Pacific Journal of Mathematics

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Volume 289 No. 2 August 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 289, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.289.257

REGULAR REPRESENTATIONS OF COMPLETELY BOUNDED MAPS

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We study properties and the structure of some special classes of homomorphisms on C^*-algebras. These maps are $*$-preserving up to conjugation by a symmetry. Making use of these homomorphisms, we prove a new structure theorem for completely bounded maps from a unital C^*-algebra into the algebra of all bounded linear maps on a Hilbert space. Finally we provide alternative proofs for some of the known results about completely bounded maps and improve on them.

1. Introduction

Completely positive maps and completely bounded maps on C^* algebras are well-studied objects (see [Arveson 1969; Paulsen 1986; Pisier 2001]). We look at two well-known structure theorems for completely bounded maps. The first one, which we call the *fundamental representation theorem* of completely bounded maps (Paulsen 2002; Wittstock 1981; 1984; Haagerup 1980) says that all completely bounded maps from a unital C^*-algebra $\mathcal{A}$ into the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ can be obtained from a unital representation of $\mathcal{A}$ on another Hilbert space, by composing it with two bounded operators. Unlike Stinespring's [1955] representation theorem for completely positive maps, there is no minimality condition on the representing Hilbert space, and hence the fundamental representation is not unique. The second structure theorem, namely *commutant representation theorem* (proved by Paulsen and Suen [1985]) says that all such completely bounded maps can be obtained from a unital representation, by first multiplying by an element in the commutant and then conjugating by a bounded operator. Later, using the theory of Hilbert C^*-modules, Heo [1999] proved analogues of these structure theorems for the case when the range algebra is an injective C^*-algebra.

We prove a new structure theorem for completely bounded maps (Theorems 3.2, 3.6). The idea is that to represent completely positive maps in Stinespring's theorem,

MSC2010: primary 46L07, 46L08; secondary 46L05.
Keywords: C^*-algebra, completely bounded map, Hilbert C^*-module, Stinespring's theorem, $*$-homomorphism.

one requires $*$-homomorphisms, and for a similar representation of completely bounded maps we need to consider homomorphisms which are not necessarily $*$-preserving. This is necessary because conjugation of a $*$-homomorphism by a bounded operator is always a completely positive map. However, it is possible to impose symmetry or some other additional restrictions on the homomorphisms in order to realize all completely bounded maps. That is what we do here.

To begin with, symmetries are self-adjoint unitaries. A homomorphism is symmetric if it preserves adjoints modulo conjugation by a symmetry. We define *regular homomorphisms* and *ternary homomorphisms*. We demonstrate that these homomorphisms are symmetric. Regular homomorphisms are essentially direct sums or direct integrals of ternary homomorphisms.

In Section 3A we prove that (Theorem 3.2) every completely bounded map from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ can be obtained by composing a regular homomorphism with a single bounded operator. We will call this the *regular representation theorem* for completely bounded maps. Moreover, we show that there exists a universal regular representation which can be used to generate all completely bounded maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. Since all representations (i.e., $*$-homomorphisms) are regular homomorphisms, we may consider the regular representation theorem as an immediate generalization of Stinespring's representation theorem for completely positive maps. We look at Hilbert C^*-module versions of these results. We also provide (Section 3B) new proofs of some results due to Paulsen and Suen [1985] and Heo [1999]. In Section 3C we study natural relationships between different representation theorems of completely bounded maps.

1A. ***Basic definitions and results:*** Throughout $\mathcal{H}$, $\mathcal{K}$ are complex Hilbert spaces. Our inner products are linear in the second variable and conjugate linear in the first variable. Typically $\mathcal{A}, \mathcal{B}, \mathcal{C}$ denote unital C^*-algebras. For an element a of a unital C^*-algebra $\mathcal{A}$, $\sigma(a)$ denote the spectrum of a. For a subset $\mathcal{X}$ of $\mathcal{A}$, the "commutant" is defined as $\mathcal{X}' = \{a \in \mathcal{A} : xa = ax \quad \text{for all } x \in \mathcal{X}\}$.

Suppose $\mathcal{A}, \mathcal{B}$ are C*-algebras. A multiplicative linear map $\pi : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism*. By a $*$-*homomorphism* we mean a homomorphism which is also $*$-preserving, i.e., $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$. If a $*$-homomorphism is mapping into the algebra of all bounded operators on a Hilbert space, or into the algebra of all bounded adjointable operators on a Hilbert C^*-module we call it a representation. If $\mathcal{A}, \mathcal{B}$ are unital C^*-algebras and $\pi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, then π is said to be *unital*.

Recall that a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras is said to be

(i) a *completely positive map* (CP-map) if for all $n \geq 1$, $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by $\varphi_n([a_{ij}]) = [\varphi(a_{ij})]$ is positive, i.e., $\varphi_n([a_{ij}]) \geq 0$ for all $0 \leq [a_{ij}] \in M_n(\mathcal{A})$,

(ii) a *completely bounded map* (CB-map) if $\|\varphi\|_{cb} := \sup\|\varphi_n\| < \infty$,

(iii) a *completely contractive map* (CC-map) if $\|\varphi\|_{cb} \leq 1$.

We let $CB(\mathcal{A}, \mathcal{B})$ denote the space of all CB-maps from $\mathcal{A}$ into $\mathcal{B}$. Given a map $\psi : \mathcal{A} \to \mathcal{B}$ define $\psi^* : \mathcal{A} \to \mathcal{B}$ by $\psi^*(a) := \psi(a^*)^*$ for all $a \in \mathcal{A}$. Note that if $\psi \in CB(\mathcal{A}, \mathcal{B})$, then $\psi^* \in CB(\mathcal{A}, \mathcal{B})$ with $\|\psi^*\|_{cb} = \|\psi\|_{cb}$. See [Paulsen 2002] for details on basic results.

Stinespring [1955] proved that if $\mathcal{A}$ is a unital C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CP-map, then there exists a triple $(\mathcal{K}, \pi, V)$, called Stinespring's dilation for φ, consisting of a unital representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ of $\mathcal{A}$ on a Hilbert space $\mathcal{K}$ and a bounded linear map $V : \mathcal{H} \to \mathcal{K}$ with $\|\varphi\|_{cb} = \|V\|^2$ such that $\varphi(\,\cdot\,) = V^*\pi(\,\cdot\,)V$. Moreover, $\mathcal{K}$ can be chosen to be "minimal" in the sense that $\mathcal{K} = \overline{\mathsf{span}}\,\pi(\mathcal{A})V\mathcal{H}$, and in such case the triple is unique up to unitary equivalence. Conversely, $\varphi(\,\cdot\,) := V^*\pi(\,\cdot\,)V$ is a CP-map if π is a representation. If π is a J-homomorphism (but not a $*$-homomorphism), then $\varphi(\,\cdot\,) := V^*\pi(\,\cdot\,)V$ need not be CP. In fact it need not even be positive. But if π is in a special class of J-homomorphisms, called regular homomorphisms, then φ is a CB-map. Theorem 3.2 says that all CB-maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ are of this form.

Here we recall basics of Hilbert C^*-module theory we will use. Given a C^*-algebra $\mathcal{B}$, by an *inner product $\mathcal{B}$-module* we mean a complex vector space E with a right $\mathcal{B}$-module structure and a $\mathcal{B}$-valued inner product $\langle\,\cdot\,,\,\cdot\,\rangle : E \times E \to \mathcal{B}$ satisfying

(i) $\langle x, x\rangle \geq 0$,

(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$,

(iii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$,

(iv) $\langle x, y\rangle = \langle y, x\rangle^*$,

(v) $\langle x, yb\rangle = \langle x, y\rangle b$,

for all $x, y, z \in E,\ b \in \mathcal{B},\ \lambda \in \mathbb{C}$. If E is complete with respect to the norm $\|x\| := \|\langle x, x\rangle\|^{1/2}$, then E is called a *Hilbert $\mathcal{B}$-module*. An inner product $\mathcal{B}$-module for which the condition (ii) does not hold is called a *semi-inner product $\mathcal{B}$-module*. For a semi-inner product $\mathcal{B}$-module we have

$$\langle x, y\rangle^*\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle.$$

A linear map $T : E \to F$ between two Hilbert $\mathcal{B}$-modules is said to be *adjointable* if there exists a linear map $T^* : F \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E,\ y \in F$. Adjointable maps are bounded and $\mathcal{B}$-linear (i.e., $T(\lambda x_1 + x_2 b) = \lambda T(x_1) + T(x_2)b$ for all $\lambda \in \mathbb{C},\ x_i \in E,\ b \in \mathcal{B}$). But the converse may not be true. We denote the space of all bounded and adjointable maps from E into F by $\mathcal{B}^{\mathrm{a}}(E, F)$, which is a Banach space under the operator norm. In particular $\mathcal{B}^{\mathrm{a}}(E) := \mathcal{B}^{\mathrm{a}}(E, E)$ forms a C^*-algebra with natural algebraic operations.

In the following E, F are Hilbert C^*-modules. An inner product preserving map $V : E \to F$ is called an *isometry*, and a surjective isometry is called a *unitary*. An isometry with complemented range is adjointable. In general, closed submodules of Hilbert C^*-modules are complemented only if there is an adjointable projection onto that submodule.

If x, y are the elements of a Hilbert $\mathcal{B}$-module E we let $|x\rangle\langle y|$ denote the adjointable operator $z \mapsto x\langle y, z\rangle$. Note that $(|x\rangle\langle y|)^* = |y\rangle\langle x|$. We let $\mathcal{K}(E)$ denote the completion of $\mathcal{F}(E) := \overline{\mathsf{span}}\{|x\rangle\langle y| : x, y \in E\}$ (Always in the module context "span" would mean $\mathcal{B}$-linear span). A Hilbert $\mathcal{B}$-module E is said to be a *Hilbert $\mathcal{A}$-$\mathcal{B}$-module* if there exists (a left module action of $\mathcal{A}$ on E, i.e.,) a $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ such that $\overline{\mathsf{span}}\{\vartheta(a)x : a \in \mathcal{A}, x \in E\} = E$ (or equivalently ϑ is unital if $\mathcal{A}$ is unital). Clearly any Hilbert $\mathcal{B}$-module E is a Hilbert $\mathcal{B}^{\mathrm{a}}(E)$-$\mathcal{B}$-module with left module action given by the identity map. In fact, E is a $\mathcal{K}(E)$-$\mathcal{B}$-module with inclusion map as the left module action. Given $x \in E$ we let $x^* \in \mathcal{B}^{\mathrm{a}}(E, \mathcal{B})$ denote the adjointable map $y \mapsto \langle x, y\rangle$. Note that $E^* := \overline{\mathsf{span}}\{x^* : x \in E\}$ forms a Hilbert $\mathcal{B}^{\mathrm{a}}(E)$-module with inner-product $\langle x_1^*, x_2^*\rangle := |x_1\rangle\langle x_2|$ and right module action $x^*a := (a^*x)^*$ for all $a \in \mathcal{B}^{\mathrm{a}}(E)$. Moreover, $\vartheta : \mathcal{B} \to \mathcal{B}^{\mathrm{a}}(E^*)$ given by $\vartheta(b)x^* := (xb^*)^*$ is a $*$-homomorphism such that $E^* = \overline{\mathsf{span}}\{\vartheta(\mathcal{B})E^*\}$ so that E^* forms a Hilbert $\mathcal{B}$-$\mathcal{B}^{\mathrm{a}}(E)$-module. If F is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module, then by $\mathcal{B}^{\mathrm{a,bil}}(F)$ we mean the space of all adjointable, bilinear (i.e., preserves both left and right module actions) maps on F.

Suppose E is a Hilbert $\mathcal{B}$-module and F is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module with left action given by the $*$-homomorphism $\vartheta : \mathcal{B} \to \mathcal{B}^{\mathrm{a}}(F)$. We let $E \odot_\vartheta F$ (or $E \odot_{\mathcal{B}} F$ or simply $E \odot F$) denote the completion of the algebraic tensor product $E \otimes F$ with respect to the $\mathcal{C}$-valued semi-inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle := \langle y_1, \vartheta(\langle x_1, x_2\rangle)y_2\rangle.$$

We let $x \odot y \in E \odot_\vartheta F$ denote the equivalence class containing $x \otimes y \in E \otimes F$. Note that $E \odot_\vartheta F$ forms a Hilbert $\mathcal{C}$-module with right module action $(x \odot y)c := x \odot yc$. In addition if E has a left action of $\mathcal{A}$ via a $*$-homomorphism $\vartheta' : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$, then $E \odot_\vartheta F$ also has a left module action $\tilde{\vartheta} : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E \odot_\vartheta F)$ given by $\tilde{\vartheta}(a)(x \odot y) := \vartheta'(a)x \odot y$. Thus $E \odot_\vartheta F$ forms a Hilbert $\mathcal{A}$-$\mathcal{C}$-module. Note that $\|xb \odot y - x \odot \vartheta(b)y\| = 0$ so that $xb \odot y = x \odot \vartheta(b)y$ for all $x \in E, y \in F, b \in \mathcal{B}$. We may identify $\mathcal{B} \odot_\vartheta F = F$ via the unitary isomorphism $b \odot y \mapsto \vartheta(b)y$. If $a \in \mathcal{B}^{\mathrm{a}}(E)$ and $\mathfrak{a} \in \mathcal{B}^{\mathrm{a,bil}}(F)$, then $a \odot I_F \in \mathcal{B}^{\mathrm{a}}(E \odot F)$ and $I_E \odot \mathfrak{a} \in \mathcal{B}^{\mathrm{a}}(E \odot F)$ are the maps defined by $x \odot y \mapsto ax \odot y$ and $x \odot y \mapsto x \odot \mathfrak{a}y$, respectively.

