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SHUICHIRO TAKEDA

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We had previously constructed a metaplectic tensor product of automorphic representations of covers of GL_r . To be precise, let $M = GL_{r_1} \times \cdots \times GL_{r_k} \subseteq GL_r$ be a Levi subgroup of GL_r , where $r = r_1 + \cdots + r_k$, and \tilde{M} its metaplectic preimage in the n -fold metaplectic cover \tilde{GL}_r of GL_r . For automorphic representations π_1, \dots, π_k of $\tilde{GL}_{r_1}(\mathbb{A}), \dots, \tilde{GL}_{r_k}(\mathbb{A})$, we had constructed (under certain technical assumptions, which are always satisfied when $n = 2$) an automorphic representation π of \tilde{M} that can be considered as the “tensor product” of the representations π_1, \dots, π_k .

Here we significantly simplify and generalize our previous construction without the technical assumptions mentioned above.

1. Introduction

Let F be a number field and \mathbb{A} be the ring of adeles. For a partition $r = r_1 + \cdots + r_k$ of r , one has the Levi subgroup

$$M(\mathbb{A}) := GL_{r_1}(\mathbb{A}) \times \cdots \times GL_{r_k}(\mathbb{A}) \subseteq GL_r(\mathbb{A})$$

of the (r_1, \dots, r_k) -parabolic. Let π_1, \dots, π_k be automorphic representations of $GL_{r_1}(\mathbb{A}), \dots, GL_{r_k}(\mathbb{A})$, respectively. It is a trivial construction to obtain the automorphic representation $\pi_1 \otimes \cdots \otimes \pi_k$ of the Levi $M(\mathbb{A})$ simply by taking the usual tensor product. Though highly trivial, this construction is of great importance in the theory of automorphic forms, especially when one would like to formulate Eisenstein series.

Now if one considers the metaplectic n -fold cover $\tilde{GL}_r(\mathbb{A})$ constructed by Kazhdan and Patterson [1984], the analogous construction turns out to be far from trivial. Namely for the metaplectic preimage $\tilde{M}(\mathbb{A})$ of $M(\mathbb{A})$ in $\tilde{GL}_r(\mathbb{A})$ and automorphic representations π_1, \dots, π_k of the metaplectic n -fold covers $\tilde{GL}_{r_1}(\mathbb{A}), \dots, \tilde{GL}_{r_k}(\mathbb{A})$, one cannot construct a representation of $\tilde{M}(\mathbb{A})$ simply by taking the tensor product

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$\pi_1 \otimes \cdots \otimes \pi_k$, because $\tilde{M}(\mathbb{A})$ is not the direct product of $\tilde{\mathrm{GL}}_{r_1}(\mathbb{A}), \dots, \tilde{\mathrm{GL}}_{r_k}(\mathbb{A})$, namely

$$\tilde{M}(\mathbb{A}) \not\cong \tilde{\mathrm{GL}}_{r_1}(\mathbb{A}) \times \cdots \times \tilde{\mathrm{GL}}_{r_k}(\mathbb{A}),$$

and even worse there is no natural map between them.

For the local case, P. Mezo [2004], whose work, we believe, is based on the work by Kable [2001], carried out a construction of an irreducible admissible representation of the Levi \tilde{M} starting with representations π_1, \dots, π_k of $\tilde{\mathrm{GL}}_{r_1}, \dots, \tilde{\mathrm{GL}}_{r_k}$, which can be called the “metaplectic tensor product” of π_1, \dots, π_k , and characterized it uniquely up to certain character twists.

In [Takeda 2016], we carried out an analogous construction for the global case and defined the global metaplectic tensor product. Further, we showed that the global metaplectic tensor product satisfies various expected properties. We, however, needed to impose certain technical assumptions for the group \tilde{M} , most notably Hypothesis (*) in [Takeda 2016, p. 202]. In this paper, we will modify the construction of that work so that the metaplectic tensor product can be defined without those technical assumptions and show that the new version also satisfies all the expected properties. Indeed, it seems our previous construction was unnecessarily complicated, and here we will give a simpler construction. To be more precise:

Main Theorem. *Let $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$ be a Levi subgroup of GL_r , and let π_1, \dots, π_k be automorphic subrepresentations of $\tilde{\mathrm{GL}}_{r_1}(\mathbb{A}), \dots, \tilde{\mathrm{GL}}_{r_k}(\mathbb{A})$. Then there exists an automorphic subrepresentation π of $\tilde{M}(\mathbb{A})$ such that*

$$\pi \cong \bigotimes'_v \pi_v,$$

where each π_v is the local metaplectic tensor product of Mezo. Moreover, if π_1, \dots, π_k are cuspidal (alternatively, square-integrable modulo center), then so is π . Further the metaplectic tensor product satisfies various expected properties.

In the above theorem, \bigotimes'_v indicates the metaplectic restricted tensor product, the meaning of which will be explained later in the paper. Also we require π_i be an automorphic subrepresentation, so that it is realized in a subspace of automorphic forms and hence each element in π_i is indeed an automorphic form. (Note that in general an automorphic representation is a subquotient.)

As we will see, strictly speaking the metaplectic tensor product of π_1, \dots, π_k might not be unique even up to equivalence but is dependent on a character ω on the center $Z_{\tilde{\mathrm{GL}}_r}$ of $\tilde{\mathrm{GL}}_r$. Hence we write

$$\pi_\omega := (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$$

for the metaplectic tensor product to emphasize the dependence on ω .

Notation. Throughout the paper, F is a number field and \mathbb{A} is the ring of adeles of F . For each place v , F_v is the corresponding local field and \mathcal{O}_{F_v} is the ring of integers of F_v . For each algebraic group G over a global F , and $g \in G(\mathbb{A})$, by g_v we mean the v -th component of g , and so $g_v \in G(F_v)$. For any group G , we denote its center by Z_G .

For a positive integer r , we denote by I_r the $r \times r$ identity matrix. Throughout we fix an integer $n \geq 2$, and we let μ_n be the group of n -th roots of unity in the algebraic closure of the prime field. We always assume that $\mu_n \subseteq F$.

We fix an ordered partition $r_1 + \dots + r_k = r$ of r , and we let

$$M = GL_{r_1} \times \dots \times GL_{r_k} \subseteq GL_r$$

and assume it is embedded diagonally as usual.

If π is a representation of a group G , we denote the space of π by V_π , though we often conflate π with V_π when there is no danger of confusion. We say π is unitary if V_π is equipped with a Hermitian structure invariant under the action of G , but we do not necessarily assume that the space V_π is complete. Now assume that the space V_π is a space of functions or maps on the group G and π is the representation of G on V_π defined by right-translation. (This is the case, for example, if π is an automorphic subrepresentation.) Let $H \subseteq G$ be a subgroup. We define $\pi \parallel_H$ to be the representation of H realized in the space

$$V_{\pi \parallel_H} := \{f|_H : f \in V_\pi\}$$

of restrictions of $f \in V_\pi$ to H , on which H acts by right translation. Namely $\pi \parallel_H$ is the representation obtained by restricting the functions in V_π . Occasionally, we confuse $\pi \parallel_H$ with its space when there is no danger of confusion. Note that there is an H -intertwining surjection $\pi|_H \rightarrow \pi \parallel_H$, where $\pi|_H$ is the (usual) restriction of π to H . Also for any subset $X \subseteq G$ and any $f \in V_\pi$, we denote by $\pi(X)f$ the vector space generated by $\pi(x)f$ for all $x \in X$. If X is a subgroup, this gives rise to a representation of X , which is a subrepresentation of $\pi|_X$.

2. The metaplectic cover \widetilde{GL}_r of GL_r

The groups. In this subsection, we set up our notations for the metaplectic n -fold cover \widetilde{GL}_r of GL_r for both local and global cases. Most of the time, we work both locally and globally at the same time. Hence we let

$$R = \begin{cases} F_v & \text{in the local case,} \\ \mathbb{A} & \text{in the global case.} \end{cases}$$

By the metaplectic n -fold cover $\widetilde{GL}_r(R)$ of $GL_r(R)$ with a fixed parameter $c \in \{0, \dots, n-1\}$, we mean the central extension of $GL_r(R)$ by μ_n as constructed

by Kazhdan and Patterson in [1984]. More concretely, as a set,

$$\tilde{\mathrm{GL}}_r(R) = \mathrm{GL}_r(R) \times \mu_n = \{(g, \xi) : g \in \mathrm{GL}_r(R), \xi \in \mu_n\},$$

whereas the multiplication is defined by

$$(g, \xi) \cdot (g', \xi') = (gg', \tau_r(g, g')\xi\xi'),$$

where τ_r is a certain 2-cocycle. (See [Takeda 2016, Sections 2 and 3] more about various issues on cocycles.)

If P is a parabolic subgroup of GL_r whose Levi part is $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$, we often write

$$\tilde{M}(R) = \tilde{\mathrm{GL}}_{r_1}(R) \tilde{\times} \cdots \tilde{\times} \tilde{\mathrm{GL}}_{r_k}(R)$$

for the metaplectic preimage of $M(R)$. Next let

$$\mathrm{GL}_r^{(n)}(R) = \{g \in \mathrm{GL}_r(R) : \det g \in R^{\times n}\},$$

and $\tilde{\mathrm{GL}}_r^{(n)}(R)$ be its metaplectic preimage. Also we define

$$M^{(n)}(R) = \{(g_1, \dots, g_k) \in M(R) : \det g_i \in R^{\times n}\}$$

and often denote its preimage by

$$\tilde{M}^{(n)}(R) = \tilde{\mathrm{GL}}_{r_1}^{(n)}(R) \tilde{\times} \cdots \tilde{\times} \tilde{\mathrm{GL}}_{r_k}^{(n)}(R).$$

The groups $M^{(n)}(R)$ and $\tilde{M}^{(n)}(R)$ are normal subgroups of $M(R)$ and $\tilde{M}(R)$, respectively. Indeed, if we define

$$(2.1) \quad \mathrm{Det}_M : M(R) = \mathrm{GL}_{r_1}(R) \times \cdots \times \mathrm{GL}_{r_k}(R) \rightarrow \underbrace{R^\times \times \cdots \times R^\times}_{k \text{ times}}$$

to be the map given by determinant on each factor GL_{r_i} , then $M^{(n)}(R)$ is the kernel of the composition of Det_M with projection to $R^{\times n} \setminus R^\times \times \cdots \times R^{\times n} \setminus R^\times$. Hence for the local case ($R = F_v$), the groups $M^{(n)}(R)$ and $\tilde{M}^{(n)}(R)$ are of finite index.

An important observation is that the metaplectic preimage of the center $Z_{\mathrm{GL}_r}(R)$ of $\mathrm{GL}_r(R)$ does not in general coincide with the center of $\tilde{\mathrm{GL}}_r(R)$. (It might not be even commutative for $n > 2$.) The center, which we denote by $Z_{\tilde{\mathrm{GL}}_r(R)}$, is

$$(2.2) \quad \begin{aligned} Z_{\tilde{\mathrm{GL}}_r(R)} &= \{(aI_r, \xi) : a^{r-1+2rc} \in R^{\times n}, \xi \in \mu_n\} \\ &= \{(aI_r, \xi) : a \in R^{\times n/d}, \xi \in \mu_n\}, \end{aligned}$$

where $d = \gcd(r-1+2rc, n)$. The second equality is proven in [Chinta and Offen 2013, Lemma 1].

Also the center $Z_{\tilde{M}(R)}$ of $\tilde{M}(R)$ is described as

$$Z_{\tilde{M}(R)} = \left\{ \begin{pmatrix} a_1 I_{r_1} & & \\ & \ddots & \\ & & a_k I_{r_k} \end{pmatrix} : a_i^{r-1+2cr} \in R^{\times n} \text{ and } a_1 \equiv \cdots \equiv a_r \pmod{R^{\times n}} \right\}.$$

See Proposition 3.10 of [Takeda 2016]. Let us mention that the above descriptions of $Z_{\tilde{\mathrm{GL}}_r(R)}$ and $Z_{\tilde{M}(R)}$ give

$$(2.3) \quad Z_{\tilde{\mathrm{GL}}_r(R)} \tilde{M}^{(n)}(R) = Z_{\tilde{M}(R)} \tilde{M}^{(n)}(R).$$

Let π be a representation of a subgroup $H \subseteq \tilde{\mathrm{GL}}_r(R)$ containing μ_n . We say π is “genuine” if each element $(1, \xi) \in H$ acts as multiplication by ξ , where we view ξ as an element of \mathbb{C} in the natural way.

