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EXACT LAGRANGIAN FILLINGS OF LEGENDRIAN (2, *n*) TORUS LINKS

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Ekholm, Honda, and Kálmán constructed C_n exact Lagrangian fillings for a Legendrian (2, *n*) torus knot or link with maximal Thurston–Bennequin number, where C_n is the *n*-th Catalan number. We show that these exact Lagrangian fillings are pairwise nonisotopic through exact Lagrangian isotopy. To do that, we compute the augmentations induced by the exact Lagrangian fillings L to $\mathbb{Z}_2[H_1(L)]$ and distinguish the resulting augmentations.

1. Introduction

A Legendrian submanifold Λ in the standard contact manifold $(\mathbb{R}^3, \xi = \ker\alpha)$, where $\alpha = dz - y\,dx$, is a 1-dimensional closed manifold such that $T\Lambda \subset \xi$ everywhere. An exact Lagrangian filling L of Λ in the symplectization manifold $(\mathbb{R}_t \times \mathbb{R}^3, \omega = d(e^t\alpha))$ is a 2-dimensional surface that is cylindrical over Λ when t is sufficiently large. See Definition 2.5 for more detail, and Figure 1 for a picture.

In this paper, we study oriented exact Lagrangian fillings of the Legendrian $(2, n)$ torus links Λ with maximal Thurston–Bennequin number $(n > 0)$. When n is even, we also require the link to have the right Maslov potential such that Reeb chords $b_1, \dots, b_n$ in Figure 2 are in degree 0 (see Section 2A for detailed definitions). Ekholm, Honda, and Kálmán [Ekholm et al. 2016] gave an algorithm (which we refer to later as the EHK algorithm) to construct exact Lagrangian fillings of the Legendrian $(2, n)$ torus link Λ as follows. Starting with a *Lagrangian projection* (a projection from $\mathbb{R}^3$ to the xy-plane) of Λ as shown in Figure 2, we can successively resolve crossings b_i in any order through pinch moves (see Figure 3), which correspond to saddle cobordisms. As a result, we get two Legendrian unknots, which admit minimum cobordisms as shown in Figure 3. Concatenating the n saddle cobordisms with these two minimum cobordisms, we get an exact Lagrangian filling of Λ.

Different orders of resolving crossings $b_1, \dots, b_n$ may give different exact Lagrangian fillings of Λ up to exact Lagrangian isotopy. Given a permutation $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ of $\{1, \dots, n\}$, write L_σ for the exact Lagrangian filling achieved by using n successive pinch moves at $b_{\sigma(1)}, b_{\sigma(2)}, \dots, b_{\sigma(n)}$, respectively,

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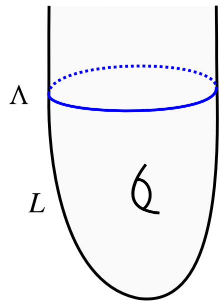


Figure 1. An exact Lagrangian filling.

and then concatenating with the two minimum cobordisms. Observe that two permutations may give isotopic exact Lagrangian fillings. For instance, let Λ be the Legendrian $(2, 3)$ torus knot and consider the exact Lagrangian fillings of Λ that correspond to permutations $(1, 3, 2)$ and $(3, 1, 2)$, respectively. Since the saddles corresponding to the pinch moves at b_1 and b_3 are disjoint when projected to $\mathbb{R}^3$, one can use a Hamiltonian vector field in the t direction to exchange the heights of these two saddles. Therefore, the two fillings $L_{(1,3,2)}$ and $L_{(3,1,2)}$ are Hamiltonian isotopic and thus are exact Lagrangian isotopic. In general, for the Legendrian $(2, n)$ torus link Λ, given any numbers i, j, k such that $i < k < j$, two permutations

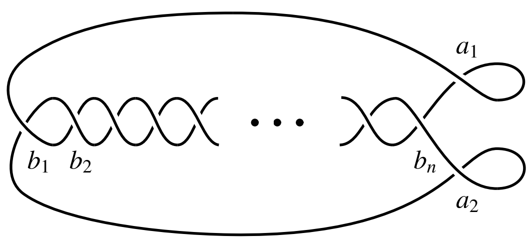


Figure 2. The Lagrangian projection of the Legendrian $(2, n)$ torus knot.

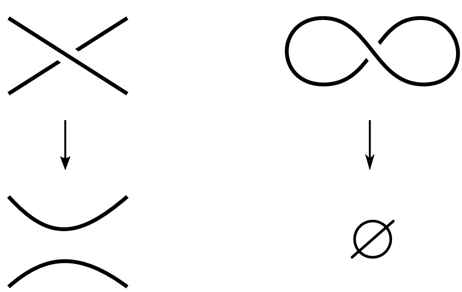


Figure 3. The pinch move (left) and the minimum cobordism (right) between Lagrangian projections of links.

$(\dots, i, j, \dots, k, \dots)$ and $(\dots, j, i, \dots, k, \dots)$, where only i and j are interchanged, give the same exact Lagrangian fillings of Λ up to exact Lagrangian isotopy. Taking all the permutations of $\{1, \dots, n\}$ modded out by this relation, we obtain C_n exact Lagrangian fillings of Λ, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the n-th Catalan number. In this paper, we prove the following theorem:

Theorem 1.1 (see Theorem 3.11 and Corollary 3.12). *The C_n exact Lagrangian fillings that come from the EHK algorithm are all of different exact Lagrangian isotopy classes. In other words, the Legendrian $(2, n)$ torus link has at least C_n exact Lagrangian fillings up to exact Lagrangian isotopy.*

Shende, Treumann, Williams and Zaslow [Shende et al. 2015] have also constructed C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot using cluster varieties and shown that they are distinct up to Hamiltonian isotopy. They remarked that these are presumably the same as fillings obtained in [Ekholm et al. 2016], but we do not resolve this issue here.

Remark 1.2. We will see from Corollary 3.12 that the conclusion of Theorem 1.1 for the case when n is even can be derived from the result for the case when n is odd. Therefore, for most of the paper, we focus on the case when n is odd, which means Λ is a knot.

Inspired by [Ekholm et al. 2016], we use augmentations to distinguish the C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot Λ. In order to talk about augmentations, we first introduce the Chekanov–Eliashberg differential graded algebra (DGA) of a Legendrian knot Λ, which is a chain complex $(\mathcal{A}(\Lambda), \partial)$. This is an invariant of Legendrian submanifolds introduced by Chekanov [2002] and Eliashberg [1998] in the spirit of symplectic field theory [Eliashberg et al. 2000]. The underlying algebra $\mathcal{A}(\Lambda)$ of the Chekanov–Eliashberg DGA is freely generated by Reeb chords of Λ over a commutative ring $\mathbb{Z}_2[H_1(\Lambda)] = \mathbb{Z}_2[s, s^{-1}]$, where Reeb chords of Λ correspond to double points of the Lagrangian projection of Λ. The differential is defined by a count of rigid holomorphic disks with boundary on Λ, taken with coefficients in $\mathbb{Z}_2[H_1(\Lambda)]$. In general, the Chekanov–Eliashberg DGA of Λ is defined with $\mathbb{Z}[H_1(\Lambda)]$ coefficients. For our purpose, it suffices to consider the DGA with $\mathbb{Z}_2[H_1(\Lambda)]$ coefficients, which means ignoring the orientations of moduli spaces of holomorphic disks. An *augmentation* ϵ of $\mathcal{A}(\Lambda)$ to a commutative ring $\mathbb{F}$ is a DGA map $\epsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$. As shown in [Ekholm et al. 2016], an exact Lagrangian filling L of Λ gives an augmentation of $\mathcal{A}(\Lambda)$ by counting rigid holomorphic disks with boundary on L. Moreover, by Theorem 1.3 of the same paper, exact Lagrangian isotopic fillings give homotopic augmentations. Therefore, in order to distinguish two fillings, we only need to show their induced augmentations are not chain homotopic.

Ekholm et al. [2016] distinguished all the exact Lagrangian fillings from the EHK algorithm when $n = 3$ by computing all the augmentations of the Legendrian $(2, 3)$ torus knot to $\mathbb{Z}_2$ and finding that they are pairwise non-chain-homotopic. However, when $n \geq 5$, a computation shows that the number of augmentations of the DGA to $\mathbb{Z}_2$ is much less than the Catalan number C_n.

In this paper, for an exact Lagrangian filling L of the Legendrian $(2, n)$ torus knot Λ, we consider its induced augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$, where $H_1(L)$ is the singular homology of L. Note that $H_1(L) \cong H_2(\mathbb{R} \times \mathbb{R}^3, L)$ and thus it is natural to count the rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ with boundary on L with $\mathbb{Z}_2[H_1(L)]$ coefficients. However, the computation of augmentations is not as easy as for the case with $\mathbb{Z}_2$ coefficients. For each exact Lagrangian filling L from the EHK algorithm, we give a combinatorial formula of the induced augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$. From the formula, we find a combinatorial invariant to show that the augmentations are not pairwise chain homotopic. In this way, we distinguish all of the C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot Λ up to exact Lagrangian isotopy.

Outline. In Section 2, we review the Chekanov–Eliashberg DGA of a Legendrian submanifold and the DGA maps induced by an exact Lagrangian cobordism. In Section 3, we compute all the augmentations of the Legendrian $(2, n)$ torus knot to $\mathbb{Z}_2[H_1(L)]$ induced by the exact Lagrangian fillings L and prove that all the resulting augmentations are distinct up to chain homotopy. In the end, we prove Theorem 1.1 for the case n even as a corollary.

