$

Next, we apply the QC Bochner formula [Ivanov et al. 2013; Ivanov et al. 2014b]

$$\begin{aligned} \tfrac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 + g(\nabla(\triangle f), \nabla f) + 2(n+2)S|\nabla f|^2 \\ + 2(n+2)T^0(\nabla f, \nabla f) + 4(n+1)U(\nabla f, \nabla f) + 4R_f(\nabla f), \end{aligned}$$

where

$$R_f(Z) = \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s Z).$$

Therefore,

$$\begin{aligned} \tfrac{1}{2}(\partial_t - \Delta)(uw) = [-|\nabla^2 f|^2 - 2(n+2)S|\nabla f|^2 - 2(n+2)T^0(\nabla f, \nabla f) \\ - 4(n+1)U(\nabla f, \nabla f) - 4R_f(\nabla f)]u. \end{aligned} \tag{2-13}$$

The next step is the computation of $\int_M R_f(\nabla f) u \,\mathrm{Vol}_\eta$ in two ways as was done in [Ivanov et al. 2013; Ivanov et al. 2014b] for the Lichnerowicz-type first eigenvalue lower bound but integrating with respect to Vol_η rather than $u\,\mathrm{Vol}_\eta$ as we need to do here. For ease of reading we will follow closely [Ivanov and Vassilev 2015, Section 8.1.1] but notice the opposite convention of the sub-Laplacian in their Section 8.1.1. First with the help of the P-function, working similarly to [Ivanov et al. 2013, Lemma 3.2], where the integration was with respect to Vol_η, we have

$$\int_M R_f(\nabla f)u\,\mathrm{Vol}_\eta = \int_M \Big[-\frac{1}{4n}P_f(\nabla f) - \frac{1}{4n}(\triangle f)^2 - S|\nabla f|^2 + \frac{n+1}{n-1}U(\nabla f,\nabla f)\Big]u\,\mathrm{Vol}_\eta + \frac{1}{4n}\int_M |\nabla f|^2(\Delta f)u\,\mathrm{Vol}_\eta, \tag{2-14}$$

with the convention that in the case $n=1$ the formula is understood by formally dropping the term involving (the vanishing) tensor U. Notice the appearance of a "new" term in the last integral in comparison to the analogous formula in [Ivanov and Vassilev 2015, Section 8.1.1, p. 310]. Indeed, taking into account the $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ invariance of $R_f(\nabla f)$ and Ricci's identities we have, cf., [Ivanov et al. 2013, Lemma 3.2],

$$R_f(X) = -\frac{1}{4n}\sum_{s=1}^{3}\sum_{a=1}^{4n}\nabla^3 f(I_sX,\, e_a,\, I_se_a) + [T^0(X,\,\nabla f) - 3U(X,\,\nabla f)]$$

hence (1-5) gives

$$uR_f(\nabla f) = \Big[-\frac{1}{4n}P_n(\nabla f) - S|\nabla f|^2 + \frac{n+1}{n-1}U(\nabla f,\nabla f)\Big]u + \frac{1}{4n}\sum_{a=1}^{4n}\nabla^3 f(\nabla f, e_a, e_a)u.$$

An integration by parts shows the validity of (2-14).

On the other hand, we have

$$\int_M R_f(\nabla f)u\,\mathrm{Vol}_\eta = -\int_M \Big[\frac{1}{4n}\sum_{s=1}^{3}g(\nabla^2 f,\omega_s)^2 + T^0(\nabla f,\,\nabla f) - 3U(\nabla f,\,\nabla f)\Big]u\,\mathrm{Vol}_\eta, \tag{2-15}$$

which other than using different volume forms is identical to the second formula in [Ivanov and Vassilev 2015, Section 8.1.1, p. 310]. Indeed, following [Ivanov et al. 2014b, Lemma 3.4], using Ricci's identity

$$\nabla^2 f(X,\,\xi_s) - \nabla^2 f(\xi_s,\,X) = T(\xi_s,\,X,\,\nabla f)$$

and (2-2), we have

$$R_f(\nabla f) = \Big(\sum_{s=1}^{3}\nabla^2 f(I_s\nabla f,\,\xi_s)\Big) - [T^0(\nabla f,\,\nabla f) - 3U(\nabla f,\,\nabla f)]$$

An integration by parts gives (2-15), noting that $\sum_{s=1}^{3} df(\xi_s)\, df(I_s\nabla f) = 0$ and taking into account that by Ricci's identity

$$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = -2\sum_{s=1}^{3}\omega_s(X,Y)\, df(\xi_s)$$

we have $g(\nabla^2 f, \omega_s) = \sum_{a=1}^{4n} \nabla^2 f(e_a, I_s e_a) = -4n\, df(\xi_s)$.

Now, working as in [Ivanov and Vassilev 2015, Section 8.1.1, p. 310], we subtract (2-15) and three times formula (2-14) from (2-13) which brings us to the identity

$$\frac{1}{2}\mathcal{E}'(t) = \int_M \Big[-|(\nabla^2 f)_0|^2 - \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f)\Big] u \operatorname{Vol}_\eta + \frac{1}{4n}\int_M [3P_f(\nabla f) + 2(\Delta f)^2 - 3|\nabla f|^2 \Delta f] u \operatorname{Vol}_\eta, \tag{2-16}$$

where $|(\nabla^2 f)_0|^2$ is the square of the norm of the traceless part of the horizontal Hessian

$$|(\nabla^2 f)_0|^2 = |\nabla^2 f|^2 - \frac{1}{4n}\Big[(\triangle f)^2 + \sum_{s=1}^{3}[g(\nabla^2 f, \omega_s)]^2\Big].$$

Next, we consider $\int_M [2(\Delta f)^2 - 3|\nabla f|^2 \Delta f] u \operatorname{Vol}_\eta$. Using the heat equation we have the relation, identical to the Riemannian case, see (2-10),

$$\mathcal{E}'(t) = \frac{d}{dt}\int_M w \Delta u \operatorname{Vol}_\eta = \int_M (-2(\Delta f)^2 + 3|\nabla f|^2 \Delta f - |\nabla f|^4) u \operatorname{Vol}_\eta,$$

hence

$$\int_M (2(\Delta f)^2 - 3|\nabla f|^2 \Delta f) u \operatorname{Vol}_\eta = -\frac{d}{dt}\mathcal{E}(t) - \int_M |\nabla f|^4 u \operatorname{Vol}_\eta\,. \tag{2-17}$$

A substitution of the above formula in (2-16) gives

$$\frac{2n+1}{4n}\frac{d}{dt}\mathcal{E}(t) = \int_M \Big[-|(\nabla^2 f)_0|^2 - \frac{2n+1}{2}\mathcal{L}(\nabla f, \nabla f)\Big] u \operatorname{Vol}_\eta + \frac{1}{4n}\int_M [3P_f(\nabla f) - |\nabla f|^4] u \operatorname{Vol}_\eta\,.$$

Finally, we invoke Proposition 1.3 in order to complete the proof. □

3. The CR case

In this section, following the method we employed in the QC case, we prove the monotonicity formula in the CR case stated in Proposition 1.4. This implies the monotonicity of the entropy-like energy which was proved earlier in [Chang and Wu 2010].

Throughout the section M will be a $(2n+1)$-dimensional strictly pseudoconvex (integrable) CR manifold with a fixed pseudo-Hermitian structure defined by a contact form η and a complex structure J on the horizontal space $H = \operatorname{Ker}\eta$. The fundamental 2-form is defined by $\omega = \frac{1}{2}d\eta$ and the Webster metric is $g(X, Y) = -\omega(JX, Y)$ which is extended to a Riemannian metric on M by declaring that the Reeb vector field associated to η is of length one and orthonormal to the horizontal space. We shall denote by ∇ the associated Tanaka–Webster connection [Tanaka 1975] and [Webster 1975; 1978], while $\triangle u = \operatorname{tr}^g(\nabla^2 u)$ will be the negative sub-Laplacian with the trace taken with respect to an orthonormal basis of the horizontal $2n$-dimensional space. Finally, we define the Ricci-type tensor

$$\mathcal{L}(X, Y) = \rho(JX, Y) + 2nA(JX, Y), \tag{3-1}$$

recalling that on a CR manifold we have

$$\operatorname{Ric}(X, Y) = \rho(JX, Y) + 2(n-1)A(JX, Y), \tag{3-2}$$

where ρ is the $(1, 1)$-part of the pseudo-Hermitian Ricci tensor (the Webster Ricci tensor) while the $(2, 0) + (0, 2)$-part is the Webster torsion A; see [Ivanov and Vassilev 2011, Chapter 7] for the expressions in real coordinates of these known formulas [Webster 1975; 1978]; see also [Dragomir and Tomassini 2006].

With the above convention in place, as in [Chang and Wu 2010], for a positive solution of (1-1) we consider the entropy (1-2) and energy (1-3), where

$$\mathrm{Vol}_\eta = \eta \wedge (d\eta)^{2n}. \tag{3-3}$$

For a function f we define the one-form,

$$P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f) \tag{3-4}$$

so that the fourth order CR-Paneitz operator is given by

$$\begin{aligned} C(f) &= -\nabla^* P = (\nabla_{e_a} P)(e_a) \\ &= \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b) \\ &\qquad\qquad -4n\nabla^* A(J\nabla f) - 4ng(\nabla^2 f, JA). \end{aligned} \tag{3-5}$$

By [Graham and Lee 1988], when $n > 1$ a function $f \in \mathcal{C}^3(M)$ satisfies the equation $Cf = 0$ if and only if f is CR-pluriharmonic. Furthermore, the CR-Paneitz operator is nonnegative,

$$\int_M f \cdot Cf \,\mathrm{Vol}_\eta = -\int_M P_f(\nabla f)\,\mathrm{Vol}_\eta \geq 0.$$

On the other hand, in the three dimensional case the positivity condition is a CR invariant since it is independent of the choice of the contact form by the conformal invariance of C proven in [Hirachi 1993]. The nonnegativity of the

CR-Paneitz operator is relevant in the embedding problem for a three dimensional strictly pseudoconvex CR manifold. As shown in [Chanillo et al. 2012], if the pseudo-Hermitian scalar curvature of M is positive and C is nonnegative, then M is embeddable in some $\mathbb{C}^n$

We turn to the proof of Proposition 1.4. Taking into account (2-12) and the CR Bochner formula [Greenleaf 1985],

$$\tfrac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 + g(\nabla(\triangle f),\nabla f) + \mathrm{Ric}(\nabla f,\nabla f) + 2A(J\nabla f,\nabla f) + 4R_f(\nabla f), \tag{3-6}$$

where $R_f(Z) = \nabla\, df(\xi, JZ)$, see [Ivanov and Vassilev 2015, Section 7.1] and references therein but note the opposite sign of the sub-Laplacian, we obtain the next identity:

$$\tfrac{1}{2}(\partial_t - \Delta)(uw) = [-|\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f) - 2A(\nabla f, \nabla\nabla f) - 4R_f(\nabla f)]u. \tag{3-7}$$

Since (2-11) still holds, working as in the QC case we compute $\int_M R_F(\nabla f)u\,\mathrm{Vol}_\eta$ in two ways [Greenleaf 1985, Lemma 4] and [Ivanov and Vassilev 2012, Lemma 8.7] following the exposition [Ivanov and Vassilev 2015].

From Ricci's identity $\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = -2\omega(X, Y)\, df(\xi)$, it follows that $df(\xi) = -\frac{1}{2n} g(\nabla^2 f, \omega)$. Hence

$$\nabla^2 f(JZ, \xi) = -\frac{1}{2n}\sum_{b=1}^{2n} \nabla^3 f(JZ, e_b, Je_b),$$

where $\{e_b\}_{b=1}^{2n}$ is an orthonormal basis of the horizontal space. Applying Ricci's identity $\nabla^2 f(X, \xi) - \nabla^2 f(\xi, X) = A(X, \nabla f)$, it follows that

$$R_f(Z) = \nabla^2 f(\xi, JZ) = -\frac{1}{2n}\sum_{b=1}^{2n} \nabla^3 f(JZ, e_b, Je_b) - A(JZ, \nabla f). \tag{3-8}$$

Taking into account (3-4), the last formula gives

$$R_f(Z) = -\frac{1}{2n} P_f(Z) + A(JZ, \nabla f) + \frac{1}{2n}\sum_{b=1}^{2n} \nabla^3 f(Z, e_b, e_b).$$

Now, an integration by parts shows the next identity

$$\begin{aligned}\int_M R_f(\nabla f) u \mathrm{Vol}_\eta \\ = \int_M \Big[-\frac{1}{2n} P_f(\nabla f) + A(J\nabla f, \nabla f) - \frac{1}{2n}(\Delta f)^2 + \frac{1}{2n}|\nabla f|^2(\Delta f)\Big] u \mathrm{Vol}_\eta.\end{aligned} \tag{3-9}$$

On the other hand, using again (3-8) but now integrating and then using integration by parts, we have

$$\int_M R_f(\nabla f)u \,\mathrm{Vol}_\eta = \int_M \Big[-\frac{1}{2n} g(\nabla^2 f, \omega)^2 - A(J\nabla f, \nabla f)\Big] u \,\mathrm{Vol}_\eta \,. \tag{3-10}$$

At this point, exactly as in the QC case, we subtract (3-10) and three times formula (3-9) from (3-7), which gives

$$\mathcal{E}'(t) = -\int_M \big[|(\nabla^2 f)_0|^2 + \mathcal{L}(\nabla f, \nabla f)\big] u \mathrm{Vol}_\eta + \frac{1}{2n}\int_M [3P_f(\nabla f) + 2(\Delta f)^2 - 3|\nabla f|^2 \Delta f] u \mathrm{Vol}_\eta,$$

where $|(\nabla^2 f)_0|^2$ is the square of the norm of the traceless part of the horizontal Hessian

$$|(\nabla^2 f)_0|^2 = |\nabla^2 f|^2 - \frac{1}{2n}[(\triangle f)^2 + g(\nabla^2 f, \omega)^2].$$

Taking into account that the formulas in Proposition 1.3 and (2-17) hold unchanged, we complete the proof. □

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ON HANDLEBODY STRUCTURES OF RATIONAL BALLS

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It is known that for coprime integers $p > q \geq 1$, the lens space $L(p^2, pq-1)$ bounds a rational ball, $B_{p,q}$, arising as the 2-fold branched cover of a (smooth) surface in B^4 bounding the associated 2-bridge knot or link. Lekili and Maydanskiy give handle decompositions for each $B_{p,q}$; whereas, Yamada gives an alternative definition of rational balls, $A_{m,n}$, bounding $L(p^2, pq-1)$ by their handlebody decompositions alone. We show that these two families coincide, answering a question of Kadokami and Yamada. To that end, we show that each $A_{m,n}$ admits a Stein filling of the universally tight contact structure, $\bar{\xi}_{st}$, on $L(p^2, pq-1)$ investigated by Lisca. Furthermore, we construct boundary diffeomorphisms between these families. Using the carving process, pioneered by Akbulut, we show that these boundary maps can be extended to diffeomorphisms between the spaces $B_{p,q}$ and $A_{m,n}$.

1. Introduction

For $p > q \geq 1$ relatively prime, let $B_{p,q}$ be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing $pq-1$ to B^4 by wrapping the attaching circle of the 2-handle p-times around the 1-handle with a q/p-twist; see Figure 1.

From this description, it is immediate that $B_{p,q}$ is always a rational homology ball. Lekili and Maydanskiy [2014] show that each such $B_{p,q}$ arises as the 2-fold branched cover of B^4 branched over a properly embedded surface bounding the 2-bridge link associated to the fraction $-p^2/(pq-1)$. That is, the family $B_{p,q}$ represents handle decompositions of the rational balls introduced by Casson and Harer [1981]. As such, $\partial B_{p,q} \approx L(p^2, pq-1)$, where $\approx$ denotes diffeomorphism of two manifolds throughout. Lekili and Maydanskiy go on to prove that each $B_{p,q}$ supports a Stein structure (see Figure 7) filling the universally tight contact structure on $L(p^2, pq-1)$ [Lekili and Maydanskiy 2014].

In a similar direction, Yamada [2007] defines a family of $X \,||\, Y$ rational balls bounding $L(p^2, pq-1)$ via their handle decompositions: For $n, m \geq 1$ relatively prime, let $A_{m,n}$ be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing mn to B^4 by attaching the 2-handle along a simple closed

MSC2010: primary 57R65; secondary 57R17.
Keywords: 4-manifolds, handle calculus, rational blow-down.

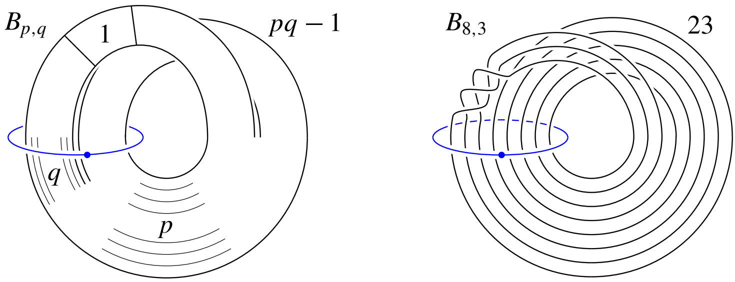

Figure 1. The rational ball $B_{p,q}$ (left); e.g., $B_{8,3}$ (right).

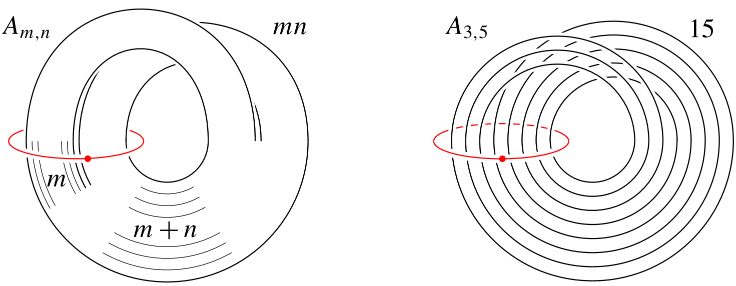

Figure 2. The rational ball $A_{m,n}$ (left); e.g., $A_{3,5}$ (right).

curve embedded on a once-punctured torus viewed in $S^1 \times S^2$ so that the attaching circle traverses the two 1-handles of the torus m and n times respectively (Figure 2).

Yamada goes on to define an involutive symmetric function A on the set of coprime pairs of positive integers such that if $A(p-q, q) = (m, n)$ then $\partial A_{m,n} \approx L(p^2, pq-1)$. Here $m+n = p$ and $mq = \pm 1 \mod p$; Remark 2.4 gives a definition of A.

Given these two constructions of rational balls with coincident boundaries, one arrives at a natural question posed by Kadokami and Yamada:

Question 1.1 [Kadokami and Yamada 2014, Problem 1.9]. Are $A_{m,n}$ and $B_{p,q}$ diffeomorphic, homeomorphic, or even homotopic relative to their boundaries as 4-manifolds?

The goal herein is to provide a complete answer to this question by proving the following theorem.

Theorem 1.2. *For each pair of relatively prime positive integers* (m, n), $A_{m,n}$ *carries a Stein structure* $\tilde{J}_{m,n}$ *filling the universally tight contact structure on the lens space* $\partial A_{m,n}$. *In particular, each* $A_{m,n} \approx B_{p,q}$ *if and only if* $\partial A_{m,n} \approx \partial B_{p,q}$.

The proof of Theorem 1.2 follows by first explicitly writing down a Stein structure on $A_{m,n}$ using Eliashberg's characterization of handle decompositions of Stein

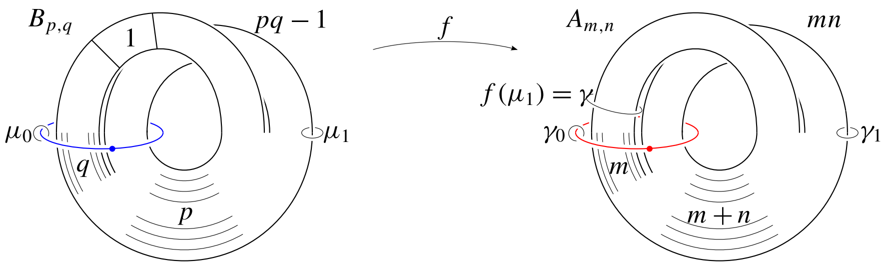

Figure 3. The spaces $B_{p,q}$ and $A_{m,n}$.

domains [Eliashberg 1990; Gompf 1998]. As the homotopy invariants of the induced contact structures on the boundary agree with those of $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$, the two structures are homotopic as 2-plane fields. Work of Honda [2000] and independently Giroux [2000] proves that this is sufficient to conclude that these two contact structures are contactomorphic. Lisca's classification [2008] of the diffeomorphism types of symplectic fillings of $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$ then gives that $A_{m,n} \approx B_{p,q}$. To provide insight into the aforementioned diffeomorphisms, we construct boundary diffeomorphisms which can be extended to explicit diffeomorphisms between $B_{p,q}$ and $A_{m,n}$ through the carving process introduced by Akbulut [1977]. In fact, we have the following result:

Theorem 1.3. *Let $(m, n) = A(p-q, q)$ for some $p > q > 0$ relatively prime. Then there exists a diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$ such that f carries the belt sphere, μ_1, of the single 2-handle in $B_{p,q}$ to a slice knot in $\partial A_{m,n}$ (see Figure 3). Moreover, carving $A_{m,n}$ along $f(\mu_1)$ gives $S^1 \times B^3$.*

Corollary 1.4. *f extends to a diffeomorphism $\tilde{f} : B_{p,q} \to A_{m,n}$.*

Further motivation. Fintushel and Stern [1997] define a smooth operation, the rational blow-down, on 4-manifolds containing certain configurations of spheres by removing a neighborhood of those spheres and replacing them by the rational ball $B_{p,1}$. Park [1997] generalized the operation to a larger set of configurations at the expense of having to glue in $B_{p,q}$ for q other than 1. In the presence of a symplectic structure and a symplectic configuration of spheres, both operations can be performed symplectically [Symington 1998; 2001]. Moreover, under mild assumptions (see [Fintushel and Stern 1997; Park 1997] for details), nontrivial solutions to the Seiberg–Witten equations on the original 4-manifold induce nontrivial solutions on the surgered manifold and vice versa.