We will revisit the question, considered in [Takeda 2016], of how the metaplectic tensor product behaves under restriction to a smaller Levi. Some relevant notation: if

$$I = \{i_1, \dots, i_l\} \subseteq \{1, \dots, k\}$$

is a nonempty subset with $i_1 < \cdots < i_l$, we set

$$(2.4) \quad M_I(R) = \mathrm{GL}_{r_{i_1}}(R) \times \cdots \times \mathrm{GL}_{r_{i_l}}(R)$$

which is embedded into $M(R)$ in the obvious way and hence viewed as a subgroup of $M(R)$. Let $\tilde{M}_I(R)$ be the metaplectic preimage of $M_I(R)$, so we have

$$\tilde{M}_I(R) \subseteq \tilde{M}(R).$$

Also set

$$\tilde{M}_I^{(n)}(R) := \tilde{M}_I(R) \cap \tilde{M}^{(n)}(R).$$

The global metaplectic cover $\tilde{\mathrm{GL}}_r(\mathbb{A})$. In this subsection we only consider the global case, i.e., $R = \mathbb{A}$.

First let us mention that both the F -rational points $\mathrm{GL}_r(F)$ and the unipotent radical $N_B(\mathbb{A})$ of the Borel subgroup B split in $\tilde{\mathrm{GL}}_r(\mathbb{A})$ via a certain partial map $s : \mathrm{GL}_r(\mathbb{A}) \rightarrow \tilde{\mathrm{GL}}_r(\mathbb{A})$. Via this splitting we identify $\mathrm{GL}_r(F)$ with a subgroup of $\tilde{\mathrm{GL}}_r(\mathbb{A})$. Let us mention, however, that this partial map is not given by the map $g \mapsto (g, 1)$ for our choice of cocycle τ_r . But rather the map $g \mapsto (g, 1)$ splits some compact subgroup. For our purpose here, we have only to mention the following. Let S be a finite set of places containing all Archimedean places and those v with $v \mid n$. Then we have a group homomorphism

$$(2.5) \quad \prod_{v \notin S} \mathrm{GL}_r(\mathcal{O}_{F_v}) \rightarrow \tilde{\mathrm{GL}}_r(\mathbb{A})$$

under the map $g \mapsto (g, 1)$.

We can also describe $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ as a quotient of a restricted direct product of the groups $\widetilde{\mathrm{GL}}_r(F_v)$ as follows. Consider the restricted direct product $\prod'_v \widetilde{\mathrm{GL}}_r(F_v)$ with respect to the groups K_v for all v with $v \nmid n$ and $v \nmid \infty$. If we denote each element in this restricted direct product by $\Pi'_v(g_v, \xi_v)$ so that $g_v \in K_v$ and $\xi_v = 1$ for almost all v , we have the surjection

$$(2.6) \quad \rho : \prod'_v \widetilde{\mathrm{GL}}_r(F_v) \rightarrow \widetilde{\mathrm{GL}}_r(\mathbb{A}), \quad \Pi'_v(g_v, \xi_v) \mapsto (\Pi'_v g_v, \Pi_v \xi_v),$$

where the product $\Pi_v \xi_v$ is literally the product inside μ_n . This is indeed a group homomorphism and

$$\prod'_v \widetilde{\mathrm{GL}}_r(F_v) / \ker \rho \cong \widetilde{\mathrm{GL}}_r(\mathbb{A}),$$

where $\ker \rho$ consists of the elements of the form $(1, \xi)$ with $\xi \in \prod'_v \mu_n$ and $\Pi_v \xi_v = 1$.

We set

$$\widetilde{\prod}'_v \widetilde{\mathrm{GL}}_r(F_v) := \prod'_v \widetilde{\mathrm{GL}}_r(F_v) / \ker \rho$$

and call it the metaplectic restricted direct product. Let us note that each $\widetilde{\mathrm{GL}}_r(F_v)$ has a natural embedding into $\prod'_v \widetilde{\mathrm{GL}}_r(F_v)$. By composing it with ρ , we have the natural inclusion

$$(2.7) \quad \widetilde{\mathrm{GL}}_r(F_v) \hookrightarrow \widetilde{\mathrm{GL}}_r(\mathbb{A}),$$

which allows us to view $\widetilde{\mathrm{GL}}_r(F_v)$ as a subgroup of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$.

Let us mention that all the discussions above on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ can be generalized to $\widetilde{M}(\mathbb{A})$, though there is a subtle issue on cocycles for $\widetilde{M}(\mathbb{A})$, which is discussed in detail in [Takeda 2016, Sect. 3]. This issue will not play any role in this paper.

We have the notion of automorphic representations as well as automorphic forms on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ or $\widetilde{M}(\mathbb{A})$. In this paper, by an automorphic form, we mean a smooth automorphic form instead of a K -finite one, namely an automorphic form is K_f -finite, \mathcal{Z} -finite and of uniformly moderate growth; see [Cogdell 2004, p. 17]. Hence if π is an automorphic representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$), the full group $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$) acts on π . An automorphic form f on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$) is said to be genuine if $f(g, \xi) = \xi f(g, 1)$ for all $(g, \xi) \in \widetilde{\mathrm{GL}}_r(\mathbb{A})$ (or $\widetilde{M}(\mathbb{A})$). In particular every automorphic form in the space of a genuine automorphic representation is genuine. We denote the space of genuine automorphic forms on $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (resp. $\widetilde{M}(\mathbb{A})$) by $\mathcal{A}(\widetilde{\mathrm{GL}}_r)$ (resp. $\mathcal{A}(\widetilde{M})$).

Suppose we are given a collection of irreducible admissible representations π_v of $\widetilde{\mathrm{GL}}_r(F_v)$ such that π_v is K_v -spherical for almost all v . Then we can form an irreducible admissible representation of $\prod'_v \widetilde{\mathrm{GL}}_r(F_v)$ by taking a restricted tensor product $\bigotimes'_v \pi_v$ as usual. Suppose further that $\ker \rho$ acts trivially on $\bigotimes'_v \pi_v$, which is always the case if each π_v is genuine. Then it descends to an irreducible admissible

representation of $\widetilde{GL}_r(\mathbb{A})$, which we denote by $\widetilde{\otimes}'_v \pi_v$, and call it the “metaplectic restricted tensor product”. Let us emphasize that the space for $\widetilde{\otimes}'_v \pi_v$ is the same as that for $\otimes'_v \pi_v$. Conversely, if π is an irreducible genuine admissible representation of $\widetilde{GL}_r(\mathbb{A})$, it is written as $\widetilde{\otimes}'_v \pi_v$ where π_v is an irreducible genuine admissible representation of $\widetilde{GL}_r(F_v)$, and for almost all v , π_v is K_v -spherical. (To see this, view π as a representation of the restricted direct product $\prod'_v \widetilde{GL}_r(F_v)$ by pulling it back by ρ in (2.6) and apply the usual tensor product theorem for the restricted direct product. This gives the restricted tensor product $\otimes'_v \pi_v$, where each π_v is genuine, and hence it descends to $\widetilde{\otimes}'_v \pi_v$.)

We now list some important properties of various groups we consider.

Lemma 2.8 [Takeda 2016, Lemma 14]. *Let S be a finite set of places containing all the Archimedean ones, and set*

$$\mathcal{O}_S^\times := \prod_{v \notin S} \mathcal{O}_{F_v}^\times.$$

Then the set $F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times / \mathcal{O}_S^\times$ is finite.

Lemma 2.9. *The group $F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times$ is compact.*

Proof. Let S be any finite set of places containing all the Archimedean ones. By the above lemma, we know $F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times$ is a finite union of sets of the form $F^\times \mathbb{A}^{\times n} a \mathcal{O}_S^\times$ for $a \in \mathbb{A}^\times$. But this set, which is the image of the compact set $a \mathcal{O}_S^\times$ under the quotient map $\mathbb{A}^\times \rightarrow F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times$, is compact in the topology of $F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times$. Hence the lemma follows. □

This in turn implies:

Lemma 2.10. *The group $M(F) \widetilde{M}^{(n)}(\mathbb{A})$ is a closed normal subgroup of $\widetilde{M}^{(n)}(\mathbb{A})$ whose quotient $M(F) \widetilde{M}^{(n)}(\mathbb{A}) \setminus \widetilde{M}^{(n)}(\mathbb{A})$ is a compact abelian group. Indeed, we have an isomorphism*

$$M(F) \widetilde{M}^{(n)}(\mathbb{A}) \setminus \widetilde{M}^{(n)}(\mathbb{A}) \cong \underbrace{F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times \times \cdots \times F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times}_{k \text{ times}}$$

of topological groups.

Proof. That it is closed is [Takeda 2016, Proposition A.4]. To show it is normal, one can check that the group $M(F) \widetilde{M}^{(n)}(\mathbb{A})$ is indeed the kernel of the composite

$$\widetilde{M}(\mathbb{A}) \rightarrow \mathbb{A}^\times \times \cdots \times \mathbb{A}^\times \rightarrow F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times \times \cdots \times F^\times \mathbb{A}^{\times n} \setminus \mathbb{A}^\times$$

where the first map is the determinant map Det_M as in (2.1). By the previous lemma, the last group on the right-hand side is compact. □

Lemma 2.11. *Let S be a finite set of places containing all the Archimedean ones and those v with $v \mid n$. Define*

$$(2.12) \quad K^S := \prod_{v \notin S} M(\mathcal{O}_{F_v}),$$

which can be viewed as a subgroup of $\tilde{M}(\mathbb{A})$ as in (2.5). Then the set

$$M(F)\tilde{M}^{(n)}(\mathbb{A}) \backslash \tilde{M}(\mathbb{A})/K^S$$

is finite.

Proof. This is immediate from Lemma 2.8, because $M(F)\tilde{M}^{(n)}(\mathbb{A}) \backslash \tilde{M}(\mathbb{A})/K^S$ is a product of k copies of $F^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times / \mathcal{O}_S^\times$ □

We next state a lemma from general topology and an important consequence of it.

Lemma 2.13. *Let A be a Hausdorff compact abelian group, and m_1, \dots, m_k be positive integers. Define*

$$H := \{(a^{m_1}, \dots, a^{m_k}) : a \in A\} = A^{m_1} \times \dots \times A^{m_k} \subseteq \underbrace{A \times \dots \times A}_{k \text{ times}}.$$

Then H is a closed subgroup of $A \times \dots \times A$.

Proof. Note that for each $i \in \{1, \dots, k\}$, the m_i -th power map $A \rightarrow A^{m_i} \subseteq A$ is continuous, and hence the image A^{m_i} of the compact A is compact. Recall that in a Hausdorff topological group, every compact subgroup is closed by, say, [Deitmar and Echterhoff 2009, Lemma 1.1.4]. So each A^{m_i} is closed. Hence H is closed. □

Proposition 2.14. *We have*

$$M(F)Z_{\tilde{M}(\mathbb{A})}\tilde{M}^{(n)}(\mathbb{A}) = M(F)Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{M}^{(n)}(\mathbb{A})$$

and this group is a closed (hence locally compact) subgroup of $\tilde{M}(\mathbb{A})$.

Proof. The equality is immediate from (2.3).

To prove this group is closed, it suffices to show that the image of $Z_{\tilde{M}(\mathbb{A})}$ in the quotient $M(F)\tilde{M}^{(n)}(\mathbb{A}) \backslash \tilde{M}(\mathbb{A})$ is closed. But one can see that the image of $Z_{\tilde{\text{GL}}_r(\mathbb{A})}$ under the isomorphism

$$M(F)\tilde{M}^{(n)}(\mathbb{A}) \backslash \tilde{M}(\mathbb{A}) = F^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times \times \dots \times F^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times$$

is the subgroup of the form

$$\{(a^{nr_1/d}, \dots, a^{nr_k/d}) : a \in F^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times\},$$

where $d = \gcd(r - 1 + 2rc, n)$. By Lemma 2.9, we know that $F^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times$ is compact, and hence by the previous lemma, this is closed. □

3. The metaplectic tensor product

In this section, after reviewing the local metaplectic tensor product of Mezo [2004] with the modification made by the author in [Takeda 2016], we will construct the global metaplectic tensor product.