2. Preliminaries

In Section 2A, we review the definition of the Chekanov–Eliashberg DGA of Legendrian submanifolds in $(\mathbb{R}^3, \ker\alpha)$ and its extension to the setting of multiple base points. For the purpose of computing augmentations in Section 3A, the definition of DGA we use here is slightly different from the versions in [Ng 2010] and [Ng et al. 2015], where the underlying algebra is completely noncommutative. In our definition, we allow elements in the coefficient ring to commute with the elements corresponding to Reeb chords. This is a generalization of the definition of Chekanov–Eliashberg DGA from [Etnyre et al. 2002]. See [Ekholm et al. 2013, Section 2.3.2] for further discussions. In Section 2B, we review the DGA map induced by an exact Lagrangian cobordism and revise coefficients of this map for the purpose of computing augmentations in Section 3A.

2A. ***The Chekanov–Eliashberg DGA.*** Let Λ be a Legendrian submanifold in $(\mathbb{R}^3, \ker\alpha)$, where $\alpha = dz - y\,dx$. There are two projection diagrams associated

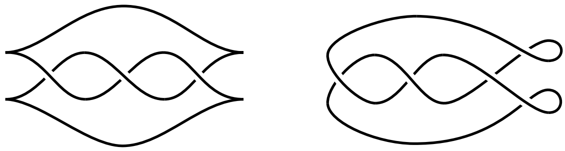


Figure 4. A front projection (left) and a Lagrangian projection (right) of the Legendrian trefoil.

Figure 5. Ng's algorithm to transfer a front projection to a Lagrangian projection by smoothing the left cusp directly and smoothing the right cusp with an additional crossing.

to Λ via the *Lagrangian projection* $\Pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, y)$ and the *front projection* $\Pi_{xz} : \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, z)$, respectively. As an example, a front projection and a Lagrangian projection of the Legendrian trefoil are shown in Figure 4. Moreover, starting from a front projection of Λ, Ng [2003] gave an algorithm to get a Lagrangian projection of Λ by smoothing the cusps of the front projection in a way shown in Figure 5.

Let $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$ be an oriented Legendrian link with k connected components. Now let us define the Chekanov–Eliashberg DGA $(\mathcal{A}(\Lambda\,; \mathbb{Z}_2[H_1(\Lambda)]), \partial)$ of Λ. To simplify the definition of grading, we assume throughout the paper that the rotation number of Λ is 0. Note that all the Legendrian $(2, n)$ torus links we consider have rotation number 0.

The underlying algebra. The underlying algebra $\mathcal{A}(\Lambda\,; \mathbb{Z}_2[H_1(\Lambda)])$ is a unital graded algebra freely generated by Reeb chords of Λ over

$$\mathbb{Z}_2[H_1(\Lambda)] = \mathbb{Z}_2[s_1^{\pm 1}, s_2^{\pm 1}, \ldots, s_k^{\pm 1}],$$

where $\{s_1, s_2, \ldots, s_k\}$ is any basis of $H_1(\Lambda)$. A *Reeb chord* of Λ in $(\mathbb{R}^3, \ker\alpha)$ is a vertical line segment (z direction) with both ends on Λ endowed with an orientation in the positive z direction. Reeb chords of Λ are in one-to-one correspondence to double points of $\Pi_{xy}(\Lambda)$, which by Ng's algorithm correspond to the crossings and right cusps of $\Pi_{xz}(\Lambda)$.

To define the grading of Reeb chords, we work on the front projection $\Pi_{xz}(\Lambda)$. Write $C(\Pi_{xz}(\Lambda))$ for the set of cusps of $\Pi_{xz}(\Lambda)$, which divides $\Pi_{xz}(\Lambda)$ into strands (ignoring double points). The *Maslov potential* is a function μ that assigns an integer to each strand such that around each cusp, the Maslov potential of the lower strand is one less than that of the upper strand. This is well defined up to a global shift on

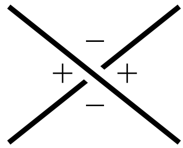


Figure 6. At each crossing, the quadrants labeled with + sign are *positive quadrants* and the ones labeled with − sign are *negative quadrants*.

each component of Λ. Once the Maslov potential is fixed, the grading of a Reeb chord c that corresponds to a crossing of $\Pi_{xz}(\Lambda)$ can be defined by

$$|c| := \mu(u) - \mu(l),$$

where u is the upper strand of the crossing and l is the lower strand of the crossing. The grading of Reeb chords that correspond to right cusps of $\Pi_{xz}(\Lambda)$ are defined to be 1. Extend the definition of grading to $\mathcal{A}(\Lambda\,; \mathbb{Z}_2[H_1(\Lambda)])$ by setting $|s_i| = 0$ for $i = 1, \dots, k$ and using the relation $|ab| = |a| + |b|$.

In the special case of Legendrian $(2, n)$ torus links, when n is odd, the degree is well defined. When n is even, we can choose a Maslov potential of the Legendrian $(2, n)$ torus link such that for any Reeb chord b_i as labeled in Figure 2, the upper strand and the lower strand of b_i have the same Maslov potential. In this setting, for a Legendrian $(2, n)$ torus link (n is either odd or even) whose Lagrangian projection is like Figure 2, we have that $|a_1| = |a_2| = 1$ and $|b_i| = 0$ for $i = 1, \dots, n$.

Differential. The differential ∂ is defined by counting rigid holomorphic disks in $\mathbb{R}^2_{xy}$ with boundary on $\Pi_{xy}(\Lambda)$.

For any Reeb chords $a, b_1, \dots, b_m$ of Λ, define $\mathcal{M}^\Lambda(a\,; b_1, \dots, b_m)$ to be the moduli space of holomorphic disks

$$u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R}^2, \Pi_{xy}(\Lambda))$$

with the following properties:

- D_{m+1} is a 2-dimensional unit disk with $m+1$ points $s, t_1, \dots, t_m$ removed from the boundary and the points $s, t_1, \dots, t_m$ are labeled in counterclockwise order.
- $\lim_{r\to s} u(r) = a$ and the image of a neighborhood of s under u covers exactly one positive quadrant of the crossing a (see Figure 6).
- $\lim_{r\to t_i} u(r) = b_i$, for $i = 1, \dots, m$, and the image of a neighborhood of t_i under u covers exactly one negative quadrant of the crossing b_i (see Figure 6).

We occasionally abbreviate $(a, b_1, \dots, b_m)$ to $(a\,; \boldsymbol{b})$, where $\boldsymbol{b}$ represents a sequence of Reeb chords, $b_1, \dots, b_m$. According to [Chekanov 2002], we have the

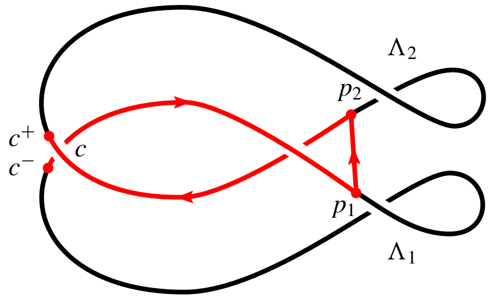


Figure 7. The Legendrian Hopf link $\Lambda_1 \cup \Lambda_2$. For a Reeb chord c from $c^- \in \Lambda_1$ to $c^+ \in \Lambda_2$, the red curve is a capping path γ_c.

following dimension formula:

$$\dim \mathcal{M}^\Lambda(a\,; b_1, \ldots, b_m) = |a| - \sum_{i=1}^{m} |b_i| - 1.$$

When $\dim \mathcal{M}^\Lambda(a\,; b_1, \ldots, b_m) = 0$, the disk $u \in \mathcal{M}^\Lambda(a\,; b_1, \ldots, b_m)$ is called *rigid*. There are finitely many rigid holomorphic disks and hence we can count them.

In order to count with $\mathbb{Z}_2[H_1(\Lambda)]$ coefficients, we want to take the homology class of the boundary of rigid disks in $H_1(\Lambda)$. However, for any rigid holomorphic disk u, the boundary $\Pi_{xy}^{-1}(u(\partial D_{m+1}))$ is not closed. Therefore, we first introduce capping paths. Equip each connected component Λ_i with a reference point p_i, for $i = 1, \ldots, k$. For each $i \neq 1$, pick a path δ_{1i} in $\mathbb{R}^3 \setminus \Lambda$ that goes from p_1 to p_i. For each Reeb chord c of Λ from $c^- \in \Lambda_{i^-}$ to $c^+ \in \Lambda_{i^+}$, the *capping path* γ_c is defined by concatenating

- a path on Λ_{i^-} from c^- to p_{i^-},
- the chosen path $-\delta_{1i^-}$ connecting p_{i^-} to p_1,
- the chosen path δ_{1i^+} connecting p_1 to p_{i^+}, and
- a path on Λ_{i^+} from p_{i^+} to c^+.

See Figure 7 for an example of a capping path.

After associating each Reeb chord with a capping path, for any rigid holomorphic disk $u \in \mathcal{M}^\Lambda(a\,; b_1, \ldots, b_m)$, the curve

$$\tilde{u} = \Pi_{xy}^{-1}(u(\partial D_{m+1})) \cup \gamma_a \cup -\gamma_{b_1} \cup \cdots \cup -\gamma_{b_m}$$

is a loop in $\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1k}$. Notice that $H_1(\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1k}) \cong H_1(\Lambda)$. Thus we can view the homology class $[\tilde{u}]$ as in $H_1(\Lambda)$.

Now we can define the differential of the Chekanov–Eliashberg DGA of Λ.