Suppose ρ is a representation of a C^*-algebra $\mathcal{B}$ on a Hilbert space $\mathcal{G}$. Given a Hilbert $\mathcal{B}$-module E, by considering $\mathcal{G}$ as a Hilbert $\mathcal{B}$-$\mathbb{C}$-module with left module action given by ρ, we let $\eta : \mathcal{B}^{\mathrm{a}}(E) \to \mathcal{B}(E \odot_\rho \mathcal{G})$ denote the unital $*$-homomorphism $a \mapsto a \odot I_{\mathcal{G}}$, that is, $\eta(a)(x \odot g) := ax \odot g$ for all $a \in \mathcal{B}^{\mathrm{a}}(E),\ x \in E,\ g \in \mathcal{G}$.

We refer to [Lance 1994; Paschke 1973; Skeide 2000] for the basic theory of Hilbert C^*-modules.

2. Symmetric homomorphisms

In this section we study homomorphisms between C^*-algebras which are not necessarily $*$-homomorphisms. These homomorphisms need not be contractive.

2A. *Symmetries.*

Definition 2.1. An element J in a unital C^*-algebra $\mathcal{B}$ satisfying $J = J^* = J^{-1}$ is called a *symmetry*. A homomorphism $\tau : \mathcal{A} \to \mathcal{B}$ is called a *J-homomorphism* if $J\tau(a)^*J = \tau(a^*)$ for all $a \in \mathcal{A}$. A homomorphism $\tau : \mathcal{A} \to \mathcal{B}$ is said to be a *symmetric homomorphism* if τ is a J-homomorphism for some symmetry $J \in \mathcal{B}$.

Clearly all $*$-homomorphisms are symmetric homomorphisms. But the converse is not true. For example, $\tau : \mathbb{C} \to \mathcal{B}(\mathbb{C}^2)$ given by $a \mapsto \left[\begin{smallmatrix} a/2 & a/4 \\ a & a/2 \end{smallmatrix}\right]$ is a J-homomorphism where $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. But τ is not $*$-preserving. It is easily seen that τ is neither positive nor contractive.

Example 2.2. Define $\tau : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by $\tau(a) = sas^{-1}$, where $s = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Clearly τ is a homomorphism. But it is not a symmetric homomorphism. For, suppose $J \in M_2(\mathbb{C})$ is a symmetry; then $J\tau(a^*)J = \tau(a)^*$ implies that $(s^*Js)a^* = a^*(s^*Js)$ for all $a \in M_2(\mathbb{C})$. Hence there exists $\lambda \in \mathbb{C}$ such that $s^*Js = \lambda I$, so that $J = \lambda(ss^*)^{-1}$ which is not a symmetry. Exactly the same situation arises when s is an invertible element which is not a scalar multiple of a unitary.

The next proposition answers the question of uniqueness of symmetry J in a symmetric homomorphism.

Proposition 2.3. *Suppose $\tau : \mathcal{A} \to \mathcal{B}$ is a symmetric homomorphism. If there exist symmetries J, $J' \in \mathcal{B}$ such that τ is both a J- and J'-homomorphism, then there exists a unitary $U \in \tau(\mathcal{A})' \subseteq \mathcal{B}$ such that $J = UJU$ and $J' = JU$.*

Proof. We have $J\tau(a)J = \tau(a^*)^* = J'\tau(a)J'$ for all $a \in \mathcal{A}$. Multiplying on the left and right side of this equation by J and J' respectively, we get $\tau(a)JJ' = JJ'\tau(a)$ for all $a \in \mathcal{A}$. Hence there exists a $U \in \tau(\mathcal{A})'$ such that $JJ' = U$. Clearly $U^*U = I = UU^*$ and $J' = JU$. Further, $(J')^* = J'$, yields $J = UJU$. $\square$

Now we show that given any homomorphism, we can associate a symmetry J in a very natural way. The usefulness of this symmetry will be seen later.

Proposition 2.4. *Suppose $\tau : \mathcal{A} \to \mathcal{B}$ is a homomorphism. Then there exists a symmetry $J \in \mathcal{B}$ such that: $J\tau(a)J = \tau(a)^*$ for all $a \in \mathcal{A}$ satisfying $\tau(a)^*\tau(1) = \tau(1)^*\tau(a)$ and $\tau(a)\tau(1)^* = \tau(1)\tau(a)^*$.*

Proof. Suppose $T := \tau(1) = R + iS$ is the cartesian decomposition of $\tau(1)$. Since $\tau(1)^2 = \tau(1)$ we have $R^2 - S^2 = R$ and $RS + SR = S$. Also since $R = R^*$ with $R^2 - R = S^2 \geq 0$ we have $\sigma(R) \subseteq \mathbb{R} \setminus (0, 1)$. Define $\hat{f} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{f}(t) = \begin{cases} -1 & \text{if } t \leq 0, \\ 2t - 1 & \text{if } 0 < t < 1, \\ 1 & \text{if } t \geq 1, \end{cases}$$

which is clearly a continuous function. Set $f = \hat{f}|_{\sigma(R)} \in C(\sigma(R)) \cong C^*(\{1, R\})$ and $J = f(R) \in \mathcal{B}$. Clearly $J^2 = I$ and $J = J^*$.

Step 1: First we prove that $JTJ = T^*$. It is enough to show that $JR = RJ$ and $JS = -SJ$. Clearly $JR = RJ$. Now

$$RS + SR = S \Longrightarrow RS = S(1 - R) \Longrightarrow R^n S = S(1 - R)^n \quad \text{for all} \quad n \geq 0.$$

Approximating f by polynomials, from $C(\sigma(R))$ we get $f(R)S = Sf(1 - R)$. But since

$$\hat{f}(1 - x) = \begin{cases} 1 & \text{if } x < 1, \\ -1 & \text{if } x \geq 1, \end{cases}$$

we have $\hat{f}(1 - x) = -\hat{f}(x)$ for all $x \in \sigma(R)$, and hence $f(1 - R) = -f(R)$. Thus

$$JS = f(R)S = Sf(1 - R) = -Sf(R) = -SJ.$$

Step 2: Fix $a \in \mathcal{A}$. Let $\tau(a) = X + iY$ be the Cartesian decomposition of $\tau(a)$. Then

$$X = \tfrac{1}{2}(\tau(a) + \tau(a)^*) = \tfrac{1}{2}(\tau(a)T + (T\tau(a))^*) = XR - YS,$$
$$Y = \tfrac{1}{2i}(\tau(a) - \tau(a)^*) = \tfrac{1}{2i}(T\tau(a) - (\tau(a)T)^*) = RY + SX.$$

Since X and Y are self-adjoint we have

$$XR - YS = X = RX - SY, \tag{2-1}$$
$$XS + YR = Y = RY + SX. \tag{2-2}$$

Now if $\tau(a)^*\tau(1) = \tau(1)^*\tau(a)$ and $\tau(a)\tau(1)^* = \tau(1)\tau(a)^*$, then we get

$$(XR + YS) + i(XS - YR) = (RX + SY) + i(RY - SX),$$
$$(XR + YS) - i(XS - YR) = (RX + SY) - i(RY - SX).$$

Adding and subtracting above two equations we get

$$XR + YS = RX + SY, \tag{2-3}$$
$$XS - YR = RY - SX. \tag{2-4}$$

Adding equations (2-1) and (2-3) we get $XR = RX$, hence $Xf(R) = f(R)X$, i.e., $XJ = JX$. Adding equations (2-2) and (2-4) we get $XS = RY$. Now since $JS = -SJ$, from equation (2-2), we have

$$YJ = (XS+YR)J = (XS+SX)J = -J(XS+SX) = -J(XS+YR) = -JY.$$

Now a direct computation shows that $J\tau(a)J = \tau(a)^*$. □

Now we introduce two subfamilies of symmetric homomorphisms, and study their structure and properties. Later, in terms of these maps we prove a new structure theorem for completely bounded maps. Before proceeding, we give a definition.

Definition 2.5. A linear map $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is said to be

(i) *nondegenerate* if $\overline{\mathsf{span}}\{\tau(a)x : a \in \mathcal{A},\ x \in E\} = E$;

(ii) $*$-*nondegenerate* if $\overline{\mathsf{span}}\{\tau(a_1)x_1, \tau(a_2)^*x_2 : x_i \in E,\ a_i \in \mathcal{A},\ i = 1, 2\} = E$.

Remark 2.6. (i) If τ is a homomorphism, then $\tau(a) = \tau(1)\tau(a)$ and $\tau(a)^* = \tau(1)^*\tau(a)^*$, therefore τ is $*$-nondegenerate if and only if

$$\overline{\mathsf{span}}\{\tau(1)x, \tau(1)^*x' : x, x' \in E\} = E.$$

A $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is $*$-nondegenerate if and only if τ is nondegenerate or equivalently τ is unital.

(ii) Suppose a Hilbert space $\mathcal{H}$ plays the role of E. Then a linear map $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is $*$-nondegenerate if and only if $\{h \in \mathcal{H} : \tau(a)h = 0 = \tau(a)^*h \text{ for all } a \in \mathcal{A}\} = \{0\}$. If τ is a homomorphism, then the above conditions are equivalent to the condition $\{h \in \mathcal{H} : \tau(1)h = 0 = \tau(1)^*h\} = \{0\}$.

2B. *Regular homomorphisms.*

Definition 2.7. A map $\tau : \mathcal{A} \to \mathcal{B}$ is said to be *regular* if $\tau(u)^*\tau(u) = \tau(1)^*\tau(1)$ and $\tau(u)\tau(u)^* = \tau(1)\tau(1)^*$ for all unitary $u \in \mathcal{A}$.

Example 2.8. (i) The map $\tau : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined in Example 2.2 is a homomorphism but it is not regular. Because $\tau(u)^*\tau(u) \neq \tau(1)^*\tau(1)$ for the unitary $u = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

(ii) All $*$-homomorphisms are regular. But the converse is not true. For example $\tau : \mathcal{A} \to M_2(\mathcal{A})$ given by $\tau(a) = \left[\begin{smallmatrix} a & 0 \\ a & 0 \end{smallmatrix}\right]$ is a regular homomorphism but it is not $*$-preserving.

Proposition 2.9. *Suppose $\tau : \mathcal{A} \to \mathcal{B}$ is a unital homomorphism. Then τ is regular if and only if it is $*$-preserving.*

Proof. Suppose τ is a unital regular homomorphism. Then for all unitary $u \in \mathcal{A}$,

$$\tau(u)^* = \tau(u)^*\tau(uu^*) = \tau(u)^*\tau(u)\tau(u^*) = \tau(1)^*\tau(1)\tau(u^*) = \tau(u^*).$$

Since any $a \in \mathcal{A}$ is a linear combination of at most four unitaries it follows that $\tau(a)^* = \tau(a^*)$. The converse is obvious. $\square$

The following theorem says that all regular homomorphisms preserve conjugation $*$ up to a symmetry. This is one of the reasons to study regular homomorphisms.

Theorem 2.10. *Every regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}$ is symmetric.*

Proof. Suppose $J \in \mathcal{B}$ is the symmetry given by Proposition 2.4. Since τ is regular, given any unitary $u \in \mathcal{A}$, we have

$$\tau(u)^*\tau(1) = \tau(u)^*\tau(u)\tau(u^*) = \tau(1)^*\tau(1)\tau(u^*) = \tau(1)^*\tau(u^*).$$

Since u^* is also a unitary, we also get $\tau(u^*)^*\tau(1) = \tau(1)^*\tau(u)$. In a similar fashion, by regularity,

$$\tau(1)\tau(u)^* = \tau(u^*)\tau(u)\tau(u)^* = \tau(u^*)\tau(1)\tau(1)^* = \tau(u^*)\tau(1)^*,$$

and replacing u by u^*, $\tau(1)\tau(u^*)^* = \tau(u)\tau(1)^*$. So if we let $u_1 := u + u^*$ and $u_2 := u - u^*$, then $\tau(u_1)^*\tau(1) = \tau(1)^*\tau(u_1)$ and $\tau(u_1)\tau(1)^* = \tau(1)\tau(u_1)^*$, so that Proposition 2.4 is applicable and we get $J\tau(u_1)J = \tau(u_1)^*$. On the other hand, as $u_2^* = -u_2$ and Proposition 2.4 can be applied to iu_2. Then we get $J\tau(u_2)J = -\tau(u_2)^*$. Thus $J\tau(u^*)J = \tau(u)^*$ for every unitary u. Since every element in a C^*-algebra can be written as a linear combination of at most four unitaries it follows that $J\tau(a^*)J = \tau(a)^*$ for all $a \in \mathcal{A}$. $\square$

Example 2.11. Suppose $\tau : \mathcal{A} \to \mathcal{B}$ is a J-homomorphism for some symmetry $J \in \mathcal{B}$. Then it can be seen that $\tau(u)^*\tau(u) = \tau(1)^*\tau(1)$ for all unitary $u \in \mathcal{A}$ if and only if $\tau(u)\tau(u)^* = \tau(1)\tau(1)^*$ for all unitary $u \in \mathcal{A}$. But for general homomorphisms this is not true. For example, let $\mathcal{A} = M_2(\mathbb{C})$ and let $v = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \mathcal{A}$. Define a homomorphism $\tau : \mathcal{A} \to M_2(\mathcal{A})$ by $\tau(a) = \left[\begin{smallmatrix} a & av \\ 0 & 0 \end{smallmatrix}\right]$. Then τ satisfies $\tau(u)^*\tau(u) = \tau(1)^*\tau(1)$ for all unitary u. But $\tau(u)\tau(u)^* \neq \tau(1)\tau(1)^*$ for the unitary $u = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Example 2.12. Suppose $v \in \mathcal{A} = \mathcal{B}(\mathcal{H})$ is a nonscalar unitary. Define a homomorphism $\tau : \mathcal{A} \to M_2(\mathcal{A})$ by $\tau(a) = \left[\begin{smallmatrix} a & \sqrt{3}(va-av) \\ 0 & a \end{smallmatrix}\right]$. Then τ is symmetric with symmetry $J = \frac{1}{2}\left[\begin{smallmatrix} -1 & \sqrt{3}v \\ \sqrt{3}v^* & 1 \end{smallmatrix}\right]$. But τ is not regular since $\tau(u)^*\tau(u) \neq \tau(1)^*\tau(1)$ for any unitary $u \in \mathcal{A}$ not commuting with v.