Mezo’s local metaplectic tensor product. In this subsection, all the groups are over the local field F_v , and accordingly we simply write $\widetilde{\mathrm{GL}}_r$, \widetilde{M} , etc, instead of $\widetilde{\mathrm{GL}}_r(F_v)$, $\widetilde{M}(F_v)$, etc.

Let π_1, \dots, π_k be irreducible admissible genuine representations of $\widetilde{\mathrm{GL}}_{r_1}, \dots, \widetilde{\mathrm{GL}}_{r_k}$, respectively. For each $i = 1, \dots, k$, let

$$\sigma_i := \pi_i|_{\widetilde{\mathrm{GL}}_{r_i}^{(n)}}.$$

Note that σ_i , as a representation of $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$, is completely reducible, and the multiplicities of all the irreducible constituents are all equal. Namely, we have

$$(3.1) \quad \sigma_i = m_i \bigoplus_j \tau_{i,j},$$

where $\tau_{i,j}$ is an irreducible representation of $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$ such that $\tau_{i,j} \not\cong \tau_{i,k}$ for $j \neq k$, and m_i is a positive multiplicity which is independent of $\tau_{i,j}$. For the non-Archimedean case, this is precisely [Gelbart and Knapp 1982, Lemma 2.1], and the Archimedean case can be proven in the same way as the non-Archimedean case because the index of $\widetilde{\mathrm{GL}}_{r_i}^{(n)}$ in $\widetilde{\mathrm{GL}}_{r_i}$ is at most 2. Mezo [2004] first picks up an irreducible constituent τ_i of σ_i and considers the (usual) tensor product

$$V_{\tau_1} \otimes \cdots \otimes V_{\tau_k},$$

which, of course, gives a representation of the direct product $\widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)}$. The genuineness of the representations τ_1, \dots, τ_k implies that this tensor product representation descends to a representation of the group $\widetilde{M}^{(n)} := \widetilde{\mathrm{GL}}_{r_1}^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{\mathrm{GL}}_{r_k}^{(n)}$, i.e., the representation factors through the natural surjection

$$\widetilde{\mathrm{GL}}_{r_1}^{(n)} \times \cdots \times \widetilde{\mathrm{GL}}_{r_k}^{(n)} \twoheadrightarrow \widetilde{\mathrm{GL}}_{r_1}^{(n)} \widetilde{\times} \cdots \widetilde{\times} \widetilde{\mathrm{GL}}_{r_k}^{(n)}.$$

We denote this representation of $\widetilde{M}^{(n)}$ by

$$\tau := \tau_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \tau_k.$$

Let us emphasize that the space V_τ of τ is the usual tensor product $V_{\tau_1} \otimes \cdots \otimes V_{\tau_k}$.

In this paper, however, we will take a different approach. Instead of picking up a τ_i , we will consider all of σ_i at the same time and define the representation

$$\sigma := \sigma_1 \widetilde{\otimes} \cdots \widetilde{\otimes} \sigma_k$$

of $\tilde{M}^{(n)}$ in the same way as τ . Let us again emphasize that the space V_σ of σ is the usual tensor product $V_{\sigma_1} \otimes \cdots \otimes V_{\sigma_k}$. Further note that because of (3.1), we have

$$(3.2) \quad \sigma = m \bigoplus_{\tau} \tau,$$

where $m = m_1 \cdots m_k$ and the sum is over all possible equivalence classes of representations of the form $\tau = \tau_1 \tilde{\otimes} \cdots \tilde{\otimes} \tau_k$.

Then we define

$$\Pi = \Pi(\pi_1, \dots, \pi_k) := \text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \sigma.$$

By (3.2), we have

$$\Pi = m \bigoplus_{\tau} \text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau,$$

where the sum is over all the equivalence classes of irreducible subrepresentations τ of σ . Note that since σ is completely reducible and the index of $\tilde{M}^{(n)}$ in \tilde{M} is finite, one can see that the representation Π is completely reducible. Certainly, it is highly unlikely that Π is irreducible. Rather it contains all the metaplectic tensor products constructed by Mezo. To see it, we need to take the action of the center $Z_{\tilde{\text{GL}}_r}$ into account. For this purpose, let us first define

$$(3.3) \quad \Omega = \Omega(\pi_1, \dots, \pi_k) := \{\omega : \omega \text{ is a character on } Z_{\tilde{\text{GL}}_r} \text{ which appears in } \sigma\},$$

where we say “ ω appears in σ ” if there is an irreducible constituent $\tau \subseteq \sigma$ that agrees with ω on the overlap, namely

$$\omega|_{Z_{\tilde{\text{GL}}_r} \cap \tilde{M}^{(n)}} = \tau|_{Z_{\tilde{\text{GL}}_r} \cap \tilde{M}^{(n)}}.$$

Now Mezo’s construction can be summarized as follows. Let $\tau \subseteq \sigma$ be an irreducible representation, so $\tau = \tau_1 \tilde{\otimes} \cdots \tilde{\otimes} \tau_k$ for some τ_i , and let $\omega \in \Omega$ be such that it agrees with τ on the overlap. Then we can extend τ to the representation

$$\tau_\omega := \omega \tau$$

of $Z_{\tilde{\text{GL}}_r} \tilde{M}^{(n)}$ by letting $Z_{\tilde{\text{GL}}_r}$ act by ω . Then if we induce it to the group \tilde{M} , it is isotypic (though possibly reducible), and we denote this isomorphism class by

$$(\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$$

and call it the metaplectic tensor product of π_1, \dots, π_k with respect to ω . With this notation, we have

$$\text{Ind}_{Z_{\tilde{\text{GL}}_r} \tilde{M}^{(n)}}^{\tilde{M}} \tau_\omega = m' (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$$

for some finite multiplicity m' , which will be seen to be independent of τ and ω but only dependent on the representations π_1, \dots, π_k .

Clearly we have the inclusions

$$(3.4) \quad \text{Ind}_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}^{\tilde{M}} \tau_\omega \hookrightarrow \text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau \hookrightarrow \text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \sigma = \Pi(\pi_1, \dots, \pi_k) = m \bigoplus_{\tau} \text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau,$$

because $\tau \subseteq \sigma$. Further we have:

Proposition 3.5. *For each fixed τ , let*

$$\Omega(\tau) := \{\omega \in \Omega : \omega|_{Z_{\tilde{GL}_r} \cap \tilde{M}^{(n)}} = \tau|_{Z_{\tilde{GL}_r} \cap \tilde{M}^{(n)}}\}.$$

Then

$$\text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau = \bigoplus_{\omega \in \Omega(\tau)} \text{Ind}_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}^{\tilde{M}} \tau_\omega = m' \bigoplus_{\omega \in \Omega(\tau)} (\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega,$$

where m' is the positive multiplicity of $(\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega$ in $\text{Ind}_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}^{\tilde{M}} \tau_\omega$, which is independent of τ and ω but is only dependent on π_1, \dots, π_k .

Proof. The proof is an elementary exercise in representation theory. But we will give a brief explanation for each equality. First, by inducing in stages, we have

$$\text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau = \text{Ind}_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}^{\tilde{M}} \text{Ind}_{\tilde{M}^{(n)}}^{Z_{\tilde{GL}_r} \tilde{M}^{(n)}} \tau.$$

Then similarly to [Mezo 2004, Lemma 4.1], one can see

$$\text{Ind}_{\tilde{M}^{(n)}}^{Z_{\tilde{GL}_r} \tilde{M}^{(n)}} \tau = \bigoplus_{\omega \in \Omega(\tau)} \tau_\omega,$$

because the quotient $\tilde{M}^{(n)} \setminus Z_{\tilde{GL}_r} \tilde{M}^{(n)} = Z_{\tilde{GL}_r} \cap \tilde{M}^{(n)} \setminus Z_{\tilde{GL}_r}$ has the same size as $\Omega(\tau)$. To be more precise, for a fixed $\omega \in \Omega(\tau)$ we can write

$$\Omega(\tau) = \{\omega\chi : \chi \text{ is in the dual of } Z_{\tilde{GL}_r} \cap \tilde{M}^{(n)} \setminus Z_{\tilde{GL}_r}\}.$$

To show the next equality, the only nontrivial part is to show that the multiplicity m' is independent of τ and ω . The independence from ω follows from the fact that the restrictions $(\text{Ind}_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}^{\tilde{M}} \tau_\omega)|_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}$ and $(\text{Ind}_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}^{\tilde{M}} \tau_\omega)|_{\tilde{M}^{(n)}}$ have the same number of constituents and the latter is independent of ω , and further the restrictions $(\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega|_{Z_{\tilde{GL}_r} \tilde{M}^{(n)}}$ and $(\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega|_{\tilde{M}^{(n)}}$ have the same number of constituents and again the latter is independent of ω . To show it is independent of τ , let us note that m' is indeed equal to $[\tilde{H} : Z_{\tilde{GL}_r} \tilde{M}^{(n)}]$, where \tilde{H} is a maximal subgroup of \tilde{M} such that τ_ω can be extended to \tilde{H} so that Mackey's irreducible criterion is satisfied as constructed in [Mezo 2004, pp. 89–90]. This can be proven in the same way as in [Takeda 2016, Proposition 4.7]. From the construction of \tilde{H} , one can see that \tilde{H} is independent of the choice of τ_1, \dots, τ_k but only dependent on π_1, \dots, π_k . Also see [Cai 2016, Section 3.4] for this issue. \square

The main theorem for local metaplectic tensor product follows easily:

Theorem 3.6. *Keeping the above notation, we have*

$$\Pi = \Pi(\pi_1, \dots, \pi_k) = \bigoplus_{\omega \in \Omega} m(\omega) (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega,$$

where Ω is as in (3.3) and $m(\omega)$ is the positive multiplicity of $(\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$.

Proof. By the previous proposition, we have

$$\Pi = mm' \bigoplus_{\tau} \bigoplus_{\omega \in \Omega(\tau)} (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega,$$

which implies the theorem. \square

Remark 3.7. In the above theorem, one may wonder if $m(\omega) = mm'$. This is certainly the case if the $\Omega(\tau)$ are all distinct for distinct τ . But it may be the case that $\Omega(\tau) \cap \Omega(\tau') \neq \emptyset$ even when $\tau \neq \tau'$. Also it should be mentioned that if $\Omega(\tau) \cap \Omega(\tau') \neq \emptyset$, then necessarily $\Omega(\tau) = \Omega(\tau')$

With this theorem, one can tell that the presentation Π contains all the metaplectic tensor products, and one can call each irreducible constituent of Π a metaplectic tensor product.

Next we consider the behavior of metaplectic tensor products upon restriction to a smaller Levi. Let I, \tilde{M}_I , etc., be as on page 203.

