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We study harmonic functions on weighted manifolds and harmonic maps from weighted manifolds into Hadamard spaces introduced by Korevaar and Schoen. We prove several Liouville-type theorems for these harmonic maps.

1. Introduction

Weighted Riemannian manifolds, also called manifolds with density or smooth metric measure spaces in the literature, are Riemannian manifolds equipped with weighted measures. Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, weighted manifolds have been proven to be nontrivial generalizations of Riemannian manifolds. There are many geometric investigations of weighted manifolds; see Morgan [2005], Wei and Wylie [2009] and many others. In this paper, we investigate various Liouville-type theorems for harmonic functions on weighted manifolds as well as harmonic maps from weighted manifolds into Hadamard spaces, i.e., globally nonpositively curved spaces in the sense of Alexandrov (also called CAT(0) spaces), see, e.g., [Jost 1997b; Burago et al. 2001].

A weighted Riemannian manifold is a triple $(M, g, e^{-f} dV_g)$, where (M, g) is an n-dimensional Riemannian manifold, dV_g is the Riemannian volume element induced by the metric g and f is a smooth positive function on M. The f-Laplacian

$$\Delta_f = \Delta - \nabla f \cdot \nabla$$

is a natural generalization of Laplace–Beltrami operator Δ as it is self-adjoint with

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respect to the weighted measure $e^{-f}dV_g$, i.e.,

$$\int_M \Delta_f uve^{-f}\,dV_g = \int_M u\Delta_f ve^{-f}\,dV_g \quad \text{for } u, v \in C_0^\infty(M).$$

A function $u \in W^{1,2}_{\mathrm{loc}}(M)$ is called f-harmonic (f-subharmonic, f-superharmonic resp.) if it satisfies $\Delta_f u = 0$ (≥ 0, ≤ 0 resp.) in the weak sense, i.e.,

$$\int_M \langle \nabla u, \nabla \varphi\rangle e^{-f}\,dV_g = 0 \quad (\leq 0,\ \geq 0 \text{ resp.}) \text{ for any } 0 \leq \varphi \in C_0^\infty(M).$$

The Dirichlet f-energy of u is defined by

$$D^f(u) = \int_M |\nabla u|^2 e^{-f}\,dV_g.$$

On the other hand, f-harmonic maps from weighted manifolds $(M, g, e^{-f}dV_g)$ to (singular) metric spaces (Y, d) have wide geometric applications. Harmonic maps into metric spaces were initiated by Gromov and Schoen [1992] and then investigated independently by Korevaar and Schoen [1993] and Jost [1994]. In particular, when the domain is a Riemannian manifold, Korevaar and Schoen [1993; 1997] gave a complete exposition. In this paper we call a map $u : M \to Y$ f-harmonic if u locally minimizes the f-energy functional E^f in the sense of Korevaar and Schoen. For a detailed definition and its properties, we refer to [Korevaar and Schoen 1993] or Section 4 below. For the special case, f-harmonic maps from the Gaussian spaces, $(\mathbb{R}^n, |\cdot|, e^{-|x|^2/4}\,dx)$, to Riemannian manifolds are called quasiharmonic spheres, which emerge in the blowup analysis of harmonic heat flow [Lin and Wang 1999; Li and Tian 2000]. In this paper, we study Liouville theorems for f-harmonic maps into metric spaces, which generalize the previous results for harmonic maps in both aspects of domain manifolds and target spaces.

Analysis on weighted manifolds and the corresponding f-Laplacian have been extensively studied recently. We refer to [Munteanu and Wang 2011, 2012; Brighton 2013; Li 2005, 2016] for the f-harmonic functions on weighted manifolds, to [Li and Wang 2009; Zhu and Wang 2010; Li and Zhu 2010; Li and Yang 2012] for f-harmonic functions on the Gaussian spaces, to [Grigor'yan 2006, 2009] for heat kernel estimates, and to [Lin and Wang 1999; Wang and Xu 2012; Chen et al. 2012; Rimoldi and Veronelli 2013; Sinaei 2014, 2016] for f-harmonic maps.

In the first part of the paper we are concerned with Liouville-type theorems for f-harmonic functions on weighted manifolds. Several Liouville-type theorems for f-harmonic functions on the Gaussian spaces, also called quasiharmonic functions, have been proved in [Zhu and Wang 2010; Li and Wang 2009], in which the main techniques adopted are gradient estimates and separation of variables coupled with ODE results. In this paper, we propose another approach, which seems to be overlooked in the literature, to reprove many previous results. This method can be easily generalized, so that we may obtain Liouville theorems for f-harmonic functions for a large class of weighted manifolds; see Section 2.

