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GLOBAL EXISTENCE OF SMOOTH SOLUTIONS TO EXPONENTIAL WAVE MAPS IN FLRW SPACETIMES

CHANG-HUA WEI AND NING-AN LAI

We consider the Cauchy problem of exponential wave maps in Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes. Using a weighted energy estimate, we show that the smooth solution of the Cauchy problem for exponential wave maps in FLRW spacetimes exists globally for small initial data.

1. Introduction

Wave maps are the Lorentzian counterparts of harmonic maps. Due to their significance in geometry and physics, wave maps have attracted much attention, and many achievements have been made in recent decades [Choquet-Bruhat 2000; Gu 1980; Klainerman and Machedon 1993; 1995; Shatah and Struwe 1998; 2002; Tao 2001a; 2001b; 2008]. In this paper, we investigate exponential wave maps which were introduced by Eells and Lemaire [1992]. The relationship between wave maps and exponential wave maps has been studied by Chiang and Yang [2007] from the geometric point of view. In this work we study the PDE aspect of the exponential wave maps on a curved Lorentzian manifold.

In this paper, we consider exponential wave maps on a special class of FLRW spacetimes, whose metric takes the form $g = -dt^2 + a^2(t)\, d\sigma^2$ for suitable $a(t)$. The FLRW metric is an exact solution of Einstein's field equations; it describes a homogeneous and isotropic universe. For more details on this metric, we refer to the book [Hawking and Ellis 1973]. In particular, in local coordinates $(t, x_1, \dots, x_m)$, with $m \geq 1$, we consider the following metric

$$g = -dt^2 + t^l \sum_{i=1}^{m} dx_i^2, \tag{1-1}$$

where $l > 2$ and $t > 0$. Here we remark that for the case $a(t) = e^t$, the metric g turns to be the de Sitter metric and much progress has been made on the study of wave type equations on de Sitter spacetimes; see [Kong and Wei 2013; Yagdjian 2012; 2009; Yagdjian and Galstian 2008; 2009].

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Before our main result, we give a brief introduction to exponential wave maps.

Basic equation and the main result.

Definition 1.1. An exponential wave map $u : M^{1+m} \to N$ from a $(1+m)$-dimensional Lorentzian manifold $(M^{1+m}, g_{\mu\nu})$ to an n-dimensional manifold $(N, h_{\alpha\beta})$ is a critical point of the exponential energy functional

$$E(u) := \int_M \exp(|du|^2)\, d\,\mathrm{Vol}_M = \int_M \exp(h_{\alpha\beta}\partial_\mu u^\alpha\, \partial_\nu u^\beta g^{\mu\nu})\, d\,\mathrm{Vol}_M, \tag{1-2}$$

where $d\,\mathrm{Vol}_M$ is the volume element of M.

We consider the exponential wave map taking M^{1+m} as the background manifold and $\mathbb{R}$ as the target manifold N in Definition 1.1. Therefore, the exponential energy functional takes the explicit form

$$E(u) = \int_M \exp\Bigl(-(\partial_t u)^2 + t^{-l}\sum_{i=1}^{m}(\partial_i u)^2\Bigr)\, t^{lm/2}\, dt\, dx_1 \cdots dx_m, \tag{1-3}$$

where $\partial_t u = \frac{\partial u}{\partial t}$ and $\partial_i u = \frac{\partial u}{\partial x_i}$; the corresponding Euler–Lagrange equation reads

$$\begin{aligned}\frac{\partial}{\partial t}\Bigl[-2\partial_t u \exp\Bigl(-(\partial_t u)^2 + t^{-l}\sum_{i=1}^{m}(\partial_i u)^2\Bigr)\, t^{lm/2}\Bigr]& \\ +\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\Bigl[2t^{-l}\partial_i u \exp\Bigl(-(\partial_t u)^2 + t^{-l}\sum_{i=1}^{m}(\partial_i u)^2\Bigr)\, t^{lm/2}\Bigr] &= 0,\end{aligned} \tag{1-4}$$

i.e., with $\Delta := \sum_{i=1}^{m} \partial_i^2$,

$$\begin{aligned}-\partial_t^2 u + t^{-l}\Delta u - \frac{lm}{2t}\partial_t u = \partial_t u\, \partial_t&\Bigl[-(\partial_t u)^2 + t^{-l}\sum_{i=1}^{m}(\partial_i u)^2\Bigr] \\ &-\sum_{i=1}^{m} t^{-l}\partial_i u\, \partial_i\Bigl[-(\partial_t u)^2 + t^{-l}\sum_{i=1}^{m}(\partial_i u)^2\Bigr].\end{aligned} \tag{1-5}$$

Remark 1.2. When $l = 0$, (1-5) is exactly the exponential wave map on Minkowski spacetime. Huh [2013] obtained a global existence result for sufficiently small initial data.

Define the wave operator as

$$\Box_g = \frac{1}{\sqrt{|\det g|}}\partial_\mu\bigl(\sqrt{|\det g|}\, g^{\mu\nu}\partial_\nu\bigr) = -\partial_t^2 - \frac{lm}{2t}\partial_t + t^{-l}\sum_{i=1}^{m}\partial_i^2, \tag{1-6}$$

where the Greek index μ ranges from 0 to m and $\partial_0 := \partial_t$. In (1-6) and throughout the paper, we use the Einstein summation convention, which says that one always takes the summation over the repeated upper and lower indices.

Let

$$Q(\varphi, \psi) = -(\partial_t \varphi)(\partial_t \psi) + t^{-l} \sum_{i=1}^{m} (\partial_i \varphi)(\partial_i \psi). \tag{1-7}$$

Then (1-5) can be rewritten as

$$\Box_g u = -Q(u, Q(u, u)). \tag{1-8}$$

From now on, we consider (1-8) with the following initial data

$$t = 1: \quad u(1, x) = \epsilon f(x), \quad u_t(1, x) = \epsilon q(x), \tag{1-9}$$

where ϵ is a small parameter, f and q are smooth functions with compact support.

The main result of this paper is as follows:

Theorem 1.3. *There exists a positive constant ϵ_0, such that the Cauchy problem (1-8)-(1-9) has a unique smooth solution on $[1, +\infty) \times \mathbb{R}^m$ for any $\epsilon \in [0, \epsilon_0]$.*

The structure of the paper. The paper is organized as follows. In Section 2, we attain pointwise decay estimates of the smooth solution to the linear wave equation by weighted energy estimates. The main theorem is proved in Section 3 by the continuity method and a careful analysis of the nonlinearities. Several further discussions are presented in Section 4.

2. Decay estimates for the linear wave equation

In this section, we investigate the pointwise decay estimates of smooth solution to the following linear wave equation on FLRW spacetimes

$$\Box_g \varphi = 0, \tag{2-1}$$

where the metric g is given by (1-1). It will play an important role in the study of nonlinear (1-8).

For multi-index $I = (I_1, \dots, I_m)$ with $|I| = \sum_{j=1}^{m} |I_j|$, denote

$$D = \{\partial_1, \dots, \partial_m\} \quad \text{and} \quad D^I = \partial_1^{I_1} \cdots \partial_m^{I_m}.$$

Furthermore, $\tilde{I} \le I$ means $\tilde{I}_i \le I_i$ for $i = 1, \dots, m$.