Proposition 3.8. *Let $\pi \subseteq \Pi(\pi_1, \dots, \pi_k)$ be a metaplectic tensor product. Then the restriction $\pi|_{\tilde{M}_I}$ is completely reducible (with most likely infinite multiplicity). Further each constituent of $\pi|_{\tilde{M}_I}$ is of the form $(\pi_{i_1} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{i_l})_{\omega'}$ for some $\omega' \in \Omega(\pi_{i_1}, \dots, \pi_{i_l})$.*

Proof. Note that $\pi \hookrightarrow \text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau$ for some irreducible representation τ of $\tilde{M}^{(n)}$. Hence it suffices to show the restriction $(\text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau)|_{\tilde{M}_I}$ is completely irreducible. But since the group $\tilde{M}^{(n)} \setminus \tilde{M}$ is finite, one has the following Mackey type theorem:

$$(\text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau)|_{\tilde{M}_I} = \bigoplus_{g \in \tilde{M}^{(n)} \setminus \tilde{M}/\tilde{M}_I} \text{Ind}_{\tilde{M}_I \cap g\tilde{M}^{(n)}g^{-1}}^{\tilde{M}_I} (\tau^g),$$

where, as usual, τ^g is the representation of τ twisted by g viewed as a representation of $\tilde{M}_I \cap g\tilde{M}^{(n)}g^{-1}$ by restriction. But note that $\tilde{M}^{(n)} \setminus \tilde{M}/\tilde{M}_I = M^{(n)} \setminus M/M_I$ and each element in this double coset is represented by an element in M which has the identity on all the components for the GL_{r_i} factors for $i \in I$. Hence $\tilde{M}_I \cap g\tilde{M}^{(n)}g^{-1} = \tilde{M}_I^{(n)}$, and $\tau^g = \tau$ as a representation of $\tilde{M}_I^{(n)}$. Hence we have

$$(\text{Ind}_{\tilde{M}^{(n)}}^{\tilde{M}} \tau)|_{\tilde{M}_I} = \bigoplus_{g \in \tilde{M}^{(n)} \setminus \tilde{M}/\tilde{M}_I} \text{Ind}_{\tilde{M}_I^{(n)}}^{\tilde{M}_I} (\tau|_{\tilde{M}_I^{(n)}}).$$

But note that the space V_τ of τ is of the form $V_{\tau_1} \otimes \cdots \otimes V_{\tau_k}$ for some irreducible representations τ_1, \dots, τ_k of $\tilde{\text{GL}}_{r_1}^{(n)}, \dots, \tilde{\text{GL}}_{r_k}^{(n)}$, which are irreducible constituents

of the restrictions $\pi_1|_{\tilde{GL}_{r_1}^{(n)}}, \dots, \pi_k|_{\tilde{GL}_{r_k}^{(n)}}$. Hence when it is restricted to $\tilde{M}_I^{(n)}$, it is completely reducible. (Yet note that the multiplicity is infinite unless all the σ_i for $i \notin I$ are one dimensional.) Indeed $\tau|_{\tilde{M}_I^{(n)}}$ is isotypic with all the irreducible constituents equivalent to $\tau_{i_1} \tilde{\otimes} \dots \tilde{\otimes} \tau_{i_l}$. Hence first of all, $\pi|_{\tilde{M}_I}$ is completely reducible. Second of all, each irreducible constituent in $\pi|_{\tilde{M}_I}$ is contained in the induced representation

$$\text{Ind}_{\tilde{M}_I^{(n)}}^{\tilde{M}} \tau_{i_1} \tilde{\otimes} \dots \tilde{\otimes} \tau_{i_l}.$$

This, together with Theorem 3.6 applied to the group \tilde{M}_I , implies that each constituent of $\pi|_{\tilde{M}_I}$ is of the form $(\pi_{i_1} \tilde{\otimes} \dots \tilde{\otimes} \pi_{i_l})_{\omega'}$. \square

The global metaplectic tensor product. By essentially following the local metaplectic tensor product, the global metaplectic tensor product was constructed in [Takeda 2016] with some technical assumptions, most notably Hypothesis (*) on page 202 of that work. Here we will simplify our previous construction and remove the technical assumptions imposed there. Throughout this subsection, let π_1, \dots, π_k be automorphic subrepresentations of the groups $\tilde{GL}_{r_1}(\mathbb{A}), \dots, \tilde{GL}_{r_k}(\mathbb{A})$ realized in the spaces of automorphic forms. Namely we assume

$$V_{\pi_i} \subseteq \mathcal{A}(\tilde{GL}_{r_i}).$$

Also let

$$H_i := GL_{r_i}(F)\tilde{GL}_{r_i}^{(n)}(\mathbb{A}).$$

Note that by Lemma 2.10, H_i is a closed normal subgroup of $GL_{r_i}(\mathbb{A})$ whose quotient is a compact abelian group.

First let

$$\sigma_i := \pi_i \parallel_{H_i},$$

where we recall the notation \parallel from the notation section. Each element φ in the space of V_{σ_i} is a restriction to H_i of an automorphic form on $\tilde{GL}_{r_i}(\mathbb{A})$, and hence we may view it as a function on H_i with the property that $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in GL_{r_i}(F)$ and $g \in \tilde{GL}_{r_i}^{(n)}(\mathbb{A})$. Namely the representation σ_i is a representation of the group H_i realized in a space of “automorphic forms on H_i ”.

We should mention

Proposition 3.9. *Let π be an irreducible smooth representation of $\tilde{GL}_r(\mathbb{A})$. Then the restriction $\pi|_{GL_r(F)\tilde{GL}_r^{(n)}(\mathbb{A})}$ is completely reducible, and hence $\pi \parallel_{GL_r(F)\tilde{GL}_r^{(n)}(\mathbb{A})}$ is a subrepresentation of $\pi|_{GL_r(F)\tilde{GL}_r^{(n)}(\mathbb{A})}$.*

Proof. In this proof, let us write $H = GL_r(F)\tilde{GL}_r^{(n)}(\mathbb{A})$. We will prove the proposition by modifying the proof of [Gelbart and Knapp 1982, Lemma 2.1].

First we will show that the restriction $\pi|_H$ has an irreducible subrepresentation. For this, consider the contragredient $\hat{\pi}$ of π . Since π is irreducible, so is $\hat{\pi}$. Let

$\varphi \in \hat{\pi}$ be nonzero. Then $\hat{\pi}$ is generated by φ as a representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$. One of the key points is that the restriction $\hat{\pi}|_H$ is also finitely generated as a representation of H . To see it, let S be a sufficiently large finite set of places such that φ is fixed by $K^S := \prod_{v \notin S} \mathrm{GL}_r(\mathcal{O}_{F_v})$. We know that the set $H \backslash \widetilde{\mathrm{GL}}(\mathbb{A})/K^S$ is finite by Lemma 2.11. Let $\{g_1, \dots, g_l\}$ be a complete set of representatives of this finite set. Then one can see that the vectors $\hat{\pi}(g_i)\varphi$ generate $\hat{\pi}|_H$, i.e., $\hat{\pi}|_H$ is finitely generated. Hence $\hat{\pi}|_H$ has an irreducible quotient. (It is an elementary exercise of Zorn's lemma to show that every finitely generated representation of any group has an irreducible quotient.) Let W be the kernel of the surjection from $V_{\hat{\pi}}$ to this irreducible quotient. Let

$$\mathrm{Ann}(W) := \{f \in V_{\pi} : \langle f, \varphi \rangle = 0 \text{ for all } \varphi \in W\}$$

be the annihilator of W , which gives a representation of H . Then one can see $\mathrm{Ann}(W)$ is an irreducible subrepresentation of $\pi|_H$ as follows. Let $X \subseteq \mathrm{Ann}(W)$ be any nonzero subrepresentation of $\mathrm{Ann}(W)$. Consider the annihilator $\mathrm{Ann}(X)$ of X , which is a subrepresentation of $V_{\hat{\pi}}$. Note that $W \subseteq \mathrm{Ann}(X) \subseteq V_{\hat{\pi}}$. But since the pairing $V_{\pi} \times V_{\hat{\pi}} \rightarrow \mathbb{C}$ is nondegenerate, we have $\mathrm{Ann}(X) \neq V_{\hat{\pi}}$. Hence $W = \mathrm{Ann}(X)$ by the irreducibility of $V_{\hat{\pi}}/W$. Hence the pairing $V_{\pi} \times V_{\hat{\pi}} \rightarrow \mathbb{C}$ gives rise to a nondegenerate pairing

$$X \times V_{\hat{\pi}}/W \rightarrow \mathbb{C},$$

which is H invariant. This implies that X is canonically isomorphic to the representation realized in the space

$$\{\langle f, - \rangle : f \in X\},$$

where $\langle f, - \rangle : V_{\hat{\pi}}/W \rightarrow \mathbb{C}$ is the functional given by $\varphi \mapsto \langle f, \varphi \rangle$. But this space is independent of X . Hence $X = \mathrm{Ann}(W)$, which shows $\mathrm{Ann}(W)$ is irreducible.

Now let V be an irreducible subrepresentation of $\pi|_H$ and let $f \in V$ be a fixed nonzero vector. As above there exists a sufficiently large S , possibly (most likely) different from the above one, such that the group K^S fixes f . Again let $\{g_1, \dots, g_l\}$ be a complete set of representatives of $H \backslash \widetilde{\mathrm{GL}}(\mathbb{A})/K^S$, which is most likely different from the one above. Then one can see

$$V_{\pi} = \sum_{i=1}^l \pi(Hg_i K^S) f = \sum_{i=1}^l \pi(Hg_i) f.$$

Note that each space $\pi(Hg_i) f$ gives rise to a representation of H , which is equivalent to the g_i twist of $\pi(H) f$. But $\pi(H) f = V$ because V is a space of an irreducible representation of H , and hence each $\pi(Hg_i) f$ is irreducible.

Let $\{g_{i_1}, \dots, g_{i_N}\}$ be the smallest subset of $\{g_1, \dots, g_l\}$ such that

$$V_\pi = \sum_{j=1}^N \pi(Hg_{i_j})f.$$

This is actually a direct sum because for each $k \in \{1, \dots, N\}$, if the intersection

$$\pi(Hg_{i_k})f \cap \sum_{j \neq k} \pi(Hg_{i_j})f,$$

which is a representation of H , is nonzero, then it is actually equal to $\pi(Hg_{i_k})f$ by irreducibility, which contradicts to the minimality of the set $\{g_{i_1}, \dots, g_{i_N}\}$. This completes the proof. \square

Next note that each element in H_i is of the form (h_i, ξ_i) for $h_i \in GL_{r_i}(F) GL_{r_i}(\mathbb{A})$ and $\xi_i \in \mu_n$. As in [Takeda 2016, p. 215], we have the natural surjection

$$(3.10) \quad H_1 \times \dots \times H_k \rightarrow M(F)\tilde{M}^{(n)}(\mathbb{A})$$

given by the map $((h_1, \xi_1), \dots, (h_k, \xi_k)) \mapsto (h_1 \cdots h_k, \xi_1 \cdots \xi_k)$. Then consider the space

$$V_{\sigma_1} \otimes \dots \otimes V_{\sigma_k}$$

of functions on the direct product $H_1 \times \dots \times H_k$, which gives rise to a representation of the direct product $H_1 \times \dots \times H_k$. But each element in V_σ , which is a function on this direct product, descends to a function on $M(F)\tilde{M}^{(n)}(\mathbb{A})$, which is “automorphic” in the sense that it is left-invariant on $M(F)$. (It should be mentioned that this is not as immediate as it looks, especially due to some issues on cocycles. See [Takeda 2016, Proposition 5.2] for details.) If $\varphi_i \in V_{\sigma_i}$ for $i = 1, \dots, k$, we denote this function by

$$\varphi_1 \tilde{\otimes} \dots \tilde{\otimes} \varphi_k,$$

and denote the space generated by those functions by V_σ . We call each function in V_σ an “automorphic form on $M(F)\tilde{M}^{(n)}(\mathbb{A})$ ”. The group $M(F)\tilde{M}^{(n)}(\mathbb{A})$ acts on V_σ by right-translation, and denote this representation by σ . We define

$$\sigma_1 \tilde{\otimes} \dots \tilde{\otimes} \sigma_k := \sigma.$$

Proposition 3.11. *With the above notation, σ is completely reducible. Further, if all of π_1, \dots, π_k are unitary, so is σ .*

Proof. Each σ_i is completely reducible by Proposition 3.9. Hence one can see σ is completely reducible. If π_1, \dots, π_k are unitary and each σ_i is a subrepresentation of $\pi_i|_{H_i}$, the unitary structure on π_i descends to σ_i . Hence one can define a unitary structure on $\sigma_1 \otimes \dots \otimes \sigma_k$, which descends to σ . \square

Now just as we did for the local case, consider the smooth induced representation

$$(3.12) \quad \Pi = \Pi(\pi_1, \dots, \pi_k) := \text{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \sigma.$$

Then we have the obvious map

$$(3.13) \quad \text{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \sigma \rightarrow \mathcal{A}(\tilde{M}), \quad f \mapsto \tilde{f}$$

where \tilde{f} is defined by

$$\tilde{f}(m) = f(m)(1) \quad \text{for } m \in \tilde{M}(\mathbb{A})$$

and $\mathcal{A}(\tilde{M})$ is the space of automorphic forms on $\tilde{M}(\mathbb{A})$. Further this map is one-to-one, and one can identify Π as a subspace of $\mathcal{A}(\tilde{M})$, namely we have

$$\Pi \subseteq \mathcal{A}(\tilde{M}).$$

Proposition 3.14. *If all of π_1, \dots, π_k are cuspidal, then Π , viewed as a subspace of $\mathcal{A}(\tilde{M})$, is in the space of cusp forms. If all of π_1, \dots, π_k are realized in the spaces of square integrable automorphic forms, then Π is also in the space of square integrable automorphic forms. Also if all of π_1, \dots, π_k are unitary, so is Π .*

Proof. The proofs for the first two parts (cuspidality and square-integrability) are simple modifications of the proofs of Theorems 5.12 and 5.13 of [Takeda 2016]. But let us repeat the key points. For this purpose, note that for $f \in \Pi$ and $m \in \tilde{M}(\mathbb{A})$, we have $f(m) \in V_\sigma$, and hence $f(m)$ is a sum of functions of the form $\varphi_1 \tilde{\otimes} \dots \tilde{\otimes} \varphi_k$, where each φ_i is a restriction of a function in V_{π_i} to H_i .

