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DISTINGUISHED THETA REPRESENTATIONS FOR CERTAIN COVERING GROUPS

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To Professor Freydoon Shahidi on his 70th birthday

For Brylinski–Deligne covering groups of an arbitrary split reductive group, we consider theta representations attached to certain exceptional genuine characters. The goal of the paper is to study the dimension of the space of Whittaker functionals of a theta representation. In particular, we investigate when the dimension is exactly one, in which case the theta representation is called distinguished. For this purpose, we first give effective lower and upper bounds for the dimension of Whittaker functionals for general theta representations. Consequently, the dimension in many cases can be reduced to simple combinatorial computations, e.g., the Kazhdan–Patterson covering groups of the general linear groups, or covering groups whose complex dual groups (à la Finkelberg, Lysenko, McNamara and Reich) are of adjoint type. In the second part of the paper, we consider coverings of certain semisimple simply connected groups and give necessary and sufficient conditions for the theta representation to be distinguished. There are subtleties arising from the relation between the rank and the degree of the covering group. However, in each case we will determine the exceptional character whose associated theta representation is distinguished.

1. Introduction and main results

1A. ***Introduction.*** Let F be a nonarchimedean local field of characteristic 0 and residue characteristic p. Let $\mathbb{G}$ be a connected split reductive group over F, and let $G := \mathbb{G}(F)$ be its rational points. One of the central ingredients in the study of irreducible admissible representation of G is the uniqueness of Whittaker functionals (see [Rodier 1973; Shalika 1974]). For instance, this uniqueness property is crucial in the Langlands–Shahidi theory of L-functions [Shahidi 2010] for the so-called generic representations of G, i.e., those with nontrivial Whittaker functionals.

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For a natural number $n \geq 1$, we assume that $F^\times$ contains the full subgroup of the n-th roots of unity, which is then denoted by μ_n. In this paper, we work with the Brylinski–Deligne n-fold covering groups $\overline{G}^{(n)}$ of G, see Section 2A for a description on such covering groups. We may write $\overline{G}^{(n)}$ and $\overline{G}$ interchangeably if no confusion arises. For simplicity, the phrase *covering groups* in this paper is used to refer to the Brylinski–Deligne covering groups. For this purpose, it is noteworthy to mention that the Brylinski–Deligne framework is quite encompassing and contains almost all classically interesting covering groups [Steinberg 1962; Moore 1968; Matsumoto 1969], in particular the Matsumoto covering groups of semisimple simply connected groups [Moore 1968] and the Kazhdan–Patterson covering groups $\overline{\mathrm{GL}}_r^{(n)}$ of GL_r [Kazhdan and Patterson 1984].

For covering groups, the uniqueness of Whittaker functionals for genuine representations of $\overline{G}^{(n)}$ holds rarely and one nontrivial example is the classical double cover $\overline{\mathrm{Sp}}_{2r}^{(2)}$ of the symplectic group Sp_{2r}, see [Szpruch 2007]. This uniqueness plays a pivotal role in the work of Szpruch [2009b; 2013] generalizing the method of Langlands and Shahidi to $\overline{\mathrm{Sp}}_{2r}^{(2)}$. Besides this special family of examples, the uniqueness of Whittaker functionals fails widely, and one almost never expects such a uniform property for all genuine representations of a general covering group. For example, it is well known that certain theta representations for the Kazhdan–Patterson coverings $\overline{\mathrm{GL}}_r^{(n)}$ of GL_r could have high dimensional space of Whittaker functionals [Kazhdan and Patterson 1984]. In fact, such theta representations show that the analogous standard module conjecture (which is a theorem for linear algebraic groups from [Casselman and Shahidi 1998]) does not hold for covering groups.

The failure of the uniqueness of Whittaker functionals for general genuine representations of covering groups, however, has been the source of both obstacles and inspirations to some advancement of the representation theory of such groups. On the one hand, for instance, it is not a priori clear how to generalize the Langlands–Shahidi theory of L-functions to covering groups because of the nonuniqueness of Whittaker functionals for unramified principal series representations. Equivalently, the difficulty for such generalization is essentially due to the fact that the analogous Casselman–Shalika formula for covering groups as in [Chinta and Offen 2013; McNamara 2016] is vector-valued, whereas for linear algebraic groups it is scalar-valued; see [Casselman and Shalika 1980].

On the other hand, there are various streams of rich theories stemming from the nonexistence or multidimensionality of Whittaker functionals. For instance, for genuine representations of covering groups without Whittaker functionals, one may consider semi-Whittaker functionals as in [Takeda 2014] or degenerate Whittaker-functionals [Mœglin and Waldspurger 1987], which interact fruitfully with the arithmetic and character theory of the representations. Meanwhile, the theory of unipotent orbit as discussed in [Ginzburg 2006; Friedberg and Ginzburg 2014;

Friedberg and Ginzburg 2016a] for instance also rectify the situation in the absence of Whittaker functionals. In the latter case where multidimensionality holds, the theory of multiple Weyl Dirichlet series makes deep and fascinating connections between representation theory of covering groups, quantum physics and statistical mechanics etc, see [Brubaker et al. 2011; Bump et al. 1990; 2012] for some of the ideas involved. In particular, the book [Bump et al. 2012] contains several excellent expository articles on multiple Dirichlet series.

Nevertheless, in this paper we consider only the so-called theta representations $\Theta(\overline{G}^{(n)}, \overline{\chi})$ which appear as the local representations for the residue of the Borel Eisenstein series (see Definition 2.1). Moreover, we are mostly interested in determining when the space of Whittaker functionals for $\Theta(\overline{G}^{(n)}, \overline{\chi})$ has dimension one, in which case $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is called distinguished following Suzuki [1998]. Here $\overline{\chi}$ is an exceptional genuine character (see Definition 2.1) of the center $Z(\overline{T})$ of the covering torus $\overline{T} \subseteq \overline{G}$. The reason for considering this problem is two-fold.

First, $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is in a certain sense the simplest family of genuine representations of a general covering group $\overline{G}^{(n)}$. Indeed, if $n = 1$, then it follows from definition that $\Theta(\overline{G}^{(n)}, \overline{\chi})$ could be the trivial representation of the linear group $\overline{G} = \overline{G}^{(1)}$, depending on a proper choice of the exceptional character $\overline{\chi}$. Therefore, for the genericity question regarding Whittaker functionals of genuine representations, it is reasonable to consider this family first. Moreover, theta representations for the Kazhdan–Patterson covering groups of GL_r, to which we have just alluded, are already studied in depth in the seminal paper [Kazhdan and Patterson 1984]. Despite the fact that the idea therein could be applicable for general covering groups, to the best of our knowledge, it seems that there is no systematic treatment on theta representations for general covering groups in the literature. Perhaps this gap is caused by the tedious cocycle computation to be carried out by any potential author. However, the Brylinski–Deligne framework enables us to compute by invoking some neat structural fact of the covering groups of interest, and to handle only a minimized usage of a cocycle on the torus. In brief, we wish to fill in the gap by generalizing the relevant work of Kazhdan and Patterson to Brylinski–Deligne covering groups.

Second, distinguished theta representations have important and emergingly wider applications. Theta representations are the representation-theoretic analogues of theta functions, one of the early applications of which was given by Riemann in his seminal paper to prove the functional equation of the Riemann zeta function. In the language of modern theory of representations, theta representations for $\overline{\mathrm{Sp}}_{2r}^{(2)}$ gain deep applications in the Shimura correspondence [Shimura 1973; Gelbart 1976]. On the other hand, following the work of Kazhdan and Patterson, theta representations for $\overline{\mathrm{GL}}_r^{(n)}$ are also studied extensively in [Bump and Hoffstein 1987; Suzuki 1998; 2012], to mention a few. In particular, these authors made some deep conjectures and also provided evidence for a generalized Shimura correspondence regarding $\overline{\mathrm{GL}}_r^{(n)}$, and the distinguishedness property is exploited to achieve the goals in their work. Another significant direction of applications is the Rankin–Selberg integral representation for the symmetric square and cube L-functions [Bump and Ginzburg 1992; Bump et al. 1996; Takeda 2014; Kaplan 2016]. Evidently, it should be mentioned that for distinguished theta representations, the theory of L-functions could be developed as in the linear algebraic case, since the Casselman–Shalika formula is then scalar-valued. More recently, the work of E. Kaplan [2015a; 2015b], and S. Friedberg and D. Ginzburg [2014; 2016a] also relies heavily on the local and global theta representations in their consideration of Fourier coefficient, Rankin–Selberg L-function and descent integral etc. Notably in their work, distinguishedness is responsible for proving that a global integral admits an Euler factorization into local factors. Besides these, the problem on global cuspidal theta representations is important and many problems are open (see [Friedberg and Ginzburg 2016a; Suzuki 1998]). In any case, we believe that distinguished theta representations are objects of great interest and significance, and we hope that our paper could shed some light on the relevant questions.

1B. ***Main results.*** We consider a Brylinski–Deligne n-fold covering group $\overline{G}^{(n)}$. Let $\overline{\chi}$ be an exceptional character for $\overline{G}^{(n)}$. Fix an unramified additive character ψ of F and consider the space $\mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ of ψ-Whittaker functionals of the theta representation $\Theta(\overline{G}^{(n)}, \overline{\chi})$. The pair $(\overline{G}^{(n)}, \overline{\chi})$ such that

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$$

is quite unique, and the goal is to investigate when $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is distinguished. We remark that for fixed $\overline{G}^{(n)}$, the set of unramified exceptional characters $\overline{\chi}$ is a torsor over $Z(\overline{G}^\vee)$, the center of the complex dual group $\overline{G}^\vee$ of $\overline{G}$. For details on $\overline{G}^\vee$, see [Finkelberg and Lysenko 2010; McNamara 2012; Reich 2012; Weissman 2015].

We outline the structure of the paper and state the main results.

In Section 2, we recall the basic structural facts on a Brylinski–Deligne covering group $\overline{G}^{(n)}$ which will be crucial for our computations. In this paper, we consider exclusively unramified covering group $\overline{G}^{(n)}$ and unramified exceptional character $\overline{\chi}$. In Section 3, the space $\mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}), \overline{\chi})$ is analyzed following the strategy in [Kazhdan and Patterson 1984] closely. In particular, it relies crucially on the Shahidi local coefficient matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ for covering groups. Note that $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ is also referred to as the scattering matrix in [Brubaker et al. 2016] and transition matrix in [Chinta and Offen 2013]. Since the matrix is an analogue (and in fact the reciprocal) of Shahidi's local coefficient in the linear algebraic case [Shahidi 2010, Chapter 5], we call it the Shahidi local coefficient matrix in this paper. See also [Budden 2006; Szpruch 2016]. In the unramified setting, the matrix is computed in [McNamara 2016]; it is also computed for ramified places in [Goldberg and Szpruch 2015].

The first main result is Theorem 3.14 from Section 3:

Theorem 1.1. *Let $\overline{G}^{(n)}$ be an arbitrary unramified Brylinski–Deligne covering group. Let $\overline{\chi}$ be an unramified exceptional genuine character of $\overline{G}^{(n)}$ with associated theta representation $\Theta(\overline{G}^{(n)}, \overline{\chi})$. Then,*

$$|\wp_{Q,n}(\mathcal{O}^{F}_{Q,n})| \leq \dim \mathrm{Wh}_{\psi}(\Theta(\overline{G}^{(n)}, \overline{\chi})) \leq |\wp_{Q,n}(\mathcal{O}^{F}_{Q,n,\mathrm{sc}})|.$$

These two bounds are combinatorial quantities involving certain Weyl-action on lattices. The readers are referred to Section 2 for details. We highlight here some consequences from the above theorem.

Firstly, Theorem 1.1 recovers the results of Kazhdan and Patterson. More precisely, for covering groups $\overline{\mathrm{GL}}_r^{(n)}$ studied in [Kazhdan and Patterson 1984], the authors determine that $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{GL}}_r^{(n)}, \overline{\chi})) = 1$ if and only if

(1) $n = r$ and $\overline{\mathrm{GL}}_r^{(n)}$ is any Kazhdan–Patterson covering group, or

(2) $n = r + 1$ and $\overline{\mathrm{GL}}_r^{(n)}$ belongs to a special type of degree n Kazhdan–Patterson covering groups.

In fact, for any covering group $\overline{\mathrm{GL}}_r^{(n)}$ studied in [Kazhdan and Patterson 1984], one has $\mathcal{O}^{F}_{Q,n} = \mathcal{O}^{F}_{Q,n,\mathrm{sc}}$. Therefore $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{GL}}_r^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}^{F}_{Q,n,\mathrm{sc}})|$. In particular, the dimension does not depend on the choice of the exceptional character $\overline{\chi}$ and can be computed effectively. For details, see Example 3.16.

In general, for cases where the two bounds in Theorem 1.1 actually agree, the computation of the dimension is reduced to a purely combinatorial problem, and thus amenable to a straightforward calculation. This includes the case where $Y_{Q,n} = Y^{\mathrm{sc}}_{Q,n}$, or equivalently $Z(\overline{G}^{\vee}) = 1$. For example, odd degree coverings of simply connected groups of type B_r, C_r have this property. See Sections 5 and 6.

Secondly in contrast, when the two bounds in Theorem 1.1 do not agree, $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{G}, \overline{\chi}))$ becomes sensitive to the choice of the exceptional character $\overline{\chi}$. The second half of this paper is devoted to investigating this. This phenomenon already occurs for the degree two metaplectic covering $\overline{\mathrm{SL}}_2^{(2)}$, see Example 4.7. In this case $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi})$ is the even Weil representation. Consider $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$, where $\overline{\chi}_{\psi_a}$ is an exceptional character defined by using the twisted additive character ψ_a, where $a \in F^{\times}$. It is well known that $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})) \leq 1$ and the equality holds if and only if $a \in (F^{\times})^2$. Our analysis shows that similar phenomenon occurs for higher rank groups, see Section 4B, in particular Corollary 4.5.

In any case, we summarize our results for certain coverings of simply connected groups as follows. We write for instance $\overline{A}_r^{(n)}$ for the degree n covering of the simply connected group of type A_r of rank r. Here the covering group arises from a quadratic form Q on the coroot lattice $Y = Y^{\mathrm{sc}}$ such that $Q(\alpha^{\vee}) = 1$ for any short coroot $\alpha^\vee$. The following theorem is an amalgam of Theorems 4.10, 5.3, 6.2 and 7.1. Only for $\overline{A}_r^{(n)}$, we impose the condition $n \le r+2$ for technical reasons.

Theorem 1.2. *Let $\overline{G}^{(n)}$ be an unramified Brylinski–Deligne degree n covering of a simply connected semisimple group of type A_r, B_r, C_r or G_2. If $\overline{G}^{(n)} = \overline{A}_r^{(n)}$, we further assume $n \le r+2$. Let $\overline{\chi}$ be an unramified exceptional character for $\overline{G}^{(n)}$. In each case for $\overline{G}^{(n)}$ below, if* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$, *then the following relations between r and n must hold*:

$$\begin{cases} \overline{A}_r^{(n)}, r \ge 1, n \le r+2, & n = r+2 \ \text{or} \ r+1; \\ \overline{C}_r^{(n)}, r \ge 2, & n = 4r-2,\ 4r,\ 4r+2 \ \text{or} \ 2r+1; \\ \overline{B}_r^{(n)}, r \ge 3, & n = 2r+1 \ \text{or} \ 2r+2; \\ \overline{G}_2^{(n)}, & n = 7 \ \text{or} \ 12. \end{cases}$$

Conversely, suppose that r and n satisfy the above relations; then for every case above except $\overline{C}_r^{(4r)}$, there exists a unique exceptional character $\overline{\chi}$ such that $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$ *for above $\overline{G}^{(n)}$.*

We actually determine the unique exceptional character specified in Theorem 1.2, see Theorems 4.10, 5.3, 6.2 and 7.1. In the $\overline{A}_r^{(r+1)}$ case, our result generalizes the result for the even Weil representation of $\overline{\mathrm{SL}}_2^{(2)}$ mentioned above. As noted, the collection of unramified exceptional characters is a torsor over $Z(\overline{G}^\vee)$. Moreover, for covering groups of simply connected groups, the choice of ψ actually gives a base point for this torsor. Thus, any exceptional character $\overline{\chi}$ gives rise to an element in $Z(\overline{G}^\vee)$, depending on the choice of ψ. That is, the explicit requirement given in those theorems could be viewed as determining the corresponding element in $Z(\overline{G}^\vee)$.

We note that for classical groups and similitude groups, an extensive study is included in [Friedberg et al. $\ge$ 2017]. Our result from Theorem 1.2 also agrees with the pertinent discussion in [Friedberg and Ginzburg 2016b] for symplectic groups. For example, the local statement for the second part of Conjecture 1 in Friedberg and Ginzburg's paper follows from our Proposition 5.1 here. Moreover, the factorizability property of the Whittaker function in that paper for $\overline{\mathrm{Sp}}_{2n}^{(4n-2)}$ also agrees with our result for the $\overline{C}_r^{(n)}$ case in Theorem 1.2.

Finally, we remark that groups of type D_r, E_6, E_7, E_8, F_4 could be analyzed by the same procedure. In principle, Theorem 1.1 coupled with the analogous argument for Theorem 1.2 enable one to determine completely $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ for arbitrary $(\overline{G}^{(n)}, \overline{\chi})$.

2. Basic setup

2A. ***Structural facts on $\overline{G}$.*** For ease of reading, we first recall some structural facts on $\overline{G}$. The main references are [Brylinski and Deligne 2001; Finkelberg and Lysenko 2010; Reich 2012; McNamara 2012; 2016; Weissman 2015; Gan and Gao 2016].

In this paper, we concentrate exclusively on unramified Brylinski–Deligne covering groups $\overline{G}$ (to be explained below). We follow the notations in [Gan and Gao 2016].

Let F be a nonarchimedean field of characteristic 0, with residual characteristic p. Fix a uniformizer ϖ of F. Let $\mathbb{G}$ be a split linear algebraic group over F with maximal split torus $\mathbb{T}$. Write $(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee)$ for the root data of $\mathbb{G}$. Here X (respectively, Y) is the character lattice (respectively, cocharacter lattice) for $(\mathbb{G}, \mathbb{T})$. Choose a set $\Delta \subseteq \Phi$ of simple roots from the set of roots Φ, and $\Delta^\vee$ the corresponding simple coroots from $\Phi^\vee$. Let $\mathbb{B}$ be the Borel subgroup associated with Δ. Write $Y^{\mathrm{sc}} \subseteq Y$ for the lattice generated by $\Phi^\vee$.

Fix a Chevalley system of pinnings for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$. That is, fix an isomorphism $e_\alpha : \mathbb{G}_{\mathrm{a}} \to \mathbb{U}_\alpha$ for each $\alpha \in \Phi$, where $\mathbb{U}_\alpha \subseteq \mathbb{G}$ is the root subgroup associated with α. Moreover, for each $\alpha \in \Phi$, there is a unique morphism $\phi_\alpha : \mathrm{SL}_2 \to \mathbb{G}$ which restricts to $e_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices of SL_2.

Consider the algebro-geometric covering $\overline{\mathbb{G}}$ of $\mathbb{G}$ by $\mathbb{K}_2$, which is categorically equivalent to the pairs $\{(D, \eta)\}$ (see [Gan and Gao 2016]). Here $\eta : Y^{\mathrm{sc}} \to F^\times$ is a homomorphism. On the other hand, D is a bisector associated to a Weyl-invariant quadratic form $Q : Y \to \mathbb{Z}$. That is, let B_Q be the Weyl-invariant bilinear form associated to Q such that $B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$, then D is a bilinear form on Y satisfying

$$D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2).$$

The bisector D is not necessarily symmetric. Any $\overline{\mathbb{G}}$ is, up to isomorphism, incarnated by (i.e., categorically associated to) (D, η) for a bisector D and some η.

Let $n \geq 1$ be a natural number. Assume that $F^\times$ contains the full group μ_n of n-th roots of unity and $p \nmid n$. Let $\overline{\mathbb{G}}$ be incarnated by (D, η). One naturally obtains degree n topological covering groups $\overline{G}, \overline{T}, \overline{B}$ of the rational points $G := \mathbb{G}(F)$, $T := \mathbb{T}(F)$, $B := \mathbb{B}(F)$, such as

$$\mu_n \hookrightarrow \overline{G} \twoheadrightarrow G.$$

We may write $\overline{G}^{(n)}$ for $\overline{G}$ to emphasize the degree of covering. For any set $H \subseteq G$, we write $\overline{H} \subseteq \overline{G}$ for the preimage of H with respect to the quotient map $\overline{G} \to G$. The Bruhat–Tits theory gives a maximal compact subgroup $K \subseteq G$, which depends on the fixed pinnings. We assume that $\overline{G}$ splits over K and fixes such a splitting; call $\overline{G}$ an unramified Brylinski–Deligne covering group in this case. We remark that if the derived group of $\mathbb{G}$ is simply connected, then $\overline{G}$ splits over K (see [Gan and Gao 2016, Theorem 4.2]). On the other hand, we refer the reader to [Gan and Gao 2016, § 4.6] for a counterexample from a certain double cover of PGL_2 where the splitting does not exist.