For any function $\varphi(x) = \varphi(x_1, \dots, x_m)$, we define

$$\|\varphi(x)\|_{L^2} = \left(\int_{\mathbb{R}^m} |\varphi(x)|^2 dx \right)^{1/2}, \quad \|\varphi(x)\|_{L^\infty} := \operatorname{ess}\sup_{x \in \mathbb{R}^m} |\varphi(x)|,$$

and

$$\|\varphi(x)\|_{H^s} = \left(\sum_{i=0}^{s} \left(\|D^i \varphi(x)\|_{L^2} \right)^2 \right)^{1/2},$$

where s is an integer and $dx = dx^1 \cdots dx^m$.

From here on, we also use $A \lesssim B$ to denote $A \le CB$ for some positive constant C.

Corresponding to the (2-1) and the specific metric g given by (1-1), we define the energy momentum tensor as

$$T_{\mu\nu}(\varphi) = (\partial_\mu\varphi)(\partial_\nu\varphi) - \frac{1}{2} g_{\mu\nu} |\nabla\varphi|^2, \tag{2-2}$$

where

$$|\nabla\varphi|^2 = g^{\kappa\lambda}\partial_\kappa\varphi\,\partial_\lambda\varphi = -(\partial_t\varphi)^2 + \sum_{i=1}^{m} t^{-l}(\partial_i\varphi)^2.$$

Given a vector field $V = V^\mu\partial_\mu$, define the compatible currents

$$J^V_\mu(\varphi) = T_{\mu\nu}(\varphi)\, V^\nu \tag{2-3}$$

and

$$K^V(\varphi) = \Pi^V_{\mu\nu} T^{\mu\nu}(\varphi), \tag{2-4}$$

where $\Pi^V_{\mu\nu}$ denotes the deformation tensor defined by

$$\Pi^V_{\mu\nu} = \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu), \tag{2-5}$$

and

$$\nabla_\mu V_\nu = g(\nabla_\mu V, \partial_\nu).$$

In a constant t-slice, the induced volume form is

$$d\,\mathrm{Vol}_t = t^{lm/2} dx_1 \cdots dx_m. \tag{2-6}$$

With above notations, we have for $i, j = 1, \ldots, m$ and $i \neq j$

$$T_{00}(\varphi) = \frac{1}{2}\left((\partial_t\varphi)^2 + \sum_{i=1}^{m} t^{-l}(\partial_i\varphi)^2\right), \tag{2-7}$$

$$T_{0i} = \partial_t\varphi\,\partial_i\varphi, \quad T_{ij} = \partial_i\varphi\,\partial_j\varphi, \tag{2-8}$$

and

$$T_{ii} = \frac{1}{2}\left(t^l(\partial_t\varphi)^2 + (\partial_i\varphi)^2 - \sum_{j\neq i}(\partial_j\varphi)^2\right). \tag{2-9}$$

We recall the following energy identity without proof; see [Alinhac 2010].

Lemma 2.1. *For a solution φ of the equation $\Box_g\varphi = f$, we have*

$$\nabla^\mu T_{\mu\nu} = (\Box_g\varphi)(\partial_\nu\varphi) \quad \textit{and} \quad \nabla^\mu J^V_\mu(\varphi) = K^V(\varphi) + \Box_g\varphi \cdot V(\varphi). \tag{2-10}$$

The energy density $e(V, \upsilon)$ of the function φ at time t with respect to a past pointed timelike vector field V is the nonnegative number given by

$$e(V,\upsilon) = J^V_\alpha \upsilon^\alpha = T_{\alpha\beta}(\varphi) V^\beta \upsilon^\alpha, \tag{2-11}$$

where υ^α is the α-th component of the past oriented unit normal vector $\upsilon = -\partial_t$.

Given a past pointed vector field V, by Lemma 2.1 and the divergence theorem, we easily get the following lemma:

Lemma 2.2. *Integrating (2-10) over the spacetime domain* $D = \{[1, t] \times \mathbb{R}^m\}$,

$$\int_{\Sigma_t} J^V_\alpha \upsilon^\alpha \, d\,\mathrm{Vol}_t - \int_{\Sigma_1} J^V_\alpha \upsilon^\alpha \, d\,\mathrm{Vol}_1 = \int_1^t \int_{\Sigma_\tau} (K^V(\varphi) + \Box_g \varphi V(\varphi)) \, d\,\mathrm{Vol}_\tau \, d\tau, \tag{2-12}$$

where Σ_t *denotes the spacelike hypersurface with* $t =$ constant.

From now on, we take $V = -\partial_t$ so that

$$\Pi^V_{\mu\nu} = -\frac{1}{2}(g_{\nu\kappa}\Gamma^\kappa_{\mu t} + g_{\mu\kappa}\Gamma^\kappa_{\nu t}). \tag{2-13}$$

Then for $i = 1, \dots, m$,

$$\Pi^V_{ii} = -\frac{l}{2} t^{l-1}, \tag{2-14}$$

and for $i \neq j$,

$$\Pi^V_{ij} = 0, \quad \Pi^V_{0i} = 0, \quad \text{and} \quad \Pi^V_{00} = 0, \tag{2-15}$$

where we have assumed that $x_0 = t$. By (2-4), we have

$$K^{-\partial_t}(\varphi) = \frac{l}{4}\left[(m-2)\sum_{i=1}^m \frac{1}{t^{l+1}}(\partial_i\varphi)^2 - \frac{m}{t}(\partial_t\varphi)^2\right]. \tag{2-16}$$

By (2-11), we have

$$e(V,\upsilon) = \frac{1}{2}\left((\partial_t\varphi)^2 + \sum_{i=1}^m t^{-l}(\partial_i\varphi)^2\right). \tag{2-17}$$

For a constant t-slice, we define

$$\begin{cases} E_0^{I,I_0}(t) = \dfrac{1}{2}\displaystyle\int_{\Sigma_t} (\partial_t \partial_t^{I_0} D^I \varphi)^2 \, d\,\mathrm{Vol}_t, \\ E_1^{I,I_0}(t) = \dfrac{1}{2}\displaystyle\int_{\Sigma_t} \left(\sum_{i=1}^m t^{-l}(\partial_i \partial_t^{I_0} D^I \varphi)^2\right) d\,\mathrm{Vol}_t \end{cases} \tag{2-18}$$

and

$$E^{I,I_0}(t) = E_0^{I,I_0}(t) + E_1^{I,I_0}(t), \tag{2-19}$$

where I_0 and $|I|$ are nonnegative integers, $|I|$ is the magnitude of any multi-index I.

Then, by above calculations and Lemma 2.2, the following zeroth-order energy identity holds:

Lemma 2.3. *The energy identity (2-12) can be rewritten as*

$$E^{0,0}(t)-E^{0,0}(1)=\int_1^t\Big[-\frac{l}{\tau}\Big(\frac{m}{2}\Big)E_0^{0,0}(\tau)+\frac{l}{\tau}\Big(\frac{m-2}{2}\Big)E_1^{0,0}(\tau)\Big]\,d\tau. \tag{2-20}$$

Corollary 2.4. *We have*

$$\frac{d}{dt}E^{0,0}(t)=-\frac{lm}{2t}E_0^{0,0}(t)+\frac{l(m-2)}{2t}E_1^{0,0}(t). \tag{2-21}$$

According to the explicit expression of FLRW metric (1-1), it is easy to see that the operator D is a Killing vector field, which means

$$\Pi_{\mu\nu}^{D}=0,$$

where D denotes the spacial derivatives. Thus, the structure of the equation (2-1) will not change if we take D^J as a commutator, i.e.,

$$\Box_g(D^J\varphi)=0. \tag{2-22}$$

By (2-22) and Corollary 2.4, for $I_0=0$, we have the following corollary.

