Pacific Journal of Mathematics

A DIRECT METHOD OF MOVING PLANES FOR THE SYSTEM OF THE FRACTIONAL LAPLACIAN

CHUNXIA CHENG, ZHONGXUE LÜ AND YINGSHU LÜ

Volume 290 No. 2 October 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.301

A DIRECT METHOD OF MOVING PLANES FOR THE SYSTEM OF THE FRACTIONAL LAPLACIAN

CHUNXIA CHENG, ZHONGXUE LÜ AND YINGSHU LÜ

We establish a direct method of moving planes for systems of fractional Laplacian equations. By using this direct method of moving planes, we obtain symmetry and nonexistence of positive solutions for the following system of fractional Laplacian equations:

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n. \end{cases}$$

1. Introduction

In this paper, we consider the following system of fractional Laplacian equations:

(1-1)
$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n. \end{cases}$$

When $\alpha = 2$, system (1-1) is an important model, the Lane–Emden system. It is conjectured that if $1/(p+1) + 1/(q+1) > (n-2)/n$, then there are no nontrivial classical solutions of (1-1) in $\mathbb{R}^N$ with $N \geq 3$. The conjecture has been proved to be true for radial solutions in all dimensions in [Mitidieri 1996]. The cases of $N = 3, 4$ for the conjecture in general have also been solved recently in [Poláčik et al. 2007] and [Souplet 2009], respectively. The interested reader can refer to the above papers for detailed descriptions (see also the works [Busca and Manásevich 2002; Serrin and Zou 1998], etc.).

More generally, Troy [1981] used the *maximum principle* and the *method of moving parallel planes* to investigate symmetry properties of solutions of systems of semilinear elliptic equations $\Delta u_i + f_i(u_1, \dots, u_n) = 0,\ i = 1, \dots, n$, in a domain of $\mathbb{R}^n$.

The work was partially supported by NSFC(No.11271166), sponsored by Qing Lan Project.
MSC2010: 35B09, 35B50, 35B53, 35J61.
Keywords: the fractional Laplacian, maximum principles for antisymmetric functions, narrow region principle, decay at infinity, method of moving planes, radial symmetry, nonexistence of positive solutions.

In the special case $p=q,\ u=v$, (1-1) changes to

$$(-\Delta)^{\alpha/2}u(x)=u^p(x),\quad x\in\mathbb{R}^n. \tag{1-2}$$

Here the fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudodifferential operator assuming the form

$$(-\Delta)^{\alpha/2}u(x)=C_{n,\alpha}\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{u(x)-u(z)}{|x-z|^{n+\alpha}}\,dz, \tag{1-3}$$

where α is any real number between 0 and 2. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^n$. In this space, it can also be equivalently defined in terms of the Fourier transform

$$\widehat{(-\Delta)^{\alpha/2}}u(\xi)=|\xi|^\alpha\hat{u}(\xi),$$

where $\hat{u}$ is the Fourier transform of u. One can extend this operator to a wider space of functions.

Let

$$L_\alpha=\left\{u:\mathbb{R}^n\to\mathbb{R}\;\middle|\;\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+\alpha}}\,dx<\infty\right\}.$$

Then it is easy to verify that for $u\in L_\alpha\cap C^{1,1}_{\mathrm{loc}}$, the integral on the right-hand side of (1-3) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

The nonlocality of the fractional Laplacian makes it difficult to study. To circumvent this difficulty, Caffarelli and Silvestre [2007] introduced the *extension method*, which reduced this nonlocal problem into a local one in higher dimensions. For a function $u:\mathbb{R}^n\to\mathbb{R}$, consider the extension $U:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ that satisfies

$$\begin{cases}\operatorname{div}(y^{1-\alpha}\nabla U)=0, & (x,y)\in\mathbb{R}^n\times[0,\infty),\\ U(x,0)=u(x).\end{cases}$$

Then

$$(-\Delta)^{\alpha/2}u=-C_{n,\alpha}\lim_{y\to 0^+}y^{1-\alpha}\frac{\partial U}{\partial y}.$$

This extension method has been applied successfully to study equations involving the fractional Laplacian, and a series of fruitful results have been obtained (see the references in that work).

In [Busca and Manásevich 2002], among many interesting results, when the authors considered the properties of the positive solutions for (1-2), they first used the above extension method to reduce the nonlocal problem into a local one for $U(x,y)$ in one higher dimensional half space $\mathbb{R}^n\times[0,\infty)$, then applied the method of moving planes to show the symmetry of $U(x,y)$ in x, and hence derived the nonexistence in the subcritical case.

Proposition 1.1. *Let* $1 \leq \alpha < 2$. *Then the problem*

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ -\lim\limits_{y\to 0^+} y^{1-\alpha}\frac{\partial U}{\partial y} = U^p(x, 0), & x \in \mathbb{R}^n, \end{cases}$$

has no positive bounded solution provided $p < (n+\alpha)/(n-\alpha)$.

They then took trace to obtain:

Corollary 1.2. *Assume that* $1 \leq \alpha < 2$ *and* $1 < p < (n-\alpha)/(n-\alpha)$. *Then equation* (1-2) *possesses no bounded positive solution.*

A similar extension method was adapted in [Chen and Zhu 2016] to obtain the nonexistence of positive solutions for an indefinite fractional problem.

Proposition 1.3. *Let* $1 \leq \alpha < 2$ *and* $1 < p < \infty$. *Then the equation*

$$(-\Delta)^{\alpha/2} = x_1 u^p, \quad x \in \mathbb{R}^n,$$

possesses no positive bounded solutions.

