# Pacific Journal of Mathematics 

EXACT LAGRANGIAN FILLINGS OF LEGENDRIAN $(2, n)$ TORUS LINKS<br>Yu Pan

# EXACT LAGRANGIAN FILLINGS OF LEGENDRIAN $(2, n)$ TORUS LINKS 

Yu Pan


#### Abstract

Ekholm, Honda, and Kálmán constructed $\boldsymbol{C}_{\boldsymbol{n}}$ exact Lagrangian fillings for a Legendrian ( $2, n$ ) torus knot or link with maximal Thurston-Bennequin number, where $C_{n}$ is the $n$-th Catalan number. We show that these exact Lagrangian fillings are pairwise nonisotopic through exact Lagrangian isotopy. To do that, we compute the augmentations induced by the exact Lagrangian fillings $L$ to $\mathbb{Z}_{2}\left[H_{1}(L)\right]$ and distinguish the resulting augmentations.


## 1. Introduction

A Legendrian submanifold $\Lambda$ in the standard contact manifold ( $\mathbb{R}^{3}, \xi=\operatorname{ker} \alpha$ ), where $\alpha=d z-y d x$, is a 1-dimensional closed manifold such that $T \Lambda \subset \xi$ everywhere. An exact Lagrangian filling $L$ of $\Lambda$ in the symplectization manifold $\left(\mathbb{R}_{t} \times \mathbb{R}^{3}, \omega=d\left(e^{t} \alpha\right)\right.$ ) is a 2-dimensional surface that is cylindrical over $\Lambda$ when $t$ is sufficiently large. See Definition 2.5 for more detail, and Figure 1 for a picture.

In this paper, we study oriented exact Lagrangian fillings of the Legendrian (2, $n$ ) torus links $\Lambda$ with maximal Thurston-Bennequin number $(n>0)$. When $n$ is even, we also require the link to have the right Maslov potential such that Reeb chords $b_{1}, \ldots, b_{n}$ in Figure 2 are in degree 0 (see Section 2A for detailed definitions). Ekholm, Honda, and Kálmán [Ekholm et al. 2016] gave an algorithm (which we refer to later as the EHK algorithm) to construct exact Lagrangian fillings of the Legendrian $(2, n)$ torus link $\Lambda$ as follows. Starting with a Lagrangian projection (a projection from $\mathbb{R}^{3}$ to the $x y$-plane) of $\Lambda$ as shown in Figure 2, we can successively resolve crossings $b_{i}$ in any order through pinch moves (see Figure 3), which correspond to saddle cobordisms. As a result, we get two Legendrian unknots, which admit minimum cobordisms as shown in Figure 3. Concatenating the $n$ saddle cobordisms with these two minimum cobordisms, we get an exact Lagrangian filling of $\Lambda$.

Different orders of resolving crossings $b_{1}, \ldots, b_{n}$ may give different exact Lagrangian fillings of $\Lambda$ up to exact Lagrangian isotopy. Given a permutation $\sigma=(\sigma(1), \sigma(2), \ldots, \sigma(n))$ of $\{1, \ldots, n\}$, write $L_{\sigma}$ for the exact Lagrangian filling achieved by using $n$ successive pinch moves at $b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(n)}$, respectively,

[^0]

Figure 1. An exact Lagrangian filling.
and then concatenating with the two minimum cobordisms. Observe that two permutations may give isotopic exact Lagrangian fillings. For instance, let $\Lambda$ be the Legendrian $(2,3)$ torus knot and consider the exact Lagrangian fillings of $\Lambda$ that correspond to permutations $(1,3,2)$ and $(3,1,2)$, respectively. Since the saddles corresponding to the pinch moves at $b_{1}$ and $b_{3}$ are disjoint when projected to $\mathbb{R}^{3}$, one can use a Hamiltonian vector field in the $t$ direction to exchange the heights of these two saddles. Therefore, the two fillings $L_{(1,3,2)}$ and $L_{(3,1,2)}$ are Hamiltonian isotopic and thus are exact Lagrangian isotopic. In general, for the Legendrian $(2, n)$ torus link $\Lambda$, given any numbers $i, j, k$ such that $i<k<j$, two permutations


Figure 2. The Lagrangian projection of the Legendrian ( $2, n$ ) torus knot.


Figure 3. The pinch move (left) and the minimum cobordism (right) between Lagrangian projections of links.
$(\ldots, i, j, \ldots, k, \ldots)$ and $(\ldots, j, i, \ldots, k, \ldots)$, where only $i$ and $j$ are interchanged, give the same exact Lagrangian fillings of $\Lambda$ up to exact Lagrangian isotopy. Taking all the permutations of $\{1, \ldots, n\}$ modded out by this relation, we obtain $C_{n}$ exact Lagrangian fillings of $\Lambda$, where

$$
C_{n}=\frac{1}{n+1}\binom{2 n}{n}
$$

is the $n$-th Catalan number. In this paper, we prove the following theorem:
Theorem 1.1 (see Theorem 3.11 and Corollary 3.12). The $C_{n}$ exact Lagrangian fillings that come from the EHK algorithm are all of different exact Lagrangian isotopy classes. In other words, the Legendrian $(2, n)$ torus link has at least $C_{n}$ exact Lagrangian fillings up to exact Lagrangian isotopy.

Shende, Treumann, Williams and Zaslow [Shende et al. 2015] have also constructed $C_{n}$ exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot using cluster varieties and shown that they are distinct up to Hamiltonian isotopy. They remarked that these are presumably the same as fillings obtained in [Ekholm et al. 2016], but we do not resolve this issue here.
Remark 1.2. We will see from Corollary 3.12 that the conclusion of Theorem 1.1 for the case when $n$ is even can be derived from the result for the case when $n$ is odd. Therefore, for most of the paper, we focus on the case when $n$ is odd, which means $\Lambda$ is a knot.

Inspired by [Ekholm et al. 2016], we use augmentations to distinguish the $C_{n}$ exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot $\Lambda$. In order to talk about augmentations, we first introduce the Chekanov-Eliashberg differential graded algebra (DGA) of a Legendrian knot $\Lambda$, which is a chain complex $(\mathcal{A}(\Lambda), \partial)$. This is an invariant of Legendrian submanifolds introduced by Chekanov [2002] and Eliashberg [1998] in the spirit of symplectic field theory [Eliashberg et al. 2000]. The underlying algebra $\mathcal{A}(\Lambda)$ of the Chekanov-Eliashberg DGA is freely generated by Reeb chords of $\Lambda$ over a commutative ring $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]=\mathbb{Z}_{2}\left[s, s^{-1}\right]$, where Reeb chords of $\Lambda$ correspond to double points of the Lagrangian projection of $\Lambda$. The differential is defined by a count of rigid holomorphic disks with boundary on $\Lambda$, taken with coefficients in $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]$. In general, the Chekanov-Eliashberg DGA of $\Lambda$ is defined with $\mathbb{Z}\left[H_{1}(\Lambda)\right]$ coefficients. For our purpose, it suffices to consider the DGA with $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]$ coefficients, which means ignoring the orientations of moduli spaces of holomorphic disks. An augmentation $\epsilon$ of $\mathcal{A}(\Lambda)$ to a commutative ring $\mathbb{F}$ is a DGA map $\epsilon:(\mathcal{A}(\Lambda), \partial) \rightarrow(\mathbb{F}, 0)$. As shown in [Ekholm et al. 2016], an exact Lagrangian filling $L$ of $\Lambda$ gives an augmentation of $\mathcal{A}(\Lambda)$ by counting rigid holomorphic disks with boundary on $L$. Moreover, by Theorem 1.3 of the same paper, exact Lagrangian isotopic fillings give homotopic augmentations. Therefore,
in order to distinguish two fillings, we only need to show their induced augmentations are not chain homotopic.

Ekholm et al. [2016] distinguished all the exact Lagrangian fillings from the EHK algorithm when $n=3$ by computing all the augmentations of the Legendrian $(2,3)$ torus knot to $\mathbb{Z}_{2}$ and finding that they are pairwise non-chain-homotopic. However, when $n \geq 5$, a computation shows that the number of augmentations of the DGA to $\mathbb{Z}_{2}$ is much less than the Catalan number $C_{n}$.

In this paper, for an exact Lagrangian filling $L$ of the Legendrian $(2, n)$ torus knot $\Lambda$, we consider its induced augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_{2}\left[H_{1}(L)\right]$, where $H_{1}(L)$ is the singular homology of $L$. Note that $H_{1}(L) \cong H_{2}\left(\mathbb{R} \times \mathbb{R}^{3}, L\right)$ and thus it is natural to count the rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^{3}$ with boundary on $L$ with $\mathbb{Z}_{2}\left[H_{1}(L)\right]$ coefficients. However, the computation of augmentations is not as easy as for the case with $\mathbb{Z}_{2}$ coefficients. For each exact Lagrangian filling $L$ from the EHK algorithm, we give a combinatorial formula of the induced augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_{2}\left[H_{1}(L)\right]$. From the formula, we find a combinatorial invariant to show that the augmentations are not pairwise chain homotopic. In this way, we distinguish all of the $C_{n}$ exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot $\Lambda$ up to exact Lagrangian isotopy.
Outline. In Section 2, we review the Chekanov-Eliashberg DGA of a Legendrian submanifold and the DGA maps induced by an exact Lagrangian cobordism. In Section 3, we compute all the augmentations of the Legendrian $(2, n)$ torus knot to $\mathbb{Z}_{2}\left[H_{1}(L)\right]$ induced by the exact Lagrangian fillings $L$ and prove that all the resulting augmentations are distinct up to chain homotopy. In the end, we prove Theorem 1.1 for the case $n$ even as a corollary.

## 2. Preliminaries

In Section 2A, we review the definition of the Chekanov-Eliashberg DGA of Legendrian submanifolds in $\left(\mathbb{R}^{3}, \operatorname{ker} \alpha\right)$ and its extension to the setting of multiple base points. For the purpose of computing augmentations in Section 3A, the definition of DGA we use here is slightly different from the versions in [ Ng 2010 ] and [ Ng et al. 2015], where the underlying algebra is completely noncommutative. In our definition, we allow elements in the coefficient ring to commute with the elements corresponding to Reeb chords. This is a generalization of the definition of Chekanov-Eliashberg DGA from [Etnyre et al. 2002]. See [Ekholm et al. 2013, Section 2.3.2] for further discussions. In Section 2B, we review the DGA map induced by an exact Lagrangian cobordism and revise coefficients of this map for the purpose of computing augmentations in Section 3A.

2A. The Chekanov-Eliashberg DGA. Let $\Lambda$ be a Legendrian submanifold in $\left(\mathbb{R}^{3}, \operatorname{ker} \alpha\right)$, where $\alpha=d z-y d x$. There are two projection diagrams associated


Figure 4. A front projection (left) and a Lagrangian projection (right) of the Legendrian trefoil.