Therefore, having well understood handle decompositions for $B_{p,q}$ allows one to construct explicit examples of rationally blown-down 4-manifolds. For instance, Stipsicz and Szabó [2005] take advantage of such decompositions to construct an exotic $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$. Corollary 1.4 and Theorem 1.2 are then useful, since either the decomposition $B_{p,q}$ or $A_{m,n}$ can conceivably be used interchangeably.

Organization. The paper is organized as follows: In Section 2, we dispense with notation and necessary calculations involving lens spaces. Then, in Section 3, we bring in the relevant symplectic topology and construct Stein handle decompositions on each $A_{m,n}$, proving Theorem 1.2. Finally, in Section 4, we recall the carving procedure and construct boundary diffeomorphisms from $\partial B_{p,q}$ and $\partial A_{m,n}$ to their lens space boundaries, proving Theorem 1.3.

2. Preliminaries

Conventions and assumptions. Unless specifically stated to the contrary, throughout the paper, we assume $p-q > q \geq 1$, $n > m \geq 1$, and that both pairs are relatively prime. As $B_{p,q} \approx B_{p,p-q}$ and $A_{m,n} \approx A_{n,m}$, this assumption does not represent a restriction. We adopt the standard orientation convention that $L(p,q)$ is the result of $-p/q$-surgery on the unknot in S^3. It is well known that $L(p,q)$ is also given as the boundary of a linear plumbing of D^2-bundles over S^2 with Euler classes chosen according to a continued fraction associated to $-p/q$:

$$[c_1, \dots, c_n] \doteq c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_n}}} = -\frac{p}{q}$$

where the c_i are uniquely determined provided each $c_i \leq -2$ (see Figure 4). Where convenient, we will use weighted trees to describe these plumbings. We will often forgo the uniqueness of the c_i in favor of shorter continued fraction expansions and thus smaller bounding 4-manifolds. In spite of this, we make the following definition.

Definition 2.1. Given $p > 0$ and q coprime, let $C_{p,q}$ be the 4-manifold bounding $L(p,q)$ obtained by plumbing D^2-bundles over S^2 according to a linear graph with weights $c_i \leq -2$ chosen so that $[c_1, \dots, c_n] = -p/q$ (see Figure 4). For conciseness, we denote $C_{p^2,pq-1}$ by $\mathcal{C}_{p,q}$.

In Section 3, we need to perform calculations in the group $H_1(L(p^2, pq-1; \mathbb{Z})$. The following lemma will prove useful.

Lemma 2.2. *Suppose that $L(p,q)$ is given by the linear plumbing of Figure 4 where the η_i are meridians spanning $H_1(L(p,q), \mathbb{Z})$. Then*

$$H_1(L(p,q), \mathbb{Z}) = \langle \eta_1 : (\det C_n)\eta_1 = 0 \rangle$$

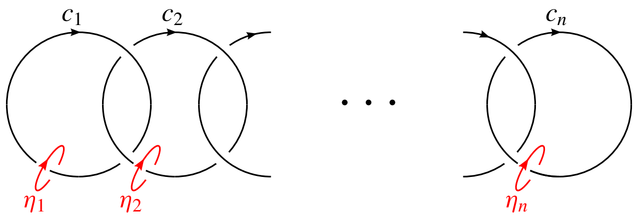

Figure 4. A linear plumbing bounding $L(p,q)$. Elements spanning $H_1(L(p,q))$ are shown in red.

where $C_i \doteq \begin{pmatrix} c_1 & 1 & & \\ 1 & c_2 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & c_i \end{pmatrix}$ *and* $\eta_i = (-1)^{i-1}(\det C_{i-1})\eta_1$ *for* $i \in \{2,\dots,n\}$.

Proof. Given a Dehn surgery description of a 3-manifold, one obtains a presentation for the first homology in terms of the right handed meridians of the (oriented) framed link; see [Gompf and Stipsicz 1999]. In the above case, we find that

$$H_1(L(p,q),\mathbb{Z}) \\ = \langle \eta_1,\dots,\eta_n : \eta_2 = -c_1\eta_1,\ \{\eta_{i+1} = -c_i\eta_i - \eta_{i-1}\}_{i=2}^{n-1},\ c_n\eta_n = -\eta_{n-1}\rangle$$

As $\eta_2 = -c_1\eta_1 = (-1)^{2-1}(\det C_{2-1})\eta_1$, the result follows by induction using that

$$\det C_k = c_k \det C_{k-1} - \det C_{k-2}. \qquad \square$$

Determining $\boldsymbol{\mathcal{C}_{p,q}}$. The continued fraction associated to $-p^2/(pq-1)$ involves the Euclidean algorithm; see [Casson and Harer 1981; Yamada 2007] as well as Proposition 2.5 below. Therefore, we use the Euclidean algorithm to define sequences of remainders and divisors of p and q as follows:

Definition 2.3. For $p > q \geq 1$, relatively prime, let $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ be defined recursively by setting $r_{-1} \doteq p$, $r_0 \doteq q$. and

$$r_{i+1} = r_{i-1} \mod r_i, \qquad r_{i-1} = r_i s_i + r_{i+1}.$$

Let ℓ be the last index where $r_\ell > 1$ so that $r_{\ell+1} = 1$ and $r_{\ell+2} \doteq 0$.

Remark 2.4. For bookkeeping purposes, we will differentiate between the above sequences for p and q and the analogously defined sequences $\{\rho_i\}_{i=-1}^{\ell+2}$ and $\{\sigma_i\}_{i=0}^{\ell+1}$ associated to $n > m \geq 1$. Furthermore, provided that $p-q > q$, and that $A(p-q,q)$ either equals (m,n) or (n,m), the four sequences are related by the following recursive dictionary:

$$
\begin{array}{rcr}
r_{-1} = p & s_0 \longleftrightarrow \rho_\ell & 1 = \rho_{\ell+1} \\
r_0 = q & s_1 \longleftrightarrow \sigma_\ell & \\
 & & \\
r_{i+1} = r_{i-1} - r_i s_i & s_j \longleftrightarrow \sigma_{\ell-j+1} & \rho_i\sigma_i + \rho_{i+1} = \rho_{i-1} \\
 & & \\
 & s_\ell \longleftrightarrow \sigma_1 & m = \rho_0 \\
r_{\ell+1} = 1 & r_\ell - 1 \longleftrightarrow \sigma_0 & n = \rho_{-1}
\end{array}
$$

That is, given the sequences associated to p and q, we get the associated sequences for m and n by declaring $\rho_{\ell+1} = 1$, $\rho_\ell = s_0$ and making the indicated identifications for the σ_j in order to recursively recover each ρ_j; ultimately determining $m = \rho_0$ and $n = \rho_{-1}$. Similarly, we may start from m and n to recover p and q. In fact, we will take this correspondence as our definition of the function A defined by Yamada [2007]. It is straightforward to verify that formulation is equivalent to Yamada's definition. As we will independently see in Section 4, this correspondence ensures that $\partial B_{p,q} \approx \partial A_{m,n}$ (see Remark 4.8); so, nothing is lost.

We can explicitly write down $\mathcal{C}_{p,q}$ in terms of these Euclidean sequences. The following is proved in Section 4 as Corollary 4.3.

Proposition 2.5. *For $p > q > 0$ coprime, the lens space $L(p^2, pq-1)$ bounds the linear plumbing $X(\Gamma)$ where Γ is the weighted graph of Figure 5 and where $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ are as in Definition 2.3.*

$X(\Gamma)$ defined in Proposition 2.5 has spheres of positive self-intersection and is therefore *not* $\mathcal{C}_{p,q}$. Given a sphere in $X(\Gamma)$ with self-intersection $s > 0$, by blowing up $s-1$ of these intersections we get a sphere with one positive self-intersection — which can be blown-down. This allows the exchange of each positive Euler-class disk bundle for, possibly many negative Euler-class bundles without altering the boundary. By applying this process at each sphere with positive self-intersection we arrive at $\mathcal{C}_{p,q}$.

Corollary 2.6. *For $p > q \geq 1$, coprime, let $\{s_i\}_{i=0}^{\ell}$ and $\{r_i\}_{i=-1}^{\ell+1}$ be as defined in Definition 2.3, the space $\mathcal{C}_{p,q}$ is given by one of the linear plumbings of Figure 6 (depending upon the parity of ℓ).*

Remark 2.7. By Definition 2.1, Figure 6 specifies $\mathcal{C}_{p,q}$. This follows since each s_i is at least 1, ensuring that each weight in the graphs of Figure 6 is less than or equal to -2. The meridians (in red) of Figure 6 are used in homological calculations in

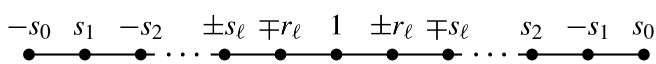

Figure 5. A linear plumbing bounding $L(p^2, pq-1)$.

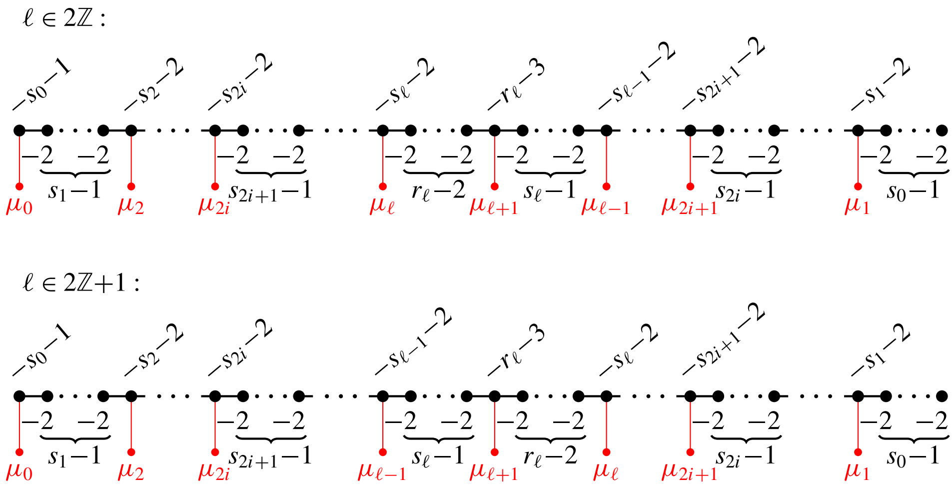

Figure 6. $\mathcal{C}_{p,q}$ when $\ell \in 2\mathbb{Z}$ and when $\ell \in 2\mathbb{Z}+1$ with relevant meridians used in homology calculations (in red).

Section 3. It is worth noting that combining Lemma 2.2 with the following lemma, we find that $\mu_i = (-1)^i \rho_{\ell-i+1}\mu_0 \in H_1(L(p^2, pq-1); \mathbb{Z})$.

Lemma 2.8. *Let $\{\rho_i\}_{i=-1}^{\ell+2}$ and $\{\sigma_i\}_{i=0}^{\ell+1}$ be as defined in Definition 2.3 (associated to n and m). Then for each $i \leq \ell+1$,*

$$\det\begin{pmatrix} -\rho_\ell & 1 & & \\ 1 & \sigma_\ell & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & (-1)^{\ell+1-i}\sigma_{\ell+1-i} \end{pmatrix} = -\Big(\sin\Big(\frac{\pi}{2}i\Big)+\cos\Big(\frac{\pi}{2}i\Big)\Big)\rho_{\ell-i}.$$

Proof. Induct on i, using that $\rho_{\ell+1}=1$ and that $\rho_{\ell-i} = \rho_{\ell-i+1}\sigma_{\ell-i+1} + \rho_{\ell-i+2}$. □

3. Stein structures on $A_{m,n}$

We are now ready to show that $A_{m,n}$ admits a Stein structure. To accomplish this, we use Eliashberg's handle characterization of Stein surfaces [Eliashberg 1990; Gompf 1998]. The reader should consult [Gompf and Stipsicz 1999] as well as [Ozbagci and Stipsicz 2004] for thoughtful treatments of the subject. Such a Stein structure induces a (tight) contact structure on $\partial A_{m,n}$. Tight contact structures on lens spaces are well understood; Honda [2000], and independently Giroux [2000], completely classify them. Moreover, Lisca classifies the diffeomorphism types of symplectic fillings of $(L(p,q), \bar{\xi}_{st})$ where $\bar{\xi}_{st}$ is the universally tight contact structure $L(p,q)$ inherits from the unique tight contact structure on S^3 via the cyclic group action. In particular, Lisca defines collections of 4-manifolds $W_{p,q}(\boldsymbol{n})$, such that

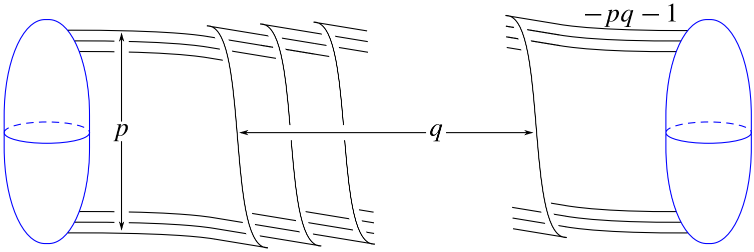

Figure 7. $(B_{p,q}, J_{p,q})$.

Theorem 3.1 [Lisca 2008, Theorem 1.1]. *Let $p > q \geq 1$ be relatively prime. Then each symplectic filling (W, ω) of $(L(p, q), \bar{\xi}_{\mathrm{st}})$ is orientation preserving diffeomorphic to a smooth blowup of $W_{p,q}(\boldsymbol{n})$ for some $\boldsymbol{n} \in \mathbf{Z}_{p,q}$. Moreover, if $b_2(W) = 0$, then W is unique.*

In light of Theorem 3.1, if we show that not only does $A_{m,n}$ admit a Stein structure, but that such a structure gives a symplectic filling of $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$, then we immediately have that $A_{m,n} \approx B_{p,q}$ since it is known that $B_{p,q}$ admits a Stein structure giving such a filling. Indeed, by sliding the 2-handle of $B_{p,q}$ under the 1-handle q-times one arrives at the Stein domain, $(B_{p,q}, J_{p,q})$, investigated by Lekili and Maydanskiy [2014] given in Figure 7. There, they prove that $(B_{p,q}, J_{p,q})$ fills the standard contact structure on $L(p^2, pq-1)$.

Tight contact structures on lens spaces. Before we explicitly construct a Stein handle decomposition for $A_{m,n}$, we note that *any* Stein structure on $A_{m,n}$ necessarily induces a tight contact structure which is contactomorphic to $\bar{\xi}_{\mathrm{st}}$ (see Proposition 3.4). When identifying tight contact structures on lens spaces, it is enough to know that the two contact structures in question are homotopic up to contactomorphism.

Theorem 3.2 [Honda 2000, Proposition 4.24; Giroux 2000, Theorem 1.1]. *The homotopy classes of the tight contact structures of $L(p, q)$ are all distinct. Moreover, if $q < p-1$, then all but exactly two tight contact structures on $L(p, q)$ are virtually overtwisted.*

The two universally tight contact structures are both contactomorphic to $\bar{\xi}_{\mathrm{st}}$. Furthermore, the problem of determining the homotopy type of the underlying 2-plane field of a given tight contact structure is completely solved by Gompf [1998].

In fact, for contact structures with c_1 torsion (which is always satisfied for 3-manifolds with $b_1 = 0$; e.g., lens spaces) two homotopy invariants d_3 and Γ completely determine their homotopy classes as 2-plane fields.

Theorem 3.3 [Gompf 1998, Theorem 4.16]. *If (Y^3, ξ_i) for $i = 1, 2$, satisfies that $c_1(\xi_1)$ is torsion and $\Gamma(\xi_1, \boldsymbol{s}) = \Gamma(\xi_2, \boldsymbol{s})$ for some spin structure $\boldsymbol{s}$, then ξ_1 is homotopic to ξ_2 if and only if their d_3 invariants coincide.*

According to Theorem 3.3, two 2-plane fields (with torsion c_1) are homotopic if and only if they have the same Γ *and* d_3 invariants. Lisca [2001] proves that in the case of tight contact structures on a lens space, the Γ invariant alone is enough — that is, if $\Gamma(\xi_x, \mathbf{s}) = \Gamma(\xi_y, \mathbf{s})$, then ξ_x is homotopic to ξ_y (and their d_3 invariants necessarily coincide). One cannot expect the same result to hold with d_3 in place of Γ. However, the d_3-invariant does detect the universally tight structures on $L(p^2, pq-1)$. In fact by using the "correction terms" from Heegaard Floer homology to determine which spin$^{\mathbb{C}}$-structures on $L(p^2, pq-1)$ induced from a tight contact structure therein can extend across a rational ball bounding the lens space we arrive at the following proposition known to experts:

Proposition 3.4. *Every tight contact structure ξ on $L(p^2, pq-1)$ with $d_3(\xi) = -\frac{1}{2}$ is universally tight.*

For completeness, we include a proof of Proposition 3.4 below. Before dispatching with that, we first recall the definitions of d_3 and Γ. For the three-dimensional invariant, d_3, we use the normalized definition [Ozbagci and Stipsicz 2004] — but note that it is equivalent to the definition of θ originally defined by Gompf [1998] which relies on the fact that each contact 3-manifold can be realized as the J-convex boundary of an almost complex 4-manifold as well as the fact that for (X^4, J), a closed almost complex 4-manifold, the quantity $c_1^2(X, J) - 3\sigma(X) - 2\chi(X) = 0$ where $\sigma(X)$ and $\chi(X)$ are the signature and Euler characteristic of X respectively.

Definition 3.5 [Gompf 1998, Definition 4.2]. For a contact 3-manifold (Y, ξ) with $c_1(\xi)$ torsion, the three-dimensional invariant

$$d_3(\xi) = \tfrac{1}{4}\big(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)\big) \in \mathbb{Q}$$

for any almost complex 4-manifold (X, J) with $\partial X = Y$ satisfying $TY \cap JTY = \xi$.

The function Γ associates to each spin structure on (Y, ξ) an element of $H_1(Y; \mathbb{Z})$. This is accomplished by noting that $\mathrm{Spin}^{\mathbb{C}}(Y)$ is an $H^2(Y; \mathbb{Z})$-torsor. So any two $\mathfrak{t}_0, \mathfrak{t}_1 \in \mathrm{Spin}^{\mathbb{C}}(Y)$, satisfy that their difference $\mathfrak{t}_1 - \mathfrak{t}_0$ is a well defined element of $H^2(Y; \mathbb{Z})$. A spin structure on Y can be canonically viewed as a spin$^{\mathbb{C}}$-structure. Then $\Gamma(\xi, \mathbf{s})$ is Poincaré dual to the difference $\mathfrak{t}_\xi - \mathbf{s}$. Furthermore, if (Y, ξ) is the boundary of a Stein 4-manifold (X, J), Gompf provides a combinatorial formula for Γ (we state it only in the case when X lacks 1-handles; we also suppress the definition of a characteristic sublink associated to $\mathbf{s} \in \mathrm{Spin}(Y)$ as we will not make use of it herein — the interested reader can refer to [Gompf 1998; Kaplan 1979] for details).

Proposition 3.6 [Gompf 1998, Theorem 4.12]. *Let (X, J) be obtained from B^4 by attaching Stein 2-handles along Legendrian knots $K_1, \ldots, K_k$ such that $\partial X = Y$ and $\xi = TY \cap JTY$. Orient $K_1 \cup \cdots \cup K_k$ to obtain a spanning set for $H_2(X; \mathbb{Z})$. Then*

$\Gamma(\xi, \mathbf{s}) \in H_1(\partial X; \mathbb{Z})$ is Poincaré dual to the restriction of the class $\rho \in H^2(X; \mathbb{Z})$ whose value on each $[K_i]$ is given by

$$\rho([K_i]) = \tfrac{1}{2}\big(\mathrm{rot}(K_i) + \ell k(K_i, L)\big) \in \mathbb{Z}$$

where L is the characteristic sublink associated to $\mathbf{s}$.

Honda [2000] and Giroux [2000] prove that each tight contact structure on $L(p, q)$ is induced by a Stein filling of $C_{p,q}$. In general, $\mathcal{C}_{p,q}$ admits numerous Stein fillings. Each is obtained by attaching the 2-handles of $\mathcal{C}_{p,q}$ along Legendrian unknots whose Seifert framings are one less than the their respective Thurston–Bennequin framings. For each $n < -1$, by stabilizing the standard Legendrian unknot positively and or negatively as needed, there are exactly $|n| - 1$ distinct rotation numbers for Legendrian unknots with Thurston–Bennequin framing equal to $n+1$: namely $n+2, n+4, \ldots, -n-2$. In particular, each unknot in the handle decomposition of $\mathcal{C}_{p,q}$ with Seifert framing -2 necessarily has rotation number zero for any Stein handle attachment. Therefore, if we let K_i denote the attaching circle of the 2-handle in $\mathcal{C}_{p,q}$ whose belt-sphere is the meridian given by μ_i as labeled in Figure 6, we see that specifying rotation numbers only for K_i fixes a Stein structure on $\mathcal{C}_{p,q}$. With this in mind, for each $x = (x_0, \ldots, x_{\ell+1})$ chosen so that

$$\begin{aligned} x_0 &\in \{1 - s_0, 3 - s_0, \ldots, s_0 - 1\}, \\ x_i &\in \{-s_i, 2 - s_i, \ldots, s_i\}, \quad \text{for } i \in \{1, \ldots, \ell\} \\ x_{\ell+1} &\in \{-1 - r_\ell, 1 - r_\ell, \ldots, r_\ell + 1\}, \end{aligned}$$

we get a unique Stein structure on $\mathcal{C}_{p,q}$ inducing a distinct (up to isotopy) tight contact structure on $L(p^2, pq-1)$. In an abuse of notation, we ignore the obvious dependence on p and q and choose to call this structure J_x.