The data (D, η) play the following role for the structural fact on $\overline{G}$:

- The group $\overline{G}$ splits canonically over any unipotent element of G. In particular, we write $\bar{e}_\alpha(u) \in \overline{G},\ \alpha \in \Phi,\ u \in F$ for the canonical lifting of $e_\alpha(u) \in G$. For any $\alpha \in \Phi$, there is a natural representative $w_\alpha := e_\alpha(1)e_{-\alpha}(-1)e_\alpha(1) \in K$ (and therefore $\bar{w}_\alpha \in \overline{G}$ by the splitting of K) of the Weyl element $\mathbb{w}_\alpha \in W$. Moreover, for $h_\alpha(a) := \alpha^\vee(a) \in G,\ \alpha \in \Phi,\ a \in F^\times$, there is a natural lifting $\bar{h}_\alpha(a) \in \overline{G}$ of $h_\alpha(a)$, which depends only on the pinning and the canonical unipotent splitting. For details, see [Gan and Gao 2016].
- There is a section $\boldsymbol{s}$ of $\overline{T}$ over T such that the group law on $\overline{T}$ is given by

$$\boldsymbol{s}(y_1(a)) \cdot \boldsymbol{s}(y_2(b)) = (a, b)_n^{D(y_1,y_2)} \cdot \boldsymbol{s}(y_1(a) \cdot y_2(b)). \tag{1}$$

  Moreover, for the natural lifting $\bar{h}_\alpha(a)$, one has

$$\bar{h}_\alpha(a) = (\eta(\alpha^\vee), a)_n \cdot \boldsymbol{s}(h_\alpha(a)) \in \overline{T}. \tag{2}$$

- Let $w_\alpha \in G$ be the natural representative of $\mathbb{w}_\alpha \in W$. For any $\overline{y(a)} \in \overline{T}$,

$$w_\alpha \cdot \overline{y(a)} \cdot w_\alpha^{-1} = \overline{y(a)} \cdot \bar{h}_\alpha(a^{-\langle y,\alpha\rangle}), \tag{3}$$

  where $\langle -, - \rangle$ is the pairing between Y and X.

Consider the sublattice $Y_{Q,n} := \{y \in Y : B_Q(y, y') \in n\mathbb{Z}\}$ of Y. For every $\alpha^\vee \in \Phi^\vee$, define $n_\alpha := n/\gcd(n, Q(\alpha^\vee))$. Write $\alpha_{Q,n}^\vee := n_\alpha \alpha^\vee$ and $\alpha_{Q,n} := n_\alpha^{-1}\alpha$. Let $Y_{Q,n}^{\mathrm{sc}} \subseteq Y$ be the sublattice generated by $\{\alpha_{Q,n}^\vee\}_{\alpha\in\Phi}$. The complex dual group $\overline{G}^\vee$ for $\overline{G}$ as given in [Finkelberg and Lysenko 2010; McNamara 2012; Reich 2012] has root data $(Y_{Q,n}, \{\alpha_{Q,n}^\vee\}, \mathrm{Hom}(Y_{Q,n}, \mathbb{Z}), \{\alpha_{Q,n}\})$. In particular, $Y_{Q,n}^{\mathrm{sc}}$ is the root lattice for $\overline{G}^\vee$. What is most pertinent to our paper is that the center $Z(\overline{G}^\vee)$ could be identified as

$$Z(\overline{G}^\vee) := \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times).$$

2B. ***Theta representations*** $\boldsymbol{\Theta(\overline{G}, \overline{\chi})}$**.** Fix an embedding $\iota : \mu_n \hookrightarrow \mathbb{C}^\times$. A representation of $\overline{G}$ is called ι-genuine if μ_n acts via ι. We consider throughout the paper ι-genuine (or simply genuine) representations of $\overline{G}$.

Let U be the unipotent subgroup of $B = TU$. As U splits canonically in $\overline{G}$, we have $\overline{B} = \overline{T}U$. The covering torus $\overline{T}$ is a Heisenberg group with center $Z(\overline{T})$. The image of $Z(\overline{T})$ in T is equal to the image of the isogeny $Y_{Q,n} \otimes F^\times \to T$ induced from $Y_{Q,n} \to Y$.

Let $\overline{\chi} \in \mathrm{Hom}_\iota(Z(\overline{T}), \mathbb{C}^\times)$ be a genuine character of $Z(\overline{T})$, write $i(\overline{\chi}) := \mathrm{Ind}_{\overline{A}}^{\overline{T}}\, \overline{\chi}'$ for the induced representation on $\overline{T}$, where $\overline{A}$ is any maximal abelian subgroup of $\overline{T}$, and $\overline{\chi}'$ is any extension of $\overline{\chi}$. By the Stone–von Neumann theorem (see [Weissman 2009, Theorem 3.1; McNamara 2012, Theorem 3]), the construction $\overline{\chi} \mapsto i(\overline{\chi})$ gives a bijection between isomorphism classes of genuine representations of $Z(\overline{T})$ and $\overline{T}$. Since we consider an unramified covering group $\overline{G}$ in this paper, we take $\overline{A}$ to be $Z(\overline{T}) \cdot (K \cap T)$ from now.

View $i(\overline{\chi})$ as a genuine representation of $\overline{B}$ by inflation from the quotient map $\overline{B} \to \overline{T}$. Write $I(i(\overline{\chi})) := \mathrm{Ind}_{\overline{B}}^{\overline{G}}\ i(\overline{\chi})$ for the normalized induced principal series representation of $\overline{G}$. For simplicity, we may also write $I(\overline{\chi})$ for $I(i(\overline{\chi}))$. One knows that $I(\overline{\chi})$ is unramified (i.e., $I(\overline{\chi})^K \neq 0$) if and only if $\overline{\chi}$ is unramified, i.e., $\overline{\chi}$ is trivial on $Z(\overline{T}) \cap K$. We consider in this paper only unramified genuine representations (and characters). In fact, one has the naturally arising abelian extension

$$\mu_n \hookrightarrow \overline{Y}_{Q,n} \twoheadrightarrow Y_{Q,n} \tag{4}$$

such that unramified genuine characters of $\overline{\chi}$ of $Z(\overline{T})$ correspond to genuine characters of $\overline{Y}_{Q,n}$. Here $\overline{Y}_{Q,n} := Z(\overline{T})/Z(\overline{T}) \cap K$. Since $\overline{A}/(T \cap K) \simeq \overline{Y}_{Q,n}$ as well, there is a canonical extension (also denoted by $\overline{\chi}$) of an unramified character $\overline{\chi}$ of $Z(\overline{T})$ to $\overline{A}$, by composing $\overline{\chi}$ with $\overline{A} \twoheadrightarrow \overline{Y}_{Q,n}$. Therefore, we will identify $i(\overline{\chi})$ as $\mathrm{Ind}_{\overline{A}}^{\overline{T}}\ \overline{\chi}$ for this $\overline{\chi}$.

For any $\mathsf{w} \in W$, the intertwining operator $T_{\mathsf{w},\chi} : I(\overline{\chi}) \to I({}^{\mathsf{w}}\overline{\chi})$ is defined by

$$(T_{\mathsf{w},\overline{\chi}} f)(\bar{g}) = \int_{U_w} f(w^{-1} u \bar{g})\, du$$

whenever it is absolutely convergent. Moreover, it can be meromorphically continued for all $\overline{\chi}$ (see [McNamara 2012, § 7]). For $I(\overline{\chi})$ unramified and $\mathsf{w} = \mathsf{w}_\alpha$ with $\alpha \in \Delta$, $T_{\mathsf{w}_\alpha,\chi}$ is determined by

$$T_{\mathsf{w}_\alpha,\chi}(f_0) = c(\mathsf{w}_\alpha, \overline{\chi}) \cdot f_0' \ \text{ with } \ c(\mathsf{w}_\alpha, \overline{\chi}) = \frac{1 - q^{-1}\overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))}{1 - \overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha})},$$

where $f_0 \in I(\overline{\chi})$ and $f_0' \in I({}^{\mathsf{w}_\alpha}\overline{\chi})$ are the unramified vectors. Moreover, $T_{\mathsf{w},\overline{\chi}}$ satisfies the cocycle condition as in the linear case. The coefficient $c(\mathsf{w}_\alpha, \overline{\chi})$ was determined in [McNamara 2016, Theorem 12.1] and later reformulated in [Gao $\geq$ 2017]. We use the latter formalism which is more suitable for our needs in this paper.

The following definition mimics that in [Kazhdan and Patterson 1984, § I.2].

Definition 2.1. An unramified genuine character $\overline{\chi}$ of $Z(\overline{T})$ is called exceptional if $\overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha})) = q^{-1}$ for all $\alpha \in \Delta$. The theta representation $\Theta(\overline{G}, \overline{\chi})$ associated to an exceptional character $\overline{\chi}$ is the unique Langlands quotient (see [Ban and Jantzen 2013]) of $I(\overline{\chi})$, which is also equal to the image of the intertwining operator $T_{\mathsf{w}_0,\overline{\chi}} : I(\overline{\chi}) \to I({}^{\mathsf{w}_0}\overline{\chi})$, where $\mathsf{w}_0 \in W$ is the longest Weyl element.

The extension $\overline{Y}_{Q,n}$ gives rise to an extension $\overline{Y}_{Q,n}^{\mathrm{sc}}$ of $Y_{Q,n}^{\mathrm{sc}}$ by restriction. All exceptional characters agree on $\overline{Y}_{Q,n}^{\mathrm{sc}}$, and therefore the set of exceptional characters is a torsor over $Z(\overline{G}^\vee)$.

2C. ***Unitary distinguished characters.*** Depending on the choice of a nontrivial additive character ψ' of F, a special class of the so-called distinguished genuine characters of $Z(\overline{T})$ is singled out in [Gan and Gao 2016] for the consideration of the L-group extension for $\overline{G}$. Distinguished characters, in the sense of [Gan and Gao 2016], may not exist for general Brylinski–Deligne covering groups. However, if $\mathbb{G}$ has a simply connected derived group or if the composition

$$\eta : Y^{\mathrm{sc}} \to F^{\times} \to F/(F^{\times})^n$$

is trivial, such characters exist. One special property of a distinguished character is its Weyl-invariance, and thus it could serve as a *distinguished* base point in the set of genuine characters of $Z(\overline{T})$.

For the purpose of Sections 4 to 7, we recall the explicit construction in [Gan and Gao 2016] when a distinguished character exists. In particular, we make the above assumption on $\overline{G}$, which is clearly satisfied in the simply connected case in Sections 4 to 7.

First, let $\{y_i\}$ be a basis of $Y_{Q,n}$ such that $\{k_i y_i\}$ is a basis for the lattice $J = nY + Y_{Q,n}^{\mathrm{sc}}$ for some $k_i \in \mathbb{Z}$. Let ψ' be a nontrivial additive character of F. Let $\boldsymbol{\gamma}_{\psi'}$ be the Weil index valued in μ_4 satisfying

$$\boldsymbol{\gamma}_{\psi'}(b^2) = 1, \quad \boldsymbol{\gamma}_{\psi'}(b)^2 = (b, b)_2, \quad \boldsymbol{\gamma}_{\psi'}(bc) = \boldsymbol{\gamma}_{\psi'}(b)\boldsymbol{\gamma}_{\psi'}(c) \cdot (b, c)_2.$$

For any $a \in F^{\times}$, let $\psi'_a : x \mapsto \psi'(ax)$ be the twisted additive character. Then

$$\boldsymbol{\gamma}_{\psi'_a}(b) = \boldsymbol{\gamma}_{\psi'}(b) \cdot (a, b)_2.$$

By definition, a unitary distinguished character $\overline{\chi}_{\psi'}^{\,0}$ of $Z(\overline{T})$ is given by

$$\overline{\chi}_{\psi'}^{\,0}(y_i(a)) = \boldsymbol{\gamma}_{\psi'}(a)^{2(k_i-1)Q(y_i)/n},$$

and for $y = \sum_i n_i y_i$ and $a \in F^{\times}$,

$$\overline{\chi}_{\psi'}^{\,0}(y(a)) = (a, a)_n^{\sum_{i<j} n_i n_j D(y_i, y_j)} \cdot \prod_i \overline{\chi}_{\psi'}^{\,0}(y_i(a^{n_i}))^{2(k_i-1)Q(y_i)/n}. \tag{5}$$

Note that in [Gan and Gao 2016], the exponent of $\boldsymbol{\gamma}_{\psi'}(a)$ in the formula of $\overline{\chi}_{\psi'}^{\,0}(y_i(a))$ is the negative of what we use here. However, both give rise to distinguished characters.

2D. ***Conventions and notations.*** Let $2\rho := \sum_{\alpha^\vee > 0} \alpha^\vee$ be the sum of all positive coroots of $\mathbb{G}$. Consider the affine translation $\ell_\rho : Y \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$ given by $y \mapsto y - \rho$. Write $\mathsf{w}(y)$ for the natural Weyl group action on Y and $Y \otimes \mathbb{Q}$. Endow the codomain of ℓ_ρ with this action. By transport of structure, one has an induced action of W on the domain of ℓ_ρ (i.e., the first $Y \otimes \mathbb{Q}$), which we denote by $\mathsf{w}[y]$. That is,

$$\mathsf{w}[y] := \mathsf{w}(y - \rho) + \rho.$$

Clearly Y is stable under this action. Write $y_\rho := y - \rho$ for any $y \in Y$, then $\mathsf{w}[y] - y = \mathsf{w}(y_\rho) - y_\rho$. From now, by Weyl orbits in Y or $Y \otimes \mathbb{Q}$ we always refer to the ones with respect to the action $\mathsf{w}[y]$. Write $\mathcal{O}$ (respectively $\mathcal{O}^F$) for the set of W-orbits (respectively, free W-orbits) in Y.

We remark that for $\mathbb{GL}_r$, the Weyl-action considered by Kazhdan and Patterson [1984, page 78] is actually $\mathsf{w}(y+\rho)-\rho$. However, the indexing of Whittaker functionals also differs from ours by taking an "inverse", thus our terminology is different but equivalent to that of [Kazhdan and Patterson 1984].

Definition 2.2. For any subgroup $\Lambda \subseteq Y$, a free orbit $\mathcal{O}_y \in \mathcal{O}^F$ is called Λ-free if the quotient map $Y \to Y/\Lambda$ is injective on $\mathcal{O}_y$. We write $\mathcal{O}^F_\Lambda \subseteq \mathcal{O}^F$ for the set of Λ-free orbits of Y.

Note that Λ-free orbits are assumed to be free by definition. For simplicity, we write $\mathcal{O}^F_{Q,n,\mathrm{sc}}$ and $\mathcal{O}^F_{Q,n}$ for the set of $Y^{\mathrm{sc}}_{Q,n}$ and $Y_{Q,n}$-free orbits of Y, respectively. Clearly, the inclusions $\mathcal{O} \supseteq \mathcal{O}^F \supseteq \mathcal{O}^F_{Q,n,\mathrm{sc}} \supseteq \mathcal{O}^F_{Q,n}$ hold.

Generally, notations will be either self-explanatory or explained the first time they occur. For convenience, we list some notations which appear frequently in the text:

ε: the element $\iota((-1,\varpi)_n) \in \mathbb{C}^\times$. In particular, for n odd, $\varepsilon = 1$. We use the following identity freely in the paper:

$$\varepsilon^{D(y,y')} = \varepsilon^{D(y',y)} \text{ for any } y \in Y_{Q,n},\ y' \in Y.$$

$\wp_{Q,n}$: the projection $Y \to Y/Y_{Q,n}$.

$\wp^{\mathrm{sc}}_{Q,n}$: the projection $Y \to Y/Y^{\mathrm{sc}}_{Q,n}$.

ψ: a fixed additive character of F into $\mathbb{C}^\times$ with conductor O_F. For any $a \in F^\times$, the twisted character ψ_a is given by $\psi_a : x \mapsto \psi(ax)$.

$\boldsymbol{s}_y$: for any $y \in Y$, we write $\boldsymbol{s}_y := \boldsymbol{s}(\varpi^y) \in \overline{T}$.

$\lceil x \rceil$: the minimum integer such that $\lceil x \rceil \geq x$ for a real number x.

3. Bounds for $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$

3A. *Whittaker functionals.* We follow the notations in Section 2B. Consider, in particular, the principal series $I(\overline{\chi}) := I(i(\overline{\chi}))$ for an unramified character $\overline{\chi} \in \mathrm{Hom}_\iota(Z(\overline{T}), \mathbb{C}^\times)$.

Let $\mathrm{Ftn}(i(\overline{\chi}))$ be the vector space of functions $\boldsymbol{c}$ on $\overline{T}$ satisfying

$$\boldsymbol{c}(\bar{t} \cdot \bar{z}) = \boldsymbol{c}(\bar{t}) \cdot \overline{\chi}(\bar{z}), \qquad \bar{t} \in \overline{T} \text{ and } \bar{z} \in \overline{A}.$$

The support of any $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ is a disjoint union of cosets in $\overline{T}/\overline{A}$. Moreover, $\dim(\mathrm{Ftn}(i(\overline{\chi}))) = |Y/Y_{Q,n}|$ since $\overline{T}/\overline{A}$ has the same size as $Y/Y_{Q,n}$.

There is a natural isomorphism of vector spaces $\mathrm{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee$, where $i(\overline{\chi})^\vee$ is the complex dual space of functionals of $i(\overline{\chi})$. More explicitly, letting $\{\gamma_i\} \subseteq \overline{T}$ be a chosen set of representatives of $\overline{T}/\overline{A}$, consider $\boldsymbol{c}_{\gamma_i} \in \mathrm{Ftn}(i(\overline{\chi}))$ which has support $\gamma_i \cdot \overline{A}$ and $\boldsymbol{c}_{\gamma_i}(\gamma_i) = 1$. It gives rise to a linear functional $\lambda^{\overline{\chi}}_{\gamma_i} \in i(\overline{\chi})^\vee$ such that $\lambda^{\overline{\chi}}_{\gamma_i}(f_{\gamma_j}) = \delta_{ij}$, where $f_{\gamma_j} \in i(\overline{\chi})$ is the unique element such that $\mathrm{supp}(f_{\gamma_j}) = \bar{A}\cdot\gamma_j^{-1}$ and $f_{\gamma_j}(\gamma_j^{-1}) = 1$. That is, $f_{\gamma_j} = i(\bar{\chi})(\gamma_j)\phi_0$, where $\phi_0 \in i(\bar{\chi})$ is the normalized unramified vector of $i(\bar{\chi})$ such that $\phi_0(1_{\bar{T}}) = 1$. Thus, the isomorphism $\mathrm{Ftn}(i(\bar{\chi})) \simeq i(\bar{\chi})^\vee$ is given explicitly by

$$\boldsymbol{c} \mapsto \lambda_{\boldsymbol{c}}^{\bar{\chi}} := \sum_{\gamma_i \in \bar{T}/\bar{A}} \boldsymbol{c}(\gamma_i)\lambda_{\gamma_i}^{\bar{\chi}}.$$

It can be checked easily that the isomorphism does not depend on the choice of representatives for $\bar{T}/\bar{A}$.

Let $\psi_U : U \to \mathbb{C}^\times$ be the character on U such that its restriction to every U_α, $\alpha \in \Delta$ is given by $\psi \circ e_\alpha^{-1}$. We may write ψ for ψ_U if no confusion arises.

Definition 3.1. For any genuine representation $(\bar{\sigma}, V_{\bar{\sigma}})$ of $\bar{G}$, a linear functional $\ell : V_{\bar{\sigma}} \to \mathbb{C}$ is called a ψ-Whittaker functional if $\ell(\bar{\sigma}(u)v) = \psi(u)\cdot v$ for all $u \in U$ and $v \in V_{\bar{\sigma}}$. Write $\mathrm{Wh}_\psi(\bar{\sigma})$ for the space of ψ-Whittaker functionals for $\bar{\sigma}$.