Assume that π_1, \dots, π_k are cuspidal, and $N := N_1 \times \dots \times N_k$ is a unipotent radical of a parabolic of M , where each N_i is a unipotent radical of a parabolic of GL_{r_i} . We view $N(\mathbb{A})$ as a subgroup of $\tilde{M}(\mathbb{A})$ via the splitting $N(\mathbb{A}) \rightarrow \tilde{M}(\mathbb{A})$. Noting that $N(\mathbb{A}) \subseteq M(F)\tilde{M}^{(n)}(\mathbb{A})$, we have

$$\begin{aligned} \int_{N(F)\backslash N(\mathbb{A})} \tilde{f}(nm) \, dn &= \int_{N(F)\backslash N(\mathbb{A})} f(nm)(1) \, dn \\ &= \int_{N(F)\backslash N(\mathbb{A})} f(m)(n) \, dn \\ &= \sum \int_{N(F)\backslash N(\mathbb{A})} (\varphi_1 \tilde{\otimes} \dots \tilde{\otimes} \varphi_k)(n) \, dn \\ &= \sum \int_{N(F)\backslash N(\mathbb{A})} \varphi_1(n) \dots \varphi_k(n) \, dn \\ &= \sum \int_{N_1(F)\backslash N_1(\mathbb{A})} \varphi_1(n_1) \, dn_1 \dots \int_{N_k(F)\backslash N_k(\mathbb{A})} \varphi_k(n_k) \, dn_k \\ &= 0, \end{aligned}$$

where the last equality follows from the cuspidality of the φ_i .

Next let us show the square integrability. By [Takeda 2016, Lemma 5.17], it suffices to show that

$$\int_{Z_{M(\mathbb{A})}^{(n)} M(F) \backslash M(\mathbb{A})} |\tilde{f}(m)|^2 dm < \infty$$

for each $\tilde{f} \in \Pi$, where

$$Z_{M(\mathbb{A})}^{(n)} = \left\{ \begin{pmatrix} a_1^n I_{r_1} & & \\ & \ddots & \\ & & a_k^n I_{r_k} \end{pmatrix} : a_i \in \mathbb{A}^\times \right\}.$$

But

$$\begin{aligned} & \int_{Z_{M(\mathbb{A})}^{(n)} M(F) \backslash M(\mathbb{A})} |\tilde{f}(m)|^2 dm \\ &= \int_{Z_{M(\mathbb{A})}^{(n)} M(F) M^{(n)}(\mathbb{A}) \backslash M(\mathbb{A})} \int_{Z_{M(\mathbb{A})}^{(n)} M^{(n)}(F) \backslash M^{(n)}(\mathbb{A})} |\tilde{f}(m'm)|^2 dm' dm \\ &= \int_{Z_{M(\mathbb{A})}^{(n)} M(F) M^{(n)}(\mathbb{A}) \backslash M(\mathbb{A})} \int_{Z_{M(\mathbb{A})}^{(n)} M^{(n)}(F) \backslash M^{(n)}(\mathbb{A})} |f(m'm)(1)|^2 dm' dm \\ &= \int_{Z_{M(\mathbb{A})}^{(n)} M(F) M^{(n)}(\mathbb{A}) \backslash M(\mathbb{A})} \int_{Z_{M(\mathbb{A})}^{(n)} M^{(n)}(F) \backslash M^{(n)}(\mathbb{A})} |f(m)(m')|^2 dm' dm, \end{aligned}$$

Note that the outer integral is over a compact set, and hence we only need to show the convergence of the inner integral. But this follows from the square integrability of the function $f(m) \in V_\sigma$ as an “automorphic form on $M(F) \tilde{M}^{(n)}(\mathbb{A})$ ”.

Finally, assume π_1, \dots, π_k are unitary. By Proposition 3.11, we know σ is unitary. But by Lemma 2.11, we know the induction defining Π is a compact induction, which makes Π unitary. \square

Remark 3.15. In the above proof for square integrability, we implicitly used the fact that the group $M(F) Z_{M(\mathbb{A})}^{(n)} M^{(n)}(\mathbb{A})$ is closed, which can be shown by the same argument as Proposition 2.14. This justifies the existence of the quotient measure for $Z_{M(\mathbb{A})}^{(n)} M(F) M^{(n)}(\mathbb{A}) \backslash M(\mathbb{A})$. The author has to admit that this subtle point was not addressed in the proof of [Takeda 2016, Theorem 5.13]. Also there we, for some reason, did not realize that the group $Z_{M(\mathbb{A})}^{(n)} M(F) M^{(n)}(\mathbb{A}) \backslash M(\mathbb{A})$ is compact when writing our previous paper, which made the proof there unnecessarily long.

We would like to have that the representation $\Pi = \Pi(\pi_1, \dots, \pi_k)$ is completely reducible, as in the local case. And this is immediate if π_1, \dots, π_k are cuspidal because then Π is in the space of cusp forms. We do not know if this is true in general, but the following formulation is enough for our purposes:

Proposition 3.16. *Let $\tau \subseteq \sigma$ be an irreducible subspace. Then the space*

$$\mathrm{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \tau$$

has an irreducible subrepresentation. Hence Π has an irreducible subrepresentation.

Proof. In this proof, let us write $H = M(F)\tilde{M}^{(n)}(\mathbb{A})$. First note that since σ is completely reducible by Proposition 3.9, an irreducible $\tau \subseteq \sigma$ always exists. Let $\varphi \in V_\tau$ be nonzero. Since π_1, \dots, π_k are smooth, there exists a finite set of places such that φ is fixed by the group $H \cap K^S$, where $K^S = \prod_{v \notin S} M(\mathcal{O}_{F_v})$. Let g_1, \dots, g_l be a complete set of representatives of the double cosets $H \backslash \tilde{M}(\mathbb{A})/K^S$, which, we know, is finite by Lemma 2.11, where we assume $g_1 = 1$. Hence each vector in $\mathrm{Ind}_H^{\tilde{M}(\mathbb{A})} \tau$ fixed by K^S is completely determined by its values at g_1, \dots, g_l . With this said, let us define an element $f : \tilde{M}(\mathbb{A}) \rightarrow V_\tau$ in $\mathrm{Ind}_H^{\tilde{M}(\mathbb{A})} \tau$ by setting

$$f(hg_i k) = \begin{cases} \tau(h)\varphi & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $h \in H$ and $k \in K^S$. This is well defined because φ is fixed by $H \cap K^S$, and has the property that $f(hm) = \tau(h)f(m)$ for all $h \in H$ and $m \in \tilde{M}(\mathbb{A})$. To show f is indeed in $\mathrm{Ind}_H^{\tilde{M}(\mathbb{A})} \tau$, we need to show that f is smooth. This can be checked at each v by viewing $\tilde{M}(F_v)$ as a subgroup of $\tilde{M}(\mathbb{A})$ as in (2.7) (or its $\tilde{M}(\mathbb{A})$ analogue). Namely for each $v \notin S$, clearly f is fixed by $M(\mathcal{O}_{F_v})$ and hence f is smooth at v . If v is Archimedean, then since the Lie algebra of $\tilde{M}(F_v)$ is the same as that of $\tilde{M}^{(n)}(F_v)$, the smoothness follows from that of φ . Finally let $v \in S$ be a non-Archimedean place in S . Then by the smoothness of φ , there is an open compact subgroup U of $\tilde{M}^{(n)}(F_v)$. Since $\tilde{M}^{(n)}(F_v)$ is an open subgroup of $\tilde{M}(F_v)$, U is also an open compact subgroup of $\tilde{M}(F_v)$. Then one can see that the intersection of all $g_i^{-1}Ug_i$, which is also an open compact subgroup of $\tilde{M}(F_v)$, fixes f . Hence f is smooth and indeed in $\mathrm{Ind}_H^{\tilde{M}(\mathbb{A})} \tau$.

Now consider the space $\Pi(\tilde{M}(\mathbb{A}))f$ generated by f . Then we can write

$$\Pi(\tilde{M}(\mathbb{A}))f = \sum_{i=1}^l \Pi(Hg_i)f,$$

where each space $\Pi(Hg_i)f$ is H invariant and hence a subrepresentation of $\Pi(\tilde{M}(\mathbb{A}))f|_H$. Now to prove the proposition, it suffices to show that $\Pi(Hg_i)f$ is irreducible, because, then, $\Pi(\tilde{M}(\mathbb{A}))f|_H$ has only finite length, and hence *a fortiori* $\Pi(\tilde{M}(\mathbb{A}))f$ is of finite length, which implies that $\Pi(\tilde{M}(\mathbb{A}))f$ has an irreducible subrepresentation. Moreover, one can see that, as abstract representations, each $\Pi(Hg_i)f$ is equivalent to the g_i twist of $\Pi(H)f$. Hence it suffices to show that $\Pi(H)f$ is irreducible.

To show each $\Pi(H)f$ is irreducible, consider the evaluation map at 1, namely

$$\Pi(H)f \rightarrow \tau, \quad f' \mapsto f'(1)$$

for $f' \in \Pi(H)f$, which is H -intertwining. Since $f(1) = \varphi \neq 0$, this map is nonzero. But note that each nonzero f' is supported on HK^S , which implies $f'(1) \neq 0$ for all nonzero $f' \in \Pi(H)f$. Therefore $\Pi(H)f \cong \tau$, which shows $\Pi(H)f$ is irreducible. \square

Now as in the local case, $\Pi(\pi_1, \dots, \pi_k)$ essentially contains all the possible metaplectic tensor products. To see it, we need to carry out a construction analogous to the local metaplectic tensor product of Mezo by taking the central character into account. Namely, we now need to consider an irreducible subrepresentation of σ and extend it to a representation of $Z_{\tilde{GL}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$ by letting the center $Z_{\tilde{GL}_r(\mathbb{A})}$ act as a character.

First note that since σ is completely reducible by Proposition 3.11, it has an irreducible subrepresentation

$$\tau \subseteq \sigma.$$

Fix such τ from now on. We need

Lemma 3.17. *For each irreducible $\tau \subseteq \sigma$, the abelian group*

$$Z_{\tilde{GL}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A})$$

acts as a character, which we denote by ω_τ .

Proof. By Proposition 3.16, there exists an irreducible subrepresentation π of $\text{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{M(\mathbb{A})} \tau$. Let ω be the central character of π . Now by Frobenius reciprocity we have an $M(F)\tilde{M}^{(n)}(\mathbb{A})$ -intertwining map $\pi \rightarrow \tau$, which shows that the group $Z_{\tilde{GL}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A})$ acts via the character ω on τ . \square

Remark 3.18. Of course, if τ is unitary, which is the case if π_1, \dots, π_k are, then τ actually has a central character because $M(F)\tilde{M}^{(n)}(\mathbb{A})$ is locally compact. But the author does not know if τ admits a central character in general.