The common restriction $\alpha \geq 1$ is due to the approach that they need to carry out the method of moving planes on the solutions U of the extended problem

$$\operatorname{div}(y^{1-\alpha}\nabla U) = 0, \quad (x, y) \in \mathbb{R}^n \times [0, \infty). \tag{1-4}$$

Because of the monotonicity requirement, they have to assume that $\alpha \geq 1$.

Jarohs and Weth [2016] without going through the extended equation (1-4), introduced *antisymmetric maximum principles* and applied them to carry on the method of moving planes directly on nonlocal problems to show the symmetry of solutions. The operators they considered are quite general; however, their maximum principles only apply to bounded regions.

Chen, Li and Li [Chen et al. 2017] developed a systematic approach to carry out the method of moving planes for nonlocal problems, either on bounded or unbounded domains, corresponding to approaches for local elliptic operators that were introduced more than twenty years ago in the paper [Chen and Li 1991] and then summarized in the book [Chen and Li 2010].

In this paper, we will establish the direct method of moving planes for the system of the fractional Laplacian equations. This will be accomplished in Section 2, in which the main results are the following:

Theorem 2.1 (maximum principle for antisymmetric functions). *Let* T *be a hyperplane in* $\mathbb{R}^n$. *Without loss of generality, we may assume that*

$$T = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \quad \textit{for some } \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \ldots, x_n)$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \quad and \quad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded domain in H. Assume that $u \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ and is lower semicontinuous on $\bar{\Omega}$. If

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) \geq 0 & in\ \Omega, \\ (-\Delta)^{\alpha/2} v(x) \geq 0 & in\ \Omega, \\ u(x) \geq 0 \quad and \quad v(x) \geq 0 & in\ H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & in\ H, \\ v(\tilde{x}) = -v(x) & in\ H, \end{cases}$$

then

$$u(x) \geq 0 \quad and \quad v(x) \geq 0 \quad in \quad \Omega.$$

This conclusion holds for unbounded region Ω if we further assume that

$$\varliminf_{|x|\to\infty} u(x) \geq 0 \quad and \quad \varliminf_{|x|\to\infty} v(x) \geq 0.$$

If $u = 0$ and $v = 0$ at some point in Ω, then

$$u(x) = 0 \quad and \quad v(x) = 0 \quad almost\ everywhere\ in\ \mathbb{R}^n.$$

Theorem 2.2 (narrow region principle). *Let T be a hyperplane in $\mathbb{R}^n$. Without loss of generality,we may assume that*

$$T = \{x = (x^1, x') \in \mathbb{R}^n \mid x_1 = \lambda \quad for\ some\ \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \dots, x_n),$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \quad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded narrow region in H such that it is contained in $\{x \mid \lambda - l < x_1 < \lambda\}$ with small l. Suppose that $u, v \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ and both are lower semicontinuous on $\bar{\Omega}$. If $c_1(x)$ and $c_2(x)$ are both bounded from below in Ω, $c_1(x) \leq 0$ and $c_2(x) \leq 0$ and

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x) v(x) \geq 0 & in\ \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x) u(x) \geq 0 & in\ \Omega, \\ u(x) \geq 0 \quad and \quad v(x) \geq 0 & in\ H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & in\ H, \\ u(\tilde{x}) = -u(x) & in\ H, \end{cases}$$

then for sufficiently small l, we have

$$u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 \quad \text{in} \quad \Omega.$$

This conclusion holds for unbounded regions Ω if we further assume that

$$\varliminf_{|x|\to\infty} u(x) \geq 0 \quad \text{and} \quad \varliminf_{|x|\to\infty} v(x) \geq 0.$$

Theorem 2.3 (decay at infinity). *Let $H = \{x \in \mathbb{R}^n \mid x_1 < \lambda \quad \text{for some} \quad \lambda \in \mathbb{R}\}$ and let Ω be an unbounded region in H. Assume*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x) v(x) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x) u(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ v(\tilde{x}) = -v(x) & \text{in } H, \end{cases}$$

with

$$\varliminf_{|x|\to\infty} |x|^\alpha c_1(x) = 0, \quad c_1(x) \leq 0,$$

and

$$\varliminf_{|x|\to\infty} |x|^\alpha c_2(x) = 0, \quad c_2(x) \leq 0,$$

then there exists a constant R_0 such that if

$$u(x^0) = \min_\Omega u(x) < 0 \quad \text{or} \quad v(x^0) = \min_\Omega v(x) < 0,$$

then

$$|x^0| \leq R_0.$$

As a simple application, we consider system (1-1).

Theorem 3.1. *Assume that $0 < \alpha < 2$ and $u, v \in L_\alpha \cup C^{1,1}_{\mathrm{loc}}$ is a nonnegative solution of equation* (1-1). *Then*

(i) *in the subcritical case $1 < p, q < (n+\alpha)/(n-\alpha)$, $(u, v) \equiv (0, 0)$;*

(ii) *in the critical case $p = q = (n+\alpha)/(n-\alpha)$, (u, v) is radially symmetric about some point.*

2. Various maximum principles

Maximum principle for antisymmetric functions.

Theorem 2.1. *Let T be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that*

$$T = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for some } \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \ldots, x_n)$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \quad \text{and} \quad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded domain in H. Assume that $u \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ is lower semicontinuous on $\bar{\Omega}$. If

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ v(\tilde{x}) = -v(x) & \text{in } H, \end{cases} \tag{2-1}$$

then

$$u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 \quad \text{in } \Omega. \tag{2-2}$$

This conclusion holds for unbounded region Ω if we further assume that

$$\varliminf_{|x| \to \infty} u(x) \geq 0 \quad \text{and} \quad \varliminf_{|x| \to \infty} v(x) \geq 0.$$

If $u = 0$ and $v = 0$ at some point in Ω, then

$$u(x) = 0 \quad \text{and} \quad v(x) = 0 \quad \text{almost everywhere in } \mathbb{R}^n.$$

Proof. If (2-2) does not hold, then the lower semicontinuity of u and v on $\bar{\Omega}$ indicates that there exists a $x^0 \in \bar{\Omega}$ such that

$$u(x^0) = \min_{\bar{\Omega}} u < 0$$

or

$$v(x^0) = \min_{\bar{\Omega}} v < 0,$$

and one can further deduce from condition (2-1) that x^0 is in the interior of Ω.