It is known that $J_{x_{\min}}$ and $J_{x^{\max}}$ induce the two universally tight contact structures on $L(p^2, pq-1)$, where $x^{\max}$ fixes the largest allowable rotation number on each K^i and $x_{\min} = -x^{\max}$. Let ξ_x, $\xi_{\min}$ and $\xi^{\max}$ be the contact structures induced by J_x, $J_{\min}$ and $J^{\max}$ respectively; similarly define the spin$^{\mathbb{C}}$-structures $\mathfrak{t}_x$, $\mathfrak{t}_{\min}$ and $\mathfrak{t}^{\max}$. As shown by Lekili and Maydanskiy [2014], $\xi_{\min}$ and $\xi^{\max}$ are also induced by the Stein structures $(B_{p,q}, J_{p,q})$ and $(B_{p,p-q}, J_{p,p-q})$ specified in Figure 7. Therefore, the spin$^{\mathbb{C}}$-structures $\mathfrak{t}_{\min}$ and $\mathfrak{t}^{\max}$ both extend over $B_{p,q}$ to $\mathfrak{s}_{\min}, \mathfrak{s}^{\max} \in \mathrm{Spin}^{\mathbb{C}}(B_{p,q})$. No other $\mathfrak{t}_x$ has this property:

Proposition 3.7. *Let $\Xi_{p,q}$ denote the set of homotopy classes of 2-plane fields induced by tight contact structures on $L(p^2, pq-1)$ and let*

$$\mathcal{S} = \left\{\mathfrak{t}_\xi \in \mathrm{Spin}^{\mathbb{C}}(L(p^2, pq-1)) : \xi \in \Xi_{p,q}\right\},$$

then $\mathcal{S}$ contains exactly two spin$^{\mathbb{C}}$*-structures that extend across the ball $B_{p,q}$, both of which arise from contact structures contactomorphic to $\bar{\xi}_{\mathrm{st}}$.*

Before we prove Proposition 3.7 we recall the obstruction to extending a given spin$^{\mathbb{C}}$-structure $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(L(p^2, pq-1))$ across a rational ball bounding the space $L(p^2, pq-1)$. We can measure this obstruction against any fixed spin$^{\mathbb{C}}$-structure which is known to extend. As every 4-manifold admits a spin$^{\mathbb{C}}$-structure (which extends its restriction to the boundary), we always have such an element to measure against. A standard obstruction theoretic proof gives the following lemma:

Lemma 3.8. *Suppose that $\mathcal{B}$ is a rational ball bounding $L(p^2, pq-1)$. For each pair $\mathfrak{t}_0, \mathfrak{t}_1 \in \mathrm{Spin}^{\mathbb{C}}(\partial\mathcal{B})$ such that $\mathfrak{t}_0$ extends across $\mathcal{B}$ to some $\mathfrak{s}_0 \in \mathrm{Spin}^{\mathbb{C}}(\mathcal{B})$, $\mathfrak{t}_1$ extends across $\mathcal{B}$ if and only if p divides the difference $\mathfrak{t}_0 - \mathfrak{t}_1 \in H^2(\partial\mathcal{B}; \mathbb{Z})$.*

We can use Lemma 3.8 to determine which other spin$^{\mathbb{C}}$-structures induced by some J_x extend over $B_{p,q}$. Note that for any spin-structure $\boldsymbol{s} \in \mathrm{Spin}(L(p^2, pq-1))$ the difference

$$\mathrm{PD}(\Gamma(\xi_y, \boldsymbol{s})) - \mathrm{PD}(\Gamma(\xi_x, \boldsymbol{s})) = (\mathfrak{t}_y - \boldsymbol{s}) - (\mathfrak{t}_x - \boldsymbol{s}) = \mathfrak{t}_y - \mathfrak{t}_x$$

doesn't depend on the choice of spin-structure. Using Proposition 3.6, we calculate

$$\mathrm{PD}(\mathfrak{t}_y - \mathfrak{t}_x) = \sum_{i=0}^{\ell+1} \frac{y_i - x_i}{2}\mu_i = \sum_{i=0}^{\ell+1} (-1)^i \frac{y_i - x_i}{2}\rho_{\ell-i+1}\mu_0$$

where the last equality follows from Remark 2.7.

Proof of Proposition 3.7. Suppose that $\mathfrak{t} \in \mathcal{S}$ extends across $B_{p,q}$. We can assume that $\mathfrak{t} = \mathfrak{t}_x$ for some Stein structure $(\mathcal{C}_{p,q}, J_x)$ on $\mathcal{C}_{p,q}$. Lemma 3.8 gives that $\mathfrak{t}_x$ extends if and only if p divides the difference $\mathrm{PD}(\mathfrak{t}^{\max} - \mathfrak{t}_x)$ in $H_1(L(p^2, pq-1))$. Write $x = x^{\max} - 2c$ where $c = (c_0, c_1, \dots, c_{\ell+1})$ necessarily satisfies $c_0 \in \{0, 1, \dots, s_0 - 1\}$, $c_i \in \{0, 1, \dots, s_i\}$ for each $i \in \{1, 2, \dots, \ell\}$ and $c_{\ell+1} = \{0, 1, \dots, r_\ell + 1\}$. Then

$$\mathrm{PD}(\mathfrak{t}^{\max} - \mathfrak{t}_x) = \sum_{i=0}^{\ell+1} (-1)^i \frac{x_i^{\max} - x_i}{2}\rho_{\ell-i+1}\mu_0 = \sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1}\mu_0.$$

Therefore, we investigate solutions to $\sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1} \equiv 0 \bmod p$. We will prove in Corollary 3.12 that there are exactly two solutions — namely $c = 0$ and $2c = x^{\max}$ (giving that the only spin$^{\mathbb{C}}$-structures which extend correspond to $x^{\max}$ and $x_{\min} = -x^{\max}$) which are known to induce the universally tight contact structures on $L(p^2, pq-1)$. $\square$

To finish the proof of Proposition 3.4, recall that Ozsváth and Szabó [2004b; 2004a] define relatively $\mathbb{Z}$-graded homology groups $\mathrm{HF}^{\pm}$, HF^{∞} associated to each 3-manifold endowed with a spin$^{\mathbb{C}}$-structure. If the spin$^{\mathbb{C}}$-structure is torsion, they obtain absolute $\mathbb{Q}$-gradings [Ozsváth and Szabó 2006]. Using this grading, they define the correction term $d(Y, \mathfrak{t})$ of any rational homology spin$^{\mathbb{C}}$ 3-sphere $(Y, \mathfrak{t})$ as the minimal degree of the image of a nontorsion element of $\mathrm{HF}^{\infty}(Y, \mathfrak{t})$ in $\mathrm{HF}^{+}(Y, \mathfrak{t})$

[Ozsváth and Szabó 2003]. Of interest to the present problem, is the following result of Ozsváth, Stipsicz and Szabó:

Proposition 3.9 [Ozsváth et al. 2005, Corollary 1.7]. *Suppose (Y,ξ) is a rational homology 3-sphere equipped with a symplectically fillable contact structure ξ supported by a planar open book, then*

$$d_3(\xi)+\tfrac{1}{2}=-d(Y,\mathfrak{t}_\xi).$$

As every tight contact structure on a lens space is supported by a planar open book [Schönenberger 2007], we gain knowledge about the three-dimensional invariant d_3 from the correction term and vice versa. In particular, compare Lemma 3.8 with the following result of Jabuka, Robins and Wang:

Proposition 3.10 [Jabuka et al. 2013]. *Suppose that $\mathfrak{t}_0$ and $\mathfrak{t}_1$ are spin-c structures on $L(p^2, pq-1)$ such that their respective correction terms vanish. Then p divides $\mathfrak{t}_0-\mathfrak{t}_1 \in H^2(L(p^2, pq-1))$.*

Proof of Proposition 3.4. As ξ is symplectically fillable and supported by a planar open book, Proposition 3.9 gives that

$$d(L(p^2, pq-1),\mathfrak{t}_\xi)=-d_3(\xi)-\tfrac{1}{2}=0.$$

Proposition 3.10 then gives that p divides $\mathfrak{t}_{\bar{\xi}_{\mathrm{st}}}-\mathfrak{t}_\xi$; and thus $\mathfrak{t}_\xi$ extends across $B_{p,q}$ as $\mathfrak{t}_{\bar{\xi}_{\mathrm{st}}}$ does. Clearly $\xi\in\Xi_{p,q}$, so by Proposition 3.7, ξ is contactomorphic to $\bar{\xi}_{\mathrm{st}}$. □

Finally, Proposition 3.7 relies on the observation that there are exactly two integral solutions to $\sum_{i=0}^{\ell+1}(-1)^i c_i\rho_{\ell-i+1}\equiv 0 \bmod p$ under the appropriate restrictions of the c_i. The following lemma gives bounds that imply this fact as a corollary.

Lemma 3.11. *Fix integers $c_0\in[0,s_0-1]$, $c_i\in[0,s_i]$ for all $1\le i\le\ell$, and $c_{\ell+1}\in[0,r_\ell-1]$. Then for each $k<\ell+1$,*

$$1-\rho_{\ell-2\lfloor(k+1)/2\rfloor+1}\le\sum_{i=0}^{k}(-1)^i c_i\rho_{\ell-i+1}\le -1+\rho_{\ell-\lfloor k/2\rfloor},$$

and

$$-p<1-\rho_0\le(-1)^{\ell+1}\sum_{i=0}^{\ell+1}(-1)^i c_i\rho_{\ell-i+1}\le\rho_{-1}+2\rho_0-1<2p.$$

Consequently, $\sum_{i=0}^{\ell+1}(-1)^i c_i\rho_{\ell-i+1}=0$ if and only if each $c_i=0$.

Proof. First, assume the inequalities; note $c_0\rho_{\ell+1}=0$ if and only if $c_0=0$. By way of induction, suppose the only solution to $\sum_{i=0}^{k}(-1)^i c_i\rho_{\ell-i+1}=0$ is the trivial solution. Any purported nontrivial solution to $\sum_{i=0}^{k+1}(-1)^i c_i \rho_{\ell-i+1}=0$, has $c_{k+1}>0$ by induction; however,

$$c_{k+1}\rho_{\ell-k} > \rho_{\ell-k}-1 \geq (-1)^k \sum_{i=0}^{k}(-1)^i c_i \rho_{\ell-i+1},$$

contradicting $\sum_{i=0}^{k+1}(-1)^i c_i \rho_{\ell-i+1}=0$. The lower bounds follow by noting that the sum minimizes by taking the c_i maximal for odd indices and zero otherwise: when $k<\ell+1$,

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i c_i \rho_{\ell-i+1} &\geq \sum_{i=1}^{\lfloor (k+1)/2\rfloor} -\sigma_{\ell-2i+2}\,\rho_{\ell-2i+2}\\ &= \sum_{i=1}^{\lfloor (k+1)/2\rfloor}(\rho_{\ell-2i+3}-\rho_{\ell-2i+1}) = \rho_{\ell+1}-\rho_{\ell-2\lfloor (k+1)/2\rfloor+1}\end{aligned}$$

here we use that $s_i=\sigma_{\ell-i+1}$ and that $\rho_{i+1}\sigma_{i+1}=\rho_i-\rho_{i+2}$. The arguments are similar for the upper bounds and those when $k=\ell+1$. □

Corollary 3.12. *For the c_i as in Lemma 3.11, there are exactly two solutions to*

$$\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \equiv 0 \mod p.$$

Proof. By Lemma 3.11, $\left|\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1}\right| < 2p$, so we only need to consider solutions with

$$\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \in \{0, \pm p\}.$$

The last inequality in Lemma 3.11 implies that if there is a solution summing to $\pm p$ then there is not one summing to $\mp p$. Lemma 3.11 also gives that there is exactly one solution summing to zero. Note that choosing the c_i maximal gives

$$\sum_{i=0}^{\ell+1}(-1)^i c_i^{\max}\rho_{\ell-i+1} = s_0-1+\sum_{i=1}^{\ell}(-1)^i s_i \rho_{\ell-i+1}+(-1)^{\ell+1}(r_\ell-1)\rho_0=(-1)^{\ell+1}p.$$

This solution is necessarily unique; whenever $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1}=(-1)^{\ell+1}p$,

$$\sum_{i=1}^{\ell+1}(-1)^i (c_i^{\max}-c_i)\rho_{\ell-i+1}=0,$$

forcing each $c_i=c_i^{\max}$. Thus, there are exactly two solutions: $c_{\min}\equiv 0$ and $c^{\max}$. □

A Stein handle decomposition. Here we prove that each rational ball $A_{m,n}$ admits a Stein structure filling the universally tight contact structure on the lens space $\partial A_{m,n}$, thereby proving Theorem 1.2. By Proposition 3.4, it is sufficient to find *any* Stein handle decomposition giving $A_{m,n}$, as all such Stein structures will induce contact structures with three-dimensional homotopy invariant equal to $-\frac{1}{2}$.

The 2-handle attachment in $A_{m,n}$ defined by Yamada (Figure 2) is Legendrian. However, the 2-handle is attached via the zero framing when measured against the resulting contact framing. We prove that there exists an ambient isotopy within $S^1 \times S^2$ of the attaching circle to a different Legendrian isotopy class satisfying that the 2-handle is attached with framing one less than the contact framing induced from this new Legendrian embedding. To that end, we have the following proposition.

Proposition 3.13. *Each $A_{m,n}$ admits a Stein structure, $\tilde{J}_{m,n}$, specified by the Stein handle decomposition of Figure 8, where we assume $\{\rho_i\}_{i=-1}^{\ell+1}$ and $\{\sigma_i\}_{i=0}^{\ell}$ are as in Definition 2.3.*

This isotopy is performed in two steps. First the 2-handle is slid under the 1-handle (around a hemisphere of a $\{\mathrm{pt}\} \times S^2$) once, then the 2-handle is dragged over the 1-handle (winding in the $S^1 \times \{\mathrm{pt}\}$ direction) repeatedly to arrive at the desired Legendrian knot specified in Figure 8. Proposition 3.13, is proved inductively. To motivate the proof as well as set up the base cases for induction we first slide the 2-handle of $A_{m,n}$ once under the 1-handle as shown in the upper left of Figure 9. Referring to the portion of the attaching circle K passing behind the central plane of the two attaching balls of the 1-handle as the "bad" strand, we can pair off negative crossings in the bad strand with positive crossings in K by "unraveling" the 2-handle. To accomplish this, begin by dragging the bad strand once over the 1-handle (bottom of Figure 9). By dragging the bad strand another $\sigma_0 - 1$ times over the 1-handle we find the bad strand now involves $\rho_1 - 1$ strands rather than the original $\rho_{-1} - 1$ strands (upper right of Figure 9). In fact, if $\rho_1 = 1$, then we immediately have the Stein structure $(A_{m,n}, \tilde{J}_{m,n})$ of Proposition 3.13.

Remark 3.14. We cannot assume $\rho_1 = 1$. That said, the same principle holds far more generally; there exist isotopies of K taking the bad strand from involving $\rho_{2i-1} - 1$ strands to involving only $\rho_{2i+1} - 1$ strands. This is the content of the following proposition.

Proposition 3.15. *For each integer k such that $0 \leq 2k \leq \ell$, $A_{m,n}$ is specified by attaching a 2-handle with framing $mn + 2(m+n)$ along (the closure across the 1-handle of) the braid B_k defined in Figure 10.*

Proposition 3.15 immediately gives Proposition 3.13 in the case $\ell \in 2\mathbb{Z}$ since $\rho_{\ell+1} - 1 = 0$ and the central band vanishes at the ℓ-th stage. Proposition 3.15 is proved by investigating how long bands of blackboard parallel strands remain

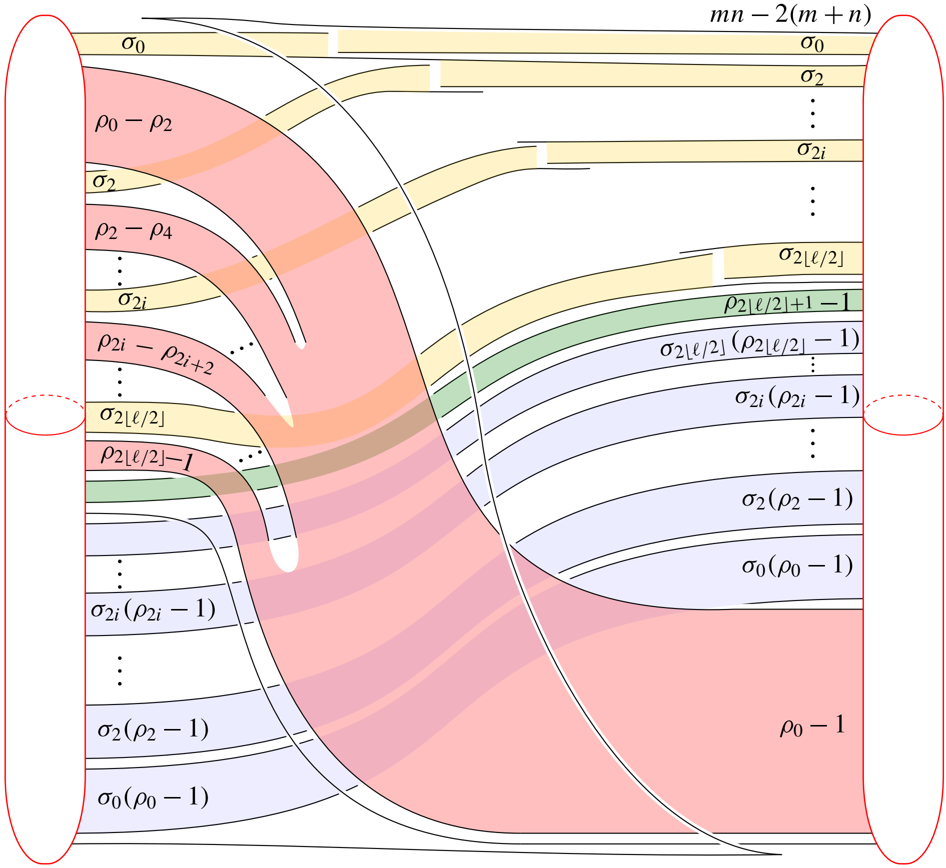

Figure 8. The Legendrian 2-handle attachment specifying the Stein structure $(A_{m,n}, \tilde{J}_{m,n})$. Here an integer superimposed on a given colored band indicates the number of blackboard parallel strands running within the band. *Warning: The vertical scaling is nonlinear and differs between the left and right foot of the* 1*-handle.*

together as they wrap around the braid B_k. To that end, we will denote the bands moving downward in B_k by D_i and those moving upward by U_i (as in Figure 10). Notice that we suppress the dependence on k for these bands since for each $i<k$, D_i (respectively U_i) persists for larger values of k. The only labeled band that changes when passing from B_k to B_{k+1} is D_k, which splits off D_{k+1}. Whereas, U_{k+1} consists of strands coming from the central band in B_k. With this notation in place we have the following lemma:

Lemma 3.16. *In the braid B_k, the D_i band returns to itself shifted down exactly ρ_{2i+1} strands and the U_i band returns to itself shifted up exactly $\rho_{2i}-1$ strands* (*e.g.*, *see Figure 11 for the case when* $k=0$).

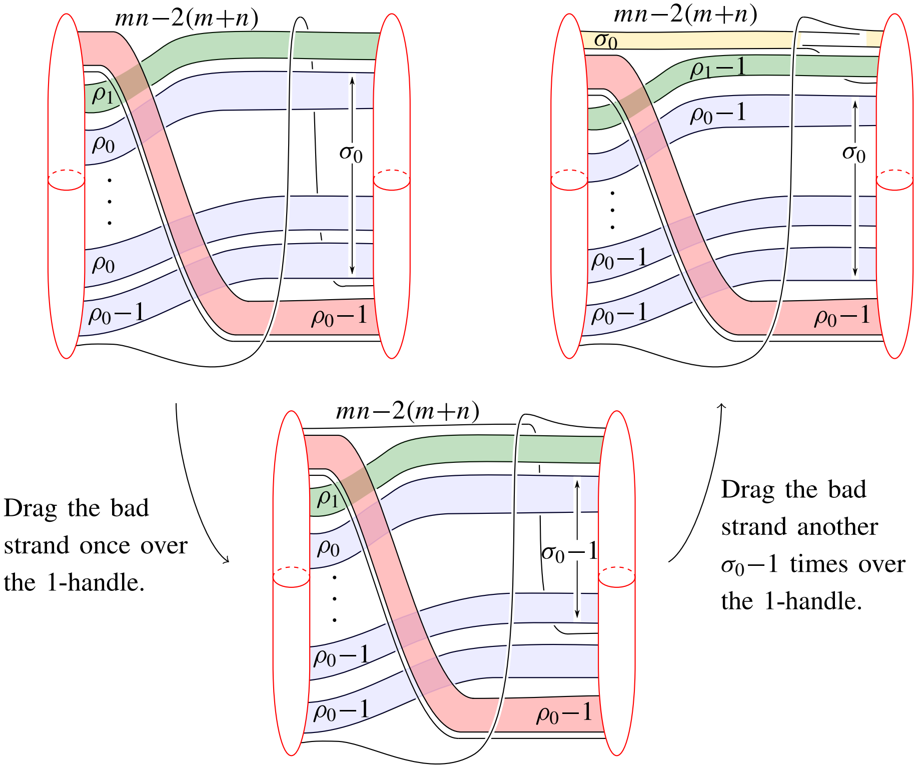

Figure 9. The result of sliding the attaching circle K once under the 1-handle, followed by isotopies of K as described.