Definition 2.1. For any Reeb chord a of Λ, the differential ∂ is defined by:

$$\partial(a)=\sum_{\dim\mathcal{M}^{\Lambda}(a;\boldsymbol{b})=0}\ \sum_{u\in\mathcal{M}^{\Lambda}(a;\boldsymbol{b})}[\tilde{u}]\,b_1\cdots b_m. \tag{2-1}$$

The definition of differential can be extended to $\mathcal{A}(\Lambda\,;\mathbb{Z}_2[H_1(\Lambda)])$ by setting $\partial(s_i)=0$ for $i=1,\dots,k$, and using the Leibniz rule

$$\partial(ab)=\partial(a)b+a\partial(b).$$

According to [Chekanov 2002], the map ∂ is a differential in degree -1, and up to stable tame isomorphism, the Chekanov–Eliashberg DGA $(\mathcal{A}(\Lambda\,;\mathbb{Z}_2[H_1(\Lambda)]),\partial)$ is an invariant of Λ under Legendrian isotopy.

Remark 2.2. In general, for any commutative ring R and a ring homomorphism $\mathbb{Z}_2[H_1(\Lambda)]\to R$, we define the Chekanov–Eliashberg DGA $(\mathcal{A}(\Lambda\,;R),\partial)$ as a tensor product of the DGA $\mathcal{A}(\Lambda\,;\mathbb{Z}_2[H_1(\Lambda)])$ with the ring R:

$$\mathcal{A}(\Lambda\,;R)=\mathcal{A}(\Lambda\,;\mathbb{Z}_2[H_1(\Lambda)])\otimes_{\mathbb{Z}_2[H_1(\Lambda)]}R,$$

where the ring homomorphism gives R the structure of a module over $\mathbb{Z}_2[H_1(\Lambda)]$.

We give a combinatorial definition of the differential of $(\mathcal{A}(\Lambda\,;\mathbb{Z}_2[H_1(\Lambda)]),\partial)$. Assign Λ an orientation and label each component Λ_i, for $i=1,\dots,k$, with a base point s_i, which is different from the reference point and ends of Reeb chords. For a union of oriented curves γ in $\Lambda\cup\delta_{12}\cup\cdots\cup\delta_{1k}$, we associate it with a monomial $w(\gamma)$ in $\mathbb{Z}_2[H_1(\Lambda)]$:

$$w(\gamma)=\prod_{i=1}^{k}s_i^{n_i(\gamma)}, \tag{2-2}$$

where $n_i(\gamma)$ is the number of times γ goes through s_i counted with sign. The sign is positive if γ goes through s_i following the link orientation and is negative if γ goes through s_i against the link orientation. In particular, for a rigid holomorphic disk $u\in\mathcal{M}^{\Lambda}(a\,;b_1,\dots,b_m)$, we have

$$[\tilde{u}]=w(\tilde{u})=w(u)w(\gamma_a)\prod_{i=1}^{m}w(\gamma_{b_i})^{-1}, \tag{2-3}$$

where $w(u)$ is short for $w(\Pi_{xy}^{-1}(u(\partial D_{m+1})))$. Plugging it into the formula (2-1), we get a combinatorial definition of the differential. It seems to depend on the choice of capping paths. However, we have the following well-known proposition.

Proposition 2.3. *Let Λ be a Legendrian link and γ, γ' be two families of capping paths of Reeb chords of Λ. The corresponding DGAs $(\mathcal{A}^{\gamma}(\Lambda),\partial)$ and $(\mathcal{A}^{\gamma'}(\Lambda),\partial')$ are isomorphic.*

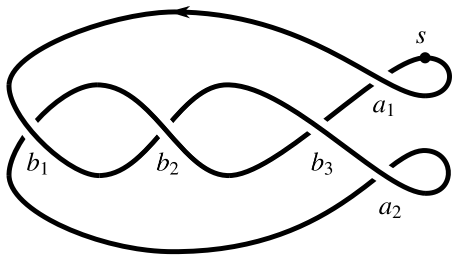


Figure 8. The Lagrangian projection of the Legendrian (2, 3) torus knot with a single base point.

Proof. For a Reeb chord a of Λ, we have

$$\partial(a) = \sum_{\dim \mathcal{M}^{\Lambda}(a;\boldsymbol{b})=0} \ \sum_{u\in\mathcal{M}^{\Lambda}(a;\boldsymbol{b})} \Big(w(u)w(\gamma_a)\prod_{i=1}^{m} w(\gamma_{b_i})^{-1}\Big) b_1\cdots b_m,$$

$$\partial'(a) = \sum_{\dim \mathcal{M}^{\Lambda}(a;\boldsymbol{b})=0} \ \sum_{u\in\mathcal{M}^{\Lambda}(a;\boldsymbol{b})} \Big(w(u)w(\gamma'_a)\prod_{i=1}^{m} w(\gamma'_{b_i})^{-1}\Big) b_1\cdots b_m.$$

For each Reeb chord c, concatenate $-\gamma'_c$ with γ_c and get a closed curve, denoted by $-\gamma'_c\cup\gamma_c$. It is not hard to check that the map

$$f:(\mathcal{A}^{\gamma}(\Lambda),\partial)\to(\mathcal{A}^{\gamma'}(\Lambda),\partial'),\qquad c\mapsto[-\gamma'_c\cup\gamma_c]\,c = w(\gamma'_c)^{-1}w(\gamma_c)\,c$$

is a chain map and is an isomorphism. □

Note that for an oriented link Λ with minimal base points (i.e., each component has exactly one base point), we can choose a family of capping paths such that none of them pass through any base point. Therefore, we only need to count intersections of the disk boundary and base points. Thanks to Proposition 2.3, we can define the Chekanov–Eliashberg DGA of Λ to be a unital graded algebra over $\mathbb{Z}_2[H_1(\Lambda)] = \mathbb{Z}_2[s_1^{\pm1},\dots,s_k^{\pm1}]$ generated by Reeb chords of Λ endowed with a differential given by

$$\partial(a) = \sum_{\dim \mathcal{M}^{\Lambda}(a;\boldsymbol{b})=0} \ \sum_{u\in\mathcal{M}^{\Lambda}(a;\boldsymbol{b})} w(u) b_1\cdots b_m,$$

$$\partial(s_i)=0,\quad i=1,\dots,k,$$

with $w(u)$ defined as in (2-3). This DGA is denoted by $(\mathcal{A}(\Lambda,\{s_1,\dots,s_k\}),\partial)$ too.

Example 2.4. Consider the Legendrian (2, 3) torus knot Λ with a single base point s as shown in Figure 8. The underlying algebra $\mathcal{A}(\Lambda,\{s\})$ is generated by Reeb chords a_1, a_2, b_1, b_2, b_3 over $\mathbb{Z}_2[s,s^{-1}]$. Reeb chords a_1 and a_2 are in degree 1 and the rest of the Reeb chords are in degree 0. The differential is given by

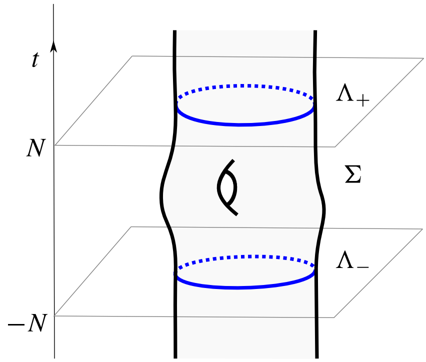


Figure 9. A schematic picture of an exact Lagrangian cobordism.

$$\begin{aligned} \partial(a_1) &= s^{-1} + b_1 + b_3 + b_1 b_2 b_3, \\ \partial(a_2) &= 1 + b_1 + b_3 + b_3 b_2 b_1, \\ \partial(b_i) &= 0, \quad i = 1, 2, 3, \\ \partial(s) &= \partial(s^{-1}) = 0. \end{aligned}$$

The definition of the DGA of a Legendrian link can be generalized to the case where there is more than one base point on some components of the link. Let Λ be an oriented Legendrian link and $\{s_1, \dots, s_l\}$ be a set of points on Λ such that each component of Λ has at least one point in the set and the set does not include any end of any Reeb chord of Λ. For a union of paths γ, associate it with a monomial $w(\gamma) = \prod_{j=1}^{l} s_j^{n_j(\gamma)}$ in $\mathbb{Z}_2[s_1^{\pm 1}, \dots, s_l^{\pm 1}]$, where n_j is defined much as above. The DGA

$$(\mathcal{A}(\Lambda, \{s_1, \dots, s_l\}), \partial)$$

is a unital graded algebra generated by Reeb chords of Λ over $\mathbb{Z}_2[s_1^{\pm 1}, \dots, s_l^{\pm 1}]$ endowed with a differential given by

$$\begin{aligned} \partial(a) &= \sum_{\dim \mathcal{M}^{\Lambda}(a;\boldsymbol{b})=0} \; \sum_{u \in \mathcal{M}^{\Lambda}(a;\boldsymbol{b})} w(u) b_1 \cdots b_m, \\ \partial(s_i) &= 0, \quad i = 1, \dots, l. \end{aligned}$$

2B. ***The DGA map induced by exact Lagrangian cobordisms.*** The Chekanov–Eliashberg DGA acts functorially on exact Lagrangian cobordisms, according to [Ekholm et al. 2016]. We first recall the definition of exact Lagrangian cobordisms.

Definition 2.5. Let Λ_+ and Λ_- be Legendrian submanifolds in $(\mathbb{R}^3, \ker\alpha)$, where $\alpha = dz - y\,dx$. An *exact Lagrangian cobordism* Σ from Λ_- to Λ_+ is a 2-dimensional surface in $(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$ such that there exists $T > 0$ such that Σ is

- cylindrical over Λ_+ on the positive end, i.e., $\Sigma \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+$;

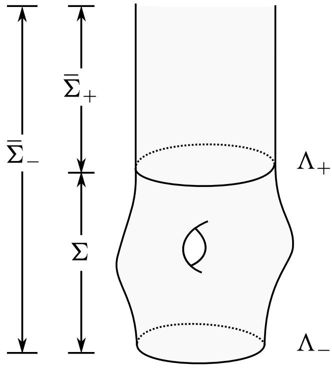


Figure 10. The relation among cobordisms $\bar{\Sigma}_+$, $\bar{\Sigma}_-$, and Σ.