Proposition 2.13. *Let $\tau : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then τ is regular if and only if for all $a, b \in \mathcal{A}$,*

(i) $\tau(a)^*\tau(b) = \tau(1)^*\tau(a^*b) = \tau(b^*a)^*\tau(1)$,

(ii) $\tau(a)\tau(b)^* = \tau(ab^*)\tau(1)^* = \tau(1)\tau(ba^*)^*$.

Proof. Assume that τ is regular. Suppose u is a unitary. Then for any $b, c \in \mathcal{A}$,

$$\tau(u)^*\tau(b) = \tau(u)^*\tau(u)\tau(u^*b) = \tau(1)^*\tau(1)\tau(u^*b) = \tau(1)^*\tau(u^*b), \tag{2-5}$$

$$\tau(c)\tau(u)^* = \tau(cu^*)\tau(u)\tau(u)^* = \tau(cu^*)\tau(1)\tau(1)^* = \tau(cu^*)\tau(1)^*. \tag{2-6}$$

Since any element in $\mathcal{A}$ is a linear combination of at most four unitaries from equations (2-5) and (2-6) we have $\tau(a)^*\tau(b) = \tau(1)^*\tau(a^*b)$ and $\tau(c)\tau(d)^* = \tau(cd^*)\tau(1)^*$, respectively for all $a, d \in \mathcal{A}$. Taking adjoints of these equalities proves (i) and (ii). The converse is obvious. □

Note that given a (unital) $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ and an idempotent operator $T \in \vartheta(\mathcal{A})' \subseteq \mathcal{B}^{\mathrm{a}}(E)$ the map $a \mapsto \vartheta(a)T$ always defines a bounded ($*$-nondegenerate) regular homomorphism from $\mathcal{A}$ into $\mathcal{B}^{\mathrm{a}}(E)$. Now we will prove that all $*$-nondegenerate regular homomorphisms can be represented this way.

Theorem 2.14. *Suppose $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a $*$-nondegenerate, regular homomorphism. Then there exists a unique unital $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ such that $\tau(a) = \vartheta(a)\tau(1) = \tau(1)\vartheta(a)$ for all $a \in \mathcal{A}$. Consequently τ is completely bounded with $\|\tau\|_{cb} = \|\tau(1)\|$.*

Proof. If τ is unital, then it is $*$-preserving and in such case we take $\vartheta = \tau$. Otherwise let $E_0 = \mathsf{span}\{\tau(\mathcal{A})E, \tau(\mathcal{A})^*E\} = \mathsf{span}\{\tau(1)E, \tau(1)^*E\}$. Now for each unitary $u \in \mathcal{A}$, define $\vartheta(u) : E_0 \to E_0$ by $\vartheta(u)\big(\sum_i \tau(1)x_i + \tau(1)^*y_i\big) = \sum_i (\tau(u)x_i + \tau(u^*)^*y_i)$ for all $x_i, y_i \in E$. Since

$$
\begin{aligned}
&\Big\|\vartheta(u)\Big(\sum_i \tau(1)x_i + \tau(1)^*y_i\Big)\Big\|^2 \\
&\quad= \Big\|\Big\langle \sum_i \tau(u)x_i + \tau(u^*)^*y_i, \sum_j \tau(u)x_j + \tau(u^*)^*y_j \Big\rangle\Big\| \\
&\quad= \Big\|\sum_{i,j}(\langle x_i, \tau(u)^*\tau(u)x_j\rangle + \langle x_i, \tau(u)^*\tau(u^*)^*y_j\rangle \\
&\qquad\qquad + \langle y_i, \tau(u^*)\tau(u)x_j\rangle + \langle y_i, \tau(u^*)\tau(u^*)^*y_j\rangle)\Big\| \\
&\quad= \Big\|\sum_{i,j}(\langle x_i, \tau(1)^*\tau(1)x_j\rangle + \langle x_i, \tau(1)^*y_j\rangle + \langle y_i, \tau(1)x_j\rangle + \langle y_i, \tau(1)\tau(1)^*y_j\rangle)\Big\| \\
&\quad= \Big\|\Big\langle \sum_i \tau(1)x_i + \tau(1)^*y_i, \sum_j \tau(1)x_j + \tau(1)^*y_j \Big\rangle\Big\| \\
&\quad= \Big\|\sum_i \tau(1)x_i + \tau(1)^*y_i\Big\|^2,
\end{aligned}
$$

we see that $\vartheta(u)$ is well defined and norm preserving on E_0. It is also $\mathcal{B}$-linear. Hence $\vartheta(u)$ is an isometry. Note that $\operatorname{ran}(\vartheta(u)) = E_0$ so that $\vartheta(u) : E_0 \to E_0$ is a unitary. Now given a linear combination of unitaries, say $a = \sum \lambda_i u_i \in \mathcal{A}$, we define $\vartheta(a) := \sum \lambda_i \vartheta(u_i)$. Note that if $\sum \lambda_i u_i = 0$, then

$$\begin{aligned}\vartheta\Big(\sum_i \lambda_i u_i\Big)\Big(\sum_j \tau(1)x_j + \tau(1)^* y_j\Big) \\ &= \sum_i \lambda_i \vartheta(u_i)\Big(\sum_j \tau(1)x_j + \tau(1)^* y_j\Big) \\ &= \sum_{i,j} (\lambda_i \tau(u_i)x_j + \lambda_i \tau(u_i^*)^* y_j) \\ &= \sum_j \bigg(\tau\Big(\sum_i \lambda_i u_i\Big)\tau(1)x_j + \tau\Big(\sum_i \bar{\lambda}_i u_i^*\Big)^* \tau(1)^* y_j\bigg) \\ &= 0,\end{aligned}$$

so that $\vartheta\big(\sum \lambda_i u_i\big) = 0$. Thus the definition of $\vartheta(a) : E_0 \to E_0$ is independent of the choice of linear combination $\sum \lambda_i u_i$. Note that if $a = \sum \lambda_i u_i \in \mathcal{A}$, then

$$\begin{aligned}\vartheta(a)(\tau(a_1)x_1 + \tau(a_2)^* x_2) \\ &= \sum \lambda_i \vartheta(u_i)\tau(1)\tau(a_1)x_1 + \sum \lambda_i \vartheta(u_i)\tau(1)^* \tau(a_2)^* x_2 \\ &= \sum \lambda_i \tau(u_i a_1)x_1 + \sum \lambda_i \tau(a_2 u_i^*)^* x_2 \\ &= \tau\Big(\sum \lambda_i u_i a_1\Big)x_1 + \tau\Big(a_2 \sum \bar{\lambda}_i u_i^*\Big)^* x_2 \\ &= \tau(a a_1)x_1 + \tau(a_2 a^*)^* x_2\end{aligned} \tag{2-7}$$

for all $x_i \in E$. Now it follows that $\vartheta(1) = I_{E_0}$, $\vartheta(a+b) = \vartheta(a) + \vartheta(b)$ and $\vartheta(a)\vartheta(b) = \vartheta(ab)$ for all $a, b \in \mathcal{A}$. Since any element in $\mathcal{A}$ is a linear combination of at most four unitaries and $\|\vartheta(u)\| \le 1$ for all unitary $u \in \mathcal{A}$, we have $\|\vartheta(a)\| < \infty$ for all $a \in \mathcal{A}$. Hence each $\vartheta(a) : E_0 \to E_0$ can be extended to a bounded operator, again denoted by $\vartheta(a)$, on $E = \overline{E_0}$. Also, from Proposition 2.13, for all $a \in \mathcal{A}$, $x_i \in E$ we have

$$\begin{aligned}&\langle \vartheta(a)(\tau(1)x_1 + \tau(1)^* x_2), \tau(1)x_3 + \tau(1)^* x_4\rangle \\ &\quad = \langle x_1, \tau(a)^*\tau(1)x_3\rangle + \langle x_1, \tau(a)^*\tau(1)^* x_4\rangle + \langle x_2, \tau(a^*)\tau(1)x_3\rangle + \langle x_2, \tau(a^*)\tau(1)^* x_4\rangle \\ &\quad = \langle x_1, \tau(1)^*\tau(a^*)x_3\rangle + \langle x_1, \tau(a)^* x_4\rangle + \langle x_2, \tau(a^*)x_3\rangle + \langle x_2, \tau(1)\tau(a)^* x_4\rangle \\ &\quad = \langle \tau(1)x_1 + \tau(1)^* x_2, \tau(a^*)x_3 + \tau(a)^* x_4\rangle \\ &\quad = \langle \tau(1)x_1 + \tau(1)^* x_2, \vartheta(a^*)(\tau(1)x_3 + \tau(1)^* x_4)\rangle,\end{aligned}$$

so that $\langle \vartheta(a)x, x'\rangle = \langle x, \vartheta(a^*)x'\rangle$ for all $x, x' \in E$, that is, $\vartheta(a)$ is adjointable with $\vartheta(a)^* = \vartheta(a^*)$. Thus $a \mapsto \vartheta(a)$ defines a unital $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$. Moreover, from (2-7) we have $\vartheta(a)\tau(1)x = \tau(a)x$ and $\vartheta(a)\tau(1)^* x = \tau(a^*)^* x$ for all $x \in E$. Hence we get $\vartheta(a)\tau(1) = \tau(a) = \tau(1)\vartheta(a)$ for all $a \in \mathcal{A}$, therefore $\|\tau\|_{cb} \le \|\vartheta\|_{cb}\|\tau(1)\| \le \|\tau(1)\|$.

Uniqueness: If $\vartheta' : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is any other such $*$-homomorphism, then

$$\vartheta'(a)\tau(1)x = \tau(a)x = \vartheta(a)\tau(1)x,$$
$$\vartheta'(a)\tau(1)^*x' = (\tau(1)\vartheta'(a^*))^*x' = \tau(a^*)^*x' = (\tau(1)\vartheta(a^*))^*x' = \vartheta(a)\tau(1)^*x'.$$

Hence $\vartheta(a)x = \vartheta'(a)x$ for all $a \in \mathcal{A},\ x \in E = \overline{\mathsf{span}}\{\tau(1)E, \tau(1)^*E\}$. □

Remark 2.15. Suppose $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a regular homomorphism, not necessarily $*$-nondegenerate. Then also $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(\overline{E_0})$ given as in the proof is a well-defined unital $*$-homomorphism. Note that $\overline{E_0}$ is a $\tau(a)$-reducing closed $\mathcal{B}$-submodule of E. Thus the proof says that: *If $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a regular homomorphism, then there exists a closed $\mathcal{B}$-submodule $E_0 \subseteq E$, which reduces all $\tau(a)$; and a unique unital $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E_0)$ such that $\tau(a)|_{E_0} = \tau(1)\vartheta(a) = \vartheta(a)\tau(1)|_{E_0}$ for all $a \in \mathcal{A}$. Moreover, if E_0 is complemented in E, then $\tilde{\vartheta} = \left[\begin{smallmatrix} \vartheta & 0 \\ 0 & 0 \end{smallmatrix}\right] : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a $*$-homomorphism such that $\tau(a) = \tau(1)\tilde{\vartheta}(a) = \tilde{\vartheta}(a)\tau(1)$.*

Corollary 2.16. *Suppose $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a $*$-nondegenerate regular homomorphism with $\tau(1) = \tau(1)^*$. Then τ is a $*$-homomorphism.*

Proof. Suppose $\vartheta : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is the unique unital $*$-homomorphism such that $\tau(a) = \vartheta(a)\tau(1) = \tau(1)\vartheta(a)$. Then

$$\tau(a)^* = (\vartheta(a)\tau(1))^* = \tau(1)^*\vartheta(a^*) = \tau(1)\vartheta(a^*) = \tau(a^*).$$ □

Note that a unital C^*-algebra $\mathcal{B}$ is a Hilbert $\mathcal{B}$-module with inner product $\langle b, b'\rangle := b^*b'$. Moreover, $\mathcal{B} \cong \mathcal{B}^{\mathrm{a}}(\mathcal{B})$ as C^*-algebras under the unital isometric $*$-isomorphism $b \mapsto T_b$ where $T_b \in \mathcal{B}^{\mathrm{a}}(\mathcal{B})$ is given by $T_b(b') = bb'$ for all $b \in \mathcal{B}$. (Note that adjointable maps preserves module action so that $T(b) = T(1b) = T(1)b$ for all $b \in \mathcal{B},\ T \in \mathcal{B}^{\mathrm{a}}(\mathcal{B})$.) So given a linear map $\tau : \mathcal{A} \to \mathcal{B}$ we say that τ is $*$-nondegenerate if $\overline{\mathsf{span}}\{\tau(\mathcal{A})\mathcal{B}, \tau(\mathcal{A})^*\mathcal{B}\} = \mathcal{B}$.

Corollary 2.17. *Suppose $\tau : \mathcal{A} \to \mathcal{B}$ is a $*$-nondegenerate regular homomorphism. Then there exists a unique unital $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}$ such that $\tau(a) = \vartheta(a)\tau(1) = \tau(1)\vartheta(a)$ for all $a \in \mathcal{A}$. Consequently τ is completely bounded with $\|\tau\|_{cb} = \|\tau(1)\|$. Moreover, if $\tau(1) = \tau(1)^*$, then τ is a $*$-homomorphism.*

Example 2.18. Let $\mathcal{A}$ be the C^*-algebra of continuous functions on the interval $[0, 1]$. Let $\mathcal{B} = M_2(\mathcal{A})$ and let $E = \mathcal{B}$, with usual inner product, so that $B^a(E) \cong \mathcal{B}$. Let $g : [0, 1] \to \mathbb{C}$ be the function defined by $g(x) = x$ for all $x \in [0, 1]$. Define $\tau : \mathcal{A} \to B^a(E)$ by

$$\tau(f) = \begin{bmatrix} f & gf \\ 0 & 0 \end{bmatrix}.$$

Then it is easily seen that τ is a regular homomorphism, E_0 defined as above is

$$\begin{aligned} E_0 &= \mathsf{span}\{\tau(\mathcal{A})E, \tau(\mathcal{A})^* E\} \\ &= \left\{ \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} : f_{ij} \in \mathcal{B},\ 1 \le i,\ j \le 2, \quad \text{where } f_{2j}(0) = 0 \right\}, \end{aligned}$$

and is not complemented in E.