Proof. We proceed by induction on k. The fact that the U_0 and D_0 bands return to themselves shifted up $\rho_0 - 1$ and down ρ_1 strands respectively is evident when looking at the closure of B_0 shown in Figure 11.

Suppose the result holds for each $0 \le i \le k-1$ in B_{k-1}. It is immediate that these shifts persist in B_k for each of the U_i and D_i bands provided $i < k$. Therefore, we only need to understand how the U_k and D_k bands return to themselves in B_k. We investigate how the U_k band returns first. To do this, we trace the U_k band as it enters and subsequently exits each of the D_i bands.

The key observation here is that the D_i band consists of a multiple of ρ_{2i+1} strands as $\rho_{2i} - \rho_{2i+2} = \sigma_{2i+1}\rho_{2i+1}$. When $i < k$, by induction, this is precisely the number of strands by which D_i shifts down when returning to itself. So the uppermost ρ_{2i+1} strands of D_i remain within D_i for a total of $\sigma_{2i+1} - 1$ returns before exiting directly below the D_i band entirely on the σ_{2i+1}-th return. We prove that the U_k band enters D_i within the uppermost ρ_{2i+1} strands. This is at least feasible since the U_k band has few enough strands to fit into uppermost ρ_{2i+1} strands of D_i as

$$\begin{aligned}\rho_{2i+1} &= \rho_{2k+1} + \sum_{j=i+1}^{k} (\rho_{2j-1} - \rho_{2j+1}) \\ &= \rho_{2k+1} + \sum_{j=i+1}^{k} \rho_{2j}\sigma_{2j} = \rho_{2k+1} + \sum_{j=i+1}^{k} (\sigma_{2j} + \sigma_{2j}(\rho_{2j} - 1)).\end{aligned}$$

When $i = 0$, we find that the U_k band indeed enters the D_0 band entirely within the uppermost ρ_1 strands as shown in the left side of Figure 12 (see also the upper right corner of Figure 10).

From above, we know that after σ_1 returns, these ρ_1 strands will have been shifted directly below the D_0 band. Of these ρ_1 strands, the uppermost σ_2 of them then pair off with those between the D_0 and D_1 bands and U_k is seen to enter the D_1 band within the first ρ_3 strands (e.g., see the center of Figure 12 taking $i = 1$). This process repeats and we find that for each $0 < i < k$, the U_k band enters the D_i

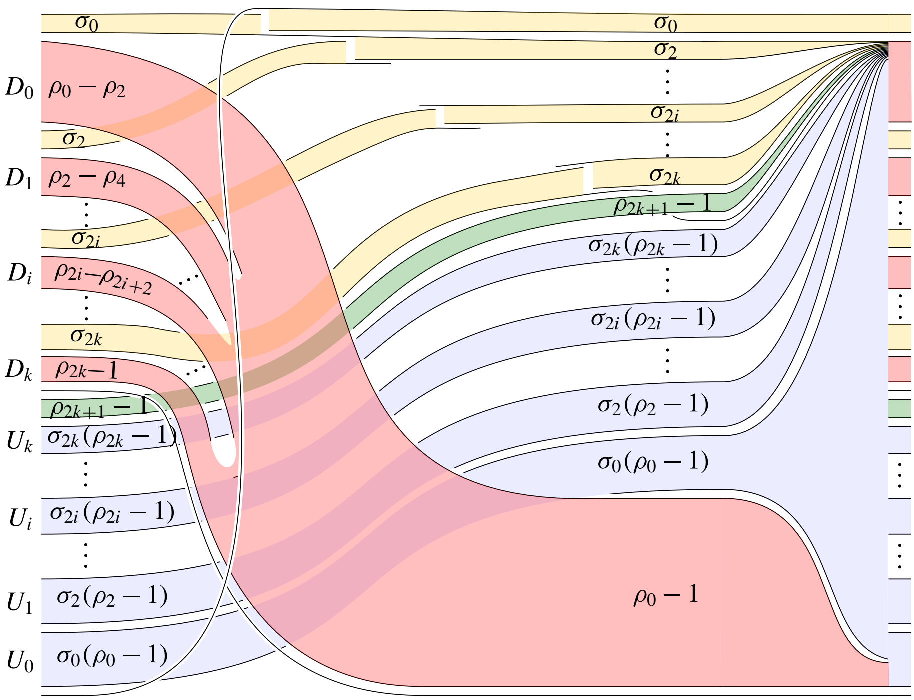

Figure 10. The braid B_k: Isotoping away the “bad strand” of the attaching circle for the 2-handle in $A_{m,n}$. The bands labeled D_i and U_i are those described in Lemma 3.16. *Warning: the* 1*-handle of* $A_{m,n}$ *has been suppressed and the vertical scaling is nonlinear.*

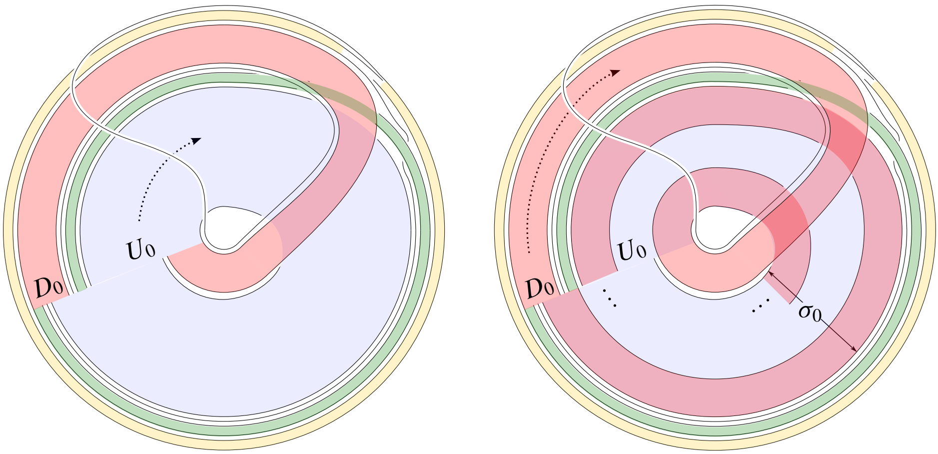

Figure 11. Left: The U_0 band in the braid B_0 returns to itself shifted up by $\rho_0 - 1$ the number of strands in the D_0 band. Right: The D_0 band in the braid B_0 returns to itself shifted down ρ_1 the number of strands in the central band.

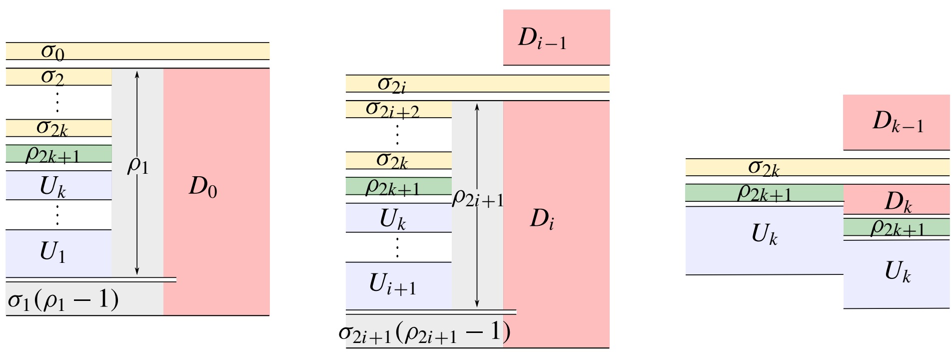

Figure 12. Left: The U_k band entering the D_0 band. Center: The U_k band entering the D_i band for $0 < i < k$. Right: The U_k band meeting the D_k band. Notice that the U_k band has returned to itself shifted up by exactly $\rho_{2k} - 1$ strands — the number of strands in the D_k band.

band as in the center of Figure 12. Therefore, we see that in B_k the strands in the U_k band remain blackboard parallel through each of the D_i bands for $i < k$. When the U_k band exits the D_{k-1} band, the U_k has returned to itself shifted up by the number of strands in the D_k band, that is, up by exactly $\rho_{2k} - 1$ strands, as claimed (see the right side of Figure 12).

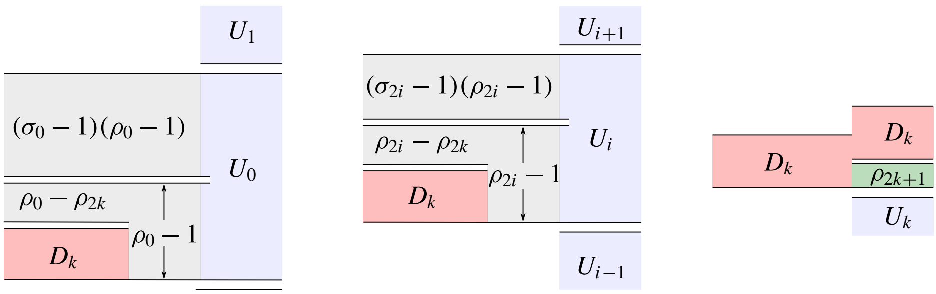

Figure 13. Left: The D_k band entering the U_0 band. Center: The D_k band entering the U_i band for $0 < i < k$. Right: The D_k band returning to the D_k band. Notice that the D_k band has shifted down by exactly ρ_{2k+1} strands.

Knowing that within the braid B_k, each U_i band returns to itself shifted up by exactly $\rho_{2i}-1$ strands, for each i less than or *equal* to k, now allows us to show that D_k returns to itself shifted down ρ_{2k+1} strands. The approach is the same as above; we make use of the fact that the number of strands in the U_i band is a multiple of the number of strands by which the U_i band shifts up when first returning to itself within B_k. Our induction hypothesis then ensures that the lower most $\rho_{2i}-1$ strands in U_i can be shifted up and out of U_i to the $\rho_{2i}-1$ strands above.

We follow the D_k band as it enters and exits each of the U_i bands. First, notice that the D_k band enters the U_0 band as the lowermost $\rho_{2k}-1$ strands as in the right side of Figure 13 (see also the lower right corner of Figure 10).

By induction, we know that when tracing the U_0 band as it returns to itself, the lowermost ρ_0-1 strands are shifted up by ρ_0-1 strands. So D_k enters U_0 a second time shifted up by ρ_0-1 strands. This process repeats a total of σ_0 times before D_k exits U_0 and enters U_1 as the lowermost $\rho_{2k}-1$ strands. Continuing by induction, for each $0 < i \leq k$, we find that D_k enters U_i as in center of Figure 13. From above, we know that the U_k band returns to itself shifted up $\rho_{2k}-1$ strands, so the D_k band continues through the U_k band σ_{2k} times before exiting directly above the U_k band (right side of Figure 13). At this point, D_k has come back to itself shifted down by ρ_{2k+1} strands, giving the result. $\square$

Proof of Proposition 3.15. We proceed by induction on k. Figure 9 gives the case when $k=0$. Suppose K has been isotoped to B_k for some k with $2k < \ell-2$. We view the "bad" strand as a tangle on ρ_{2k+1} strands. We begin to push this tangle over the 1-handle repeatedly. Notice that anytime the bad strand enters D_i, Lemma 3.16 ensures that it can be moved down ρ_{2i+1} strands. The bad strand initially enters the D_0 band within the uppermost ρ_1 strands (see the upper left of Figure 14).

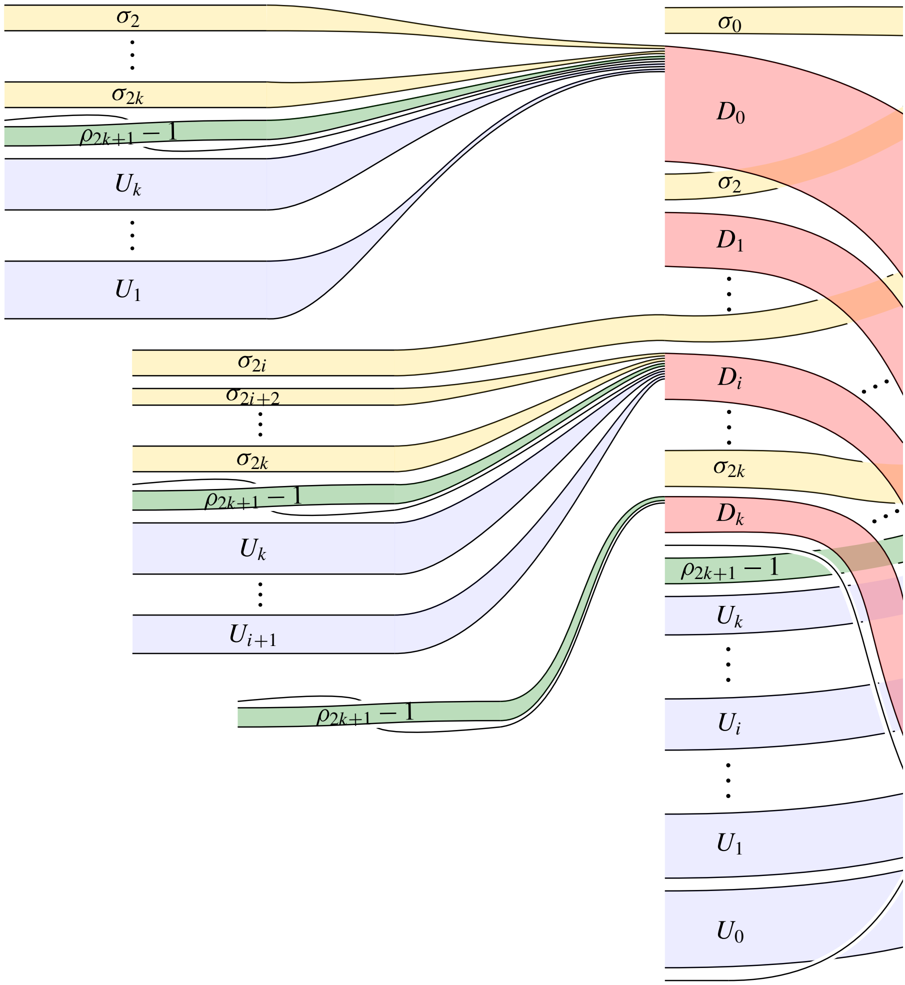

Figure 14. Pushing the bad strand into D_0 (upper left). Repeated application of Lemma 3.16 proves that the bad strand can be pushed into each D_i for $i < k$ (center left) and for $i = k$ (bottom left).

Applying Lemma 3.16, σ_1 times to the D_0 band shows that the bad strand can be isotoped (by pushing it along the blackboard parallel strands of D_0) into the uppermost ρ_3 strands of the D_2 band. By applying Lemma 3.16 to each D_j band, we can position the bad strand within the uppermost ρ_{2i+1} strands of the D_i band (see Figure 14) for each $i \leq k$.

As D_k consists of $\rho_{2k}-1 = \rho_{2k+1}\sigma_{2k+1}+\rho_{2k+2}-1$ strands, applying Lemma 3.16 to D_k, we can move the bad tangle down a total of σ_{2k+1} times before it begins to leave D_k. At this point, we find that the bad strand only involves $\rho_{2k+1}-(\rho_{2k+2}-1)$ strands. This occurs at the expense of splitting the lowermost $\rho_{2k+2}-1$ strands

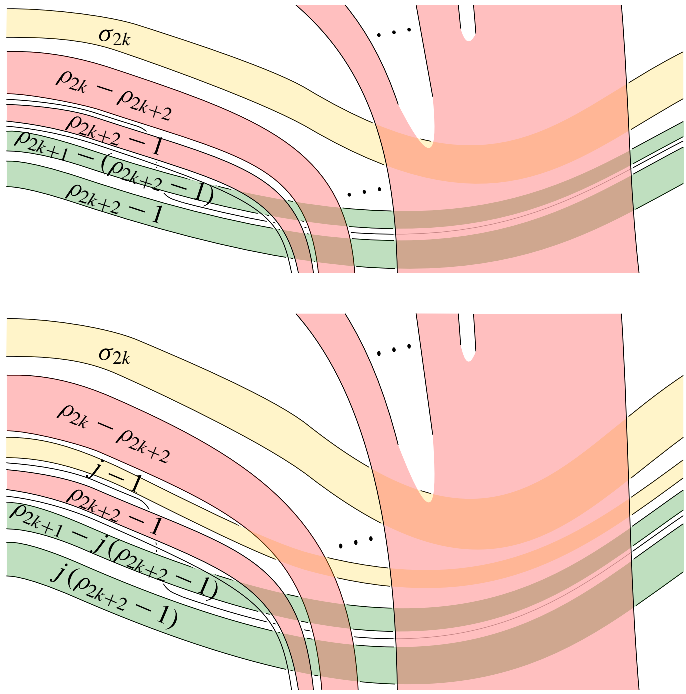

Figure 15. Top: The result of pushing the bad strand once through each of the D_i bands. Bottom: The result of pushing the bad strand j-times through each of the D_i bands. When $j = \sigma_{2k+2}$, we have the completed the isotopy from B_k to B_{k+1} claimed in Proposition 3.15.

from D_k, thereby forming what will be the D_{k+1} band within the braid B_{k+1} (top of Figure 15).

This process is repeated, each time the bad strand involving $\rho_{2k+2} - 1$ fewer strands. Repeating the process j times results in the bottom of Figure 15. Taking $j = \sigma_{2k+2}$ then gives B_{k+1}. □

Proof of Proposition 3.13. By Proposition 3.15, the 2-handle attachment of Figure 8 is isotopic to the 2-handle attachment defined by Yamada (Figure 2). Indeed if $\ell \in 2\mathbb{Z}$ then B_ℓ agrees with Figure 8. When $\ell \in 2\mathbb{Z} + 1$, one applies the induction step of Proposition 3.15 a final time to arrive at Figure 8. Therefore, Figure 8 specifies $A_{m,n}$.

Moreover, as each isotopy from B_k to B_{k+1} is writhe preserving. The writhe of B_k is that of B_0 which equals $mn - 2(m + n) + 2$. Therefore, the 2-handle attachment of Theorem 1.2 is Stein since K's induced contact framing is

$$\text{writhe}(K) - \#(\text{left cusps}) = (mn - 2(m + n) + 2) - 1.$$

Eliashberg's characterization of handle decompositions of Stein domains [Eliashberg 1990; Gompf 1998] then gives that $A_{m,n}$ is realized as a Stein domain. □

Proof of Theorem 1.2. The fact that $(\partial A_{m,n}, \xi_{\tilde{J}_{m,n}})$ is contactomorphic to the universally tight lens space $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$ follows by noting that any almost complex structure on the rational ball $A_{m,n}$ (indeed any rational ball) satisfies

$$\frac{c_1^2(A_{m,n}, J) - 2\chi(A_{m,n}) - 3\sigma(A_{m,n})}{4} = -\frac{1}{2},$$

thus $d_3(\xi_{\tilde{J}_{m,n}}) = -\frac{1}{2}$. By Proposition 3.4, $\xi_{\tilde{J}_{m,n}}$ is universally tight. Since $(A_{m,n}, \tilde{J}_{m,n})$ gives a symplectic filling of the space $(L(p^2, pq-1), \bar{\xi}_{\text{st}})$, Lisca's classification then gives that $A_{m,n} \approx B_{p,q}$. □

4. Boundary diffeomorphisms

From here, we pursue a handle-theoretic approach to understanding the diffeomorphisms $B_{p,q} \approx A_{m,n}$ ensured by Theorem 1.2. To that end, we define maps from $\partial B_{p,q}$ and $\partial A_{m,n}$ to the same linear plumbing of S^1-bundles.

It is worth noting that such diffeomorphisms have been known previously. Yamada [2007] produces similar diffeomorphisms from $\partial A_{m,n}$ to $L(p^2, pq-1)$ expressed as the boundary of $\mathcal{C}_{p,q}$. To accomplish this, one must carefully keep track of every stage of the Euclidean algorithm applied to $(p-q, q) = 1$. We perform a courser bookkeeping of the Euclidean algorithm via Definition 2.3, which allows for arguably clearer definitions. However, we do this at the expense of arriving at the plumbing of Proposition 2.5 rather than $\mathcal{C}_{p,q}$. This approach has the added advantage of applying to $\partial B_{p,q}$ in a structurally similar way.

Composing these maps gives a diffeomorphism from $\partial B_{p,q}$ to $\partial A_{m,n}$ that can be seen as a restriction of a diffeomorphism between the 4-manifolds $B_{p,q}$ and $A_{m,n}$ through carving, introduced by Akbulut [1977]; see also [Akbulut 2016]. By doing so, we will prove Theorem 1.3 as well as Corollary 1.4. For convenience we briefly outline the carving procedure.

Carving 4-manifolds. Suppose we have two 4-manifolds X and X' and a diffeomorphism $f : \partial X \to \partial X'$ where X admits a handle decomposition consisting of a single 0-handle, k 1-handles, and N 2-handles, where the i-th 2-handle h_i is attached along a knot K_i in $\#k(S^1 \times S^2)$. Let μ_i denote the belt-sphere of h_i (i.e., a meridian of K_i).

If f extends to a diffeomorphism between X and X', then in particular it extends across a neighborhood of the collection of cocores of the 2-handles in X. Thus, a necessary condition for f to extend is the property that the image of the belt-spheres $f(\mu_1) \cup \cdots \cup f(\mu_N)$ must be a slice link in $\partial X'$. That is, there exists a collection of properly embedded disks $D_i \subset X'$ such that $D_i \cap D_j = \varnothing$ and $\partial D_i = f(\mu_i)$.