Figure 5. Ng's algorithm to transfer a front projection to a Lagrangian projection by smoothing the left cusp directly and smoothing the right cusp with an additional crossing.
to $\Lambda$ via the Lagrangian projection $\Pi_{x y}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2},(x, y, z) \mapsto(x, y)$ and the front projection $\Pi_{x z}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2},(x, y, z) \mapsto(x, z)$, respectively. As an example, a front projection and a Lagrangian projection of the Legendrian trefoil are shown in Figure 4. Moreover, starting from a front projection of $\Lambda, \mathrm{Ng}$ [2003] gave an algorithm to get a Lagrangian projection of $\Lambda$ by smoothing the cusps of the front projection in a way shown in Figure 5.

Let $\Lambda=\Lambda_{1} \cup \Lambda_{2} \cup \cdots \cup \Lambda_{k}$ be an oriented Legendrian link with $k$ connected components. Now let us define the Chekanov-Eliashberg $\operatorname{DGA}\left(\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right), \partial\right)$ of $\Lambda$. To simplify the definition of grading, we assume throughout the paper that the rotation number of $\Lambda$ is 0 . Note that all the Legendrian $(2, n)$ torus links we consider have rotation number 0 .

The underlying algebra. The underlying algebra $\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right)$ is a unital graded algebra freely generated by Reeb chords of $\Lambda$ over

$$
\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]=\mathbb{Z}_{2}\left[s_{1}^{ \pm 1}, s_{2}^{ \pm 1}, \ldots, s_{k}^{ \pm 1}\right]
$$

where $\left\{s_{1}, s_{2}, \ldots, s_{k}\right\}$ is any basis of $H_{1}(\Lambda)$. A Reeb chord of $\Lambda$ in $\left(\mathbb{R}^{3}, \operatorname{ker} \alpha\right)$ is a vertical line segment ( $z$ direction) with both ends on $\Lambda$ endowed with an orientation in the positive $z$ direction. Reeb chords of $\Lambda$ are in one-to-one correspondence to double points of $\Pi_{x y}(\Lambda)$, which by Ng's algorithm correspond to the crossings and right cusps of $\Pi_{x z}(\Lambda)$.

To define the grading of Reeb chords, we work on the front projection $\Pi_{x z}(\Lambda)$. Write $C\left(\Pi_{x z}(\Lambda)\right)$ for the set of cusps of $\Pi_{x z}(\Lambda)$, which divides $\Pi_{x z}(\Lambda)$ into strands (ignoring double points). The Maslov potential is a function $\mu$ that assigns an integer to each strand such that around each cusp, the Maslov potential of the lower strand is one less than that of the upper strand. This is well defined up to a global shift on


Figure 6. At each crossing, the quadrants labeled with + sign are positive quadrants and the ones labeled with - sign are negative quadrants.
each component of $\Lambda$. Once the Maslov potential is fixed, the grading of a Reeb chord $c$ that corresponds to a crossing of $\Pi_{x z}(\Lambda)$ can be defined by

$$
|c|:=\mu(u)-\mu(l),
$$

where $u$ is the upper strand of the crossing and $l$ is the lower strand of the crossing. The grading of Reeb chords that correspond to right cusps of $\Pi_{x z}(\Lambda)$ are defined to be 1 . Extend the definition of grading to $\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right)$ by setting $\left|s_{i}\right|=0$ for $i=1, \ldots, k$ and using the relation $|a b|=|a|+|b|$.

In the special case of Legendrian ( $2, n$ ) torus links, when $n$ is odd, the degree is well defined. When $n$ is even, we can choose a Maslov potential of the Legendrian $(2, n)$ torus link such that for any Reeb chord $b_{i}$ as labeled in Figure 2, the upper strand and the lower strand of $b_{i}$ have the same Maslov potential. In this setting, for a Legendrian $(2, n)$ torus link ( $n$ is either odd or even) whose Lagrangian projection is like Figure 2, we have that $\left|a_{1}\right|=\left|a_{2}\right|=1$ and $\left|b_{i}\right|=0$ for $i=1, \ldots, n$.

Differential. The differential $\partial$ is defined by counting rigid holomorphic disks in $\mathbb{R}_{x y}^{2}$ with boundary on $\Pi_{x y}(\Lambda)$.

For any Reeb chords $a, b_{1}, \ldots, b_{m}$ of $\Lambda$, define $\mathcal{M}^{\Lambda}\left(a ; b_{1}, \ldots, b_{m}\right)$ to be the moduli space of holomorphic disks

$$
u:\left(D_{m+1}, \partial D_{m+1}\right) \rightarrow\left(\mathbb{R}^{2}, \Pi_{x y}(\Lambda)\right)
$$

with the following properties:

- $D_{m+1}$ is a 2-dimensional unit disk with $m+1$ points $s, t_{1}, \ldots, t_{m}$ removed from the boundary and the points $s, t_{1}, \ldots, t_{m}$ are labeled in counterclockwise order.
- $\lim _{r \rightarrow s} u(r)=a$ and the image of a neighborhood of $s$ under $u$ covers exactly one positive quadrant of the crossing $a$ (see Figure 6).
- $\lim _{r \rightarrow t_{i}} u(r)=b_{i}$, for $i=1, \ldots, m$, and the image of a neighborhood of $t_{i}$ under $u$ covers exactly one negative quadrant of the crossing $b_{i}$ (see Figure 6).

We occasionally abbreviate $\left(a, b_{1}, \ldots, b_{m}\right)$ to $(a ; \boldsymbol{b})$, where $\boldsymbol{b}$ represents a sequence of Reeb chords, $b_{1}, \ldots, b_{m}$. According to [Chekanov 2002], we have the


Figure 7. The Legendrian Hopf link $\Lambda_{1} \cup \Lambda_{2}$. For a Reeb chord $c$ from $c^{-} \in \Lambda_{1}$ to $c^{+} \in \Lambda_{2}$, the red curve is a capping path $\gamma_{c}$.
following dimension formula:

$$
\operatorname{dim} \mathcal{M}^{\Lambda}\left(a ; b_{1}, \ldots, b_{m}\right)=|a|-\sum_{i=1}^{m}\left|b_{i}\right|-1
$$

When $\operatorname{dim} \mathcal{M}^{\Lambda}\left(a ; b_{1}, \ldots, b_{m}\right)=0$, the disk $u \in \mathcal{M}^{\Lambda}\left(a ; b_{1}, \ldots, b_{m}\right)$ is called rigid. There are finitely many rigid holomorphic disks and hence we can count them.

In order to count with $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]$ coefficients, we want to take the homology class of the boundary of rigid disks in $H_{1}(\Lambda)$. However, for any rigid holomorphic disk $u$, the boundary $\Pi_{x y}^{-1}\left(u\left(\partial D_{m+1}\right)\right)$ is not closed. Therefore, we first introduce capping paths. Equip each connected component $\Lambda_{i}$ with a reference point $p_{i}$, for $i=1, \ldots, k$. For each $i \neq 1$, pick a path $\delta_{1 i}$ in $\mathbb{R}^{3} \backslash \Lambda$ that goes from $p_{1}$ to $p_{i}$. For each Reeb chord $c$ of $\Lambda$ from $c^{-} \in \Lambda_{i^{-}}$to $c^{+} \in \Lambda_{i^{+}}$, the capping path $\gamma_{c}$ is defined by concatenating

- a path on $\Lambda_{i^{-}}$from $c^{-}$to $p_{i^{-}}$,
- the chosen path $-\delta_{1 i^{-}}$connecting $p_{i^{-}}$to $p_{1}$,
- the chosen path $\delta_{1 i^{+}}$connecting $p_{1}$ to $p_{i+}$, and
- a path on $\Lambda_{i+}$ from $p_{i^{+}}$to $c^{+}$.

See Figure 7 for an example of a capping path.
After associating each Reeb chord with a capping path, for any rigid holomorphic disk $u \in \mathcal{M}^{\Lambda}\left(a ; b_{1}, \ldots, b_{m}\right)$, the curve

$$
\tilde{u}=\Pi_{x y}^{-1}\left(u\left(\partial D_{m+1}\right)\right) \cup \gamma_{a} \cup-\gamma_{b_{1}} \cup \cdots \cup-\gamma_{b_{m}}
$$

is a loop in $\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1 k}$. Notice that $H_{1}\left(\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1 k}\right) \cong H_{1}(\Lambda)$. Thus we can view the homology class $[\tilde{u}]$ as in $H_{1}(\Lambda)$.

Now we can define the differential of the Chekanov-Eliashberg DGA of $\Lambda$.

Definition 2.1. For any Reeb chord $a$ of $\Lambda$, the differential $\partial$ is defined by:

$$
\begin{equation*}
\partial(a)=\sum_{\operatorname{dim} \mathcal{M}^{\wedge}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\wedge}(a ; \boldsymbol{b})}[\tilde{u}] b_{1} \cdots b_{m} . \tag{2-1}
\end{equation*}
$$

The definition of differential can be extended to $\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right)$ by setting $\partial\left(s_{i}\right)=0$ for $i=1, \ldots, k$, and using the Leibniz rule

$$
\partial(a b)=\partial(a) b+a \partial(b)
$$

According to [Chekanov 2002], the map $\partial$ is a differential in degree -1 , and up to stable tame isomorphism, the Chekanov-Eliashberg DGA $\left(\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right), \partial\right)$ is an invariant of $\Lambda$ under Legendrian isotopy.

Remark 2.2. In general, for any commutative ring $R$ and a ring homomorphism $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right] \rightarrow R$, we define the Chekanov-Eliashberg DGA $(\mathcal{A}(\Lambda ; R), \partial)$ as a tensor product of the DGA $\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right)$ with the ring $R$ :

$$
\mathcal{A}(\Lambda ; R)=\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right) \otimes_{\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]} R
$$

where the ring homomorphism gives $R$ the structure of a module over $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]$.
We give a combinatorial definition of the differential of $\left(\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right), \partial\right)$. Assign $\Lambda$ an orientation and label each component $\Lambda_{i}$, for $i=1, \ldots, k$, with a base point $s_{i}$, which is different from the reference point and ends of Reeb chords. For a union of oriented curves $\gamma$ in $\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1 k}$, we associate it with a monomial $w(\gamma)$ in $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]:$

$$
\begin{equation*}
w(\gamma)=\prod_{i=1}^{k} s_{i}^{n_{i}(\gamma)} \tag{2-2}
\end{equation*}
$$

where $n_{i}(\gamma)$ is the number of times $\gamma$ goes through $s_{i}$ counted with sign. The sign is positive if $\gamma$ goes through $s_{i}$ following the link orientation and is negative if $\gamma$ goes through $s_{i}$ against the link orientation. In particular, for a rigid holomorphic disk $u \in \mathcal{M}^{\Lambda}\left(a ; b_{1}, \ldots, b_{m}\right)$, we have

$$
\begin{equation*}
[\tilde{u}]=w(\tilde{u})=w(u) w\left(\gamma_{a}\right) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}\right)^{-1} \tag{2-3}
\end{equation*}
$$

where $w(u)$ is short for $w\left(\Pi_{x y}^{-1}\left(u\left(\partial D_{m+1}\right)\right)\right)$. Plugging it into the formula (2-1), we get a combinatorial definition of the differential. It seems to depend on the choice of capping paths. However, we have the following well-known proposition.