In the following E, F are Hilbert C^*-modules over possibly different C^*-algebras $\mathcal{B}$, $\mathcal{C}$ respectively. We wish to obtain a structure theorem for strictly continuous regular homomorphisms from $\mathcal{B}^a(E)$ to $\mathcal{B}^a(F)$. Recall (see [Lance 1994]) that a net $\{a_\alpha\}$ in $\mathcal{B}^a(E)$ is said to converge *strictly* (or $*$-*strongly*) to $a \in \mathcal{B}^a(E)$ if, for all $x, x' \in E$, the nets $\{a_\alpha x\}$ and $\{a_\alpha^* x'\}$ converge to ax and $a^* x'$, respectively. Note that $\{a_\alpha\}$ converges to a strictly if and only if $\{a_\alpha^*\}$ converges to a^* strictly. A bounded linear map $\tau : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is said to be *strict* if $\tau(a_\alpha)$ converges strictly to $\tau(a)$ in $\mathcal{B}^a(F)$ whenever a net $\{a_\alpha\}$ in the unit ball of $\mathcal{B}^a(E)$ converges strictly to $a \in \mathcal{B}^a(E)$.

Remark 2.19. If the map τ given in Theorem 2.14 is also a strict map (so bounded by definition), then ϑ is a strict map. For, suppose $\{a_\alpha\}$ is a net in the unit ball of $\mathcal{A} = \mathcal{B}^a(\mathcal{A})$ which converges strictly to $a \in \mathcal{A}$. Then for all $x_1, x_2 \in E$ we have

$$\vartheta(a_\alpha)\tau(1)x_1 = \tau(a_\alpha)x_1 \xrightarrow{\alpha} \tau(a)x_1 = \vartheta(a)\tau(1)x_1,$$

and

$$\begin{aligned} \vartheta(a_\alpha)\tau(1)^* x_2 = (\tau(1)\vartheta(a_\alpha^*))^* x_2 = \tau(a_\alpha^*)^* x_2 \xrightarrow{\alpha} \tau(a^*)^* x_2 &= (\tau(1)\vartheta(a^*))^* x_2 \\ &= \vartheta(a)\tau(1)^* x_2, \end{aligned}$$

so that $\{\vartheta(a_\alpha)x\}$ and $\{\vartheta(a_\alpha)^* x'\}$ converge to $\vartheta(a)x$ and $\vartheta(a)^* x'$ respectively, for all $x, x' \in E$. Thus ϑ is a strict unital $*$-homomorphism. In particular, if $\tau : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a $*$-nondegenerate, strict, regular homomorphism then the strict, unital $*$-homomorphism $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ has a factorization $\vartheta(a) = U(a \odot I)U^*$ where $U : E \odot F_\vartheta \to F$ is a unitary on a suitable Hilbert $\mathcal{B}$-$\mathcal{C}$-module F_ϑ (see [Muhly et al. 2006, Theorem 1.4]). In fact, if we consider F as a Hilbert $\mathcal{B}^a(E)$-$\mathcal{C}$-module with left action given by ϑ, then $F_\vartheta = E^* \odot_\vartheta F$ and $U \in \mathcal{B}^{a,\text{bil}}(E \odot F_\vartheta, F)$. A more generalized version says that: if $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a strict CP-map, then $\vartheta(a) = W(a \odot I)W^*$ for some bounded adjointable operator $W : E \odot F_\vartheta \to F$ on a suitable Hilbert $\mathcal{B}$-$\mathcal{C}$-module F_ϑ (see [Skeide and Sumesh 2014, Theorem 3.2]). In this case, $F_\vartheta = E^* \odot \mathcal{E} \odot F$ where $\mathcal{E}$ is the Hilbert $\mathcal{B}^a(E)$-$\mathcal{B}^a(F)$-module obtained from the *GNS construction* ([Paschke 1973, Theorem 5.2]) of the CP-map ϑ.

Recall that a Hilbert $\mathcal{B}$-module E is said to be *full* if $\overline{\mathsf{span}}\{\langle x, y\rangle : x, y \in E\} = \mathcal{B}$. The following lemma is a known result. But for the sake of completeness of the note we include a proof here.

Lemma 2.20. *Suppose F is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Then for any full Hilbert $\mathcal{B}$-module E the relative commutant of $\mathcal{B}^a(E) \odot I_F$ in $\mathcal{B}^a(E \odot F)$ is $I_E \odot \mathcal{B}^{a,bil}(F)$.*

Proof. If $T \in \mathcal{B}^{a,bil}(F)$, then $I_E \odot T \in \mathcal{B}^a(E \odot F)$ commutes with all elements of the form $a \odot I_F$ for all $a \in \mathcal{B}^a(E)$ and hence we have $I_E \odot \mathcal{B}^{a,bil}(F) \subseteq (\mathcal{B}^a(E) \odot I_F)'$. For the reverse inclusion assume that $\mathfrak{a} \in (\mathcal{B}^a(E) \odot I_F)' \subseteq \mathcal{B}^a(E \odot F)$. Since E is full and $F = \overline{\mathsf{span}}\, \mathcal{B}F = \overline{\mathsf{span}}\{\langle x_1, x_2\rangle y : x_i \in E,\ y \in F\}$ we have $F = E^* \odot_{\mathcal{B}^a(E)} E \odot_{\mathcal{B}} F$ under the identification $\langle x_1, x_2\rangle y \mapsto x_1^* \odot x_2 \odot y$. Set $T = (I_{E^*} \odot \mathfrak{a}) \in \mathcal{B}^{a,bil}(F)$. Then, since $E \odot_{\mathcal{B}} E^* \cong \mathcal{K}(E)$ via $x_1 \odot x_2^* \mapsto |x_1\rangle\langle x_2|$, and $\mathcal{K}(E) \odot_{\mathcal{K}(E)} E \odot_{\mathcal{B}} F \cong E \odot_{\mathcal{B}} F$ via $|x_1\rangle\langle x_2| \odot x \odot y \mapsto x_1\langle x_2, x\rangle \odot y$, we get

$$
\begin{aligned}
(I_E \odot T)(x_1 \odot \langle x_2, x_3\rangle y) &= (I_E \odot I_{E^*} \odot \mathfrak{a})(x_1 \odot x_2^* \odot x_3 \odot y) \\
&= x_1 \odot x_2^* \odot \mathfrak{a}(x_3 \odot y) \\
&= |x_1\rangle\langle x_2| \odot \mathfrak{a}(x_3 \odot y) \\
&= |x_1\rangle\langle x_2| \mathfrak{a}(x_3 \odot y) \\
&= (|x_1\rangle\langle x_2| \odot id_F)\mathfrak{a}(x_3 \odot y) \\
&= \mathfrak{a}(|x_1\rangle\langle x_2| \odot id_F)(x_3 \odot y) \\
&= \mathfrak{a}(|x_1\rangle\langle x_2| x_3 \odot y) \\
&= \mathfrak{a}(x_1 \odot \langle x_2, x_3\rangle y).
\end{aligned}
$$

for all $x_i \in E$, $y \in F$. Thus $T \in \mathcal{B}^{a,bil}(F)$ is such that $\mathfrak{a} = id_E \odot T$. Hence $(\mathcal{B}^a(E) \odot I_F)' \subseteq I_E \odot \mathcal{B}^{a,bil}(F)$. □

Theorem 2.21. *Suppose E is a full Hilbert $\mathcal{B}$-module, F is a Hilbert $\mathcal{B}$-$\mathcal{C}$-module and $\tau : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is a $*$-nondegenerate, strict, regular homomorphism. Then there exists a Hilbert $\mathcal{B}$-$\mathcal{C}$-module F_τ, an idempotent operator $T \in \mathcal{B}^{a,bil}(F_\tau)$ and a unitary $U : E \odot F_\tau \to F$ such that*

$$\tau(a) = U(a \odot T)U^*$$

for all $a \in \mathcal{B}^a(E)$. Moreover, the triple (F_τ, T, U) is unique up to a unitary isomorphism.

Proof. Suppose $\vartheta : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is the unique unital $*$-homomorphism such that $\tau(a) = \vartheta(a)\tau(1) = \tau(1)\vartheta(a)$. Since τ is strict we have ϑ is strict, and hence there exists a Hilbert $\mathcal{B}$-$\mathcal{C}$-module F_ϑ and a unitary $U \in \mathcal{B}^{a,bil}(E \odot F_\vartheta, F)$ such that $\vartheta(a) = U(a \odot I)U^*$. Take $F_\tau = F_\vartheta$. Then $\vartheta(a)\tau(1) = \tau(1)\vartheta(a)$ implies that $(a \odot I)U^*\tau(1)U = U^*\tau(1)U(a \odot I)$ for all $a \in \mathcal{B}^a(E)$ so that $U^*\tau(1)U \in (\mathcal{B}^a(E) \odot I_{F_\tau})'$. Hence there exists a $T \in \mathcal{B}^{a,bil}(F_\tau)$ such that $\tau(1) = U(I_E \odot T)U^*$.

Clearly, $\tau(a) = \vartheta(a)\tau(1) = U(a \odot T)U^*$. Now

$$\begin{aligned}\tau(1)^2 = \tau(1) &\Longrightarrow I_E \odot T^2 = I_E \odot T \\ &\Longrightarrow \langle (I_E \odot T^2)(x_1 \odot y_1), x_2 \odot y_2\rangle = \langle (I_E \odot T)(x_1 \odot y_1), x_2 \odot y_2\rangle \\ &\Longrightarrow \langle T^2 y_1, \langle x_1, x_2\rangle y_2\rangle = \langle T y_1, \langle x_1, x_2\rangle y_2\rangle\end{aligned}$$

for all $x_1, x_2 \in E$ and $y_1, y_2 \in F_\tau$. But since E is full and F_τ has a nondegenerate left action of $\mathcal{B}$, from equation above, we have $\langle T^2 y, y'\rangle = \langle T y, y'\rangle$ for all $y, y' \in F_\tau$, so that $T^2 = T$.

Uniqueness: Suppose (F'_τ, T', U') is another such triple. Then we have $E \odot F_\tau \cong F \cong E \odot F'_\tau$ via the unitary isomorphism U'^*U. Since E is full we identify $E^* \odot_{\mathcal{B}^a(E)} E = \mathcal{B}$ via the unitary isomorphism $x^* \odot x' \mapsto \langle x, x'\rangle$. Then

$$F_\tau = \mathcal{B} \odot F_\tau = E^* \odot E \odot F_\tau \cong E^* \odot E \odot F'_\tau = \mathcal{B} \odot F'_\tau = F'_\tau,$$

where the isomorphism is given by the unitary $\widehat{U} = (I_{E^*} \odot U'^*)(I_{E^*} \odot U) : F_\tau \to F'_\tau$. Observe that

$$U = I_E \odot I_{E^*} \odot U = I_E \odot (I_{E^*} \odot U')\widehat{U} = (I_E \odot I_{E^*} \odot U')(I_{E^*} \odot \widehat{U}) = U'(I_{E^*} \odot \widehat{U}).$$

Also since $U(I_E \odot T)U^* = \tau(I_E) = U'(I_E \odot T')U'^*$ we have

$$T = I_{E^*} \odot I_E \odot T = I_{E^*} \odot \{U^*U'(I_E \odot T')U'^*U\} = \widehat{U}^*(I_{E^*} \odot I_E \odot T')\widehat{U} = \widehat{U}^* T' \widehat{U}.$$

Thus $\widehat{U}$ gives the required unitary equivalence. □

Corollary 2.22. *Suppose τ and T are as in the theorem above. Then τ is a $*$-homomorphism if and only if $T = T^*$.*

Proof. Clearly if $T = T^*$, then τ is a $*$-homomorphism. Conversely assume that τ is a $*$-homomorphism. Then $U(I \odot T)U^* = \tau(I) = \tau(I)^* = U(I \odot T^*)U^*$. Since U is a unitary we get $I \odot T = I \odot T^*$. Since E is full, this implies $T = T^*$. □

Proposition 2.23. *Suppose $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a regular homomorphism. Then there exists a $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\tau(a) = \vartheta(a)\tau(1) = \tau(1)\vartheta(a)$ for all $a \in \mathcal{A}$. Consequently τ is completely bounded with $\|\tau\|_{cb} = \|\tau(1)\|$. If $\tau(1) = \tau(1)^*$, then τ is a $*$-homomorphism. If τ is $*$-nondegenerate, then ϑ is unique and it is unit-preserving.*

Proof. Follows from Remark 2.15. □

Suppose $\mathcal{A}$ is a von Neumann algebra and $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a normal, regular homomorphism. Then it can be verified that ϑ given by the Proposition 2.23 is a normal $*$-homomorphism. In particular if $\mathcal{A} = \mathcal{B}(\mathcal{H}')$, where $\mathcal{H}'$ is another Hilbert space, then it is well known that ϑ has a factorization $\vartheta(a) = V(a \odot I)V^*$ for some isometry $V : \mathcal{H}' \odot \mathcal{H}_\vartheta \to \mathcal{H}$ on a suitable Hilbert space $\mathcal{H}_\vartheta$. Again

$\vartheta(a)\tau(1) = \tau(1)\vartheta(a)$ for all $a \in \mathcal{B}(\mathcal{H}')$ implies that $\tau(1) = V(I \odot T)V^*$ for some $T \in \mathcal{B}(\mathcal{K}_\vartheta)$. Thus we have:

Theorem 2.24. *Suppose $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a normal, regular homomorphism. Then there exists a Hilbert space $\mathcal{K}_\tau$, an idempotent operator $T \in \mathcal{B}(\mathcal{K}_\tau)$ and an isometry $V : \mathcal{H} \odot \mathcal{K}_\tau \to \mathcal{K}$ such that*

$$\tau(a) = V(a \odot T)V^*.$$

Moreover, there exists a symmetry $J_0 \in \mathcal{B}(\mathcal{K}_\tau)$ such that τ is a J-homomorphism with $J = V(I \odot J_0)V^*$. If τ is $*$-nondegenerate, then V is a unitary and (K_τ, T, V) is unique up to unitary equivalence. Further, τ is a $*$-homomorphism if and only if $T = T^*$.