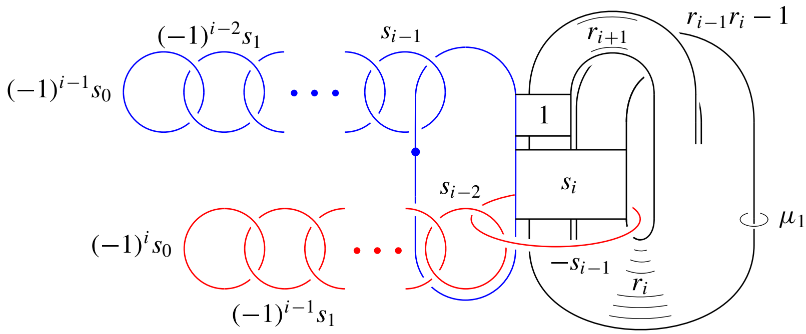

Figure 16. The 4-manifold $B^i_{p,q}$

Assuming this, if f carries the 0-framing of each μ_i (induced by the cocore) to the framing of $f(\mu_i)$ induced by the slice disk, then f extends across the neighborhoods of the cocores of the 2-handles in X. In order to extend f across the rest of X, we are left needing to extend a map $f_0 : \#k(S^1 \times S^2) \to \#k(S^1 \times S^2)$. Laudenbach and Poenaru [1972] prove that every self diffeomorphism of $\partial(\natural k(S^1 \times B^3))$ extends. Therefore, f_0 extends provided that

$$X' - \nu(D_1 \cup \cdots \cup D_N) \approx \natural k(S^1 \times B^3)$$

as obviously removing neighborhoods of the cocores of the 2-handles in X gives $\natural k(S^1 \times B^3)$.

Boundary diffeomorphisms: $\partial B_{p,q}$. The key observation to build such maps is that if $p = qs + r$, then $\partial B_{p,q}$ is obtained from $\partial B_{q,r}$ via integral surgeries on two unknotted circles. The boundary maps that we are after are obtained by iterating this process. As we define these maps, we trace the belt-sphere of the single 2-handle of $B_{p,q}$.

Proposition 4.1. *Let $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ be as defined in Definition 2.3. Then for each $i \in \{0, \ldots, \ell+1\}$, $\partial B_{p,q} \approx \partial B^i_{p,q}$ where $B^i_{p,q}$ is the 4-manifold specified by Figure 16.*

Proof. We induct on i. When $i = 0$, the result is immediate since $B^0_{p,q} \approx B_{p,q}$. Therefore, the proposition holds provided that $\partial B^i_{p,q} \approx \partial B^{i+1}_{p,q}$. Let K^i_1 be the attaching circle of the $r_{i-1}r_i - 1$-framed 2-handle in $B^i_{p,q}$. Suppose the result holds for some $i \le \ell$. For $i+1$, first, surger the single 1-handle and introduce a canceling pair of 1- and 2-handles to remove the s_i-full twists between K^i_1 and the, now surgered, 1-handle (Figure 17).

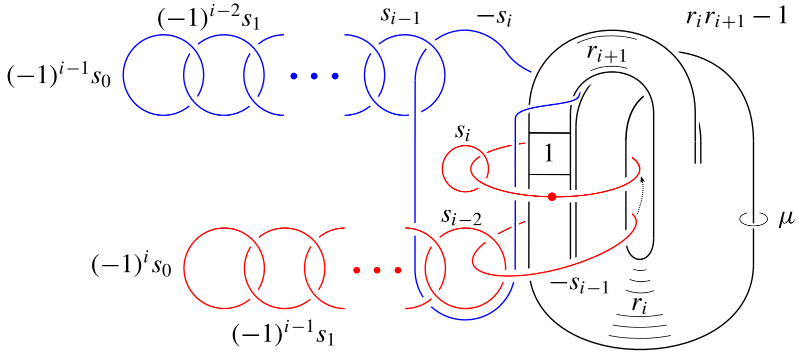

Figure 17. Introducing a canceling pair after surgery.

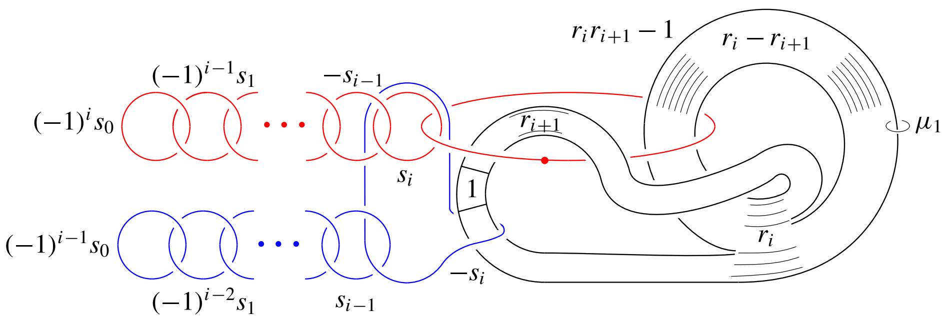

Figure 18. Isotoping K_1^i.

Since K_1^i links the new 1-handle r_i times, the framing on K_1^i decreases by $s_i r_i^2$ and the new framing on K_1^i is

$$r_{i-1}r_i - 1 - s_i r_i^2 = r_i(r_{i-1} - s_i r_i) - 1 = r_i r_{i+1} - 1.$$

Sliding the $-s_{i-1}$-framed 2-handle under the new 1-handle as indicated in Figure 17, and isotoping the r_{i+1}-stranded band (see Figure 18), we find that the r_{i+1}-stranded band traverses the 1-handle (positively) s_{i+1}-times as a complete band, while r_{i+2} strands traverse an additional one time to make up the complete $s_{i+1}r_{i+1} + r_{i+2} = r_i$ linking. With this view in mind, we isotope K_1^i into a closed braid on r_{i+1} strands appropriately linking the carving disk of the 1-handle; see Figure 19. The result holds by induction. $\square$

Remark 4.2. At no point does μ_1, the meridian of K_1^i, get damaged under the boundary diffeomorphisms defined in Proposition 4.1. In particular, for each i, μ_1 bounds a disk in $B_{p,q}^i$ and the image of a collar neighborhood of μ_1 arising from such a disk persists under the boundary diffeomorphisms defined above. So, each diffeomorphism preserves the 0-framing on μ_1.

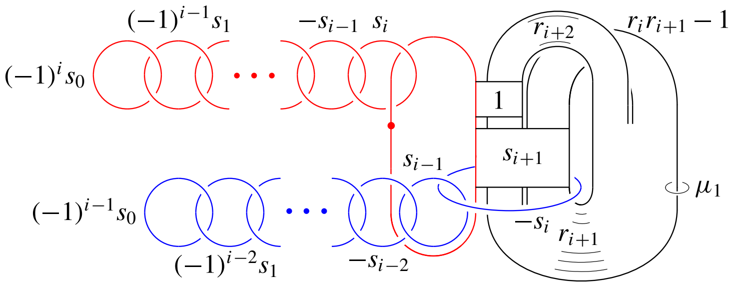

Figure 19. Further isotopy of K_1^i to K_1^{i+1}

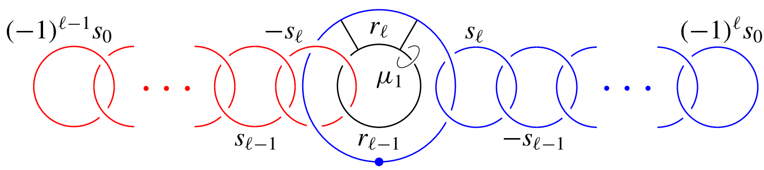

Figure 20. The space $B_{p,q}^{\ell+1}$.

Since $r_{\ell+1} = 1$ and $r_{\ell+2} = 0$, by definition, $s_{\ell+1} = s_{\ell+1}r_{\ell+1} + r_{\ell+2} = r_\ell$. By looking at $B_{p,q}^{\ell+1}$ we arrive at the following result of Casson and Harer [1981].

Corollary 4.3. $\partial B_{p,q} \approx L(p^2, pq-1)$.

Proof. By Proposition 4.1, we have that $\partial B_{p,q} \approx \partial B_{p,q}^{\ell+1}$ (Figure 20). The boundary diffeomorphism from $\partial B_{p,q}^{\ell+1}$ to a linear plumbing of S^1-bundles over S^2 is contained in Figure 21. □

Remark 4.4. It is an easy exercise to verify that the linear plumbing in Figure 21 bounds $L(p^2, pq-1)$. Indeed, one finds that

$$[-s_0, s_1, \dots, \pm r_\ell, 1, \mp r_\ell, \dots, -s_1, s_0] = -\frac{p^2}{pq-1}.$$

Boundary Diffeomorphisms: $\partial A_{m,n}$. As in the previous section, we exhibit explicit diffeomorphisms, this time from $\partial A_{m,n}$ to $L(p^2, pq-1)$. As the image of μ_1 is given as the 0-framed push-off of the attaching circle of the central 1-framed unknot at the bottom of Figure 21. We will trace where the curve, γ in Figure 3, goes as well — finding that it too goes to the 0-framed push-off of the central 1-framed unknot via an appropriately defined diffeomorphism. We want to define these diffeomorphisms similarly to those of Proposition 4.1.

Lemma 4.5. *$A_{m,n}$ is given by Figure 22.*

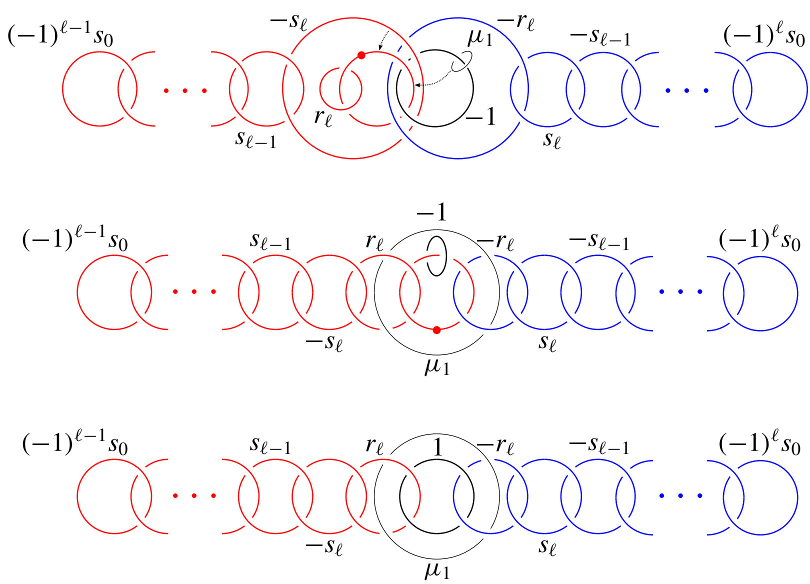

Figure 21. From top to bottom: The introduction of a canceling pair to $B^{\ell+1}_{p,q}$ after surgery; the result of the indicated slides; a linear plumbing associated to $\partial B_{p,q}$.

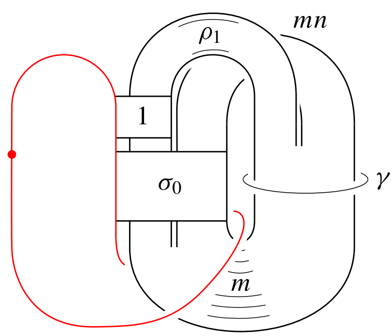

Figure 22. An alternative description of $A_{m,n}$.

Proof. The result follows from an isotopy of the 2-handle's attaching circle. First, view the $m+n$ strands of the attaching circle in Figure 2 as a band of n strands going over the 1-handle once with the remaining m strands going over twice (left side of Figure 23). Viewing the band of m strands going over the 1-handle completely σ_0 times with ρ_1 strands traversing an extra time (right side of Figure 23) gives the result. □

Using Lemma 4.5, we prove the analog of Proposition 4.1 in the $\partial A_{m,n}$ case.

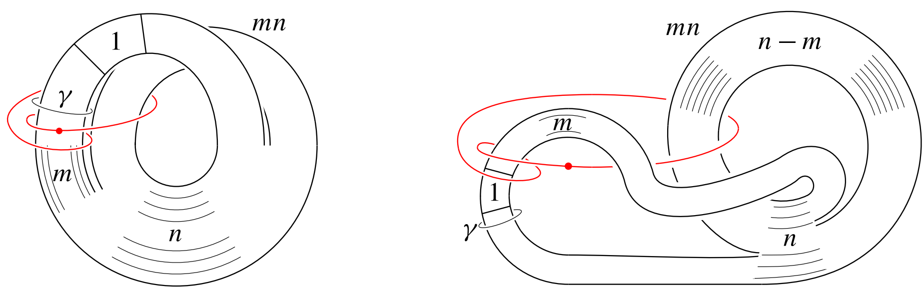

Figure 23. The isotopy of the 2-handle in $A_{m,n}$ used in the proof of Lemma 4.5.

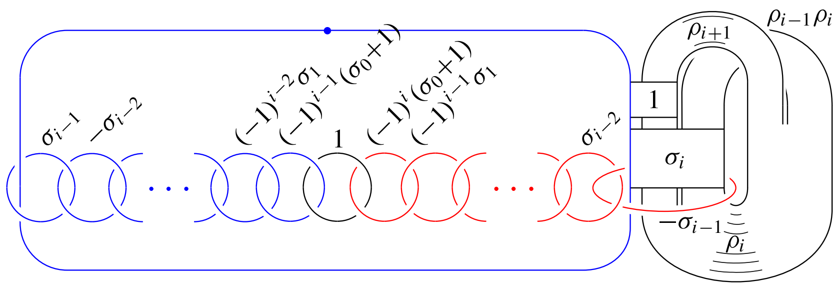

Figure 24. The 4-manifold $A^i_{m,n}$

Proposition 4.6. *Let* $\{\rho_i\}_{i=-1}^{\ell+2}$ *and* $\{\sigma_i\}_{i=0}^{\ell+1}$ *be as defined in Definition 2.3 (associated to* $n > m \geq 1$*). Then for each* $i \in \{0, \dots, \ell+1\}$,

$$A_{m,n} \overset{\partial}{\approx} A^i_{m,n}$$

where $A^i_{m,n}$ *is the 4-manifold given by Figure 24.*

Proof. We induct on i, treating the base case and the induction step simultaneously. For the base case, start with the handle decomposition from Lemma 4.5. For the induction step, suppose that the result holds for some $i \leq \ell$. Let K^i_1 be the attaching circle of the $\rho_{i-1}\rho_i$-framed 2-handle in $A^i_{m,n}$. Surger the 1-handle and introduce a canceling 1- and 2-handle (for the base case see the left side of Figure 25, for the induction step see Figure 26). Notice, similar to Proposition 4.1 the framing of K^i_1 changes from $\rho_{i-1}\rho_i$ to $\rho_i\rho_{i+1}$.

Slide the now surgered 1-handle as indicated in the respective figures and, for the base case, blow-up once (right side of Figure 25). From here the base case follows similarly to the induction step; both of which are similar to Proposition 4.1. Indeed, isotope K^i_1 to view a band with ρ_{i+1} stands traversing the 1-handle σ_{i+1}-times along with ρ_{i+2} of those strands traversing an extra time as in Figure 27.

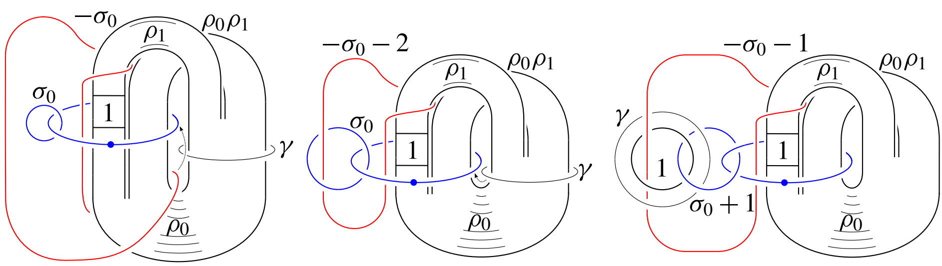

Figure 25. The base case of Proposition 4.6.

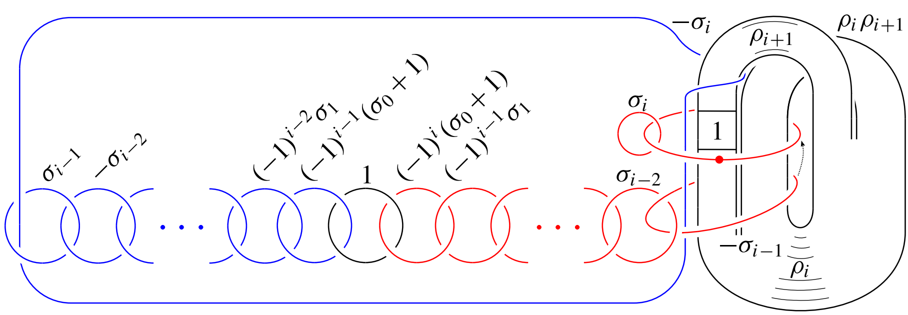

Figure 26. Introducing a canceling pair.

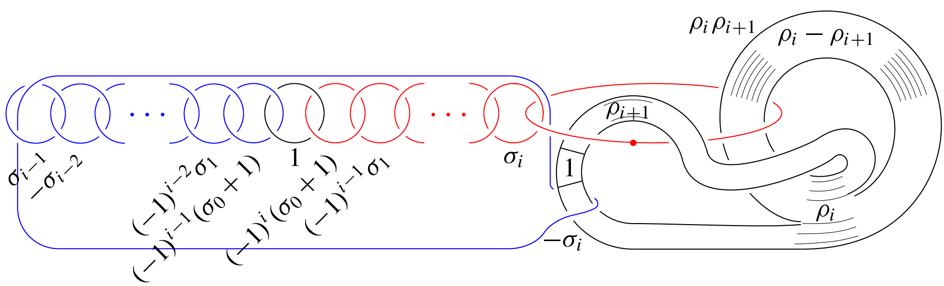

Figure 27. Isotoping K_1^i in $A_{m,n}^i$.

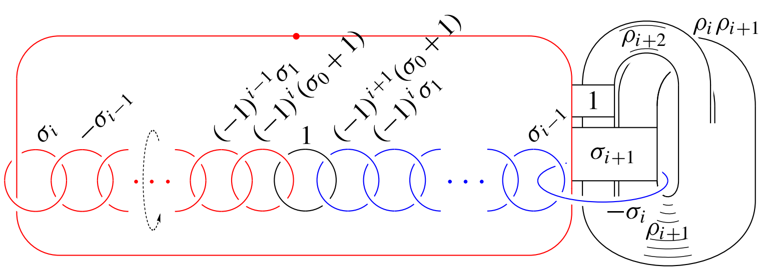

Figure 28. Further isotopy of K_1^i to K_1^{i+1} in $A_{m,n}^{i+1}$.

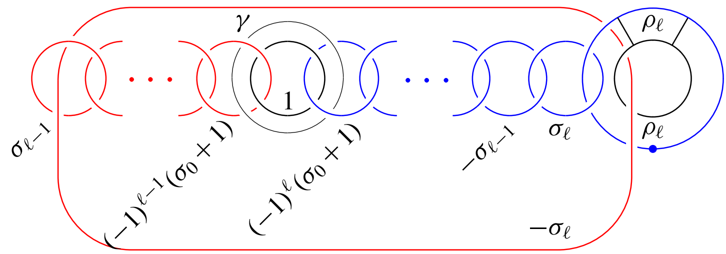

Figure 29. The space $A^{\ell+1}_{m,n}$.

A further isotopy of K^i_1 gives a closed braid on ρ_{i+1} strands geometrically linking the carving disk of the new 1-handle ρ_i-times. Finally, notice that to get the appropriate linking on the chain of unknots, we have to wind the chain (as indicated in Figure 28) to add a total of i positive half-twists to the left of the disk bundle of Euler class 1 along with i negative half-twists to the right. The result follows by induction. □

Corollary 4.7 [Yamada 2007, Theorem 1.1]. *$\partial A_{m,n} \approx L(p^2, pq-1)$ for $(p-q, q) = A(m, n)$.*

Proof. By Proposition 4.6, $\partial A_{m,n} \approx \partial A^{\ell+1}_{m,n}$; see Figure 29. We proceed as in Corollary 4.3. The boundary diffeomorphism from $\partial A^{\ell+1}_{m,n}$ to a linear plumbing of S^1-bundles over S^2 is contained in Figure 30. □

Remark 4.8. The fact that $\partial A_{m,n}$ is $L(p^2, pq-1)$ for $A(m, n) = (p-q, q)$ follows by noting that given p and q, or equivalently m and n, we can define the other pair by an appropriate identification of the linear plumbings in Corollaries 4.3 and 4.7, provided that $s_0 > 1$ (that is, provided that $p-q > q$, which we have assumed all along). In fact, as we have chosen to do in Remark 2.4, this can be taken as the definition of the function A defined by Yamada [2007]. Notice also that γ bounds a disk in each $\partial A^i_{m,n}$ as well as in the linear plumbing of Figure 30. Furthermore, each boundary diffeomorphism defined in Proposition 4.6 and those of Corollary 4.7 preserve the 0-framing of γ specified by those disks. Therefore, we can employ the carving method provided that carving along γ gives $S^1 \times B^3$, which it does:

Proposition 4.9 Proof of Corollary 1.4. *Carving $A_{m,n}$ along γ gives $S^1 \times B^3$.*

Proof. Carving $A_{m,n}$ along the curve γ means removing a neighborhood of the disk γ bounds inside $A_{m,n}$. The resulting handlebody decomposition is given by that of $A_{m,n}$ along with an extra 1-handle whose carving disk is γ. If we let γ_i be the analogous curve in A_{ρ_{i-1},ρ_i}, then the result of carving A_{ρ_{i-1},ρ_i} along γ_i is given in Figure 31.