If $u(x^0) < 0$, it follows that

$$\begin{aligned}
(-\Delta)^{\alpha/2}u(x^0) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy + \int_{\tilde{H}} \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy + \int_H \frac{u(x^0)-u(\tilde{y})}{|x^0-\tilde{y}|^{n+\alpha}}\,dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy + \int_H \frac{u(x^0)+u(y)}{|x^0-\tilde{y}|^{n+\alpha}}\,dy \right\} \\
&\le C_{n,\alpha} \int_H \left\{ \frac{u(x^0)-u(y)}{|x^0-\tilde{y}|^{n+\alpha}} + \frac{u(x^0)+u(y)}{|x^0-\tilde{y}|^{n+\alpha}} \right\} dy \\
&= C_{n,\alpha} \int_H \frac{2u(x^0)}{|x^0-\tilde{y}|^{n+\alpha}}\,dy \\
&< 0,
\end{aligned}$$

which contradicts inequality (2-1).

Similarly, if $v(x^0) < 0$, we also get a contradiction with (2-1). This verifies (2-2).

Now we show that $u \ge 0$ and $v \ge 0$ in H. If there is some point $x^0 \in \Omega$, such that $u(x^0) = 0$ and $v(x^0) = 0$, then from

$$\begin{aligned}
0 \le (-\Delta)^{\alpha/2}u(x^0) &= C_{n,\alpha} PV \int_H \frac{-u(y)}{|x^0-y|^{n+\alpha}}\,dy, \\
0 \le (-\Delta)^{\alpha/2}v(x^0) &= C_{n,\alpha} PV \int_H \frac{-v(y)}{|x^0-y|^{n+\alpha}}\,dy,
\end{aligned}$$

we derive immediately that

$$u(x) = 0 \quad \text{and} \quad v(x) = 0 \quad \text{almost everywhere in} \quad \mathbb{R}^n.$$

This completes the proof. □

Narrow region principle.

Theorem 2.2. *Let T be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that*

$$T = \{x = (x^1, x') \in \mathbb{R}^n \mid x_1 = \lambda \quad \text{for some} \quad \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \ldots, x_n),$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \qquad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded narrow region in H such that it is contained in $\{x \mid \lambda - l < x_1 < \lambda\}$ with small l. Suppose that $u, v \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ and both are lower semicontinuous on $\bar{\Omega}$. If $c_1(x)$ and $c_2(x)$ are both bounded from below in Ω, $c_1(x) \le 0$ and $c_2(x) \le 0$ and

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x) v(x) \ge 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x) u(x) \ge 0 & \text{in } \Omega, \\ u(x) \ge 0 \quad \text{and} \quad v(x) \ge 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ v(\tilde{x}) = -v(x) & \text{in } H, \end{cases} \tag{2-3}$$

then for sufficiently small l, we have

$$u(x) \ge 0 \quad \text{and} \quad v(x) \ge 0 \quad \text{in } \Omega. \tag{2-4}$$

This conclusion holds for unbounded regions Ω if we further assume that

$$\varliminf_{|x| \to \infty} u(x) \ge 0 \quad \text{and} \quad \varliminf_{|x| \to \infty} v(x) \ge 0.$$

Proof. If (2-4) does not hold, then the lower semicontinuity of u and v on $\bar{\Omega}$ indicates that there exists an $x^0 \in \bar{\Omega}$ such that

$$u(x^0) = \min_{\bar{\Omega}} u < 0 \quad \text{or} \quad v(x^0) = \min_{\bar{\Omega}} v < 0,$$

and one can further deduce from condition (2-3) that x^0 is in the interior of Ω.

Next we discuss the problem in three different cases.

Case i. ($u(x^0) = \min_\Omega u < 0$ and $v(x^0) \ge 0$).

It follows that

$$\begin{aligned} (-\Delta)^{\alpha/2} u(x^0) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy \\ &= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy + \int_{\tilde{H}} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy \right\} \\ &= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy + \int_H \frac{u(x^0) - u(\tilde{y})}{|x^0 - \tilde{y}|^{n+\alpha}}\, dy \right\} \\ &= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy + \int_H \frac{u(x^0) + u(y)}{|x^0 - \tilde{y}|^{n+\alpha}}\, dy \right\} \\ &\le C_{n,\alpha} \int_H \left\{ \frac{u(x^0) - u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} + \frac{u(x^0) + u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} \right\} dy \\ &= C_{n,\alpha} \int_H \frac{2u(x^0)}{|x^0 - \tilde{y}|^{n+\alpha}}\, dy. \end{aligned} \tag{2-5}$$

Let $D = \{y \mid l < y_1 - x_1^0 < 1,\ |y' - (x^0)'| < 1\}$, $s = y_1 - x_1^0$, $\tau = y' - (x^0)'$ and $\omega_{n-2} = |B_1(0)|$ in $\mathbb{R}^{n-2}$. Now we have

$$
\begin{aligned}
\int_H \frac{1}{|x^0 - \tilde{y}|^{n+\alpha}}\, dy &\geq \int_D \frac{1}{|x^0 - y|^{n+\alpha}}\, dy \\
&= \int_l^1 \int_0^1 \frac{\omega_{n-2}\tau^{n-2}}{(s^2+\tau^2)^{\frac{n+\alpha}{2}}}\, d\tau\, ds \\
&= \int_l^1 \int_0^{\frac{1}{s}} \frac{\omega_{n-2}(st)^{n-2}s}{s^{n+\alpha}(1+t^2)^{\frac{n+\alpha}{2}}}\, dt\, ds \qquad (2\text{-}6)\\
&= \int_l^1 \frac{1}{s^{1+\alpha}} \int_0^{\frac{1}{s}} \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{\frac{n+\alpha}{2}}}\, dt\, ds \\
&\geq \int_l^1 \frac{1}{s^{1+\alpha}} \int_0^1 \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{\frac{n+\alpha}{2}}}\, dt\, ds \\
&\geq C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds \to \infty, \qquad (2\text{-}7)
\end{aligned}
$$

where (2-6) follows from the substitution $\tau = st$ and (2-7) is true when $l \to 0$.