2C. *Ternary homomorphisms.*

Definition 2.25. Let $t \in \mathbb{R}$. A map $\tau : \mathcal{A} \to \mathcal{B}$ is said to be *t-ternary* if

$$\tau(a)\tau(b)^*\tau(c) = t\tau(ab^*c) \quad \text{for all } a, b, c \in \mathcal{A}.$$

A 1-ternary map is simply called a *ternary* map. Note that all $*$-homomorphisms are ternary maps. In fact if $\mathcal{A}, \mathcal{B}$ are unital C^*-algebras, then a unital linear map $\tau : \mathcal{A} \to \mathcal{B}$ is ternary if and only if τ is a $*$-homomorphism. Here is a typical example of a t-ternary homomorphism:

Example 2.26. Clearly, $\tau : \mathcal{A} \to M_2(\mathcal{A})$ given by $\tau(a) = \left[\begin{smallmatrix} a & 0 \\ (\sqrt{t-1})a & 0 \end{smallmatrix}\right]$ is a t-ternary homomorphism for all $t \in (1, \infty)$.

We are only interested in t-ternary maps which are homomorphisms. In this context, we have the following basic observation.

Proposition 2.27. *Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras and let $\tau : \mathcal{A} \to \mathcal{B}$ be a nonzero t-ternary homomorphism. Then $1 \leq t = \|\tau(1)\|^2$. If $t = 1$, then τ is a $*$-homomorphism.*

Proof. For convenience, without loss of generality, we assume $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Take $T = \tau(1)$. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ be the orthogonal decomposition of $\mathcal{H}$, where $\mathcal{H}_0 = \overline{T(\mathcal{H})}$. Since $T^2 = T$, $Th = h$ for $h \in \mathcal{H}_0$, as a consequence the operator T decomposes as

$$T = \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}$$

for some N, with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$. Now computing, $TT^*T = tT$, we see $I + NN^* = tI$. In particular $t \geq 1$. Also since $P = (1/t)T^*T$ is a nonzero projection we have $\|T\|^2 = t$.

If $t = 1$, we get $N = 0$ and hence $\tau(1)^* = \tau(1)$. Taking $a = c = 1$ in the definition of 1-ternary, we get $\tau(b)^* = \tau(1)^*\tau(b)^*\tau(1)^* = \tau(1)\tau(b)^*\tau(1) = 1.\tau(1.b^*.1) = \tau(b^*)$. Therefore, τ is a $*$-homomorphism. $\square$

Proposition 2.28. *All t-ternary homomorphisms $\tau : \mathcal{A} \to \mathcal{B}$ are regular.*

Proof. An easy computation using t-ternary and homomorphism properties yields $(\tau(1)^*\tau(1) - \tau(u)^*\tau(u))^2 = 0$ and $(\tau(1)\tau(1)^* - \tau(u)\tau(u)^*)^2 = 0$ for any unitary $u \in \mathcal{A}$. □

The converse of this proposition is not true. For example, the direct sum of t_1 and t_2-ternaries as in Example 2.26, is easily seen to be regular but not a t-ternary for any t. The direct sum of two t-ternary homomorphisms on a common domain algebra is again a t-ternary homomorphism.

From the proposition above, since all regular homomorphisms are symmetric, all t-ternary homomorphisms $\tau : \mathcal{A} \to \mathcal{B}$ are symmetric homomorphisms. Since 1-ternary homomorphisms are already $*$-homomorphisms, we will assume that $t > 1$. Now we will show that for a $*$-nondegenerate t-ternary homomorphism $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$, $t \in (1, \infty)$, a possible choice of symmetry can be written down explicitly as $(1/\sqrt{t})(\tau(1) + \tau(1)^* - I)$.

Proposition 2.29. *Suppose $t \in (1, \infty)$ and $\tau : \mathcal{A} \to \mathcal{B}$ is a t-ternary homomorphism. Take $T = \tau(1)$ and $J_t = (1/\sqrt{t})(T + T^* - I)$. Then*

(i) *$J_t\tau(a)^*J_t = \tau(a^*)$ and $J_t\tau(a^*)J_t = \tau(a)^*$ for all $a \in \mathcal{A}$;*

(ii) *$\sigma(J_t) \subseteq \{1, -1, -1/\sqrt{t}\}$.*

Proof. We have $T^2 = T$ and $TT^*T = tT$. To prove (i),

$$\begin{aligned}
J_t\tau(a)^*J_t =& \frac{1}{\sqrt{t}}(T + T^* - I)\tau(a)^*\frac{1}{\sqrt{t}}(T + T^* - I)\\
=&\frac{1}{t}(T\tau(a)^* + T^*\tau(a)^* - \tau(a)^*)(T + T^* - I)\\
=&\frac{1}{t}T\tau(a)^*(T + T^* - I)\\
=&\frac{1}{t}T\tau(a)^*T\\
=&\tau(a^*).
\end{aligned}$$

Similarly we can prove that $J_t\tau(a^*)J_t = \tau(a)^*$. To see (ii), observe,

$$\begin{aligned}
(J_t + I)(J_t - I)(\sqrt{t}J_t + I) &= \sqrt{t}J_t^3 + J_t^2 - \sqrt{t}J_t - I\\
&= J_t(T + T^* - I)J_t + J_t^2 - \sqrt{t}J_t - I\\
&= (T^* + T - J_t^2) + J_t^2 - \sqrt{t}J_t - I\\
&= 0.
\end{aligned}$$

Since $J_t = J_t^*$ the proof is complete. □

In this Proposition, as $J_t = J_t^*$ by spectral theorem $J_t = P_1 - P_2 + (-1/\sqrt{t})P_3$, where P_1, P_2, P_3 are orthogonal projections with $P_1 + P_2 + P_3 = I$. Note that due to finiteness of the spectrum of J_t, P_1, P_2, P_3 are in the C^*-algebra $\mathcal{B}$. Furthermore, J_t is a symmetry if and only if $P_3 = 0$.

Proposition 2.30. *Suppose $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a t-ternary homomorphism with $t \in (1, \infty)$. Then $E_0 = \overline{\mathrm{span}}\{\tau(a_1)x_1, \tau(a_2)^* x_2 : x_i \in E,\ a_i \in \mathcal{A}, i = 1, 2\}$ is complemented in E. Moreover, the following are equivalent*:

(i) *τ is $*$-nondegenerate*;

(ii) $\ker(\tau(1) + \tau(1)^*) = \{0\}$;

(iii) *J_t is a symmetry.*

Proof. Set $T = \tau(1)$, $E_0 = \mathrm{span}\{T(E), T^*(E)\}$. We wish to construct an orthogonal projection onto E_0. Take $Q = (1/(t-1))[TT^* + T^*T - T - T^*]$. From $T^2 = T$ and $TT^*T = tT$, simple algebra shows $QT = T$, $QT^* = T^*$ and $Q = Q^* = Q^2$. So Q is a projection whose range contains $T(E)$ and $T^*(E)$. From the definition of Q, $Q(E) \subseteq E_0$. This proves $Q(E) = E_0$. Clearly then $(I - Q)$ is the projection onto $E_0^{\perp}$ and $E = E_0 \oplus E_0^{\perp}$.

To show the equivalence of (i) to (ii), we show $\ker(T + T^*) = \ker Q$. If $(T + T^*)x = 0$, then

$$\begin{aligned} Qx = \frac{1}{t-1}[TT^* + T^*T - T - T^*]x &= \frac{1}{t-1}[TT^*x + T^*Tx] \\ &= \frac{1}{t-1}[T(-Tx) + T^*(-T^*)x] \\ &= \frac{-1}{t-1}[Tx + T^*x] \\ &= 0. \end{aligned}$$

Conversely if $Qx = 0$, then $x \in E_0^{\perp}$; hence $(T + T^*)x = 0$. The equivalence of (ii) and (iii) is obvious as $\ker(T+T^*) = \ker(\sqrt{t}\,J_t + 1) = \left\{x : J_t x = \frac{-1}{\sqrt{t}}x\right\} = \mathrm{ran}(P_3)$. $\square$

Theorem 2.31. *Suppose $t \in (1, \infty)$ and $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ is a $*$-nondegenerate linear map. Take $J_t = (1/\sqrt{t})[\tau(1) + \tau(1)^* - 1]$. Then the following are equivalent*:

(i) *τ is a t-ternary homomorphism.*

(ii) *τ is a J_t-homomorphism.*

Proof. (i) $\Rightarrow$ (ii) : We have already seen this.

(ii) $\Rightarrow$ (i) : For all $a, b, c \in \mathcal{A}$ we have

$$\begin{aligned} t\tau(ab^*c) &= t\tau(a)\tau(b^*)\tau(c) \\ &= t\tau(a)J_t\tau(b)^*J_t\tau(c) \\ &= t\tau(a)\frac{\tau(1)\tau(b)^*\tau(1)}{t}\tau(c) \\ &= \tau(a)\tau(b)^*\tau(c). \end{aligned}$$

□

Theorem 2.32. *Suppose $\tau : \mathcal{A} \to \mathcal{B}^a(E)$ is a t-ternary homomorphism, where $t \in (1, \infty)$. Then there exists a closed, complemented, $\mathcal{B}$-submodule $E_1 \subseteq E$, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}^a(E_1)$ and isometries $V_1, V_2 \in \mathcal{B}^a(E_1, E)$ with $V_2^* V_1 = (1/\sqrt{t}) I_{E_1}$ such that*

$$\tau(\cdot) = \sqrt{t} V_1 \pi(\cdot) V_2^*. \tag{2-8}$$

Consequently τ is completely bounded with $\|\tau\|_{cb} = \|\tau(1)\| = \sqrt{t}$. Moreover, (2-8) always defines a t-ternary homomorphism.

Proof. Let E_1 be the range of the orthogonal projection $P = (1/t) T^* T$ where $T = \tau(1)$. Define linear maps $V_i : E_1 \to E$ by $V_1 = (1/\sqrt{t}) T|_{E_1}$ and $V_2 = I|_{E_1}$. Note that the V_i's are adjointable isometries with $V_1^* = (1/\sqrt{t}) P T^*$ and $V_2^* = P$. Now for each $a \in \mathcal{A}$ define $\pi(a) : E_1 \to E_1$ by $\pi(a) = P\tau(a)|_{E_1}$. Clearly $\pi(1) = P\tau(1)|_{E_1} = P|_{E_1} = I_{E_1}$. Also for all $a, b \in \mathcal{A}$,

$$\begin{aligned}\pi(a)\pi(b) &= P\tau(a)\frac{\tau(1)^*\tau(1)}{t}\tau(b)|_{E_1}\\ &= P\frac{\tau(a)\tau(1)^*\tau(b)}{t}|_{E_1}\\ &= P\tau(ab)|_{E_1}\\ &= \pi(ab).\end{aligned}$$

Now since

$$P\tau(a)^*P = \frac{1}{t^2}\tau(1)^*\tau(1)\tau(a)^*\tau(1) = \frac{1}{t}\tau(1)^*\tau(a^*) = P\tau(a^*)$$

for all $x, x' \in E_1$ we have

$$\langle \pi(a)x, x'\rangle = \langle P\tau(a)x, x'\rangle = \langle x, \tau(a)^* P x'\rangle = \langle x, P\tau(a)^* P x'\rangle = \langle x, \pi(a^*)x'\rangle,$$

so that $\pi(a)^* = \pi(a^*)$. Thus $a \mapsto \pi(a)$ defines a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}^a(E_1)$. Also for $a \in \mathcal{A}$ we have

$$\sqrt{t} V_1 \pi(a) V_2^* = T P \tau(a) P = \frac{1}{t^2}\tau(1)\tau(1)^*\tau(1)\tau(a)\tau(1)^*\tau(1) = \tau(a).$$

Since the V_i are isometries, τ is completely bounded with $\|\tau\|_{cb} \le \sqrt{t} = \|\tau(1)\|$. □