- cylindrical over Λ_- on the negative end, i.e., $\Sigma \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_-$;
- compact in $[-T, T] \times \mathbb{R}^3$,

and $e^t \alpha|_{T\Sigma} = df$ for some function $f : \Sigma \to \mathbb{R}$. (See Figure 9.)

When Λ_- is empty, the surface L satisfying the conditions above is called an *exact Lagrangian filling* of Λ_+.

By [Ekholm et al. 2016], an exact Lagrangian cobordism Σ from Λ_- to Λ_+ gives a DGA map from $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$ with $\mathbb{Z}_2[H_1(\Sigma)]$ coefficients. Thus, an exact Lagrangian filling L of a Legendrian submanifold Λ, which can be viewed as a cobordism from the empty set to Λ, gives a DGA map from $\mathcal{A}(\Lambda)$ to the trivial DGA

$$(\mathbb{Z}_2[H_1(L)], 0),$$

which is an augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$.

For the purpose of computing augmentations of the Legendrian $(2, n)$ torus knots in Section 3A, we revise the coefficient ring of the DGA map induced by exact Lagrangian cobordisms from [Ekholm et al. 2016]. Instead of using $\mathbb{Z}_2[H_1(\Sigma)]$ coefficients, we will show the following proposition:

Proposition 2.6. *Let Λ_+ and Λ_- be Legendrian submanifolds in $(\mathbb{R}^3, \ker \alpha)$ and Σ be a connected exact Lagrangian cobordism from Λ_- to Λ_+. Assume that $\bar{\Sigma}_+$ is a connected exact Lagrangian cobordism from Λ_+ to some other Legendrian link and $\bar{\Sigma}_-$ is the concatenation of $\bar{\Sigma}_+$ and Σ as shown in Figure 10. The exact Lagrangian cobordism Σ induces a DGA map*

$$\Phi : (\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\bar{\Sigma}_+)]), \partial_+) \to (\mathcal{A}(\Lambda_- ; \mathbb{Z}_2[H_1(\bar{\Sigma}_-)]), \partial_-)$$

with $\mathbb{Z}_2[H_1(\bar{\Sigma}_-)]$ coefficients.

Note that when $\overline{\Sigma}_+$ is an exact Lagrangian cylinder over Λ_+, this map agrees with the DGA map introduced in [Ekholm et al. 2016]. The proof of Proposition 2.6 follows Section 3 of that paper. Our revision of the coefficient ring is based on a different choice of capping paths of Λ_+ and Λ_-. Ekholm et al. choose capping paths of Λ_+ and Λ_- on Σ, while we choose capping paths of Λ_+ on $\overline{\Sigma}_+$ and capping paths of Λ_- on $\overline{\Sigma}_-$. For the rest of the section, we will describe this DGA map.

The inclusion map $\Lambda_+ \hookrightarrow \overline{\Sigma}_+$ makes it natural to define the DGA

$$(\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\overline{\Sigma}_+)]), \partial_+).$$

The underlying algebra

$$\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\overline{\Sigma}_+)]) = \mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\Lambda_+)]) \otimes_{\mathbb{Z}_2[H_1(\Lambda_+)]} \mathbb{Z}_2[H_1(\overline{\Sigma}_+)]$$

is generated by Reeb chords of Λ_+ over the ring $\mathbb{Z}_2[H_1(\overline{\Sigma}_+)]$. Given that $\overline{\Sigma}_+$ is connected, we can choose a family of capping paths for Λ_+ on $\overline{\Sigma}_+$. Thus, for any rigid holomorphic disk u_+ counted by ∂_+, it is natural to take the homology class of $\tilde{u}_+$ in $H_1(\overline{\Sigma}_+)$. Hence the differential coefficients of ∂_+ are in $\mathbb{Z}_2[H_1(\overline{\Sigma}_+)]$. In addition, the DGA $(\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\overline{\Sigma}_+)]), \partial_+)$ does not depend on the choice of capping paths on $\overline{\Sigma}_+$ for a similar reason as in Proposition 2.3. The DGA $(\mathcal{A}(\Lambda_- ; \mathbb{Z}_2[H_1(\overline{\Sigma}_-)]), \partial_-)$ is defined similarly.

The DGA map Φ induced by Σ is a composition of two maps. The first map

$$\psi : (\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\overline{\Sigma}_+)]), \partial_+) \to (\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\overline{\Sigma}_-)]), \partial_+)$$

is induced by the inclusion map $\overline{\Sigma}_+ \hookrightarrow \overline{\Sigma}_-$. It is not hard to show ψ is a DGA map. The second map

$$\phi : (\mathcal{A}(\Lambda_+ ; \mathbb{Z}_2[H_1(\overline{\Sigma}_-)]), \partial_+) \to (\mathcal{A}(\Lambda_- ; \mathbb{Z}_2[H_1(\overline{\Sigma}_-)]), \partial_-)$$

is defined by counting rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ with boundary on Σ.

Fix an almost complex structure J on $\mathbb{R} \times \mathbb{R}^3$ which is adjusted to the symplectic form ω (see [Ekholm et al. 2016, Section 3.2] for details). For a Reeb chord a of Λ_+ and Reeb chords $b_1, \dots, b_m$ of Λ_-, define $\mathcal{M}^{\Sigma}(a ; b_1, \dots, b_m)$ to be the moduli space of J-holomorphic disks

$$u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R} \times \mathbb{R}^3, \Sigma)$$

with the following properties:

- D_{m+1} is a 2-dimensional unit disk with $m+1$ points $r, s_1, s_2, \dots, s_m$ removed. The points $r, s_1, s_2, \dots, s_m$ are arranged counterclockwise on the boundary of the disk.
- The image of u is asymptotic to a strip $\mathbb{R}_+ \times a$ around r.
- The image of u is asymptotic to a strip $\mathbb{R}_- \times b_i$ around s_i for $i = 1, \dots, m$.

By [Cieliebak et al. 2010], there is a corresponding dimension formula:

$$\dim \mathcal{M}^{\Sigma}(a\,; b_1, \dots, b_m) = |a| - \sum_{i=1}^{m} |b_i|.$$

If $\dim \mathcal{M}^{\Sigma}(a\,; b_1, \dots, b_m) = 0$, the J-holomorphic disk $u \in \mathcal{M}^{\Sigma}(a\,; b_1, \dots, b_m)$ is called *rigid*. For each rigid J-holomorphic disk u, concatenate the image of the disk boundary with the capping paths of corresponding Reeb chords on $\bar{\Sigma}_-$ and get

$$\tilde{u} = u(\partial D_{m+1}) \cup \gamma_a \cup -\gamma_{b_1} \cup \cdots \cup -\gamma_{b_m},$$

which is a loop in $\bar{\Sigma}_-$. Hence we can take the homology class of $\tilde{u}$ in $H_1(\bar{\Sigma}_-)$, denoted by $[\tilde{u}]_{\bar{\Sigma}_-}$. The map

$$\phi : (\mathcal{A}(\Lambda_+\,; \mathbb{Z}_2[H_1(\bar{\Sigma}_-)]), \partial_+) \to (\mathcal{A}(\Lambda_-\,; \mathbb{Z}_2[H_1(\bar{\Sigma}_-)]), \partial_-)$$

is defined as follows. For any Reeb chord a of Λ_+, the map ϕ maps a to

$$\phi(a) = \sum_{\dim \mathcal{M}^{\Sigma}(a;\boldsymbol{b})=0} \; \sum_{u \in \mathcal{M}^{\Sigma}(a;\boldsymbol{b})} [u]_{\bar{\Sigma}_-} b_1 \cdots b_m.$$

The map ϕ is the identity on $\mathbb{Z}_1[H_1(\bar{\Sigma}_-)]$. By [Ekholm et al. 2016, Section 3.5], the map ϕ is a DGA map.

Therefore, the exact Lagrangian cobordism Σ induces a DGA map, $\Phi = \phi \circ \psi$

$$\Phi : (\mathcal{A}(\Lambda_+\,; \mathbb{Z}_2[H_1(\bar{\Sigma}_+)]), \partial_+) \to (\mathcal{A}(\Lambda_-\,; \mathbb{Z}_2[H_1(\bar{\Sigma}_-)]), \partial_-).$$

3. Main results

We consider the exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot constructed from the EHK algorithm. Each filling can be achieved by concatenating n successive saddle cobordisms with two minimum cobordisms. In Section 3A, we combine results in [Ekholm et al. 2016] and Proposition 2.6 to write down combinatorial formulas for the DGA maps induced by a pinch move and a minimum cobordism. Composing all the DGA maps induced by n ordered pinch moves and the two minimum cobordisms, we obtain a combinatorial formula for augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$ induced by exact Lagrangian fillings L. In Section 3B, we find a combinatorial invariant to distinguish these resulting augmentations and hence we show that the C_n exact Lagrangian fillings are distinct up to exact Lagrangian isotopy. As a corollary, we extend the result to the case n is even.

3A. ***Computation of augmentations.*** Consider the Lagrangian projection of the Legendrian $(2, n)$ torus knot Λ with a base point $\tilde{s}_0$ and label the n crossings in degree 0 from left to right by $b_1, \dots, b_n$ as shown in Figure 11.