Hence $c_1(x) \leq 0$ leads to

$$
\begin{aligned}
(-\Delta)^{\alpha/2}u(x^0) + c_1(x)v(x^0) &\leq C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds\, u(x^0) + c_1(x^0)v(x^0) \\
&= u(x^0)\left[C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0)\frac{v(x^0)}{u(x^0)}\right] \\
&< 0,
\end{aligned}
$$

when l sufficiently small. This is a contradiction with condition (2-3).

Case ii ($v(x^0) = \min\limits_{\Omega} v < 0$ and $u(x^0) \geq 0$). Similarly to Case i, $c_2(x) \leq 0$ leads to

$$
(-\Delta)^{\alpha/2}v(x^0) + c_2(x^0)u(x^0) \leq v(x^0)\left[C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^0)\frac{u(x^0)}{v(x^0)}\right] < 0,
$$

when l sufficiently small. This is a contradiction with condition (2-3).

Case iii ($u(x^0) = \min\limits_{\Omega} u < 0$ and $v(x^0) < 0$). Similarly to Case i, by (2-3), we have

$$
0 \leq (-\Delta)^{\alpha/2}u(x^0) + c_1(x^0)v(x^0) \leq C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds\, u(x^0) + c_1(x^0)v(x^0). \qquad (2\text{-}8)
$$

By $v(x^0) < 0$, there exists $x^1 \in \Omega$ such that $v(x^1) = \min_\Omega v < 0$. Similarly to Case ii, by (2-3) and $c_2(x) \le 0$, we have

$$0 \le (-\Delta)^{\alpha/2} v(x^1) + c_2(x^1)u(x^1) \le C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds\, v(x^0) + c_2(x^1)u(x^0). \tag{2-9}$$

Adding (2-8) to (2-9), we get

$$\left[C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1)\right] u(x^0) + \left[C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0)\right] v(x^0) \ge 0. \tag{2-10}$$

As $u(x^0) < 0$ and $v(x^0) < 0$, if (2-10) holds, then at least one of

$$C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1) \le 0 \quad \text{or} \quad C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0) \le 0$$

holds.

Equivalently,

$$C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1) \le 0 \quad \text{or} \quad C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0) \le 0. \tag{2-11}$$

However, when l sufficiently small, from the fact that $c_1(x)$ and $c_2(x)$ are both bounded from below in Ω, we have

$$C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1) > 0 \quad \text{and} \quad C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0) > 0.$$

which is a contradiction with (2-11).

Similarly, we can prove the case $v(x^0) = \min_\Omega v < 0$ and $u(x^0) < 0$.

Therefore, (2-4) must be true. This completes the proof. □

Decay at infinity.

Theorem 2.3. *Let $H = \{x \in \mathbb{R}^n \mid x_1 < \lambda \quad \text{for some } \lambda \in \mathbb{R}\}$ and let Ω be an unbounded region in H. Assume*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x)v(x) \ge 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x)u(x) \ge 0 & \text{in } \Omega, \\ u(x) \ge 0 \quad \text{and} \quad v(x) \ge 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H \\ v(\tilde{x}) = -v(x) & \text{in } H \end{cases} \tag{2-12}$$

with

$$\varliminf_{|x|\to\infty} |x|^\alpha c_1(x) = 0, \quad c_1(x) \le 0, \tag{2-13}$$

and

$$\varliminf_{|x|\to\infty} |x|^{\alpha} c_2(x) = 0, \quad c_2(x) \le 0. \tag{2-14}$$

Then there exists a constant R_0 such that if

$$u(x^0) = \min_{\Omega} u(x) < 0 \quad or \quad v(x^0) = \min_{\Omega} v(x) < 0, \tag{2-15}$$

then

$$|x^0| \le R_0. \tag{2-16}$$

Proof. Following from (2-15), there are three different cases for this proof.

Case i ($u(x^0) < 0$ and $v(x^0) \ge 0$). It follows that

$$\begin{aligned}
(-\Delta)^{\alpha/2} u(x^0) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy + \int_{\tilde H} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy + \int_H \frac{u(x^0) - u(\tilde y)}{|x^0 - \tilde y|^{n+\alpha}}\, dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}}\, dy + \int_H \frac{u(x^0) + u(y)}{|x^0 - \tilde y|^{n+\alpha}}\, dy \right\} \\
&\le C_{n,\alpha} \int_H \left\{ \frac{u(x^0) - u(y)}{|x^0 - \tilde y|^{n+\alpha}} + \frac{u(x^0) + u(y)}{|x^0 - \tilde y|^{n+\alpha}} \right\} dy \\
&= C_{n,\alpha} \int_H \frac{2u(x^0)}{|x^0 - \tilde y^{n+\alpha}|}\, dy.
\end{aligned}$$