Now we show that every regular homomorphism is essentially a direct sum or direct integral of t-ternary homomorphisms through spectral integration. Let $\tau : \mathcal{A} \to \mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-nondegenerate regular homomorphism. In view of Proposition 2.23, it suffices to know the structure of $\tau(1)$. As before, take $T = \tau(1) = R + iS$ and let $J = f(R)$ be the symmetry constructed in Proposition 2.4. In the following, we decompose the Hilbert space $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$\mathcal{H}_+ = \{h \in \mathcal{H} : Jh = h\}$ and $\mathcal{H}_- = \{h \in \mathcal{H} : Jh = -h\}$. With respect to this decomposition, decompose the operator $T = \tau(1)$ as

$$T = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}.$$

Now $JTJ = T^*$ yields $X = X^*$, $W = W^*$, $Z = -Y^*$. Furthermore, from $T^2 = T$, we get $X^2 - YY^* = X$, $XY + YW = Y$, $-Y^*Y + W^2 = W$. So $Y^*Y = W(W - I)$. Let $Y = V[W(W-I)]^{1/2}$, be the polar decomposition of Y. Suppose $0 \in \sigma(W)$. Let $0 \neq h_- \in \mathcal{H}_-$ such that $Wh_- = 0$. Set $h = \left[\begin{smallmatrix} 0 \\ h_- \end{smallmatrix}\right]$. Then $Yh_- = 0$, hence $\tau(1)h = 0$. Since τ is $*$-nondegenerate this implies that $h = 0$, which is a contradiction. Again, suppose $1 \in \sigma(W)$, choose $0 \neq h_- \in \mathcal{H}_-$ such that $Wh_- = h_-$. Then $Yh_- = 0$. Since τ is t-ternary, $\tau(1)\tau(1)^*\tau(1)h = t\tau(1)h$ for $t \in (1, \infty)$. But $\tau(1)\tau(1)^*\tau(1)h = \tau(1)h$. Thus $\tau(1)h = t\tau(1)h \Longrightarrow \tau(1)h = 0$. Since τ is $*$-nondegenerate this implies that $h = 0$, which is a contradiction. Thus $0, 1 \notin \sigma(W)$. To prove V is unitary it is enough to show that $\overline{\operatorname{ran}(Y)} = \mathcal{H}_+$, i.e, we have to show that $\ker(Y^*) = 0$. Let $0 \neq h_+ \in \mathcal{H}_+$. We have $X^2 = X - YY^*$ and $Y^*h_+ = 0$, which implies $X^2h_+ = Xh_+$. Now $-ZXh_+ = Y^*X^*h_+ = (XY)^*h_+ = (Y^* - W^*Y^*)h_+ = 0$. Set $h = \left[\begin{smallmatrix} h_+ \\ 0 \end{smallmatrix}\right]$. Then $\tau(1)\tau(1)^*\tau(1)h = \tau(1)h$. Since τ is t-ternary, where $t \in (1, \infty)$, we will get $\tau(1)h = t\tau(1)h$. Thus $Xh_+ = 0$ and hence $\tau(1)h = 0 = \tau(1)^*h$. Since τ is $*$-nondegenerate, $h = 0$. Thus to avoid degenerate cases assume that V is a unitary and $0, 1 \notin \sigma(W)$. Now $X^2 - X = YY^* = V[W(W - I)]V^*$. Also from $XY = Y(I - W)$, we get $XV[W(W - I)]^{1/2} = V[W(W - I)]^{1/2}(I - W)$, which yields, $X = V(I - W)V^*$. Now T decomposes as

$$T = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (I - W) & [W(W - I)]^{1/2} \\ -[W(W - I)]^{1/2} & W \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix}.$$

Observe that, for any real number $w \notin [0, 1]$

$$T_w = \begin{bmatrix} (1 - w) & [w(w - 1)]^{1/2} \\ -[w(w - 1)]^{1/2} & w \end{bmatrix}$$

satisfies $T_w = T_w^2$ and $J_2T_wJ_2 = T_w^*$, where

$$J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

and also $T_wT_w^*T_w = (1 - 2w)^2T_w$. In other words $z \mapsto zT_w$ is a $(1 - 2w)^2$-ternary of complex numbers in $M_2(\mathbb{C})$.

3. Representations of completely bounded maps

In this section we give a new structure theorem for CB-maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. We study some known structure theorems (see [Paulsen and Suen 1985; Suen 1991]) and make a comparison. We repeatedly use the following well-known theorem (see [Paulsen 2002, Theorem 8.3]):

Theorem 3.1. *Suppose $\mathcal{A}$ is a unital C^*-algebra and $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CB-map. Then there exist CP-maps $\varphi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ with $\|\varphi_i\|_{cb} = \|\psi\|_{cb}$ such that $\Phi : M_2(\mathcal{A}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ defined by*

$$\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \varphi_1(a) & \psi(b) \\ \psi^*(c) & \varphi_2(d) \end{bmatrix}$$

is a CP-map. Moreover, if $\|\psi\|_{cb} \leq 1$, it is possible to take $\varphi_i(1) = I_{\mathcal{H}}$.

3A. ***Regular representations.*** Observe that if $\mathcal{K}$ is another Hilbert space and $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a regular homomorphism, then $\psi(\,\cdot\,) := W^*\tau(\,\cdot\,)W$ defines a CB-map from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ for all $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. We prove all CB-maps arise this way.

Theorem 3.2. *Suppose $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CB-map. Then there exists a Hilbert space $\mathcal{K}$, a $*$-nondegenerate regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and a bounded linear map $W : \mathcal{H} \to \mathcal{K}$ such that $\psi(\,\cdot\,) = W^*\tau(\,\cdot\,)W$. Moreover, given any $t \in (1, \infty)$ we can choose τ and W such that τ is t-ternary and W satisfies*

$$\left(\frac{(t-1)\sqrt{t-1}}{2\sqrt{t-1}+2t-1}\right)\|W\|^2 \leq \|\psi\|_{cb} \leq \sqrt{t}\,\|W\|^2.$$

Proof. Since ψ is a CB-map, by Theorem 3.1, there exists CP-maps $\varphi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\Phi = \left[\begin{smallmatrix} \varphi_1 & \psi \\ \psi^* & \varphi_2 \end{smallmatrix}\right] : M_2(\mathcal{A}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is a CP-map. Suppose $(\mathcal{K}, \Pi, V)$ is the (minimal) Stinespring dilation for Φ. Given $t \in (1, \infty)$ set $t' = \sqrt{t-1}$. Then for $a \in \mathcal{A}$ we have

$$\begin{aligned}
\psi(a) &= \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} \begin{bmatrix} 0 & \psi(a) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix} \\
&= \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} \Phi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix} \\
&= \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} V^*\Pi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) V \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix} \qquad (3\text{-}1) \\
&= \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} V^*\Pi\left(\begin{bmatrix} 0 & \frac{1}{\sqrt{t'}} \\ \frac{1}{\sqrt{t'}} & \frac{-1}{t'\sqrt{t'}} \end{bmatrix}\right) \Pi\left(\begin{bmatrix} a & 0 \\ t'a & 0 \end{bmatrix}\right) \Pi\left(\begin{bmatrix} 0 & \frac{1}{\sqrt{t'}} \\ \frac{1}{\sqrt{t'}} & \frac{-1}{t'\sqrt{t'}} \end{bmatrix}\right) V \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix}. \qquad (3\text{-}2)
\end{aligned}$$

Define $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ by $\tau(a) = \Pi\left(\left[\begin{smallmatrix} a & 0 \\ t'a & 0 \end{smallmatrix}\right]\right)$, which is a t-ternary homomorphism. Note that if $k \in \mathcal{K}$ is such that $\tau(1)k = 0 = \tau(1)^*k$, then

$$\begin{aligned} k = \Pi(1)k &= \Pi\left(\begin{bmatrix} 0 & \frac{1}{t'} \\ 0 & \frac{-1}{t'^2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ t' & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{t'} & 0 \end{bmatrix}\begin{bmatrix} 1 & t' \\ 0 & 0 \end{bmatrix}\right)k \\ &= \Pi\left(\begin{bmatrix} 0 & \frac{1}{t'} \\ 0 & \frac{-1}{t'^2} \end{bmatrix}\right)\tau(1)k + \Pi\left(\begin{bmatrix} 0 & 0 \\ \frac{1}{t'} & 0 \end{bmatrix}\right)\tau(1)^*k \\ &= 0. \end{aligned}$$

Thus τ is a $*$-nondegenerate t-ternary homomorphism. Set

$$W = \Pi\left(\begin{bmatrix} 0 & 1/\sqrt{t'} \\ 1/\sqrt{t'} & -1/t'\sqrt{t'} \end{bmatrix}\right) V \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix} \in \mathcal{B}(\mathcal{H}, \mathcal{K}).$$

Then from (3-2) we get $\psi(\,\cdot\,) = W^*\tau(\,\cdot\,)W$; hence $\|\psi\|_{cb} \le \|W\|^2\|\tau\|_{cb} = \sqrt{t}\|W\|^2$. Also note that

$$\begin{aligned} \|W\|^2 &= \|W^*W\| \\ &= \left\| \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} V^*\Pi\left(\begin{bmatrix} 0 & 1/\sqrt{t'} \\ 1/\sqrt{t'} & -1/t'\sqrt{t'} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{t'} \\ 1/\sqrt{t'} & -1/t'\sqrt{t'} \end{bmatrix}\right) V \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix}\right\| \\ &= \left\| \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} \Phi\left(\begin{bmatrix} 1/t' & -1/t'^2 \\ -1/t'^2 & (t'^2+1)/t'^3 \end{bmatrix}\right) \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix}\right\| \\ &= \left\| \begin{bmatrix} I_{\mathcal{H}} & I_{\mathcal{H}} \end{bmatrix} \begin{bmatrix} \varphi_1(1/t') & \psi(-1/t'^2) \\ \psi^*(-1/t'^2) & \varphi_2((t'^2+1)/t'^3) \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}} \\ I_{\mathcal{H}} \end{bmatrix}\right\| \\ &= \left\| \varphi_1\left(\frac{1}{t'}\right) - \psi\left(\frac{1}{t'^2}\right) - \psi^*\left(\frac{1}{t'^2}\right) + \varphi_2\left(\frac{t'+1}{t'^2}\right)\right\| \\ &\le \frac{1}{t'}\|\varphi_1\|_{cb} + \frac{1}{t'^2}\|\psi\|_{cb} + \frac{1}{t'^2}\|\psi\|_{cb} + \left(\frac{t'^2+1}{t'^3}\right)\|\varphi_2\|_{cb} \\ &= \left(\frac{2t'+2t'^2+1}{t'^3}\right)\|\psi\|_{cb} \\ &= \frac{2\sqrt{t-1}+2t-1}{(t-1)\sqrt{t-1}}\|\psi\|_{cb}. \end{aligned}$$

□

Theorem 3.3. *Suppose $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ is a Hilbert space. Then there exists a Hilbert space $\mathcal{K}$ and a $*$-nondegenerate regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that given any $\psi \in CB(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ there exists an operator $W_\psi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\psi(\,\cdot\,) = W_\psi^*\tau(\,\cdot\,)W_\psi$. Moreover, given any $t \in (1, \infty)$ we can choose τ and W_ψ such that τ is t-ternary and W_ψ satisfies*

$$\left(\frac{(t-1)\sqrt{t-1}}{2\sqrt{t-1}+2t-1}\right)\|W_\psi\|^2 \le \|\psi\|_{cb} \le \sqrt{t}\|W_\psi\|^2.$$

Proof. Suppose $t \in (1, \infty)$. For each $\psi \in CB(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ fix a $*$-nondegenerate regular representation $(\mathcal{K}_\psi, \tau_\psi, W_\psi)$ as in Theorem 3.2. Take $\mathcal{K} = \oplus_\psi \mathcal{K}_\psi$ and $\tau = \oplus_\psi \tau_\psi$. Note that $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a well defined $*$-nondegenerate t-ternary homomorphism since each τ_ψ is a $*$-nondegenerate t-ternary homomorphism with $\|\tau_\psi\| = \sqrt{t}$. Now given any CB-map ψ we have the corresponding $W_\psi \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\psi)$. Considering $\mathcal{K}_\psi \subseteq \mathcal{K}$ via the natural inclusion map we have $W_\psi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with

$$\left(\frac{(t-1)\sqrt{t-1}}{2\sqrt{t-1}+2t-1}\right)\|W_\psi\|^2 \leq \|\psi\|_{cb} \leq \sqrt{t}\|W_\psi\|^2$$

and $\psi(\cdot) = W_\psi^* \tau_\psi(\cdot) W_\psi = W_\psi^* \tau(\cdot) W_\psi$. □

Theorem 3.4. *Suppose $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CC-map. Then there exists a Hilbert space $\mathcal{K}$, a (not necessarily $*$-nondegenerate) regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and an isometry $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\psi(\cdot) = V^* \tau(\cdot) V$. Moreover, we can choose τ to be t-ternary for t large enough ($t \geq 18$).*

Proof. As in the proof of Theorem 3.2 consider the CP-extension of ψ given by $\Phi = \left[\begin{smallmatrix} \varphi_1 & \psi \\ \psi^* & \varphi_2 \end{smallmatrix}\right] : M_2(\mathcal{A}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and the corresponding (minimal) Stinespring dilation $(\mathcal{K}', \Pi', V')$ for Φ. Given $t \in (1, \infty)$ set $t' = \sqrt{t-1}$ and define

$$W' = \Pi'\left(\begin{bmatrix} 0 & 1/\sqrt{t'} \\ 1/\sqrt{t'} & -1/(t'\sqrt{t'}) \end{bmatrix}\right) V' \begin{bmatrix} I_\mathcal{H} \\ I_\mathcal{H} \end{bmatrix} \in \mathcal{B}(\mathcal{H}, \mathcal{K}')$$

and define $\tau' : \mathcal{A} \to \mathcal{B}(\mathcal{K}')$ by $\tau'(a) = \Pi'\left(\left[\begin{smallmatrix} a & 0 \\ t'a & 0 \end{smallmatrix}\right]\right)$, which is a $*$-nondegenerate t-ternary homomorphism. Clearly $\psi(a) = W'^* \tau'(a) W'$. Note that since ψ is CC-map V' can be chosen to be an isometry. Hence

$$\|W'\|^2 \leq \left\|\begin{bmatrix} 0 & 1/\sqrt{t'} \\ 1/\sqrt{t'} & -1/(t'\sqrt{t'}) \end{bmatrix}\right\|^2 \left\|\begin{bmatrix} I_\mathcal{H} \\ I_\mathcal{H} \end{bmatrix}\right\|^2 \leq 2\left(\frac{1}{t'} + \frac{1}{t'} + \frac{1}{t'^3}\right),$$

because $\|[a_{ij}]\|^2 \leq \sum_{ij} |a_{ij}|^2$. So we can assume that $W' \in \mathcal{B}(\mathcal{H}, \mathcal{K}')$ is a contraction by taking t large enough ($t \geq 18$). Let $\mathcal{K}'' = \mathcal{H} \oplus \mathcal{K}'$ and $W'' := \left[\begin{smallmatrix} 0 & 0 \\ W' & 0 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{K}'')$. Define $\tau'' : \mathcal{A} \to \mathcal{B}(\mathcal{K}'')$ by $a \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & \tau'(a) \end{smallmatrix}\right]$, which is a t-ternary homomorphism. Suppose U is Halmos's unitary dilation of W'', that is,

$$U = \begin{bmatrix} W'' & (1 - W''W''^*)^{1/2} \\ (1 - W''^*W'')^{1/2} & -W''^* \end{bmatrix} \in \mathcal{B}(\mathcal{K}'' \oplus \mathcal{K}'').$$

Set $\mathcal{K} = \mathcal{K}'' \oplus \mathcal{K}''$ and define $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ by $\tau(a) = \left[\begin{smallmatrix} \tau''(a) & 0 \\ 0 & 0 \end{smallmatrix}\right]$ which is a t-ternary homomorphism. Let V_0 be the inclusion map of $\mathcal{H}$ in $\mathcal{K} = \mathcal{K}'' \oplus \mathcal{K}'' = \mathcal{H} \oplus \mathcal{K}' \oplus \mathcal{K}''$.