An isomorphism exists between $i(\bar{\chi})^\vee$ and the space $\mathrm{Wh}_\psi(I(\bar{\chi}))$ of ψ-Whittaker functionals on $I(\bar{\chi})$ (see [McNamara 2016, § 6]), given by $\lambda \mapsto W_\lambda$ with

$$W_\lambda : I(\bar{\chi}) \to \mathbb{C}, \quad f \mapsto \lambda\left(\int_U f(w_0^{-1}u)\psi(u)^{-1}\mu(u)\right),$$

where $f \in I(\bar{\chi})$ is an $i(\bar{\chi})$-valued function on $\bar{G}$. Here U^- is the unipotent subgroup opposite to U; also, $w_0 = w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_k} \in K$ is a representative of w_0, where $\mathsf{w}_0 = \mathsf{w}_{\alpha_1}\mathsf{w}_{\alpha_2}\cdots\mathsf{w}_{\alpha_k}$ is a minimum decomposition of w_0. For any $\boldsymbol{c} \in \mathrm{Ftn}(i(\bar{\chi}))$, by abuse of notation, we will write $\lambda_{\boldsymbol{c}}^{\bar{\chi}} \in \mathrm{Wh}_\psi(I(\bar{\chi}))$ for the resulting ψ-Whittaker functional of $I(\bar{\chi})$ from the isomorphism $\mathrm{Ftn}(i(\bar{\chi})) \simeq i(\bar{\chi})^\vee \simeq \mathrm{Wh}_\psi(I(\bar{\chi}))$. An easy consequence is

$$\dim \mathrm{Wh}_\psi(I(\bar{\chi})) = |Y/Y_{Q,n}|.$$

Let $J(\mathsf{w}, \bar{\chi})$ be the image of $T_{\mathsf{w},\bar{\chi}}$. The operator $T_{\mathsf{w},\bar{\chi}}$ induces a homomorphism $T^*_{\mathsf{w},\bar{\chi}}$ of vectors spaces with image $\mathrm{Wh}_\psi(J(\mathsf{w}, \bar{\chi}))$:

$$\begin{tikzcd} T^*_{\mathsf{w},\bar{\chi}} : \mathrm{Wh}_\psi(I({}^{\mathsf{w}}\bar{\chi})) \arrow[r] \arrow[dr, two heads] & \mathrm{Wh}_\psi(I(\bar{\chi})) \\ & \mathrm{Wh}_\psi(J(\mathsf{w}, \bar{\chi})) \arrow[u, hook] \end{tikzcd}$$

given by $\langle \lambda_{\boldsymbol{c}}^{{}^{\mathsf{w}}\bar{\chi}}, - \rangle \mapsto \langle \lambda_{\boldsymbol{c}}^{{}^{\mathsf{w}}\bar{\chi}}, T_{\mathsf{w},\bar{\chi}}(-)\rangle$ for any $\boldsymbol{c} \in \mathrm{Ftn}(i({}^{\mathsf{w}}\bar{\chi}))$. Letting $\{\lambda_\gamma^{{}^{\mathsf{w}}\bar{\chi}}\}_{\gamma \in \bar{T}/\bar{A}}$ be a basis for $\mathrm{Wh}_\psi(I({}^{\mathsf{w}}\bar{\chi}))$, and $\{\lambda_{\gamma'}^{\bar{\chi}}\}$ a basis for $\mathrm{Wh}_\psi(I(\bar{\chi}))$, the map $T^*_{\mathsf{w},\bar{\chi}}$ is then determined by the square matrix $[\tau(\bar{\chi}, \mathsf{w}, \gamma, \gamma')]_{\gamma,\gamma' \in \bar{T}/\bar{A}}$ of size $|Y/Y_{Q,n}|$ such that

$$T^*_{\mathsf{w},\bar{\chi}}(\lambda_\gamma^{{}^{\mathsf{w}}\bar{\chi}}) = \sum_{\gamma' \in \bar{T}/\bar{A}} \tau(\bar{\chi}, \mathsf{w}, \gamma, \gamma')\cdot\lambda_{\gamma'}^{\bar{\chi}}.$$

Some immediate properties are as follows.

Lemma 3.2. *For* $\mathbb{w} \in W$ *and* $\bar{z}, \bar{z}' \in \bar{A}$, *the following identity holds*:

$$\tau(\bar{\chi}, \mathbb{w}, \gamma \cdot \bar{z}, \gamma' \cdot \bar{z}') = ({}^{\mathbb{w}}\bar{\chi})^{-1}(\bar{z}) \cdot \tau(\bar{\chi}, \mathbb{w}, \gamma, \gamma') \cdot \bar{\chi}(\bar{z}').$$

Moreover, for $\mathbb{w}_1, \mathbb{w}_2 \in W$ *such that* $l(\mathbb{w}_2\mathbb{w}_1) = l(\mathbb{w}_2) + l(\mathbb{w}_1)$, *one has*

$$\tau(\bar{\chi}, \mathbb{w}_2\mathbb{w}_1, \gamma, \gamma') = \sum_{\gamma'' \in \bar{T}/\bar{A}} \tau({}^{\mathbb{w}_1}\bar{\chi}, \mathbb{w}_2, \gamma, \gamma'') \cdot \tau(\bar{\chi}, \mathbb{w}_1, \gamma'', \gamma'),$$

which is referred to as the cocycle relation.

Proof. The first equality follows from a change of basis formula from a different choice of representations for $\bar{T}/\bar{A}$. The second equality follows from the cocycle relation of intertwining operators. □

3B. *Reduction of* $\mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi}))$. Let $\mathbb{w}_0$ be the longest Weyl element of $\mathbb{G}$. Consider the theta representation $\Theta(\bar{G}, \bar{\chi}) = T_{\mathbb{w}_0, \bar{\chi}}(I(\bar{\chi}))$ attached to an unramified exceptional character $\bar{\chi}$ (see Definition 2.1).

Definition 3.3. A theta representation $\Theta(\bar{G}, \bar{\chi})$ attached to an unramified exceptional genuine character $\bar{\chi}$ is called distinguished if

$$\dim \mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) = 1.$$

The distinguishedness of a theta representation here is not to be confused with that of a distinguished genuine character as given in Section 2C.

Proposition 3.4. *Let* $\bar{\chi}$ *be an unramified exceptional character of* $\bar{G}$, *and* Δ *the set of simple roots. Then*

$$\mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) = \bigcap_{\alpha \in \Delta} \mathrm{Ker}(T^*_{\mathbb{w}_\alpha, {}^{\mathbb{w}_\alpha}\bar{\chi}} : \mathrm{Wh}_\psi(I(\bar{\chi})) \to \mathrm{Wh}_\psi(I({}^{\mathbb{w}_\alpha}\bar{\chi}))),$$

where $T_{\mathbb{w}_\alpha, {}^{\mathbb{w}_\alpha}\bar{\chi}}$ *is the intertwining operator from* $I({}^{\mathbb{w}_\alpha}\bar{\chi})$ *to* $I(\bar{\chi})$.

Proof. The same proof for [Kazhdan and Patterson 1984, Theorem I.2.9] applies here mutatis mutandis. □

Let $\lambda^{\bar{\chi}}_\gamma \in \mathrm{Wh}_\psi(I(\bar{\chi}))$ and $\alpha \in \Delta$, then

$$T^*_{\mathbb{w}_\alpha, {}^{\mathbb{w}_\alpha}\bar{\chi}}(\lambda^{\bar{\chi}}_\gamma) = \sum_{\gamma'} \tau\left({}^{\mathbb{w}_\alpha}\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\right) \cdot \lambda^{{}^{\mathbb{w}_\alpha}\bar{\chi}}_{\gamma'}.$$

In general, let $\boldsymbol{c} \in \mathrm{Ftn}(i(\bar{\chi}))$, and write

$$\lambda^{\bar{\chi}}_{\boldsymbol{c}} = \sum_{\gamma \in \bar{T}/\bar{A}} \boldsymbol{c}(\gamma) \lambda^{\bar{\chi}}_\gamma.$$

Then,

$$T^*_{\mathbb{w}_\alpha, {}^{\mathbb{w}_\alpha}\overline{\chi}}(\lambda^{\overline{\chi}}_{\boldsymbol{c}}) = \sum_{\gamma} \boldsymbol{c}(\gamma)\Big(\sum_{\gamma'} \tau\big({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\big) \cdot \lambda^{{}^{\mathbb{w}_\alpha}\overline{\chi}}_{\gamma'}\Big)$$
$$= \sum_{\gamma'}\Big(\sum_{\gamma} \boldsymbol{c}(\gamma)\tau\big({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\big)\Big)\lambda^{{}^{\mathbb{w}_\alpha}\overline{\chi}}_{\gamma'}.$$

As an immediate consequence of Proposition 3.4, one has (see also [Kazhdan and Patterson 1984, page 76]):

Corollary 3.5. *A function* $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ *gives rise to a functional in* $\mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ *(i.e.,* $\lambda^{\overline{\chi}}_{\boldsymbol{c}} \in \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$*) if and only if for all* $\alpha \in \Delta$,

$$\sum_{\gamma \in \overline{T}/\overline{A}} \boldsymbol{c}(\gamma)\tau\big({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\big) = 0 \textit{ for all } \gamma'.$$

The left-hand side is independent of the choice of representatives for $\overline{T}/\overline{A}$ *by Lemma 3.2.*

3C. ***The Shahidi local coefficient matrix.*** We would like to compute the matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ for any unramified character $\overline{\chi}$ (not necessarily exceptional) and simple reflection $\mathbb{w}_\alpha$, $\alpha \in \Delta$.

For Kazhdan–Patterson coverings $\overline{\mathrm{GL}}^{(n)}_r$, the matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ is first studied in [Kazhdan and Patterson 1984]. It also appears in the work of Suzuki [1998], Chinta and Offen [2013] among others. For a subclass of Brylinski–Deligne covering groups, the study of matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ is conducted in [McNamara 2016] for unramified characters χ, generalizing that of Kazhdan and Patterson. Meanwhile, for ramified characters, it is included in the work of [Goldberg and Szpruch 2015]. However, in order to work with the full class of Brylinski–Deligne covering groups and also remove the assumption $\mu_{2n} \subseteq F^\times$ in [McNamara 2016], we refine the computation in [McNamara 2016] slightly. This is achieved by invoking the structural facts of Brylinski–Deligne covering groups, in particular those from Section 2A. We also note that interesting phenomena dissipate when the assumption $\mu_{2n} \subseteq F^\times$ is imposed, for example for the type A_r case in Section 4. There are subtleties arising from the fact that -1 is not a square root. For this purpose, it is important to rigidify the formula for the matrix and express its entries in terms of naturally defined elements of the group.

Consider the Haar measure μ of F such that $\mu(O_F) = 1$. Thus,

$$\mu(O_F^\times) = 1 - 1/q.$$

The Gauss sum is given by

$$G_\psi(a, b) = \int_{O_F^\times} (u, \varpi)_n^a \cdot \psi(\varpi^b u)\mu(u), \quad a, b \in \mathbb{Z}.$$

It is known that

$$G_\psi(a,b)=\begin{cases}0 & \text{if } b<-1,\\ 1-1/q & \text{if } n|a, b\geq 0,\\ 0 & \text{if } n\nmid a, b\geq 0,\\ -1/q & \text{if } n|a, b=-1,\\ G_\psi(a,-1) \text{ with } |G_\psi(a,-1)|=q^{-1/2} & \text{if } n\nmid a, b=-1.\end{cases}$$

Recall $\varepsilon := \iota((-1,\varpi)_n)\in\mathbb{C}^\times$. One has $\overline{G_\psi(a,b)}=\varepsilon^a\cdot G_\psi(-a,b)$. For any $k\in\mathbb{Z}$, we write

$$\boldsymbol{g}_\psi(k):=G_\psi(k,-1).$$

As in [McNamara 2016, § 9], let $f_{\gamma'}\in I(\overline{\chi})$ be the function with $\mathrm{supp}(f_{\gamma'})=\overline{B}w_0K_1$, and $f_{\gamma'}(w_0^{-1})=i(\overline{\chi})(\gamma')\phi_0$ for a certain compact open subgroup K_1. Here $\phi_0\in i(\overline{\chi})^{T\cap K}$ is the unramified vector in $i(\overline{\chi})$. From [McNamara 2016, Corollary 9.2], one has $\tau(\overline{\chi},\mathsf{w}_\alpha,\gamma,\gamma')=\langle\lambda_\gamma^{{}^{\mathsf{w}_\alpha}\overline{\chi}}, T_{\mathsf{w}_\alpha,\overline{\chi}}(f_{\gamma'})\rangle/|U^-\cap K_1|$. More precisely, from equality (9.3) of [McNamara 2016] one could evaluate $\tau(\overline{\chi},\mathsf{w}_\alpha,\gamma,\gamma')$ by applying $\lambda_\gamma^{{}^{\mathsf{w}_\alpha}\overline{\chi}}\in i\big({}^{\mathsf{w}_\alpha}\overline{\chi}\big)^\vee$ to the integral

$$\int_F f_{\gamma'}\big(\bar{h}_\alpha(x^{-1})\cdot\bar{e}_\alpha(-x)\cdot w_0^{-1}\big)\cdot\psi^{-1}\big(\bar{e}_\alpha(x^{-1})\big)\mu(x)\in i\big({}^{\mathsf{w}_\alpha}\overline{\chi}\big). \tag{6}$$

Note that the integrand of (6) takes values in $i(\overline{\chi})$. However, on the one hand, as vector spaces of functions on $\overline{T}$, the underlying space $i(\overline{\chi})$ is identical to that of ${}^{w_\alpha}i(\overline{\chi})$ (see [Gao $\geq$ 2017]); on the other hand, it follows from the Stone–von Neumann theorem that ${}^{w_\alpha}i(\overline{\chi})\simeq i({}^{\mathsf{w}_\alpha}\overline{\chi})$ as representations of $\overline{T}$. Therefore, there is a canonical vector space isomorphism $i(\overline{\chi})\simeq i({}^{\mathsf{w}_\alpha}\overline{\chi})$. For the computation below, we will follow [McNamara 2016] closely and adopt this viewpoint implicitly.

To ease notations, write $\pi=i(\overline{\chi})$. Use the partition $F=\bigcup_{m\in\mathbb{Z}}\varpi^{-m}O_F^\times$ and write $x=\varpi^{-m}u^{-1}$, where $u\in O_F^\times$. Then $\mu(x)=|\varpi|^{-m}\mu(u)$ and the integral in (6) is equal to

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}|\varpi|^{-m}\int_{O_F^\times} f_{\gamma'}(\bar{h}_\alpha(\varpi^m\cdot u)\cdot\bar{e}_\alpha(-\varpi^{-m}u^{-1})\cdot w_0^{-1})\cdot\psi^{-1}(\bar{e}_\alpha(\varpi^m\cdot u))\mu(u)\\ &\quad=\sum_{m\in\mathbb{Z}}\int_{O_F^\times}(u,\varpi)_n^{mQ(\alpha^\vee)}\cdot\pi(\bar{h}_\alpha(\varpi^m))\cdot\pi(\bar{h}_\alpha(u))\cdot\pi(\gamma')\phi_0\cdot\psi^{-1}(\varpi^m\cdot u)\mu(u).\end{aligned}$$

Suppose $\gamma'=\boldsymbol{s}_y\in\overline{T}$ for some $y\in Y$. (We write $\boldsymbol{s}_y:=\boldsymbol{s}(\varpi^y)\in\overline{T}$ for $y\in Y$, see Section 2 for notations.) Then the above is equal to

$$\sum_{m\in\mathbb{Z}}\int_{O_F^\times}(u,\varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee,y)}\cdot\pi(\bar{h}_\alpha(\varpi^m))\cdot\pi(\boldsymbol{s}_y)\phi_0\cdot\psi^{-1}(\varpi^m\cdot u)\mu(u). \tag{7}$$

From now, we write $\Gamma(m, y, \alpha^\vee) := \varepsilon^{(m+\langle y,\alpha\rangle)D(y,\alpha^\vee)}$ and $\mathbf{\Gamma}(y, \alpha^\vee) := \Gamma(-1, y, \alpha^\vee)$, which lie in $\{\pm 1\}$. Following (3), $\bar{h}_\alpha(\varpi^m) \cdot \boldsymbol{s}_y = w_\alpha \cdot (\Gamma(m, y, \alpha^\vee) \cdot \boldsymbol{s}_{y+m\alpha^\vee}) \cdot w_\alpha^{-1}$. Therefore (7) is equal to

$$\sum_{m\in\mathbb{Z}} \Gamma(m, y, \alpha^\vee) \cdot {}^{w_\alpha}\pi\left(\boldsymbol{s}_{\mathbb{w}_\alpha(y+m\alpha^\vee)}\right)\phi_0 \cdot \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee,y)} \psi^{-1}(\varpi^m \cdot u)\mu(u).$$

There are three cases for each term in the sum:

- For $m \le -2$, the integral over $O_F^\times$ vanishes, and thus the contribution to $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ is 0.
- For $m = -1$, the contribution $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ is nonzero only when $\mathbb{w}_\alpha(y_1) \equiv y - \alpha^\vee \bmod Y_{Q,n}$ where $\gamma = \boldsymbol{s}_{y_1}$, $\gamma' = \boldsymbol{s}_y$. When $\mathbb{w}_\alpha(y_1) = y - \alpha^\vee$, the contribution to $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ is

$$\mathbf{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(B(\alpha^\vee, y) - Q(\alpha^\vee)) = \mathbf{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee)).$$

- For any $x \in \mathbb{R}$, recall that we denote by $\lceil x \rceil$ the minimum integer such that $\lceil x \rceil \ge x$. The sum for $m \ge 0$ is equal to

$$\begin{aligned}
&\sum_{m\ge 0} \Gamma(m, y, \alpha^\vee) \cdot {}^{w_\alpha}\pi(\boldsymbol{s}_{\mathbb{w}_\alpha(y+m\alpha^\vee)})\phi_0 \cdot \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee,y)}\mu(u) \\
&\quad = \sum_{\substack{m=k\cdot n_\alpha - B(\alpha^\vee,y)/Q(\alpha^\vee) \\ k\ge\lceil B(\alpha^\vee,y)/n_\alpha Q(\alpha^\vee)\rceil}} \begin{aligned}[t] &\Gamma(m, y, \alpha^\vee) \cdot \varepsilon^{(m+\langle y,\alpha,\rangle)D(\alpha^\vee,y)} \\ &\cdot {}^{w_\alpha}\pi(\boldsymbol{s}_{(-m-\langle y,\alpha\rangle)\alpha^\vee})\, {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0 \cdot (1-q^{-1})\end{aligned} \\
&\quad = (1-q^{-1}) \sum_{k\ge\lceil\langle y,\alpha^\vee\rangle/n_\alpha\rceil} \varepsilon^{kn_\alpha B(\alpha^\vee,y)} \cdot {}^{w_\alpha}\pi(\bar{h}_\alpha(\varpi^{-kn_\alpha})) \cdot {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0 \\
&\quad = (1-q^{-1}) \sum_{k\ge\lceil\langle y,\alpha^\vee\rangle/n_\alpha\rceil} \bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^k \cdot {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0 \\
&\quad = (1-q^{-1}) \frac{\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1-\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))} \cdot {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0, \quad \text{where } k_{y,\alpha} = \lceil\langle y,\alpha\rangle/n_\alpha\rceil.
\end{aligned}$$

The contribution is nonzero only for $\gamma = \boldsymbol{s}_{y_1}$ with $y_1 \equiv y \bmod Y_{Q,n}$. In particular, if $y_1 = y$, then the contribution to $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ (for $\gamma = \gamma' = \boldsymbol{s}_y$) is

$$(1-q^{-1}) \frac{\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1-\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))}, \quad \text{where } k_{y,\alpha} = \left\lceil \frac{\langle y,\alpha\rangle}{n_\alpha} \right\rceil.$$

To summarize, we state the following theorem by McNamara which generalizes [Kazhdan and Patterson 1984, Lemma I.3.3]:

Theorem 3.6 [McNamara 2016, Theorem 13.1]. *Suppose that $\gamma = \boldsymbol{s}_{y_1}$ is represented by y_1 and $\gamma' = \boldsymbol{s}_y$ by y. Then we can write $\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = \tau^1(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') + \tau^2(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ with the following properties*:

- $\tau^i(\overline{\chi}, \mathbb{w}_\alpha, \gamma \cdot \bar{z}, \gamma' \cdot \bar{z}') = ({}^{\mathbb{w}_\alpha}\overline{\chi})^{-1}(\bar{z}) \cdot \tau^i(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') \cdot \overline{\chi}(\bar{z}'), \qquad \bar{z}, \bar{z}' \in \overline{A}$;
- $\tau^1(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = 0$ *unless* $y_1 \equiv y \mod Y_{Q,n}$;
- $\tau^2(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = 0$ *unless* $y_1 \equiv \mathbb{w}_\alpha[y] \mod Y_{Q,n}$.

Moreover,

- *If $y_1 = y$, then*
$$\tau^1(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = (1 - q^{-1}) \frac{\overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1 - \overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))}, \quad \text{where } k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil.$$
- *If $y_1 = \mathbb{w}_\alpha[y]$, then*
$$\tau^2(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee)).$$

As an analogue of [Kazhdan and Patterson 1984, Corollary I.3.4], we have the following result.

Corollary 3.7. *Let $\overline{\chi}$ be an unramified exceptional character. Let $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(I(\overline{\chi}))$ be the ψ-Whittaker functional of $I(\overline{\chi})$ associated to some $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$. Then, $\lambda_{\boldsymbol{c}}^{\overline{\chi}}$ lies in $\mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ if and only if for any simple root $\alpha \in \Delta$ one has*

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]})) = q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \text{ for all } y. \tag{8}$$

Proof. By Corollary 3.5, for all $\alpha \in \Delta$, we have

$$\boldsymbol{c}(\boldsymbol{s}_y) \cdot \tau({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \boldsymbol{s}_y, \boldsymbol{s}_y) + \boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) \cdot \tau({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \boldsymbol{s}_{\mathbb{w}_\alpha[y]}, \boldsymbol{s}_y) = 0,$$

where $y \in Y$ is any element. The preceding theorem gives

$$\begin{aligned}\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) &= -(1-q^{-1}) \frac{(\overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha})))^{-k_{y,\alpha}}}{1-\overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{-1}} \cdot \boldsymbol{\Gamma}(y,\alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \\ &= q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y,\alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y).\end{aligned}$$ □

From now on, for $y \in Y$ and $\alpha \in \Delta$, we write

$$\boldsymbol{t}(\mathbb{w}_\alpha, y) := q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1}, \tag{9}$$

where

$$k_{y,\alpha} = \left\lceil \frac{\langle y_\rho, \alpha \rangle + 1}{n_\alpha} \right\rceil \text{ and } \boldsymbol{\Gamma}(y, \alpha^\vee) = \varepsilon^{\langle y_\rho, \alpha \rangle \cdot D(y, \alpha^\vee)}.$$

It is clear $\boldsymbol{t}(\mathbb{w}_\alpha, y) \neq 0$.

Definition 3.8. For $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$, we say that $\boldsymbol{c}$ vanishes on $y \in Y$ if and only if $\boldsymbol{c}(\boldsymbol{s}_y) = 0$. It is said to vanish on the orbit $\mathcal{O}_{y_0} \subset Y$ if and only if it vanishes on all $y \in \mathcal{O}_{y_0}$, in which case we write $\boldsymbol{c}(\mathcal{O}_{y_0}) = 0$.