Corollary 2.5. *For arbitrary nonnegative multi-index J, we have*

$$\frac{d}{dt}E^{J,0}(t)=-\frac{lm}{2t}E_0^{J,0}(t)+\frac{(m-2)l}{2t}E_1^{J,0}(t). \tag{2-23}$$

Define

$$\begin{cases} f^{I,I_0}(t)=E^{I,I_0}(t)t^{(2-m)l/2},\\ f_0^{I,I_0}(t)=E_0^{I,I_0}(t)t^{(2-m)l/2},\\ f_1^{I,I_0}(t)=E_1^{I,I_0}(t)t^{(2-m)l/2}. \end{cases} \tag{2-24}$$

Then we obtain the lemma:

Lemma 2.6. *The function $f^{I,0}(t)$ is uniformly bounded, provided that $E^{I,0}(1)$ is bounded for arbitrary multi-index I.*

Proof. By (2-24), since $m\geq 1$,

$$\begin{aligned}\frac{d}{dt}f^{I,0}(t)&=\Big(\frac{d}{dt}E^{I,0}(t)\Big)t^{(2-m)l/2}+\frac{(2-m)l}{2}E^{I,0}(t)t^{(2-m)l/2}t^{-1}\\ &=t^{(2-m)l/2}\Big[-\frac{lm}{2t}E_0^{I,0}(t)+\frac{(m-2)l}{2t}E_1^{I,0}(t)+\frac{(2-m)l}{2t}E^{I,0}(t)\Big]\\ &=-\frac{l(m-1)}{t}t^{(2-m)l/2}E_0^{I,0}(t)\leq 0.\end{aligned} \tag{2-25}$$

Thus, $f^{I,0}(t)$ is monotonically decreasing and we have

$$f^{I,0}(t)\leq f^{I,0}(1)=E^{I,0}(1).$$

□

Next, we consider the equation with higher-order derivatives on t.

Lemma 2.7. *For $I_0 > 0$, differentiating (2-22) by $\partial_t^{I_0}$, we have*

$$\Box_g(\partial_t^{I_0} D^J \varphi) = \sum_{I_0'=1}^{I_0} C_{I_0',l,m} t^{-(I_0-I_0'+2)} \partial_t^{I_0'} D^J \varphi + \sum_{i=1}^{m} \sum_{I_0''=0}^{I_0-1} C_{I_0'',l,m} t^{-l-(I_0-I_0'')} \partial_i^2 \partial_t^{I_0''} D^J \varphi, \tag{2-26}$$

where $C_{I_0',l,m}$ and $C_{I_0'',l,m}$ are constants depending on I_0', I_0'', l, and m.

Proof. With the notation $D^J\varphi = v$, it suffices to prove

$$\Box_g(\partial_t^{I_0} v) = \sum_{I_0'=1}^{I_0} C_{I_0',l,m} t^{-(I_0-I_0'+2)} \partial_t^{I_0'} v + \sum_{i=1}^{m} \sum_{I_0''=0}^{I_0-1} C_{I_0'',l,m} t^{-l-(I_0-I_0'')} \partial_i^2 \partial_t^{I_0''} v. \tag{2-27}$$

By (2-22), we know that

$$\Box_g v = 0. \tag{2-28}$$

Since ∂_t is not a Killing vector field, it does not commute with the operator $\Box_g$, and a direct calculation gives

$$[\Box_g, \partial_t] = -\frac{lm}{2} t^{-2} \partial_t + l t^{-l-1} \sum_{i=1}^{m} \partial_i^2. \tag{2-29}$$

Now, we can prove the lemma by induction on I_0.

When $I_0 = 1$,

$$\Box_g(\partial_t v) = [\Box_g, \partial_t] v + \partial_t(\Box_g v) = -\frac{lm}{2} t^{-2} \partial_t v + l t^{-l-1} \sum_{i=1}^{m} \partial_i^2 v. \tag{2-30}$$

Thus, the lemma holds for $I_0 = 1$.

Suppose the lemma holds for $I_0 - 1$. Then, for I_0,

$$\begin{aligned}\Box_g(\partial_t^{I_0} v) &= [\Box_g, \partial_t](\partial_t^{I_0-1} v) + \partial_t(\Box_g \partial_t^{I_0-1} v) \\ &= \Big(-\frac{lm}{2} t^{-2} \partial_t + l t^{-l-1} \sum_{i=1}^{m} \partial_i^2\Big) \partial_t^{I_0-1} v \\ &\quad + \partial_t\Big(\sum_{I_0'=1}^{I_0-1} C_{I_0',l,m} t^{-(I_0-1-I_0'+2)} \partial_t^{I_0'} v \\ &\quad + \sum_{i=1}^{m} \sum_{I_0''=0}^{I_0-2} C_{I_0'',l,m} t^{-l-(I_0-1-I_0'')} \partial_i^2 \partial_t^{I_0''} v\Big),\end{aligned} \tag{2-31}$$

which we rewrite as

$$\Box_g(\partial_t^{I_0}v) = -\frac{lm}{2}t^{-2}\partial_t^{I_0}v + lt^{-l-1}\sum_{i=1}^{m}\partial_i^2\partial_t^{I_0-1}v + \sum_{I_0'=1}^{I_0} C_{I_0',l,m}t^{-(I_0-I_0'+2)}\partial_t^{I_0'}v + \sum_{i=1}^{m}\sum_{I_0''=0}^{I_0-1} C_{I_0'',l,m}t^{-l-(I_0-I_0'')}\partial_i^2\partial_t^{I_0''}v.$$

By grouping together the last two terms on the right-hand-side, we see that (2-27) holds for arbitrary I_0. □

We define the total energy of M-th order as

$$F^M(t) = \sum_{|I|+I_0\le M} f^{I,I_0}(t). \tag{2-32}$$

Lemma 2.8. *We have*

$$F^M(t) \le C_M F^M(1), \tag{2-33}$$

where C_M is a positive constant depending only on M, and $F^M(1)$ is determined by the initial data.

Proof. For arbitrary $|I|+I_0 \le M$ and $I_0 \ge 1$, by Lemmas 2.2, 2.6, 2.7, and Corollary 2.5,

$$\frac{d}{dt}E^{I,I_0}(t) = -\frac{lm}{2t}E_0^{I,I_0}(t) + \frac{l(m-2)}{2t}E_1^{I,I_0}(t) - \int_{\Sigma_t}\Bigg(\sum_{I_0'=1}^{I_0} C_{I_0',l,m}t^{-(I_0-I_0'+2)}\partial_t^{I_0'}D^J\varphi + \sum_{i=1}^{m}\sum_{I_0''=0}^{I_0-1}C_{I_0'',l,m}t^{-l-(I_0-I_0'')}\partial_i^2\partial_t^{I_0''}\varphi\Bigg)\partial_t^{I_0+1}D^I\varphi\, d\,\mathrm{Vol}_t. \tag{2-34}$$

Then, by Lemma 2.6

$$\begin{aligned}\frac{d}{dt}f^{I,I_0}(t) &= \Big(\frac{d}{dt}E^{I,I_0}(t)\Big)t^{(2-m)l/2} + \frac{(2-m)l}{2}E^{I,I_0}(t)t^{(2-m)l/2}t^{-1}\\ &\le \Bigg|t^{(2-m)l/2}\int_{\Sigma_t}\Bigg(\sum_{I_0'=1}^{I_0}C_{I_0',l,m}t^{-(I_0-I_0'+2)}\partial_t^{I_0'}D^J\varphi + \sum_{i=1}^{m}\sum_{I_0''=0}^{I_0-1}C_{I_0'',l,m}t^{-l-(I_0-I_0'')}\partial_i^2\partial_t^{I_0''}\varphi\Bigg)\partial_t^{I_0+1}D^I\varphi\, d\,\mathrm{Vol}_t\Bigg|.\end{aligned} \tag{2-35}$$

By Hölder's inequality, for arbitrary $|I|+I_0 \le M$,

$$\frac{d}{dt}f^{I,I_0}(t) \le C_M\big(t^{-(I_0-I_0'+2)}F^M(t) + t^{-l/2-(I_0-I_0'')}F^M(t)\big), \tag{2-36}$$

where $1 \le I_0' \le I_0$, $0 \le I_0'' \le I_0 - 1$, which is given by Lemma 2.7.