Then $V_0 = \begin{bmatrix} I_{\mathcal{K}''} \\ 0_{\mathcal{K}''} \end{bmatrix}\begin{bmatrix} I_{\mathcal{H}} \\ 0_{\mathcal{K}'} \end{bmatrix}$. Set $V = UV_0 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, which is an isometry. Then

$$\begin{aligned}
V^*\tau(a)V &= V_0^*U^*\tau(a)UV_0 \\
&= V_0^*\begin{bmatrix} W''^*\tau''(a)W'' & * \\ * & * \end{bmatrix} V_0 \\
&= \begin{bmatrix} I_{\mathcal{H}} & 0_{\mathcal{K}'} \end{bmatrix} W''^*\tau''(a)W'' \begin{bmatrix} I_{\mathcal{H}} \\ 0_{\mathcal{K}'} \end{bmatrix} \\
&= \begin{bmatrix} I_{\mathcal{H}} & 0_{\mathcal{K}'} \end{bmatrix} \begin{bmatrix} 0 & W'^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \tau'(a) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ W' & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}} \\ 0_{\mathcal{K}'} \end{bmatrix} \\
&= W'^*\tau'(a)W' \\
&= \psi(a).
\end{aligned}$$

Note that τ is a t-ternary for $t \geq 18$. □

Theorem 3.5. *Suppose $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ is a Hilbert space. There exists a Hilbert space $\mathcal{K}$ and a regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that given any completely contractive map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ there exists an isometry $V_\psi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\psi(\,\cdot\,) = V_\psi^*\tau(\,\cdot\,)V_\psi.$$

Moreover, we can choose τ to be t-ternary for t large enough ($t \geq 18$).

Proof. This follows by considering the direct sum of all representations given by Theorem 3.4. □

Now we prove analogues of above theorems for the case when the range algebra is an injective C^*-algebra.

Theorem 3.6. *Suppose $\mathcal{B}$ is an injective C^*-algebra and $\psi : \mathcal{A} \to \mathcal{B}$ is a CB-map. Then there exists a Hilbert $\mathcal{B}$-module E, a $*$-nondegenerate regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ and a vector $z \in E$ such that*

$$\psi(\,\cdot\,) = \langle z, \tau(\,\cdot\,)z\rangle.$$

Moreover, given any $t \in (1, \infty)$ we can choose τ and z such that τ is t-ternary and z satisfies $((t-1)\sqrt{t-1}/(2\sqrt{t-1}+2t-1))\|z\|^2 \leq \|\psi\|_{cb} \leq \sqrt{t}\|z\|^2$.

Proof. Suppose $t \in (1, \infty)$. Let $\rho : \mathcal{B} \to \mathcal{B}(\mathcal{G})$ be a faithful unital $*$-homomorphism $\rho : \mathcal{B} \to \mathcal{B}(\mathcal{G})$ of $\mathcal{B}$ on some Hilbert space $\mathcal{G}$ satisfying $\overline{\operatorname{span}}\,\rho(\mathcal{B})\mathcal{G} = \mathcal{G}$. As $\mathcal{B}$ is injective there exists a conditional expectation $P_{\mathcal{B}} : \mathcal{B}(\mathcal{G}) \to \rho(\mathcal{B})$, i.e., $P_{\mathcal{B}}$ is a CP-map satisfying $P_{\mathcal{B}}(b_1Tb_2) = b_1P_{\mathcal{B}}(T)b_2$ for all $b_i \in \rho(\mathcal{B})$, $T \in \mathcal{B}(\mathcal{G})$. Consider the CB-map $\tilde{\psi} = \rho \circ \psi : \mathcal{A} \to \mathcal{B}(\mathcal{G})$ with a $*$-nondegenerate regular representation

$(\mathcal{K}, \tilde{\tau}, W)$, as in Theorem 3.2, where $\tilde{\tau} : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a $*$-nondegenerate t-ternary homomorphism. Set $E_0 = \mathcal{B}(\mathcal{G}, \mathcal{K})$. Given $x, y \in E_0$ and $b \in \mathcal{B}$ define

$$xb := x \circ \rho(b) \in E_0 \text{ and } \langle x, y \rangle := \rho^{-1} P_{\mathcal{B}}(x^* \circ y) \in \mathcal{B}.$$

Here x^* is the adjoint of $x \in \mathcal{B}(\mathcal{G}, \mathcal{K})$. It is easy to verify that with the above operations E_0 forms a semi-inner product $\mathcal{B}$-module. Let E be the completion of the inner product $\mathcal{B}$-module E_0/N where

$$N := \{x \in E_0 : \langle x, x \rangle = 0\} = \{x \in E_0 : \langle x, y \rangle = 0 \text{ for all } y \in E_0\}.$$

Note that for $x+N, x'+N \in E$ the inner product is given by $\langle x+N, x'+N \rangle := \langle x, x' \rangle$. We denote the equivalence classes $x + N$ by x itself. Now for each $a \in \mathcal{A}$ define $\tau(a) : E \to E$ by $\tau(a)x := \tilde{\tau}(a) \circ x$ for all $x \in E_0$. Note that

$$\begin{aligned}
\|\tau(a)x\|^2 &= \|\langle \tilde{\tau}(a) \circ x, \tilde{\tau}(a) \circ x \rangle\| \\
&= \|\rho^{-1} P_{\mathcal{B}}(x^* \circ \tilde{\tau}(a)^* \circ \tilde{\tau}(a) \circ x)\| \\
&\le \|\tilde{\tau}(a)\|^2 \|\rho^{-1} P_{\mathcal{B}}(x^* \circ x)\| \\
&= \|\tilde{\tau}(a)\|^2 \|\langle x, x \rangle\| \\
&= \|\tilde{\tau}(a)\|^2 \|x\|^2,
\end{aligned}$$

so that $\tau(a)$ is a well defined bounded linear map. Also for all $x, y \in E$ we have

$$\langle \tau(a)x, y \rangle = \rho^{-1} P_{\mathcal{B}}((\tilde{\tau}(a) \circ x)^* \circ y) = \rho^{-1} P_{\mathcal{B}}(x^* \circ \tilde{\tau}(a)^* \circ y) = \langle x, \tilde{\tau}(a)^* \circ y \rangle,$$

so that $\tau(a) \in \mathcal{B}^{\mathrm{a}}(E)$ with $\tau(a)^* y = \tilde{\tau}(a)^* \circ y$. Also for all $a, b, c \in \mathcal{A}$ and $x \in E$ we have

$$\tau(a)\tau(b)x = \tau(a)(\tilde{\tau}(b) \circ x) = \tilde{\tau}(a) \circ \tilde{\tau}(b) \circ x = \tilde{\tau}(ab) \circ x = \tau(ab)x$$

and

$$\tau(a)\tau(b)^*\tau(c)x = \tilde{\tau}(a) \circ \tilde{\tau}(b)^* \circ \tilde{\tau}(c) \circ x = t\tilde{\tau}(ab^*c) \circ x = t\tau(ab^*c)x,$$

so that $\tau(a)\tau(b) = \tau(ab)$ and $\tau(a)\tau(b)^*\tau(c) = t\tau(ab^*c)$. Thus $a \mapsto \tau(a)$ defines a t-ternary homomorphism $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$. Now if we set $z = W \in E$, then for $a \in \mathcal{A}$ we have

$$\begin{aligned}
\langle z, \tau(a)z \rangle &= \rho^{-1} P_{\mathcal{B}}(W^* \circ \tilde{\tau}(a) \circ W) \\
&= \rho^{-1} P_{\mathcal{B}}(\tilde{\psi}(a)) \\
&= \rho^{-1} P_{\mathcal{B}}(\rho \circ \psi(a)) \\
&= \rho^{-1} \circ \rho \circ \psi(a) \\
&= \psi(a)
\end{aligned}$$

and hence $\|\psi\|_{cb} \le \|\tau\|_{cb}\|z\|^2 = \sqrt{t}\|z\|^2$. Also

$$\left(\frac{(t-1)\sqrt{t-1}}{2\sqrt{t-1}+2t-1}\right)\|z\|^2 = \left(\frac{(t-1)\sqrt{t-1}}{2\sqrt{t-1}+2t-1}\right)\|W\|^2 \le \|\tilde{\psi}\|_{cb} \le \|\psi\|_{cb}.$$

Also since $\tilde{\tau}$ is a $*$-nondegenerate t-ternary homomorphism, from Proposition 2.30 it follows that τ is also $*$-nondegenerate. □

In this case also we can have a universal representation. Fixing one regular representation for each $\psi \in CB(\mathcal{A}, \mathcal{B})$ and considering the direct sum of all such representations as in the proof of Theorem 3.3 we can have the following.

Theorem 3.7. *Suppose $\mathcal{B}$ is an injective C^*-algebra. There exists a Hilbert $\mathcal{B}$-module E, and a $*$-nondegenerate regular homomorphism $\tau : \mathcal{A} \to \mathcal{B}^{\mathrm{a}}(E)$ such that given any $\psi \in CB(\mathcal{A}, \mathcal{B})$ there exists a vector $z_\psi \in E$ such that*

$$\psi(\cdot) = \langle z_\psi, \tau(\cdot) z_\psi \rangle.$$

Moreover, given any $t \in (1, \infty)$ we can choose τ and z_ψ such that τ is t-ternary and z_ψ satisfies $((t-1)\sqrt{t-1}/(2\sqrt{t-1}+2t-1))\|z_\psi\|^2 \le \|\psi\|_{cb} \le \sqrt{t}\|z_\psi\|^2$.

3B. ***Commutant representations.*** In this section we provide new and possibly simpler proofs of some known results for completely bounded maps. To begin with we give a different proof of the following result due to Paulsen and Suen [1985, Theorem 2.2]. Our proof involves mainly matrix manipulation.

Theorem 3.8. *Suppose $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CB-map. Then there exists a Hilbert space $\mathcal{K}$, a unital representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, an isometry $V : \mathcal{H} \to \mathcal{K}$ and a unique operator $T \in \pi(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{K})$ such that*

$$\psi(\cdot) = V^* T \pi(\cdot) V \quad \text{and} \quad \overline{\mathrm{span}}\, \pi(\mathcal{A}) V \mathcal{H} = \mathcal{K}.$$

Furthermore, $\|\psi\|_{cb} \le \|T\| \le 2\|\psi\|_{cb}$. If $\psi = \psi^*$, then $T = T^*$ and $\|\psi\|_{cb} = \|T\|$.

Proof. For nonzero ψ, replacing ψ by $\psi/\|\psi\|_{cb}$ if necessary, we may assume that $\|\psi\|_{cb} = 1$. Construct Φ as in Theorem 3.1 and let $(\widetilde{\mathcal{K}}, \Pi, \widetilde{V})$ be the minimal Stinespring dilation for Φ with $\widetilde{V}$ an isometry. Then from equation (3-1) we have

$$\psi(a) = \begin{bmatrix} \frac{I_{\mathcal{H}}}{\sqrt{2}} & \frac{I_{\mathcal{H}}}{\sqrt{2}} \end{bmatrix} \widetilde{V}^* \Pi\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) \Pi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) \widetilde{V} \begin{bmatrix} I_{\mathcal{H}}/\sqrt{2} \\ I_{\mathcal{H}}/\sqrt{2} \end{bmatrix} = V^* \widetilde{T} \tilde{\pi}(a) V, \tag{3-3}$$

where $V = \widetilde{V}\left[\begin{smallmatrix} I_{\mathcal{H}}/\sqrt{2} \\ I_{\mathcal{H}}/\sqrt{2} \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{K}})$ is an isometry, $\widetilde{T} = \Pi\left(\left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right]\right) \in \mathcal{B}(\mathcal{K})$ and $\tilde{\pi} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{K}})$ is the unital representation given by $\tilde{\pi}(a) := \Pi\left(\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]\right)$. Clearly

$$\widetilde{T}\tilde{\pi}(a) = \Pi\left(\begin{bmatrix} 0 & 2a \\ 0 & 0 \end{bmatrix}\right) = \tilde{\pi}(a)\widetilde{T}$$

for all $a \in \mathcal{A}$ so that $\widetilde{T} \in \tilde{\pi}(\mathcal{A})' \subseteq \mathcal{B}(\widetilde{\mathcal{K}})$. Set $\mathcal{K} = \overline{\operatorname{span}}\, \tilde{\pi}(\mathcal{A})V\mathcal{H} \subseteq \widetilde{\mathcal{K}}$ and $T = P_{\mathcal{K}}\widetilde{T}|_{\mathcal{K}} \in \mathcal{B}(\mathcal{K})$ where $P_{\mathcal{K}}$ is the orthogonal projection of $\widetilde{\mathcal{K}}$ onto $\mathcal{K}$. Note that $\tilde{\pi}(a)$ reduces $\mathcal{K}$ for all $a \in \mathcal{A}$. Then $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ given by $\pi(a) = \tilde{\pi}(a)|_{\mathcal{K}}$ defines a unital representation such that

$$\tilde{\pi}(a) = \begin{bmatrix} \pi(a) & 0 \\ 0 & * \end{bmatrix} \in \mathcal{B}(\widetilde{\mathcal{K}}) = \mathcal{B}(\mathcal{K} \oplus \mathcal{K}^{\perp}).$$

So $\widetilde{T} = \left[\begin{smallmatrix} T & * \\ * & * \end{smallmatrix}\right] \in \tilde{\pi}(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{K} \oplus \mathcal{K}^{\perp})$ implies that $T\pi(a) = \pi(a)T$ for all $a \in \mathcal{A}$. That is, $T \in \pi(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{K})$. Since $\tilde{\pi}$ is unital we have $V\mathcal{H} \subseteq \mathcal{K}$, i.e., $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, hence $\mathcal{K} = \overline{\operatorname{span}}\, \tilde{\pi}(\mathcal{A})V\mathcal{H} = \overline{\operatorname{span}}\, \pi(\mathcal{A})V\mathcal{H}$. Also,

$$\psi(a) = V^*\widetilde{T}\tilde{\pi}(a)V = \begin{bmatrix} V^* & 0 \end{bmatrix} \begin{bmatrix} T & * \\ * & * \end{bmatrix} \begin{bmatrix} \pi(a) & 0 \\ 0 & * \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} = V^*T\pi(a)V$$

for all $a \in \mathcal{A}$. Clearly $\|\psi\|_{cb} \le \|T\| \le \|\widetilde{T}\| \le 2$ gives the required bounds.