Proposition 2.3. Let $\Lambda$ be a Legendrian link and $\gamma, \gamma^{\prime}$ be two families of capping paths of Reeb chords of $\Lambda$. The corresponding DGAs $\left(\mathcal{A}^{\gamma}(\Lambda), \partial\right)$ and $\left(\mathcal{A}^{\gamma^{\prime}}(\Lambda), \partial^{\prime}\right)$ are isomorphic.


Figure 8. The Lagrangian projection of the Legendrian (2, 3) torus knot with a single base point.

Proof. For a Reeb chord $a$ of $\Lambda$, we have

$$
\begin{aligned}
\partial(a) & =\sum_{\operatorname{dim} \mathcal{M}^{\wedge}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\wedge}(a ; \boldsymbol{b})}\left(w(u) w\left(\gamma_{a}\right) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}\right)^{-1}\right) b_{1} \cdots b_{m} \\
\partial^{\prime}(a) & =\sum_{\operatorname{dim} \mathcal{M}^{\wedge}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\wedge}(a ; \boldsymbol{b})}\left(w(u) w\left(\gamma_{a}^{\prime}\right) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}^{\prime}\right)^{-1}\right) b_{1} \cdots b_{m}
\end{aligned}
$$

For each Reeb chord $c$, concatenate $-\gamma_{c}^{\prime}$ with $\gamma_{c}$ and get a closed curve, denoted by $-\gamma_{c}^{\prime} \cup \gamma_{c}$. It is not hard to check that the map

$$
f:\left(\mathcal{A}^{\gamma}(\Lambda), \partial\right) \rightarrow\left(\mathcal{A}^{\gamma^{\prime}}(\Lambda), \partial^{\prime}\right), \quad c \mapsto\left[-\gamma_{c}^{\prime} \cup \gamma_{c}\right] c=w\left(\gamma_{c}^{\prime}\right)^{-1} w\left(\gamma_{c}\right) c
$$

is a chain map and is an isomorphism.
Note that for an oriented link $\Lambda$ with minimal base points (i.e., each component has exactly one base point), we can choose a family of capping paths such that none of them pass through any base point. Therefore, we only need to count intersections of the disk boundary and base points. Thanks to Proposition 2.3, we can define the Chekanov-Eliashberg DGA of $\Lambda$ to be a unital graded algebra over $\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]=\mathbb{Z}_{2}\left[s_{1}^{ \pm 1}, \ldots, s_{k}^{ \pm 1}\right]$ generated by Reeb chords of $\Lambda$ endowed with a differential given by

$$
\begin{aligned}
& \partial(a)=\sum_{\operatorname{dim} \mathcal{M}^{\wedge}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\wedge}(a ; \boldsymbol{b})} w(u) b_{1} \cdots b_{m}, \\
& \partial\left(s_{i}\right)=0, \quad i=1, \ldots, k
\end{aligned}
$$

with $w(u)$ defined as in (2-3). This DGA is denoted by $\left(\mathcal{A}\left(\Lambda,\left\{s_{1}, \ldots, s_{k}\right\}\right), \partial\right)$ too.
Example 2.4. Consider the Legendrian $(2,3)$ torus $\operatorname{knot} \Lambda$ with a single base point $s$ as shown in Figure 8. The underlying algebra $\mathcal{A}(\Lambda,\{s\})$ is generated by Reeb chords $a_{1}, a_{2}, b_{1}, b_{2}, b_{3}$ over $\mathbb{Z}_{2}\left[s, s^{-1}\right]$. Reeb chords $a_{1}$ and $a_{2}$ are in degree 1 and the rest of the Reeb chords are in degree 0 . The differential is given by


Figure 9. A schematic picture of an exact Lagrangian cobordism.

$$
\begin{aligned}
& \partial\left(a_{1}\right)=s^{-1}+b_{1}+b_{3}+b_{1} b_{2} b_{3}, \\
& \partial\left(a_{2}\right)=1+b_{1}+b_{3}+b_{3} b_{2} b_{1}, \\
& \partial\left(b_{i}\right)=0, \quad i=1,2,3, \\
& \partial(s)=\partial\left(s^{-1}\right)=0 .
\end{aligned}
$$

The definition of the DGA of a Legendrian link can be generalized to the case where there is more than one base point on some components of the link. Let $\Lambda$ be an oriented Legendrian link and $\left\{s_{1}, \ldots, s_{l}\right\}$ be a set of points on $\Lambda$ such that each component of $\Lambda$ has at least one point in the set and the set does not include any end of any Reeb chord of $\Lambda$. For a union of paths $\gamma$, associate it with a monomial $w(\gamma)=\prod_{j=1}^{l} s_{j}^{n_{j}(\gamma)}$ in $\mathbb{Z}_{2}\left[s_{1}^{ \pm 1}, \ldots, s_{l}^{ \pm 1}\right]$, where $n_{j}$ is defined much as above. The DGA

$$
\left(\mathcal{A}\left(\Lambda,\left\{s_{1}, \ldots, s_{l}\right\}\right), \partial\right)
$$

is a unital graded algebra generated by Reeb chords of $\Lambda$ over $\mathbb{Z}_{2}\left[s_{1}^{ \pm 1}, \ldots, s_{l}^{ \pm 1}\right]$ endowed with a differential given by

$$
\begin{aligned}
& \partial(a)=\sum_{\operatorname{dim} \mathcal{M}^{\wedge}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\wedge}(a ; \boldsymbol{b})} w(u) b_{1} \cdots b_{m}, \\
& \partial\left(s_{i}\right)=0, \quad i=1, \ldots, l .
\end{aligned}
$$

2B. The DGA map induced by exact Lagrangian cobordisms. The ChekanovEliashberg DGA acts functorially on exact Lagrangian cobordisms, according to [Ekholm et al. 2016]. We first recall the definition of exact Lagrangian cobordisms.
Definition 2.5. Let $\Lambda_{+}$and $\Lambda_{-}$be Legendrian submanifolds in $\left(\mathbb{R}^{3}, \operatorname{ker} \alpha\right.$ ), where $\alpha=d z-y d x$. An exact Lagrangian cobordism $\Sigma$ from $\Lambda_{-}$to $\Lambda_{+}$is a 2-dimensional surface in $\left(\mathbb{R} \times \mathbb{R}^{3}, d\left(e^{t} \alpha\right)\right)$ such that there exists $T>0$ such that $\Sigma$ is

- cylindrical over $\Lambda_{+}$on the positive end, i.e., $\Sigma \cap\left((T, \infty) \times \mathbb{R}^{3}\right)=(T, \infty) \times \Lambda_{+}$;


Figure 10. The relation among cobordisms $\bar{\Sigma}_{+}, \bar{\Sigma}_{-}$, and $\Sigma$.

- cylindrical over $\Lambda_{-}$on the negative end, i.e., $\Sigma \cap\left((-\infty,-T) \times \mathbb{R}^{3}\right)=$ $(-\infty,-T) \times \Lambda_{-}$;
- compact in $[-T, T] \times \mathbb{R}^{3}$,
and $\left.e^{t} \alpha\right|_{T \Sigma}=d f$ for some function $f: \Sigma \rightarrow \mathbb{R}$. (See Figure 9.)
When $\Lambda_{-}$is empty, the surface $L$ satisfying the conditions above is called an exact Lagrangian filling of $\Lambda_{+}$.

By [Ekholm et al. 2016], an exact Lagrangian cobordism $\Sigma$ from $\Lambda_{-}$to $\Lambda_{+}$ gives a DGA map from $\mathcal{A}\left(\Lambda_{+}\right)$to $\mathcal{A}\left(\Lambda_{-}\right)$with $\mathbb{Z}_{2}\left[H_{1}(\Sigma)\right]$ coefficients. Thus, an exact Lagrangian filling $L$ of a Legendrian submanifold $\Lambda$, which can be viewed as a cobordism from the empty set to $\Lambda$, gives a DGA map from $\mathcal{A}(\Lambda)$ to the trivial DGA

$$
\left(\mathbb{Z}_{2}\left[H_{1}(L)\right], 0\right),
$$

which is an augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_{2}\left[H_{1}(L)\right]$.
For the purpose of computing augmentations of the Legendrian $(2, n)$ torus knots in Section 3A, we revise the coefficient ring of the DGA map induced by exact Lagrangian cobordisms from [Ekholm et al. 2016]. Instead of using $\mathbb{Z}_{2}\left[H_{1}(\Sigma)\right]$ coefficients, we will show the following proposition:

Proposition 2.6. Let $\Lambda_{+}$and $\Lambda_{-}$be Legendrian submanifolds in $\left(\mathbb{R}^{3}, \operatorname{ker} \alpha\right)$ and $\Sigma$ be a connected exact Lagrangian cobordism from $\Lambda_{-}$to $\Lambda_{+}$. Assume that $\bar{\Sigma}_{+}$ is a connected exact Lagrangian cobordism from $\Lambda_{+}$to some other Legendrian link and $\bar{\Sigma}_{-}$is the concatenation of $\bar{\Sigma}_{+}$and $\Sigma$ as shown in Figure 10. The exact Lagrangian cobordism $\Sigma$ induces a DGA map

$$
\Phi:\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]\right), \partial_{+}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{-} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{-}\right)
$$

with $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]$coefficients.

Note that when $\bar{\Sigma}_{+}$is an exact Lagrangian cylinder over $\Lambda_{+}$, this map agrees with the DGA map introduced in [Ekholm et al. 2016]. The proof of Proposition 2.6 follows Section 3 of that paper. Our revision of the coefficient ring is based on a different choice of capping paths of $\Lambda_{+}$and $\Lambda_{-}$. Ekholm et al. choose capping paths of $\Lambda_{+}$and $\Lambda_{-}$on $\Sigma$, while we choose capping paths of $\Lambda_{+}$on $\bar{\Sigma}_{+}$and capping paths of $\Lambda_{-}$on $\bar{\Sigma}_{-}$. For the rest of the section, we will describe this DGA map.