Our observation is that the weighted version of L^p-Liouville theorem for weighted manifolds can be used to derive various Liouville theorems concerning the growth of f-harmonic functions. Yau [1976] first proved the L^p-Liouville theorem (for $1 < p < \infty$) for harmonic functions on any complete Riemannian manifold. Later, Karp [1982] obtained a quantitative version of this result. Li and Schoen [1984] proved other L^p-Liouville theorems (e.g., $0 < p < 1$) under the curvature assumption of manifolds. Karp's version of L^p-Liouville theorem has been generalized by Sturm [1994] to the setting of strongly local regular Dirichlet forms. In particular, our f-harmonic functions lie in this setting. By applying Sturm's L^p-Liouville theorem to f-harmonic functions, we immediately obtain several consequences which generalize previous results of [Zhu and Wang 2010; Li and Wang 2009; Li and Zhu 2010; Li and Yang 2012]. Although the proof of L^p-Liouville theorem is quite general and only involves integration by parts and the Caccioppoli inequality (thus it holds for all reasonable spaces), it is surprisingly powerful to obtain various Liouville theorems for weighted manifolds with slow volume growth, especially for the Gaussian spaces; see Corollaries 2.5 and 2.6 in Section 2. This does provide another approach to derive Liouville theorems without using any gradient estimate.

In the second part, we study Liouville-type theorems for harmonic maps from weighted manifolds to Hadamard spaces. For applications of f-harmonic maps with singular targets we refer to Gromov and Schoen [1992]. Our first result is an analogue to Kendall's theorem [1990, Theorem 3.2]. The essence of Kendall's theorem is that validity of a Liouville theorem for f-harmonic maps into Hadamard spaces, a priori a nonlinear problem, is reduced to that of a Liouville theorem of f-harmonic functions, a linear problem. Kendall [1990] proved this theorem for harmonic maps between Riemannian manifolds, by using probabilistic methods and potential theory. Kuwae and Sturm [2008] generalized Kendall's method to a class of harmonic maps between general metric spaces in the framework of Markov processes. Note that the harmonic maps they were concerned with are different from those of Korevaar and Schoen [1993] when targets are singular. In this paper, we consider harmonic maps into Hadamard spaces in the sense of Korevaar and Schoen. Following the argument by Li and Wang [1998], we are able to prove the following Kendall-type theorem by assuming local compactness of the targets. Recall that a geodesic space (Y, d) is called locally compact if every closed geodesic ball is compact.

Theorem 1.1. *Let $(M, g, e^{-f}\,dV_g)$ be a complete weighted Riemannian manifold satisfying that any bounded f-harmonic function is constant. Let (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y having bounded image is a constant map.*

In the same spirit as Kendall's theorem, Cheng, Tam and Wan [Cheng et al. 1996] proved a Liouville-type theorem for harmonic maps with finite energy. Our second result is a generalization of their theorem to f-harmonic maps into Hadamard spaces.

Theorem 1.2. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying that any f-harmonic function with finite Dirichlet f-energy is bounded. Let (Y, d) be an Hadamard space. Then any f-harmonic map from M to Y with finite f-energy has bounded image.*

We will follow the line of Cheng, Tam and Wan's reasoning, but using the techniques in potential theory, especially the theory of Royden and Nakai's decomposition on Riemannian manifolds [Royden 1952; Nakai 1960; Sario and Nakai 1970]. This possible approach of potential theory was implicitly suggested by Lyons in [Cheng et al. 1996, pp. 278]. We figure out the detailed arguments of this insight and apply them to Liouville theorems of f-harmonic maps. The Royden–Nakai decomposition theorem and Virtanen's theorem, see, e.g., Section 5 for weighted versions, play important roles in the classification theory of Riemannian manifolds developed by Royden, Nakai, Sario et al. many years ago. We shall dwell on these theories in the framework of weighted manifolds in Section 5 and utilize them to prove Theorem 1.2.

The following theorem is, more or less, a consequence of the combination of Theorems 1.1 and 1.2.

Theorem 1.3. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying that any bounded f-harmonic functions is constant. Let (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y with finite f-energy is a constant map.*

This theorem has an interesting application which motivates our studies in some sense. Bakry and Émery [1985] introduced weighted Ricci curvature for weighted manifolds. In particular, the so-called ∞-Bakry–Émery Ricci curvature

$$\mathrm{Ric}_f := \mathrm{Ric} + \nabla^2 f$$

turns out to be a suitable and important curvature quantity for weighted manifolds. The nonnegativity of Ric_f corresponds to the curvature-dimension condition $CD(0, \infty)$ on metric measure spaces via optimal transport, in the sense of Lott and Villani [2009] and Sturm [2006a; 2006b]. By a theorem of Brighton [2013], see also [Li 2016], the weighted manifold $(M, g, e^{-f} dV_g)$ satisfying $\mathrm{Ric}_f \geq 0$ admits no nonconstant bounded f-harmonic functions. Hence by Theorem 1.3 we immediately have:

Theorem 1.4. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying $\mathrm{Ric}_f \geq 0$ and (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y with finite f-energy is a constant map.*

The novelty of the result lies in the generality of targets, i.e., including singular metric spaces. In the smooth setting, Hadamard spaces are in fact Cartan–Hadamard manifolds, i.e., simply connected Riemannian manifolds with nonpositive sectional curvature. On Riemannian manifolds, Theorem 1.4 has been proved by Wang and Xu [2012] and Rimoldi and Veronelli [2013] independently under an additional assumption of $\int_M e^{-f}\,dV_g = \infty$ for domain manifolds, while simply-connectedness of the targets is not needed. Note that the weighted volume assumption here cannot be derived from the curvature condition $\mathrm{Ric}_f \geq 0$ in general. In addition, there is a nontrivial f-harmonic map from a domain manifold with $\mathrm{Ric}_f \geq 0$ and $\int_M e^{-f}\,dV_g < \infty$ to a nonpositively curved target manifold, constructed by Rimoldi and Veronelli [2013, Remark 3.7]. Our contribution is to drop the weighted volume assumption by assuming simply-connectedness of the targets and to extend the result to singular spaces.