Summing over all $|I| + I_0 \le M$ and using (2-36),

$$\frac{d}{dt} F^M(t) \le C_M \big(t^{-(I_0 - I_0' + 2)} F^M(t) + t^{-l/2 - (I_0 - I_0'')} F^M(t) \big). \tag{2-37}$$

By virtue of (2-37), one gets

$$F^M(t) \le F^M(1) \exp\bigg(\int_1^t C_M \big(t^{-(I_0 - I_0' + 2)} + t^{-l/2 - (I_0 - I_0'')} \big)\, dt \bigg) \le C_M F^M(1), \tag{2-38}$$

where for the second inequality we use the facts that $I_0 - I_0' + 2 \ge 2$, $I_0 - I_0'' \ge 1$, and $l > 0$. □

Remark 2.9. The constant C_M may vary from line to line in the above proof.

We define

$$e_0^{I,I_0}(t) = \frac{1}{2} \| \partial_t (\partial_t^{I_0} D^I \varphi) \|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{R}^m} (\partial_t \partial_t^{I_0} D^I \varphi)^2 \, dx, \tag{2-39}$$

$$e_1^{I,I_0}(t) = \frac{1}{2} \bigg\| \sum_{i=1}^m \partial_i (\partial_t^{I_0} D^I \varphi) \bigg\|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{R}^m} \sum_{i=1}^m (\partial_i \partial_t^{I_0} D^I \varphi)^2 \, dx \tag{2-40}$$

and

$$e^{I,I_0}(t) = e_0^{I,I_0}(t) + e_1^{I,I_0}(t). \tag{2-41}$$

By (2-18), (2-39), and (2-40),

$$E_0^{I,I_0}(t) = 1/2 \int_{\mathbb{R}^m} (\partial_t \partial_t^{I_0} D^I \varphi)^2 t^{lm/2} \, dx = t^{lm/2} e_0^{I,I_0}(t) \tag{2-42}$$

and

$$E_1^{I,I_0}(t) = \frac{1}{2} \int_{\mathbb{R}^m} \sum_{i=1}^m (\partial_i \partial_t^{I_0} D^I \varphi)^2 t^{-l} t^{lm/2} \, dx = t^{(m-2)l/2} e_1^{I,I_0}(t). \tag{2-43}$$

We obtain the following lemma directly from Lemma 2.8 and (2-42)–(2-43):

Lemma 2.10. *The following decay estimates hold*:

$$e_0^{I,I_0}(t) \le t^{-l} f^{I,I_0}(1) \quad \textit{and} \quad e_1^{I,I_0}(t) \le f^{I,I_0}(1). \tag{2-44}$$

By Lemma 2.10 and the Sobolev embedding, one easily obtain the following L^∞ estimates:

Lemma 2.11. *For any* $|I| \ge |J| + \lceil \frac{m}{2} + 1 \rceil$ *and* $i = 1, \dots, m$,

$$\| \partial_t (\partial_t^{I_0} D^J \varphi)(t) \|_{L^\infty} \le C_I t^{-l/2} \bigg(\sum_{K=0}^{I} f^{K,I_0}(t) \bigg)^{1/2} \quad \textit{for } I_0 \ge 0 \tag{2-45}$$

and

$$\|\partial_i(D^J\varphi)(t)\|_{L^\infty} \le C_I\Big(\sum_{K=0}^{I} f^{K,0}(t)\Big)^{1/2}, \tag{2-46}$$

provided that $f^{K,I_0}(t)$ $(K=0,\dots,I)$ *is bounded.*

3. The proof of Theorem 1.3

In this section, we will prove the global existence of smooth solutions to exponential wave maps (1-8) for small initial data (1-9). Since (1-8) can be reduced to a symmetric hyperbolic system, via standard results [Majda 1984; Alinhac 2010], there exists a unique local solution in the function space H^s, for $s > 1+\frac{m}{2}$. Thus, for the global existence result, it suffices to prove a priori estimate; see Lemma 3.7.

To obtain the global existence of smooth solution to Eq. (1-8), one needs to derive the equation satisfied by the higher-order derivatives of the solution first.

Lemma 3.1. *Differentiating* (1-8) *by the spacial derivatives* D^I,

$$\Box_g(D^Iu) = \sum_{I_1+I_2+I_3=I} -Q(D^{I_1}u, Q(D^{I_2}u, D^{I_3}u)). \tag{3-1}$$

Proof. Since D is a Killing vector field, it suffices to prove

$$D^IQ(u,v) = \sum_{I_1+I_2=I} C_{I_1,I_2}Q(D^{I_1}u, D^{I_2}v). \tag{3-2}$$

By Leibniz's rule,

$$\begin{aligned} D^IQ(u,v) &= D^I\Big(-\partial_tu\,\partial_tv + t^{-l}\sum_{i=1}^{m}(\partial_iu\partial_iv)\Big) \\ &= \sum_{I_1+I_2=I} C_{I_1,I_2}\Big(\partial_tD^{I_1}u\,\partial_tD^{I_2}v + t^{-l}\sum_{i=1}^{m}(\partial_iD^{I_1}u)(\partial_iD^{I_2}v)\Big) \\ &= \sum_{I_1+I_2=I} C_{I_1,I_2}Q(D^{I_1}u, D^{I_2}v). \end{aligned} \tag{3-3}$$

Let

$$v = Q(u,u). \tag{3-4}$$

By (3-2), we have

$$\begin{aligned} D^IQ(u,Q(u,u)) &= \sum_{I_1+I_2=I} C_{I_1,I_2}Q(D^{I_1}u, D^{I_2}Q(u,u)) \\ &= \sum_{I_1+I_{21}+I_{22}=I_1+I_2} C_{I_1,I_{21},I_{22}}Q(D^{I_1}u, Q(D^{I_{21}}u, D^{I_{22}}u)). \end{aligned} \tag{3-5}$$

Thus,

$$\Box_g(D^I u) = [\Box_g, D^I]u + D^I(\Box_g u) = -D^I Q(u, Q(u, u)). \tag{3-6}$$

From (3-5) and (3-6), we complete the proof of the lemma. □

For higher-order derivatives with respect to t, we have the following lemmas:

Lemma 3.2. *For any smooth functions $v(t, x)$ and $w(t, x)$,*

$$\begin{aligned}\partial_t^{I_0} Q(v, w) \\ &= \sum_{I_{01}+I_{02}=I_0} Q(\partial_t^{I_{01}} v, \partial_t^{I_{02}} w) + \sum_{i=1}^{m} \sum_{I_{01}+I_{02}<I_0} t^{-l-(I_0-I_{01}-I_{02})} \partial_i \partial_t^{I_{01}} v \, \partial_i \partial_t^{I_{02}} w.\end{aligned} \tag{3-7}$$

Proof. We prove it by induction. For $I_0 = 1$, we have

$$\begin{aligned}&\partial_t Q(v, w) \\ &\quad= \partial_t\Big[-\partial_t v \, \partial_t w + \sum_{i=1}^{m} t^{-l} \partial_i v \, \partial_i w\Big] \\ &\quad= -\partial_t^2 v \, \partial_t w - \partial_t v \, \partial_t^2 w + \sum_{i=1}^{m} t^{-l} \partial_{ti}^2 v \, \partial_i w + \sum_{i=1}^{m} t^{-l} \partial_i v \, \partial_{ti}^2 w - \sum_{i=1}^{m} l t^{-l-1} \partial_i v \, \partial_i w \\ &\quad= Q(\partial_t v, w) + Q(v, \partial_t w) - \sum_{i=1}^{m} t^{-l-(1-0)} \partial_i v \, \partial_i w.\end{aligned}$$

Therefore, the lemma holds for $I_0 = 1$.