The inclusion map $\Lambda_{+} \hookrightarrow \bar{\Sigma}_{+}$makes it natural to define the DGA

$$
\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]\right), \partial_{+}\right)
$$

The underlying algebra

$$
\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]\right)=\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\Lambda_{+}\right)\right]\right) \otimes_{\mathbb{Z}_{2}\left[H_{1}\left(\Lambda_{+}\right)\right]} \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]
$$

is generated by Reeb chords of $\Lambda_{+}$over the ring $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]$. Given that $\bar{\Sigma}_{+}$is connected, we can choose a family of capping paths for $\Lambda_{+}$on $\bar{\Sigma}_{+}$. Thus, for any rigid holomorphic disk $u_{+}$counted by $\partial_{+}$, it is natural to take the homology class of $\tilde{u}_{+}$in $H_{1}\left(\bar{\Sigma}_{+}\right)$. Hence the differential coefficients of $\partial_{+}$are in $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]$. In addition, the DGA $\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]\right), \partial_{+}\right)$does not depend on the choice of capping paths on $\bar{\Sigma}_{+}$for a similar reason as in Proposition 2.3. The DGA $\left(\mathcal{A}\left(\Lambda_{-} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{-}\right)$is defined similarly.

The DGA map $\Phi$ induced by $\Sigma$ is a composition of two maps. The first map

$$
\psi:\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]\right), \partial_{+}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{+}\right)
$$

is induced by the inclusion map $\bar{\Sigma}_{+} \hookrightarrow \bar{\Sigma}_{-}$. It is not hard to show $\psi$ is a DGA map. The second map

$$
\phi:\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{+}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{-} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{-}\right)
$$

is defined by counting rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^{3}$ with boundary on $\Sigma$.
Fix an almost complex structure $J$ on $\mathbb{R} \times \mathbb{R}^{3}$ which is adjusted to the symplectic form $\omega$ (see [Ekholm et al. 2016, Section 3.2] for details). For a Reeb chord $a$ of $\Lambda_{+}$and Reeb chords $b_{1}, \ldots, b_{m}$ of $\Lambda_{-}$, define $\mathcal{M}^{\Sigma}\left(a ; b_{1}, \ldots, b_{m}\right)$ to be the moduli space of $J$-holomorphic disks

$$
u:\left(D_{m+1}, \partial D_{m+1}\right) \rightarrow\left(\mathbb{R} \times \mathbb{R}^{3}, \Sigma\right)
$$

with the following properties:

- $D_{m+1}$ is a 2 -dimensional unit disk with $m+1$ points $r, s_{1}, s_{2}, \ldots, s_{m}$ removed. The points $r, s_{1}, s_{2}, \ldots, s_{m}$ are arranged counterclockwise on the boundary of the disk.
- The image of $u$ is asymptotic to a strip $\mathbb{R}_{+} \times a$ around $r$.
- The image of $u$ is asymptotic to a strip $\mathbb{R}_{-} \times b_{i}$ around $s_{i}$ for $i=1, \ldots, m$.

By [Cieliebak et al. 2010], there is a corresponding dimension formula:

$$
\operatorname{dim} \mathcal{M}^{\Sigma}\left(a ; b_{1}, \ldots, b_{m}\right)=|a|-\sum_{i=1}^{m}\left|b_{i}\right|
$$

If $\operatorname{dim} \mathcal{M}^{\Sigma}\left(a ; b_{1}, \ldots, b_{m}\right)=0$, the $J$-holomorphic disk $u \in \mathcal{M}^{\Sigma}\left(a ; b_{1}, \ldots, b_{m}\right)$ is called rigid. For each rigid $J$-holomorphic disk $u$, concatenate the image of the disk boundary with the capping paths of corresponding Reeb chords on $\bar{\Sigma}_{-}$and get

$$
\tilde{u}=u\left(\partial D_{m+1}\right) \cup \gamma_{a} \cup-\gamma_{b_{1}} \cup \cdots \cup-\gamma_{b_{m}},
$$

which is a loop in $\bar{\Sigma}_{-}$. Hence we can take the homology class of $\tilde{u}$ in $H_{1}\left(\bar{\Sigma}_{-}\right)$, denoted by $[\tilde{u}]_{\bar{\Sigma}_{-}}$. The map

$$
\phi:\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{+}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{-} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{-}\right)
$$

is defined as follows. For any Reeb chord $a$ of $\Lambda_{+}$, the map $\phi$ maps $a$ to

$$
\phi(a)=\sum_{\operatorname{dim} \mathcal{M}^{\Sigma}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\Sigma}(a ; \boldsymbol{b})}[u]_{\bar{\Sigma}_{-}} b_{1} \cdots b_{m}
$$

The map $\phi$ is the identity on $\mathbb{Z}_{1}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]$. By [Ekholm et al. 2016, Section 3.5], the map $\phi$ is a DGA map.

Therefore, the exact Lagrangian cobordism $\Sigma$ induces a DGA map, $\Phi=\phi \circ \psi$

$$
\Phi:\left(\mathcal{A}\left(\Lambda_{+} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{+}\right)\right]\right), \partial_{+}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{-} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{-}\right)\right]\right), \partial_{-}\right) .
$$

## 3. Main results

We consider the exact Lagrangian fillings of the Legendrian (2, $n$ ) torus knot constructed from the EHK algorithm. Each filling can be achieved by concatenating $n$ successive saddle cobordisms with two minimum cobordisms. In Section 3A, we combine results in [Ekholm et al. 2016] and Proposition 2.6 to write down combinatorial formulas for the DGA maps induced by a pinch move and a minimum cobordism. Composing all the DGA maps induced by $n$ ordered pinch moves and the two minimum cobordisms, we obtain a combinatorial formula for augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_{2}\left[H_{1}(L)\right]$ induced by exact Lagrangian fillings $L$. In Section 3B, we find a combinatorial invariant to distinguish these resulting augmentations and hence we show that the $C_{n}$ exact Lagrangian fillings are distinct up to exact Lagrangian isotopy. As a corollary, we extend the result to the case $n$ is even.

3A. Computation of augmentations. Consider the Lagrangian projection of the Legendrian $(2, n)$ torus knot $\Lambda$ with a base point $\tilde{s}_{0}$ and label the $n$ crossings in degree 0 from left to right by $b_{1}, \ldots, b_{n}$ as shown in Figure 11.


Figure 11. The Lagrangian projection of the Legendrian (2,n) torus knot with a base point.

For each permutation $\sigma$ of $\{1, \ldots, n\}$, the corresponding exact Lagrangian filling $L_{\sigma}$ of the Legendrian $(2, n)$ torus $\operatorname{knot} \Lambda$ is achieved in the following way:

- Start with an exact Lagrangian cylinder over $\Lambda$, denoted by $\bar{\Sigma}_{0}$. Label $\Lambda$ as $\Lambda_{0}$.
- For $i=1, \ldots, n$, concatenate $\bar{\Sigma}_{i-1}$ from the bottom with a saddle cobordism $\Sigma_{i}$ corresponding to the pinch move at crossing $b_{\sigma(i)}$ and get a new exact Lagrangian cobordism $\bar{\Sigma}_{i}$. Label the new Legendrian submanifold after the pinch move as $\Lambda_{i}$.
- Finally, use two minimal cobordisms, denoted by $\Sigma_{n+1}$, to close up $\bar{\Sigma}_{n}$ from the bottom and get the exact Lagrangian filling $L_{\sigma}$. To be consistent, let $\Lambda_{n+1}$ be the empty set.
By Proposition 2.6, for $i=1, \ldots, n+1$, each exact Lagrangian cobordism $\Sigma_{i}$ induces a DGA map:

$$
\Phi_{i}:\left(\mathcal{A}\left(\Lambda_{i-1} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i-1}\right)\right]\right), \partial_{i-1}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}\right)
$$

The map $\Phi_{n+1}$ that is induced by minimum cobordisms is well understood while the maps $\Phi_{i}$ for $i=1, \ldots, n$ that correspond to pinch moves are not. We will first study $H_{1}\left(\bar{\Sigma}_{n}\right)$ and give a geometric description of the DGA map that corresponds to a pinch move. Combining this with [Ekholm et al. 2016], we will write down an explicit combinatorial formula for each $\Phi_{i}$, for $i=1, \ldots, n+1$.

To describe $H_{1}\left(\bar{\Sigma}_{n}\right)$ easily, we chop off the cylindrical top of $\bar{\Sigma}_{n}$ and view it as a surface with boundary $\Lambda \cup \Lambda_{n}$, also denoted by $\bar{\Sigma}_{n}$. By Poincaré duality, we have $H^{1}\left(\bar{\Sigma}_{n}\right) \cong H_{1}\left(\bar{\Sigma}_{n}, \Lambda \cup \Lambda_{n}\right)$. In particular, for each oriented curve $\alpha$ in $\bar{\Sigma}_{n}$ with ends on $\Lambda \cup \Lambda_{n}$, which is an element in $H_{1}\left(\bar{\Sigma}_{n}, \Lambda \cup \Lambda_{n}\right)$, there exists an element $\theta_{\alpha} \in H^{1}\left(\bar{\Sigma}_{n}\right)$ such that for any oriented loop $\beta$ in $\bar{\Sigma}_{n}$, the intersection number of $\alpha$ and $\beta$ is $\theta_{\alpha}(\beta)$. Thus, in order to know the homology class of a loop $\beta$ in $H_{1}\left(\bar{\Sigma}_{n}\right)$, we only need to count the intersection number of each generator curve of $H_{1}\left(\bar{\Sigma}_{n}, \Lambda \cup \Lambda_{n}\right)$ with $\beta$.

We choose the set of generator curves of $H_{1}\left(\bar{\Sigma}_{n}, \Lambda \cup \Lambda_{n}\right)$ as follows. Use the $t$ coordinate to slice $\bar{\Sigma}_{n}$ into a movie of diagrams (some of them may not be


Figure 12. As an example, assume $\Lambda$ is the Legendrian $(2,3)$ torus knot and the first pinch move is taken at $b_{2}$. The blue curve and the red curve are $\alpha_{2}$ and $\alpha_{0}$ restricted on $\bar{\Sigma}_{1}$, respectively.