Assume that $\boldsymbol{c}$ gives rise to $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$. Since $\boldsymbol{t}(\mathbb{w}_\alpha, y) \neq 0$ for all y and $\alpha \in \Delta$, it follows from Corollary 3.7 that $\boldsymbol{c}$ vanishes on $\mathcal{O}_{y_0}$ if and only if it vanishes on any $y \in \mathcal{O}_{y_0}$. It is therefore easy to see that

$$\text{(10)}\quad \dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = \left| \left\{ \wp_{Q,n}(\mathcal{O}_{y_0}) : \begin{array}{l} \mathcal{O}_{y_0} \in \mathcal{O} \text{ is a } W\text{-orbit in } Y, \text{ and there} \\ \text{exists } \boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi})) \text{ satisfying (8) for} \\ \text{all } \alpha \in \Delta,\ y \in \mathcal{O}_{y_0}. \text{ Also } \boldsymbol{c}(\mathcal{O}_{y_0}) \neq 0. \end{array} \right\} \right|$$

In the remaining part of this section we will prove an effective lower and upper bound for $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$.

3D. *A lower bound for* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$.

The Weyl group W of $\mathbb{G}$ has the presentation

$$W = \langle \mathbb{w}_\alpha : (\mathbb{w}_\alpha \mathbb{w}_\beta)^{m_{\alpha\beta}} = 1 \text{ for } \alpha, \beta \in \Delta \rangle.$$

Lemma 3.9. *Let $\mathcal{O}_y \in \mathcal{O}^F_{Q,n,\mathrm{sc}}$ be a $Y^{\mathrm{sc}}_{Q,n}$-free orbit in Y. Then the following holds*:

$$\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) = 1 \ \textit{for all}\ \alpha \in \Delta.$$

Proof. Note that $\mathbb{w}_\alpha[y] = \mathbb{w}_\alpha(y) + \alpha^\vee = y + (1 - \langle y, \alpha \rangle)\alpha^\vee$. It follows that $\langle \mathbb{w}_\alpha[y], \alpha \rangle = 2 - \langle y, \alpha \rangle$. Therefore

$$\begin{aligned} &\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \\ &\quad = q^{\lceil \langle \mathbb{w}_\alpha[y], \alpha \rangle / n_\alpha \rceil - 1} \cdot \boldsymbol{\Gamma}(\mathbb{w}_\alpha[y], \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha^\vee)(\langle \mathbb{w}_\alpha[y], \alpha \rangle - 1))^{-1} \\ &\quad = q^{\lceil (2 - \langle y, \alpha \rangle)/n_\alpha \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle \cdot (D(y, \alpha^\vee) - \langle y_\rho, \alpha^\vee \rangle Q(\alpha^\vee))} \cdot \boldsymbol{g}_{\psi^{-1}}(-Q(\alpha^\vee)\langle y_\rho, \alpha \rangle)^{-1} \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) \\ &\qquad = q^{\lceil (2 - \langle y, \alpha \rangle)/n_\alpha \rceil + \lceil \langle y, \alpha \rangle / n_\alpha \rceil - 2} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle^2 \cdot Q(\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha^\vee) \cdot \langle y_\rho, \alpha \rangle)^{-1} \\ &\qquad\qquad \cdot \boldsymbol{g}_{\psi^{-1}}(-Q(\alpha^\vee) \cdot \langle y_\rho, \alpha \rangle)^{-1}. \end{aligned}$$

However, it follows from $\boldsymbol{g}_{\psi^{-1}}(k) = \varepsilon^k \cdot \overline{\boldsymbol{g}_{\psi^{-1}}(-k)}$ that $|\boldsymbol{g}_{\psi^{-1}}(k)| = q^{-1/2}$. Moreover, since $\mathcal{O}_y$ is a $Y^{\mathrm{sc}}_{Q,n}$-free orbit, $\mathbb{w}_\alpha[y] - y \notin Y^{\mathrm{sc}}_{Q,n}$. Therefore, $n_\alpha \nmid (1 - \langle y, \alpha \rangle)$ and so

$$\left\lceil \frac{2 - \langle y, \alpha \rangle}{n_\alpha} \right\rceil + \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil = 1.$$

Now it can be checked easily that $\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) = 1$. $\square$

Consider adjacent $\alpha, \beta \in \Delta$ from the Dynkin diagram. We would like to show that for the $Y_{Q,n}^{\mathrm{sc}}$-free orbit $\mathcal{O}_y$ the equality

$$\prod_{i=1}^{m_{\alpha\beta}} \boldsymbol{t}(\mathbb{w}_\alpha \mathbb{w}_\beta, (\mathbb{w}_\alpha \mathbb{w}_\beta)^i[y]) = 1$$

holds, where $\boldsymbol{t}(\mathbb{w}_\alpha \mathbb{w}_\beta, y) := \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y)$. This will follow from a case by case discussion. We will give the details for $m_{\alpha\beta} = 3, 4$ below and leave the case for $m_{\alpha\beta} = 6$ to the reader.

Case $m_{\alpha\beta} = 3$: The relation $(\mathbb{w}_\alpha \mathbb{w}_\beta)^{m_{\alpha\beta}} = 1$ is equivalent to $\mathbb{w}_\alpha \mathbb{w}_\beta \mathbb{w}_\alpha = \mathbb{w}_\beta \mathbb{w}_\alpha \mathbb{w}_\beta$. By Lemma 3.9, it suffices to show

$$\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) = \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y). \tag{11}$$

We first note that

$$\boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle + 1}{n_\alpha} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle \cdot D(y, \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(B_Q(y_\rho, \alpha^\vee))^{-1}.$$

We also have $\langle \mathbb{w}_\beta \mathbb{w}_\alpha(y_\rho), \alpha\rangle = \langle y_\rho, \beta\rangle$ since the pairing $\langle -, -\rangle$ is W-equivariant and $\mathbb{w}_\alpha \mathbb{w}_\beta(\alpha) = \beta$. Similarly, $\langle \mathbb{w}_\alpha \mathbb{w}_\beta(y_\rho), \beta\rangle = \langle y_\rho, \alpha\rangle$. A simple computation gives

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle + 1}{n_\alpha} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle D(y, \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \alpha+\beta\rangle + 1}{n_\beta} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha+\beta\rangle D(\mathbb{w}_\alpha[y], \beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \beta\rangle + 1}{n_\alpha} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \beta\rangle D(\mathbb{w}_\beta \mathbb{w}_\alpha[y], \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \beta\rangle Q(\alpha^\vee))^{-1}. \end{cases}$$

Meanwhile,

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\beta, y) = q^{\lceil \frac{\langle y_\rho, \beta\rangle + 1}{n_\beta} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \beta\rangle D(y, \beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) = q^{\lceil \frac{\langle y_\rho, \alpha+\beta\rangle + 1}{n_\alpha} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha+\beta\rangle D(\mathbb{w}_\beta[y], \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+\beta\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta \mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle + 1}{n_\beta} \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle D(\mathbb{w}_\alpha \mathbb{w}_\beta[y], \beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\beta^\vee))^{-1}. \end{cases}$$

Since $Q(\alpha^\vee) = Q(\beta^\vee)$ and thus $n_\alpha = n_\beta$, to show that (11) holds, it suffices to check that the powers of ε on the two sides of (11) are equal. However, a straightforward computation shows that this is indeed the case, and we may omit the details.

Case $m_{\alpha\beta} = 4$: Let $\alpha, \beta \in \Delta$ be two adjacent roots such that $m_{\alpha\beta} = 4$. We assume that α is the longer one. Thus, $\langle \alpha^\vee, \beta\rangle = -1$, $\langle \beta^\vee, \alpha\rangle = -2$, and $Q(\beta^\vee) = 2Q(\alpha^\vee)$. As in the preceding case, we want to show

$$\begin{aligned} &\boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) \\ &\qquad = \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y). \end{aligned} \tag{12}$$

A simple computation yields

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle+1}{n_\alpha}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\alpha\rangle D(y,\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \alpha+\beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\alpha+\beta\rangle D(\mathbb{w}_\alpha[y],\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha+\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta\mathbb{w}_\alpha[y]) \\ \qquad = q^{\lceil \frac{\langle y_\rho, \alpha+2\beta\rangle+1}{n_\alpha}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\alpha+2\beta\rangle D(\mathbb{w}_\beta\mathbb{w}_\alpha[y],\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha+2\beta\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha\mathbb{w}_\beta\mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\beta\rangle D(\mathbb{w}_\alpha\mathbb{w}_\beta\mathbb{w}_\alpha[y],\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\beta\rangle Q(\beta^\vee))^{-1}. \end{cases}$$

On the other hand, for the right-hand side of (12), one has

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\beta, y) = q^{\lceil \frac{\langle y_\rho, \beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\beta\rangle D(y,\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) = q^{\lceil \frac{\langle y_\rho, \alpha+2\beta\rangle+1}{n_\alpha}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\alpha+2\beta\rangle D(\mathbb{w}_\beta[y],\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha+2\beta\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha\mathbb{w}_\beta\mathbb{w}_\alpha[y]) \\ \qquad = q^{\lceil \frac{\langle y_\rho, \alpha+\beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho,\alpha+\beta\rangle D(\mathbb{w}_\alpha\mathbb{w}_\beta[y],\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha+\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta\mathbb{w}_\alpha\mathbb{w}_\beta[y]) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle+1}{n_\alpha}\rceil -1} \varepsilon^{\langle y_\rho,\alpha\rangle D(\mathbb{w}_\beta\mathbb{w}_\alpha\mathbb{w}_\beta[y],\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho,\alpha\rangle Q(\alpha^\vee))^{-1}. \end{cases}$$

To show equality (12), again it suffices to show that the powers of ε of the two sides have the same parities, which is achieved from a straightforward check.

Analogous argument for $m_{\alpha\beta} = 6$ works, and we give a summary.

Proposition 3.10. *Let $\mathcal{O}_y$ be a $Y_{Q,n}^{\mathrm{sc}}$-free orbit. For all adjacent α, $\beta \in \Delta$, one has*

$$\prod_{i=1}^{m_{\alpha\beta}} \boldsymbol{t}(\mathbb{w}_\alpha\mathbb{w}_\beta, (\mathbb{w}_\alpha\mathbb{w}_\beta)^i[y]) = 1,$$

where $\boldsymbol{t}(\mathbb{w}_\alpha\mathbb{w}_\beta, y) := \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y)$.

Definition 3.11. Let $\mathcal{O}_y \in \mathcal{O}_{Q,n,\mathrm{sc}}^{F}$ be a $Y_{Q,n}^{\mathrm{sc}}$-free orbit. For any

$$\mathbb{w} = \mathbb{w}_k\mathbb{w}_{k-1}\cdots\mathbb{w}_2\mathbb{w}_1 \in W$$

written as a minimum expansion, write

$$\boldsymbol{T}(\mathbb{w}, y) := \prod_{i=1}^{k} \boldsymbol{t}(\mathbb{w}_i, \mathbb{w}_{i-1}\cdots\mathbb{w}_1[y]),$$

which, by Lemma 3.9 and Proposition 3.10, is independent of the choice of minimum expansion of $\mathbb{w}$.

Let $\mathcal{O}_y \in \mathcal{O}_{Q,n}^{F}$ be a $Y_{Q,n}$-free orbit (and therefore $Y_{Q,n}^{\mathrm{sc}}$-free). We define a nonzero $\boldsymbol{c}$ with support $\mathcal{O}_y$ as follows. First, let $\boldsymbol{c}(\boldsymbol{s}_y) = 1$, and for any $\alpha \in \Delta$, let

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) := \boldsymbol{t}(\mathbb{w}_\alpha, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y).$$

Inductively, one can define $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]}) := \boldsymbol{T}(\mathbb{w}, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$ for any $\mathbb{w} \in W$. It is well defined and independent of the minimum decomposition of $\mathbb{w}$. Second, extend $\boldsymbol{c}$ by

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]} \cdot \bar{z}) = \boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]}) \cdot \bar{\chi}(\bar{z}), \quad \bar{z} \in \bar{A},$$

and

$$\boldsymbol{c}(\bar{t}) = 0 \text{ if } \bar{t} \notin \bigcup_{\mathbb{w} \in W} \boldsymbol{s}_{\mathbb{w}[y]} \cdot \bar{A}.$$

By using the property that $\boldsymbol{T}(\mathbb{w}, y)$ and $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]})$ are independent of the minimum decomposition of $\mathbb{w}$, we see that equality (8) is satisfied. It follows that $\wp_{Q,n}(\mathcal{O}_y)$ belongs to the right-hand side of (10). Therefore,

$$\dim \mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) \geq |\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})|. \tag{13}$$

3E. *An upper bound for* $\dim \mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi}))$. First we show a result in the general setting regarding the usual Weyl action. Let Ψ be a root system and Ψ_s be a fixed choice of simple roots. Write $\mathrm{L} := \langle \Psi \rangle$ for the lattice generated by Ψ and $V = \mathrm{L} \otimes \mathbb{R}$. The Weyl group W associated to Ψ acts on V naturally by the usual linear transformation generated by simple reflections. Recall that we write $\mathbb{w}(v)$, $\mathbb{w} \in W$, $v \in V$ for this action.

Lemma 3.12. *Let $v \in V$ be any vector such that $\mathbb{w}(v) \equiv v \bmod \mathrm{L}$. Then there exist $\mathbb{w}' \in W$ and $\alpha \in \Psi_s$ such that $\mathbb{w}_\alpha(\mathbb{w}'(v)) \equiv \mathbb{w}'(v) \bmod \mathrm{L}$.*

Proof. Let $W_{\mathrm{aff}} = \mathrm{L} \rtimes W$ be the affine Weyl group, and denote any element of W_{aff} by $\mathbb{w}_{\mathrm{a}} = (y, \mathbb{w})$. We call $\mathbb{w}$ the Weyl component of $\mathbb{w}_{\mathrm{a}}$. The congruence $\mathbb{w}(v) \equiv v \bmod \mathrm{L}$ is equivalent to $\mathbb{w}_{\mathrm{a}}(v) = v$ for some $\mathbb{w}_{\mathrm{a}}$ which projects to $\mathbb{w} \in W$.

If $\mathbb{w}_{\mathrm{a}}(v) = v$, it then follows that $v \in V$ lies on the boundary of $\bar{C}$, where C is an alcove (i.e., a fundamental domain) of the action of W_{aff} on V, see [Bourbaki 2002]. Note that $\bar{C}$ is a simplicial complex whose boundary consists of $|\Psi_s| + 1$ walls $\{\mathrm{E}_i\}$. Moreover, we may assume that for $1 \leq i \leq |\Psi_s|$, the wall E_i lies in the hyperplane fixed by $\mathbb{w}_{\mathrm{a}}$ whose Weyl component is $\mathbb{w}_{\alpha_i}$ for some $\alpha_i \in \Psi_s$. In this case, one also knows that $\mathrm{E}_{|\Psi_s|+1}$ is fixed by $(y, \mathbb{w}_\beta) \in W_{\mathrm{aff}}$ for some $\beta \in \Psi - \Psi_s$.

Since $v \in \bigcup_i \mathrm{E}_i$, there are two cases. First, suppose $v \in \mathrm{E}_i$ for some $1 \leq i \leq |\Psi_s|$; then clearly $\mathbb{w}_{\alpha_i}(v) \equiv v \bmod \mathrm{L}$ for some $\alpha_i \in \Psi_s$. Otherwise, suppose $v \in \mathrm{E}_{|\Psi_s|+1}$. Let $\mathbb{w}' \in W$ be such that $\mathbb{w}'(\beta) \in \Psi_s$. It follows that $\mathbb{w}'(\mathrm{E}_{|\Psi_s|+1})$ is fixed by some $\mathbb{w}_{\mathrm{a}} = (y, \mathbb{w}_\alpha)$ with $\alpha \in \Psi_s$. That is, $\mathbb{w}_\alpha(\mathbb{w}'(v)) \equiv \mathbb{w}'(v) \bmod \mathrm{L}$. The proof is completed. □

Proposition 3.13. *Consider $\boldsymbol{c} \in \mathrm{Ftn}(i(\bar{\chi}))$ such that $\lambda_{\boldsymbol{c}}^{\bar{\chi}}$ is a ψ-Whittaker functional on $\Theta(\bar{G}, \bar{\chi})$. If $\mathcal{O}_{y^0}$ is not $Y^{\mathrm{sc}}_{Q,n}$-free, then $\boldsymbol{c}$ is zero on $\mathcal{O}_{y^0}$. It follows that $\dim \mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) \leq |\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}})|$.*

Proof. Write $V = Y \otimes \mathbb{R}$. One has $V = (Y^{\mathrm{sc}} \otimes \mathbb{R}) \oplus V_0$ where $V_0 \subseteq V$ is fixed by W pointwise with respect to the usual action, i.e., the action $\mathbb{w}(v)$ of W. In general

$y_\rho^0 \in V$; however, without loss of generality, we may assume $y_\rho^0 \in Y^{\mathrm{sc}} \otimes \mathbb{R}$ now. There is a canonical W-equivariant isomorphism $Y_{Q,n}^{\mathrm{sc}} \otimes \mathbb{R} \simeq Y^{\mathrm{sc}} \otimes \mathbb{R}$ with respect to that usual action. Moreover, $\{\alpha_{Q,n}^\vee\}_{\alpha \in \Phi}$ forms a root system.

If $\mathcal{O}_{y^0}$ is not $Y_{Q,n}^{\mathrm{sc}}$-free, there exists $\mathbb{w} \in W$ such that $\mathbb{w}[y^0] \equiv y^0 \bmod Y_{Q,n}^{\mathrm{sc}}$, i.e., $\mathbb{w}(y_\rho^0) \equiv y_\rho^0 \bmod Y_{Q,n}^{\mathrm{sc}}$. By the preceding Lemma, there exist $y \in \mathcal{O}_{y^0}$ and $\alpha \in \Delta$ such that $\mathbb{w}_\alpha(y_\rho) \equiv y_\rho \bmod Y_{Q,n}^{\mathrm{sc}}$. Now it suffices to show that $\boldsymbol{c}$ vanishes on y.

By Corollary 3.7, $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) = \boldsymbol{t}(\mathbb{w}_\alpha, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$. Since $\mathbb{w}_\alpha(y_\rho) \equiv y_\rho \bmod Y_{Q,n}^{\mathrm{sc}}$, it follows that $n_\alpha | \langle y_\rho, \alpha \rangle$. Write $\langle y_\rho, \alpha \rangle = k \cdot n_\alpha$. Since

$$\boldsymbol{s}_{\mathbb{w}_\alpha[y]} = \boldsymbol{s}_y \cdot \boldsymbol{s}_{-\langle y_\rho, \alpha \rangle) \alpha^\vee} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle \cdot D(\alpha^\vee, y)},$$

one has

$$\begin{aligned}\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) &= \overline{\chi}(\boldsymbol{s}_{-k n_\alpha \alpha^\vee}) \cdot \boldsymbol{c}(\boldsymbol{s}_y) \cdot \varepsilon^{\langle y_\rho, \alpha \rangle \cdot D(\alpha^\vee, y)} \\ &= q^k \cdot \varepsilon^{k n_\alpha \cdot D(\alpha^\vee, y)} \cdot \boldsymbol{c}(\boldsymbol{s}_y).\end{aligned}$$

On the other hand,

$$\begin{aligned}\boldsymbol{t}(\mathbb{w}_\alpha, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y) &= q^{k_{y,\alpha} - 1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \\ &= q^k \cdot (-1, \varpi)_n^{k n_\alpha \cdot D(y, \alpha^\vee)} \cdot (-q^{-1}) \cdot \boldsymbol{c}(\boldsymbol{s}_y).\end{aligned}$$

It follows that $\boldsymbol{c}(\boldsymbol{s}_y) = -q^{-1} \cdot \varepsilon^{k n_\alpha B(y, \alpha^\vee)} \cdot \boldsymbol{c}(\boldsymbol{s}_y) = (-q^{-1}) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$. Therefore $\boldsymbol{c}(\boldsymbol{s}_y) = 0$. The proof is completed. $\square$

Theorem 3.14. *Let $\overline{G}$ be an unramified Brylinski–Deligne covering group incarnated by (D, η). Let $\overline{\chi}$ be an unramified exceptional character and $\Theta(\overline{G}, \overline{\chi})$ the theta representation associated with $\overline{\chi}$. Then*

$$|\wp_{Q,n}(\mathcal{O}_{Q,n}^{\digamma})| \le \dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) \le |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{\digamma})|.$$

The group $\mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times)$ is identified with $Z(\overline{G}^\vee)$, the center of the dual group $\overline{G}^\vee$ of $\overline{G}$, so $Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n}$ if and only if $Z(\overline{G}^\vee) = \{1\}$. Immediately it follows that:

Corollary 3.15. *If the dual group $\overline{G}^\vee$ of $\overline{G}$ is of adjoint type, i.e., $Z(\overline{G}^\vee) = 1$, then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^{\digamma})|$.

For groups of type E_8, F_4 and G_2, the complex dual group of their covering group has trivial center and thus Corollary 3.15 applies.

More generally, if $\mathcal{O}_{Q,n}^{\digamma} = \mathcal{O}_{Q,n,\mathrm{sc}}^{\digamma}$, then the dimension of $\mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ can be uniquely determined. We will illustrate below that Theorem 3.14 recovers the result of Kazhdan and Patterson in this case.