For harmonic maps into singular Hadamard spaces, the arguments in [Wang and Xu 2012; Rimoldi and Veronelli 2013], both following Schoen and Yau [1976], do not work any more since we cannot apply Bochner techniques as in those works due to the singularity of targets. Although a weak Bochner formula can also be derived following Korevaar and Schoen [1993], it is insufficient for our purpose. Fortunately, we can circumvent these technical problems by proving Theorem 1.3, which follows from Kendall-type theorems. This does provide another approach to Liouville theorems for f-harmonic maps without using Bochner techniques. This is one of the main points of the paper.

The rest of the paper is organized as follows. In Section 2, we study L^p Liouville theorem for f-harmonic functions and give some applications. In Section 3, we consider harmonic maps with smooth targets. In Section 4, we define f-harmonic maps into Hadamard spaces and prove Theorem 1.1. In Section 5, we dwell on the Royden-Nakai theory and prove Theorems 1.2 and 1.3.

2. f-harmonic functions

In this section, we study L^p-Liouville theorems for f-harmonic functions and their applications. We will show that L^p-Liouville theorems are quite powerful for weighted manifolds with finite volume.

The L^p-Liouville theorem, $1 < p < \infty$, for harmonic functions (or nonnegative subharmonic functions) was initiated by Yau [1976] on complete Riemannian manifolds. Karp [1982] obtained a quantitative version of this Liouville theorem. Later, Sturm [1994] proved an L^p-Liouville theorem for strongly local regular Dirichlet forms. The following theorem is a special case of Sturm's result for f-harmonic functions. We denote by $B_r := B_r(x_0)$ the closed geodesic ball of radius r centered at a fixed point $x_0 \in M$.

Theorem 2.1 [Sturm 1994, Theorem 1]. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold and u be a nonnegative f-subharmonic function (or an f-harmonic function). For $1 < p < \infty$, set $v(r) := \int_{B_r} |u|^p e^{-f}\, dV_g$. Then either*

$$\inf_{a>0} \int_a^\infty \frac{r}{v(r)}\, dr < \infty,$$

or u is a constant.

We state several consequences of Theorem 2.1.

A quite useful consequence is about f-parabolicity of M. Recall that a weighted manifold $(M, g, e^{-f} dV_g)$ is called *f-parabolic* if there are no nonconstant nonnegative f-superharmonic functions on M. For a compact set $K \subset M$, the *f-capacity* of K is defined as

$$\mathrm{cap}^f(K) := \inf_{\substack{\varphi \in \mathrm{Lip}_0(M) \\ \varphi|_K = 1}} \int_M |\nabla \varphi|^2 e^{-f}\, dV_g,$$

where $\mathrm{Lip}_0(M)$ is the space of compactly supported Lipschitz functions on M.

Proposition 2.2 (f-parabolicity). *Let $(M, g, e^{-f} dV_g)$ be a complete weighted manifold. Then the following are equivalent:*

(i) *M is f-parabolic;*

(ii) $\mathrm{cap}^f(K) = 0$ *for some (then any) compact set $K \subset M$;*

(iii) *any bounded f-superharmonic function on M is constant.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from [Grigor'yan 1985, Proposition 3]; see also Proposition 2.1 of [Grigor'yan 1999].

(i) $\Leftrightarrow$ (iii). This follows from the fact that any nonnegative f-superharmonic function u can be approximated by bounded f-superharmonic functions $u_n = \min\{u, n\}$, $n \in \mathbb{N}$. □

We say a weighted manifold $(M, g, e^{-f} dV_g)$ has the *moderate volume growth property* if

$$\int_1^\infty \frac{r}{V_f(B_r)}\, dr = \infty, \tag{1}$$

where $V_f(B_r) := \int_{B_r} e^{-f}\, dV_g$.

Corollary 2.3. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold satisfying the moderate volume growth property. Then M is f-parabolic.*

Proof. Let u be a bounded f-superharmonic function on M. Then for any $a > 0$,

$$\int_a^\infty \frac{r}{v(r)}\, dr \ge C \int_a^\infty \frac{r}{V_f(B_r)}\, dr = \infty.$$

Theorem 2.1 yields that u is a constant. This proves the corollary. □

Remark 2.4. Corollary 2.3 slightly generalizes [Wang and Xu 2012, Theorem 1.4]. In particular, this corollary implies [Zhu and Wang 2010, Theorem 2].

We can also derive several Liouville-type theorems for f-harmonic functions from Theorem 2.1.