To see uniqueness of T, suppose there exists another operator $S \in \pi(\mathcal{A})'$ such that $\psi(\cdot) = V^*S\pi(\cdot)V$. Then

$$\begin{aligned} \langle \pi(a_1)Vh_1, (T-S)\pi(a_2)Vh_2\rangle &= \langle h_1, V^*\pi(a_1^*)T\pi(a_2)Vh_2\rangle - \langle h_1, V^*\pi(a_1^*)S\pi(a_2)Vh_2\rangle \\ &= \langle h_1, V^*T\pi(a_1^*a_2)Vh_2\rangle - \langle h_1, V^*S\pi(a_1^*a_2)Vh_2\rangle \\ &= \langle h_1, \psi(a_1^*a_2)h_2\rangle - \langle h_1, \psi(a_1^*a_2)h_2\rangle \\ &= 0 \end{aligned}$$

for all $a_i \in \mathcal{A},\ h_i \in \mathcal{H}$ so that $T - S = 0$.

Finally if $\psi = \psi^*$, observe

$$V^*T\pi(a)V = \psi(a) = \psi^*(a) = \psi(a^*)^* = (V^*T\pi(a^*)V)^* = V^*T^*\pi(a)V,$$

and by the uniqueness property we have $T = T^*$. Note that $\frac{1}{2}(\widetilde{T} + \widetilde{T}^*) = \left[\begin{smallmatrix} T & * \\ * & * \end{smallmatrix}\right]$, so that $\|T\| \le \left\|\frac{1}{2}(\widetilde{T} + \widetilde{T}^*)\right\| = \frac{1}{2}\left\|\Pi\left(\left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right]\right) + \Pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right]\right)\right\| = 1 = \|\psi\|_{cb}$. □

For CB-maps from unital C^*-algebras into injective C^*-algebras Heo [1999] gave an analogue of Theorem 3.8. For a Hilbert $\mathcal{B}$ module E, consider $E^\sharp := \mathcal{B}_{\mathcal{B}}(E, \mathcal{B})$, the set of all bounded $\mathcal{B}$-module maps from E into $\mathcal{B}$. It forms a right $\mathcal{B}$-module with the following operations:

$$(\phi_1 + \phi_2)(x) := \phi_1(x) + \phi_2(x), \quad (\lambda\phi)(x) := \bar{\lambda}\phi(x), \quad (\phi b)(x) := b^*\phi(x)$$

for all $x \in E, b \in \mathcal{B}, \lambda \in \mathbb{C}$ and $\phi, \phi_i \in E^\sharp$. Also the operator norm makes $E^\sharp$ a Banach $\mathcal{B}$-module. With this notation, the theorem of Heo states the following.

Theorem 3.9. *Suppose $\mathcal{B}$ is an injective C^*-algebra and $\psi : \mathcal{A} \to \mathcal{B}$ is a CB-map. Then there exists a Hilbert $\mathcal{B}$-module E, a vector $z \in E$, a representation*

$\pi : \mathcal{A} \to \mathcal{B}^{a}(E)$ *and a unique operator* $T \in \mathcal{B}_{\mathcal{B}}(E, E^{\sharp}) \cap \pi(\mathcal{A})'$ *such that*

$$\psi(\,\cdot\,) = \langle z, T\pi(\,\cdot\,)z\rangle \quad \text{and} \quad \overline{\mathrm{span}}\, \pi(\mathcal{A})z\mathcal{B} = E.$$

Heo proved this result using a structure theorem for so-called "*completely multi-positive*" linear maps. Our proof is straightforward and we also get some norm estimates.

Theorem 3.10. *Suppose $\mathcal{B}$ is an injective C^*-algebra and $\psi : \mathcal{A} \to \mathcal{B}$ is a CB-map. Then there exists a quadruple (E, z, π, T) consisting of a Hilbert $\mathcal{B}$-module E, a unit vector $z \in E$, a unital representation $\pi : \mathcal{A} \to \mathcal{B}^{a}(E)$, and an operator $T \in \pi(\mathcal{A})' \subseteq \mathcal{B}^{a}(E)$ with $\|\psi\|_{cb} \le \|T\| \le 2\|\psi\|_{cb}$ such that*

$$\psi(\,\cdot\,) = \langle z, T\pi(\,\cdot\,)z\rangle.$$

If $\psi = \psi^*$, then $T = T^*$ and $\|\psi\|_{cb} = \|T\|$. Furthermore, if $\overline{\mathrm{span}}\, \pi(\mathcal{A})z\mathcal{B} = E$, then T is unique.

Proof. Consider a unital faithful representation $\rho : \mathcal{B} \to \mathcal{B}(\mathcal{G})$ of $\mathcal{B}$ on some Hilbert space $\mathcal{G}$ satisfying $\overline{\mathrm{span}}\, \rho(\mathcal{B})\mathcal{G} = \mathcal{G}$. By the assumption of injectivity, there is a conditional expectation map $P_{\mathcal{B}} : \mathcal{B}(\mathcal{G}) \to \rho(\mathcal{B})$. Suppose $(\mathcal{K}, \tilde{\pi}, \widetilde{T}, V)$ is a commutant representation of the CB-map $\tilde{\psi} := \rho \circ \psi : \mathcal{A} \to \mathcal{B}(\mathcal{G})$ given by Theorem 3.8. From $\mathcal{B}(\mathcal{G}, \mathcal{K})$, with $z = V$, construct the triple (E, π, z) as in the proof of Theorem 3.6. Note that since $\tilde{\pi}$ is a unital representation so is π. Also z is a unit vector since

$$\langle z, z\rangle = \rho^{-1} P_{\mathcal{B}}(V^* \circ V) = \rho^{-1} P_{\mathcal{B}}(I) = \rho^{-1} P_{\mathcal{B}}\rho(1) = \rho^{-1}\rho(1) = 1.$$

Define $T : E \to E$ by $T(x) = \widetilde{T} \circ x$ for all $x \in E$. It can be verified that T is well defined and $T \in \mathcal{B}^{a}(E)$ with $T^*(x) = \widetilde{T}^* \circ x$ for all $a \in E$. Note that

$$T\pi(a)x = \widetilde{T} \circ \tilde{\pi}(a) \circ x = \tilde{\pi}(a) \circ \widetilde{T} \circ x = \pi(a)Tx$$

for all $a \in \mathcal{A}$, $x \in E$ so that $T \in \pi(\mathcal{A})' \subseteq \mathcal{B}^{a}(E)$. Also,

$$\langle z, T\pi(a)z\rangle = \rho^{-1} P_{\mathcal{B}}(V^* \circ \widetilde{T} \circ \tilde{\pi}(a) \circ V) = \rho^{-1} P_{\mathcal{B}}(\tilde{\psi}(a)) = \psi(a)$$

for all $a \in \mathcal{A}$. Now it follows that $\|\psi\|_{cb} \le \|T\| \le \|\widetilde{T}\| \le 2\|\tilde{\psi}\|_{cb} \le 2\|\psi\|_{cb}$ since z is a unit vector and π is a unital representation. Now if $\psi = \psi^*$, then for all $a \in \mathcal{A}$,

$$\tilde{\psi}^*(a) = \tilde{\psi}(a^*)^* = \rho(\psi(a^*))^* = \rho(\psi(a^*)^*) = \rho(\psi^*(a)) = \rho(\psi(a)) = \tilde{\psi}(a),$$

so that $\tilde{\psi} = \tilde{\psi}^*$ and hence $\widetilde{T} = \widetilde{T}^*$. Therefore $T = T^*$. Also $\|T\|_{cb} \le \|\widetilde{T}\|_{cb} = \|\tilde{\psi}\|_{cb} = \|\psi\|_{cb}$.

Uniqueness: Suppose $\overline{\mathrm{span}}\, \pi(\mathcal{A})z\mathcal{B} = E$. Now if $S \in \pi(A)'$ any other operator such that $\pi(\,\cdot\,) = \langle z, S\pi(\,\cdot\,)z\rangle$, then $\langle \pi(a)zb, (T - S)\pi(a')zb'\rangle = 0$ for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, so that $T - S = 0$. $\square$

3C. *Representations of CB-maps: One from another.* In this section we see how different representations of CB-maps $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ are related each other. Since the results are straightforward we do not provide proofs.

Proposition 3.11 (Commutant representation from regular representation I). *Let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CB-map with a regular representation $(\mathcal{K}, \tau, W)$, that is, $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a $*$-nondegenerate regular homomorphism and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\psi(\cdot) = W^*\tau(\cdot)W$. Suppose $\vartheta : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is the unique unital $*$-homomorphism such that $\tau(\cdot) = \vartheta(\cdot)T = T\vartheta(\cdot)$, where $T = \tau(1)$. Then $(\mathcal{K}, \vartheta, T, W)$ is a commutant representation for ψ.*

Note that W of this proposition may not be an isometry. This can be taken care of as follows:

Proposition 3.12 (Commutant representation from regular representation II). *Let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a nonzero CB-map. Let $(\mathcal{K}, \tau, V)$ be a regular representation for $\hat{\psi} = \psi/\|\psi\|_{cb}$ with V as an isometry. Choose a (not necessarily unital) $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that $\tau(\cdot) = \vartheta(\cdot)\tau(1) = \tau(1)\vartheta(\cdot)$. Then $T = \|\psi\|_{cb}\tau(1) \in \tau(\mathcal{A})' \subseteq \mathcal{B}(\mathcal{K})$ is such that $\psi(\cdot) = \|\psi\|_{cb} V^*\tau(\cdot)V = V^*T\vartheta(\cdot)V$, so that $(\mathcal{K}, \vartheta, T, V)$ is a commutant representation for ψ.*

The drawback of the previous representation is that the $*$-homomorphism ϑ may not be unital.

Proposition 3.13 (Regular representation from commutant representation I). *Suppose (K, π, T, V) is a commutant representation of a CB-map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Set $\widehat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$. Define $\tau : \mathcal{A} \to \mathcal{B}(\widehat{\mathcal{K}})$ by*

$$\tau(a) = \begin{bmatrix} \pi(a) & (2T - I)\pi(a) \\ 0 & 0 \end{bmatrix} \quad \textit{and set} \quad W = \begin{bmatrix} V/\sqrt{2} \\ V/\sqrt{2} \end{bmatrix} \in \mathcal{B}(\mathcal{H}, \widehat{\mathcal{K}}).$$

Then τ is a regular homomorphism and W is an isometry such that $\psi(\cdot) = W^*\tau(\cdot)W$.

We may prefer to get a t-ternary representation instead of just a regular representation. This can be achieved as follows:

Proposition 3.14 (Regular representation from commutant representation II). *Suppose (K, π, T, V) is a commutant representation of a CB-map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Set $\widehat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$. Given any $t \in (1, \infty)$ define $\tau : \mathcal{A} \to \mathcal{B}(\widehat{\mathcal{K}})$ by*

$$\tau(a) = \begin{bmatrix} \pi(a) & 0 \\ -\sqrt{t-1}\,\pi(a) & 0 \end{bmatrix} \quad \textit{and set} \quad W = \begin{bmatrix} 0 & I \\ \frac{I-T^*}{\sqrt{t-1}} & 0 \end{bmatrix} \begin{bmatrix} V \\ V \end{bmatrix} \in \mathcal{B}(\widehat{\mathcal{K}}).$$

Then τ is a $*$-nondegenerate t-ternary homomorphism. Also $\psi(\cdot) = W^*\tau(\cdot)W$, so that $(\widehat{\mathcal{K}}, \tau, W)$ is a regular representation of ψ.

Finally we show that any regular representation also gives another familiar representation called the fundamental representation for completely bounded maps (Theorem 8.4 of [Paulsen 2002]):

Proposition 3.15 (Fundamental representation from regular representation). *Suppose $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a CB-map with regular representation $(\mathcal{K}, \tau, W)$. Let $\vartheta : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be the $*$-homomorphism such that $\tau(\cdot) = \vartheta(\cdot)\tau(1) = \tau(1)\vartheta(\cdot)$. Then $V_1 := W$ and $V_2 := \tau(1)W$ are elements of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\psi(\cdot) = W^*\tau(\cdot)W = V_1^*\vartheta(\cdot)V_2.$$

Acknowledgement

Bhat and Mallick are supported by UGC under India–Israel Joint Research Project 2014: *E_0-semigroups: classification and invariants.* Mallick is thankful to the National Board for Higher Mathematics (NBHM), India for a senior research fellowship. This work is a part of the PhD thesis she submitted to the Indian Statistical Institute. This research work was done when Sumesh was at IMSc Chennai, and he thanks the Department of Atomic Energy (DAE), Government of India for financial support and IMSc Chennai for providing necessary facilities.

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Received October 6, 2015.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 289 No. 2 August 2017