For each fixed λ, when $|x^0| \ge \lambda$, we have $B_{|x^0|}(x^1) \subset \tilde H$ with $x^1 = (3|x^0| + x_1^0, (x^0)')$, and it follows that

$$\begin{aligned}
\int_H \frac{1}{|x^0 - \tilde y|^{n+\alpha}}\, dy &= \int_{\tilde H} \frac{1}{|x^0 - y|^{n+\alpha}}\, dy \\
&\ge \int_{B_{|x^0|}(x^1)} \frac{1}{|x^0 - y|^{n+\alpha}}\, dy \\
&\ge \int_{B_{|x^0|}(x^1)} \frac{1}{4^{n+\alpha}|x^0|^{n+\alpha}}\, dy = \frac{\omega_n}{4^{n+\alpha}|x^0|^{n+\alpha}}, \qquad (2\text{-}17)
\end{aligned}$$

where (2-17) follows from $|x^0 - y| \le |x^0 - x_1| + |x^0| = 4|x^0|$ for all $y \in B_{|x^0|}(x^1)$. Then we have

$$0 \le (-\Delta)^{\alpha/2} u(x^0) + c_1(x^0) v(x^0) \le \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^{\alpha}} u(x^0) + c_1(x^0) v(x^0). \tag{2-18}$$

Following from (2-13), $c_1(x^0) \le 0$ for all $x^0 \in H$, we have

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} u(x^0) + c_1(x^0)v(x^0) < 0.$$

This contradicts (2-18).

Case ii ($v(x^0) < 0$ and $u(x^0) \ge 0$). Using the same method as Case i, we have

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} v(x^0) + c_2(x^0)u(x^0) \ge 0,$$

which is a contradiction with

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} v(x^0) + c_2(x^0)u(x^0) < 0,$$

for $c_2(x^0) \le 0$ for all $x^0 \in H$.

Case iii ($u(x^0) < 0$ and $v(x^0) < 0$). We have

$$0 \le (-\Delta)^{\alpha/2} u(x^0) + c_1(x^0)v(x^0) \le \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} u(x^0) + c_1(x^0)v(x^0), \tag{2-19}$$

$$0 \le (-\Delta)^{\alpha/2} v(x^0) + c_2(x^0)u(x^0) \le \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} v(x^0) + c_2(x^0)u(x^0). \tag{2-20}$$

Adding (2-19) to (2-20), we get

$$\left[\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0)\right] u(x^0) + \left[\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0)\right] v(x^0) \ge 0. \tag{2-21}$$

As $u(x^0) < 0$ and $v(x^0) < 0$, if (2-21) holds, at least one of

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0) \le 0 \quad \text{or} \quad \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0) \le 0$$

holds. Equivalently,

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0) \le 0 \quad \text{or} \quad \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0) \le 0. \tag{2-22}$$

However, if $|x^0|$ is sufficiently large, following from (2-13) and (2-14), we have

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0) > 0 \quad \text{and} \quad \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0) > 0.$$

which is a contradiction with (2-22).

Therefore, (2-16) holds. This completes the proof. $\square$

3. Method of moving planes and its applications

Theorem 3.1. *Assume that $u, v \in C^{1,1}_{\mathrm{loc}} \cap L_\alpha$ and*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = v^q(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2} v(x) = u^p(x), & x \in \mathbb{R}^n. \end{cases} \tag{3-1}$$

Then

(i) *in the subcritical case $1 < p, q < (n+\alpha)/(n-\alpha)$,* (3-1) *has no positive solution*;

(ii) *in the critical case $p = q = (n+\alpha)/(n-\alpha)$, the positive solutions must be radially symmetric about some point in $\mathbb{R}^n$.*

Proof. Because no decay condition on u near infinity is assumed, we are not able to carry out the method of moving planes on u directly. To circumvent this difficulty, we make a Kelvin transform.

Let x^0 be a point in $\mathbb{R}^n$, and let

$$\bar{u}(x^0) = \frac{1}{|x-x^0|^{n-\alpha}} u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right), \quad \bar{v}(x^0) = \frac{1}{|x-x^0|^{n-\alpha}} v\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right)$$

be the Kelvin transform of (u, v) centered at x^0. Then it follows that

$$\begin{aligned} \bar{u}(x) &= \frac{1}{|x-x^0|^{n-\alpha}} u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right) \\ &= \frac{1}{|x-x^0|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{v^q(y)}{|y - \frac{x-x^0}{|x-x^0|^2} - x^0|^{n-\alpha}}\, dy \\ &= \frac{1}{|x-x^0|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{\left(\frac{1}{|y-x^0|^{n-\alpha}}\right)^q \bar{v}^q\left(\frac{y-x^0}{|y-x^0|^2} + x^0\right)}{\left|y - x^0 - \frac{x-x^0}{|x-x^0|^2}\right|^{n-\alpha}}\, dy \\ &= \frac{1}{|x-x^0|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{|z-x^0|^{q(n-\alpha)} \bar{v}^q(z)}{\left|\frac{z-x^0}{|z-x^0|^2} - \frac{x-x^0}{|x-x^0|^2}\right|^{n-\alpha}} \frac{1}{|z-x^0|^{2n}}\, dz \qquad (3\text{-}2) \\ &= \int_{\mathbb{R}^n} \frac{\bar{v}^q(z)}{|z-x^0|^\tau |x-z|^{n-\alpha}}\, dz, \end{aligned}$$

where the step (3-2) follows from the substitution $z = (y-x^0)/|y-x^0|^2 + x^0$ and $\tau = n + \alpha - q(n-\alpha)$.

This means

$$(-\Delta)^{\alpha/2} \bar{u}(x) = \frac{\bar{v}^q(x)}{|x-x^0|^\tau}. \tag{3-3}$$

Similarly, we have

$$(-\Delta)^{\alpha/2}\bar{v}(x) = \frac{\bar{u}^p(x)}{|x-x^0|^\gamma}, \tag{3-4}$$

with $\gamma = n+\alpha-p(n-\alpha)$. Obviously, $\tau=\gamma=0$ in the critical case.