Corollary 2.5. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold and u be a nonnegative f-subharmonic function (or f-harmonic function). Assume one of the following holds:*

(i) *$u = O(w^\alpha)$ for some nonnegative function w with $\int_M w d^{-2}(\cdot, x_0) e^{-f} dV_g < \infty$ and some $\alpha \in (0, 1)$;*

(ii) *$\int_M d^k(\cdot, x_0) e^{-f} dV_g < \infty$ for some $k > -2$ and $u = O(d^\beta(\cdot, x_0))$ for some $\beta \in (0, k+2)$;*

(iii) *$\int_M e^{-f} dV_g < \infty$ and $u = O(d^\beta(\cdot, x_0))$ for $\beta \in (0, 2)$;*

(iv) *$f \geq Cd(\cdot, x_0)^\beta$ for some $C > 0$, $\beta > 0$ and $\int_M e^{-\delta f} dV_g < \infty$ for some $0 < \delta < 1$ and u has polynomial growth;*

(v) *$f \geq Cd(\cdot, x_0)^\beta$ for some $C > 0$, $\beta > 0$ and the Riemannian volume has polynomial volume growth and $u = O(e^{\alpha C d(\cdot, x_0)^\beta})$, $\alpha \in (0, 1)$.*

Then u is a constant.

Proof. For (i), we see that there exists $p \in (1, \infty)$ such that $|u|^p = O(w)$. Hence

$$\begin{aligned}\frac{1}{r^2 \log r} v(r) &= \frac{1}{r^2 \log r} \int_{B_r} |u|^p e^{-f} \, dV_g \\ &\leq \frac{C}{\log r} \int_{B_r} \frac{w(x)}{d^2(x, x_0)} e^{-f(x)} \, dV_g(x) = o(1).\end{aligned}$$

It follows from Theorem 2.1 that u is a constant. The case (ii) follows from (i) by letting $w = d^{k+2}(\cdot, x_0)$. The case (iii) follows from (ii) by letting $k = 0$.

For (iv), let us observe for any $1 < p < \infty$,

$$\int_M |u|^p e^{-f} \, dV_g \leq C \int_M d^{sp}(x, x_0) e^{-f(x)} \, dV_g(x) \leq C \int_M e^{-\delta f} \, dV_g < \infty,$$

where $s > 0$. Then the statement also follows from Theorem 2.1. The case (v) can be proved in a similar way. □

The following result is a direct corollary of the above (v).

Corollary 2.6. *Let u be an f-harmonic function on the Gaussian space, i.e.,*

$$\Delta u - \tfrac{1}{2}\langle x, \nabla u \rangle = 0.$$

If $u = O(e^{\alpha |x|^2/4})$ as $x \to \infty$, for some $0 < \alpha < 1$, then u is a constant.

Remark 2.7. Corollary 2.6 implies that there are no nonconstant polynomial growth f-harmonic functions on the Gaussian space. This improves the result in [Li and Wang 2009, Theorem 4.2]. By Caccioppoli's inequality, Corollary 2.6 can be also derived from Li and Yang [2012, Corollary 1.2] .

In the remaining part of this section, we study the L^p-Liouville theorem introduced by Zhu and Wang [2010] using a different measure from ours. We shall explain why the critical exponent of the L^p-Liouville theorem in [Zhu and Wang 2010, Theorem 3] is $p = n/(n-2)$ $(n \ge 3)$ by applying our result. Let $(M, g, e^{-f} dV_g)$ be an n-dimensional $(n \ge 3)$ complete weighted manifold. In fact, they consider the L^p space with respect to the Riemannian volume in a modified Riemannian manifold $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$, denoted by $L^p(\widetilde{M}, dV_{\tilde{g}})$, where $\tilde{g}$ is a conformal change of g given by $\tilde{g} = e^{-2f/(n-2)} g$. Since this new manifold $\widetilde{M}$ may be incomplete, e.g., Gaussian space, Yau's L^p-Liouville theorem fails in this setting. In the following, we use the L^p-Liouville theorem on weighted manifolds to show the one on modified Riemannian manifolds.

Theorem 2.8. *Let $(M, g, e^{-f} dV_g)$ be an n-dimensional $(n \ge 3)$ complete weighted manifold, $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ be the modified Riemannian manifold and u be a nonnegative f-subharmonic function (or f-harmonic function) on M. For any $p > n/(n-2)$, there exists a constant $\delta = \delta(p, n) \in (0, 1)$ such that if $\int_M e^{-\delta f} dV_g < \infty$ and $u \in L^p(\widetilde{M}, dV_{\tilde{g}})$, then u is a constant.*

Proof. For any $p > n/(n-2)$, let $q = 2p/(p + n/(n-2)) > 1$, $\alpha = p/q > n/(n-2)$ and $\alpha^* = \alpha/(\alpha-1) \in (1, n/2)$. Set $\delta = (n - 2\alpha^*)/(n-2) \in (0, 1)$. By Hölder's inequality, we can verify that

$$\begin{aligned}\int_M u^q e^{-f} dV_g = \int_M u^q e^{\frac{2f}{n-2}} dV_{\tilde{g}} &\le \left(\int_M u^{q\alpha} dV_{\tilde{g}}\right)^{\frac{1}{\alpha}} \left(\int_M e^{\frac{2\alpha^* f}{n-2}} dV_{\tilde{g}}\right)^{\frac{1}{\alpha^*}} \\ &= \left(\int_M u^p dV_{\tilde{g}}\right)^{\frac{1}{\alpha}} \left(\int_M e^{-\delta f} dV_g\right)^{\frac{1}{\alpha^*}} < \infty.\end{aligned}$$

The statement follows from Theorem 2.1. □

This yields a direct corollary which generalizes [Zhu and Wang 2010, Theorem 3], which is restricted to the Gaussian spaces, to general weighted manifolds. The Riemannian manifold (M, g, dV_g) is said to be *of subexponential volume growth* if $V_g(r) := V_g(B_r(x_0)) = e^{o(r)}$ for some (then all) $x_0 \in M$.