Assume that the lemma holds for $I_0 - 1$ for arbitrary $I_0 > 1$; then for I_0,

$$\begin{aligned}&\partial_t^{I_0} Q(v, w) \\ &\quad= \partial_t(\partial_t^{I_0-1} Q(v, w)) \\ &\quad= \partial_t\Big(\sum_{I_{01}+I_{02}=I_0-1} Q(\partial_t^{I_{01}} v, \partial_t^{I_{02}} w) + \sum_{i=1}^{m} \sum_{I_{01}+I_{02}<I_0-1} t^{-l-(I_0-I_{01}-I_{02})} \partial_t^{I_{01}} \partial_i v \, \partial_t^{I_{02}} \partial_i w\Big) \\ &\quad= \sum_{I_{01}+I_{02}=I_0-1} \big(Q(\partial_t^{I_{01}+1} v, \partial_t^{I_{02}} w) + Q(\partial_t^{I_{01}} v, \partial_t^{I_{02}+1} w)\big) \\ &\qquad\quad + \sum_{i=1}^{m} \sum_{I_{01}+I_{02}<I_0-1} t^{-1-(I_0+1-I_{01}-I_{02})} \partial_t^{I_{01}} \partial_i v \, \partial_t^{I_{02}} \partial_i w \\ &\qquad\quad + \sum_{i=1}^{m} \sum_{I_{01}+I_{02}<I_0-1} t^{-1-(I_0+1-I_{01}-I_{02})} (\partial_t^{I_{01}+1} \partial_i v \, \partial_t^{I_{02}} \partial_i w + \partial_t^{I_{01}} \partial_i v \, \partial_t^{I_{02}+1} \partial_i w) \\ &\quad= \sum_{I_{01}+I_{02}=I_0} Q(\partial_t^{I_{01}} v, \partial_t^{I_{02}} w) + \sum_{i=1}^{m} \sum_{I_{01}+I_{02}<I_0} t^{-l-(I_0-I_{01}-I_{02})} \partial_t^{I_{01}} \partial_i v \, \partial_t^{I_{02}} \partial_i w.\end{aligned}$$

Thus, the lemma holds for arbitrary I_0. □

We define

$$S(I_0)=\{(I_0,I_{01},I_{02},I_{021},I_{022}) \mid I_{01}+I_{02}<I_0,\ I_{021}+I_{022}<I_{02}\}.$$

Lemma 3.3. *For any smooth functions $u(t,x)$, $v(t,x)$, and $w(t,x)$, and any nonnegative integer I_0,*

$$\begin{aligned}(3\text{-}8)\quad &\partial_t^{I_0}Q(u,Q(v,w))\\ &=\sum_{I_{01}+I_{02}+I_{03}=I_0} Q(\partial_t^{I_{01}}u,Q(\partial_t^{I_{02}}v,\partial_t^{I_{03}}w))\\ &\quad+\sum_{i=1}^{m}\sum_{S(I_0)} t^{-l-(I_{02}-I_{021}-I_{022})}Q(\partial_t^{I_{01}}u,\partial_t^{I_{021}}\partial_i v\,\partial_t^{I_{022}}\partial_i w)\\ &\quad+\sum_{i=1}^{m}\sum_{S(I_0)} t^{-l-1-(I_{02}-I_{021}-I_{022})}\partial_t^{I_{01}}\partial_t u\,\partial_t^{I_{021}}\partial_i v\,\partial_t^{I_{022}}\partial_i w\\ &\quad+\sum_{i=1}^{m}\sum_{S(I_0)} t^{-l-(I_0-I_{01}-I_{02})}\partial_t^{I_{01}}\partial_i u\,\partial_i Q(\partial_t^{I_{021}}v,\partial_t^{I_{022}}w)\\ &\quad+\sum_{i=1}^{m}\sum_{S(I_0)} t^{2l+(I_0-I_{01}-I_{02})+(I_{02}-I_{021}-I_{022})}\partial_t^{I_{01}}\partial_i u\,\partial_i(\partial_t^{I_{021}}\partial_i v\,\partial_t^{I_{022}}\partial_i w).\end{aligned}$$

Proof. By Lemma 3.2, let $p=Q(v,w)$, so

$$\begin{aligned}(3\text{-}9)\quad &\partial_t^{I_0}Q(u,Q(v,w))\\ &=\partial_t^{I_0}Q(u,p)\\ &=\sum_{I_{01}+I_{02}=I_0} Q(\partial_t^{I_{01}}u,\partial_t^{I_{02}}p)+\sum_{I_{01}+I_{02}<I_0} t^{-l-(I_0-I_{01}-I_{02})}\partial_t^{I_{01}}\partial_i u\,\partial_t^{I_{02}}\partial_i p,\end{aligned}$$

since

$$(3\text{-}10)\quad \partial_t^{I_{02}}p=\sum_{I_{021}+I_{022}=I_{02}} Q(\partial_t^{I_{021}}v,\partial_t^{I_{022}}w)+\sum_{I_{021}+I_{022}<I_{02}} t^{-l-(I_{02}-I_{021}-I_{022})}\partial_t^{I_{021}}\partial_i v\,\partial_t^{I_{022}}\partial_i w.$$

Inserting (3-10) into (3-9), we obtain the lemma. □

Combining Lemmas 2.1, 3.1, 3.2 and 3.3, we obtain an important lemma:

Lemma 3.4. *Differentiating (1-8) by $\partial_t^{I_0}D^I$ for any nonnegative integer I_0 and multi-index I, we have*

$$
\begin{aligned}
(3\text{-}11)\quad &\Box_g(\partial_t^{I_0} D^I u) \\
&= \sum_{I_0'=1}^{I_0} t^{-(I_0-I_0'+2)}\partial_t^{I_0'} D^I u + \sum_{i=1}^{m}\sum_{I_0''=0}^{I_0-1} t^{-l-(I_0-I_0'')}\partial_i^2\partial_t^{I_0''} D^I u \\
&\quad + \sum_{I_{01}+I_{02}+I_{03}=I_0} Q(\partial_t^{I_{01}} D^{I_1} u, Q(\partial_t^{I_{02}} D^{I_2} u, \partial_t^{I_{03}} D^{I_3} u)) \\
&\quad + \sum_{S(I_0)} t^{-l-(I_{02}-I_{021}-I_{022})} Q(\partial_t^{I_{01}} D^{I_1} u, \partial_t^{I_{021}}\partial_i(D^{I_2}u)\,\partial_t^{I_{022}}\partial_i(D^{I_3}u)) \\
&\quad + \sum_{S(I_0)} t^{-l-1-(I_{02}-I_{021}-I_{022})}\partial_t^{I_{01}}(D^{I_1}\partial_t u)\,\partial_t^{I_{021}}(D^{I_2}\partial_i u)\,\partial_t^{I_{022}}(D^{I_3}\partial_i u) \\
&\quad + \sum_{S(I_0)} t^{-l-(I_0-I_{01}-I_{02})}\partial_t^{I_{01}}(D^{I_1}\partial_i u)\,\partial_i Q(\partial_t^{I_{021}} D^{I_2} u, \partial_t^{I_{022}}(D^{I_3} u)) \\
&\quad + \sum_{S(I_0)} t^{-E}\partial_t^{I_{01}}\partial_i(D^{I_1}u)\,\partial_i(\partial_t^{I_{021}}(D^{I_2}\partial_i u)\partial_t^{I_{022}}(D^{I_3}\partial_i u)) \\
&:= \sum_{I_0'=1}^{I_0} t^{-(I_0-I_0'+2)}\partial_t^{I_0'} D^I u + \sum_{i=1}^{m}\sum_{I_0''=0}^{I_0-1} t^{-l-(I_0-I_0'')}\partial_i^2\partial_t^{I_0''} D^I u \\
&\quad + Q(u, Q(u, \partial_t^{I_0} D^I u)) + R,
\end{aligned}
$$

where $E = 2l + (I_0 - I_{01} - I_{02}) + (I_{02} - I_{021} - I_{022})$, $I_1 + I_2 + I_3 = I$, *and* R *denotes the remaining terms.*

Remark 3.5. For convenience, in Lemmas 3.1–3.4, we have omitted the numerical constants in front of each term at the right hand of (3-1), (3-7), (3-8), and (3-11). It does not affect the estimates in the sequel since they are all universal constants.

Remark 3.6. In the following estimates, we distinguish the $Q(u, Q(u, \partial_t^{I_0} D^I u))$ term from others, since it contains the highest-order derivatives of $\partial_t^{I_0} D^I u$.

Based on Lemma 3.4, we have the following energy estimate. It plays a key role in the proof of Theorem 1.3.

Lemma 3.7. *The following generalized energy inequality holds in the maximal development of the smooth solution of Cauchy problem* (1-8), (1-9):

$$
\begin{aligned}
(3\text{-}12)\quad &F^M(t) \\
&\lesssim F^M(1) + \int_1^t \big(\max\{\tau^{-(I_0-I_0'+2)}, \tau^{-l/2-(I_0-I_0'')}\} + \tau^{-l/2} F^M(\tau)\big) F^M(\tau)\, d\tau.
\end{aligned}
$$

provided $F^M(t) \ll 1$, *where* $F^M(t)$ *is defined by* (2-32) *with* $M \geq m + 2$.

Proof. We will prove the lemma in three steps.

Energy estimates based on (3-11). Taking $V = -\partial_t$ in Lemma 2.2, one gets

(3-13)
$$E^{I,J}(t) - E^{I,J}(1) = \int_1^t \int_{\Sigma_\tau} \big(K^V(\partial_\tau^J D^I u) + \Box_g(\partial_\tau^J D^I u) V(\partial_\tau^J D^I u)\big)\, d\,\mathrm{Vol}_\tau \; d\tau,$$

which is equivalent to

$$\frac{d}{dt} E^{I,J}(t) = \int_{\Sigma_t} \big(K^V(\partial_t^J D^I u(t)) + \Box_g(\partial_t^J D^I u) V(\partial_t^J D^I u)\big)\, d\,\mathrm{Vol}_t. \tag{3-14}$$

By (3-14) and (2-24),

$$\begin{aligned}
\frac{d}{dt} f^{I,J}(t)
&= t^{(2-m)l/2}\Big(\frac{d}{dt} E^{I,J}(t)\Big) + \frac{(2-m)l}{2} t^{(2-m)l/2} t^{-1} E^{I,J}(t) \\
&= t^{(2-m)l/2} \int_{\Sigma_t} \big(K^V(\partial_t^J D^I u(t)) + \Box_g(\partial_t^J D^I u) V(\partial_t^J D^I u)\big)\, d\,\mathrm{Vol}_t \\
&\qquad + \frac{(2-m)l}{2} t^{-1} t^{(2-m)l/2} E^{I,J}(t) \\
&\le t^{(2-m)l/2} \int_{\Sigma_t} \bigg(\sum_{J'=1}^{J} t^{-(J-J'+2)} \partial_t^{J'} D^I u + \sum_{i=1}^{m} \sum_{J''=0}^{J-1} t^{-l-(J-J'')} \partial_i^2 \partial_t^{J''} D^I u \bigg) \\
&\qquad\qquad \times V(\partial_t^J D^I u)\, d\,\mathrm{Vol}_t \\
&\qquad + t^{(2-m)l/2} \int_{\Sigma_t} \big(Q(u, Q(u, \partial_t^J D^I u)) + R\big) V(\partial_t^J D^I u)\, d\,\mathrm{Vol}_t.
\end{aligned} \tag{3-15}$$

We first estimate the term that contains the second-order derivatives of $\partial_t^J D^I u$, i.e.,

$$\int_{\mathbb{R}^m} Q(u, Q(u, \partial_t^J D^I u)) V(\partial_t^J D^I u)\, t^{(2-m)l/2} t^{ml/2}\, dx. \tag{3-16}$$

Energy estimates based on (3-16). Let

$$v = \partial_t^J D^I u. \tag{3-17}$$

Expanding the term $Q(u, Q(u, v))$, one obtains

$$\begin{aligned}
&Q(u, Q(u, v)) \\
&= -\partial_t u\, \partial_t\Big(-\partial_t u\, \partial_t v + t^{-l} \sum_{i=1}^m \partial_i u\, \partial_i v\Big) + t^{-l} \sum_{j=1}^m \partial_j u\, \partial_j \Big(-\partial_t u\, \partial_t v + t^{-l} \sum_{i=1}^m \partial_i u\, \partial_i v\Big)
\end{aligned}$$

which yields

$$
\begin{aligned}
&Q(u, Q(u, v)) \\
&\quad = (\partial_t u)^2 \partial_t^2 v - 2t^{-l} \sum_{i=1}^{m} \partial_t u\, \partial_i u\, \partial_{ti}^2 v + \sum_{i,j=1}^{m} t^{-2l} \partial_i u\, \partial_j u\, \partial_{ij}^2 v + \partial_t u\, \partial_t^2 u\, \partial_t v \\
&\qquad\qquad - t^{-l} \sum_{i=1}^{m} \partial_t u\, \partial_{ti}^2 u\, \partial_i v + l t^{-l-1} \sum_{i=1}^{m} \partial_t u\, \partial_i u\, \partial_i v \\
&\qquad\qquad - t^{-l} \sum_{j=1}^{m} \partial_j u\, \partial_{jt}^2 u\, \partial_t v + t^{-2l} \sum_{i,j=1}^{m} \partial_j u\, \partial_{ij}^2 u\, \partial_i v \\
&\quad = A + B + C + L,
\end{aligned}
$$

where

$$
A = (\partial_t u)^2 \partial_t^2 v, \quad B = -2t^{-l} \sum_{i=1}^{m} \partial_t u\, \partial_i u\, \partial_{ti}^2 v, \quad C = \sum_{i,j=1}^{m} t^{-2l} \partial_i u\, \partial_j u\, \partial_{ij}^2 v, \tag{3-18}
$$

and

$$
\begin{aligned}
L = \partial_t u\, \partial_t^2 u\, \partial_t v - t^{-l} \sum_{i=1}^{m} \partial_t u\, \partial_{ti}^2 u\, \partial_i v + l t^{-l-1} \sum_{i=1}^{m} \partial_t u\, \partial_i u\, \partial_i v \\
- t^{-l} \sum_{j=1}^{m} \partial_j u\, \partial_{jt}^2 u\, \partial_t v + t^{-2l} \sum_{i,j=1}^{m} \partial_j u\, \partial_{ij}^2 u\, \partial_i v.
\end{aligned} \tag{3-19}
$$