Example 3.16. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for the cocharacter lattice Y of GL_r. The simple coroots $\Delta^\vee$ of GL_r are $\Delta^\vee = \{\alpha_i^\vee := e_i - e_{i+1}\}_{1 \le i \le r-1}$. The isomorphism class of (D, η) in the incarnation category corresponds to a Weyl-invariant quadratic form Q, or equivalently, to the bilinear form B_Q. Let $B_Q(e_i, e_j)$ be the Weyl-invariant bilinear form determined by

$$B_Q(e_i, e_i) = 2\boldsymbol{p}, \qquad B_Q(e_i, e_j) = \boldsymbol{q} \qquad \text{if } i \neq j.$$

For any root α, one has $Q(\alpha^\vee) = 2\boldsymbol{p} - \boldsymbol{q}$. We assume $2\boldsymbol{p} - \boldsymbol{q} = -1$ and therefore $n_\alpha = n$. The covering groups $\overline{\mathrm{GL}}_r^{(n)}$ arising from such B_Q are exactly those studied by Kazhdan and Patterson. The parameter $\boldsymbol{p}$ corresponds to the twisting parameter c in [Kazhdan and Patterson 1984].

From B_Q, the lattice $Y_{Q,n}$ is given by

$$\left\{ \sum_i x_i e_i \in \bigoplus_{i=1}^r \mathbb{Z} e_i : x_1 \equiv x_2 \equiv \cdots \equiv x_r \bmod n, \text{ and } n \mid (\boldsymbol{q} r - 1) x_i \right\}.$$

The lattice $Y_{Q,n}^{\mathrm{sc}}$ is generated by $\{\alpha_{Q,n}^\vee\}_{\alpha \in \Phi}$. It is easy to check $Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n} \cap Y^{\mathrm{sc}}$, and this has the following implications:

Suppose that $\mathcal{O}_y$ is not $Y_{Q,n}$-free, i.e., $\mathsf{w}[y] - y \in Y_{Q,n}$ for some $\mathsf{w} \neq 1 \in W$. Clearly $\mathsf{w}[y] - y \in Y^{\mathrm{sc}}$ as well. It follows that $\mathsf{w}[y] - y \in Y_{Q,n}^{\mathrm{sc}}$, that is, $\mathcal{O}_y$ is not $Y_{Q,n}^{\mathrm{sc}}$-free. Therefore, for the Kazhdan–Patterson covering group $\overline{\mathrm{GL}}_r^{(n)}$, one has that $\mathcal{O}_{Q,n}^F$ is equal to $\mathcal{O}_{Q,n,\mathrm{sc}}^F$. Consequently, for the covering group $\overline{\mathrm{GL}}_r^{(n)}$ with parameter $(\boldsymbol{p}, \boldsymbol{q})$ such that $2\boldsymbol{p} - \boldsymbol{q} = -1$, Theorem 3.14 yields

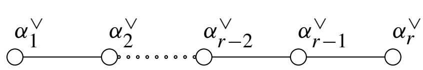
$$\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)|,$$

which is the content of [Kazhdan and Patterson 1984, Theorem I.3.5]. Moreover, distinguished theta representations (see Definition 3.3) for $\overline{\mathrm{GL}}_r^{(n)}$ are completely determined in [Kazhdan and Patterson 1984, Corollary I.3.6].

In the remaining part of the paper, we will determine the distinguished theta representations for coverings of simply connected groups of type A_r, B_r, C_r and G_2. To ease the computations, we will use the standard coordinates for the coroot system of each type as in [Bourbaki 2002, pages 265–290].

4. The A_r, $r \geq 1$ case

Consider the Dynkin diagram for the simple coroots of A_r:



The cocharacter lattice is $Y = Y^{\mathrm{sc}} = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$. As in [Bourbaki 2002, page 265], consider the embedding $\boldsymbol{i}_A : \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i$, which is given by

$$\boldsymbol{i}_A : y = (x_1, x_2, \ldots, x_r) \mapsto \boldsymbol{i}_A(y) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_r - x_{r-1}, -x_r).$$

In particular, we can identify the image of $\boldsymbol{i}_A$: any $(y_1, y_2, \ldots, y_r, y_{r+1}) \in \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i$ is equal to $\boldsymbol{i}_A(y)$ for some y if and only if $\sum_{i=1}^{r+1} y_i = 0$.

Meanwhile, $\rho = \sum_{i=1}^{r} \frac{i}{2}(r-i+1)\alpha_i^\vee$. We use $\boldsymbol{i}_A : \bigoplus_{i=1}^{r} \mathbb{Q}\alpha_i^\vee \to \bigoplus_{i=1}^{r+1} \mathbb{Q}e_i$ to denote the canonical extension of $\boldsymbol{i}_A$. Then,

$$\boldsymbol{i}_A(\rho) = \Big(\frac{r}{2}, \frac{r-2}{2}, \ldots, \frac{-(r-2)}{2}, \frac{-r}{2}\Big) \in \bigoplus_{i=1}^{r+1} \mathbb{Q}e_i.$$

It follows that for any $y \in Y$,

$$\begin{aligned}\boldsymbol{i}_A(y_\rho) &= \Big(x_1 - \frac{r}{2}, \ldots, x_i - x_{i-1} + (i-1) - \frac{r}{2}, \ldots, -x_r + r - \frac{r}{2}\Big), \qquad 1 \le i \le r \\ &= \big(x_1, x_2 - x_1 + 1, \ldots, x_i - x_{i-1} + (i-1), \ldots, -x_r + r\big) + \Big(\frac{-r}{2}, \frac{-r}{2}, \ldots, \frac{-r}{2}\Big).\end{aligned}$$

From now, we write $\boldsymbol{i}_A^*(y_\rho) := (x_1^*, x_2^*, \ldots, x_r^*, x_{r+1}^*)$ for

$$\big(x_1, x_2 - x_1 + 1, \ldots, x_i - x_{i-1} + (i-1), \ldots, -x_r + r\big) \in \bigoplus_i \mathbb{Z}e_i.$$

Thus,

$$\boldsymbol{i}_A(y_\rho) = \boldsymbol{i}_A^*(y_\rho) + \Big(\frac{-r}{2}, \frac{-r}{2}, \ldots, \frac{-r}{2}\Big).$$

Meanwhile, any $(x_1^*, x_2^*, \ldots, x_r^*, x_{r+1}^*) \in \bigoplus_i \mathbb{Z}e_i$ is equal to $\boldsymbol{i}_A^*(y_\rho)$ for some y if and only if $\sum_{i=1}^{r+1} x_i^* = r(r+1)/2$.

Consider the quadratic form Q on $Y = \langle \alpha_i^\vee, 1 \le i \le r\rangle$ with $Q(\alpha_i^\vee) = 1$ for all i. Then

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } j = i+1, \\ 0, & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.}\end{cases}$$

This gives rise to the degree n covering group $\overline{\mathrm{SL}}_{r+1}^{(n)}$. Any element $\sum_{i=1}^{r} x_i\alpha_i^\vee \in Y$ lies in $Y_{Q,n}$ if and only if

$$2x_1 - x_2, \quad -x_1 + 2x_2 - x_3, \quad -x_2 + 2x_3 - x_4, \quad \ldots \; -x_{r-2} + 2x_{r-1} - x_r, \quad -x_{r-1} + 2x_r$$

are in $n\mathbb{Z}$.

By using $\boldsymbol{i}_A$, we see

$$Y_{Q,n} = \Big\{(y_1, y_2, \ldots, y_r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z}e_i \;:\; \sum_{i=1}^{r+1} y_i = 0, \text{ and } y_1 \equiv \cdots \equiv y_r \equiv y_{r+1} \bmod n\Big\}$$

and

$$Y_{Q,n}^{\mathrm{sc}} = \Big\{(y_1, y_2, \ldots, y_r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z}e_i \;:\; \sum_{i=1}^{r+1} y_i = 0, \text{ and } n | y_i \text{ for all } i.\Big\}$$

The Weyl group $W = S_{r+1}$ acts as permutations on $\bigoplus_{i=1}^{r+1} \mathbb{Z}e_i$. In particular, $\mathbb{w}_{\alpha_i}$ for $\alpha_i \in \Delta$ acts by exchanging e_i and e_{i+1}.

4A. ***Case I:*** $\overline{\mathbf{SL}}^{(n)}_{r+1}$**,** $n \leq r$**.** Suppose $n \leq r$, then for any $y \in Y$ with $\boldsymbol{i}_A^*(y_\rho) = (x_1^*, x_2^*, \ldots, x_{r+1}^*)$, there exists $x_i^*, x_j^*, i \neq j$ such that $n|(x_i^* - x_j^*)$. Then clearly $\mathbb{w}(y_\rho) - y_\rho \in Y_{Q,n}^{\mathrm{sc}}$ for some $\mathbb{w} \in W$. That is, $\mathcal{O}_y \notin \mathcal{O}^F_{Q,n,\mathrm{sc}}$ and one has in this case

$$\mathcal{O}^F_{Q,n,\mathrm{sc}} = \varnothing.$$

Therefore, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_{r+1}, \overline{\chi})) = 0$ for $n \leq r$.

4B. ***Case II:*** $\overline{\mathbf{SL}}^{(n)}_{r+1}$**,** $n = r+1$**.** In this case, the dual group for $\overline{\mathrm{SL}}^{(n)}_n$ is SL_n, see [Weissman 2015]. Consider $\mathcal{O}_y \in \mathcal{O}^F_{Q,n,\mathrm{sc}}$ such that

$$\boldsymbol{i}_A^*(y_\rho) = (0, 1, 2, \ldots, r-1, r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i.$$

It is easy to check $\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}_y)\}$, and this implies $|\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = 1$. However, $\mathcal{O}_y \notin \mathcal{O}^F_{Q,n}$. For example, let $\mathbb{w}_\natural$ be such that $\boldsymbol{i}_A^*(\mathbb{w}_\natural(y_\rho)) = (1, 2, \ldots, r, 0)$, then $\boldsymbol{i}_A(\mathbb{w}_\natural(y_\rho)) - \boldsymbol{i}_A(y_\rho) = (1, 1, \ldots, 1, -r) \in Y_{Q,n}$. That is, $\mathbb{w}_\natural[y] - y \in Y_{Q,n}$. Therefore,

$$|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| = 0.$$

It follows that $0 \leq \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_n, \overline{\chi})) \leq 1$. In this case, determining $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_n, \overline{\chi}))$ is delicate, and there are additional constraints on the exceptional character $\overline{\chi}$ such that $\Theta(\overline{\mathrm{SL}}^{(n)}_n, \overline{\chi})$ is distinguished. The analysis below is devoted to this.

4B1. *The reduction step.* It is clear that $\boldsymbol{i}_A^*(y_\rho) = (0, 1, 2, \ldots, r-1, r)$ if and only if $y = 0$. Moreover, $\boldsymbol{i}_A^*(\mathbb{w}_\natural(y_\rho)) = (1, 2, 3, \ldots, r, 0)$ for $\mathbb{w}_\natural = \mathbb{w}_{\alpha_r}\mathbb{w}_{\alpha_{r-1}} \cdots \mathbb{w}_{\alpha_2}\mathbb{w}_{\alpha_1}$. As above,

$$\mathbb{w}_\natural[0] - 0 = \sum_{i=1}^{r} i \cdot \alpha_i^\vee \in Y_{Q,n}.$$

Write $y_{Q,n} := \sum_{i=1}^r i \cdot \alpha_i^\vee$. In fact, the set $\{n\alpha_i^\vee : 2 \leq i \leq r\} \cup \{y_{Q,n}\}$ forms a basis for $Y_{Q,n}$, whereas $\{n\alpha_i^\vee : 2 \leq i \leq r\} \cup \{n \cdot y_{Q,n}\}$ is a basis for $Y^{\mathrm{sc}}_{Q,n}$. It follows that any exceptional character $\overline{\chi}$ is determined by its value at $\boldsymbol{s}_{y_{Q,n}}$.

We choose the bisector D on Y^{sc} such that $D(\alpha_i^\vee, \alpha_j^\vee)$ is given by

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} Q(\alpha_i^\vee) & \text{if } i = j, \\ 0 & \text{if } i < j, \\ B_Q(\alpha_i^\vee, \alpha_j^\vee) & \text{if } i > j. \end{cases}$$

Recall from Corollary 3.7 that $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ gives rise to a ψ-Whittaker functional of $\Theta(\overline{\mathrm{SL}}^{(n)}_n, \overline{\chi})$ if and only if for all y and $\alpha \in \Delta$,

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) = q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(B(\alpha^\vee, y_\rho))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y).$$

For $1 \le i \le r$, write $y_{\langle i\rangle} = \mathbb{w}_{\alpha_i}\mathbb{w}_{\alpha_{i-1}} \cdots \mathbb{w}_{\alpha_1}[0]$ and we set $y_{\langle 0\rangle} = 0$. Recall that $\boldsymbol{t}(\mathbb{w}_\alpha, y)$ is the coefficient in the above formula. In this case, it reads $\boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle)^{-1}$ since $Q(\alpha^\vee) = 1$ (and therefore $n_\alpha = n$) for all $\alpha \in \Delta$. In order to have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})) = 1$, we must have the equality

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) = \boldsymbol{T}(\mathbb{w}_\natural, 0) \;\text{ where }\; \boldsymbol{T}(\mathbb{w}_\natural, 0) = \prod_{i=1}^{r} \boldsymbol{t}(\mathbb{w}_{\alpha_i}, y_{\langle i-1\rangle}). \tag{14}$$

We would like to show that the equality (14) is also sufficient. Consider any $\mathbb{w}' \in W$, $y \in \mathcal{O}_0$, one has $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}'[y]}) = \boldsymbol{T}(\mathbb{w}', y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$. Now assume $\mathbb{w}'[y] - y \in Y_{Q,n}$, we have

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}'[y]-y+y}) = \overline{\chi}(\boldsymbol{s}_{\mathbb{w}'[y]-y}) \cdot \boldsymbol{c}(\boldsymbol{s}_y) \cdot \varepsilon^{D(\mathbb{w}'[y]-y,\, y)}.$$

To show $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})) = 1$, it suffices to show $\boldsymbol{c}(\boldsymbol{s}_y)$ to be nonzero for all $y \in \mathcal{O}_0$ such that $\mathbb{w}'[y] - y \in Y_{Q,n}$. That is, it requires

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}'[y]-y}) = \varepsilon^{D(\mathbb{w}'[y]-y,\, y)} \cdot \boldsymbol{T}(\mathbb{w}', y). \tag{15}$$

Write $\mathbb{w}'[y]-y = \sum_{i=2}^{r} k_i \cdot \alpha_{i,Q,n}^\vee + k_1 \cdot y_{Q,n}$. Note that $\mathcal{O}_0$ is $Y_{Q,n}^{\mathrm{sc}}$-free, thus $k_1 \neq 0$. We may reduce the negative case to the positive case by a simple computation, and therefore we can assume that $k_1 \ge 1$. Furthermore, we may apply induction on k_1, and thus it suffices to: i) prove the inductive step, ii) check the equality (15) when $\mathbb{w}'[y] - y = \sum_{i=2}^{r} k_i \alpha_{i,Q,n}^\vee + y_{Q,n}$. The assertion i) can be checked easily, and thus we will only outline the proof of ii).

For ii), if $\mathbb{w}'[y] - y = \sum_{i=2}^{r} k_i \alpha_{i,Q,n}^\vee + y_{Q,n}$, then it is not hard to see that $\mathbb{w}'[y] - y = \mathbb{w}(y_{Q,n})$, i.e., $\mathbb{w}^{-1}\mathbb{w}'[y] - \mathbb{w}^{-1}[y] = y_{Q,n}$ for some $\mathbb{w} \in W$. We may change $\mathbb{w}$ if necessary such that $\mathbb{w}^{-1}[y] = 0$. With this assumption, $\mathbb{w}^{-1}\mathbb{w}'\mathbb{w} = \mathbb{w}_\natural$, i.e., $\mathbb{w}' = \mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1}$. Therefore, we need only show that for any $\mathbb{w} \in W$,

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}[0]}) = \varepsilon^{D(\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}[0],\, \mathbb{w}[0])} \cdot \boldsymbol{T}(\mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1}, \mathbb{w}[0]). \tag{16}$$

To show (16), we would like to apply induction on the length of $\mathbb{w}$. When $\mathbb{w} = 1$, it is just the equality (14). For the induction step, assuming the equality (16), we would like to prove that for $\alpha \in \Delta$ the following equality holds:

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}_\alpha\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}[0]}) = \varepsilon^{D(\mathbb{w}_\alpha\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}_\alpha\mathbb{w}[0],\, \mathbb{w}_\alpha\mathbb{w}[0])} \cdot \boldsymbol{T}(\mathbb{w}_\alpha\mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1}\mathbb{w}_\alpha^{-1}, \mathbb{w}_\alpha\mathbb{w}[0]). \tag{17}$$

For this purpose, write $x := \mathbb{w}\mathbb{w}_\natural[0] - \mathbb{w}[0] \in Y_{Q,n}$. We have $n_\alpha | \langle x, \alpha\rangle$. Write $\langle x, \alpha\rangle = k \cdot n_\alpha$.

The left-hand side of (17) is

$$\begin{aligned}\overline{\chi}(\boldsymbol{s}_{x-\langle x,\alpha\rangle\alpha^\vee}) &= \overline{\chi}(\boldsymbol{s}_x) \cdot \overline{\chi}(\boldsymbol{s}_{-kn_\alpha\alpha^\vee}) \cdot \varepsilon^{D(x,-kn_\alpha\alpha^\vee)} \\ &= \overline{\chi}(\boldsymbol{s}_x) \cdot \overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{-k} = q^k \cdot \overline{\chi}(\boldsymbol{s}_x).\end{aligned}$$

The right-hand side of (17) is

$$
\begin{aligned}
&\varepsilon^{D(\mathbb{w}_\alpha(x),\mathbb{w}_\alpha\mathbb{w}[0])}\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}\mathbb{w}_\natural[0])\cdot \boldsymbol{T}(\mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1},\mathbb{w}[0])\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}_\alpha\mathbb{w}[0])\\
&\qquad=\varepsilon^{D(x,\mathbb{w}_\alpha\mathbb{w}[0])-\mathbb{w}[0])}\cdot\overline{\chi}(\boldsymbol{s}_x)\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}\mathbb{w}_\natural[0])\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}_\alpha\mathbb{w}[0]) \qquad \text{by (16)}\\
&\qquad=\varepsilon^{D(x,\mathbb{w}_\alpha\mathbb{w}[0])-\mathbb{w}[0])}\cdot\overline{\chi}(\boldsymbol{s}_x)\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}\mathbb{w}_\natural[0])\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}[0])^{-1}\\
&\qquad=\varepsilon^{D(x,\mathbb{w}_\alpha\mathbb{w}[0])-\mathbb{w}[0])}\cdot q^{\lceil\frac{\langle\mathbb{w}[0],\alpha\rangle}{n_\alpha}\rceil-1+k}\cdot \boldsymbol{g}_{\psi^{-1}}(\langle\mathbb{w}(0_\rho),\alpha\rangle Q(\alpha^\vee))^{-1}\\
&\qquad\quad\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle\cdot D(\mathbb{w}\mathbb{w}_\natural[0],\alpha^\vee)}\cdot\overline{\chi}(\boldsymbol{s}_x)\cdot q^{-\lceil\frac{\langle\mathbb{w}[0],\alpha\rangle}{n_\alpha}\rceil+1}\\
&\qquad\qquad\cdot \boldsymbol{g}_{\psi^{-1}}(\langle\mathbb{w}(0_\rho),\alpha\rangle Q(\alpha^\vee))\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle\cdot D(\mathbb{w}[0],\alpha^\vee)}\\
&\qquad=\overline{\chi}(\boldsymbol{s}_x)\cdot q^k\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle D(x,\alpha^\vee)}\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle D(x,\alpha^\vee)}\\
&\qquad=\overline{\chi}(\boldsymbol{s}_x)\cdot q^k,
\end{aligned}
$$

which is clearly equal to the left-hand side. To summarize, we have:

Proposition 4.1. *Let* $\overline{\chi}\in\mathrm{Hom}_\iota(Z(\overline{T}),\mathbb{C}^\times)$ *be an exceptional character of* $\overline{\mathrm{SL}}_n^{(n)}$. *Then*

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)},\overline{\chi}))=1$$

if and only if $\overline{\chi}$ *is the unique exceptional character satisfying* (14).

We would like to explicate the condition given by (14).