Corollary 2.9. *Let $(M, g, e^{-f} dV_g)$ be an n-dimensional $(n \ge 3)$ complete weighted manifold satisfying that $f \ge C d^\beta(\cdot, x_0)$ for some $C > 0$, $\beta > 0$ and $V_g(r) = e^{o(r^\beta)}$. Let $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ be the modified Riemannian manifold. Then for any $p > n/(n-2)$, the f-harmonic function in $L^p(\widetilde{M}, dV_{\tilde{g}})$ is constant. In particular, for $\beta = 1$, it suffices to assume (M, g, dV_g) has subexponential volume growth.*

Proof. By virtue of Theorem 2.8, it is sufficient to prove $\int_M e^{-\delta f}\,dV_g < \infty$ where δ is the constant in Theorem 2.8. We see by the coarea formula that

$$\begin{aligned}\int_M e^{-\delta f}\,dV_g &= \int_0^1 \int_{S_r(x_0)} e^{-\delta f}\,dA_r\,dr + \int_1^\infty \int_{S_r(x_0)} e^{-\delta f}\,dA_r\,dr\\ &\le C_0 + \int_1^\infty \int_{S_r(x_0)} e^{-\delta C r^\beta}\\ &= C_0 + \int_1^\infty e^{-\delta C r^\beta}\frac{d}{dr}V_g(r)\,dr\\ &= C_0 + e^{-\delta C r^\beta}V_g(r)\big|_1^\infty + \delta C\int_1^\infty \beta r^{\beta-1}e^{-\delta C r^\beta}V_g(r)\,dr.\end{aligned}$$

Since $V_g(r) = e^{o(r^\beta)}$, there exists R large such that

$$V_g(r) \le e^{\frac{1}{2}\delta C r^\beta} \quad \text{for } r > R.$$

It follows that $\lim_{r\to\infty} e^{-\delta C r^\beta}V_g(r) = 0$ and $\int_1^\infty \beta r^{\beta-1}e^{-\delta C r^\beta}V_g(r)\,dr < \infty$. It follows that $\int_M e^{-\delta f}\,dV_g < \infty$. This completes the proof. □

3. f-harmonic maps into Cartan–Hadamard manifolds

In this section, we prove Theorem 1.4 in the case that the target $Y = N$ is a Cartan–Hadamard manifold.

Theorem 3.1. *Let $(M, g, e^{-f}dV_g)$ be a complete weighted Riemannian manifold which is f-parabolic and N be a Cartan–Hadamard manifold. Then any f-harmonic map with finite f-energy, i.e., $E^f(u) := \int_M |\nabla u|^2 e^{-f}\,dV_g < \infty$, is a constant map.*

Proof. We use a construction by Rimoldi and Veronelli [2013] which associates an f-harmonic map with a harmonic map on some higher dimensional warped product manifold.

Precisely, let $\overline{M} := M \times_{e^{-f}} \mathbb{S}^1$ denote a warped product, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with $\mathrm{Vol}(\mathbb{S}^1) = 1$, with the metric on $\overline{M}$ given by $\bar{g}(x,t) = g(x) + e^{-2f(x)}dt^2$. Note that $\overline{M}$ is complete. It follows from [Rimoldi and Veronelli 2013, Proposition 2.5 and Lemma 2.6] that $\overline{M}$ is parabolic and the map $\bar{u} : \overline{M} \to N$, defined by $\bar{u}(x,t) = u(x)$ is a harmonic map. Moreover, $E_{\overline{M}}(\bar{u}) = E^f_M(u) < \infty$.

Now by applying [Cheng et al. 1996, Proposition 2.1 and Theorem 3.1] to $\bar{u}$ and $\overline{M}$, we know that the image of $\bar{u}$, $\bar{u}(\overline{M}) = u(M)$, is bounded in N. Since N is a Cartan–Hadamard manifold, $d^2(\bar{u}(\cdot), Q)$ is a subharmonic function for any $Q \in N$, which is also bounded. By the parabolicity of $\overline{M}$, we know that $d^2(\bar{u}(\cdot), Q)$ is constant for any $Q \in N$. This proves the theorem. □

Theorem 3.2. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold satisfying* $\mathrm{Ric}_f \geq 0$ *and N be a Cartan–Hadamard manifold. Then any f-harmonic map with finite f-energy $E^f(u) < \infty$ is a constant map.*

Proof. We divide the theorem into two cases:

(a) $\int_M e^{-f} dV_g = \infty$,

(b) $\int_M e^{-f} dV_g < \infty$.