By the ‘‘automorphy’’ of each element in V_τ , we can see that the character ω_τ in the above lemma is ‘‘automorphic’’ in the sense that

$$\omega_\tau(\gamma z) = \omega_\tau(z)$$

for all $z \in Z_{\tilde{GL}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A})$ and $\gamma \in M(F) \cap (Z_{\tilde{GL}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A}))$. Then we can find a ‘‘Hecke character’’ ω on $Z_{\tilde{GL}_r(\mathbb{A})}$ by extending ω_τ ; namely ω is a character on $Z_{\tilde{GL}_r(\mathbb{A})}$ such that

$$\omega(z) = \omega_\tau(z) \quad \text{for all } z \in Z_{\tilde{GL}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A}).$$

Such ω always exists because both $Z_{\tilde{\text{GL}}_r(\mathbb{A})}$ and $Z_{\tilde{\text{GL}}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A})$ are locally compact abelian groups. Also note that any such ω is indeed a ‘‘Hecke character’’ in the sense that

$$\omega(\gamma z) = \omega(z)$$

for all $z \in Z_{\tilde{\text{GL}}_r(\mathbb{A})}$ and $\gamma \in \text{GL}_r(F) \cap Z_{\tilde{\text{GL}}_r(\mathbb{A})}$, simply because

$$\text{GL}_r(F) \cap Z_{\tilde{\text{GL}}_r(\mathbb{A})} \subseteq Z_{\tilde{\text{GL}}_r(\mathbb{A})} \cap M(F)\tilde{M}^{(n)}(\mathbb{A}).$$

For each $f \in V_\tau$, which is a function on $M(F)\tilde{M}^{(n)}(\mathbb{A})$, we can extend it to a function

$$f_\omega : Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A}) \rightarrow \mathbb{C}$$

by

$$f_\omega(zm) = \omega(z)f(m) \quad \text{for all } z \in Z_{\tilde{\text{GL}}_r(\mathbb{A})} \text{ and } m \in M(F)\tilde{M}^{(n)}(\mathbb{A}).$$

This is well defined because of our choice of ω , and is considered as an ‘‘automorphic form on the group $Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$ ’’. We define

$$V_{\tau_\omega} := \{f_\omega : f \in V_\tau\}.$$

The group $Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$ irreducibly acts on this space, giving rise to an ‘‘automorphic representation’’ τ_ω of $Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$. As abstract representations, we have

$$(3.19) \quad \tau_\omega \cong \omega\tau$$

where by $\omega\tau$ we mean the representation of the group $Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$ extended from τ by letting $Z_{\tilde{\text{GL}}_r(\mathbb{A})}$ act via the character ω .

As we did before, let us consider the smooth induced representation

$$\Pi(\tau_\omega) := \text{Ind}_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \tau_\omega.$$

Note that we have the obvious inclusion

$$\text{Ind}_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \tau_\omega \hookrightarrow \text{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \sigma,$$

which allows us to view $\Pi(\tau_\omega)$ as a subrepresentation of Π realized in the space of automorphic forms on $\tilde{M}(\mathbb{A})$, namely

$$\Pi(\tau_\omega) \subseteq \Pi \subseteq \mathcal{A}(\tilde{M}).$$

Then we have

Proposition 3.20. *The representation $\Pi(\tau_\omega)$ has an irreducible subrepresentation.*

Proof. This can be proven identically to Proposition 3.16. □

Finally, we can define our metaplectic tensor product as follows.

Definition 3.21. Keeping the above notations, let $\pi_\omega \subseteq \Pi(\tau_\omega)$ be an irreducible subrepresentation. Then we write

$$\pi_\omega = (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$$

and call it a metaplectic tensor product of π_1, \dots, π_k with respect to the character ω .

The definition of metaplectic tensor product along with Proposition 3.14 immediately implies the following.

Proposition 3.22. *If all of π_1, \dots, π_k are cuspidal (alternatively, unitary, or square integrable modulo center), then so is $(\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$.*

This π_ω is precisely the metaplectic tensor product we want:

Theorem 3.23. *The representation π_ω constructed above has the desired local-global compatibility. Namely if we write $\pi_\omega = \tilde{\otimes}'_v \pi_{\omega,v}$, then for each v we have*

$$\pi_{\omega,v} = (\pi_{1,v} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v})_{\omega_v}.$$

Thus π_ω is unique up to equivalence, and depends only on π_1, \dots, π_k and ω .

Proof. Note that the uniqueness assertion follows from the corresponding local statement that the local metaplectic tensor product only depends on $\pi_{1,v}, \dots, \pi_{k,v}$ and ω_v . Hence we have only to show the local-global compatibility.

First, note that, since $\pi_\omega \subseteq \text{Ind}_{Z_{\tilde{GL}_r(\mathbb{A})} M(F) \tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \tau_\omega$, we have the natural surjection

$$\pi_\omega|_{Z_{\tilde{GL}_r(\mathbb{A})} M(F) \tilde{M}^{(n)}(\mathbb{A})} \rightarrow \tau_\omega.$$

Recall that as abstract representations, we have $\tau_\omega \cong \omega\tau$, where τ is an irreducible representation of $M(F) \tilde{M}^{(n)}(\mathbb{A})$. So by restricting further down to $Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})$, we have

$$\pi_\omega|_{Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})} \rightarrow \omega\tau|_{Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})}.$$

Now by Lemma A.1 in the Appendix, $\pi_\omega|_{Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})}$ is completely reducible. Hence $\omega\tau|_{Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})}$ is completely reducible. Let

$$\omega\pi^{(n)} \subseteq \omega\tau|_{Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})}$$

be an irreducible subrepresentation, where $\pi^{(n)}$ is an irreducible representation of $\tilde{M}^{(n)}(\mathbb{A})$. By complete reducibility, this is also a quotient, and hence we have a surjection

$$\pi_\omega|_{Z_{\tilde{GL}_r(\mathbb{A})} \tilde{M}^{(n)}(\mathbb{A})} \rightarrow \omega\pi^{(n)}.$$

Recall that τ is realized as a space of “automorphic forms on $M(F)\tilde{M}^{(n)}(\mathbb{A})$ ” and is written as

$$V_\tau = V_{\tau_1} \tilde{\otimes} \cdots \tilde{\otimes} V_{\tau_k},$$

where each V_{τ_i} is a space of restrictions of automorphic forms in the space V_{π_i} . By the automorphy, one can see that

$$\tau_\omega|_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{M}^{(n)}(\mathbb{A})} = \tau_\omega|_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{M}^{(n)}(\mathbb{A})}.$$

Hence we have

$$V_{\pi^{(n)}} = V_{\pi_1^{(n)}} \tilde{\otimes} \cdots \tilde{\otimes} V_{\pi_k^{(n)}},$$

where $V_{\pi_i^{(n)}} \subseteq V_{\tau_i}$ for each i , and indeed we have

$$\pi_i^{(n)} \subseteq \tau_i|_{\tilde{\text{GL}}_{r_i}^{(n)}(\mathbb{A})} = \tau_i|_{\tilde{\text{GL}}_{r_i}^{(n)}(\mathbb{A})} \subseteq \pi_i|_{\tilde{\text{GL}}_{r_i}^{(n)}(\mathbb{A})}.$$

Therefore we can write

$$\omega\pi^{(n)} = \tilde{\otimes}'_v \omega_v(\pi_{1,v}^{(n)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v}^{(n)}),$$

where $\omega_v(\pi_{1,v}^{(n)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v}^{(n)})$ is the irreducible representation of $Z_{\tilde{\text{GL}}_r(F_v)}\tilde{M}^{(n)}(F_v)$ constructed from $\pi_{1,v}^{(n)}, \dots, \pi_{k,v}^{(n)}$ as is done for the local metaplectic tensor product.

Then if we write

$$\pi_\omega = \bigotimes'_v \pi_{\omega,v},$$

where $\pi_{\omega,v}$ is an irreducible representation of $\tilde{M}(F_v)$, we have the surjection

$$\left(\bigotimes'_v \pi_{\omega,v} \right) |_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{M}^{(n)}(\mathbb{A})} \rightarrow \bigotimes'_v \omega_v(\pi_{1,v}^{(n)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v}^{(n)}).$$

Hence by Lemma 5.5 of [Takeda 2016], we conclude that at each place v , the representation $\omega_v(\pi_{1,v}^{(n)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v}^{(n)})$ is a quotient of $\pi_{\omega,v}|_{Z_{\tilde{\text{GL}}_r(F_v)}\tilde{M}^{(n)}(F_v)}$. By Frobenius reciprocity, we have

$$\pi_{\omega,v} \subseteq \text{Ind}_{Z_{\tilde{\text{GL}}_r(F_v)}\tilde{M}^{(n)}(F_v)}^{\tilde{M}(F_v)} \omega_v(\pi_{1,v}^{(n)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v}^{(n)}).$$

Thus by the definition of local metaplectic tensor product, we have

$$\pi_{\omega,v} = (\pi_{1,v} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{k,v})_{\omega_v}.$$

Hence we have the desired local-global compatibility. \square

Remark 3.24. With the theorem, we can say that the notation $(\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega$ is unambiguous in the sense that it only depends on π_1, \dots, π_k and ω as long as we consider the metaplectic tensor product as an equivalence class of representations, which we usually do.

This theorem immediately implies the following.

Corollary 3.25. *For fixed ω , all the irreducible subrepresentations of*

$$\mathrm{Ind}_{Z_{\widetilde{GL}_r(\mathbb{A})}M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \tau_\omega$$

are equivalent.

Next we will show that $\Pi(\pi_1, \dots, \pi_k)$ contains all the possible metaplectic tensor products of π_1, \dots, π_k . For this purpose, let us define

$$\Omega = \Omega(\pi_1, \dots, \pi_k) := \{\omega : \omega \text{ is a Hecke character on } Z_{\widetilde{GL}_r(\mathbb{A})} \text{ which appears in } \sigma\},$$

where we say “ ω appears in σ ” if there exists a nonzero function $\varphi \in \sigma$ such that

$$(3.26) \quad \varphi(zm) = \omega(z)\varphi(m)$$

for all $z \in Z_{\widetilde{GL}_r(\mathbb{A})} \cap M(F)\widetilde{M}^{(n)}(\mathbb{A})$ and $m \in M(F)\widetilde{M}^{(n)}(\mathbb{A})$.

We need

Proposition 3.27. *Let $\omega \in \Omega$ be as above, i.e., ω appears in σ . Then there exists a metaplectic tensor product $\pi_\omega = (\pi_1 \widetilde{\otimes} \dots \widetilde{\otimes} \pi_k)_\omega$ such that $\pi_\omega \subseteq \Pi$.*

Proof. Since ω appears in σ , there exists $\varphi \in V_\sigma$ with the property (3.26). Consider the space $\sigma(M(F)\widetilde{M}^{(n)}(\mathbb{A}))\varphi$ generated by φ inside V_σ . Because each $z \in Z_{\widetilde{GL}_r(\mathbb{A})} \cap M(F)\widetilde{M}^{(n)}(\mathbb{A})$ is in the center of $M(F)\widetilde{M}^{(n)}(\mathbb{A})$, one can see that $\sigma(z)\varphi' = \omega(z)\varphi'$ for all $\varphi' \in \sigma(M(F)\widetilde{M}^{(n)}(\mathbb{A}))\varphi$. Hence if we pick up an irreducible $\tau \subseteq \sigma(M(F)\widetilde{M}^{(n)}(\mathbb{A}))\varphi$, we can extend it to τ_ω , and an irreducible subrepresentation of $\mathrm{Ind}_{Z_{\widetilde{GL}_r(\mathbb{A})}M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \tau_\omega$ is the desired metaplectic tensor product. \square

With this proposition, we can state the global analogue of Proposition 3.5 as follows:

Proposition 3.28. *First we have the decomposition*

$$\Pi(\pi_1, \dots, \pi_k) = \bigoplus_{\tau} m(\tau) \mathrm{Ind}_{M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \tau,$$

where the sum is over all the equivalence classes $\tau \subseteq \sigma$ and $m(\tau)$ is the positive multiplicity of τ in σ . Further for each fixed τ , let

$$\Omega(\tau) := \{\omega \in \Omega : \omega|_{Z_{\widetilde{GL}_r(\mathbb{A})} \cap M(F)\widetilde{M}^{(n)}(\mathbb{A})} = \tau|_{Z_{\widetilde{GL}_r(\mathbb{A})} \cap M(F)\widetilde{M}^{(n)}(\mathbb{A})}\}.$$

Then we have

$$\mathrm{Ind}_{M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \tau \supseteq \bigoplus_{\omega \in \Omega(\tau)} \mathrm{Ind}_{Z_{\widetilde{GL}_r(\mathbb{A})}M(F)\widetilde{M}^{(n)}(\mathbb{A})}^{\widetilde{M}(\mathbb{A})} \tau_\omega \supseteq \bigoplus_{\omega \in \Omega(\tau)} m(\tau, \omega) (\pi_1 \widetilde{\otimes} \dots \widetilde{\otimes} \pi_k)_\omega,$$

where $m(\tau, \omega)$ is the positive multiplicity.