Lemma 4.2. *One has*

$$\boldsymbol{T}(\mathbb{w}_\natural,0)=\begin{cases} q^{-r/2} & \text{if } n \text{ is odd},\\ \varepsilon^{n(n-2)/8}\cdot q^{-n/2}\cdot \boldsymbol{g}_{\psi^{-1}}(-n/2)^{-1} & \text{if } n \text{ is even}.\end{cases}$$

Proof. We compute each $\boldsymbol{t}(\mathbb{w}_{\alpha_i},y_{\langle i-1\rangle})$ for $1\le i\le r$. First, one can check easily that $y_{\langle i\rangle}=\sum_{j=1}^i i\cdot\alpha_i^\vee=\alpha_1^\vee+2\alpha_2^\vee+\cdots+i\cdot\alpha_i^\vee$. Thus, $\langle y_{\langle i-1\rangle},\alpha_i\rangle=-(i-1)$ and therefore

$$k_{y_{\langle i-1\rangle},\alpha_i}=0\ \text{ for all }\ 1\le i\le r.$$

Second, $\boldsymbol{\Gamma}(y_{\langle i-1\rangle},\alpha_i^\vee)=\varepsilon^{-i\cdot D(y_{\langle i-1\rangle},\alpha_i^\vee)}$. Since $D(\alpha_j^\vee,\alpha_i^\vee)=0$ for all $j<i$, we see $\boldsymbol{\Gamma}(y_{\langle i-1\rangle},\alpha_i^\vee)=1$. Thus, $\boldsymbol{t}(\mathbb{w}_{\alpha_i},y_{\langle i-1\rangle})=q^{-1}\cdot \boldsymbol{g}_{\psi^{-1}}(-i)^{-1}$. Now, if $1\le i,j\le n$ and $i+j=n$, one has

$$
\begin{aligned}
\boldsymbol{g}_{\psi^{-1}}(-i)^{-1}\cdot \boldsymbol{g}_{\psi^{-1}}(-j)^{-1}&=\boldsymbol{g}_{\psi^{-1}}(-i)^{-1}\cdot(\overline{\boldsymbol{g}_{\psi^{-1}}(j)}\cdot\varepsilon^j)^{-1}\\
&=|\boldsymbol{g}_{\psi^{-1}}(j)|^{-2}\cdot\varepsilon^i\\
&=q\cdot\varepsilon^i.
\end{aligned}
$$

The result then follows from simply multiplying together each term. $\square$

4B2. *Interlude: Weil-index.* Let $\boldsymbol{\gamma}_\psi$ be the Weil-index given in Section 2C.

Lemma 4.3. *Suppose $n=2m$ is an even number. Then the following equality holds*:

$$\boldsymbol{g}_{\psi^{-1}}(m)=\frac{q^{-1/2}}{\boldsymbol{\gamma}_\psi(\varpi)}.$$

Proof. By definition, $\boldsymbol{g}_{\psi^{-1}}(m)$ is equal to

$$\begin{aligned}\int_{O_F^\times}(u,\varpi)_2\cdot\psi^{-1}(\varpi^{-1}u)\mu(u)&=\int_{O_F^\times}\boldsymbol{\gamma}_\psi(\varpi u)\boldsymbol{\gamma}_\psi(\varpi)^{-1}\boldsymbol{\gamma}_\psi(u)^{-1}\cdot\psi^{-1}(\varpi^{-1}u)\mu(u)\\&=\boldsymbol{\gamma}_\psi(\varpi)^{-1}\cdot\int_{O_F^\times}\boldsymbol{\gamma}_\psi(\varpi u)\cdot\psi^{-1}(\varpi^{-1}u)\mu(u).\end{aligned}$$

However, by Equation (3.7) of [Szpruch 2009b, Lemma 3.2],

$$\boldsymbol{\gamma}_\psi(\varpi u)=q^{-1/2}\bigg(1+q\int_{O_F^\times}\psi(\varpi^{-1}v^2u)\mu(v)\bigg).$$

Thus,

$$\begin{aligned}\boldsymbol{g}_{\psi^{-1}}(m)&=q^{-1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)^{-1}\cdot\int_{O_F^\times}\bigg(1+q\int_{O_F^\times}\psi(\varpi^{-1}v^2u)\mu(v)\bigg)\psi^{-1}(\varpi^{-1}u)\mu(u)\\&=q^{-1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)^{-1}\cdot\bigg(-\frac{1}{q}+q\cdot\int_{O_F^\times}\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\mu(v)\bigg)\end{aligned}$$

Let $D=\{v\in O_F^\times:|1-v^2|=1\}$ and $H=\{v\in O_F^\times:|1-v^2|\le q^{-1}\}$. We get

$$\begin{aligned}&\int_{O_F^\times}\bigg(\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\bigg)\mu(v)\\&=\int_{v\in H}\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\mu(v)+\int_{v\in D}\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\mu(v)\\&=\mu(H)\cdot(1-q^{-1})+\mu(D)\cdot(-q^{-1})\ \text{ by (8.19) of [Szpruch 2009b, Lemma 8.6]}\\&=2q^{-1}\cdot(1-q^{-1})+(1-3q^{-1})\cdot(-q^{-1})\ \text{ by [Szpruch 2009b, Lemma 8.9]}\\&=q^{-1}+q^{-2}.\end{aligned}$$

The result follows easily by simplification. □

4B3. *An explicit criterion.* Consider the *unitary* distinguished character $\overline{\chi}^0_{\psi'}$ constructed in [Gan and Gao 2016], which we recalled and gave in (5). Then the character $\overline{\chi}_{\psi'}=\overline{\chi}^0_{\psi'}\cdot\delta_B(\cdot)^{1/2n}$ is an exceptional character. In the simply connected case, $J=Y^{\mathrm{sc}}_{Q,n}$. For the definition of $\overline{\chi}^0_{\psi'}$, we pick a basis $\{y_i\}$ for $Y_{Q,n}$ such that

$\{k_i y_i\}$ is a basis for $J = Y_{Q,n}^{\mathrm{sc}}$. Then by definition,

$$\overline{\chi}_{\psi'}^{\,0}(\boldsymbol{s}_{y_i}) = \boldsymbol{\gamma}_{\psi'}(\varpi)^{2(k_i-1)Q(y_i)/n}$$

and, for $y = \sum_i n_i y_i \in Y_{Q,n}$, one has

$$\overline{\chi}_{\psi'}^{\,0}(\boldsymbol{s}_y) = \prod_i \overline{\chi}_{\psi'}^{\,0}(\varpi^{n_i})^{2(k_i-1)Q(y_i)/n} \cdot \varepsilon^{\sum_{i<j} n_i n_j D(y_i,y_j)}.$$

For the covering group $\overline{\mathrm{SL}}_n^{(n)}$, we take $y_i = n\alpha_i^\vee$, $2 \le i \le r$ and $y_1 = y_{Q,n}$, with $k_i = 1$ for $2 \le i \le r$ and $k_1 = n$.

An easy computation shows $Q(y_{Q,n}) = r(r+1)/2$, and thus

$$\overline{\chi}_{\psi'}(\boldsymbol{s}_{y_{Q,n}}) = \overline{\chi}_{\psi'}^{\,0}(\boldsymbol{s}_{y_{Q,n}}) \cdot \delta_B(\boldsymbol{s}_{y_{Q,n}})^{\frac{1}{2n}} = \boldsymbol{\gamma}_{\psi'}(\varpi)^{(n-1)^2} \cdot q^{-(n-1)/2}. \tag{18}$$

Proposition 4.4. *For the exceptional character $\overline{\chi}_{\psi'} = \overline{\chi}_{\psi'}^{\,0} \cdot \delta_B(\cdot)^{\frac{1}{2n}}$ given above, one has that the dimension of* $\mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi}_{\psi'}))$ *equals* 1 *in the following cases, and* 0 *otherwise*:

$$\begin{cases} \text{any } \psi', & \text{if } n \text{ is odd;} \\ \boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi), & \text{if } n \equiv 0, 2 \bmod 8; \\ \boldsymbol{\gamma}_{\psi'}(\varpi) = (-1,\varpi)_4 \cdot \boldsymbol{\gamma}_{\psi}(\varpi) & \text{if } n \equiv 4 \bmod 8; \\ \boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)^{-1} & \text{if } n \equiv 6 \bmod 8. \end{cases}$$

Proof. By the value of $\overline{\chi}_{\psi'}(\boldsymbol{s}_{y_{Q,n}})$ in (18), it follows from Lemma 4.2 that the equality (14) is equivalent to

$$\boldsymbol{\gamma}_{\psi'}(\varpi)^{(n-1)^2} \cdot q^{-\frac{(n-1)}{2}} = \begin{cases} q^{-r/2} & \text{if } n \text{ is odd;} \\ (-1,\varpi)_n^{n(n-2)/8} \cdot q^{-\frac{n}{2}} \cdot \boldsymbol{g}_{\psi^{-1}}(-\tfrac{n}{2})^{-1} & \text{if } n \text{ is even.} \end{cases} \tag{19}$$

For n odd, the equality holds for any ψ'. Now we assume n even.

For $n = 4k+2$, by Lemma 4.3, the required equality in (19) becomes

$$\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)^{2k+1}.$$

In particular, if k is even, it is equivalent to $\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)$. If k is odd, it is equivalent to $\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)^{-1}$.

For $n = 4k$, applying Lemma 4.3 again, the equality in (19) reads

$$\boldsymbol{\gamma}_{\psi'}(\varpi) = (-1,\varpi)_n^k \cdot \boldsymbol{\gamma}_{\psi}(\varpi) = (-1,\varpi)_4 \cdot \boldsymbol{\gamma}_{\psi}(\varpi).$$

A special case is when k is even. In this case $(-1,\varpi)_4 = 1$ and therefore it is equivalent to $\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)$. $\square$

Corollary 4.5. *Consider $\psi' = \psi_a$ for some $a \in F^\times$. Assume ψ_a has conductor O_F, i.e., $a \in O_F^\times$. Then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi}_{\psi_a})) = 1$ *if and only if the following hold*:

$$\begin{cases} a \in O_F^\times & \text{if } n \text{ is odd}, \\ a \in (O_F^\times)^2 & \text{if } n \equiv 0, 2 \bmod 8, \\ a^2 \in -(O_F^\times)^4 & \text{if } n \equiv 4 \bmod 8, \\ a \in -(O_F^\times)^2 & \text{if } n \equiv 6 \bmod 8. \end{cases}$$

Remark 4.6. The facts that for any exceptional representation $\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})$ there exists ψ such that it is ψ-generic, and that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})) \le 1$ for all ψ also follow from the work of [Kazhdan and Patterson 1984] on $\overline{\mathrm{GL}}_n^{(n)}$ combined with the relation between $\overline{\mathrm{SL}}_n^{(n)}$ and $\overline{\mathrm{GL}}_n^{(n)}$ in [Adams 2003]. (We thank the referee for pointing this out.) However, our Corollary 4.5 gives precise information for the matching between ψ and the distinguished theta representation in terms of the distinguished character.

Example 4.7. The first nontrivial example is the metaplectic covering $\overline{\mathrm{SL}}_2^{(2)}$. In this case, we have $Y_{Q,n} = Y = \mathbb{Z} \cdot \alpha^\vee$ and $Y_{Q,n}^{\mathrm{sc}} = \mathbb{Z} \cdot (2\alpha^\vee)$. As mentioned at the beginning of Section 4B, one has that the lower and upper bounds in Theorem 3.14 are 0 and 1 respectively and thus

$$0 \le \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi})) \le 1$$

for any exceptional $\overline{\chi}$. For the character ψ_a, the representation $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$ is the even Weil representation in the following exact sequence:

$$\mathrm{St}(\overline{\chi}_{\psi_a}) \hookrightarrow I(\overline{\chi}_{\psi_a}) \twoheadrightarrow \Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a}),$$

where $\mathrm{St}(\overline{\chi}_{\psi_a})$ is the metaplectic analogue of the Steinberg representation. From Corollary 4.5, we can recover the well-known fact, which follows from the work of Gelbart and Piatetski-Shapiro [1980], that for $\overline{\mathrm{SL}}_2^{(2)}$ the even Weil representation $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$ (for unramified data) is ψ-generic if and only if $a \in (O_F^\times)^2$. We note that this also follows directly from the computation of the local coefficient for $\overline{\mathrm{SL}}_2^{(2)}$ in [Szpruch 2009a].

Example 4.8. We also discuss explicitly the example $\overline{\mathrm{SL}}_3^{(3)}$. Consider $\overline{\mathrm{SL}}_3^{(3)}$ with cocharacter lattice $Y = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$. Consider Q such that $Q(\alpha_i^\vee) = 1$. Then

$$Y_{Q,n} = \langle 2\alpha_1^\vee + \alpha_2^\vee, 3\alpha_1^\vee \rangle = \langle 2\alpha_2^\vee + \alpha_1^\vee, 3\alpha_2^\vee \rangle.$$

Note $Y = \langle 2\alpha_1^\vee + \alpha_2^\vee, \alpha_1^\vee \rangle = \langle 2\alpha_2^\vee + \alpha_1^\vee, \alpha_2^\vee \rangle$. We know $\rho = \alpha_1^\vee + \alpha_2^\vee$. For $y = 0$ one has

$$y_\rho = 0_\rho = -(\alpha_1^\vee + \alpha_2^\vee).$$

Consider $\mathbb{w}_\natural = \mathbb{w}_{\alpha_1}\mathbb{w}_{\alpha_2}$, then $\mathbb{w}_{\alpha_2}[y] = \alpha_2^\vee$ and moreover $\mathbb{w}_{\alpha_1}\mathbb{w}_{\alpha_2}[y] = 2\alpha_1^\vee + \alpha_2^\vee$. One has

$$\begin{aligned} \boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]}) &= q^{k_{\mathbb{w}_2[y],\alpha_1}-1} \cdot \boldsymbol{\Gamma}(\mathbb{w}_2[y], \alpha_1^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_1^\vee)(\langle \mathbb{w}_2[y], \alpha_1\rangle - 1))^{-1} \\ &\quad \cdot q^{k_{y,\alpha_2}-1} \cdot \boldsymbol{\Gamma}(y, \alpha_2^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_2^\vee)(\langle y, \alpha_2\rangle - 1))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \\ &= q^{\lceil \frac{\langle \alpha_2^\vee, \alpha_1\rangle}{3}\rceil + \lceil \frac{\langle y, \alpha_2\rangle}{3}\rceil - 2} \cdot \boldsymbol{\Gamma}(\alpha_2^\vee, \alpha_1^\vee) \cdot \boldsymbol{\Gamma}(0, \alpha_2^\vee) \\ &\quad \cdot \boldsymbol{g}_{\psi^{-1}}(-2)^{-1} \boldsymbol{g}_{\psi^{-1}}(-1)^{-1} \cdot \boldsymbol{c}(1_{\overline{\mathrm{SL}}_3^{(3)}}) \\ &= q^{-2} \cdot q \cdot \boldsymbol{c}(1_{\overline{\mathrm{SL}}_3^{(3)}}) = q^{-1}, \end{aligned}$$

where $\boldsymbol{c}$ is normalized to take value 1 at the $1 \in \overline{\mathrm{SL}}_3^{(3)}$. This implies that necessarily $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]}) = q^{-1}$, and thus

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]}) = q^{-1}.$$

Note, this is not a consequence of $\overline{\chi}$ being exceptional, although it is compatible. Clearly, an exceptional character $\overline{\chi}$ is such that

$$\begin{cases} \overline{\chi}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]})^3 = q^{-3}, \\ \overline{\chi}(\boldsymbol{s}_{3\alpha_1^\vee}) = q^{-1}. \end{cases}$$

In particular, if for some third root of unity $\zeta \neq 1$, $\overline{\chi}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]})$ is equal to $\zeta \cdot q^{-1}$, then $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_3^{(3)}, \overline{\chi})) = 0$ for such $\overline{\chi}$.

4C. ***Case III:*** $\boldsymbol{\overline{\mathrm{SL}}_{r+1}^{(n)}, n = r+2}$**.** For $n = r+2$, we show $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ and therefore Corollary 3.15 applies. Picking any $(y_1, y_2, \ldots, y_{r+1}) \in Y_{Q,n}$, we have

$$a \equiv y_1 \equiv y_2 \equiv \cdots \equiv y_{r+1} \bmod n,$$

where $a \in \{0, 1, 2, \ldots, r+1\}$. Write $y_i = k_i n + a$. Since $\sum_{i=1}^{r+1} y_i = 0$, one has

$$n \cdot \left(\sum_{i=1}^{r+1} k_i \right) + (r+1) \cdot a = 0.$$

In particular, $n | (r+1)a$. However, $\gcd(n, r+1) = 1$, so $n | a$ and $a = 0$. That is, $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ and therefore $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(r+2)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$. Note that, the equality $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ reflects the fact that the dual group for $\overline{\mathrm{SL}}_n^{(n+1)}$ is PGL_n (see [Weissman 2015, § 2.7.2]).

We claim that the dimension is equal to 1 in this case. Let $\mathcal{O}_y \in \mathcal{O}_{Q,n,\mathrm{sc}}^F$ be a $Y_{Q,n}^{\mathrm{sc}}$-free orbit with $\boldsymbol{i}_A^*(y_\rho) = (0, 1, \ldots, r-1, r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i$. We know that $\mathcal{O}_y$ is $Y_{Q,n}$-free (or equally, $Y_{Q,n}^{\mathrm{sc}}$-free). Moreover, one can check easily that $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Therefore $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(r+2)}, \overline{\chi})) = 1$ for the unique exceptional character $\overline{\chi}$ in this case.

4D. *Case IV:* $\overline{\mathrm{SL}}_{r+1}^{(n)}$, $n \geq r+3$.

Lemma 4.9. *Consider $y \in Y$ such that $\boldsymbol{i}_A^*(y_\rho) = (x_1^*, x_2^*, \ldots, x_r^*, x_{r+1}^*)$ with $x_i^* = i-1$. If $n \geq r+3$, the orbit $\mathcal{O}_y$ is $Y_{Q,n}$-free.*

Proof. Suppose not, then there exists $\mathbb{w} \neq 1$ such that $\mathbb{w}[y] - y \in Y_{Q,n}$. Identify $\mathbb{w}$ with a permutation, then we have

$$(x_1^*, x_2^*, \ldots, x_{r+1}^*) - (x_{\mathbb{w}(1)}^*, x_{\mathbb{w}(2)}^*, \ldots, x_{\mathbb{w}(r+1)}^*) \in Y_{Q,n}.$$

More precisely, $i - \mathbb{w}(i) \equiv j - \mathbb{w}(j) \bmod n$ for all i, j. Clearly, $n \nmid (i - \mathbb{w}(i))$ for all i, otherwise one can deduce $\mathbb{w}(i) = i$ for all i and therefore $\mathbb{w} = 1$. That is, $(i - \mathbb{w}(i))$ is either negative or positive. We reorder the terms $(i - \mathbb{w}(i))$ as

$$-r \leq (i_1 - \mathbb{w}(i_1)) \leq (i_2 - \mathbb{w}(i_2)) \leq \cdots < 0 < \cdots \leq (i_r - \mathbb{w}(i_r)) \leq (i_{r+1} - \mathbb{w}(i_{r+1})) \leq r.$$

Write $(i_1 - \mathbb{w}(i_1)) = -s,\ s \in \mathbb{N}$ and $(i_{r+1} - \mathbb{w}(i_{r+1})) = t,\ t \in \mathbb{N}$. It is easy to see that any negative $i - \mathbb{w}(i)$ must be equal to $-s$, and any positive $i - \mathbb{w}(i)$ must be equal to t.

We claim that $2 < t + s \leq r + 1$ and therefore $n \nmid (t+s)$, i.e., $\mathbb{w}[y] - y \notin Y_{Q,n}$ for all $\mathbb{w} \neq 1$. Note $0 - \mathbb{w}(0) = -s$ and $r - \mathbb{w}(r) = t$. Suppose $t + s > r + 1$, then there exists i_0 such that $r + 1 - t < i_0 < 1 + s$. However, there exists no i' such that $\mathbb{w}(i') = i_0$. This is a contradiction, and the claim follows.

Therefore $\mathcal{O}_y$ is $Y_{Q,n}$-free for the given y. □

It follows that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(n)}, \overline{\chi})) \geq 1$ for $n \geq r+3$. In principle, one could proceed as in Section 4B to analyze every element in $\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)$ and determine completely $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(n)}, \overline{\chi}))$ in this case. However, the level of complexity of the computation depends inevitably on (the center of) the dual group of $\overline{\mathrm{SL}}_r^{(n)}$ and could be quite involved for general $n \geq r+3$.

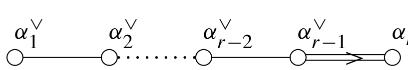
We summarize for the $n \leq r+2$ cases below.

Theorem 4.10. *Consider the Brylinski–Deligne covering $\overline{\mathrm{SL}}_{r+1}^{(n)}$, $n \leq r+2$ with $Q(\alpha^\vee) = 1$ for all coroots $\alpha^\vee$. Let $\overline{\chi}$ be an exceptional character of $\overline{\mathrm{SL}}_{r+1}^{(n)}$. Then $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(n)}, \overline{\chi})) = 1$ if and only if*

- *$n = r+2$ and $\overline{\chi}$ is the only exceptional character*, *or*
- *$n = r+1$ and $\overline{\chi}$ is the unique exceptional character satisfying* (14).

5. The C_r, $r \geq 2$ case

Consider the Dynkin diagram for the simple coroots for C_r:



Let

$$Y = Y^{\mathrm{sc}} = \langle \alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$$

be the cocharacter lattice of Sp_{2r}, where $\alpha_r^\vee$ is the short coroot. Let Q be the Weyl-invariant quadratic on Y such that $Q(\alpha_r^\vee) = 1$. Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = r, \\ 4 & \text{if } 1 \le i = j \le r-1, \\ -2 & \text{if } j = i+1, \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

A simple computation gives

$$Y_{Q,n} = \left\{ \sum_{i=1}^{n} x_i \alpha_i^\vee : n | (2x_i) \right\}.$$

We write $n_2 := n / \gcd(2, n)$. Then

$$Y_{Q,n} = \langle n_2\alpha_1^\vee, n_2\alpha_2^\vee, \ldots, n_2\alpha_{r-1}^\vee, n_2\alpha_r^\vee \rangle$$

and

$$Y_{Q,n}^{\mathrm{sc}} = \langle n_2\alpha_1^\vee, n_2\alpha_2^\vee, \ldots, n_2\alpha_{r-1}^\vee, n\alpha_r^\vee \rangle.$$

The map $\boldsymbol{i}_C : \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^r \mathbb{Z}e_i$ is given by

$$\boldsymbol{i}_C : (x_1, x_2, x_3, \ldots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{r-1} - x_{r-2}, x_r - x_{r-1}).$$

Here $\boldsymbol{i}_C$ is an isomorphism. The Weyl group is $W = S_r \rtimes (\mathbb{Z}/2\mathbb{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbb{Z}e_i$ and each $(\mathbb{Z}/2\mathbb{Z})_i$ acts by $e_i \mapsto \pm e_i$. In particular, $\mathbb{w}_{\alpha_i}$, $1 \le i \le r-1$, acts on $(y_1, y_2, \ldots, y_r) \in \bigoplus_i \mathbb{Z}e_i$ by exchanging y_i and y_{i+1}, while $\mathbb{w}_{\alpha_r}$ acts by (-1) on $\mathbb{Z}e_r$.