Legendrian diagrams). We study the trace of points on the diagram when $t$ is decreasing. For $i=1, \ldots, n$, the saddle cobordism $\Sigma_{i}$ flows all the points directly downward except ends of the Reeb chord $b_{\sigma(i)}$. According to [Lin 2016], the ends of the Reeb chord $b_{\sigma(i)}$ merge to a point $r_{\sigma(i)}$, and then split into two points, labeled as $\tilde{s}_{\sigma(i)}$ and $\tilde{s}_{\sigma(i)}^{-1}$ respectively. Now for $i=1, \ldots, n$, consider the trace of $\tilde{s}$ in $\bar{\Sigma}_{n}$, which is a flow line from $r_{i}$ to the bottom of $\bar{\Sigma}_{n}$. Concatenating it with the inverse trace of $\tilde{s}_{i}^{-1}$ in $\bar{\Sigma}_{n}$, we get a curve $\alpha_{i}$ in $\bar{\Sigma}_{n}$ as shown in Figure 12. In addition, denote the trace of the base point $\tilde{s}_{0}$ in $\bar{\Sigma}_{n}$ by $\alpha_{0}$. In this way, we have that $\alpha=\left\{\alpha_{0}, \alpha_{1}, \ldots, \alpha_{n}\right\}$ is a set of generator curves of $H_{1}\left(\bar{\Sigma}_{n}, \Lambda \cup \Lambda_{n}\right) \cong \mathbb{Z}^{n+1}$.

For each curve $\alpha_{i}$, where $i=0, \ldots, n$, Poincaré duality gives an element $\theta_{\alpha_{i}} \in H^{1}\left(\bar{\Sigma}_{n}\right)$. Denote its dual in $H_{1}\left(\Sigma_{n}\right)$ by $\tilde{s}_{i}$. Therefore, for any union of paths $\gamma$ in $\bar{\Sigma}_{n}$, the monomial $w(\gamma)$ associated to $\gamma$ in $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{n}\right)\right]$ is

$$
w(\gamma)=\prod_{i=0}^{n} \tilde{s}_{i}^{n_{i}(\gamma)}
$$

where $n_{i}(\gamma)$ is the intersection number of $\alpha_{i}$ and $\gamma$ counted with signs.
For $i<n$, the map $H_{1}\left(\bar{\Sigma}_{i}\right) \rightarrow H_{1}\left(\bar{\Sigma}_{n}\right)$ induced by the inclusion map is injective. A similar argument shows that for a union of paths $\gamma$ in $\bar{\Sigma}_{i}$, the monomial associated to $\gamma$ in $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]$ counts intersections of $\alpha_{0}, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$ with $\gamma$. Notice that the curves $\alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(n)}$ do not intersect $\bar{\Sigma}_{i}$. Hence the monomial in $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]$ agrees with $w(\gamma)$ in $\mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{n}\right)\right]$.

Pick a family of capping paths for $\Lambda_{i}$ on $\bar{\Sigma}_{i}$ for $i=0, \ldots, n$. By Proposition 2.6, for $i=1, \ldots, n+1$, each exact Lagrangian cobordism $\Sigma_{i}$ gives a DGA map $\Phi_{i}$,

$$
\Phi_{i}:\left(\mathcal{A}\left(\Lambda_{i-1} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i-1}\right)\right]\right), \partial_{i-1}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}\right)
$$

which maps any Reeb chord $a$ of $\Lambda_{i-1}$ to

$$
\begin{aligned}
\sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})=0} & \sum_{u \in \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})} w(\tilde{u}) b_{1} \cdots b_{m} \\
& =\sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}(a ; \boldsymbol{b})=0}} \sum_{u \in \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})}\left(w\left(\gamma_{a}\right) w(u) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}\right)^{-1}\right) b_{1} \cdots b_{m} .
\end{aligned}
$$

Now we show that the DGA map induced by the exact Lagrangian cobordisms is independent of the choice of capping paths.

Theorem 3.1. Let $\gamma$ and $\gamma^{\prime}$ be two families of capping paths of $\Lambda_{i}$ on $\bar{\Sigma}_{i}$ for $i=0, \ldots, n$. Denote the corresponding DGAs by $\left(\mathcal{A}^{\gamma}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}^{\gamma}\right)$ and $\left(\mathcal{A}^{\gamma^{\prime}}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}^{\gamma^{\prime}}\right)$. Assume $\Phi_{i}^{\gamma}$ and $\Phi_{i}^{\gamma^{\prime}}$ are the corresponding DGA maps induced by $\Sigma_{i}$. Then the maps

$$
\begin{aligned}
f_{i}:\left(\mathcal{A}^{\gamma}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}^{\gamma}\right) & \rightarrow\left(\mathcal{A}^{\gamma^{\prime}}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}^{\gamma^{\prime}}\right) \\
c & \mapsto w\left(\gamma_{c}^{\prime}\right)^{-1} w\left(\gamma_{c}\right) c
\end{aligned}
$$

are $D G A$ isomorphisms for $i=0, \ldots, n$. Further, the following diagram commutes:

$$
\begin{gathered}
\left(\mathcal{A}^{\gamma}\left(\Lambda_{i-1} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i-1}\right)\right]\right), \partial_{i-1}^{\gamma}\right) \xrightarrow{f_{i-1}}\left(\mathcal{A}^{\gamma^{\prime}}\left(\Lambda_{i-1} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i-1}\right)\right]\right), \partial_{i-1}^{\gamma^{\prime}}\right) \\
\Phi_{i}^{\gamma} \downarrow \\
\left(\mathcal{A}^{\gamma}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}^{\gamma}\right) \xrightarrow{f_{i}} \xrightarrow{f_{i}^{\prime}}\left(\mathcal{A}^{\gamma^{\prime}}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}^{\gamma^{\prime}}\right)
\end{gathered}
$$

Proof. The maps $f_{i}$ are DGA isomorphisms for the same reason as in Proposition 2.3. Now we prove the second part. For any Reeb chord $a$ of $\Lambda_{i-1}$ (and denoting $b_{1} \cdots b_{m}$ by $\boldsymbol{b}_{*}$ ),

$$
\begin{aligned}
f_{i} \circ \Phi_{i}^{\gamma}(a) & =f_{i}\left(\sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})=0} \sum_{u \in \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})}\left(w\left(\gamma_{a}\right) w(u) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}\right)^{-1}\right) \boldsymbol{b}_{*}\right) \\
& =\sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}(a ; \boldsymbol{b})=0}} \sum_{u \in \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})}\left(w\left(\gamma_{a}\right) w(u) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}\right)^{-1} w\left(\gamma_{b_{i}}^{\prime}\right)^{-1} w\left(\gamma_{b_{i}}\right)\right) \boldsymbol{b}_{*} \\
& =\sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}(a ; \boldsymbol{b})=0}} \sum_{u \in \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})}\left(w\left(\gamma_{a}\right) w(u) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}^{\prime}\right)^{-1}\right) \boldsymbol{b}_{*},
\end{aligned}
$$



Figure 13. A cobordism corresponding to a pinch move, where the purple disk represents a holomorphic disk with a positive puncture at $b_{\sigma(i)}$.

$$
\begin{aligned}
\Phi_{i}^{\gamma^{\prime}} \circ f_{i-1}(a) & =\Phi_{i}^{\gamma^{\prime}}\left(w\left(\gamma_{a}^{\prime}\right)^{-1} w\left(\gamma_{a}\right) a\right) \\
& =w\left(\gamma_{a}^{\prime}\right)^{-1} w\left(\gamma_{a}\right) \sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}(a ; \boldsymbol{b})=0}} \sum_{u \in \mathcal{M}^{\Sigma_{i}}(a ; \boldsymbol{b})}\left(w\left(\gamma_{a}^{\prime}\right) w(u) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}^{\prime}\right)^{-1}\right) \boldsymbol{b}_{*} \\
& =\sum_{\operatorname{dim} \mathcal{M}^{\Sigma_{i}(a ; b)=0}} \sum_{u \in \mathcal{M}^{\Sigma_{i}(a ; \boldsymbol{b})}}\left(w\left(\gamma_{a}\right) w(u) \prod_{i=1}^{m} w\left(\gamma_{b_{i}}^{\prime}\right)^{-1}\right) \boldsymbol{b}_{*} .
\end{aligned}
$$

Note that, if we cut $\bar{\Sigma}_{i}$ along the curves $\alpha_{0}, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$, the resulting surface is connected. Therefore, we can choose a family $\gamma$ of capping paths for $\Lambda_{i}$ on $\bar{\Sigma}_{i}$ such that none of them intersect the curves $\alpha_{0}, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$. Choose families of capping paths for $\Lambda_{0}, \ldots, \Lambda_{n}$ in a similar way. As a result, for any rigid holomorphic disk $u$ used in differentials of DGAs and DGA maps, we only need to count the intersections of curves in $\alpha$ with the disk boundary, i.e., $w(\tilde{u})=w(u)$.

With this selection of capping paths, we are able to write down the DGA $\left(\mathcal{A}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}\right)$ combinatorially, for $i=1, \ldots, n$. There are $2 i+1$ points on $\Lambda_{i}$ given by the intersection of $\alpha_{0}$ and $\Lambda_{i}$, labeled by $\tilde{s}_{0}$, along with the two intersections of $\alpha_{\sigma(j)}$ and $\Lambda_{i}$, labeled by $\tilde{s}_{\sigma(j)}$ (positive intersection) and $\tilde{s}_{\sigma(j)}^{\prime}$ (negative intersection), for $j=1, \ldots, i$. One then takes the DGA of $\Lambda_{i}$ with these $2 i+1$ base points, which has coefficients $\mathbb{Z}_{2}\left[\tilde{s}_{0}^{ \pm 1}, \tilde{s}_{\sigma(1)}^{ \pm 1}, \tilde{s}_{\sigma(1)}^{\prime \pm 1}, \ldots, \tilde{s}_{\sigma(i)}^{ \pm 1}, \tilde{s}_{\sigma(i)}^{\prime \pm 1}\right]$, and quotients by the relations $\tilde{s}_{\sigma(j)}^{\prime}=\tilde{s}_{\sigma(j)}^{-1}$ for $j=1, \ldots, i$, to get the DGA

$$
\left(\mathcal{A}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}\right)
$$

which is a DGA over $\mathbb{Z}_{2}\left[\tilde{s}_{0}^{ \pm 1}, \tilde{s}_{\sigma(1)}^{ \pm 1}, \ldots, \tilde{s}_{\sigma(i)}^{ \pm 1}\right]$, and $\left\{\tilde{s}_{0}, \tilde{s}_{\sigma(1)}, \ldots, \tilde{s}_{\sigma(i)}\right\}$ is a basis of $H_{1}\left(\bar{\Sigma}_{i}\right)$ that corresponds to the curves $\alpha_{0}, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$.


Figure 14. A part of the Lagrangian projection of $\Lambda_{i-1}$.