Proof. The proposition can be proven in the same way as the local case. Yet, we should mention that unlike the local case, we do not seem to know the precise information on the multiplicities. \square

From the proposition, the following is immediate.

Theorem 3.29. *We have*

$$\Pi(\pi_1, \dots, \pi_k) \supseteq \bigoplus_{\omega \in \Omega} m(\omega)(\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega,$$

where $m(\omega)$ is the multiplicity of $(\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega$. Also if all of π_1, \dots, π_k are cuspidal, then the inclusion is actually an equality.

Proof. The first part is obvious from the above proposition. The second part follows from Proposition 3.14 because if Π is in the cuspidal spectrum, it is completely reducible. \square

Let us note that if we know the multiplicity-one property for the group $\tilde{M}(\mathbb{A})$, we could set $m(\omega) = 1$. Yet, the author does not know if the multiplicity-one property holds even for the cuspidal spectrum.

Restriction to a smaller Levi. As we did for the local case, we will discuss the restriction of our metaplectic tensor products to a smaller Levi \tilde{M}_I , where M_I is as in (2.4). Recall

$$\sigma = \sigma_1 \tilde{\otimes} \dots \tilde{\otimes} \sigma_k,$$

whose space V_σ is essentially identified with $V_{\sigma_1} \otimes \dots \otimes V_{\sigma_k}$, which is a space of functions on the direct product $H_1 \times \dots \times H_k$. Hence if we let

$$\sigma_I := \sigma_{i_1} \tilde{\otimes} \dots \tilde{\otimes} \sigma_{i_l},$$

one can see that if $\varphi \in V_\sigma$, we have $\varphi|_{M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})} \in V_{\sigma_I}$. Indeed, we have an $M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})$ -intertwining surjection

$$\sigma \rightarrow \sigma_I, \quad \varphi \mapsto \varphi|_{M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})}.$$

In other words, we have $\sigma_I = \sigma|_{M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})}$.

Now we can prove

Theorem 3.30. *For each $\pi_\omega = (\pi_1 \tilde{\otimes} \dots \tilde{\otimes} \pi_k)_\omega \subseteq \Pi(\pi_1, \dots, \pi_k)$, we have*

$$\pi_\omega|_{\tilde{M}_I(\mathbb{A})} \subseteq \bigoplus_{\omega' \in \Omega_I} m(\omega')(\pi_{i_1} \tilde{\otimes} \dots \tilde{\otimes} \pi_{i_l})_{\omega'},$$

where $\Omega_I = \Omega(\pi_{i_1}, \dots, \pi_{i_l})$ is defined analogously to Ω .

Proof. Let us view π_ω as a subspace of the induced space of (3.12). Then we have the commutative diagram

$$\begin{array}{ccccc}
 \pi_\omega \subseteq \text{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \sigma & \hookrightarrow & \mathcal{A}(\tilde{M}) & & \\
 \downarrow & & \downarrow & & \\
 \pi_\omega|_{\tilde{M}_I(\mathbb{A})} \subseteq \text{Ind}_{M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})}^{\tilde{M}_I(\mathbb{A})} \sigma_I & \hookrightarrow & \mathcal{A}(\tilde{M}_I) & &
 \end{array}$$

where all the vertical arrows are restriction of functions and the hooked arrow on the top is the “automorphic realization map” as in (3.13), and the one on the bottom is its analogue for \tilde{M}_I . By Lemma A.1, we know that $\pi_\omega|_{\tilde{M}_I(\mathbb{A})}$ is completely irreducible, and hence so is $\pi_\omega|_{\tilde{M}_I(\mathbb{A})}$. Note that every irreducible subrepresentation of $\text{Ind}_{M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})}^{\tilde{M}_I(\mathbb{A})} \sigma_I$ is a metaplectic tensor product of $\pi_{i_1}, \dots, \pi_{i_l}$ with respect to some ω' . Hence the theorem follows by identifying $\text{Ind}_{M_I(F)\tilde{M}_I^{(n)}(\mathbb{A})}^{\tilde{M}_I(\mathbb{A})} \sigma_I$ with a subspace of $\mathcal{A}(\tilde{M}_I)$. \square

Other properties. In [Takeda 2016], a couple of other properties of metaplectic tensor product are discussed. To be precise, they have the expected behavior under the Weyl group action and the compatibility with parabolic induction, which are, respectively, Theorems 5.19 and 5.22 in that work. But both of them follow from the corresponding local statements, and hence they also hold in our new construction.

It should also be mentioned that recently it has been shown by W. T. Gan [2016] that the metaplectic tensor product can be interpreted as an instance of Langlands functoriality by using the L -group formalism of covering groups developed by Weissman. (See [Weissman 2016; Gan and Gao 2014] for this formalism.) This shows that the construction of the metaplectic tensor product is indeed a natural one.

Some remarks on past literature. The notion of metaplectic tensor product has been implicitly used in many of the past works on automorphic forms on $\tilde{GL}_r(\mathbb{A})$, especially when one would like to construct Eisenstein series on $\tilde{GL}_r(\mathbb{A})$. But there are various discrepancies in the past literature in this subject, which, we believe, was due to the lack of a foundation on the metaplectic tensor product. In this final subsection, let us briefly discuss some of the previous works and how they can be reconciled with the theory developed in this paper.

The first work that considered Eisenstein series on $\tilde{GL}_r(\mathbb{A})$ is, of course, the important work of Kazhdan and Patterson [1984]. There they only considered those Eisenstein series which are induced from the Borel subgroup B . Namely they only considered the case $M = GL_1 \times \dots \times GL_1$. In this case, one can show that the group $Z_{\tilde{GL}_r(\mathbb{A})} M(F)\tilde{M}^{(n)}(\mathbb{A})$ is a maximal abelian subgroup of \tilde{M} , and accordingly, from the outset they considered a character on $Z_{\tilde{GL}_r(\mathbb{A})} M(F)\tilde{M}^{(n)}(\mathbb{A})$

instead of starting with characters on $\tilde{\text{GL}}_1(\mathbb{A})$ (see their Section II.1). Yet, one can see that this is the same as constructing some τ_ω in our notation. Then Kazhdan and Patterson considered the induced representation $\text{Ind}_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})N_B}^{\tilde{\text{GL}}_r(\mathbb{A})} \tau_\omega$, to construct Eisenstein series. By inducing in stages,

$$\text{Ind}_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})N_B}^{\tilde{\text{GL}}_r(\mathbb{A})} \tau_\omega = \text{Ind}_{\tilde{B}(\mathbb{A})}^{\tilde{\text{GL}}_r(\mathbb{A})} \text{Ind}_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})N_B}^{\tilde{M}(\mathbb{A})N_B} \tau_\omega,$$

and hence the metaplectic tensor product that is implicitly used in [Kazhdan and Patterson 1984] is our $\Pi(\tau_\omega) = \text{Ind}_{Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \tau_\omega$. Further the fact that $Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$ is maximal abelian implies that $\Pi(\tau_\omega)$ is irreducible (see Section 0.3 in the same reference).

The next important set of works on this subject is probably the one by Bump and Ginzburg [1992] on the symmetric square L -function, and the work by Banks [1997] on the twisted case for GL_3 , both of which dealt with only the case $n = 2$. There are two main parabolic subgroups considered there: the Borel and the $(r - 1, 1)$ -parabolic. For the Borel, they use the same formulation as [Kazhdan and Patterson 1984]. For the $(r - 1, 1)$ -parabolic, they first start with a representation of $\tilde{\text{GL}}_{r-1}(\mathbb{A})$ viewed as a subgroup of $\tilde{\text{GL}}_r(\mathbb{A})$ and they extend it to a representation of $Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{\text{GL}}_{r-1}(\mathbb{A})$ by letting $Z_{\tilde{\text{GL}}_r(\mathbb{A})}$ act by an appropriate character. Now if r is odd (and $n = 2$), this gives a representation of $\tilde{\text{GL}}_{r-1}(\mathbb{A}) \tilde{\times} \tilde{\text{GL}}_1(\mathbb{A})$ because $Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{\text{GL}}_{r-1}(\mathbb{A}) = \tilde{\text{GL}}_{r-1}(\mathbb{A}) \tilde{\times} \tilde{\text{GL}}_1(\mathbb{A})$. But if r is even, we only have $Z_{\tilde{\text{GL}}_r(\mathbb{A})}\tilde{\text{GL}}_{r-1}(\mathbb{A}) = \tilde{\text{GL}}_{r-1}(\mathbb{A}) \tilde{\times} \tilde{\text{GL}}_1^{(2)}(\mathbb{A})$. Then, in [Bump and Ginzburg 1992], they induced the representation of $\tilde{\text{GL}}_{r-1}(\mathbb{A}) \tilde{\times} \tilde{\text{GL}}_1^{(2)}(\mathbb{A})$ to $\tilde{\text{GL}}_{r-1}(\mathbb{A}) \tilde{\times} \tilde{\text{GL}}_1(\mathbb{A})$ (see the middle of page 159 in the same reference.) However, it seems to the author that one cannot show the automorphy of this induced representation if it is simply induced from $\tilde{\text{GL}}_{r-1}(\mathbb{A}) \tilde{\times} \tilde{\text{GL}}_1^{(2)}(\mathbb{A})$, and probably this is another technical issue to be addressed in that work. At any rate, one can see that at least if r is odd this construction is also obtained as our metaplectic tensor product, say, by first restricting to $Z_{\tilde{\text{GL}}_r(\mathbb{A})}M(F)\tilde{M}^{(n)}(\mathbb{A})$ and then inducing one of the constituents to $\tilde{M}(\mathbb{A})$. It should be also mentioned that in [Bump and Ginzburg 1992; Banks 1997] various properties of metaplectic tensor product, such as the behavior of metaplectic tensor product upon restriction to a smaller Levi, are implicitly used.

From those two works, the author generalized in [Takeda 2014], in which the parabolic subgroups considered are mainly $(2, \dots, 2)$ and $(r - 1, 1)$ parabolic. For the $(2, \dots, 2)$ -parabolic, the inducing representation for each $\tilde{\text{GL}}_2$ factor is only the Weil representation, and hence by using the Schrödinger model for $\tilde{\text{GL}}_2^{(2)}$, we explicitly constructed what we called the “Weil representation of \tilde{M}_P ”. One can see that this is also an instance of our metaplectic tensor product, because for the case at hand we have $Z_{\tilde{\text{GL}}_r(\mathbb{A})} \subseteq \tilde{M}^{(n)}(\mathbb{A})$, which means that the central character does not play any role in the formation of metaplectic tensor product and hence the metaplectic tensor product only depends locally on restrictions to

$\tilde{M}^{(n)}(F_v)$. For the $(r-1, 1)$ -parabolic case in [Takeda 2014], however, depending on the parity of r , we took a different approach. For r odd, we did just as in [Bump and Ginzburg 1992; Banks 1997]. For r even, we directly constructed a representation of the Levi $\tilde{GL}_{r-1} \times \tilde{GL}_1$ as residues of Eisenstein series induced from the Borel, instead of starting with representations of \tilde{GL}_{r-1} and \tilde{GL}_1 separately. One can show that this construction is the same as our metaplectic tensor product by using the compatibility of our metaplectic tensor product with parabolic induction as discussed in the previous subsection.