For case (a), it was already proved in [Wang and Xu 2012, Theorem 1.2] or [Rimoldi and Veronelli 2013, Theorem 3.3] for general Riemannian target of nonpositive curvature (without the assumption of simply-connectedness). For case (b), we observe that M satisfies the moderate volume growth property (1). By Corollary 2.3, M is f-parabolic. Then the statement follows from Theorem 3.1. □

Remark 3.3. Comparing Theorem 3.2 with [Wang and Xu 2012, Theorem 1.2] or [Rimoldi and Veronelli 2013, Theorem 3.3], we remove the condition of the infinity of f-volume for M but add the assumption that N is simply connected.

4. f-harmonic maps into Hadamard spaces

In this section, we define f-harmonic maps from an n-dimensional complete weighted Riemannian manifold $(M, g, e^{-f} dV_g)$ to a general metric space (Y, d). For that purpose we investigate an f-energy functional E^f whose definition given here follows Korevaar and Schoen [1993], where a Sobolev space theory for maps from Riemannian domains to metric spaces was developed. Note that the energy functional has been further extended to maps from complete noncompact Riemannian manifolds, and even more generally the so-called admissible Riemannian polyhedrons with simplexwise smooth Riemannian metric, in Eells and Fuglede [2001] (see Chapter 9 therein).

We consider Borel-measurable (equivalently, measurable with respect to $e^{-f} dV_g$) maps $u : M \to Y$ (u then has separable range since M is a separable metric space; see [Dudley 2002, Problem 10 in Section 4.2]). The space $L^2_{\mathrm{loc}}(M_f, Y)$ is defined as the set of Borel-measurable maps u for which $d(u(\cdot), Q) \in L^2_{\mathrm{loc}}(M, e^{-f} dV_g)$ for some point Q (and hence for any Q by the triangle inequality) in Y. Since this space is unchanged if we use the unweighted measure dV_g instead of $e^{-f} dV_g$ in its definition, we will write $L^2_{\mathrm{loc}}(M, Y)$ for simplicity in the following. When M is compact, $L^2_{\mathrm{loc}}(M, Y)$ is a complete metric space, with distance function $\hat{d}$ defined by

$$\hat{d}^2(u, v) := \int_M d^2(u(x), v(x)) e^{-f(x)} \, dV_g(x),$$

provided that (Y, d) is complete.

The approximate energy density for a map $u \in L^2_{\mathrm{loc}}(M, Y)$ is defined for $\varepsilon > 0$ as

$$e_\varepsilon(u) := \frac{1}{\omega_n} \int_{S(x,\varepsilon)} \frac{d^2(u(x), u(y))}{\varepsilon^2} \frac{d\sigma_{x,\varepsilon}(y)}{\varepsilon^{n-1}}, \tag{2}$$

where $d\sigma_{x,\varepsilon}(y)$ is the $(n-1)$-dimensional surface measure on the sphere $S(x, \varepsilon)$ of radius ε centered at x induced by the Riemannian metric g, and ω_n is the volume of the n-dimensional unit Euclidean ball. One can check that the function $e_\varepsilon(u) \in L^1_{\mathrm{loc}}(M)$ (see [Korevaar and Schoen 1993]). Then we can define the f-energy functional E^f by

$$E^f(u) := \sup_{\substack{\eta \in C_0(M) \\ 0 \le \eta \le 1}} \left(\limsup_{\varepsilon \to 0} \int_M \eta e_\varepsilon(u) e^{-f} \, dV_g \right).$$

We say a map $u \in L^2_{\mathrm{loc}}(M, Y)$ is locally of finite energy, denoted by $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$, if $E^f(u_{|\Omega}) < \infty$ for any relatively compact domain $\Omega \subset M$.

Theorem 4.1. *If $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$, then there exists a function $e(u) \in L^1_{\mathrm{loc}}(M)$, such that for any $\eta \in C_0(M)$, the following limit exists*

$$\lim_{\varepsilon \to 0} \int_M \eta e_\varepsilon(u) e^{-f} \, dV_g =: \int_M \eta e(u) e^{-f} \, dV_g, \tag{3}$$

which serves as the definition of $e(u)$.

Proof. By definition, $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ implies that for any connected, open and relatively compact subset $\Omega \subset M$, $u_{|\Omega} \in L^2(\Omega, Y)$ and

$$\sup_{\substack{\zeta \in C_0(\Omega) \\ 0 \le \zeta \le 1}} \left(\limsup_{\varepsilon \to 0} \int_\Omega \zeta e_\varepsilon(u_{|\Omega}) \, dV_g \right) < \infty,$$

that is, $u_{|\Omega} \in W^{1,2}(\Omega, Y)$ in Korevaar and Schoen's notation [1993].