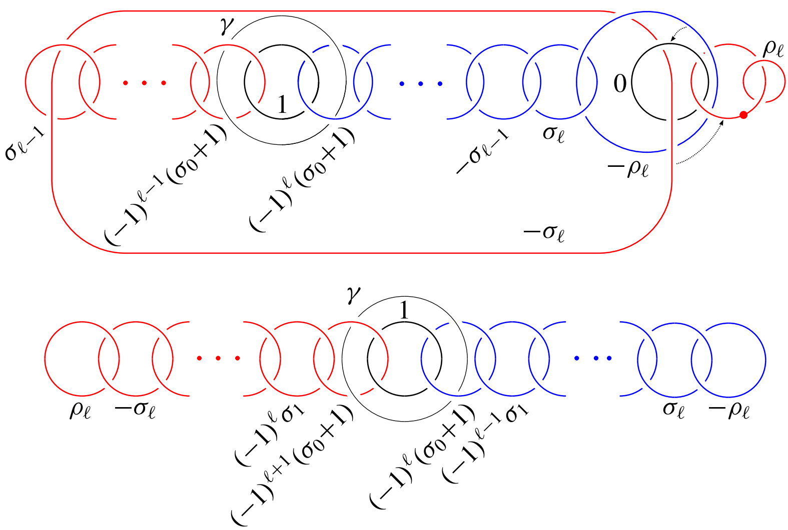

Figure 30. The result of surgering $A^{\ell+1}_{m,n}$ and introducing a canceling pair; the result of sliding and canceling as indicated gives a linear plumbing associated to $\partial A_{m,n}$.

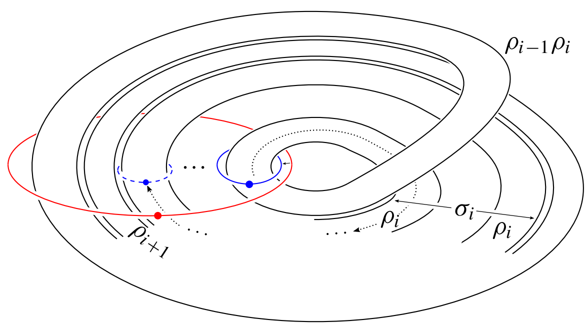

Figure 31. A_{ρ_{i-1},ρ_i} carved along γ_i.

Notice that $A_{m,n} = A_{\rho_0,\rho_{-1}}$ and $\gamma = \gamma_0$. By sliding the original 1-handle across the newly carved 1-handle σ_i times, twisting the 1-handle σ_i-times (negatively) and finally sliding as indicated in the left side of Figure 32 we arrive at $A_{\rho_i,\rho_{i+1}}$ carved along γ_{i+1} (right side of Figure 32). Therefore, the result of carving along γ_i in A_{ρ_{i-1},ρ_i} is diffeomorphic to carving along γ_{i+1} in $A_{\rho_i,\rho_{i+1}}$. As carving A_{1,ρ_ℓ} along γ_ℓ gives $S^1 \times B^3$ we have the result. $\square$

Proof of Theorem 1.3. As $A(p-q,q) = (m,n)$, we can identify the plumbings of Figures 21 and 30. By first, applying the diffeomorphisms of Proposition 4.1 we get a diffeomorphism from $\partial B_{p,q}$ to the boundary of the linear plumbing of the

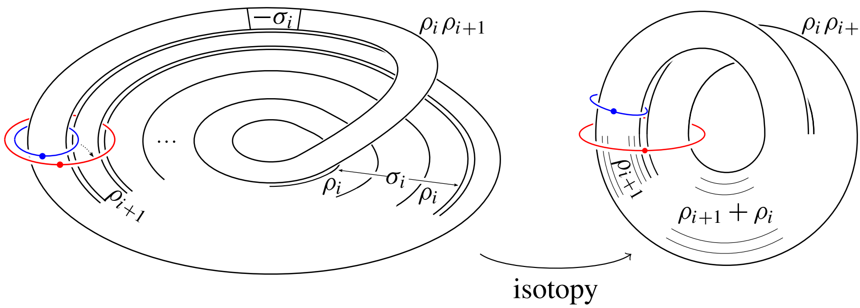

Figure 32. A_{ρ_{i-1},ρ_i} carved along γ_i after sliding and twisting σ_i-times.

bottom of Figure 21 carrying μ_1 as indicated. Applying the diffeomorphisms of Proposition 4.6 in reverse from the boundary of the linear plumbing of Figure 30 to $A_{m,n}$ gives the required diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$. □

Acknowledgments

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ON CERTAIN FOURIER COEFFICIENTS OF EISENSTEIN SERIES ON G_2

WEI XIONG

We compute certain Fourier coefficients of Eisenstein series on the split simple exceptional group G_2, and the result is a product of zeta functions and a finite product of local integrals. The method is via exceptional theta correspondence for $G_2 \times \mathrm{PGL}_3$.

1. Introduction

Let F be a number field with adele ring $\mathbb{A}$. Consider the split simple exceptional group G_2 over F. Let $P = M \cdot N$ be the maximal Heisenberg parabolic subgroup of G_2 associated to the short simple root. Then the Levi subgroup $M \cong \mathrm{GL}_2$ and the unipotent radical N is five-dimensional. For $s \in \mathbb{C}$, let $I_P(s) = \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})} |\det|^{s+3/2}$ (unnormalized induction). For $\Phi_s \in I_P(s)$, consider the Eisenstein series on $G_2(\mathbb{A})$ defined for $\mathrm{Re}(s) \gg 0$ by

$$E(\Phi_s, g) = \sum_{\gamma \in P(F)\backslash G_2(F)} \Phi_s(\gamma g), \quad \text{for all } g \in G_2(\mathbb{A}).$$

If Φ_s is holomorphic, then the Eisenstein series $E(\Phi_s, g)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to $\mathbb{C}$.

MSC2010: 11F27, 11F30.
Keywords: exceptional group, Eisenstein series, Fourier coefficients, minimal representations, exceptional theta correspondence, higher-dimensional Hensel's lemma.

Consider the Fourier coefficient of the Eisenstein series $E(\Phi_s, g)$ with respect to a character $\chi : N(F) \setminus N(\mathbb{A}) \to \mathbb{C}^\times$, which is given by

$$E_\chi(\Phi_s, g) = \int_{N(F)\setminus N(\mathbb{A})} E(\Phi_s, ng)\overline{\chi(n)}\, dn.$$

These Fourier coefficients are interesting objects of study. For example, Dihua Jiang and Stephen Rallis [1997] showed these Fourier coefficients are essentially quotients of Dedekind zeta functions of number fields, under the assumption that the base field F contains the third root of unity. Our goal is to obtain similar results without this assumption. In the following, we give more detailed description.

Write an element in N as $n(x, y, z, u, v)$, where $x, y, z, u, v \in F$. The center Z of N is given by $Z = \{n(0, 0, z, 0, 0) : z \in F\}$. In this paper, we restrict our attention to the following type of characters of N. Fix a nontrivial character $\psi : F \setminus \mathbb{A} \to \mathbb{C}^\times$. Let $b, c \in F$ be such that the cubic polynomial $x^3 + bx - c$ is irreducible over F. Let $\sigma = (1, 0, b, c) \in F^4$, and define a character $\psi_\sigma : N(F) \setminus N(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\psi_\sigma(n(x, y, z, u, v)) = \psi(x + bu + cv).$$

Note that ψ_σ is trivial on $Z(\mathbb{A})$. See [Jiang and Rallis 1997, §2.4] and [Wright 1985, §2] for more details.

Jiang and Rallis [1997] studied the Fourier coefficients of $E(\Phi_s, 1)$ with respect to the character ψ_σ for a standard decomposable section Φ_s. They first showed that if $\Phi_s = \otimes\Phi_{s,v}$ is decomposable, then $E_{\psi_\sigma}(\Phi_s, 1)$ is Eulerian:

$$E_{\psi_\sigma}(\Phi_s, 1) = \prod_v E_{\psi_\sigma,v}(\Phi_{s,v}, 1),$$

where

$$E_{\psi_\sigma,v}(\Phi_{s,v}, 1) = \int_{N(F_v)} \Phi_{s,v}(wn)\psi_{\sigma,v}(n)\, dn,$$

and w is an appropriate Weyl element.

Then Jiang and Rallis evaluated the integrals $E_{\psi_\sigma,v}(\Phi_{s,v}, 1)$ in the unramified case directly, with the help of [Igusa 1988, Lemma 6]. Let E be a cubic field extension of F generated by one of the roots of the polynomial $x^3 + bx - c$. Assuming that F contains the third root of unity, they obtained, after a rather long and complicated computation, that in the unramified case,

$$E_{\psi_\sigma,v}(\Phi_{s,v}, 1) = \frac{\zeta_{E_v}\big(s + \frac{1}{2}\big)}{\zeta_v\big(s + \frac{1}{2}\big)\zeta_v\big(s + \frac{3}{2}\big)\zeta_v(2s + 1)\zeta_v\big(3s + \frac{3}{2}\big)},$$

where $\zeta_v(s)$ and $\zeta_{E_v}(s)$ are local zeta factors (see [Jiang and Rallis 1997, §5]). Therefore, for a pure tensor $\Phi = \otimes\Phi_v$, the Fourier coefficient $E_{\psi_\sigma}(\Phi_s, 1)$ is of the following form:

$$E_{\psi_\sigma}(\Phi_s, 1) = \frac{\zeta_E\big(s+\frac{1}{2}\big)}{\zeta_F\big(s+\frac{1}{2}\big)\zeta_F\big(s+\frac{3}{2}\big)\zeta_F(2s+1)\zeta_F\big(3s+\frac{3}{2}\big)} \prod_v E^*_{\psi_\sigma,v}(\Phi_{s,v}, 1),$$

where $\zeta_F(s)$ (resp. $\zeta_E(s)$) is the complete zeta function of F (resp. of E), and the local factors $E^*_{\psi_\sigma,v}(\Phi_{s,v}, 1)$ is equal to 1 for almost all v. See [Jiang and Rallis 1997, Theorem 2 (4)].

The assumption that the base field F contains the third root of unity seems to be nonessential, as observed by Jiang and Rallis [1997]. Note that Gan, Gross and Savin calculated the Fourier coefficients for Eisenstein series on $G_2(\mathbb{Z})$ in [Gan et al. 2002, §9], assuming an extension of Jiang and Rallis's local result.

The purpose of this paper is to try to remove this assumption, and the method of proof, which was suggested to the author by Wee Teck Gan, is to reduce the computation to local integrals over PGL_3 via exceptional theta correspondence for the dual pair (G_2, PGL_3). The main result of this paper is as follows:

Theorem 1.1. *Keep the notation above. If $\Phi_s = \otimes\Phi_{s,v}$ is a pure tensor, then the Fourier coefficient $E_{\psi_\sigma}(\Phi_s, 1)$ is of the following form*:

$$E_{\psi_\sigma}(\Phi_s, 1) = \frac{\zeta_E\big(s+\frac{1}{2}\big)}{\zeta_F\big(s+\frac{1}{2}\big)\zeta_F\big(s+\frac{3}{2}\big)\zeta_F(2s+1)\zeta_F\big(3s+\frac{3}{2}\big)} \cdot J_\infty(\Phi_s) \cdot \prod_{v\nmid\infty} J_v(\Phi_s),$$

where $J_\infty(\Phi_s)$ and each of the $J_v(\Phi_s)$ are meromorphic functions of s, and $J_v(\Phi_s)$ is equal to 1 for almost all finite v.

Let us give more details on the contents of this paper. Let H be the split simple adjoint group of type E_6 over F. Then $G_2\times\mathrm{PGL}_3$ forms a dual pair in H. The group $H(\mathbb{A})$ has a distinguished representation Π called the minimal representation, which has an embedding $\theta : \Pi \to \mathcal{A}_2(H)$ into the space of square integrable automorphic forms on H (see [Ginzburg et al. 1997]). For $f \in \Pi$, as in the classical case [Weil 1965; Kudla and Rallis 1988a; 1988b], one may define the theta integral on $G_2(\mathbb{A})$ by

$$I(f)(g) = \int_{\mathrm{PGL}_3(F)\backslash \mathrm{PGL}_3(\mathbb{A})} \theta(f)(gh)\, dh, \quad \text{for all } g \in G_2(\mathbb{A}),$$

but this integral may not converge. Analogous to [Kudla and Rallis 1994; Ichino 2001; 2004], W.T. Gan [2011] found a regularization of the theta integral. For $f \in \Pi$, the regularized theta integral is defined by

$$I^{\mathrm{reg}}(f, s)(g) = \int_{\mathrm{PGL}_3(F)\backslash \mathrm{PGL}_3(\mathbb{A})} \theta(z\cdot f)(gh) E(h, s)\, dh,$$

where z is some element in the Bernstein center of PGL_3 at some finite place of F, and $E(h, s)$ is a spherical Eisenstein series on PGL_3. Note that $\theta(z \cdot f)$ is rapidly decreasing on $\mathrm{PGL}_3(F)\backslash\mathrm{PGL}_3(\mathbb{A})$, so the integral defining $I^{\mathrm{reg}}(f, s)(g)$ is convergent.

As in the classical case [Kudla and Rallis 1994], Gan showed that regularized theta integral $I^{\mathrm{reg}}(f, s)(g)$ is essentially an Eisenstein series $E(\Phi(f, s), g)$ associated with a meromorphic section $\Phi(f, s)$ in $I_P(s)$. Thus the Fourier coefficients of the regularized theta integral contains information of the Fourier coefficients of Eisenstein series.

The contents of this paper are as follows. In Section 2 we give notation and preliminaries. In Section 3, we study the Fourier coefficient of the regularized theta integral $I^{\mathrm{reg}}(f, s)(1)$ with respect to ψ_σ, which turns out to be equal to an integral over $\mathrm{PGL}_3(\mathbb{A})$ and can be decomposed as the product of an archimedean part and a finite part. In Section 4, we compute the local unramified factors in the finite part with the help of the higher dimensional Hensel's lemma. In Section 5, we give the proof of Theorem 1.1.

2. Notation and preliminaries

Groups and principal series. In this paper (except in Section 4), F is a number field with adele ring $\mathbb{A}$, b and c are two elements in F such that the cubic polynomial x^3+bx-c is irreducible over F, and E is a cubic field extension of F generated by one of the roots of the polynomial x^3+bx-c. For a place v of F, let F_v be the corresponding local field. If v is a finite place of F, let $\mathcal{O}_v$ be the ring of integers in F_v, ϖ_v a uniformizer of $\mathcal{O}_v$, $k_v = \mathcal{O}_v/\varpi_v\mathcal{O}_v$ the residue field, and q_v the cardinality of k_v.

Fix a nontrivial character $\psi : F \backslash \mathbb{A} \to \mathbb{C}^\times$. For each place v of F, take the Haar measure dx_v on F_v which is self-dual with respect to ψ_v. For a finite place v, if ψ_v is unramified (i.e., ψ_v is trivial on $\mathcal{O}_v$ but nontrivial on $\varpi_v^{-1}\mathcal{O}_v$), then $\mathrm{Vol}(\mathcal{O}_v) = 1$.

For an algebraic group G over F, denote $[G] = G(F) \backslash G(\mathbb{A})$.

Let G be an algebraic group over F, and let $P = NM$ be a parabolic F-subgroup of G with Levi subgroup M and unipotent radical N. Let $K = \prod K_v$ be a maximal compact subgroup of $G(\mathbb{A})$, where each K_v is a maximal compact subgroup of $G(F_v)$ such that $G(F_v) = P(F_v)K_v$ for every place v of F. Then there is an Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$. Let δ_P be the modulus character of P. For $s \in \mathbb{C}$ and a character $\chi : M(F) \backslash M(\mathbb{A}) \to \mathbb{C}^\times$, the *principal series representation* $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\chi \cdot \delta_P^s$ is the space of all smooth functions $\Phi_s : G(\mathbb{A}) \to \mathbb{C}$ such that $\Phi_s(nmg) = \chi(m)\delta_P(m)^s\Phi_s(g)$ for all $n \in N(\mathbb{A})$, $m \in M(\mathbb{A})$ and $g \in G(\mathbb{A})$. An element in $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\chi \cdot \delta_P^s$ is called a *section*. Because of the Iwasawa decomposition, a section is determined by its restriction to K. A section Φ_s is called *holomorphic* (resp. *meromorphic*) if $s \mapsto \Phi_s(g)$ is a holomorphic (resp. meromorphic) function in s for every $g \in G(\mathbb{A})$. A *standard* section is a holomorphic section whose restriction to K is independent of s. For a place v of F, $s \in \mathbb{C}$ and a character $\chi_v : M(F_v) \to \mathbb{C}^\times$, one can define the local *principal series representation* $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \chi_v \delta_P^s$ similarly. For a place v of F, the *spherical vector* $\Phi_{s,v}^0$ in $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \chi_v \delta_P^s$ is the one whose restriction to K_v is equal to 1. Then $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi \cdot \delta_P^s$ is the restricted tensor product of $\mathrm{Ind}_{P(F_v)}^{G(F_v)} \chi_v \delta_P^s$ with respect to $\Phi_{s,v}^0$, as defined in [Bump 1997, pp. 300–301]. A section is called *decomposable* if it is of the form $\Phi_s = \bigotimes \Phi_{s,v}$ with each $\Phi_{s,v} \in I_{P,v}(s)$. A *pure tensor* is a decomposable section $\Phi_s = \bigotimes \Phi_{s,v}$ where almost all $\Phi_{s,v}$ are spherical vectors.

Minimal representations. Let H be the split simple adjoint group of type E_6 over F, which is explicitly described in [Magaard and Savin 1997, §3] . We also simply write $H = E_6$. Then $H(\mathbb{A})$ has a distinguished representation $\Pi = \bigotimes_v \Pi_v$ called the *minimal representation*, where each Π_v is the local minimal representation of $H(F_v)$. Each Π_v is an irreducible spherical representation of $H(F_v)$ with spherical vector f_v^0, and Π is the restricted tensor product of Π_v with respect to f_v^0. See [Kazhdan and Savin 1990; Ginzburg et al. 1997; Gan and Savin 2005] for more details.

Let $P_H = N_H M_H$ be the Heisenberg parabolic subgroup of H, where M_H is the Levi subgroup and N_H is the unipotent radical, so that N_H is a Heisenberg group with one-dimensional center Z_H. For a finite place v of F, let Ω_v be the minimal nontrivial orbit of $M_H(F_v)$ on the set of unitary characters of $N_H(F_v)$, which can be noncanonically identified with $\bar{V}_H(F_v) = \bar{N}_H(F_v)/\bar{Z}_H(F_v)$, where $\bar{N}_H$ is the opposite of N_H and $\bar{Z}_H$ is the center of $\bar{N}_H$. Then there is a $P_H(F_v)$-equivariant embedding

$$i_v : (\Pi_v)_{Z_H} \hookrightarrow C^\infty(\Omega_v),$$

where $(\Pi_v)_{Z_H} = \Pi_v / \langle \Pi_v(z) f - f \mid z \in Z_H(F_v),\ f \in \Pi_v \rangle$ is the maximal Z_H-invariant quotient of Π_v, and $C^\infty(\Omega_v)$ is the space of locally constant functions on Ω_v (see [Gan 2011, §2.3]).

For a finite place v of F, let $\bar{f}_v^0$ be the image of the spherical vector f_v^0 in $(\Pi_v)_{Z_H}$. Then the action of $\bar{f}_v^0$ on Ω_v is easily described as follows. For each $n \in \mathbb{Z}$, let $\Omega_v(n) = \Omega_v \cap (\varpi_v^n \Lambda_v \setminus \varpi_v^{n+1} \Lambda_v)$, where $\Lambda_v = \bar{V}_H(\mathcal{O}_v)$ is the $\mathcal{O}_v$-lattice in $\bar{V}_H(F_v)$. Then $\bar{f}_v^0$ is constant on each $\Omega_v(n)$; more precisely, it is zero on $\Omega_v(n)$ if $n < 0$, and it takes the value $(q_v^{n+1} - 1)(q_v - 1)^{-1}$ on $\Omega_v(n)$ for $n \geq 1$. See [Kazhdan and Polishchuk 2004, Theorem 1.1.3] or [Gan 2011, §2] for more details.

3. Reduction to PGL_3

Exceptional theta correspondence. Let $H = E_6$ be the split simple adjoint group of type E_6 over F. Let $\Pi = \bigotimes_v \Pi_v$ be the minimal representation of $H(\mathbb{A})$. There is an $H(\mathbb{A})$-equivariant embedding $\theta : \Pi \hookrightarrow \mathcal{A}_2(H)$, where $\mathcal{A}_2(H)$ is the space of square integrable automorphic forms on H (see [Ginzburg et al. 1997]).

Consider the dual pair $G_2 \times \mathrm{PGL}_3$ in $H = E_6$ as in [Magaard and Savin 1997], where G_2 denotes the split simple group of type G_2 over F. See [Carter 1972] for more details on the structure of exceptional groups.

For $f \in \Pi$, the associated theta integral on $G_2(\mathbb{A})$ is defined by

$$I(f)(g) = \int_{[\mathrm{PGL}_3]} \theta(f)(gh)\, dh, \quad \text{for all } g \in G_2(\mathbb{A}),$$

where $[\mathrm{PGL}_3] = \mathrm{PGL}_3(F) \backslash \mathrm{PGL}_3(\mathbb{A})$. This integral may not converge. Similar to [Kudla and Rallis 1994; Ichino 2001; 2004], W.T. Gan [2011] found a regularization of the theta integral by using an element z of the Bernstein center of PGL_3 at some finite place v_0 of F. Precisely, z belongs to the component of the Bernstein center associated to the trivial representation of $\mathrm{PGL}_3(F_{v_0})$, which is equal to $\mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]^{S_3}/(x_1x_2x_3 - 1)$, and $z = \prod_{i=1}^{3}(x_i - q)(x_i^{-1} - q)$, where q is the order of the residue field of F at v_0. For any $f \in \Pi$, the function $\theta(z \cdot f)$ is rapidly decreasing on $\mathrm{PGL}_3(F) \backslash \mathrm{PGL}_3(\mathbb{A})$ (see [Gan 2011, Proposition 5.2]). The regularized theta integral is defined as

$$I^{\mathrm{reg}}(f, s)(g) = \int_{[\mathrm{PGL}_3]} \theta(z \cdot f)(gh) E(h, s)\, dh,$$

where $E(h, s)$ is the spherical Eisenstein series associated to the spherical vector φ_s^0 in $I_Q(s) = \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{PGL}_3(\mathbb{A})} |\det|^{s+1/2}$, and $Q = UL$ is the parabolic subgroup of PGL_3 with Levi subgroup $L \cong \mathrm{GL}_2$ and unipotent radical $U \cong F^2$.