Besides those applications to symmetric square L -functions, the works of Suzuki [1997; 1998] should be mentioned. In the first of these works, he considered the (r_1, r_2) -parabolic for $r_1 + r_2 = r$. To construct an automorphic form on the Levi part, he uses what he calls “partial Eisenstein series” (see his Sec. 5.4). This construction is essentially the same as the $r = \text{even}$ case of [Takeda 2014] mentioned above, and again our metaplectic tensor product encompasses this construction of Suzuki. Also in [Suzuki 1998], Eisenstein series induced from the (ℓ, \dots, ℓ) -parabolic are considered. There it seems that what he considers is our Π , namely the whole induced representation $\text{Ind}_{M(F)\tilde{M}^{(n)}(\mathbb{A})}^{\tilde{M}(\mathbb{A})} \sigma$. Yet, it should be mentioned that first of all he assumes that each automorphic representation of $\tilde{GL}_\ell(\mathbb{A})$ is already induced from $GL_\ell(F)\tilde{GL}_\ell^{(n)}(\mathbb{A})$ (page 750 of that work), and second of all it is claimed, without proof, that the representation on the Levi thus constructed is irreducible with local-global compatibility. (See the beginning of [Suzuki 1998, p. 752].) At any rate, since no proofs or no detailed explanations are given for his assertions, it is not completely clear to the author that what kind of construction is carried out there and even that his construction is legitimate.

Finally, more recently Brubaker and Friedberg [2015] considered metaplectic Eisenstein series not just on the group \tilde{GL}_r but on other covering groups in general. Although they use the language of “ S -integers”, what they use to construct representations of the Levi amounts to our Π , the whole induced representation. Also the same convention is used in the even more recent [Friedberg and Ginzburg 2016].

Probably which convention to use might be a matter of taste or the nature of the problem one works on. But it seems to the author that for the purpose of constructing Eisenstein series, using the whole induced space Π , which contains all the metaplectic tensor products, is an easy choice, especially because then, the inducing data is essentially the same as the usual tensor product. One should, however, be careful that usually the representation $\Pi = \Pi(\pi_1, \dots, \pi_k)$ is reducible. Hence for example we do not know if we can express it as a restricted tensor product as $\Pi = \widetilde{\bigotimes}'_v \Pi_v$, which is often crucial when one would like to find out analytic properties of intertwining operators. Therefore, it might be more convenient to pick an irreducible subrepresentation $\pi_\omega \subseteq \Pi$, although this requires one to take care of the dependence of π_ω on ω . Nonetheless, probably many of the important properties (especially

analytic ones) of Eisenstein series constructed from different π_ω might usually be independent of ω , because, after all, the characters ω differ by characters on $Z_{\widetilde{\text{GL}}(\mathbb{A})} \cap M(F) \widetilde{M}^{(n)}(\mathbb{A}) \backslash Z_{\widetilde{\text{GL}}_r(\mathbb{A})}$, which is compact, and hence it seems unlikely that a difference in a character on a compact group affects analytic properties of Eisenstein series. Indeed, for example, in [Takeda 2015] the author studied some analytic properties of some Eisenstein series using the formalism of metaplectic tensor product of [Takeda 2016], and all the results there hold independently of the choice of ω .

Appendix: A lemma on complete reducibility

We prove the result used on pages 219 and 223.

Lemma A.1. *Let $\tilde{\pi} = \bigotimes'_v \pi_v$ be an irreducible admissible representation of $\widetilde{M}(\mathbb{A})$. Let \tilde{H} be a group of the form $\tilde{H} = \prod'_v H_v$, where $H_v \subseteq \widetilde{M}(F_v)$ and the restricted direct product is with respect to the group $H_v \cap M(\mathcal{O}_{F_v})$. (The groups $\widetilde{M}^{(n)}(\mathbb{A})$, $Z_{\widetilde{\text{GL}}_r(\mathbb{A})} \widetilde{M}^{(n)}(\mathbb{A})$ and $\widetilde{M}_I(\mathbb{A})$ are such examples of \tilde{H} .) Further assume that for each v , the restriction $\pi_v|_{H_v}$ is completely reducible. Then the restriction $\tilde{\pi}|_{\tilde{H}}$ is completely reducible.*

Proof. We argue “semilocally” using the definition of the restricted metaplectic tensor product $\tilde{\pi} = \bigotimes'_v \pi_v$. First note that the space of $\bigotimes'_v \pi_v$ is actually $\bigotimes'_v V_{\pi_v}$ (usual restricted tensor product) on which not only the group $\prod'_v \widetilde{M}(F_v)$, but also $\prod'_v \widetilde{M}(F_v)$ acts. Accordingly we set

$$\pi := \bigotimes'_v \pi_v \quad (\text{usual restricted tensor product}),$$

$$H := \prod'_v H_v \quad (\text{usual restricted direct product}),$$

and it suffices to show that the restriction $\pi|_H$ is completely reducible.

Now let us recall the definition of $\bigotimes'_v \pi_v$. For almost all v , we choose a spherical vector $\xi_v^\circ \in \pi_v$. Let S be a sufficiently large finite set of places so that each π_v is spherical for $v \notin S$. Let

$$\pi_S = \bigotimes_{v \in S} \pi_v,$$

which gives a representation of $\prod_{v \in S} \widetilde{M}(F_v)$. For each $S' \supseteq S$ we have the inclusion $\pi_S \rightarrow \pi_{S'}$ by tensoring the chosen spherical vectors ξ_v° for $v \in S' \setminus S$. Then the system $\{\pi_S\}_S$ is a directed system and by definition $\bigotimes'_v \pi_v = \varinjlim_S \pi_S$. For each S , let us define $H_S := \prod_{v \in S} H_v$. Then one can see that

$$\pi|_H = \varinjlim_S \pi_S|_{H_S}.$$

For each v , the restriction $\pi_v|_{H_v}$ is completely reducible by our assumption.

Hence let us fix the decomposition

$$\pi_v|_{H_v} = \bigoplus_{i_v \in I_v} \pi_{i_v}$$

for some finite indexing set I_v , where each π_{i_v} is irreducible. (We do not assume that the restriction $\pi_v|_{H_v}$ is multiplicity free, and hence this decomposition might not be unique even up to ordering. So we “fix” the decomposition for each $\pi_v|_{H_v}$ once and for all.) We let

$$\text{pr}_{i_v} : \pi_v \rightarrow \pi_{i_v}$$

be the projection map. Further we let

$$I_v^\circ := \{i_v \in I_v : \text{pr}_{i_v}(\xi_v^\circ) \neq 0\}.$$

Note that if $i_v \in I_v^\circ$ then π_{i_v} is spherical in the sense that it contains a vector fixed by $H_v \cap M(\mathcal{O}_{F_v})$. (The author does not know if I_v° has only one element, and probably it does have more than one in general. This makes the following argument a bit delicate.) Let us define

$$I_S = \prod_{v \in S} I_v \quad \text{and} \quad I = \prod'_v I_v = \left\{ i \in \prod_v I_v : i_v \in I_v^\circ \text{ for almost all } v \right\},$$

where for each $i \in I$, we denote its v -th component by i_v . Namely I is the restricted direct product of I_v with respect to I_v° . For each $i \in I$, we write

$$i_S := (i_v)_{v \in S} \in I_S.$$

With this notation, we can write

$$\pi_S|_{H_S} = \bigoplus_{i \in I_S} \pi_{i_S},$$

where $\pi_{i_S} = \bigotimes_{v \in S} \pi_{i_v}$.

Now for each $i \in I$, let us define

$$\pi_i := \lim_{\substack{\longrightarrow \\ S}} \pi_{i_S}$$

by using $\text{pr}_{i_v}(\xi_v^\circ)$ for our spherical vector for $i_v \in I_v^\circ$. Note that each π_i is an irreducible representation of H . To prove the lemma, it suffices to show we have an isomorphism

$$(A.2) \quad \lim_{\substack{\longrightarrow \\ S}} \pi_S|_{H_S} \cong \bigoplus_{i \in I} \lim_{\substack{\longrightarrow \\ S}} \pi_{i_S},$$

namely $\pi|_H \cong \bigoplus_{i \in I} \pi_i$, which will show that $\pi|_H$ is completely reducible. To show there is such an isomorphism, first note that for each $S \subseteq S'$, the following

diagram commutes:

(A.3)

$$\begin{array}{ccc} \pi_S|_{H_S} & \longrightarrow & \pi_{S'}|_{H_{S'}} \\ \downarrow & & \downarrow \\ \bigoplus_{i_S \in I_S} \pi_{i_S} & \longrightarrow & \bigoplus_{i_{S'} \in I_{S'}} \pi_{i_{S'}} \end{array}$$

where the vertical arrows are actually an equality and the top horizontal arrow is given by tensoring with $\otimes_{v \in S' \setminus S} \xi_v^\circ$ and the bottom horizontal arrow is given as follows: for each $i_S \in I_S$, define a map

$$\pi_{i_S} \rightarrow \pi_{i_S} \otimes \bigoplus_{i_{S'} \setminus S \in I_{S'} \setminus S} \pi_{i_{S'} \setminus S} = \pi_{i_S} \otimes \bigotimes_{v \in S' \setminus S} \bigoplus_{i_v \in I_v} \pi_{i_v}$$

by

$$v_{i_S} \mapsto v_{i_S} \otimes \bigotimes_{v \in S' \setminus S} \bigoplus_{i_v \in I_v} \text{pr}_{i_v}(\xi_v^\circ),$$

where recall that pr_{i_v} is the projection from π_v to π_{i_v} . Then the bottom horizontal arrow is given by combining all those maps for all the i_S .

Next, one can see that for each S there is an obvious injection

(A.4)

$$\bigoplus_{i \in I_S} \pi_{i_S} \hookrightarrow \bigoplus_{i \in I} \lim_{\rightarrow S} \pi_{i_S},$$

which makes the diagram

$$\begin{array}{ccc} \bigoplus_{i_S \in I_S} \pi_{i_S} & \longrightarrow & \bigoplus_{i_{S'} \in I_{S'}} \pi_{i_{S'}} \\ & \searrow & \swarrow \\ & \bigoplus_{i \in I} \lim_{\rightarrow S} \pi_{i_S} & \end{array}$$

commute. This diagram and the diagram (A.3) together with the universal property of $\lim_{\rightarrow S} \pi_S|_{H_S}$ give a unique map

$$T : \pi|_H = \lim_{\rightarrow S} \pi_S|_{H_S} \longrightarrow \bigoplus_{i \in I} \lim_{\rightarrow S} \pi_{i_S} = \bigoplus_{i \in I} \pi_i,$$

which “commutes with the directed system”. This map is injective because each $\varphi \in \pi$ is in $\varphi \in \pi_S$ for some S , which maps to $\bigoplus_{i \in I} \pi_i$ via the map in (A.4), and hence there is no kernel for T . Also one can see that T is surjective because if $\varphi_i \in \pi_i$, one can find S such that $\varphi_i \in \pi_{i_S}$, which comes from some vector π_S under the vertical map in (A.3). This completes the proof. \square

Remark A.5. As our last remark, let us mention that in [Takeda 2016, Lemma 5.1], it is erroneously claimed that the complete reducibility of a unitary automorphic representation of $\widetilde{GL}_r(\mathbb{A})$ to $\widetilde{GL}_r^{(n)}(\mathbb{A})$ follows from the admissibility and unitarity, but actually the restriction to $\widetilde{GL}_r^{(n)}(\mathbb{A})$ is most likely not admissible, and hence the argument in the proof there does not work. But the proof of the above lemma, we hope, fixes the mistake.

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SHUICHIRO TAKEDA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI
202 MATH SCIENCE BLDG
COLUMBIA, MO 65211
UNITED STATES
takedas@missouri.edu

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Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

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