Now we are ready to describe the DGA map $\Phi_{i}$ induced by the exact Lagrangian cobordism $\Sigma_{i}$, for $i=1, \ldots, n$, which corresponds to a pinch move at crossing $b_{\sigma(i)}$. When we combine [Ekholm et al. 2016, Section 6.5] with Proposition 2.6, we find that the DGA map

$$
\Phi_{i}:\left(\mathcal{A}\left(\Lambda_{i-1} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i-1}\right)\right]\right), \partial_{i-1}\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{i} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{i}\right)\right]\right), \partial_{i}\right)
$$

maps the Reeb chord $b_{\sigma(i)}$ to $\tilde{s}_{\sigma(i)}$ and any other Reeb chord $c$ to

$$
c+\sum_{\operatorname{dim} \mathcal{M}\left(c, b_{\sigma(i)} ; c_{1}, \ldots, c_{m}\right)=1} \sum_{u \in \mathcal{M}\left(c, b_{\sigma(i)} ; c_{1}, \ldots, c_{m}\right)} w(u) \tilde{s}_{\sigma(i)}^{-1} c_{1} \cdots c_{m}
$$

where $\mathcal{M}\left(c, b_{\sigma(i)} ; c_{1}, \ldots, c_{m}\right)$ is the moduli space of holomorphic disks in $\mathbb{R}_{x y}^{2}$ with boundary on $\Pi_{x y}\left(\Lambda_{i-1}\right)$ that covers a positive quadrant around $c$ and $b_{\sigma(i)}$ and a negative quadrant around $c_{1}, \ldots, c_{m}$. Please see [Ekholm et al. 2016, Section 6.5] for a detailed definition.

Here we discuss why the formulas make sense. The pinch move at $b_{\sigma(i)}$ pinches the Reeb chord $b_{\sigma(i)}$ down, which gives a holomorphic disk (as shown in Figure 13) with a positive puncture at $b_{\sigma(i)}$ and intersects $\tilde{s}_{\sigma(i)}$ exactly once. For a holomorphic disk $u \in \mathcal{M}\left(c, b_{\sigma(i)} ; c_{1}, \ldots, c_{m}\right)$, one can close the puncture of $u$ at $b_{\sigma(i)}$ using the disk in Figure 13, which gives a holomorphic disk that contributes to $\Phi_{i}(c)$. Note that the boundary of this disk consists of the boundary of $u$ and $\gamma^{-1}$. Thus the homology class of the boundary is $w(u) \tilde{s}_{\sigma(i)}^{-1}$, which matches the formula above.

In our case, in order to describe $\Phi_{i}$ combinatorially, we introduce two notations:
Definition 3.2. Let $\sigma$ be a permutation of $\{1, \ldots, n\}$. For $i \in\{1, \ldots, n\}$, we define

$$
\begin{aligned}
T_{\sigma}^{i}:= & \left\{j \in\{1, \ldots, n\} \mid \sigma^{-1}(j)>\sigma^{-1}(i)\right. \\
& \left.\quad \text { and if } i<k<j \text { or } j<k<i, \text { then } \sigma^{-1}(k)<\sigma^{-1}(i)\right\}, \\
S_{\sigma}^{i}:= & \left\{j \in\{1, \ldots, n\} \mid i \in T_{\sigma}^{j}\right\} \\
= & \left\{j \in\{1, \ldots, n\} \mid \sigma^{-1}(j)<\sigma^{-1}(i)\right. \\
& \left.\quad \text { and if } i<k<j \text { or } j<k<i, \text { then } \sigma^{-1}(k)<\sigma^{-1}(j)\right\} .
\end{aligned}
$$

Here $T_{\sigma}^{i}$ captures all the Reeb chords $b_{j}$ with the property that, before performing a pinch move at $b_{i}$, one can find a holomorphic disk with exactly two positive punctures at $b_{i}$ and $b_{j}$. In other words, it gathers all the Reeb chords on which the


Figure 15. The Lagrangian projection of $\Lambda_{n}$.
DGA map induced by the pinch move at $b_{i}$ acts nontrivially. The other set $S_{\sigma}^{i}$, on the other hand, detects all the Reeb chords $b_{j}$ where a pinch move at $b_{j}$ gives a DGA map that acts nontrivially on $b_{i}$.

If $j$ is in $T_{\sigma}^{\sigma(i)}$ (an example is shown in Figure 14), the map $\Phi_{i}$ sends $b_{j}$ to

$$
\Phi_{i}\left(b_{j}\right)=b_{j}+\tilde{s}_{\sigma(i)}^{-1} \prod_{\substack{j<k<\sigma(i) \text { or } \\ \sigma(i)<k<j}} \tilde{s}_{k}^{-2}
$$

For $a_{1}, a_{2}$ and the rest of the $b_{j}$ where $j$ is not in $T_{\sigma}{ }^{\sigma(i)}$, the map $\Phi_{i}$ is identity.
Composing all the maps $\Phi_{i}$ for $i=1, \ldots, n$ together, we get a DGA map,

$$
\bar{\Phi}_{n}:\left(\mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right), \partial\right) \rightarrow\left(\mathcal{A}\left(\Lambda_{n} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{n}\right)\right]\right), \partial_{n}\right)
$$

that is the identity map on the Reeb chords $a_{1}, a_{2}$. For $i=1, \ldots, n$, in order to know $\bar{\Phi}_{n}\left(b_{i}\right)$, we consider pinch moves at $b_{j}$ such that $j \in S_{\sigma}^{i}$ together with the pinch move at $b_{i}$. These pinch moves correspond to all the DGA maps that contribute to $\bar{\Phi}_{n}$. Composing all these maps together, we have that

$$
\bar{\Phi}_{n}\left(b_{i}\right)=\Phi_{1} \circ \cdots \circ \Phi_{\sigma^{-1}(i)}\left(b_{i}\right)=\tilde{s}_{i}+\sum_{j \in S_{\sigma}^{i}}\left(\tilde{s}_{j}^{-1} \prod_{\substack{j<k<i \text { or } \\ i<k<j}} \tilde{s}_{k}^{-2}\right) .
$$

Now we describe the last DGA map,

$$
\Phi_{n+1}:\left(\mathcal{A}\left(\Lambda_{n} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{n}\right)\right]\right), \partial_{n}\right) \rightarrow\left(\mathbb{Z}_{2}\left[H_{1}\left(L_{\sigma}\right)\right], 0\right) .
$$

As shown in Figure 15, the underlying algebra of $\Lambda_{n}$ is generated by $a_{1}$ and $a_{2}$ and the differential is given by

$$
\partial_{n}\left(a_{1}\right)=\tilde{s}_{1} \tilde{s}_{2} \cdots \tilde{s}_{n}+\tilde{s}_{0}^{-1}, \quad \partial_{n}\left(a_{2}\right)=\tilde{s}_{n} \tilde{s}_{n-1} \cdots \tilde{s}_{1}+1
$$

Consider the map $\psi: H_{1}\left(\bar{\Sigma}_{n}\right) \rightarrow H_{1}\left(L_{\sigma}\right)$ induced by the inclusion map $\bar{\Sigma}_{n} \hookrightarrow L_{\sigma}$. Since the DGA map

$$
\Phi_{n+1}:\left(\mathcal{A}\left(\Lambda_{n} ; \mathbb{Z}_{2}\left[H_{1}\left(\bar{\Sigma}_{n}\right)\right]\right), \partial_{n}\right) \rightarrow\left(\mathbb{Z}_{2}\left[H_{1}\left(L_{\sigma}\right)\right], 0\right)
$$

satisfies $\Phi_{n+1} \circ \partial_{n}=0 \circ \Phi_{n+1}=0$, we have $\psi\left(\tilde{s}_{0}\right)=1$ and $\psi\left(\tilde{s}_{1}\right) \psi\left(\tilde{s}_{2}\right) \cdots \psi\left(\tilde{s}_{n}\right)=1$. Given that the map $\psi$ is surjective, we assume a basis of $H_{1}\left(L_{\sigma}\right)$ is $\left\{s_{1}, \ldots, s_{n-1}\right\}$, where $s_{i}=\tilde{s}_{i}$, for $i=1, \ldots, n-1$. The DGA map $\Phi_{n+1}$ is given by

$$
\begin{aligned}
a_{1} \mapsto 0, & a_{2} \mapsto 0, \\
\tilde{s}_{0} \mapsto 1, & \tilde{s}_{i} \mapsto s_{i}, \quad i=1, \ldots, n-1, \quad \tilde{s}_{n} \mapsto\left(s_{1} s_{2} \cdots s_{n-1}\right)^{-1}
\end{aligned}
$$

Composing $\Phi_{n+1}$ with $\bar{\Phi}_{n}$, we get the augmentation $\epsilon_{\sigma}$ induced by $L_{\sigma}$ as follows.
Theorem 3.3. Given a permutation $\sigma$ of $\{1, \ldots, n\}$, let $L_{\sigma}$ be the exact Lagrangian filling of the Legendrian $(2, n)$ torus knot $\Lambda$ constructed from the EHK algorithm. If we write

$$
\begin{aligned}
\mathbb{Z}_{2}\left[H_{1}(\Lambda)\right] & =\mathbb{Z}_{2}\left[\tilde{s}_{0}, \tilde{s}_{0}^{-1}\right], \\
\mathbb{Z}_{2}\left[H_{1}\left(L_{\sigma}\right)\right] & =\mathbb{Z}_{2}\left[s_{1}^{ \pm 1}, \ldots, s_{n-1}^{ \pm 1}\right]
\end{aligned}
$$

and set $s_{n}=\left(s_{1} s_{2} \cdots s_{n-1}\right)^{-1}$, then the augmentation

$$
\epsilon_{\sigma}: \mathcal{A}\left(\Lambda ; \mathbb{Z}_{2}\left[H_{1}(\Lambda)\right]\right) \rightarrow \mathbb{Z}_{2}\left[H_{1}\left(L_{\sigma}\right)\right]
$$

induced by $L_{\sigma}$ is given by

$$
\begin{aligned}
& \epsilon_{\sigma}\left(a_{j}\right)=0, \quad j=1,2 \\
& \epsilon_{\sigma}\left(b_{i}\right)=s_{i}+\sum_{j \in S_{\sigma}^{i}}\left(s_{j}^{-1} \prod_{\substack{j<k<i o r \\
i<k<j}} s_{k}^{-2}\right), \quad i=1, \ldots, n ; \\
& \epsilon_{\sigma}\left(\tilde{s}_{0}\right)=1
\end{aligned}
$$

Example 3.4. In Figure 16, as an example, we compute the augmentation $\epsilon_{(2,3,1)}$ of the Legendrian $(2,3)$ torus knot induced by the exact Lagrangian filling $L_{(2,3,1)}$.