Choose any direction to be the x_1 direction. For $\lambda < x_1^0$, let

$$T_\lambda = \{x\in\mathbb{R}^n \mid x_1=\lambda\}, \qquad x^\lambda = (2\lambda - x_1, x'), \quad \bar{u}_\lambda(x) = \bar{u}(x^\lambda),$$
$$w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x), \qquad \bar{v}_\lambda(x) = \bar{v}(x^\lambda), \qquad \varphi_\lambda(x) = \bar{v}_\lambda(x) - \bar{v}(x),$$

and

$$\Sigma_\lambda = \{x\in\mathbb{R}^n \mid x_1<\lambda\}, \quad \tilde{\Sigma}_\lambda = \{x^\lambda \mid x\in\Sigma_\lambda\}.$$

First, notice that, by the definition of w_λ and φ_λ, we have

$$\lim_{|x|\to\infty} w_\lambda(x) = 0, \qquad \lim_{|x|\to\infty}\varphi_\lambda(x)=0.$$

Hence, if w_λ or φ_λ is negative somewhere in Σ_λ, then the negative minima of w_λ or φ_λ was attained in the interior of Σ_λ.

From (3-3), at points where φ_λ is negative, we have

$$\begin{aligned}(-\Delta)^{\alpha/2}w_\lambda(x) &= \frac{\bar{v}_\lambda^q(x)}{|x^\lambda-x^0|^\tau} - \frac{\bar{v}^q(x)}{|x-x^0|^\tau}\\ &\ge \frac{\bar{v}_\lambda^q(x)-\bar{v}^q(x)}{|x-x^0|^\tau}\\ &\ge \frac{q\bar{v}^{q-1}(x)\varphi_{\lambda(x)}}{|x-x^0|^\tau},\end{aligned} \tag{3-5}$$

where (3-5) follows from the *mean value theorem*, that is,

$$(-\Delta)^{\alpha/2}w_\lambda(x) + c_1(x)\varphi_\lambda(x) \ge 0$$

with

$$c_1(x) = -\frac{q\bar{v}^{q-1}(x)}{|x-x^0|^\tau}. \tag{3-6}$$

From (3-4), at points where w_λ is negative, we similarly have

$$(-\Delta)^{\alpha/2}\varphi_\lambda(x) + c_2(x)w_\lambda(x) \ge 0 \tag{3-7}$$

with

$$c_2(x) = -\frac{p\bar{u}^{p-1}(x)}{|x-x^0|^\gamma}. \tag{3-8}$$

The subcritical case. For $1 < p, q < (n+\alpha)/(n-\alpha)$, we show that (3-1) admits no positive solution.

Step 1. We show that, for λ sufficiently negative,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{in } \Sigma_\lambda. \tag{3-9}$$

This is done by using Theorem 2.3 (*decay at infinity*).

It follows from (3-6) that,

$$\begin{aligned} c_1(x) &= -\frac{q\big(\frac{1}{|x-x^0|^{n-\alpha}}\big)^{q-1} v^{q-1}\big(\frac{x-x^0}{|x-x^0|^2}+x^0\big)}{|x-x^0|^{n+\alpha-q(n-\alpha)}} \\ &= -\frac{q v^{q-1}\big(\frac{x-x^0}{|x-x^0|^2}+x^0\big)}{|x-x^0|^{2\alpha}}. \end{aligned}$$

It is easy to verify that, for $|x|$ sufficiently large,

$$c_1(x) \sim \frac{1}{|x|^{2\alpha}}. \tag{3-10}$$

In the same way,

$$c_2(x) \sim \frac{1}{|x|^{2\alpha}}. \tag{3-11}$$

In addition, following from (3-6) and (3-8), we have $c_1(x) \leq 0$ and $c_2(x) \leq 0$. Hence, $c_1(x)$ and $c_2(x)$ satisfy conditions (2-13) and (2-14) respectively in Theorem 2.3. Applying Theorem 2.3 to w_λ and φ_λ with $\Omega = H = \Sigma_\lambda$, we conclude that, there exists an $R_0 > 0$ (independent of λ), such that if $\bar{x}$ is a negative minimum of w_λ or φ_λ in Σ_λ, then

$$|\bar{x}| \leq R_0. \tag{3-12}$$

Now for $\lambda \leq -R_0$, we must have

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda.$$

This verifies (3-9).

Step 2. Step 1 provides a starting point, from which we can now move the plane T_λ to the right as long as (3-9) holds to its limiting position.

Let

$$\lambda_0 = \sup\{\lambda \leq x_1^0 \mid w_\mu(x) \geq 0 \quad \text{and} \quad \varphi_\mu(x) \geq 0, \quad \text{for all } x \in \Sigma_\mu,\ \mu \leq \lambda\}.$$

In this part, we show that

$$\lambda_0 = x_1^0$$

and

$$w_{\lambda_0}(x) \equiv 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \equiv 0, \quad \text{for all } x \in \Sigma_{\lambda_0}. \tag{3-13}$$

Suppose that $\lambda_0 < x_1^0$. We show that the plane T_λ can be moved further right. To be more rigorous, there exists some $\epsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0, \quad \text{for all } x \in \Sigma_\lambda. \tag{3-14}$$

This is a contradiction with the definition of λ_0. Hence we must have

$$\lambda_0 = x_1^0. \tag{3-15}$$

Now we prove (3-14) by combining the use of the *narrow region principle* and *decay at infinity*.