Moreover, $y \in Y$ lies in $Y_{Q,n}$ if and only if all entries of $\boldsymbol{i}_C(y)$ are divisible by n_2. It is easy to obtain

$$Y_{Q,n} = \left\{ (y_1, y_2, \ldots, y_r) \in \bigoplus\nolimits_{i=1}^r \mathbb{Z}e_i \ : \ n_2 | y_i \text{ for all } i. \right\}$$

and

$$Y_{Q,n}^{\mathrm{sc}} = \left\{ (y_1, y_2, \ldots, y_r) \in \bigoplus\nolimits_{i=1}^r \mathbb{Z}e_i \ : \ n_2 | y_i \ \text{ for all } \ i, \ \text{ and } \ n \Big| \sum\nolimits_i y_i. \right\}$$

We further note

$$2\rho = \sum_{i=1}^r (2r - 2i + 1) e_i = \sum_{i=1}^r i(2r - i)\alpha_i^\vee.$$

Assume $x_0 = 0$, then

$$\boldsymbol{i}_C(y_\rho) = (x_i - x_{i-1} - (r - i + 1/2))_{1 \le i \le r}.$$

Write $x_i^* := x_i - x_{i-1} - (r-i)$, and also $\boldsymbol{i}_C^*(y_\rho) := (x_1^*, x_2^*, \dots, x_{r-1}^*, x_r^*)$. Then

$$\boldsymbol{i}_C(y_\rho) = \boldsymbol{i}_C^*(y_\rho) - \Big(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}\Big).$$

We will discuss the two cases depending on the parity of n separately.

5A. ***The case where n is odd.*** Here, $n_2 = n$ and

$$nY = Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n} = \Big\{(y_1, \dots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i \;:\; n|y_i \text{ for all } i\Big\}.$$

The complex dual group for $\overline{\mathrm{Sp}}_{2r}^{(n)}$ for n odd is SO_{2r+1}.

Proposition 5.1. *Let n be an odd number, one has*

$$\begin{cases} |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| \ge 2 & \text{if } n \ge 2r+3, \\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| = 1 & \text{if } n = 2r+1, \\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| = 0 & \text{if } n \le 2r-1. \end{cases}$$

So, we have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(n)}, \chi)) = 1$, *for n odd, if and only if* $n = 2r+1$ *for the only exceptional character of* $\overline{\mathrm{Sp}}_{2r}^{(2r+1)}$.

Proof. We have written

$$\boldsymbol{i}_C(y_\rho) = \boldsymbol{i}_C^*(y_\rho) - \Big(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}\Big).$$

Since $x_1, \dots, x_r$ are arbitrary, the associated x_i^* are also arbitrary.

First, when $n \ge 2r+3$, consider the orbits $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ where

$$\boldsymbol{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r) \quad \text{and} \quad \boldsymbol{i}_C^*(y'_\rho) = (1, 2, \dots, r-1, r+1).$$

If $r = 2$, consider $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ with $\boldsymbol{i}_C^*(y_\rho) = (1, 2)$ and $\boldsymbol{i}_C^*(y'_\rho) = (1, 3)$. Both $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are $Y_{Q,n}$-free orbits. For example, for $\mathcal{O}_y$, this follows from the fact that the entries of $\boldsymbol{i}_C(\mathrm{w}(y_\rho)) - \boldsymbol{i}_C(y_\rho)$ are either $j-i$ or $j+i-1$, for $0 \le i, j \le r-1$. One can check also that $\wp_{Q,n}(\mathcal{O}_y) \ne \wp_{Q,n}(\mathcal{O}_{y'})$, and therefore $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \ge 2$.

Second, assume $n = 2r+1$. Consider $\mathcal{O}_y$ such that $\boldsymbol{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r)$. For $r = 2$, consider $\boldsymbol{i}_C^*(y_\rho) = (1, 2)$. It can be checked easily that $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Thus, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(2r+1)}, \chi)) = 1$.

Third, assume that $n \le 2r-1$, we want to show that $\mathcal{O}_{Q,n,\mathrm{sc}}^F = \varnothing$. If $\boldsymbol{i}_C^*(y_\rho) = (x_1^*, x_2^*, \dots, x_i^*, \dots, x_r^*)$ is such that $x_i^* \equiv x_j^* \bmod n$ for some $i \ne j$, then clearly $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\mathrm{sc}}^F$. Now if $n \nmid (x_i^* - x_j^*)$ for all $i \ne j$; since $n \le 2r-1$, it is not hard to see that there always exist i, j such that $n|(x_j^* - 1/2) + (x_i^* - 1/2)$, i.e., $n|(x_j^* + x_i^* - 1)$. In this case, one also has $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\mathrm{sc}}^F$. In any case, $\mathcal{O}_{Q,n,\mathrm{sc}}^F = \varnothing$ for $n \le 2r-1$. The proof is completed. □

5B. ***The case where n is even.*** Writing $n = 2m$,

$$Y_{Q,n} = \langle m\alpha_1^\vee, m\alpha_2^\vee, \dots, m\alpha_{r-1}^\vee, m\alpha_r^\vee \rangle, \qquad Y_{Q,n}^{\mathrm{sc}} = \langle m\alpha_1^\vee, m\alpha_2^\vee, \dots, m\alpha_{r-1}^\vee, n\alpha_r^\vee \rangle.$$

Equivalently, one has:

$$Y_{Q,n} = \Big\{ (y_1, y_2, \dots, y_r) \in \bigoplus\nolimits_{i=1}^{r} \mathbb{Z}e_i \ : \ m|y_i \text{ for all } i. \Big\}$$

and

$$Y_{Q,n}^{\mathrm{sc}} = \Big\{ (y_1, y_2, \dots, y_r) \in \bigoplus\nolimits_{i=1}^{r} \mathbb{Z}e_i \ : m|y_i \text{ for all } i, \text{ and } n| \sum\nolimits_i y_i. \Big\}$$

The dual group for $\overline{\mathrm{Sp}}_{2r}^{(n)}$ with n even is Sp_{2r}.

5B1. *The case where $m \geq 2r+2$.* Here, consider the orbits $\mathcal{O}_y$, $\mathcal{O}_{y'}$ given in the proof of Proposition 5.1. They are $Y_{Q,n}$-free; moreover, $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are distinct in the image of $\wp_{Q,n}$. Thus, we have $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$.

5B2. *The case where $m \leq 2r-2$.* Here, we can easily check $\mathcal{O}_{Q,n,\mathrm{sc}}^F = \varnothing$.

5B3. *The case where $m = 2r-1$.* Consider y with $\boldsymbol{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r)$, i.e.,

$$\boldsymbol{i}_C(y_\rho) = \Big(1 - \frac{1}{2}, 2 - \frac{1}{2}, \dots, (r-1) - \frac{1}{2}, r - \frac{1}{2}\Big).$$

Consider $\mathsf{w}_{\alpha_r} \in W$, then $\boldsymbol{i}_C(\mathsf{w}_{\alpha_r}(y_\rho)) = \big(1 - \frac{1}{2}, 2 - \frac{1}{2}, \dots, (r-1) - \frac{1}{2}, -(r - \frac{1}{2})\big)$.

Note $\mathcal{O}_y$ is $Y_{Q,n}^{\mathrm{sc}}$-free, and $\wp_{Q,n}^{\mathrm{sc}}(\mathcal{O}_{Q,n,\mathrm{sc}}^F) = \{\wp_{Q,n}^{\mathrm{sc}}(\mathcal{O}_y)\} = \{\wp_{Q,n}^{\mathrm{sc}}(\mathcal{O}_0)\}$. However, it is not $Y_{Q,n}$-free, since $\boldsymbol{i}_C(y_\rho - \mathsf{w}_{\alpha_r}(y_\rho)) = (0, 0, \dots, m) \in Y_{Q,n}$. Remember that any $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ which gives rise to $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ satisfies $\boldsymbol{c}(\boldsymbol{s}_{\mathsf{w}_{\alpha_r}[y]}) = \boldsymbol{t}(\mathsf{w}_{\alpha_r}, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$ where

$$\boldsymbol{t}(\mathsf{w}_{\alpha_r}, y) = q^{k_{y,\alpha_r} - 1} \cdot \boldsymbol{\Gamma}(y, \alpha_r^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_r^\vee) \cdot \langle y_\rho, \alpha_r \rangle)^{-1}.$$

Meanwhile, in our case $\mathsf{w}_{\alpha_r}[y] - y = (-m)\alpha_r^\vee \in Y_{Q,n}$. It follows that

$$\boldsymbol{c}(\boldsymbol{s}_{\mathsf{w}_{\alpha_r}[y]}) = \varepsilon^{D(\mathsf{w}_{\alpha_r}(y_\rho) - y_\rho, y)} \cdot \overline{\chi}(\boldsymbol{s}_{\mathsf{w}_{\alpha_r}(y_\rho) - y_\rho}) \cdot \boldsymbol{c}(\boldsymbol{s}_y).$$

For $\boldsymbol{c}$ to be nonzero on $\mathcal{O}_y$, i.e., $\wp_{Q,n}(\mathcal{O}_y)$ contributes to the right-hand side of (10), one has

$$\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) = q^{k_{y,\alpha_r} - 1} \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_r^\vee) \cdot \langle y_\rho, \alpha_r \rangle)^{-1}.$$

Moreover, we can argue as in Section 4B that this condition is also sufficient. One has $\langle y, \alpha_r \rangle = 2r$ and thus $k_{y,\alpha_r} = 1$. The equality is thus simplified to

$$\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) = \boldsymbol{g}_{\psi^{-1}}(m)^{-1}. \tag{20}$$

Consider the exceptional character $\overline{\chi}_{\psi'} = \overline{\chi}_{\psi'}^0 \cdot \delta_B^{1/2n}$, which relies on the distinguished unitary character $\overline{\chi}_{\psi'}^0$ depending on a nontrivial character $\psi' : F \to \mathbb{C}^\times$

(see Section 2C). Since $\overline{\chi}^{\,0}_{\psi'}(\boldsymbol{s}_{m\alpha_r^\vee}) = \boldsymbol{\gamma}_{\psi'}(\varpi)^{mQ(\alpha_r^\vee)}$, by Lemma 4.3, equality (20) becomes $\boldsymbol{\gamma}_{\psi}(\varpi) = (-1,\varpi)_n^{m^2} \cdot \boldsymbol{\gamma}_{\psi'}(\varpi)^{-m}$, which can be further reduced to

$$\boldsymbol{\gamma}_{\psi'}(\varpi) = (-1,\varpi)_n^{r+1} \cdot \boldsymbol{\gamma}_{\psi}(\varpi) = (-1,\varpi)_2^{r+1} \cdot \boldsymbol{\gamma}_{\psi}(\varpi).$$

In particular, if $\psi' = \psi_a$ with $a \in O_F^\times$, then the equality is equivalent to $(a(-1)^{r+1}, \varpi)_2 = -1$, i.e., $a \in (-1)^{r+1} \cdot (O_F^\times)^2$.

5B4. *The case where $m = 2r$.* We claim that here $\mathcal{O}^F_{Q,n} = \mathcal{O}^F_{Q,n,\mathrm{sc}}$. Clearly it suffices to show that $\mathcal{O}^F_{Q,n} \supseteq \mathcal{O}^F_{Q,n,\mathrm{sc}}$. Equivalently, if $\mathcal{O}_y$ is not $Y_{Q,n}$-free, we would like to show that it is not $Y^{\mathrm{sc}}_{Q,n}$-free. Write $\boldsymbol{i}^*_C(y_\rho) = (x_1^*, x_2^*, \ldots, x_r^*)$. By assumption,

$$\boldsymbol{i}_C(y - \mathsf{w}[y]) = \boldsymbol{i}^*_C(y_\rho - \mathsf{w}(y_\rho)) \in Y_{Q,n}$$

for some $\mathsf{w} \in W$. Entries of $\boldsymbol{i}_C(y - \mathsf{w}[y])$ cannot be of the form $2x_i^* - 1$ since m is even; thus they are of the form 0, $x_i^* - x_j^*$ or $x_i^* + x_j^* - 1$ for $i \neq j$. In this case, it is easy to see that $\boldsymbol{i}_C(y - \mathsf{w}'[y]) \in Y^{\mathrm{sc}}_{Q,n}$ for some $\mathsf{w}' \in W$, i.e., $\mathcal{O}_y$ is not $Y^{\mathrm{sc}}_{Q,n}$-free.

Consequently,

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}^F_{Q,n})|.$$

On the other hand, consider $\mathcal{O}_y$ with $\boldsymbol{i}^*_C(y_\rho) = (1, \ldots, r-1, r)$. It is easy to see $\wp_{Q,n}(\mathcal{O}^F_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Therefore, we always have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r)}, \overline{\chi})) = 1$ for any of the two exceptional characters of $\overline{\mathrm{Sp}}_{2r}^{(4r)}$.

5B5. *The case where $m = 2r+1$.* Consider $\mathcal{O}_y$ with $\boldsymbol{i}^*_C(y_\rho) = (1, 2, \ldots, r-1, r)$. One can check $\wp_{Q,n}(\mathcal{O}^F_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ with $\mathcal{O}_y \in \mathcal{O}^F_{Q,n}$, i.e., $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| = 1$. On the other hand,

$$\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y)\} \cup \{\wp_{Q,n}(\mathcal{O}_{z_i}) : 1 \le i \le r\}$$

with z_i described as follows. Recall that we write $z_{i,\rho} := z_i - \rho$. For $1 \le i \le r-1$, z_i is such that $\boldsymbol{i}^*_C(z_{i,\rho}) = (0, 2, 3, \ldots, \widehat{i+1}, \ldots, r, r+1)$, which denotes the r-tuple obtained from the $(r+1)$-tuple $(0, 2, 3, \ldots, r-1, r, r+1)$ by removing the entry $i+1$. Meanwhile, z_r is such that $\boldsymbol{i}^*_C(z_{r,\rho}) = (2, 3, \ldots, r-1, r, r+1)$.

Note that $\mathcal{O}_{z_i} \in \mathcal{O}^F_{Q,n,\mathrm{sc}} \setminus \mathcal{O}^F_{Q,n}$, since

$$\boldsymbol{i}_C(\mathsf{w}_{\alpha_r}[z_i] - z_i) = \boldsymbol{i}_C(\mathsf{w}_{\alpha_r}(z_{i,\rho}) - z_{i,\rho}) = -(0, 0, \ldots, 0, m) = \boldsymbol{i}_C(-m\alpha_r^\vee) \in Y_{Q,n}.$$

The $r+1$ elements $\wp_{Q,n}(\mathcal{O}_y)$ and $\wp_{Q,n}(\mathcal{O}_{z_i})$ $(1 \le i \le r)$ are all distinct. It follows that $|\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = r+1$. Therefore,

$$1 \le \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) \le r+1.$$

However, because there are only two exceptional characters $\overline{\chi}$, the dimension $\mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi}))$ can take at most two values. In fact, we will determine completely the value and its dependence on $\overline{\chi}$.

Proposition 5.2. *Let $\overline{\chi}$ be an exceptional character of $\overline{\mathrm{Sp}}_{2r}^{(4r+2)}$. Then*

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = \begin{cases} 1 & \text{if } \overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) = -q^{1/2}\cdot \boldsymbol{\gamma}_\psi(\varpi), \\ r+1 & \text{if } \overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) = q^{1/2}\cdot \boldsymbol{\gamma}_\psi(\varpi). \end{cases}$$

Proof. First, we show that $\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee})$ is equal to $\pm q^{1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)$ if $\overline{\chi}$ is an exceptional character. Consider

$$\begin{aligned} \overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee})^2 &= \overline{\chi}(\boldsymbol{s}_{-n\alpha_r^\vee})\cdot \varepsilon^{m^2 Q(\alpha_r^\vee)} \\ &= \overline{\chi}(\boldsymbol{s}_{n\alpha_r^\vee})^{-1}\cdot\varepsilon \\ &= q\cdot(-1,\varpi)_2, \end{aligned}$$

which has square roots exactly $\pm q^{1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)$. That is, an exceptional character $\overline{\chi}$ of $\overline{\mathrm{Sp}}_{2r}^{(4r+2)}$ is uniquely determined by the sign.

Second, arguing as in Section 4B , we see that $\wp_{Q,n}(\mathcal{O}_{z_i})$, $1\le i\le r$ contributes to the right-hand side of equality (10) if and only if (as in equality (15))

$$\overline{\chi}(\boldsymbol{s}_{\mathrm{w}_{\alpha_r}[z_i]-z_i}) = \varepsilon^{D(\mathrm{w}_{\alpha_r}[z_i]-z_i,z_i)}\cdot \boldsymbol{t}(\mathrm{w}_{\alpha_r}, z_i). \tag{21}$$

That is, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = 1+|\{z_i :$ the equality (21) holds for $z_i\}|$. Note that, $\mathrm{w}_{\alpha_r}[z_i]-z_i = -m\alpha_r^\vee$ for all i. On the other hand, we claim that the right-hand side of (21) is independent of i. A simple computation gives $\langle z_{i,\rho},\alpha_r\rangle = m$ and therefore

$$\begin{aligned} &\varepsilon^{D(\mathrm{w}_{\alpha_r}[z_i]-z_i,z_i)}\cdot \boldsymbol{t}(\mathrm{w}_{\alpha_r}, z_i) \\ &\quad= \varepsilon^{D(\alpha_r^\vee,z_i)}\cdot q^{\lceil \frac{\langle z_{i,\rho},\alpha_r\rangle+1}{n_{\alpha_r}}\rceil-1}\cdot \varepsilon^{\langle z_{i,\rho},\alpha_r\rangle\cdot D(z_i,\alpha_r^\vee)}\cdot \boldsymbol{g}_{\psi^{-1}}(\langle z_{i,\rho},\alpha_r\rangle\cdot Q(\alpha_r^\vee))^{-1} \\ &\quad= \varepsilon^{B_Q(z_i,\alpha_r^\vee)}\cdot q^{\lceil\frac{m+1}{n}\rceil-1}\cdot \boldsymbol{g}_{\psi^{-1}}(m)^{-1} \\ &\quad= \boldsymbol{g}_{\psi^{-1}}(m)^{-1}, \quad \text{by the evenness of } B_Q. \end{aligned}$$

Thus, it follows that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = 1$ or $r+1$. Moreover, it is equal to 1 if and only if $\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) \neq \boldsymbol{g}_{\psi^{-1}}(m)^{-1}$. By Lemma 4.3,

$$\boldsymbol{g}_{\psi^{-1}}(m)^{-1} = q^{1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi).$$

Thus, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = 1$ (respectively, $r+1$) if and only if $\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee})$ is equal to $-q^{1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)$ (respectively $q^{1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)$). □

We summarize the results in this section as follows:

Theorem 5.3. *Consider the Brylinski–Deligne covering group $\overline{\mathrm{Sp}}_{2r}^{(n)}$, where $r\ge 2$, and $n\ge 1$. Let $\overline{\chi}$ be an unramified exceptional character, then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(n)}, \overline{\chi}))$ *is equal to* 1 *if and only if the following hold*:

- $n = 4r-2$ *and $\overline{\chi}$ is the unique exceptional character satisfying* (20), *or*
- $n = 4r$ *and $\overline{\chi}$ is any exceptional character of $\overline{\mathrm{Sp}}_{2r}^{(4r)}$*, *or*

- *$n = 4r + 2$ and $\overline{\chi}$ is the unique exceptional character from Proposition 5.2, or*
- *$n = 2r + 1$ and $\overline{\chi}$ is the only exceptional character of $\overline{\mathrm{Sp}}_{2r}^{(2r+1)}$.*

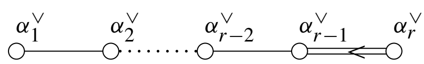
Further, consider the exceptional character $\overline{\chi}_{\psi_a} := \overline{\chi}^0_{\psi_a} \cdot \delta_B^{1/2n}$ associated with ψ_a. Assume ψ_a has conductor O_F, i.e., $a \in O_F^\times$. Then,

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r-2)}, \overline{\chi}_{\psi_a})) = 1$$

if and only if $a \in (-1)^{r+1} \cdot (O_F^\times)^2$, and $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi}_{\psi_a})) = 1$ if and only if $a \in (-1)^r \cdot (O_F^\times)^2$.