In view of (3-16) and (3-18), and integrating by parts, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^m} A \partial_t v\, t^{(2-m)l/2} t^{ml/2} dx \\
&\quad = \int_{\mathbb{R}^m} (\partial_t u)^2\, \partial_t^2 v\, \partial_t v\, t^l\, dx \\
&\quad = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} (\partial_t u)^2 (\partial_t v)^2 t^l\, dx \\
&\qquad - \int_{\mathbb{R}^m} \partial_t u\, \partial_t^2 u\, (\partial_t v)^2 t^l\, dx - \frac{l}{2} \int_{\mathbb{R}^m} (\partial_t u)^2 (\partial_t v)^2 t^{-1} t^l\, dx \\
&\quad = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} (\partial_t u)^2 (\partial_t v)^2 t^l\, dx + A_1 + A_2.
\end{aligned} \tag{3-20}
$$

and

$$\int_{\mathbb{R}^m} B\partial_t v t^l\,dx \tag{3-21}$$
$$= \int_{\mathbb{R}^m}\Bigl(2t^{-l}\sum_{i=1}^m \partial_t u\,\partial_i u\,\partial_{it}^2 v\Bigr)\partial_t v t^l\,dx$$
$$= \sum_{i=1}^m \partial_i\Bigl(\int_{\mathbb{R}^m} \partial_t u\,\partial_i u(\partial_i v)^2\,dx\Bigr)$$
$$-\sum_{i=1}^m \int_{\mathbb{R}^m} t^{-l}\partial_{ti}^2 u\,\partial_i u(\partial_t v)^2 t^l\,dx - \sum_{i=1}^m \int_{\mathbb{R}^m} t^{-l}\partial_t u\,\partial_i^2 u(\partial_t v)^2 t^l\,dx$$
$$:= \sum_{i=1}^m \partial_i\Bigl(\int_{\mathbb{R}^m} \partial_t u\,\partial_i u(\partial_i v)^2\,dx\Bigr) + B_1 + B_2.$$

Also,

$$\int_{\mathbb{R}^m} C\partial_t v t^l\,dx \tag{3-22}$$
$$= -\sum_{i,j=1}^m \int_{\mathbb{R}^m} t^{-2l}\partial_i u\,\partial_j u\,\partial_{ij}^2 v\,\partial_t v t^l\,dx$$
$$= \sum_{i,j=1}^m \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^m} t^{-2l}\partial_i u\,\partial_j u\,\partial_i v\,\partial_j v t^l\,dx$$
$$+\sum_{i,j=1}^m \int_{\mathbb{R}^m} t^{-2l}\partial_{ij}^2 u\,\partial_j u\,\partial_i v\,\partial_t v t^l\,dx + \int_{\mathbb{R}^m} t^{-2l}\partial_i u\,\partial_j^2 u\,\partial_i v\,\partial_t v t^l\,dx$$
$$-\sum_{i,j=1}^m \int_{\mathbb{R}^m} t^{-2l}\partial_{ti}^2 u\,\partial_j u\,\partial_i v\,\partial_j v t^l\,dx - \sum_{i,j=1}^m \int_{\mathbb{R}^m} \frac{l}{2}t^{-2l-1}\partial_i u\,\partial_j u\,\partial_i v\,\partial_j v t^l\,dx$$
$$= \sum_{i,j=1}^m \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^m} t^{-2l}\partial_i u\,\partial_j u\,\partial_i v\,\partial_j v t^l\,dx + C_1 + C_2 + C_3 + C_4.$$

Next, we estimate these terms via $F^M(t)$. By Lemma 2.11 and the assumption $M \geq m+2$, we obtain easily

$$|A_i| \lesssim t^{-l}[F^M(t)]^2, \qquad i = 1, 2, \tag{3-23}$$
$$|B_i| \lesssim t^{-l}[F^M(t)]^2, \qquad i = 1, 2, \tag{3-24}$$
$$|C_i| \lesssim t^{-3l/2}[F^M(t)]^2, \qquad i = 1, 2, 3, \tag{3-25}$$

and

$$|C_4| \lesssim t^{-l-1}[F^M(t)]^2. \tag{3-26}$$

For the remaining terms L, we have

$$\left|\int_{\mathbb{R}^m} L\partial_t v\, t^{(2-m)l/2} t^{lm/2}\, dx\right| \lesssim (t^{-l}+t^{-2l}+t^{-l-1})\,[F^M(t)]^2. \tag{3-27}$$

At last, we estimate the lower-order terms

$$\int_{\mathbb{R}^m} R\partial_t v\, t^{(2-m)l/2} t^{ml/2}\, dx, \tag{3-28}$$

where R is defined by (3-11).

***Energy estimates based on* (3-28).** We have

$$\int_{\mathbb{R}^m} R\partial_t v\, t^l dx = \int_{\mathbb{R}^m}\left(\sum_{i=1}^{5} R_i\right)\partial_t v\, t^l dx, \tag{3-29}$$

where R_i $(i=1,\dots,5)$ is defined by (3-11).

For R_1, we have

$$\begin{aligned}
&Q(\partial_t^{I_{01}} D^{I_1}u,\, Q(\partial_t^{I_{02}} D^{I_2}u,\, \partial_t^{I_{03}} D^{I_3}u))\\
&\quad= -\partial_t(\partial_t^{I_{01}} D^{I_1}u)\,\partial_t\Big[-\partial_t(\partial_t^{I_{02}} D^{I_2}u)\,\partial_t(\partial_t^{I_{03}} D^{I_3}u)\\
&\qquad\qquad + t^{-l}\sum_{i=1}^{m}\partial_i(\partial_t^{I_{02}} D^{I_2}u)\,\partial_i(\partial_t^{I_{03}} D^{I_3}u)\Big]\\
&\qquad + t^{-l}\sum_{j=1}^{m}\partial_j(\partial_t^{I_{01}} D^{I_1}u)\,\partial_j\Big[-\partial_t(\partial_t^{I_{02}} D^{I_2}u)\,\partial_t(\partial_t^{I_{03}} D^{I_3}u)\\
&\qquad\qquad + t^{-l}\sum_{i=1}^{m}\partial_i(\partial_t^{I_{02}} D^{I_2}u)\,\partial_i(\partial_t^{I_{03}} D^{I_3}u)\Big],
\end{aligned} \tag{3-30}$$

so

$$\left|\int_{\mathbb{R}^m} R_1\partial_t v\, t^{(2-m)l/2} t^{ml/2}\, dx\right| \lesssim (t^{-l}+t^{-l/2})[F^M(t)]^2. \tag{3-31}$$