The regularized theta integral is essentially an Eisenstein series. The following results are proved in [Gan 2011, §6].

Proposition 3.1. (i) *For $f \in \Pi$ and* $\mathrm{Re}(s) \gg 0$,

$$I^{\mathrm{reg}}(f, s)(g) = P_z(s) E(\Phi(f, s), g),$$

where $\Phi(f, s)$ is a meromorphic section in $I_P(s) = \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})} |\det|^{s+3/2}$, and

$$P_z(s) = \big(q^{-s-\frac{1}{2}} - q\big)\big(q^{s+\frac{1}{2}} - q\big)\big(q^{-s+\frac{1}{2}} - q\big)\big(q^{s-\frac{1}{2}} - q\big)\big(q^{2s} - q\big)\big(q^{-2s} - q\big),$$

where q is the order of the residue field of F at v_0.

(ii) *If $f = \bigotimes f_v$ is decomposable, then $\Phi(f, s) = \bigotimes \Phi_v(f_v, s)$ is also decomposable; if v is a finite place and f_v^0 is the spherical vector in Π_v, then*

$$\Phi_v(f_v^0, s) = \zeta_v\big(s + \tfrac{1}{2}\big)\zeta_v\big(s + \tfrac{3}{2}\big)\zeta_v(2s + 1)\Phi_v^0(s),$$

where $\Phi_v^0(s) \in I_{P,v}(s)$ is the spherical vector.

We see from this proposition that the Fourier coefficients of the regularized theta integral are closely related to those of the Eisenstein series. Next we study them.

Fourier coefficient of regularized theta integral. For $f \in \Pi$, consider the Fourier coefficient of $I^{\mathrm{reg}}(f, s)(1)$ with respect to the character ψ_σ, which is given by

$$\begin{aligned} I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) &= \int_{[N]} \overline{\psi_\sigma(n)} I^{\mathrm{reg}}(f, s)(n)\, dn \\ &= \int_{[N]} \overline{\psi_\sigma(n)} \int_{[\mathrm{PGL}_3]} \theta(z \cdot f)(nh) E(h, s)\, dh\, dn. \end{aligned}$$

We want to interchange the order of integration in the above iterated integral. By Fubini's theorem, this is possible if the integral

$$\int_{[N]} \int_{[\mathrm{PGL}_3]} \theta(z \cdot f)(nh) E(h, s)\, dh\, dn$$

is absolutely convergent. The author of this paper did not know how to prove this statement. Luckily, the referee communicated a note to the author in which the following inequality is proved:

For every integer r, there exist an integer m and a constant c such that

$$|\theta(z \cdot f)(gh)| \le c \|g\|^m \|h\|^{-r} \tag{1}$$

for all $g \in G_2(\mathbb{A})$ and $h \in S$, where $\|\cdot\|$ is the height function as defined in [Mœglin and Waldspurger 1995, page 20] and S is a Siegel domain in $\mathrm{PGL}_3(\mathbb{A})$ with $\mathrm{PGL}_3(\mathbb{A}) = \mathrm{PGL}_3(F)S$.

The absolute convergence of the integral thus follows from the above inequality, since $E(h, s)$, as an automorphic form on PGL_3, is of moderate growth, and the height function is bounded on compact sets.

So we have

$$I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) = \int_{[\mathrm{PGL}_3]} E(h, s) \int_{[N]} \overline{\psi_\sigma(n)} \theta(z \cdot f)(nh)\, dn\ dh.$$

Next we study the Fourier coefficient $\int_{[N]} \overline{\psi_\sigma(n)} \theta(z \cdot f)(nh)\, dn$ of the function $\theta(z \cdot f)$.

Let $P_H = N_H \cdot M_H$ be the Heisenberg parabolic subgroup of $H = E_6$. Let Z_H be the center of N_H. Then $Z_H \cong F$ and $V_H := N_H / Z \cong F \oplus M_3(F) \oplus M_3(F) \oplus F$ ([Magaard and Savin 1997, p. 114]). Furthermore, $P = P_H \cap G_2$, $N = N_H \cap G_2$ and $Z = Z_H$.

Let $V = N/Z$, which is abelian. Then

$$\int_{[N]} \overline{\psi_\sigma(n)} \theta(z \cdot f)(nh)\, dn = \int_{[V]} \overline{\psi_\sigma(v)} \theta(z \cdot f)_Z(vh)\, dv,$$

where

$$\theta(z \cdot f)_Z(vh) = \int_{[Z]} \theta(z \cdot f)(zvh)\, dz.$$

Let Ω be the minimal nontrivial orbit of $M_H(F)$ on the set of unitary characters of $N_H(F) \setminus N_H(\mathbb{A})$, which can be noncanonically identified with $\overline{V}_H(F) = \overline{N}_H(F)/\bar{Z}_H(F)$, where $\overline{N}_H$ is the opposite of N_H and $\bar{Z}_H$ is the center of $\overline{N}_H$.

By [Gan and Savin 2003, Proposition 5.2], the term $\theta(z \cdot f)_Z$ has the following Fourier expansion along N_H:

$$\theta(z \cdot f)_Z(h) = \sum_{\chi \in \bar{\Omega}} \theta(z \cdot f)_{N_H,\chi}(h),$$

where $\bar{\Omega}$ is the union of Ω with the trivial character, and the term $\theta(z \cdot f)_{N_H,\chi}$ is given by

$$\theta(z \cdot f)_{N_H,\chi}(h) = \int_{[N_H]} \theta(z \cdot f)(nh)\overline{\chi(n)}\, dn.$$

Moreover, since $\theta(z \cdot f)$ is rapidly decreasing on $\mathrm{PGL}_3(F) \setminus \mathrm{PGL}_3(\mathbb{A})$, this Fourier expansion converges absolutely and uniformly on compact subsets of $\mathrm{PGL}_3(\mathbb{A})$.

So,

$$\begin{aligned}\int_{[N]} \overline{\psi_\sigma(n)}\theta(z \cdot f)(nh)\, dn &= \int_{[V]} \overline{\psi_\sigma(v)}\theta(z \cdot f)_Z(vh)\, dv \\ &= \int_{[V]} \overline{\psi_\sigma(v)}\Bigg(\sum_{\chi \in \bar{\Omega}} \theta(z \cdot f)_{N_H,\chi}(vh)\Bigg)\, dv \\ &= \sum_{\chi \in \bar{\Omega}} \int_{[V]} \overline{\psi_\sigma(v)}\theta(z \cdot f)_{N_H,\chi}(vh)\, dv,\end{aligned}$$

where the order change of the integration and summation is justified by the uniform convergence of the Fourier expansion. Then

$$\begin{aligned}\int_{[N]} \overline{\psi_\sigma(n)}\theta(z \cdot f)(nh)\, dn &= \sum_{\chi \in \bar{\Omega}} \int_{[V]} \overline{\psi_\sigma(v)} \int_{[N_H]} \theta(z \cdot f)(nvh)\overline{\chi(n)}\, dn\, dv \\ &= \sum_{\chi \in \bar{\Omega}} \int_{[V]} \overline{\psi_\sigma(v)}\chi(v) \int_{[N_H]} \theta(z \cdot f)(nvh)\overline{\chi(nv)}\, dn\, dv \\ &= \sum_{\chi \in \bar{\Omega}} \int_{[V]} \overline{\psi_\sigma(v)}\chi(v)\theta(z \cdot f)_{N_H,\chi}(h)\, dv \\ &= \sum_{\chi \in \bar{\Omega}} \theta(z \cdot f)_{N_H,\chi}(h) \int_{[V]} \overline{\psi_\sigma(v)}\chi(v)\, dv.\end{aligned}$$

Note that

$$\int_{[V]} \overline{\psi_\sigma(v)}\chi(v)\, dv = \begin{cases} \mathrm{Vol}([V]) = 1 & \text{if } \chi|_{V(\mathbb{A})} = \psi_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)\,dn = \sum_{\chi\in\bar{\Omega}:\chi|_{V(\mathbb{A})}=\psi_\sigma} \theta(z\cdot f)_{N_H,\chi}(h) = \sum_{\chi\in\Omega:\chi|_{V(\mathbb{A})}=\psi_\sigma} \theta(z\cdot f)_{N_H,\chi}(h),$$

since ψ_σ is nontrivial.

Let $\Omega^0 = \{\chi \in \Omega : \chi|_{V(\mathbb{A})} = \psi_\sigma\}$. Then

$$\int_{[N]} \overline{\psi_\sigma(n)}\theta(z\cdot f)(nh)dn = \sum_{\chi\in\Omega^0} \theta(z\cdot f)_{N_H,\chi}(h).$$

It follows that the Fourier coefficient of $I^{\mathrm{reg}}(f, s)(1)$ with respect to ψ_σ is given by

$$I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) = \int_{[\mathrm{PGL}_3]} E(h, s) \sum_{\chi\in\Omega^0} \theta(z\cdot f)_{N_H,\chi}(h)\,dh.$$

Note that $\Omega \subset \overline{V}_H = \bar{N}_H/\bar{Z}_H = F \oplus M_3(F) \oplus M_3(F) \oplus F$ can be described as

$$\Omega = \{(a, x, y, b) : y^\sharp = bx,\ x^\sharp = ay,\ xy = abI_3\},$$

where I_3 is the identity matrix in $M_3(F)$, and $x^\sharp$ is the adjoint of $x \in M_3(F)$.

Also $\overline{V} = \bar{N}/\bar{Z} \subset \overline{V}_H = \bar{N}_H/\bar{Z}_H$ can be described as

$$\overline{V} = F \oplus F \oplus F \oplus F.$$

The restriction from $\overline{V}_H$ to $\overline{V}$ is given by

$$(\alpha, x, y, \beta) \mapsto (\alpha, \mathrm{Tr}(x), \mathrm{Tr}(y), \beta).$$

For $\sigma = (1, 0, b, c) \in \overline{V} = \bar{N}/\bar{Z}$, we have

$$\begin{aligned}\Omega^0 &= \{\chi \in \Omega : \chi|_{N(\mathbb{A})} = \psi_\sigma\}\\ &= \{(1, x, x^\sharp, \det(x)) : x \in M_3(F) \text{ has characteristic polynomial } \lambda^3 + b\lambda - c\}.\end{aligned}$$

Compare [Gan 2008, Lemma 2.1].

Note that there is an action of $\mathrm{PGL}_3(F)$ on Ω given by conjugation:

$$h\cdot(\alpha, x, y, \beta) = (\alpha, hxh^{-1}, hyh^{-1}, \beta).$$

The following result is easy to verify.

Lemma 3.2. $\mathrm{PGL}_3(F)$ *acts on* Ω^0 *transitively with representative*

$$\chi_0 = (1, A_0, A_0^\sharp, \det(A_0)),$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & -b \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $T(F)$ be the stabilizer of χ_0 in $\mathrm{PGL}_3(F)$. Then $T(F) = E^\times/F^\times$, where E is the cubic field generated by one of the roots of the irreducible polynomial x^3+bx-c; and

$$T(F)\setminus \mathrm{PGL}_3(F) \cong \Omega^0 \quad \text{via } T(F)h \mapsto h\cdot\chi_0.$$

So

$$\begin{aligned}
I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) &= \int_{[\mathrm{PGL}_3]} E(h,s) \sum_{\gamma\in T(F)\backslash \mathrm{PGL}_3(F)} \theta(z\cdot f)_{N_H,\gamma\cdot\chi_0}(h)\,dh \\
&= \int_{[\mathrm{PGL}_3]} E(h,s) \sum_{\gamma\in T(F)\backslash \mathrm{PGL}_3(F)} \theta(z\cdot f)_{N_H,\chi_0}(\gamma h)\,dh \\
&= \int_{T(F)\backslash \mathrm{PGL}_3(\mathbb{A})} E(h,s)\theta(z\cdot f)_{N_H,\chi_0}(h)\,dh \\
&= \int_{T(\mathbb{A})\backslash \mathrm{PGL}_3(\mathbb{A})} \int_{[T]} E(th,s)\theta(z\cdot f)_{N_H,\chi_0}(th)\,dt\,dh \\
&= \int_{T(\mathbb{A})\backslash \mathrm{PGL}_3(\mathbb{A})} \theta(z\cdot f)_{N_H,\chi_0}(h) \int_{[T]} E(th,s)\,dt\ dh,
\end{aligned}$$

where we have used the fact that $\theta(z\cdot f)_{N_H,\chi_0}(th) = \theta(z\cdot f)_{N_H,\chi_0}(h)$ for all $t\in T(\mathbb{A})$, which can be shown as follows: it is true for $t\in T(F)$, and it is true for $t\in T(F_v)$ for all finite places v by the local formulae; by weak approximation ([Platonov and Rapinchuk 1994, Theorem 7.7, p. 415]), $T(F)$ is dense in $T(F_\infty)$, where $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$, so it is true for all $t\in T(\mathbb{A})$.

Lemma 3.3. *Keep the notation above. Then*

$$\int_{[T]} E(th,s)\,dt = \int_{T(\mathbb{A})} \varphi_s^0(th)\,dt,$$

where φ_s^0 is the spherical vector in $I_Q(s)$.

Proof. We have

$$\int_{[T]} E(th,s)\,dt = \int_{T(F)\backslash T(\mathbb{A})} \sum_{\gamma\in Q(F)\backslash \mathrm{PGL}_3(F)} \varphi_s^0(\gamma th)\,dt.$$

Note that $Q(F)\backslash\mathrm{PGL}_3(F)$ is the Grassmannian $\mathbb{P}^2(F)$, which is just $E^\times/F^\times = T(F)$, where E is the cubic field extension of F generated by one of the roots of the polynomial x^3+bx-c. The lemma then follows. □

In summary, we have shown the following result:

Proposition 3.4. *We have*

$$I^{\mathrm{reg}}_{\psi_\sigma}(f,s)(1) = \int_{\mathrm{PGL}_3(\mathbb{A})} \theta(z\cdot f)_{N_H,\chi_0}(h)\varphi_s^0(h)\,dh.$$

Next we analyze $\theta(z \cdot f)_{N_H,\chi_0}(h) = \int_{[N_H]} \theta(z \cdot f)(nh)\overline{\chi_0(n)}dn$. We follow the arguments in [Gan 2008, §3].

First note that the mapping $L : \phi \mapsto \theta(\phi)_{N_H,\chi_0}(1)$ gives a nonzero element in the space $\mathrm{Hom}_{N_H(\mathbb{A})}(\Pi, \chi_0)$. Fix a finite place v of F. Since $\chi_{0,v} \in \Omega_v$, it follows from [Magaard and Savin 1997, Lemma 6.2] that

$$\dim \mathrm{Hom}_{N_H(F_v)}(\Pi_v, \chi_0) = 1.$$

Note that in this case, a nonzero element of the space $\mathrm{Hom}_{N_H(F_v)}(\Pi_v, \chi_0)$ is given by

$$L_v(\phi) = i_v(\bar{\phi})(\chi_{0,v}),$$

where $\bar{\phi}$ is the image of ϕ in $(\Pi_v)_{Z_H}$, and i_v is the mapping on page 239.

Next we consider the archimedean places. Let the archimedean part of Π be $\Pi_\infty = \bigotimes_{v|\infty} \Pi_v$. Then the global functional $L \in \mathrm{Hom}_{N_H(\mathbb{A})}(\Pi, \chi_0)$ can be decomposed as $L = L_\infty \otimes \bigotimes_{v\nmid\infty} L_v$, where $L_\infty \in \mathrm{Hom}_{N_H(F_\infty)}(\Pi_\infty, \chi_0)$ is nontrivial.

So we have the following result:

Lemma 3.5. *Keep the notation above. For a pure tensor $f = \bigotimes f_v \in \Pi$, we have*

$$\theta(f)_{N_H,\chi_0}(h) = \theta(h \cdot f)_{N_H,\chi_0}(1) = L_\infty(h_\infty \cdot f_\infty) \cdot \prod_{v\nmid\infty} L_v(h_v \cdot f_v),$$

where h_∞ (resp. f_∞) is the archimedean part of h (resp. f).

Combining Proposition 3.4 and Lemma 3.5, we obtain the following result:

Proposition 3.6. *Suppose $f = \bigotimes f_v \in \Pi$ is a pure tensor. Then*

$$I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) = I(f_\infty, s) \cdot I(z \cdot f_{v_0}, s) \cdot \prod_{v\neq v_0, v\nmid\infty} I(f_v, s),$$

where

$$I(f_\infty, s) = \int_{\mathrm{PGL}_3(F_\infty)} L_\infty(h_\infty \cdot f_\infty)\varphi^0_{s,\infty}(h_\infty)\, dh_\infty,$$

and for a finite place v and $\phi_v \in \Pi_v$,

$$I(\phi_v, s) = \int_{\mathrm{PGL}_3(F_v)} L_v(h_v \cdot \phi_v)\varphi^0_{s,v}(h_v)\, dh_v,$$

where

$$L_v(\phi_v) = i_v(\bar{\phi}_v)(\chi_0).$$

Recall v_0 is a finite place of F such that z comes from the Bernstein center of $\mathrm{PGL}_3(F_{v_0})$.

In the next section, we will compute the local integrals $I(f_v, s)$ in the finite unramified case, and we will see that they are quotients of local zeta factors. See Proposition 4.3 in the next section for the precise result.

4. Unramified computation

This section is devoted to the computation of the local unramified factors $I(f_v, s)$ of the Fourier coefficient $I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1)$. So v is a finite place of F such that v is unramified in the cubic field extension E of F, b and c are units in $\mathcal{O}_v$, the character ψ_v is unramified, f_v and $\varphi_{s,v}$ are spherical vectors.

For simplicity, we omit the subscript v from notation. So F is a p-adic local field with $p \neq 2$, $\mathcal{O}$ its ring of integers, ϖ a uniformizer of $\mathcal{O}$, $k_F = \mathcal{O}/\varpi\mathcal{O}$ the residue field of F, q the cardinality of k_F, $\nu : F^\times \to \mathbb{Z}$ the valuation given by $\nu(\varpi^n\mathcal{O}^\times) = n$, and $|\cdot| : F^\times \to \mathbb{R}_+^\times$ the absolute value given by $|x| = q^{-\nu(x)}$. The Haar measure dx on F satisfies $\mathrm{Vol}(\mathcal{O}) = 1$, and take the Haar measure $d^\times x$ on $F^\times$ such that $\mathrm{Vol}(\mathcal{O}^\times) = 1$.

Reduction to a volume computation. Let $G = \mathrm{PGL}_3(F)$, $K = \mathrm{PGL}_3(\mathcal{O})$, $Q = UL$ the parabolic subgroup of G with Levi subgroup $L \cong \mathrm{GL}_2(F)$ and unipotent radical $U \cong F^2$. Then there is an Iwasawa decomposition $G = QK$.

We want to compute the integral

$$I(s) = \int_G i(\overline{hf})(\chi)\varphi_s(h)\, dh,$$

where f is the spherical vector in Π, φ_s is the spherical vector in $I_Q(s)$, i is the mapping as on page 239, and $\chi = (1, A_0, A_0^\sharp, \det(A_0)) \in F \oplus M_3(F) \oplus M_3(F) \oplus F$, where

$$A_0 = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & -b \\ 0 & 1 & 0 \end{pmatrix}$$

with $b, c \in \mathcal{O}^\times$.

We have

$$\begin{aligned} I(s) &= \int_K \int_Q i(\overline{qkf})(\chi)\varphi_s(qk)dq\, dk \\ &= \int_Q i(\overline{qf})(\chi)\varphi_s(q)dq \\ &= \int_L \int_U i(\overline{ulf})(\chi)\varphi_s(ul)\delta_Q(ul)^{-1}du\, dl \\ &= \int_L \int_U i(\overline{ulf})(\chi)|\det(l)|^{s-1/2}du\, dl, \end{aligned}$$

since $\varphi_s(ul) = |\det(l)|^{s+1/2}$ and $\delta_Q(ul) = |\det(l)|$.