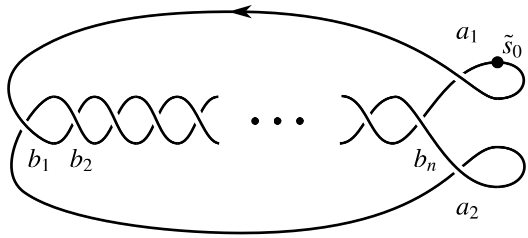


Figure 11. The Lagrangian projection of the Legendrian $(2, n)$ torus knot with a base point.

For each permutation σ of $\{1, \dots, n\}$, the corresponding exact Lagrangian filling L_σ of the Legendrian $(2, n)$ torus knot Λ is achieved in the following way:

- Start with an exact Lagrangian cylinder over Λ, denoted by $\overline{\Sigma}_0$. Label Λ as Λ_0.
- For $i = 1, \dots, n$, concatenate $\overline{\Sigma}_{i-1}$ from the bottom with a saddle cobordism Σ_i corresponding to the pinch move at crossing $b_{\sigma(i)}$ and get a new exact Lagrangian cobordism $\overline{\Sigma}_i$. Label the new Legendrian submanifold after the pinch move as Λ_i.
- Finally, use two minimal cobordisms, denoted by Σ_{n+1}, to close up $\overline{\Sigma}_n$ from the bottom and get the exact Lagrangian filling L_σ. To be consistent, let Λ_{n+1} be the empty set.

By Proposition 2.6, for $i = 1, \dots, n+1$, each exact Lagrangian cobordism Σ_i induces a DGA map:

$$\Phi_i : (\mathcal{A}(\Lambda_{i-1}\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}) \to (\mathcal{A}(\Lambda_i\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i).$$

The map Φ_{n+1} that is induced by minimum cobordisms is well understood while the maps Φ_i for $i = 1, \dots, n$ that correspond to pinch moves are not. We will first study $H_1(\overline{\Sigma}_n)$ and give a geometric description of the DGA map that corresponds to a pinch move. Combining this with [Ekholm et al. 2016], we will write down an explicit combinatorial formula for each Φ_i, for $i = 1, \dots, n+1$.

To describe $H_1(\overline{\Sigma}_n)$ easily, we chop off the cylindrical top of $\overline{\Sigma}_n$ and view it as a surface with boundary $\Lambda \cup \Lambda_n$, also denoted by $\overline{\Sigma}_n$. By Poincaré duality, we have $H^1(\overline{\Sigma}_n) \cong H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$. In particular, for each oriented curve α in $\overline{\Sigma}_n$ with ends on $\Lambda \cup \Lambda_n$, which is an element in $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$, there exists an element $\theta_\alpha \in H^1(\overline{\Sigma}_n)$ such that for any oriented loop β in $\overline{\Sigma}_n$, the intersection number of α and β is $\theta_\alpha(\beta)$. Thus, in order to know the homology class of a loop β in $H_1(\overline{\Sigma}_n)$, we only need to count the intersection number of each generator curve of $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$ with β.

We choose the set of generator curves of $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$ as follows. Use the t coordinate to slice $\overline{\Sigma}_n$ into a movie of diagrams (some of them may not be

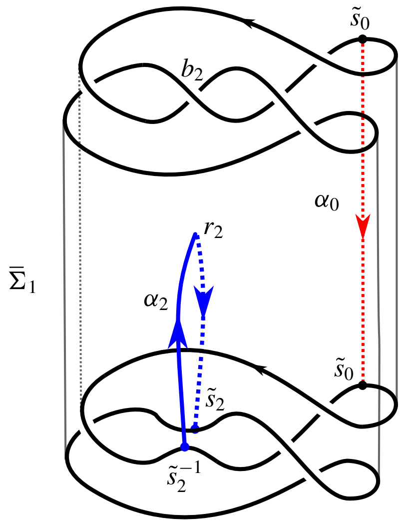


Figure 12. As an example, assume Λ is the Legendrian $(2, 3)$ torus knot and the first pinch move is taken at b_2. The blue curve and the red curve are α_2 and α_0 restricted on $\overline{\Sigma}_1$, respectively.

Legendrian diagrams). We study the trace of points on the diagram when t is decreasing. For $i = 1, \ldots, n$, the saddle cobordism Σ_i flows all the points directly downward except ends of the Reeb chord $b_{\sigma(i)}$. According to [Lin 2016], the ends of the Reeb chord $b_{\sigma(i)}$ merge to a point $r_{\sigma(i)}$, and then split into two points, labeled as $\tilde{s}_{\sigma(i)}$ and $\tilde{s}_{\sigma(i)}^{-1}$ respectively. Now for $i = 1, \ldots, n$, consider the trace of $\tilde{s}$ in $\overline{\Sigma}_n$, which is a flow line from r_i to the bottom of $\overline{\Sigma}_n$. Concatenating it with the inverse trace of $\tilde{s}_i^{-1}$ in $\overline{\Sigma}_n$, we get a curve α_i in $\overline{\Sigma}_n$ as shown in Figure 12. In addition, denote the trace of the base point $\tilde{s}_0$ in $\overline{\Sigma}_n$ by α_0. In this way, we have that $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a set of generator curves of $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n) \cong \mathbb{Z}^{n+1}$.

For each curve α_i, where $i = 0, \ldots, n$, Poincaré duality gives an element $\theta_{\alpha_i} \in H^1(\overline{\Sigma}_n)$. Denote its dual in $H_1(\Sigma_n)$ by $\tilde{s}_i$. Therefore, for any union of paths γ in $\overline{\Sigma}_n$, the monomial $w(\gamma)$ associated to γ in $\mathbb{Z}_2[H_1(\overline{\Sigma}_n)]$ is

$$w(\gamma) = \prod_{i=0}^{n} \tilde{s}_i^{n_i(\gamma)},$$

where $n_i(\gamma)$ is the intersection number of α_i and γ counted with signs.

For $i < n$, the map $H_1(\overline{\Sigma}_i) \to H_1(\overline{\Sigma}_n)$ induced by the inclusion map is injective. A similar argument shows that for a union of paths γ in $\overline{\Sigma}_i$, the monomial associated to γ in $\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]$ counts intersections of $\alpha_0, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$ with γ. Notice that the curves $\alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(n)}$ do not intersect $\overline{\Sigma}_i$. Hence the monomial in $\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]$ agrees with $w(\gamma)$ in $\mathbb{Z}_2[H_1(\overline{\Sigma}_n)]$.

Pick a family of capping paths for Λ_i on $\overline{\Sigma}_i$ for $i=0,\dots,n$. By Proposition 2.6, for $i=1,\dots,n+1$, each exact Lagrangian cobordism Σ_i gives a DGA map Φ_i,

$$\Phi_i : (\mathcal{A}(\Lambda_{i-1}\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]),\partial_{i-1}) \to (\mathcal{A}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i),$$

which maps any Reeb chord a of Λ_{i-1} to

$$\sum_{\dim\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0}\ \sum_{u\in\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})} w(\tilde{u})b_1\cdots b_m = \sum_{\dim\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0}\ \sum_{u\in\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})}\left(w(\gamma_a)w(u)\prod_{i=1}^m w(\gamma_{b_i})^{-1}\right)b_1\cdots b_m.$$

Now we show that the DGA map induced by the exact Lagrangian cobordisms is independent of the choice of capping paths.

Theorem 3.1. *Let γ and γ' be two families of capping paths of Λ_i on $\overline{\Sigma}_i$ for $i=0,\dots,n$. Denote the corresponding DGAs by $(\mathcal{A}^{\gamma}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i^{\gamma})$ and $(\mathcal{A}^{\gamma'}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i^{\gamma'})$. Assume Φ_i^{γ} and $\Phi_i^{\gamma'}$ are the corresponding DGA maps induced by Σ_i. Then the maps*

$$\begin{aligned} f_i : (\mathcal{A}^{\gamma}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i^{\gamma}) &\to (\mathcal{A}^{\gamma'}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i^{\gamma'}) \\ c &\mapsto w(\gamma'_c)^{-1}w(\gamma_c)\,c \end{aligned}$$

are DGA isomorphisms for $i=0,\dots,n$. Further, the following diagram commutes:

$$\begin{array}{ccc} \big(\mathcal{A}^{\gamma}(\Lambda_{i-1}\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]),\partial_{i-1}^{\gamma}\big) & \xrightarrow{f_{i-1}} & \big(\mathcal{A}^{\gamma'}(\Lambda_{i-1}\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]),\partial_{i-1}^{\gamma'}\big) \\ \Big\downarrow{\scriptstyle \Phi_i^{\gamma}} & & \Big\downarrow{\scriptstyle \Phi_i^{\gamma'}} \\ \big(\mathcal{A}^{\gamma}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i^{\gamma}\big) & \xrightarrow{f_i} & \big(\mathcal{A}^{\gamma'}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]),\partial_i^{\gamma'}\big) \end{array}$$

Proof. The maps f_i are DGA isomorphisms for the same reason as in Proposition 2.3. Now we prove the second part. For any Reeb chord a of Λ_{i-1} (and denoting $b_1\cdots b_m$ by $\boldsymbol{b}_*$),

$$\begin{aligned} f_i\circ\Phi_i^{\gamma}(a) &= f_i\left(\sum_{\dim\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0}\ \sum_{u\in\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})}\left(w(\gamma_a)w(u)\prod_{i=1}^m w(\gamma_{b_i})^{-1}\right)\boldsymbol{b}_*\right) \\ &= \sum_{\dim\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0}\ \sum_{u\in\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})}\left(w(\gamma_a)w(u)\prod_{i=1}^m w(\gamma_{b_i})^{-1}w(\gamma'_{b_i})^{-1}w(\gamma_{b_i})\right)\boldsymbol{b}_* \\ &= \sum_{\dim\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0}\ \sum_{u\in\mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})}\left(w(\gamma_a)w(u)\prod_{i=1}^m w(\gamma'_{b_i})^{-1}\right)\boldsymbol{b}_*, \end{aligned}$$