Again by (3-12), the negative minimum of w_λ cannot be attained outside of $B_{R_0}(0)$. Next we argue that it can neither be attained inside of $B_{R_0}(0)$. Actually, we will show that for λ sufficiently close to λ_0,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0, \quad \text{for all } x \in \Sigma_\lambda \cap B_{R_0}(0). \tag{3-16}$$

From the *narrow region principle*, there is a small $\delta > 0$, such that for $\lambda \in [\lambda_0, \lambda_0 + \delta)$, if

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0 - \delta}, \tag{3-17}$$

then

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}. \tag{3-18}$$

To see this, in Theorem 2.2, we let $H = \Sigma_\lambda$ and the narrow region $\Omega = \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}$, while the lower bound of $c_1(x)$, $c_2(x)$ can be seen from (3-10) and (3-11).

Then what is left to show is (3-17), and actually we only need

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0 - \delta} \cap B_{R_0}(0). \tag{3-19}$$

In fact, when $\lambda_0 < x_1^0$, we have

$$w_{\lambda_0}(x) > 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) > 0 \quad \text{for all } x \in \Sigma_{\lambda_0}. \tag{3-20}$$

If not, there exists some $\hat{x}$ such that

$$w_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0 \quad \text{or} \quad \varphi_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} \varphi_{\lambda_0}(x) = 0.$$

Case i ($w_{\lambda_0}(\hat{x})=0$ and $\varphi_{\lambda_0}(\hat{x})>0$). It follows that

$$\begin{aligned}(-\Delta)^{\alpha/2}w_{\lambda_0}(\hat{x}) &= C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy\\ &= C_{n,\alpha}PV\biggl[\int_{\Sigma_{\lambda_0}}\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy+\int_{\tilde{\Sigma}_{\lambda_0}}\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy\biggr]\\ &= C_{n,\alpha}PV\biggl[\int_{\Sigma_{\lambda_0}}\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy+\int_{\Sigma_{\lambda_0}}\frac{-w_{\lambda_0}(\tilde{y})}{|\hat{x}-\tilde{y}|^{n+\alpha}}\,dy\biggr]\\ &= C_{n,\alpha}PV\biggl[\int_{\Sigma_{\lambda_0}}\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy+\int_{\Sigma_{\lambda_0}}\frac{w_{\lambda_0}(y)}{|\hat{x}-\tilde{y}|^{n+\alpha}}\,dy\biggr]\\ &\le C_{n,\alpha}\int_{\Sigma_{\lambda_0}}\biggl[\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}+\frac{w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\biggr]dy\\ &= 0. \qquad (3\text{-}21)\end{aligned}$$

On the other hand,

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}w_{\lambda_0}(\hat{x}) &= \frac{\bar{v}^q_{\lambda_0}(\hat{x})}{|\hat{x}^{\lambda_0}-x^0|^{\tau}}-\frac{\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^{\tau}}\\ &> \frac{\bar{v}^q_{\lambda_0}(\hat{x})-\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^{\tau}}\\ &> \frac{q\bar{v}^{q-1}(\hat{x})\varphi_{\lambda_0}(\hat{x})}{|\hat{x}-x^0|^{\tau}}\\ &> 0,\end{aligned}$$

which is a contradiction with (3-21).

Case ii ($\varphi_{\lambda_0}(\hat{x})=0$ and $w_{\lambda_0}(\hat{x})>0$). As in Case i, there will be a contradiction.

Case iii ($w_{\lambda_0}(\hat{x})=0$ and $\varphi_{\lambda_0}(\hat{x})=0$). We have

$$(-\Delta)^{\frac{\alpha}{2}}w_{\lambda_0}(\hat{x})=\frac{\bar{v}^q_{\lambda_0}(\hat{x})}{|\hat{x}^{\lambda_0}-x^0|^{\tau}}-\frac{\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^{\tau}}=\frac{\bar{v}^q(\hat{x})}{|\hat{x}^{\lambda_0}-x^0|^{\tau}}-\frac{\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^{\tau}}>0,$$

a contradiction with (3-21).

These three cases prove (3-20). It follows from (3-20) that there exists a constant $c_0>0$, such that

$$w_{\lambda_0}(x)\ge c_0 \quad\text{and}\quad \varphi_{\lambda_0}(x)\ge c_0 \quad\text{for all } x\in\overline{\Sigma_{\lambda_0-\delta}\cap B_{R_0}(0)}.$$

Since w_λ and φ_λ both depend on λ continuously, there exist $\epsilon>0$ and $\epsilon<\delta$, such that for all $\lambda\in(\lambda_0,\lambda_0+\epsilon)$, we have

$$w_{\lambda_0}(x)\ge 0 \quad\text{and}\quad \varphi_{\lambda_0}(x)\ge 0 \quad\text{for all } x\in\overline{\Sigma_{\lambda_0-\delta}\cap B_{R_0}(0)}. \qquad (3\text{-}22)$$

Combining (3-18), (3-12) and (3-22), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda. \tag{3-23}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = x_1^0 \quad \text{and} \quad w_{\lambda_0} \geq 0,\ \varphi_{\lambda_0} \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0}.$$

Similarly, one can move the plane T_λ from $+\infty$ to the left and show that

$$w_{\lambda_0}(x) \geq 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0}. \tag{3-24}$$

Now we have shown that

$$\lambda_0 = x_1^0 \quad \text{and} \quad w_{\lambda_0}(x) \equiv 0,\ \varphi_{\lambda_0}(x) \equiv 0 \quad \text{for all } x \in \Sigma_{\lambda_0}.$$

This completes Step 2.

So far, we have proved that $(\bar{u}, \bar{v})$ is symmetric about the plane $T_{x_1^0}$. Since the x_1 direction can be chosen arbitrarily, we have actually shown that $(\bar{u}, \bar{v})$ is radially symmetric about x^0.