Similarly, one can compute the augmentation for each permutation of $\{1,2,3\}$ and get the following table:

| $\epsilon$ | $\epsilon\left(b_{1}\right)$ | $\epsilon\left(b_{2}\right)$ | $\epsilon\left(b_{3}\right)$ |
| :--- | :--- | :--- | :--- |
| $\epsilon_{(1,2,3)}$ | $s_{1}$ | $s_{2}+s_{1}^{-1}$ | $s_{1}^{-1} s_{2}^{-1}+s_{2}^{-1}$ |
| $\epsilon_{(1,3,2)}=\epsilon_{(3,1,2)}$ | $s_{1}$ | $s_{2}+s_{1}^{-1}+s_{1} s_{2}$ | $s_{1}^{-1} s_{2}^{-1}$ |
| $\epsilon_{(2,1,3)}$ | $s_{1}+s_{2}^{-1}$ | $s_{2}$ | $s_{1}^{-1} s_{2}^{-1}+s_{2}^{-1}+s_{1}^{-1} s_{2}^{-2}$ |
| $\epsilon_{(2,3,1)}$ | $s_{1}+s_{2}^{-1}+s_{1} s_{2}^{-1}$ | $s_{2}$ | $s_{1}^{-1} s_{2}^{-1}+s_{2}^{-1}$ |
| $\epsilon_{(3,2,1)}$ | $s_{1}+s_{2}^{-1}$ | $s_{2}+s_{1} s_{2}$ | $s_{1}^{-1} s_{2}^{-1}$ |

3B. Proof of the main theorem. In this section, we use Theorem 3.3 to find an invariant of augmentations induced from the exact Lagrangian fillings obtained


|  | $b_{1}$ | $b_{2}$ | $b_{3}$ |
| :---: | :---: | :---: | :---: |
| $\Phi_{1}$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
|  | $b_{1}+\tilde{s}_{2}^{-1}$ | $\tilde{s}_{2}$ | $b_{3}+\tilde{s}_{2}^{-1}$ |
| $\Phi_{2}$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
|  | $b_{1}+\tilde{s}_{2}^{-1}+\tilde{s}_{2}^{-2} \tilde{s}_{3}^{-1}$ | $\tilde{s}_{2}$ | $\tilde{s}_{3}+\tilde{s}_{2}^{-1}$ |
| $\Phi_{3}$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
|  | $\tilde{s}_{1}+\tilde{s}_{2}^{-1}+\tilde{s}_{2}^{-2} \tilde{s}_{3}^{-1}$ | $\tilde{S}_{2}$ | $\tilde{s}_{3}+\tilde{s}_{2}^{-1}$ |
| $\Phi_{4}$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
|  | $s_{1}+s_{2}^{-1}+s_{1} s_{2}^{-1}$ | $s_{2}$ | $s_{1}^{-1} s_{2}^{-1}+s_{2}^{-1}$ |

Figure 16. A computation of the augmentation induced by an exact Lagrangian filling of the Legendrian $(2,3)$ torus knot. We keep track of the image of $b_{1}, b_{2}, b_{3}$ under the composition of $\Phi_{1}, \Phi_{2}, \Phi_{3}$ and $\Phi_{4}$. The last line is the image of $b_{1}, b_{2}, b_{3}$ under the augmentation $\epsilon_{(2,3,1)}$.
from the EHK algorithm. As a result, we distinguish all the augmentations in Theorem 3.3 and thus prove Theorem 1.1.

Lemma 3.5. Let $L_{1}$ and $L_{2}$ be two exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot $\Lambda$ constructed from the EHK algorithm. If $L_{1}$ and $L_{2}$ are exact Lagrangian isotopic, then there exists an invertible map $g: H_{1}\left(L_{1}\right) \rightarrow H_{1}\left(L_{2}\right)$ such that the following diagram commutes:

where $\epsilon_{L_{1}}$ and $\epsilon_{L_{2}}$ are augmentations induced by $L_{1}$ and $L_{2}$ respectively.

Proof. The isotopy between $L_{1}$ and $L_{2}$ induces an invertible map $g: H_{1}\left(L_{1}\right) \rightarrow$ $H_{1}\left(L_{2}\right)$. If we identify both $H_{1}\left(L_{1}\right)$ and $H_{1}\left(L_{2}\right)$ with $\mathbb{Z}^{n-1}$, then $g \in G L(n-1, \mathbb{Z})$. This map induces a natural map on the corresponding group rings $\mathbb{Z}_{2}\left[H_{1}\left(L_{1}\right)\right] \rightarrow$ $\mathbb{Z}_{2}\left[H_{1}\left(L_{2}\right)\right]$, also denoted by $g$. Thus, we have two augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_{2}\left[H_{1}\left(L_{2}\right)\right]: \epsilon_{1}=g \circ \epsilon_{L_{1}}$ and $\epsilon_{2}=\epsilon_{L_{2}}$. Since the two fillings $L_{1}$ and $L_{2}$ are isotopic through a family of exact Lagrangian fillings, according to [Ekholm et al. 2016, Theorem 1.3], we know that $\epsilon_{1}$ and $\epsilon_{2}$ are chain homotopic. In other words, there exists a degree 1 map $H: \mathcal{A}(\Lambda) \rightarrow \mathbb{Z}_{2}\left[H_{1}\left(L_{2}\right)\right]$ such that $H \circ \partial=\epsilon_{1}-\epsilon_{2}$ as one can see from following diagram, where $C_{i}$ denotes the degree $i$ part of $\mathcal{A}(\Lambda)$.


Note that $\Lambda$ has a Lagrangian projection (as shown in Figure 11) such that no Reeb chords are in negative degree. Hence $C_{-1}=0$ and $\epsilon_{1}-\epsilon_{2}=H \circ \partial=0$. Therefore $\epsilon_{1}=\epsilon_{2}$, i.e., the diagram (3-1) commutes.

Remark 3.6. For any DGA $\mathcal{A}$ that vanishes on the degree -1 part, by the same argument, we have that two augmentations $\epsilon_{1}$ and $\epsilon_{2}$ of $\mathcal{A}$ are chain homotopic if and only if they are identically the same. For a more general criteria of two augmentations to be chain homotopic, check [ Ng et al. 2015, Proposition 5.16].

Therefore, in order to distinguish exact Lagrangian fillings, we only need to distinguish their induced augmentations up to a $G L(n-1, \mathbb{Z})$ action. Observing the formula of the augmentation $\epsilon_{\sigma}$ in Theorem 3.3, we get a combinatorial way to define the number of terms in $\epsilon_{\sigma}\left(b_{i}\right)$ for $i=1, \ldots, n$ as follows.

Definition 3.7. For each permutation $\sigma$ of $\{1, \ldots, n\}$ and any number $i \in\{1, \ldots, n\}$, we define $C_{\sigma}:=\left(C_{\sigma}^{1}, C_{\sigma}^{2}, \ldots, C_{\sigma}^{n}\right)$, where $C_{\sigma}^{i}=\left|S_{\sigma}^{i}\right|+1$.

Example 3.8. We compute the vector $C_{\sigma}$ for all of the permutations $\sigma$ of $\{1,2,3\}$ :

| $\sigma$ | $(1,2,3)$ | $(1,3,2) \sim(3,1,2)$ | $(2,1,3)$ | $(2,3,1)$ | $(3,2,1)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $C_{\sigma}$ | $(1,2,2)$ | $(1,3,1)$ | $(2,1,3)$ | $(3,1,2)$ | $(2,2,1)$ |

Proposition 3.9. If two exact Lagrangian fillings $L_{\sigma_{1}}$ and $L_{\sigma_{2}}$ are exact Lagrangian isotopic, then $C_{\sigma_{1}}=C_{\sigma_{2}}$. In other words, the vector $C_{\sigma}$ is an invariant of the exact Lagrangian filling $L_{\sigma}$ up to exact Lagrangian isotopy.
Proof. Using the formula in Theorem 3.3, we first show that $C_{\sigma}^{i}$ is the number of terms in $\epsilon_{\sigma}\left(b_{i}\right)$. In order to do that, we need to prove that $\epsilon_{\sigma}\left(b_{i}\right)$ as a sum of monomials cannot be shorter, i.e, no terms in $\epsilon_{\sigma}\left(b_{i}\right)$ can be canceled by another
term. First, replace $s_{n}$ with $\left(s_{1} \cdots s_{n-1}\right)^{-1}$. If $i \neq n$, then each term of $\epsilon_{\sigma}\left(b_{i}\right)$ is one of the following forms:
(1) $s_{i}$,
(2) $s_{k}^{-1} \prod_{j \in S} s_{j}^{-2}$ for some $k \neq i \in\{1, \ldots, n-1\}$ and a subset $S \subset\{1, \ldots, n-1\}$ that does not contain $i, k$ (can be an empty set),
(3) $\prod_{j \in T} s_{j}^{-1} \prod_{k \notin T} s_{k}$ for some subset $T \subset\{1, \ldots, n-1\}$ that does not contain $i$ (can be an empty set).
If $i=n$, each term of $\epsilon_{\sigma}\left(b_{n}\right)$ can be either $s_{1}^{-1} \cdots s_{n-1}^{-1}$ or the form (2). Comparing degrees of $s_{1}, \ldots, s_{n-1}$ of each term, we know that no terms can be canceled.

If $L_{\sigma_{1}}$ and $L_{\sigma_{2}}$ are exact Lagrangian isotopic, by Lemma 3.5, there is a map $g: \mathbb{Z}_{2}\left[H_{1}\left(L_{1}\right)\right] \rightarrow \mathbb{Z}_{2}\left[H_{1}\left(L_{2}\right)\right]$ such that $g \circ \epsilon_{L_{1}}=\epsilon_{L_{2}}$. Note that the map $g$ on the group rings $\mathbb{Z}_{2}\left[H_{1}\left(L_{1}\right)\right] \rightarrow \mathbb{Z}_{2}\left[H_{1}\left(L_{2}\right)\right]$ is induced from an invertible map $H_{1}\left(L_{1}\right) \rightarrow H_{1}\left(L_{2}\right)$ and thus $g$ maps a monomial to a monomial. Therefore $\epsilon_{\sigma_{1}}\left(b_{i}\right)$ and $\epsilon_{\sigma_{2}}\left(b_{i}\right)$ have the same number of terms, i.e., $C_{\sigma_{1}}=C_{\sigma_{2}}$.