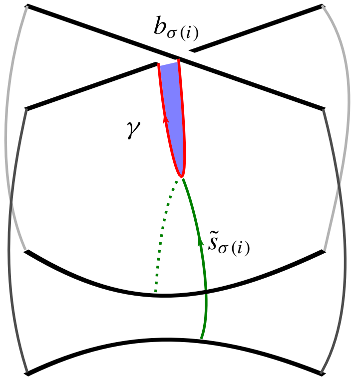


Figure 13. A cobordism corresponding to a pinch move, where the purple disk represents a holomorphic disk with a positive puncture at $b_{\sigma(i)}$.

$$\begin{aligned}\Phi_i^{\gamma'} \circ f_{i-1}(a) &= \Phi_i^{\gamma'}\big(w(\gamma_a')^{-1} w(\gamma_a) a\big) \\ &= w(\gamma_a')^{-1} w(\gamma_a) \sum_{\dim \mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0} \ \sum_{u \in \mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})} \left(w(\gamma_a') w(u) \prod_{i=1}^{m} w(\gamma_{b_i}')^{-1} \right) \boldsymbol{b}_* \\ &= \sum_{\dim \mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})=0} \ \sum_{u \in \mathcal{M}^{\Sigma_i}(a;\boldsymbol{b})} \left(w(\gamma_a) w(u) \prod_{i=1}^{m} w(\gamma_{b_i}')^{-1} \right) \boldsymbol{b}_*. \qquad \square\end{aligned}$$

Note that, if we cut $\overline{\Sigma}_i$ along the curves $\alpha_0, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(i)}$, the resulting surface is connected. Therefore, we can choose a family γ of capping paths for Λ_i on $\overline{\Sigma}_i$ such that none of them intersect the curves $\alpha_0, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(i)}$. Choose families of capping paths for $\Lambda_0, \dots, \Lambda_n$ in a similar way. As a result, for any rigid holomorphic disk u used in differentials of DGAs and DGA maps, we only need to count the intersections of curves in α with the disk boundary, i.e., $w(\tilde{u}) = w(u)$.

With this selection of capping paths, we are able to write down the DGA $(\mathcal{A}(\Lambda_i\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i)$ combinatorially, for $i = 1, \dots, n$. There are $2i+1$ points on Λ_i given by the intersection of α_0 and Λ_i, labeled by $\tilde{s}_0$, along with the two intersections of $\alpha_{\sigma(j)}$ and Λ_i, labeled by $\tilde{s}_{\sigma(j)}$ (positive intersection) and $\tilde{s}'_{\sigma(j)}$ (negative intersection), for $j = 1, \dots, i$. One then takes the DGA of Λ_i with these $2i+1$ base points, which has coefficients $\mathbb{Z}_2[\tilde{s}_0^{\pm 1}, \tilde{s}_{\sigma(1)}^{\pm 1}, \tilde{s}'^{\pm 1}_{\sigma(1)}, \dots, \tilde{s}_{\sigma(i)}^{\pm 1}, \tilde{s}'^{\pm 1}_{\sigma(i)}]$, and quotients by the relations $\tilde{s}'_{\sigma(j)} = \tilde{s}_{\sigma(j)}^{-1}$ for $j = 1, \dots, i$, to get the DGA

$$(\mathcal{A}(\Lambda_i\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i),$$

which is a DGA over $\mathbb{Z}_2[\tilde{s}_0^{\pm 1}, \tilde{s}_{\sigma(1)}^{\pm 1}, \dots, \tilde{s}_{\sigma(i)}^{\pm 1}]$, and $\{\tilde{s}_0, \tilde{s}_{\sigma(1)}, \dots, \tilde{s}_{\sigma(i)}\}$ is a basis of $H_1(\overline{\Sigma}_i)$ that corresponds to the curves $\alpha_0, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(i)}$.

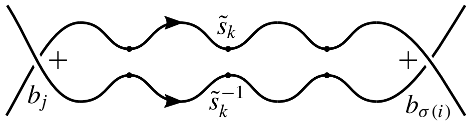


Figure 14. A part of the Lagrangian projection of Λ_{i-1}.

Now we are ready to describe the DGA map Φ_i induced by the exact Lagrangian cobordism Σ_i, for $i = 1, \dots, n$, which corresponds to a pinch move at crossing $b_{\sigma(i)}$. When we combine [Ekholm et al. 2016, Section 6.5] with Proposition 2.6, we find that the DGA map

$$\Phi_i : (\mathcal{A}(\Lambda_{i-1}\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}) \to (\mathcal{A}(\Lambda_i\,;\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i)$$

maps the Reeb chord $b_{\sigma(i)}$ to $\tilde{s}_{\sigma(i)}$ and any other Reeb chord c to

$$c + \sum_{\dim \mathcal{M}(c, b_{\sigma(i)}\,;\, c_1, \dots, c_m)=1} \;\; \sum_{u \in \mathcal{M}(c, b_{\sigma(i)}\,;\, c_1, \dots, c_m)} w(u)\tilde{s}_{\sigma(i)}^{-1} c_1 \cdots c_m,$$

where $\mathcal{M}(c, b_{\sigma(i)}\,; c_1, \dots, c_m)$ is the moduli space of holomorphic disks in $\mathbb{R}^2_{xy}$ with boundary on $\Pi_{xy}(\Lambda_{i-1})$ that covers a positive quadrant around c and $b_{\sigma(i)}$ and a negative quadrant around $c_1, \dots, c_m$. Please see [Ekholm et al. 2016, Section 6.5] for a detailed definition.

Here we discuss why the formulas make sense. The pinch move at $b_{\sigma(i)}$ pinches the Reeb chord $b_{\sigma(i)}$ down, which gives a holomorphic disk (as shown in Figure 13) with a positive puncture at $b_{\sigma(i)}$ and intersects $\tilde{s}_{\sigma(i)}$ exactly once. For a holomorphic disk $u \in \mathcal{M}(c, b_{\sigma(i)}\,; c_1, \dots, c_m)$, one can close the puncture of u at $b_{\sigma(i)}$ using the disk in Figure 13, which gives a holomorphic disk that contributes to $\Phi_i(c)$. Note that the boundary of this disk consists of the boundary of u and γ^{-1}. Thus the homology class of the boundary is $w(u)\tilde{s}_{\sigma(i)}^{-1}$, which matches the formula above.

In our case, in order to describe Φ_i combinatorially, we introduce two notations:

Definition 3.2. Let σ be a permutation of $\{1, \dots, n\}$. For $i \in \{1, \dots, n\}$, we define

$$\begin{aligned} T_\sigma^i :=&\{ j \in \{1, \dots, n\} \mid \sigma^{-1}(j) > \sigma^{-1}(i) \\ &\quad \text{and if } i < k < j \text{ or } j < k < i, \text{ then } \sigma^{-1}(k) < \sigma^{-1}(i)\}, \\ S_\sigma^i :=&\{ j \in \{1, \dots, n\} \mid i \in T_\sigma^j\} \\ =&\{ j \in \{1, \dots, n\} \mid \sigma^{-1}(j) < \sigma^{-1}(i) \\ &\quad \text{and if } i < k < j \text{ or } j < k < i, \text{ then } \sigma^{-1}(k) < \sigma^{-1}(j)\}. \end{aligned}$$

Here T_σ^i captures all the Reeb chords b_j with the property that, before performing a pinch move at b_i, one can find a holomorphic disk with exactly two positive punctures at b_i and b_j. In other words, it gathers all the Reeb chords on which the

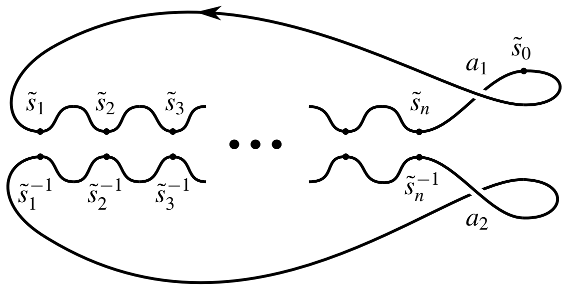


Figure 15. The Lagrangian projection of Λ_n.

DGA map induced by the pinch move at b_i acts nontrivially. The other set S^i_σ, on the other hand, detects all the Reeb chords b_j where a pinch move at b_j gives a DGA map that acts nontrivially on b_i.

If j is in $T^{\sigma(i)}_\sigma$ (an example is shown in Figure 14), the map Φ_i sends b_j to

$$\Phi_i(b_j) = b_j + \tilde{s}^{-1}_{\sigma(i)} \prod_{\substack{j<k<\sigma(i) \text{ or} \\ \sigma(i)<k<j}} \tilde{s}^{-2}_k.$$

For a_1, a_2 and the rest of the b_j where j is not in $T^{\sigma(i)}_\sigma$, the map Φ_i is identity.