6. The B_r, $r \geq 2$ case

Consider the Dynkin diagram for the simple coroots for the type B_r group Spin_{2r+1}:



Let $Y = \langle \alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$ be the cocharacter lattice of Spin_{2r+1}, where $\alpha_r^\vee$ is the long coroot. Let Q be the Weyl-invariant quadratic form on Y such that $Q(\alpha_r^\vee) = 2$, i.e., $Q(\alpha_i^\vee) = 1$ for $1 \leq i \leq r-1$. Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 4 & \text{if } i = j = r; \\ 2 & \text{if } 1 \leq i = j \leq r-1; \\ -1 & \text{if } 1 \leq i \leq r-2 \text{ and } j = i+1; \\ -2 & \text{if } i = r-1,\ j = r; \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

The map $\boldsymbol{i}_B : \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^r \mathbb{Z}e_i$ is given by

$$\boldsymbol{i}_B : (x_1, x_2, x_3, \dots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \dots, x_{r-1} - x_{r-2}, 2x_r - x_{r-1}).$$

In particular, any $(y_1, \dots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i$ is equal to $\boldsymbol{i}_B(y)$ for some y if and only if $2 | (\sum_i y_i)$.

The Weyl group is $W = S_r \rtimes (\mathbb{Z}/2\mathbb{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbb{Z}e_i$ and $(\mathbb{Z}/2\mathbb{Z})_i : e_i \mapsto \pm e_i$. In particular, $\mathbb{w}_{\alpha_i}$, $1 \leq i \leq r-1$, acts on $(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbb{Z}e_i$ by exchanging y_i and y_{i+1}. Also, $\mathbb{w}_{\alpha_r}$ acts by (-1) on $\mathbb{Z}e_r$.

A simple computation gives

$$Y_{Q,n} = \Big\{(y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i : 2 \big| \Big(\sum_{i=1}^r y_i\Big),\ y_1 \equiv \dots \equiv y_r \bmod n,\ n | 2y_i \text{ for all } i.\Big\},$$
$$Y_{Q,n}^{\mathrm{sc}} = \Big\{(y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i : 2 \big| \Big(\sum_{i=1}^r y_i\Big),\ n | y_i \text{ for all } i.\Big\}$$

In particular, if n is odd, then $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$.

We note that $2\rho = \sum_{i=1}^{r} 2(r-i+1)e_i$, and therefore $\rho = \sum_{i=1}^{r}(r-i+1)e_i$. If $y = (x_1, x_2, \ldots, x_r) \in \bigoplus_i \mathbb{Z}\alpha_i^\vee$, then

$$\begin{aligned}\boldsymbol{i}_B(y_\rho) &= (x_1 - (r-1+1), x_2 - x_1 - (r-2+1), \ldots, x_i - x_{i-1} - (r-i+1),\\ &\qquad \ldots, x_{r-1} - x_{r-2} - (r-(r-1)+1), 2x_r - x_{r-1} - (r-r+1))\\ &=: (x_1^*, x_2^*, \ldots, x_i^*, \ldots x_{r-1}^*, x_r^*).\end{aligned}$$

Any $(x_1^*, \ldots, x_r^*) \in \bigoplus_i \mathbb{Z}e_i$ such that $2 \mid \big(\sum_i x_i^* + r(r+1)/2\big)$ is equal to $\boldsymbol{i}_B(y_\rho)$ for some y.

6A. ***The case where n is odd.*** Here,

$$nY = Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n}.$$

Therefore, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}_{2r+1}^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{F})|$, where $\overline{\chi}$ is the only exceptional character of $\overline{\mathrm{Spin}}_{2r+1}^{(n)}$. For n odd, the dual group for $\overline{\mathrm{Spin}}_{2r+1}^{(n)}$ is PGSp_{2r}.

Proposition 6.1. *Letting n be an odd number, one has*

$$\begin{cases} |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{F})| \geq 2 & \text{if } n \geq 2r+3,\\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{F})| = 0 & \text{if } n \leq 2r-1,\\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{F})| = 1 & \text{if } n = 2r+1.\end{cases}$$

Therefore, when n is odd, we have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}_{2r+1}^{(n)}, \overline{\chi})) = 1$ *if and only if* $n = 2r+1$.

Proof. First, assume that $n \geq 2r+3$. We write

$$\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \ldots, x_i^*, \ldots, x_r^*), \quad \text{with } 2 \,\Big|\, \Big(\sum_{i=1}^{r} x_i^* + \frac{r(r+1)}{2}\Big).$$

For $r \geq 3$, let $y \in Y$ y' be such that $\boldsymbol{i}_B(y_\rho) = (1, 2, 3, \ldots, r-2, r-1, r)$ and y' be such that $\boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1))$. For $r = 2$, we take $(x_1^*, x_2^*) = (1, 2)$ or $(2, 3)$, and let y and y' be the associated element in Y respectively. In any case, the two orbits $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are $Y_{Q,n}$-free. Moreover, $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, for $n \geq 2r+3$, one has

$$|\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{F})| \geq 2.$$

Second, assuming that $n \leq 2r-1$, we want to show that $\mathcal{O}_{Q,n,\mathrm{sc}}^{F} = \varnothing$. If $\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \ldots, x_i^*, \ldots, x_r^*)$ is such that $x_i^* \equiv x_j^* \bmod n$ for some $i \neq j$, then clearly $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\mathrm{sc}}^{F}$. Suppose $n \nmid (x_i^* - x_j^*)$ for all $i \neq j$, then since $n \leq 2r-1$, it is not hard to see that there always exist i, j such that $n | (x_j^* + x_i^*)$. That is, $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\mathrm{sc}}^{F}$ for any $\mathcal{O}_y$.

Third, if $n = 2r + 1$, consider the orbit $\mathcal{O}_y$ with

$$\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \dots, x_{r-1}^*, x_r^*) = (1, 2, 3, \dots, r-2, r-1, r).$$

(For $r = 2$, consider $\boldsymbol{i}_B(y_\rho) = (1, 2)$.) One has $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$, and therefore $|\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = 1$ for $n = 2r + 1$. □

6B. ***The case where n is even.*** Write $n = 2m$. Here,

$$Y = \left\{(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbb{Z}e_i : 2 \mid \textstyle\sum_{i=1}^r y_i\right\}.$$

Moreover,

$$Y_{Q,n} = \left\{(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbb{Z}e_i : 2 \Big| \sum_{i=1}^r y_i, \text{ if } y_i = k_i n + m \text{ for all } i \text{ or } y_i = k_i n \text{ for all } i\right\},$$

$$Y^{\mathrm{sc}}_{Q,n} = \left\{(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbb{Z}e_i : n | y_i \text{ for all } i\right\}.$$

We see easily that for $y_i = k_i n + m$, one has $(y_1, y_2, \dots, y_r) \in Y_{Q,n}$ if and only if $2|(rm)$. In fact, for n even, the dual group for $\overline{\mathrm{Spin}}^{(n)}_{2r+1}$ is equal to SO_{2r+1} if m and r are both odd; otherwise, the dual group is Spin_{2r+1}, see [Weissman 2015]. We discuss case by case according to the parities of r and m.

6B1. *The case where m and r are odd.* In particular, one has $r \geq 3$. In this case, $Y_{Q,n} = Y^{\mathrm{sc}}_{Q,n}$, and $\wp_{Q,n}(\mathcal{O}^F_{Q,n}) = \wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})$. Consider the following situations:

- If $n > 2(r + 1)$ (i.e., $m > r + 1$ and therefore $m \geq r + 2$), consider y such that $\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \dots, x_r^*)$ is equal to

  $$(1, 2, \dots, r-2, r-1, r) \quad \text{or} \quad (1, 2, \dots, r-2, r, r+1).$$

  We can check the two orbits $\mathcal{O}_y$ for these two choices of y are $Y_{Q,n}$-free, and moreover their images with respect to the map $\wp_{Q,n}$ are distinct in $\wp_{Q,n}(\mathcal{O}^F_{Q,n})$. Thus, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$ in this case.
- If $n < 2r$ (i.e., $m < r$ and so $m \leq r-2$), one can check that $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2r$ (note $n \neq 2(r+1)$), i.e., $m = r$, one can also check $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.

Therefore, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) \neq 1$ for both r and m odd.

6B2. *The case where m is odd and $r \geq 2$ is even.* Here, $Y_{Q,n} \neq Y^{\mathrm{sc}}_{Q,n}$. One has the following situations:

- Assume $n > 2(r + 1)$ (i.e., $m > r + 1$ and thus $m \geq r + 3$).

  Case I: If $r \geq 3$, consider y and y' such that

  $$\boldsymbol{i}_B(y_\rho) = (1, 2, \dots, r-2, r-1, r) \quad \text{and} \quad \boldsymbol{i}_B(y'_\rho) = (1, 2, \dots, r-2, r, r+1).$$

  We can check the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$-free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$.

Case II: If $r = 2$ and $m \geq r + 5$, consider $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ with $\boldsymbol{i}_B(y_\rho) = (1, 2)$ and $\boldsymbol{i}_B(y'_\rho) = (2, 3)$. Then as in the preceding case, they are $Y_{Q,n}$-free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$.

Case III: If $r = 2$ and $m = 5$, consider $\mathcal{O}_y$ with $\boldsymbol{i}_B(y_\rho) = (1, 2)$. It is easy to check $\wp_{Q,n}(\mathcal{O}^F_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. On the other hand, let z, z' be such that $\boldsymbol{i}_B(z_\rho) = (1, 4)$ and $\boldsymbol{i}_B(z'_\rho) = (2, 3)$. Then

$$\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y), \wp_{Q,n}(\mathcal{O}_z), \wp_{Q,n}(\mathcal{O}_{z'})\},$$

which is a set of size 3. Note, $\mathcal{O}_z, \mathcal{O}_{z'} \in \mathcal{O}^F_{Q,n,\mathrm{sc}} \backslash \mathcal{O}^F_{Q,n}$. That is, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| = 1$ and $|\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = 3$ in this case.

Let $\mathbb{w}, \mathbb{w}' \in W$ be such that

$$\boldsymbol{i}_B(\mathbb{w}[z] - z) = \boldsymbol{i}_B(\mathbb{w}'[z'] - z') = -(5, 5) \in Y_{Q,n}.$$

Write $y_{Q,n} = \boldsymbol{i}_B(\mathbb{w}[z] - z) \in Y_{Q,n}$. Then, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(10)}_5, \overline{\chi}))$ is equal to 1, as in Section 5B5, if and only if

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) \neq \varepsilon^{D(y_{Q,n},z)} \cdot \boldsymbol{T}(\mathbb{w}, z) \quad \text{and} \quad \overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) \neq \varepsilon^{D(y_{Q,n},z')} \cdot \boldsymbol{T}(\mathbb{w}', z'). \tag{22}$$

However, as in Proposition 5.2, that $\varepsilon^{D(y_{Q,n},z)} \cdot \boldsymbol{T}(\mathbb{w}, z) = \varepsilon^{D(y_{Q,n},z')} \cdot \boldsymbol{T}(\mathbb{w}', z')$ can be easily checked, and the condition (22) is equivalent to

$$\overline{\chi}(\boldsymbol{s}_{-5\alpha_r^\vee}) = -q^{1/2} \cdot \boldsymbol{\gamma}_\psi(\varpi). \tag{23}$$

This agrees with the result from Proposition 5.2 for the $\overline{C}^{(10)}_2$ case.

- If $n < 2r$ (i.e., $m \leq r$ and therefore $m \leq r - 1$), one can check $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2(r + 1)$ (note $n \neq 2r$), i.e., $r = m - 1$, one can check $\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}_0)\}$ (and thus $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_0)\}$) is a singleton with

$$\boldsymbol{i}_B(0_\rho) = (-r, -(r - 1), \ldots, -2, -1).$$

That is, $\mathcal{O}_0$ is $Y^{\mathrm{sc}}_{Q,n}$-free. However, it is not $Y_{Q,n}$-free, since there exists $\mathbb{w} \in W$ such that $\boldsymbol{i}_B(\mathbb{w}(0_\rho)) = (1, 2, \ldots, r - 1, r)$. It follows that

$$\boldsymbol{i}_B(\mathbb{w}(0_\rho) - 0_\rho) = (m, m, \ldots, m, m) \in Y_{Q,n}.$$

Write $y_{Q,n} = \mathbb{w}(0_\rho) - 0_\rho = \mathbb{w}[0] - 0$. It follows from an analogous argument for Proposition 4.1 that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(2r+2)}_{2r+1}, \overline{\chi})) = 1$ if and only if $\overline{\chi}$ is the unique exceptional character satisfying

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) = \boldsymbol{T}(\mathbb{w}, 0). \tag{24}$$

One can explicate the equality by computing the right-hand side as in Lemma 4.2. We omit the details here.

6B3. *The case where m is even and $r \geq 3$ is odd.* Here, $Y_{Q,n} \neq Y^{\mathrm{sc}}_{Q,n}$. We have:

- If $n > 2(r+1)$ (i.e., $m > r+1$ and therefore $m \geq r+3$), consider y and y' such that

$$\boldsymbol{i}_B(y_\rho) = (1, 2, \ldots, r-2, r-1, r) \quad \text{and} \quad \boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1).$$

  We can check the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$-free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$.

- If $n < 2r$ (i.e., $m < r$ and therefore $m \leq r-1$), one can check $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2(r+1)$ (note $n \neq 2r$), i.e., $r = m-1$, then $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ is a singleton with

$$\boldsymbol{i}_B(0_\rho) = (-r, -(r-1), \ldots, -2, -1).$$

  The situation is exactly as in the third case of Section 6B2, i.e., $\mathcal{O}_0$ is $Y^{\mathrm{sc}}_{Q,n}$-free but not $Y_{Q,n}$-free. Consider $\mathbb{w} \in W$ such that $\boldsymbol{i}_B(\mathbb{w}(0_\rho)) = (1, 2, \ldots, r-1, r)$ and

$$\boldsymbol{i}_B(\mathbb{w}(0_\rho) - 0_\rho) = (m, m, \ldots, m, m) \in Y_{Q,n}.$$

Write $y_{Q,n} = \mathbb{w}(0_\rho) - 0_\rho = \mathbb{w}[0] - 0$. Then $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(2r+2)}_{2r+1}, \overline{\chi})) = 1$ if and only if $\overline{\chi}$ is the unique exceptional character satisfying

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) = \boldsymbol{T}(\mathbb{w}, 0). \tag{25}$$

6B4. *The case where m is even and $r \geq 2$ is even.* Here, $Y_{Q,n} \neq Y^{\mathrm{sc}}_{Q,n}$. One has the following situations:

- If $n > 2(r+1)$ (i.e., $m > r+1$ and therefore $m \geq r+2$), there are two cases to consider.

  Case I: $r \geq 4$. Consider y and y' such that

$$\boldsymbol{i}_B(y_\rho) = (1, 2, \ldots, r-2, r-1, r) \quad \text{and} \quad \boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1).$$

  We can check easily that the orbits $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ for these two choices are $Y_{Q,n}$-free. Note that $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$, since $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$.

  Case II: $r = 2$. Consider y and y' such that $\boldsymbol{i}_B(y_\rho) = (1, 2)$ and $\boldsymbol{i}_B(y'_\rho) = (2, 3)$. For $m \geq 4$, $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are both $Y_{Q,n}$-free. Moreover, we can check that $\wp_{Q,n}(\mathcal{O}^F_{Q,n,sc}) \subseteq \{\wp_{Q,n}(\mathcal{O}_y), \wp_{Q,n}(\mathcal{O}_{y'})\}$. Now if $m \geq 6$, then $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. On the other hand, for $m = 4$, one has $\wp_{Q,n}(\mathcal{O}_y) = \wp_{Q,n}(\mathcal{O}_{y'})$ and so $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(8)}_5, \overline{\chi})) = 1$ for any exceptional character $\overline{\chi}$ in this case.

  To summarize for the case $m \geq r+2$:

$$\begin{cases} \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) = 1 & \text{if } m = 4, r = 2, \\ \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) \geq 2 & \text{if } r \geq 4 \text{ and } m \geq r+2, \text{ or } r = 2 \text{ and } m \geq 6. \end{cases}$$

- If $n < 2r$ (i.e., $m < r$ and therefore $m \le r-2$), one can check easily that $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2r$ (note $n \neq 2(r+1)$), i.e., $r = m$, one also has $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \varnothing$.

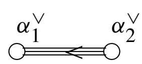
From the above discussion, we observe that for $r = 2$, the result agrees with that for covering groups of type C_2, as expected. Therefore, we just summarize our result for covering $\overline{\mathrm{Spin}}^{(n)}_{2r+1}$ with $r \ge 3$ as follows.

Theorem 6.2. *Consider Brylinski–Deligne covering* $\overline{\mathrm{Spin}}^{(n)}_{2r+1}$ *with* $r \ge 3$. *Let* $\overline{\chi}$ *be an exceptional character*, *then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) = 1$ *if and only if one of the following holds*:

- $n = 2(r+1)$ *and* $\overline{\chi}$ *is the unique exceptional character satisfying* (24) *or* (25),
- $n = 2r+1$ *and* $\overline{\chi}$ *is the only exceptional character of* $\overline{\mathrm{Spin}}^{(2r+1)}_{2r+1}$.

7. The G_2 case

Consider G_2 with Dynkin diagram for its simple coroots:



Let $Y = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$ be the cocharacter lattice of G_2, where $\alpha_1^\vee$ is the short coroot. Let Q be the Weyl-invariant quadratic on Y such that $Q(\alpha_1^\vee) = 1$ (thus $Q(\alpha_2^\vee) = 3$). Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = 1, \\ -3 & \text{if } i = 1, j = 2, \\ 6 & \text{if } i = j = 2. \end{cases}$$

A simple computation gives

$$Y_{Q,n} = Y^{\mathrm{sc}}_{Q,n} = \mathbb{Z}(n_{\alpha_1}\alpha_1^\vee) \oplus \mathbb{Z}(n_{\alpha_2}\alpha_2^\vee),$$

where $n_{\alpha_2} = n/\gcd(n,3)$ and $n_{\alpha_1} = n$.

The map $\boldsymbol{i}_G : \bigoplus_{i=1}^2 \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^3 \mathbb{Z}e_i$ is given by

$$\boldsymbol{i}_G : (x_1, x_2) \mapsto (x_1 - 2x_2, x_2 - x_1, x_2).$$

Any $(y_i)_i \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$ lies in the image of $\boldsymbol{i}_G$ if and only if $y_1 + y_2 + y_3 = 0$.

The Weyl group $W = \langle \mathbb{w}_{\alpha_1}, \mathbb{w}_{\alpha_2} \rangle$ generated by $\mathbb{w}_{\alpha_1}$ and $\mathbb{w}_{\alpha_2}$ is the dihedral group of order 12. In particular, $\mathbb{w}_{\alpha_1}(y_1, y_2, y_3) = (y_2, y_1, y_3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$, and $\mathbb{w}_{\alpha_2}(y_1, y_2, y_3) = (-y_1, -y_3, -y_2)$.

By using $\boldsymbol{i}_G$, we could identify

$$Y_{Q,n} = Y^{\mathrm{sc}}_{Q,n} = \left\{ (y_1, y_2, y_3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i : y_1 + y_2 + y_3 = 0,\ y_1 \equiv y_2 \equiv y_3 \bmod n \right\}.$$

We have $\rho = 5\alpha_1^\vee + 3\alpha_2^\vee$ with $\boldsymbol{i}_G(\rho) = (-1, -2, 3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$. It follows that for any $y = (x_1, x_2) \in \bigoplus_{i=1}^2 \mathbb{Z}\alpha_i^\vee$,

$$\boldsymbol{i}_G(y_\rho) = (x_1 - 2x_2 - 1, x_2 - x_1 - 2, x_2 + 3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i.$$

We may write $\boldsymbol{i}_G(y_\rho) = (x_1^*, x_2^*, x_3^*)$. In particular, $(x_1^*, x_2^*, x_3^*) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$ lies in the image of $\boldsymbol{i}_G$ if and only if $x_1^* + x_2^* + x_3^* = 0$.

Since $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$, it follows that $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$, where $\overline{\chi}$ is the only exceptional character of $\overline{G}_2^{(n)}$ as $Z(\overline{G}_2^\vee) = 1$.

To determine the n such that $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$, we only give an outline of the argument, the details of which consists of basic combinatorial computations:

- For $n = 7, 8$ or $n \geq 10$, the orbit $\mathcal{O}_y$ with $\boldsymbol{i}_G(y_\rho) = (-2, -1, 3)$ is $Y_{Q,n}$-free.
- For $n = 8, 10, 11$ or $n \geq 13$, the orbit $\mathcal{O}_{y'}$ with $\boldsymbol{i}_G(y'_\rho) = (-3, -1, 4)$ is $Y_{Q,n}$-free. Moreover, for $n = 8, 10, 11$ or $n \geq 13$, one has $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$ for $\boldsymbol{i}_G(y_\rho) = (-2, -1, 3)$ and $\boldsymbol{i}_G(y'_\rho) = (-3, -1, 4)$.
- If $\mathcal{O}_{Q,n,\mathrm{sc}}^F \neq \varnothing$, then necessarily $|Y/Y_{Q,n}^{\mathrm{sc}}| \geq |W|$, i.e., $n \cdot n_{\alpha_2} \geq 12$. Thus $n \geq 4$.
- One can also check by hand that $\mathcal{O}_{Q,n,\mathrm{sc}}^F = \varnothing$ for $n = 4, 5, 6, 9$.
- For $n = 7$ or 12, $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ with $\boldsymbol{i}_G(y_\rho) = (-2, -1, 3)$, i.e., $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$ for $n = 7$ or 12.

To summarize:

Theorem 7.1. *Consider the Brylinski–Deligne covering $\overline{G}_2^{(n)}$. Let $\overline{\chi}$ be the only exceptional character on $\overline{G}_2^{(n)}$, then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$ *if and only if $n = 7$ or* 12.

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