For R_2, by direct calculations,

$$\begin{aligned}
&Q(\partial_t^{I_{01}} D^{I_1}u,\, \partial_t^{I_{021}}\partial_i(D^{I_2}u)\,\partial_t^{I_{022}}\partial_i(D^{I_3}u))\\
&\quad= -\partial_t(\partial_t^{I_{01}} D^{I_1}u)\,\partial_t(\partial_t^{I_{021}}\partial_i(D^{I_2}u)\,\partial_t^{I_{022}}\partial_i(D^{I_3}u))\\
&\qquad + t^{-l}\sum_{i=1}^{m}\partial_i(\partial_t^{I_{01}} D^{I_1}u)\,\partial_i(\partial_t^{I_{021}}\partial_i(D^{I_2}u)\,\partial_t^{I_{022}}\partial_i(D^{I_3}u)),
\end{aligned} \tag{3-32}$$

so

$$\left|\int_{\mathbb{R}^m} R_2 \partial_t v\, t^{-l-(I_{02}-I_{021}-I_{022})}\, t^{(2-m)l/2}\, t^{ml/2}\, dx\right| \lesssim \left(t^{-3l/2-(I_{02}-I_{021}-I_{022})} + t^{-l/2-(I_{02}-I_{021}-I_{022})}\right)[F^M(t)]^2. \tag{3-33}$$

For R_3, we have

$$\left|\int_{\mathbb{R}^m} R_3 \partial_t v\, t^{-2l-(I_0-I_{01}-I_{02})-(I_{02}-I_{021}-I_{022})}\, t^{(2-m)l/2}\, t^{ml/2}\, dx\right| \lesssim t^{-l/2-(I_0-I_{01}-I_{02})-(I_{02}-I_{021}-I_{022})}[F^M(t)]^2. \tag{3-34}$$

For R_4,

$$Q(\partial_t^{I_{02}} D^{I_2} u, \partial_t^{I_{03}} D^{I_3} u) = -\partial_t(\partial_t^{I_{02}} D^{I_2} u)\,\partial_t(\partial_t^{I_{03}} D^{I_3} u) + t^{-l}\sum_{i=1}^{m} \partial_i(\partial_t^{I_{02}} D^{I_2} u)\partial_i(\partial_t^{I_{03}} D^{I_3} u), \tag{3-35}$$

so

$$\left|\int_{\mathbb{R}^m} R_4 \partial_t v\, t^{-l-(I_0-I_{01}-I_{02})}\, t^{(2-m)l/2}\, t^{ml/2}\, dx\right| \lesssim t^{-l/2-(I_0-I_{01}-I_{02})}[F^M(t)]^2. \tag{3-36}$$

For R_5, we obtain

$$\left|\int_{\mathbb{R}^m} R_5 \partial_t v\, t^{-2l-(I_0-I_{01}-I_{02})-(I_{02}-I_{021}-I_{022})}\, t^{(2-m)l/2}\, t^{ml/2}\, dx\right| \lesssim t^{-l/2-(I_0-I_{01}-I_{02})-(I_{02}-I_{021}-I_{022})}[F^M(t)]^2. \tag{3-37}$$

Combining these estimates above, (2-37) and (3-15)–(3-37), and summing up $|I|+J \le M$, we have

$$\begin{aligned}\frac{d}{dt}\tilde{F}^M(t) \lesssim{}& \max\{t^{-I_0-I_0'+2}, t^{-l/2-(I_0-I_0'')}\}F^M(t)\\ &+ \max\{t^{-l}, t^{-l-1}, t^{-l/2}, t^{-2l}, t^{-3l/2-(I_{02}-I_{021}-I_{022})},\\ &\quad t^{-l/2-(I_0-I_{01}-I_{02})-(I_{02}-I_{021}-I_{022})}\}[F^M(t)]^2.\end{aligned} \tag{3-38}$$

Here, we define $\tilde{F}^M(t)$ to be

$$\tilde{F}^M(t) = \sum_{|I|+J\le M} f^{I,J}(t) + \frac{1}{2}\int_{\mathbb{R}^m}\left((\partial_t u)^2(\partial_t v)^2 t^l + \partial_i u\,\partial_j u\,\partial_i v \partial_j v\, t^{-2l}\right)dx.$$

It is easy to see that, if $F(t) \ll 1$, then there exists a positive constant C such that

$$C^{-1}F^M(t) \le \tilde{F}^M(t) \le C F^M(t).$$

Integrating (3-38) over $[1, t)$,

$$F^M(t) \lesssim F^M(1) + \int_1^t \bigl(\max\{\tau^{-(I_0-I_0'+2)}, \tau^{-l/2-(I_0-I_0'')}\} + \tau^{-l/2} F^M(\tau)\bigr) F^M(\tau)\, d\tau. \tag{3-39}$$

□

Based on Lemma 3.7, we can prove Theorem 1.3 by the bootstrap method.

Proof of Theorem 1.3. Set

$$E = \{t \in [1, T) : F^M(s) \le A\epsilon^2 \quad \text{for } 1 \le s \le t\}. \tag{3-40}$$

For the proof of the main result, it suffices to show that for any $t \in E$, the assumption $F^M(t) \le A\epsilon^2 \ll 1$ will imply $F^M(t) \le (A/2)\epsilon^2$, provided that A is sufficiently large and ϵ is sufficiently small.

By (3-12) and Gronwall's lemma, there exists a positive constant C such that

$$F^M(t) \le C F^M(1) \exp \int_1^t \bigl(\max\{\tau^{-(I_0-I_0'+2)}, \tau^{-l/2-(I_0-I_0'')}\} + \tau^{-l/2} A\epsilon^2\bigr)\, d\tau. \tag{3-41}$$

Choosing

$$\epsilon_0 = \sqrt{\frac{\ln(2)}{100A}} \quad \text{and} \quad A \ge 16 C D^M(1),$$

where $D^M(1)\epsilon^2 = F^M(1)$, then for any $\epsilon \in [0, \epsilon_0]$ and $l > 2$, we have

$$\begin{aligned} F^M(t) &\le C F^M(1) \exp \int_1^\infty \bigl(\max\{\tau^{-(I_0-I_0'+2)}, \tau^{-l/2-(I_0-I_0'')}\} + \tau^{-l/2} A\epsilon\bigr)\, d\tau \\ &\le \frac{A}{2}\epsilon^2. \end{aligned} \tag{3-42}$$

Then we can argument by contradiction and get the global existence result. □

4. Some discussions

In this paper, we have proved the global existence of smooth solutions to exponential wave maps on some special FLRW spacetimes, which are important and interesting in geometry and physics. Along with the development of Lorentzian geometry, the study of classical field theory and evolution equations is generalized to curved spacetimes. We believe that this field will attract better attention in the near future. Equations on a curved background, especially the solutions to the Einstein field equation, take a much more important role in physics. Confined by our knowledge, we only consider the easy case here, but there still exist a lot of problems that are worth focusing on.

For large initial data or large initial energy, whether the smooth solution of exponential wave maps exists globally or not is an interesting problem in the field of PDEs. Since wave maps are a class of equations with good structure, the study of the large data problem is under consideration now; one can refer to [Wang and Yu 2013; Yang 2015] on Minkowski spacetime.

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CHANG-HUA WEI
DEPARTMENT OF MATHEMATICS
ZHEJIANG SCI-TECH UNIVERSITY
HANGZHOU, 310018
CHINA
changhuawei1986@gmail.com

NING-AN LAI
DEPARTMENT OF MATHEMATICS, SCIENCES COLLEGE
LISHUI UNIVERSITY
LISHUI, 323000
CHINA
hyayue@gmail.com

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