Now by their Theorem 1.5.1 and Theorem 1.10, we know that there exists a function $e(u_{|\Omega}) \in L^1(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \int_\Omega \zeta e_\varepsilon(u) \, dV_g = \int_\Omega \zeta e(u_{|\Omega}) \, dV_g \quad \text{for all } \zeta \in C_0(\Omega). \tag{4}$$

In particular, one has

$$\lim_{\varepsilon \to 0} \int_\Omega \eta e_\varepsilon(u) e^{-f} \, dV_g = \int_\Omega \eta e(u_{|\Omega}) e^{-f} \, dV_g \quad \text{for all } \eta \in C_0(\Omega). \tag{5}$$

We then define a function $e(u)$ on M by $e(u)_{|\Omega} := e(u_{|\Omega})$ for any $\Omega \subset M$ with smooth boundary. One can show that $e(u)$ is well defined. For that purpose, one only needs to check $e(u_{|\Omega}) = e(u_{|\Omega_1})$ on $\Omega_1 \subset \Omega$ where both Ω_1 and $\Omega \setminus \Omega_1$ have

Lipschitz boundary. This is true since by the trace theory [Korevaar and Schoen 1993, Theorem 1.12.3], one has

$$\int_{\Omega} e(u_{|\Omega})\,dV_g = \int_{\Omega_1} e(u_{|\Omega_1})\,dV_g + \int_{\Omega\setminus\Omega_1} e(u_{|\Omega\setminus\Omega_1})\,dV_g.$$

Then (3) follows from (5) which proves this theorem. □

Remark 4.2. By the definition of $e(u)$ and (4), we know

$$e(u)(x) = |\nabla u|^2(x),$$

where $|\nabla u|^2(x)$ is the energy density function in [Korevaar and Schoen 1993]. This function is consistent with the usual way of defining $|du|^2$ for maps between Riemannian manifolds. Therefore we use $|\nabla u|^2(x)$ instead of $e(u)(x)$ in the following.

Remark 4.3. By a polarization argument, we can check that for any two functions $h_1, h_2 \in W^{1,2}_{\mathrm{loc}}(M, e^{-f}\,dV_g)$,

$$\lim_{\varepsilon\to 0}\int_M \eta(x)\frac{1}{\omega_n}\int_{S(x,\varepsilon)} \frac{(h_1(x)-h_1(y))(h_2(x)-h_2(y))}{\varepsilon^2}\frac{d\sigma_{x,\varepsilon}(y)}{\varepsilon^{n-1}}e^{-f(x)}\,dV_g(x) \\ = \int_M \eta(x)\langle\nabla h_1(x),\nabla h_2(x)\rangle e^{-f(x)}\,dV_g(x) \quad \text{for all } \eta\in C_0(M).$$

Remark 4.4. With (3) in hand, by the definition of E^f, we can derive (see [Eells and Fuglede 2001, Theorem 9.1]),

$$E^f(u) = \int_M |\nabla u|^2 e^{-f}\,dV_g \quad \text{for all } u\in W^{1,2}_{\mathrm{loc}}(M,Y).$$

In particular, we define $D^f(u) = E^f(u)$ when u is a scalar function.

Remark 4.5. As in [Korevaar and Schoen 1993], the definition of E^f is unchanged if we replace $e_\varepsilon(x)$ by ${}_\nu e_\varepsilon(x) := \int_0^2 e_{\lambda\varepsilon}(x)\,d\nu(\lambda)$, where ν is any Borel measure on the interval $(0, 2)$ satisfying $\nu\ge 0$, $\nu((0,2))=1$, $\int_0^2\lambda^{-2}\,d\nu(\lambda)<\infty$. For example, the approximate energy density function can be chosen as follows.

(1) When $n\ge 3$, for the measure $d\nu_1(\lambda) = n\lambda^{n-1}d\lambda,\ 0<\lambda<1$,

$${}_{\nu_1}e_\varepsilon(x) = \frac{n}{\omega_n}\int_{B(x,\varepsilon)}\frac{d^2(u(x),u(y))}{d^2(x,y)}\frac{dV_g(y)}{\varepsilon^n};$$

(2) For the measure $d\nu_2(\lambda) = (n+2)\lambda^{n+1}d\lambda,\ 0<\lambda<1$,

$${}_{\nu_2}e_\varepsilon(x) = \frac{n+2}{\omega_n}\int_{B(x,\varepsilon)}\frac{d^2(u(x),u(y))}{\varepsilon^2}\frac{dV_g(y)}{\varepsilon^n}.$$

Remark 4.6. For $n\ge 3$, by introducing a conformal change of the metric $\widetilde{M} = (M, \tilde g, dV_{\tilde g})$ where $\tilde g = e^{-2f/(n-2)}g$ and employing the energy density ${}_{\nu_1}e_\varepsilon$, many problems for weighted manifolds can be reduced to those on (possibly incomplete) unweighted manifolds. However, we prefer to write the proofs in a unified way which includes the case $n = 2$.

We call a map $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ *f-harmonic* if it is a local minimizer of the energy functional E^f, i.e., for any connected, open and relatively compact domain $\Omega \subset M$, $E^f(u) \le E^f(v)$ for every map $v \in W^{1,2}_{\mathrm{loc}}(M, Y)$ such that $u = v$ in $M \setminus \Omega$.