We say that two permutations $\sigma_{1}$ and $\sigma_{2}$ of $\{1, \ldots, n\}$ are isotopy equivalent if they are equivalent via a sequence of relations of the form

$$
\begin{equation*}
(\ldots, i, j, \ldots, k, \ldots) \sim(\ldots, j, i, \ldots, k, \ldots), \quad \text { where } i<k<j \tag{3-2}
\end{equation*}
$$

By [Ekholm et al. 2016], if $\sigma_{1}$ and $\sigma_{2}$ are isotopy equivalent, the corresponding exact Lagrangian fillings $L_{\epsilon_{1}}$ and $L_{\epsilon_{2}}$ are exact Lagrangian isotopic and hence $C_{\sigma_{1}}=C_{\sigma_{2}}$. Conversely, we have the following:
Lemma 3.10. If $C_{\sigma_{1}}=C_{\sigma_{2}}$, then $\sigma_{1}$ and $\sigma_{2}$ are isotopy equivalent.
Proof. If $\sigma_{1}(1)=k$, then $C_{\sigma_{1}}^{k}=1$. So $C_{\sigma_{2}}^{k}=1$, i.e., we have that $S_{\sigma_{2}}^{k}=\varnothing$. If $\sigma_{2}(1) \neq k$, assume the element in $\sigma_{2}$ right before $k$ is $l$, i.e., $\sigma_{2}\left(\sigma_{2}^{-1}(k)-1\right)=l$. Note that $l \notin S_{\sigma_{2}}^{k}$, i.e., there exists $i$ such that $l<i<k$ or $k<i<l$ and $\sigma_{2}^{-1}(i)>\sigma_{2}^{-1}(l)$. Note that $i \neq k$ and hence $\sigma_{2}^{-1}(i)>\sigma_{2}^{-1}(k)=\sigma_{2}^{-1}(l)+1$. Thus we can use the relation (3-2) to switch $l$ and $k$. In this way we can switch $k$ to the first position in $\sigma_{2}$, i.e., $\sigma_{2}(1)=k=\sigma_{1}(1)$.

By induction, assume $\sigma_{2}(i)=\sigma_{1}(i)$ for $i<l$ and $\sigma_{1}(l)=k$. Then $S_{\sigma_{1}}^{k} \subset S_{\sigma_{2}}^{k}$. The assumption $C_{\sigma_{2}}^{k}=C_{\sigma_{1}}^{k}$ implies that $\left|S_{\sigma_{1}}^{k}\right|=\left|S_{\sigma_{2}}^{k}\right|$ and thus $S_{\sigma_{1}}^{k}=S_{\sigma_{2}}^{k}$. If $\sigma_{2}(l) \neq k$, for a similar reason to above, one can switch $k$ to the $l$-th position and get $\sigma_{2}(l)=\sigma_{1}(l)$. Therefore, $\sigma_{1}$ and $\sigma_{2}$ are isotopy equivalent.
Theorem 3.11. If $n$ is odd, the $C_{n}$ exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot $\Lambda$ from the EHK algorithm are all of different exact Lagrangian isotopy classes.
Proof. If two augmentations $\sigma_{1}$ and $\sigma_{2}$ are not isotopy equivalent, by Lemma 3.10, we have $C_{\sigma_{1}} \neq C_{\sigma_{2}}$. According to Proposition 3.9, the corresponding exact $\mathrm{La}-$ grangian fillings $L_{\sigma_{1}}$ and $L_{\sigma_{2}}$ are not exact Lagrangian isotopic. Therefore, the

Legendrian $(2, n)$ torus knot has at least $C_{n}$ exact Lagrangian fillings up to exact Lagrangian isotopy.
Corollary 3.12. When $n$ is even, the Legendrian $(2, n)$ torus link $\Lambda$ has at least $C_{n}$ exact Lagrangian fillings.
Proof. Start with the Legendrian $(2, n+1)$-knot $\Lambda_{0}$ and label its degree 0 Reeb chords from left to right by $b_{1}, \ldots, b_{n+1}$ as usual. Let $\Sigma$ be the exact Lagrangian cobordism from $\Lambda$ to $\Lambda_{0}$ that corresponds to a pinch move of $\Lambda_{0}$ at $b_{n+1}$. For any permutation $\sigma$ of $\{1, \ldots, n\}$, the exact Lagrangian filling $L_{\sigma}$ of $\Lambda$ gives an exact Lagrangian filling of $\Lambda_{0}$ by concatenating with $\Sigma$ on the top. This new exact Lagrangian filling of $\Lambda_{0}$ corresponds to the permutation $\tilde{\sigma}=(n+1, \sigma(1), \ldots, \sigma(n))$ of $\{1,2, \ldots, n+1\}$, i.e., it is the filling $L_{\tilde{\sigma}}$ of $\Lambda_{0}$. Note that $C_{\tilde{\sigma}}^{n+1}=1$. Moreover, we have that $C_{\tilde{\sigma}}^{i}=C_{\sigma}^{i}$ for $i=1, \ldots, n-1$ and $C_{\tilde{\sigma}}^{n}=C_{\sigma}^{n}+1$. Thus $C_{\tilde{\sigma}}$ is determined by $C_{\sigma}$. Therefore, by Proposition 3.9 and Lemma 3.10, if two permutations $\sigma_{1}$ and $\sigma_{2}$ of $\{1, \ldots, n\}$ are not isotopy equivalent, their induced permutations $\tilde{\sigma}_{1}$ and $\tilde{\sigma}_{2}$ of $\{1, \ldots, n+1\}$ are not isotopy equivalent. According to Theorem 3.11, the corresponding exact Lagrangian fillings $L_{\tilde{\sigma}_{1}}$ and $L_{\tilde{\sigma}_{2}}$ of $\Lambda_{0}$ are not exact Lagrangian isotopic. Hence $L_{\sigma_{1}}$ and $L_{\sigma_{2}}$ are not exact Lagrangian isotopic.

## Acknowledgements

The author would like to thank Lenhard Ng for introducing her to the problem, and for many enlightening discussions. This work was partially supported by NSF grants DMS-0846346 and DMS-1406371.

## References

[Chekanov 2002] Y. Chekanov, "Differential algebra of Legendrian links", Invent. Math. 150:3 (2002), 441-483. MR Zbl
[Cieliebak et al. 2010] K. Cieliebak, T. Ekholm, and J. Latschev, "Compactness for holomorphic curves with switching Lagrangian boundary conditions", J. Symplectic Geom. 8:3 (2010), 267-298. MR Zbl
[Ekholm et al. 2013] T. Ekholm, J. B. Etnyre, L. Ng, and M. G. Sullivan, "Knot contact homology", Geom. Topol. 17:2 (2013), 975-1112. MR Zbl
[Ekholm et al. 2016] T. Ekholm, K. Honda, and T. Kálmán, "Legendrian knots and exact Lagrangian cobordisms", J. Eur. Math. Soc. (JEMS) 18:11 (2016), 2627-2689. MR Zbl
[Eliashberg 1998] Y. Eliashberg, "Invariants in contact topology", Doc. Math. Extra Vol. 2 (1998), 327-338. MR Zbl
[Eliashberg et al. 2000] Y. Eliashberg, A. Givental, and H. Hofer, "Introduction to symplectic field theory", Geom. Funct. Anal. Special Volume, Part 2 (2000), 560-673. MR Zbl
[Etnyre et al. 2002] J. B. Etnyre, L. L. Ng, and J. M. Sabloff, "Invariants of Legendrian knots and coherent orientations", J. Symplectic Geom. 1:2 (2002), 321-367. MR Zbl
[Lin 2016] F. Lin, "Exact Lagrangian caps of Legendrian knots", J. Symplectic Geom. 14:1 (2016), 269-295. MR Zbl
[Ng 2003] L. L. Ng, "Computable Legendrian invariants", Topology 42:1 (2003), 55-82. MR Zbl
[Ng 2010] L. Ng, "Rational symplectic field theory for Legendrian knots", Invent. Math. 182:3 (2010), 451-512. MR Zbl
[Ng et al. 2015] L. Ng, D. Rutherford, V. Shende, S. Sivek, and E. Zaslow, "Augmentations are sheaves", preprint, 2015. arXiv
[Shende et al. 2015] V. Shende, D. Treumann, H. Williams, and E. Zaslow, "Cluster varieties from Legendrian knots", preprint, 2015. arXiv

Received August 1, 2016. Revised January 13, 2017.

Yu Pan<br>Mathematics Department<br>Duke University<br>Box 90320<br>DURHAM, NC 27708<br>United States<br>yp37@math.duke.edu

# PACIFIC JOURNAL OF MATHEMATICS 

Founded in 1951 by E. F. Beckenbach (1906-1982) and F. Wolf (1904-1989)
msp.org/pjm
EDITORS
Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu
Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@ math.ucla.edu
Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu
Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@ math.ucla.edu

Vyjayanthi Chari
Department of Mathematics University of California
Riverside, CA 92521-0135 chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555 liu@math.ucla.edu

Igor Pak
Department of Mathematics University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com
Paul Yang
Department of Mathematics Princeton University Princeton NJ 08544-1000 yang@math.princeton.edu

Daryl Cooper
Department of Mathematics University of California Santa Barbara, CA 93106-3080 cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

## PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

## SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY INST. DE MATEMÁTICA PURA E APLICADA KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE NEW MEXICO STATE UNIV. OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA UNIV. OF CALIFORNIA, BERKELEY UNIV. OF CALIFORNIA, DAVIS UNIV. OF CALIFORNIA, LOS ANGELES UNIV. OF CALIFORNIA, RIVERSIDE UNIV. OF CALIFORNIA, SAN DIEGO UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

[^1]The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall \#3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW ${ }^{\circledR}$ from Mathematical Sciences Publishers.
PUBLISHED BY

## E. mathematical sciences publishers

nonprofit scientific publishing
http://msp.org/
© 2017 Mathematical Sciences Publishers

## PACIFIC JOURNAL OF MATHEMATICS

$$
\text { Volume } 289 \quad \text { No. } 2 \quad \text { August } 2017
$$

Regular representations of completely bounded maps ..... 257
B. V. Rajarama Bhat, Nirupama Mallick and K. Sumesh
Ball convex bodies in Minkowski spaces ..... 287
Thomas Jahn, Horst Martini and Christian Richter
Local constancy of dimension of slope subspaces of automorphic ..... 317
forms
JOACHIM MAHNKOPF
Weakening idempotency in $K$-theory ..... 381
Vladimir Manuilov
On Langlands quotients of the generalized principal series isomorphic ..... 395
to their Aubert duals
Ivan Matić
Exact Lagrangian fillings of Legendrian $(2, n)$ torus links ..... 417
Yu Pan
Elementary calculation of the cohomology rings of real Grassmann ..... 443 manifolds
RUSTAM SADYKOV
Cluster tilting modules and noncommutative projective schemes ..... 449
Kenta Ueyama
Concentration for a biharmonic Schrödinger equation ..... 469
Dong WANG
Global existence of smooth solutions to exponential wave maps in ..... 489
FLRW spacetimesChang-Hua Wei and Ning-An Lai


[^0]:    MSC2010: 53D42, 57R17.
    Keywords: Exact Lagrangian fillings, $(2, \mathrm{n})$ torus links, augmentation, Legendrian knots.

[^1]:    See inside back cover or msp.org/pjm for submission instructions.
    The subscription price for 2017 is US $\$ 450 /$ year for the electronic version, and $\$ 625 /$ year for print and electronic.
    Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