For any two points $X^i \in \mathbb{R}^n$, $i = 1, 2$. Choose x_0 to be the midpoint, i.e., $x^0 = (X^1 + X^2)/2$. Since $(\bar{u}, \bar{v})$ is radially symmetric about x^0, so is (u, v), hence $(u(X^1), v(X^1)) = (u(X^2), v(X^2))$. This implies that u is constant. A positive constant function does not satisfy (3-1). This proves the nonexistence of positive solutions for (3-1) when $1 < p, q < (n+\alpha)/(n-\alpha)$.

The critical case. Let $(\bar{u}, \bar{v})$ be the Kelvin transform of (u, v) centered at the origin. Then

$$(-\Delta)^{\alpha/2}\bar{u}(x) = \bar{v}^q(x), \quad (-\Delta)^{\alpha/2}\bar{v}(x) = \bar{u}^p(x). \tag{3-25}$$

We will show that either $(\bar{u}, \bar{v})$ is symmetric about the origin or (u, v) is symmetric about some point.

We still use the notation as in the subcritical case. **Step 1** is entirely the same as that in the subcritical case, that is, we can show that for λ sufficiently negative,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda.$$

Let

$$\lambda_0 = \sup\{\lambda \geq 0 \mid w_\mu(x) \geq 0 \quad \text{and} \quad \varphi_\mu(x) \geq 0 \quad \text{for all } x \in \Sigma_\mu, \mu \leq \lambda\}.$$

Case i. $\lambda_0 < 0$. Similarly to the subcritical case, one can show that

$$w_{\lambda_0}(x) \equiv 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \equiv 0 \quad \text{for all } x \in \Sigma_{\lambda_0}.$$

It follows that 0 is not a singular point of $\bar{u}$ or $\bar{v}$, and hence following from Kelvin transform of $\bar{u}$ centered at the origin

$$u(x) = \frac{1}{|x|^{n-\alpha}} \bar{u}\Big(\frac{x}{|x|^2}\Big),$$

we have

$$\lim_{|x|\to\infty} |x|^{n-\alpha} u(x) = \lim_{|x|\to\infty} \bar{u}\Big(\frac{x}{|x|^2}\Big) = \bar{u}(0) > 0,$$

that is,

$$u(x) = O\Big(\frac{1}{|x|^{n-\alpha}}\Big) \quad \text{when } |x| \to \infty.$$

Similarly for v,

$$v(x) = O\Big(\frac{1}{|x|^{n-\alpha}}\Big) \quad \text{when } |x| \to \infty.$$

This enables us to apply the method of moving planes to (u, v) directly and show that (u, v) is symmetric about some point in $\mathbb{R}^n$.

Case ii. $\lambda_0 = 0$. Then by moving planes from near $x_1 = +\infty$, we derive that $(\bar{u}, \bar{v})$ is symmetric about the origin, and so is (u, v).

In any case, (u, v) is symmetric about some point in $\mathbb{R}^n$. □

References

[Busca and Manásevich 2002] J. Busca and R. Manásevich, "A Liouville-type theorem for Lane–Emden systems", *Indiana Univ. Math. J.* **51**:1 (2002), 37–51. MR Zbl

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:8 (2007), 1245–1260. MR Zbl

[Chen and Li 1991] W. X. Chen and C. Li, "Classification of solutions of some nonlinear elliptic equations", *Duke Math. J.* **63**:3 (1991), 615–622. MR Zbl

[Chen and Li 2010] W. Chen and C. Li, *Methods on nonlinear elliptic equations*, AIMS Series Diff. Equations Dyn. Syst. **4**, Amer. Inst. Math. Sci., Springfield, MO, 2010. MR Zbl

[Chen and Zhu 2016] W. Chen and J. Zhu, "Indefinite fractional elliptic problem and Liouville theorems", *J. Differential Equations* **260**:5 (2016), 4758–4785. MR Zbl

[Chen et al. 2017] W. Chen, C. Li, and Y. Li, "A direct method of moving planes for the fractional Laplacian", *Adv. Math.* **308** (2017), 404–437. MR Zbl

[Jarohs and Weth 2016] S. Jarohs and T. Weth, "Symmetry via antisymmetric maximum principles in nonlocal problems of variable order", *Ann. Mat. Pura Appl.* (4) **195**:1 (2016), 273–291. MR Zbl

[Mitidieri 1996] E. Mitidieri, "Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$", *Differential Integral Equations* **9**:3 (1996), 465–479. MR Zbl

[Poláčik et al. 2007] P. Poláčik, P. Quittner, and P. Souplet, "Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems", *Duke Math. J.* **139**:3 (2007), 555–579. MR Zbl

[Serrin and Zou 1998] J. Serrin and H. Zou, "Existence of positive solutions of the Lane–Emden system", *Atti Sem. Mat. Fis. Univ. Modena* **46**:Suppl. (1998), 369–380. MR Zbl

[Souplet 2009] P. Souplet, "The proof of the Lane–Emden conjecture in four space dimensions", *Adv. Math.* **221**:5 (2009), 1409–1427. MR Zbl

[Troy 1981] W. C. Troy, "Symmetry properties in systems of semilinear elliptic equations", *J. Differential Equations* **42**:3 (1981), 400–413. MR Zbl

Received November 29, 2015. Revised October 13, 2016.

CHUNXIA CHENG
SCHOOL OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, 221116
CHINA
860119947@qq.com

ZHONGXUE LÜ
SCHOOL OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, 221116
CHINA
6020010017@jsnu.edu.cn

YINGSHU LÜ
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAO TONG UNIVERSITY
SHANGHAI, 200240
CHINA
lvyingshu@126.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 290 No. 2 October 2017

0030-8730(201710)290:2;1-Z