Note that $i(\bar{f}) \in C^\infty(\Omega)$, where $\Omega = F \oplus M_3(F) \oplus M_3(F) \oplus F$, and it is constant on $\Omega(n)$, where $\Omega(n) = \Omega \cap (\varpi^n\Lambda - \varpi^{n+1}\Lambda)$, with

$$\Lambda = \mathcal{O} \oplus M_3(\mathcal{O}) \oplus M_3(\mathcal{O}) \oplus \mathcal{O}.$$

Now we have

$$i(\overline{ulf})(\chi) = i(\bar{f})(l^{-1}u^{-1}\chi).$$

But

$$\chi = (1, A_0, A_0^\sharp, \det(A_0)) \in \Omega(0).$$

So

$$l^{-1}u^{-1}\chi = (1, l^{-1}u^{-1}A_0ul, \dots),$$

and

$$i(\bar{f})(l^{-1}u^{-1}\chi) = \begin{cases} 1 & \text{if } l^{-1}u^{-1}A_0ul \in M_3(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} I(s) &= \int_L \int_U 1_{M_3(\mathcal{O})}(l^{-1}u^{-1}A_0ul)|\det(l)|^{s-1/2} du\, dl \\ &= \int_{\mathrm{GL}_2(F)} \int_{F^2} 1_{M_3(\mathcal{O})}(l(g)^{-1}u(x,y)^{-1}A_0u(x,y)l(g))|\det(g)|^{s-1/2}\, dx\, dy\, dg. \end{aligned}$$

Let B be the group of upper triangular matrices in $\mathrm{GL}_2(F)$. Then $\mathrm{GL}_2(F) = \mathrm{GL}_2(\mathcal{O})B$, and

$$\begin{aligned} I(s) &= \int_{\mathrm{GL}_2(\mathcal{O})} \int_B \int_{F^2} 1_{M_3(\mathcal{O})}(l(k)^{-1}l(p)^{-1}u(x,y)^{-1} \\ &\qquad\qquad A_0u(x,y)l(p)l(k))|\det(p)|^{s-1/2}\, dx\, dy\, dp\, dk \\ &= \int_B \int_{F^2} 1_{M_3(\mathcal{O})}(l(p)^{-1}u(x,y)^{-1}A_0u(x,y)l(p))|\det(p)|^{s-1/2}\, dx\, dy\, dp. \end{aligned}$$

Now $B = N_BM_B$, where $N_B \cong F$ is the unipotent radical and $M_B \cong F^\times \times F^\times$ is the Levi subgroup. So

$$\begin{aligned} I(s) &= \int_{M_B} \int_{N_B} \int_{F^2} 1_{M_3(\mathcal{O})}(l(nm)^{-1}u(x,y)^{-1}A_0u(x,y)l(nm)) \\ &\qquad\qquad \delta_B(m)^{-1}|\det(nm)|^{s-1/2}\, dx\, dy\, dn\, dm \\ &= \int_{F^{\times 2}} \int_{F^3} 1_{M_3(\mathcal{O})}(l(n(\beta)m(\alpha,\delta))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\alpha,\delta))) \\ &\qquad\qquad \cdot |\alpha/\delta|^{-1}|\alpha\delta|^{s-1/2}\, dx\, dy\, d\beta\, d^\times\alpha\, d^\times\delta, \end{aligned}$$

where

$$n(\beta) = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}, \quad m(\alpha,\delta) = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}.$$

Since $F^\times = \bigcup_{n\in\mathbb{Z}} \varpi^n \mathcal{O}^\times$, we have

$$\begin{aligned}
&I(s)\\
&= \sum \int_{F^3}\int_{\varpi^n\mathcal{O}^\times}\int_{\varpi^m\mathcal{O}^\times} 1_{M_3(\mathcal{O})}(l(n(\beta)m(\alpha,\delta))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\alpha,\delta)))\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot|\alpha/\delta|^{-1}|\alpha\delta|^{s-1/2}d^\times\alpha\, d^\times\delta\, dx\, dy\, d\beta\\
&= \sum \int_{F^3} 1_{M_3(\mathcal{O})} l(n(\beta)m(\varpi^m,\varpi^n))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\varpi^m,\varpi^n)))\\
&\qquad \cdot|\varpi|^{(m+n)(s-1/2)}|\varpi^{m-n}|^{-1}\mathrm{Vol}(\varpi^m\mathcal{O}^\times, d^\times\alpha)\mathrm{Vol}(\varpi^n\mathcal{O}^\times, d^\times\delta)\, dx\, dy\, d\beta\\
&= \sum \int_{F^3} 1_{M_3(\mathcal{O})}(l(n(\beta)m(\varpi^m,\varpi^n))^{-1}u(x,y)^{-1}A_0u(x,y)l(n(\beta)m(\varpi^m,\varpi^n)))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot q^{-(m+n)(s-1/2)+m-n}\, dx\, dy\, d\beta,
\end{aligned}$$

where the sum is taken over all $m, n \in \mathbb{Z}$, and we have used $\mathrm{Vol}(\varpi^m\mathcal{O}^\times, d^\times\alpha) = \mathrm{Vol}(\varpi^n\mathcal{O}^\times, d^\times\delta) = \mathrm{Vol}(\mathcal{O}^\times) = 1$, and $|\varpi|^{-1} = q$.

Note that

$$\begin{aligned}
&l(n(\beta)m(\varpi^m, \varpi^n))^{-1}u(x, y)^{-1}A_0u(x, y)l(n(\beta)m(\varpi^m, \varpi^n))\\
&\qquad = \begin{pmatrix} -\beta & -\varpi^{n-m}(x+\beta^2-y\beta) & \varpi^{-m}(c-xy-\beta(x-y^2-b)) \\ \varpi^{m-n} & \beta-y & \varpi^{-n}(x-y^2-b) \\ 0 & \varpi^n & y \end{pmatrix}.
\end{aligned}$$

So

$$\begin{aligned}
I(s) &= \sum_{m\ge n\ge 0} S(m, n)q^{-(m+n)(s-1/2)+m-n}\\
&= \sum_{k\ge 0} q^{-k(s+1/2)} \sum_{\substack{m+n=k\\ m\ge n\ge 0}} S(m, n)q^{2m}\\
&= 1 + \sum_{k\ge 1} q^{-k(s+1/2)} \sum_{\substack{m+n=k\\ m\ge n\ge 0}} S(m, n)q^{2m},
\end{aligned}$$

where $S(m, n)$ is the volume of the set of $(x, y, \beta) \in \mathcal{O}^3$ such that $\nu(x+\beta^2-y\beta) \ge m-n$, $\nu(x-y^2-b) \ge n$ and $\nu(c-xy-\beta(x-y^2-b)) \ge m$ (with respect to the Haar measure on F such that $\mathrm{Vol}(\mathcal{O}) = 1$).

Let r be the number of solutions of x^3+bx-c over F. Then $r = 0$, 1 or 3.

Lemma 4.1. (1) *If* $r = 0$, *then* $S(m, n) = 0$ *for* $m \ge n \ge 0$ *with* $m+n \ge 1$.

(2) *If* $r = 1$, *then*

$$S(m, n) = \begin{cases} 0 & \text{if } m > n > 0,\\ q^{-2m} & \text{if } m = n > 0,\ /\\ q^{-2m} & \text{if } m > n = 0. \end{cases}$$

(3) *If* $r = 3$, *then*

$$S(m,n) = \begin{cases} 6q^{-2m} & \text{if } m > n > 0, \\ 3q^{-2m} & \text{if } m = n > 0, \\ 3q^{-2m} & \text{if } m > n = 0. \end{cases}$$

Proof. The results follow from Lemmas 4.7, 4.8 and 4.9. □

Corollary 4.2. (1) *If* $r = 0$, *then for* $k \geq 1$,

$$\sum_{\substack{m+n=k \\ m \geq n \geq 0}} S(m,n) q^{2m} = 0.$$

(2) *If* $r = 1$, *then for* $k \geq 1$,

$$\sum_{\substack{m+n=k \\ m \geq n \geq 0}} S(m,n) q^{2m} = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ 2 & \text{if } k \text{ is even}. \end{cases}$$

(3) *If* $r = 3$, *then for* $k \geq 1$,

$$\sum_{\substack{m+n=k \\ m \geq n \geq 0}} S(m,n) q^{2m} = 3k.$$

Recall that r is the number of solutions of $x^3 + bx - c$ over F. Let E be the cubic algebra over F determined by the polynomial $x^3 + bx - c$ as follows. If $r = 0$, then E is a field generated by one of the roots of $x^3 + bx - c$ over F, which is an unramified cubic field extension of F; if $r = 1$, say

$$x^3 + bx - c = (x - \alpha) \cdot g(x),$$

then $E = F \oplus F'$, where F' is the splitting field of $g(x)$ over F, which is an unramified quadratic field extension of F; if $r = 3$, then $E = F \oplus F \oplus F$.

Proposition 4.3. *Let* E *be the cubic algebra determined by* $x^3 + bx - c$ *as above. Then*

$$I(s) = \frac{\zeta_E\left(s + \frac{1}{2}\right)}{\zeta_F\left(3s + \frac{3}{2}\right)}.$$

More precisely,

$$I(s) = \begin{cases} 1 & \text{if } r = 0, \\ \dfrac{(1 - q^{-(s+1/2)})^{-1}(1 - q^{-2(s+1/2)})^{-1}}{(1 - q^{-3(s+1/2)})^{-1}} & \text{if } r = 1, \\ \dfrac{(1 - q^{-(s+1/2)})^{-3}}{(1 - q^{-3(s+1/2)})^{-1}} & \text{if } r = 3. \end{cases}$$

Computation of the volume. In this section, we will compute the volume $S(m,n)$ in the last section by using the higher-dimensional Hensel's lemma.

Let $f=(f_1,\dots,f_n):\mathcal{O}^n\to\mathcal{O}^n$ be a polynomial map, where each f_i is a polynomial over $\mathcal{O}$. We say $f:\mathcal{O}^n\to\mathcal{O}^n$ is *strongly regular* at $x_0\in\mathcal{O}^n$ if $\det D_f(x_0)\in\mathcal{O}^\times$, where $D_f(x_0)$ is the Jacobian matrix of f at x_0.

Theorem 4.4 (higher-dimensional Hensel's lemma). *Suppose $f:\mathcal{O}^n\to\mathcal{O}^n$ is a polynomial map which is strongly regular at $x_0\in\mathcal{O}^n$, and $y\in\mathcal{O}^n$ satisfies $y\equiv f(x_0)\pmod{\varpi}$. Then there exists a unique $x\in\mathcal{O}^n$ such that $f(x)=y$ and $x\equiv x_0\pmod{\varpi}$.*

Proof. See [Green et al. 1995, Theorem 8.1, p. 71] or [Kuhlmann 2011, §4.4]. □

Theorem 4.5. *Let $f:\mathcal{O}^n\to\mathcal{O}^n$ be a polynomial map which is strongly regular at $x_0\in\mathcal{O}^n$. Then $f:x_0+(\varpi\mathcal{O})^n\to f(x_0)+(\varpi\mathcal{O})^n$ is a measure-invariant bijection.*

Corollary 4.6. *Suppose $y_0\in\mathcal{O}^n$ and $f:\mathcal{O}^n\to\mathcal{O}^n$ is strongly regular at all solutions of $f(x)=y_0$ over $\mathcal{O}$. Then for an open subset $U\subset y_0+(\varpi\mathcal{O})^n$,*

$$\mathrm{Vol}(f^{-1}(U))=N\cdot\mathrm{Vol}(U),$$

where $\mathrm{Vol}(X)$ is the volume of the subset X of $\mathcal{O}^n$, and N is the number of $x\in k_F^n$ with $f(x)\equiv y_0\pmod{\varpi}$.

Now we compute the volume $S(m,n)$.

For $m\ge n\ge 0$, let $\Omega_{m,n}$ be the set of $(x,y,\beta)\in\mathcal{O}^3$ such that

$$\begin{cases} v(x+\beta^2-y\beta)\ge m-n,\\ v(x-y^2-b)\ge n,\\ v(c-xy-\beta(x-y^2-b))\ge m.\end{cases}$$

Then $S(m,n)=\mathrm{Vol}(\Omega_{m,n})$.

Let r be the number of solutions of $x^3+bx-c=0$ over F. Then $r=0$, 1 or 3.

Lemma 4.7. *Suppose $m>n>0$. Then*

$$\mathrm{Vol}(\Omega_{m,n})=N\cdot q^{-2m},$$

where

$$N=\begin{cases}0 & \text{if } r=0,1,\\ 6 & \text{if } r=3.\end{cases}$$

Proof. Define a polynomial map $f:\mathcal{O}^3\to\mathcal{O}^3$ by

$$(x,y,\beta)\mapsto(w_1,w_2,w_3),$$

where

$$\begin{cases} w_1=x+\beta^2-y\beta,\\ w_2=x-y^2-b,\\ w_3=c-xy-\beta(x-y^2-b).\end{cases}$$

It is routine to check that f is strongly regular at all solutions of $f(x, y, \beta) = 0$.

Let

$$E_{m,n} = \{(w_1, w_2, w_3) \in \mathcal{O}^3 \mid \nu(w_1) \geq m - n, \nu(w_2) \geq n, \nu(w_3) \geq m\}.$$

Then $\Omega_{m,n} = f^{-1}(E_{m,n})$. By Corollary 4.6,

$$\mathrm{Vol}(\Omega_{m,n}) = N \cdot \mathrm{Vol}(E_{m,n}) = N \cdot q^{-2m},$$

where N is the number of $(x, y, \beta) \in k_F^3$ such that $f(x, y, \beta) = 0$.

Suppose $N \neq 0$. Take $(x_0, y_0, \beta_0) \in k_F^3$ satisfying $f(x_0, y_0, \beta_0) = 0$. It is easy to see that y_0 and $-\beta_0$ are two distinct roots of $x^3 + bx - c$ over k_F, so $r = 3$. Then $N = 6$.

If $r \neq 3$ (i.e., $r = 0$ or 1), then $N = 0$ (since $N \neq 0$ implies that $r = 3$).

The desired result follows. □

Lemma 4.8. *Suppose $m = n > 0$. Then*

$$\mathrm{Vol}(\Omega_{m,m}) = r \cdot q^{-2m}.$$

Proof. In this case, for $(x, y, \beta) \in \mathcal{O}^3$, the condition $\nu(x + \beta^2 - y\beta) \geq 0$ holds automatically. So

$$\begin{aligned}\Omega_{m,m} &= \{(x, y, \beta) \in \mathcal{O}^3 \mid \nu(x - y^2 - b) \geq m, \nu(c - xy - \beta(x - y^2 - b)) \geq m\} \\ &= \{(x, y, \beta) \in \mathcal{O}^3 \mid \nu(x - y^2 - b) \geq m, \nu(c - xy) \geq m\}.\end{aligned}$$

For $\beta \in \mathcal{O}$, let

$$\Omega_m(\beta) = \{(x, y) \in \mathcal{O}^2 \mid (x, y, \beta) \in \Omega_{m,m}\}.$$

Then $\Omega_m(\beta) = \Omega_m(0)$ for any $\beta \in \mathcal{O}$, and

$$\begin{aligned}\mathrm{Vol}(\Omega_{m,m}) &= \iiint_{(x,y,\beta)\in\Omega_{m,m}} dx\, dy\, d\beta \\ &= \int_{\beta\in\mathcal{O}} \iint_{(x,y)\in\Omega_m(\beta)} dx\, dy\, d\beta \\ &= \int_{\mathcal{O}} \mathrm{Vol}(\Omega_m(\beta))\, d\beta \\ &= \mathrm{Vol}(\Omega_m(0)) \cdot \mathrm{Vol}(\mathcal{O}) \\ &= \mathrm{Vol}(\Omega_m(0)).\end{aligned}$$

Define a polynomial map $f_\beta : \mathcal{O}^2 \to \mathcal{O}^2$ by

$$(x, y) \mapsto (w_2 = x - y^2 - b, w_3 = c - xy).$$

Then it is routine to check that f is strongly regular at all solutions of $f(x, y) = 0$.

Let

$$E_m = \{(w_2, w_3) \in \mathcal{O}^2 \mid \nu(w_2) \geq m, \nu(w_3) \geq m\}.$$

Then $\Omega_m(0) = f^{-1}(E_m)$.

By Corollary 4.6, $\mathrm{Vol}(f^{-1}(E_m)) = N \cdot \mathrm{Vol}(E_m) = N \cdot q^{-2m}$, where N is the number of $(x, y) \in k_F^2$ such that $f(x, y) = 0$. It is easy to check that $N = r$. So

$$\mathrm{Vol}(\Omega_{m,m}) = \mathrm{Vol}(\Omega_m(0)) = \mathrm{Vol}(f^{-1}(E_m)) = r \cdot q^{-2m}. \qquad \square$$

Lemma 4.9. *Suppose $m > n = 0$. Then*

$$\mathrm{Vol}(\Omega_{m,0}) = r \cdot q^{-2m}.$$

Proof. In this case, for $(x, y, \beta) \in \mathcal{O}^3$, the condition $\nu(x - y^2 - b) \geq 0$ holds automatically. So

$$\Omega_{m,0} = \{(x, y, \beta) \in \mathcal{O}^3 \mid \nu(x + \beta^2 - y\beta) \geq m, \nu(c - xy - \beta(x - y^2 - b)) \geq m\}.$$

For $y \in \mathcal{O}$, let

$$\Omega_m(y) = \{(x, \beta) \in \mathcal{O}^2 \mid (x, y, \beta) \in \Omega_{m,0}\}.$$

Then

$$\begin{aligned}\mathrm{Vol}(\Omega_{m,0}) &= \iiint_{(x,y,\beta)\in\Omega_{m,0}} dx\, d\beta\, dy \\ &= \int_{y\in\mathcal{O}} \iint_{(x,\beta)\in\Omega_m(y)} dx\, d\beta\, dy \\ &= \int_{\mathcal{O}} \mathrm{Vol}(\Omega_m(y))\, dy.\end{aligned}$$

For $y \in \mathcal{O}$, define a polynomial map $f_y : \mathcal{O}^2 \to \mathcal{O}^2$ by

$$(x, \beta) \mapsto (w_1 = x + \beta^2 - y\beta, w_3 = c - xy - \beta(x - y^2 - b)).$$

Then it is routine to check that f_y is strongly regular at all solutions of $f_y(x, \beta) = 0$.

Let

$$E_m = \{(w_1, w_3) \in \mathcal{O}^2 \mid \nu(w_1) \geq m, \nu(w_3) \geq m\}.$$

Then $\Omega_m(y) = f_y^{-1}(E_m)$. By Corollary 4.6,

$$\mathrm{Vol}(f_y^{-1}(E_m)) = N_y \cdot \mathrm{Vol}(E_m) = N_y \cdot q^{-2m},$$

where N_y is the number of $(x, \beta) \in k_F^2$ such that $f_y(x, \beta) = 0$. It is easy to check that $N_y = r$. So $\mathrm{Vol}(\Omega_m(y)) = r \cdot q^{-2m}$, and

$$\mathrm{Vol}(\Omega_{m,0}) = \int_{\mathcal{O}} \mathrm{Vol}(\Omega_m(y))\, dy = r \cdot q^{-2m} \cdot \mathrm{Vol}(\mathcal{O}) = r \cdot q^{-2m}. \qquad \square$$

5. Proof of the main result

In this section, we prove Theorem 1.1.

Take $f = f^0 = \bigotimes f_v^0 \in \Pi$ to be the spherical vector. Consider the regularized theta integral $I^{\mathrm{reg}}(f, s)(1)$. Then for $\mathrm{Re}(s) \gg 0$, $I^{\mathrm{reg}}(f, s)(1) = P_z(s)E(\Phi(f, s), 1)$.

By Proposition 3.6 and Proposition 4.3, the Fourier coefficient of $I^{\mathrm{reg}}(f, s)(1)$ with respect to ψ_σ is given by

$$I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) = I(f_\infty, s) \cdot I(z \cdot f_{v_0}, s) \cdot \prod_{\substack{v \neq v_0 \\ v \nmid \infty}} I(f_v, s) = \frac{\zeta_E\left(s + \frac{1}{2}\right)}{\zeta_F\left(3s + \frac{3}{2}\right)} \cdot \alpha_\infty(s) \cdot \prod_v \alpha_v(s),$$

where

$$\alpha_\infty(s) = I(f_\infty, s) \cdot \frac{\zeta_\infty\left(3s + \frac{3}{2}\right)}{\zeta_{E_\infty}\left(s + \frac{1}{2}\right)},$$

and for a finite place v of F,

$$\alpha_v(s) = \begin{cases} I(f_v, s) \cdot \dfrac{\zeta_v\left(3s + \frac{3}{2}\right)}{\zeta_{E_v}\left(s + \frac{1}{2}\right)} & \text{if } v \neq v_0, \\ I(z \cdot f_v, s) \cdot \dfrac{\zeta_v\left(3s + \frac{3}{2}\right)}{\zeta_{E_v}\left(s + \frac{1}{2}\right)} & \text{if } v = v_0, \end{cases}$$

and $\alpha_v(s) = 1$ for almost all finite v.

On the other hand, by Proposition 3.1, for a pure tensor $\Phi_s = \bigotimes \Phi_{s,v} \in I_P(s)$,

$$\frac{E_{\psi_\sigma}(\Phi(f,s),1)}{E_{\psi_\sigma}(\Phi_s,1)} = \prod_v \frac{E_{\psi_\sigma,v}(\Phi(f,s),1)}{E_{\psi_\sigma,v}(\Phi_s,1)} = \zeta_F\left(s+\tfrac{1}{2}\right)\zeta_F\left(s+\tfrac{3}{2}\right)\zeta_F(2s+1)\prod_v \beta_v(\Phi_s)$$

where

$$\beta_v(\Phi_s) = \frac{1}{\zeta_v\left(s + \frac{1}{2}\right)\zeta_v\left(s + \frac{3}{2}\right)\zeta_v(2s + 1)} \frac{E_{\psi_\sigma,v}(\Phi(f, s), 1)}{E_{\psi_\sigma,v}(\Phi_s, 1)},$$

and $\beta_v(\Phi_s) = 1$ for almost all finite v.

Since $I^{\mathrm{reg}}_{\psi_\sigma}(f, s)(1) = P_z(s)E_{\psi_\sigma}(\Phi(f, s), 1)$, we get

$$E_{\psi_\sigma}(\Phi_s, 1) = \frac{\zeta_E\left(s + \frac{1}{2}\right)}{\zeta_F\left(s + \frac{1}{2}\right)\zeta_F\left(s + \frac{3}{2}\right)\zeta_F(2s + 1)\zeta_F\left(3s + \frac{3}{2}\right)} \cdot J_\infty(\Phi_s) \cdot \prod_v J_v(\Phi_s),$$

where

$$J_\infty(\Phi_s) = \alpha_\infty(s)/\beta_\infty(\Phi_s),$$

and for a finite place v of F,

$$J_v(\Phi_s) = \begin{cases} \alpha_v(s)/\beta_v(\Phi_s) & \text{if } v \neq v_0, \\ P_z(s)\alpha_v(s)/\beta_v(\Phi_s) & \text{if } v = v_0. \end{cases}$$

Note that $J_v(\Phi_s)$ is equal to 1 for almost all finite v. This is the desired result.

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