Composing all the maps Φ_i for $i = 1, \ldots, n$ together, we get a DGA map,

$$\overline{\Phi}_n : (\mathcal{A}(\Lambda\,; \mathbb{Z}_2[H_1(\Lambda)]), \partial) \to (\mathcal{A}(\Lambda_n\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_n)]), \partial_n),$$

that is the identity map on the Reeb chords a_1, a_2. For $i = 1, \ldots, n$, in order to know $\overline{\Phi}_n(b_i)$, we consider pinch moves at b_j such that $j \in S^i_\sigma$ together with the pinch move at b_i. These pinch moves correspond to all the DGA maps that contribute to $\overline{\Phi}_n$. Composing all these maps together, we have that

$$\overline{\Phi}_n(b_i) = \Phi_1 \circ \cdots \circ \Phi_{\sigma^{-1}(i)}(b_i) = \tilde{s}_i + \sum_{j \in S^i_\sigma} \Bigg(\tilde{s}^{-1}_j \prod_{\substack{j<k<i \text{ or} \\ i<k<j}} \tilde{s}^{-2}_k \Bigg).$$

Now we describe the last DGA map,

$$\Phi_{n+1} : (\mathcal{A}(\Lambda_n\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_n)]), \partial_n) \to (\mathbb{Z}_2[H_1(L_\sigma)], 0).$$

As shown in Figure 15, the underlying algebra of Λ_n is generated by a_1 and a_2 and the differential is given by

$$\partial_n(a_1) = \tilde{s}_1\tilde{s}_2 \cdots \tilde{s}_n + \tilde{s}^{-1}_0, \qquad \partial_n(a_2) = \tilde{s}_n\tilde{s}_{n-1} \cdots \tilde{s}_1 + 1.$$

Consider the map $\psi : H_1(\overline{\Sigma}_n) \to H_1(L_\sigma)$ induced by the inclusion map $\overline{\Sigma}_n \hookrightarrow L_\sigma$. Since the DGA map

$$\Phi_{n+1} : (\mathcal{A}(\Lambda_n\,; \mathbb{Z}_2[H_1(\overline{\Sigma}_n)]), \partial_n) \to (\mathbb{Z}_2[H_1(L_\sigma)], 0)$$

satisfies $\Phi_{n+1} \circ \partial_n = 0 \circ \Phi_{n+1} = 0$, we have $\psi(\tilde{s}_0) = 1$ and $\psi(\tilde{s}_1)\psi(\tilde{s}_2)\cdots\psi(\tilde{s}_n) = 1$. Given that the map ψ is surjective, we assume a basis of $H_1(L_\sigma)$ is $\{s_1, \dots, s_{n-1}\}$, where $s_i = \tilde{s}_i$, for $i = 1, \dots, n-1$. The DGA map Φ_{n+1} is given by

$$a_1 \mapsto 0, \qquad a_2 \mapsto 0,$$
$$\tilde{s}_0 \mapsto 1, \qquad \tilde{s}_i \mapsto s_i, \quad i = 1, \dots, n-1, \qquad \tilde{s}_n \mapsto (s_1 s_2 \cdots s_{n-1})^{-1}.$$

Composing Φ_{n+1} with $\overline{\Phi}_n$, we get the augmentation ϵ_σ induced by L_σ as follows.

Theorem 3.3. *Given a permutation σ of $\{1, \dots, n\}$, let L_σ be the exact Lagrangian filling of the Legendrian $(2, n)$ torus knot Λ constructed from the EHK algorithm. If we write*

$$\mathbb{Z}_2[H_1(\Lambda)] = \mathbb{Z}_2[\tilde{s}_0, \tilde{s}_0^{-1}],$$
$$\mathbb{Z}_2[H_1(L_\sigma)] = \mathbb{Z}_2[s_1^{\pm 1}, \dots, s_{n-1}^{\pm 1}],$$

and set $s_n = (s_1 s_2 \cdots s_{n-1})^{-1}$, then the augmentation

$$\epsilon_\sigma : \mathcal{A}(\Lambda\,; \mathbb{Z}_2[H_1(\Lambda)]) \to \mathbb{Z}_2[H_1(L_\sigma)]$$

induced by L_σ is given by

$$\epsilon_\sigma(a_j) = 0, \quad j = 1, 2;$$
$$\epsilon_\sigma(b_i) = s_i + \sum_{j \in S_\sigma^i} \Bigl(s_j^{-1} \prod_{\substack{j<k<i \text{ or} \\ i<k<j}} s_k^{-2} \Bigr), \quad i = 1, \dots, n;$$
$$\epsilon_\sigma(\tilde{s}_0) = 1.$$

Example 3.4. In Figure 16, as an example, we compute the augmentation $\epsilon_{(2,3,1)}$ of the Legendrian (2, 3) torus knot induced by the exact Lagrangian filling $L_{(2,3,1)}$.

Similarly, one can compute the augmentation for each permutation of $\{1, 2, 3\}$ and get the following table:

ϵ	$\epsilon(b_1)$	$\epsilon(b_2)$	$\epsilon(b_3)$
$\epsilon_{(1,2,3)}$	s_1	$s_2 + s_1^{-1}$	$s_1^{-1}s_2^{-1} + s_2^{-1}$
$\epsilon_{(1,3,2)} = \epsilon_{(3,1,2)}$	s_1	$s_2 + s_1^{-1} + s_1 s_2$	$s_1^{-1}s_2^{-1}$
$\epsilon_{(2,1,3)}$	$s_1 + s_2^{-1}$	s_2	$s_1^{-1}s_2^{-1} + s_2^{-1} + s_1^{-1}s_2^{-2}$
$\epsilon_{(2,3,1)}$	$s_1 + s_2^{-1} + s_1 s_2^{-1}$	s_2	$s_1^{-1}s_2^{-1} + s_2^{-1}$
$\epsilon_{(3,2,1)}$	$s_1 + s_2^{-1}$	$s_2 + s_1 s_2$	$s_1^{-1}s_2^{-1}$

3B. ***Proof of the main theorem.*** In this section, we use Theorem 3.3 to find an invariant of augmentations induced from the exact Lagrangian fillings obtained

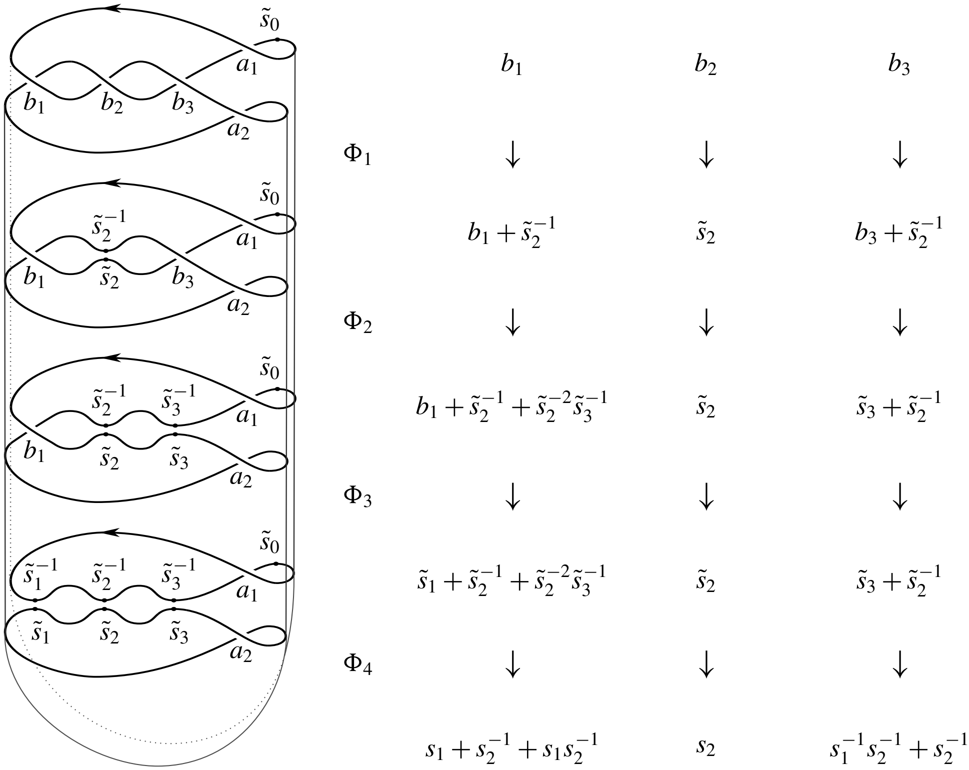


Figure 16. A computation of the augmentation induced by an exact Lagrangian filling of the Legendrian (2, 3) torus knot. We keep track of the image of b_1, b_2, b_3 under the composition of Φ_1, Φ_2, Φ_3 and Φ_4. The last line is the image of b_1, b_2, b_3 under the augmentation $\epsilon_{(2,3,1)}$.

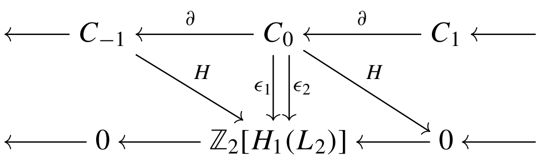
from the EHK algorithm. As a result, we distinguish all the augmentations in Theorem 3.3 and thus prove Theorem 1.1.

Lemma 3.5. *Let L_1 and L_2 be two exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot Λ constructed from the EHK algorithm. If L_1 and L_2 are exact Lagrangian isotopic, then there exists an invertible map $g : H_1(L_1) \to H_1(L_2)$ such that the following diagram commutes*:

$$\begin{array}{ccc} (\mathcal{A}(\Lambda), \partial) & \xrightarrow{Id} & (\mathcal{A}(\Lambda), \partial) \\ \downarrow{\scriptstyle \epsilon_{L_1}} & & \downarrow{\scriptstyle \epsilon_{L_2}} \\ \mathbb{Z}_2[H_1(L_1)] & \xrightarrow[g]{} & \mathbb{Z}_2[H_1(L_2)] \end{array} \tag{3-1}$$

where ϵ_{L_1} and ϵ_{L_2} are augmentations induced by L_1 and L_2 respectively.

Proof. The isotopy between L_1 and L_2 induces an invertible map $g : H_1(L_1) \to H_1(L_2)$. If we identify both $H_1(L_1)$ and $H_1(L_2)$ with $\mathbb{Z}^{n-1}$, then $g \in GL(n-1, \mathbb{Z})$. This map induces a natural map on the corresponding group rings $\mathbb{Z}_2[H_1(L_1)] \to \mathbb{Z}_2[H_1(L_2)]$, also denoted by g. Thus, we have two augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L_2)]$: $\epsilon_1 = g \circ \epsilon_{L_1}$ and $\epsilon_2 = \epsilon_{L_2}$. Since the two fillings L_1 and L_2 are isotopic through a family of exact Lagrangian fillings, according to [Ekholm et al. 2016, Theorem 1.3], we know that ϵ_1 and ϵ_2 are chain homotopic. In other words, there exists a degree 1 map $H : \mathcal{A}(\Lambda) \to \mathbb{Z}_2[H_1(L_2)]$ such that $H \circ \partial = \epsilon_1 - \epsilon_2$ as one can see from following diagram, where C_i denotes the degree i part of $\mathcal{A}(\Lambda)$.



Note that Λ has a Lagrangian projection (as shown in Figure 11) such that no Reeb chords are in negative degree. Hence $C_{-1} = 0$ and $\epsilon_1 - \epsilon_2 = H \circ \partial = 0$. Therefore $\epsilon_1 = \epsilon_2$, i.e., the diagram (3-1) commutes. $\square$

Remark 3.6. For any DGA $\mathcal{A}$ that vanishes on the degree -1 part, by the same argument, we have that two augmentations ϵ_1 and ϵ_2 of $\mathcal{A}$ are chain homotopic if and only if they are identically the same. For a more general criteria of two augmentations to be chain homotopic, check [Ng et al. 2015, Proposition 5.16].

Therefore, in order to distinguish exact Lagrangian fillings, we only need to distinguish their induced augmentations up to a $GL(n-1, \mathbb{Z})$ action. Observing the formula of the augmentation ϵ_σ in Theorem 3.3, we get a combinatorial way to define the number of terms in $\epsilon_\sigma(b_i)$ for $i = 1, \dots, n$ as follows.

Definition 3.7. For each permutation σ of $\{1, \dots, n\}$ and any number $i \in \{1, \dots, n\}$, we define $C_\sigma := (C_\sigma^1, C_\sigma^2, \dots, C_\sigma^n)$, where $C_\sigma^i = |S_\sigma^i| + 1$.

Example 3.8. We compute the vector C_σ for all of the permutations σ of $\{1, 2, 3\}$:

σ	$(1, 2, 3)$	$(1, 3, 2) \sim (3, 1, 2)$	$(2, 1, 3)$	$(2, 3, 1)$	$(3, 2, 1)$
C_σ	$(1, 2, 2)$	$(1, 3, 1)$	$(2, 1, 3)$	$(3, 1, 2)$	$(2, 2, 1)$

Proposition 3.9. *If two exact Lagrangian fillings L_{σ_1} and L_{σ_2} are exact Lagrangian isotopic, then $C_{\sigma_1} = C_{\sigma_2}$. In other words, the vector C_σ is an invariant of the exact Lagrangian filling L_σ up to exact Lagrangian isotopy.*

Proof. Using the formula in Theorem 3.3, we first show that C_σ^i is the number of terms in $\epsilon_\sigma(b_i)$. In order to do that, we need to prove that $\epsilon_\sigma(b_i)$ as a sum of monomials cannot be shorter, i.e, no terms in $\epsilon_\sigma(b_i)$ can be canceled by another term. First, replace s_n with $(s_1 \cdots s_{n-1})^{-1}$. If $i \neq n$, then each term of $\epsilon_\sigma(b_i)$ is one of the following forms:

(1) s_i ,

(2) $s_k^{-1} \prod_{j \in S} s_j^{-2}$ for some $k \neq i \in \{1, \dots, n-1\}$ and a subset $S \subset \{1, \dots, n-1\}$ that does not contain i, k (can be an empty set),

(3) $\prod_{j \in T} s_j^{-1} \prod_{k \notin T} s_k$ for some subset $T \subset \{1, \dots, n-1\}$ that does not contain i (can be an empty set).

If $i = n$, each term of $\epsilon_\sigma(b_n)$ can be either $s_1^{-1} \cdots s_{n-1}^{-1}$ or the form (2). Comparing degrees of $s_1, \dots, s_{n-1}$ of each term, we know that no terms can be canceled.

If L_{σ_1} and L_{σ_2} are exact Lagrangian isotopic, by Lemma 3.5, there is a map $g : \mathbb{Z}_2[H_1(L_1)] \to \mathbb{Z}_2[H_1(L_2)]$ such that $g \circ \epsilon_{L_1} = \epsilon_{L_2}$. Note that the map g on the group rings $\mathbb{Z}_2[H_1(L_1)] \to \mathbb{Z}_2[H_1(L_2)]$ is induced from an invertible map $H_1(L_1) \to H_1(L_2)$ and thus g maps a monomial to a monomial. Therefore $\epsilon_{\sigma_1}(b_i)$ and $\epsilon_{\sigma_2}(b_i)$ have the same number of terms, i.e., $C_{\sigma_1} = C_{\sigma_2}$. □

We say that two permutations σ_1 and σ_2 of $\{1, \dots, n\}$ are *isotopy equivalent* if they are equivalent via a sequence of relations of the form

$$(\dots, i, j, \dots, k, \dots) \sim (\dots, j, i, \dots, k, \dots), \quad \text{where } i < k < j. \tag{3-2}$$

By [Ekholm et al. 2016], if σ_1 and σ_2 are isotopy equivalent, the corresponding exact Lagrangian fillings L_{ϵ_1} and L_{ϵ_2} are exact Lagrangian isotopic and hence $C_{\sigma_1} = C_{\sigma_2}$. Conversely, we have the following:

Lemma 3.10. *If $C_{\sigma_1} = C_{\sigma_2}$, then σ_1 and σ_2 are isotopy equivalent.*

Proof. If $\sigma_1(1) = k$, then $C^k_{\sigma_1} = 1$. So $C^k_{\sigma_2} = 1$, i.e., we have that $S^k_{\sigma_2} = \varnothing$. If $\sigma_2(1) \neq k$, assume the element in σ_2 right before k is l, i.e., $\sigma_2(\sigma_2^{-1}(k) - 1) = l$. Note that $l \notin S^k_{\sigma_2}$, i.e., there exists i such that $l < i < k$ or $k < i < l$ and $\sigma_2^{-1}(i) > \sigma_2^{-1}(l)$. Note that $i \neq k$ and hence $\sigma_2^{-1}(i) > \sigma_2^{-1}(k) = \sigma_2^{-1}(l) + 1$. Thus we can use the relation (3-2) to switch l and k. In this way we can switch k to the first position in σ_2, i.e., $\sigma_2(1) = k = \sigma_1(1)$.

By induction, assume $\sigma_2(i) = \sigma_1(i)$ for $i < l$ and $\sigma_1(l) = k$. Then $S^k_{\sigma_1} \subset S^k_{\sigma_2}$. The assumption $C^k_{\sigma_2} = C^k_{\sigma_1}$ implies that $|S^k_{\sigma_1}| = |S^k_{\sigma_2}|$ and thus $S^k_{\sigma_1} = S^k_{\sigma_2}$. If $\sigma_2(l) \neq k$, for a similar reason to above, one can switch k to the l-th position and get $\sigma_2(l) = \sigma_1(l)$. Therefore, σ_1 and σ_2 are isotopy equivalent. □

Theorem 3.11. *If n is odd, the C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot Λ from the EHK algorithm are all of different exact Lagrangian isotopy classes.*

Proof. If two augmentations σ_1 and σ_2 are not isotopy equivalent, by Lemma 3.10, we have $C_{\sigma_1} \neq C_{\sigma_2}$. According to Proposition 3.9, the corresponding exact Lagrangian fillings L_{σ_1} and L_{σ_2} are not exact Lagrangian isotopic. Therefore, the

Legendrian $(2, n)$ torus knot has at least C_n exact Lagrangian fillings up to exact Lagrangian isotopy. □

Corollary 3.12. *When n is even, the Legendrian* $(2, n)$ *torus link* Λ *has at least* C_n *exact Lagrangian fillings.*

Proof. Start with the Legendrian $(2, n+1)$-knot Λ_0 and label its degree 0 Reeb chords from left to right by $b_1, \dots, b_{n+1}$ as usual. Let Σ be the exact Lagrangian cobordism from Λ to Λ_0 that corresponds to a pinch move of Λ_0 at b_{n+1}. For any permutation σ of $\{1, \dots, n\}$, the exact Lagrangian filling L_σ of Λ gives an exact Lagrangian filling of Λ_0 by concatenating with Σ on the top. This new exact Lagrangian filling of Λ_0 corresponds to the permutation $\tilde{\sigma} = (n+1, \sigma(1), \dots, \sigma(n))$ of $\{1, 2, \dots, n+1\}$, i.e., it is the filling $L_{\tilde{\sigma}}$ of Λ_0. Note that $C_{\tilde{\sigma}}^{n+1} = 1$. Moreover, we have that $C_{\tilde{\sigma}}^i = C_\sigma^i$ for $i = 1, \dots, n-1$ and $C_{\tilde{\sigma}}^n = C_\sigma^n + 1$. Thus $C_{\tilde{\sigma}}$ is determined by C_σ. Therefore, by Proposition 3.9 and Lemma 3.10, if two permutations σ_1 and σ_2 of $\{1, \dots, n\}$ are not isotopy equivalent, their induced permutations $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of $\{1, \dots, n+1\}$ are not isotopy equivalent. According to Theorem 3.11, the corresponding exact Lagrangian fillings $L_{\tilde{\sigma}_1}$ and $L_{\tilde{\sigma}_2}$ of Λ_0 are not exact Lagrangian isotopic. Hence L_{σ_1} and L_{σ_2} are not exact Lagrangian isotopic. □

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