We now investigate the properties of the function $d(u(\cdot), Q)$ on M, where $u : M \to Y$ is an f-harmonic map and $Q \in Y$. The first observation is that

$$E^f(d(u, Q)) \le E^f(u). \tag{6}$$

This can be derived from the triangle inequality

$$(d(u(x), Q) - d(u(y), Q))^2 \le d^2(u(x), u(y)).$$

Recall that an *Hadamard space* (also called global NPC space) is a complete geodesic space which is globally nonpositively curved in the sense of Alexandrov, i.e., Toponogov's triangle comparison for nonpositive curvature holds for any geodesic triangle. The class of Hadamard spaces, natural generalizations of Cartan–Hadamard manifolds, includes all simply connected local NPC spaces (see, e.g., [Burago et al. 2001]). When the target space (Y, d) is an Hadamard space, we have the following theorem.

Theorem 4.7. *If $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ is an f-harmonic map into an Hadamard space Y, then for any $Q \in Y$,*

$$-\int_M \langle \nabla\eta(x), \nabla d(u(x), Q)\rangle e^{-f}\, dV_g \ge 0 \quad \textit{for all } 0 \le \eta \in \mathrm{Lip}_0(M), \tag{7}$$

i.e., $d(u(x), Q) \in W^{1,2}_{\mathrm{loc}}(M)$ is an f-subharmonic function.

This theorem is a consequence of Jost [1997a, Lemma 5]. The subharmonicity of $d(u(\cdot), Q)$ for harmonic maps from an admissible Riemannian polyhedron with simplexwise smooth Riemannian metric to an Hadamard space was obtained by Eells and Fuglede [2001, Lemma 10.2]. Their argument essentially also works in our setting. Using Remark 4.3, Jost's lemma can be reformulated in our setting as follows.

Lemma 4.8 [Jost 1997a, Lemma 5]. *If $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ is an f-harmonic map into an Hadamard space Y, then for any $Q \in Y$ and $\eta \in \mathrm{Lip}_0(M)$, $0 \le \eta \le 1$,*

$$-\int_M \langle \nabla\eta(x), \nabla d^2(u(x), Q)\rangle e^{-f(x)}\, dV_g(x) \ge 2\int_M \eta(x)|\nabla u|^2(x) e^{-f(x)}\, dV_g(x). \tag{8}$$

In fact, (8) still holds for nonnegative functions $\eta \in W^{1,2}(M)$ with compact support. (When $E^f(u)$ is finite, (8) even holds for $0 \le \eta \in W^{1,2}_0(M)$.) Now we can prove Theorem 4.7 concerning the f-subharmonicity of $d(u(\cdot), Q)$.

Proof of Theorem 4.7. Denote $\varphi(x) := \sqrt{x^2+\varepsilon}$ for $\varepsilon > 0$. For any $0 \le \eta \in \mathrm{Lip}_0(M)$, we choose a compactly supported function

$$\eta_1(x) := \frac{\eta(x)}{2\varphi(d(u(x),\, Q))} \in W^{1,2}(M).$$

Then we calculate (we suppress the measure $e^{-f}\,dV_g$ in the notation)

$$\begin{aligned}
-\int_M \bigl\langle \nabla\eta(x), \nabla\sqrt{d^2(u(x),Q)+\varepsilon}\bigr\rangle &= -\int_M \Bigl\langle \nabla\eta(x), \frac{\nabla d^2(u(x),Q)}{2\varphi(d(u(x),Q))}\Bigr\rangle \\
&= -\int_M \langle \nabla\eta_1(x), \nabla d^2(u(x),Q)\rangle \\
&\quad -\int_M 2\eta_1 \frac{d(u(x),Q)\varphi'(d(u(x),Q))}{\varphi(d(u(x),Q))} |\nabla d(u(x),Q)|^2.
\end{aligned}$$

Note that

$$\frac{d(u(x),\, Q)\varphi'(d(u(x),\, Q))}{\varphi(d(u(x),\, Q))} = \frac{d^2(u(x),\, Q)}{d^2(u(x),\, Q)+\varepsilon} \le 1,$$

and by (6), $|\nabla d(u(x),\, Q)|^2 \le |\nabla u(x)|^2$, we obtain

$$-\int_M \bigl\langle \nabla\eta(x), \nabla\sqrt{d^2(u(x),Q)+\varepsilon}\bigr\rangle \ge -\int_M \langle \nabla\eta_1(x), \nabla d^2(u(x),Q)\rangle - 2\int_M \eta_1 |\nabla u(x)|^2. \tag{9}$$

Applying Lemma 4.8, and letting $\varepsilon \to 0$, we complete the proof. □

Now we adopt the method of Li and Wang [1998], a geometric analysis method, to prove Kendall's theorem when the target is a locally compact Hadamard space.

Proof of Theorem 1.1. By assumption, the space of bounded f-harmonic functions is of dimension one. Then by the arguments of Grigor'yan [1990], every two f-massive subsets of M have a nonempty intersection. Here by a f-massive subset, we mean an open proper subset of $\Omega \subset M$ on which there is a bounded, nonnegative, nontrivial, f-subharmonic function h such that $h_{|\partial\Omega} = 0$. Such function h is called an f-potential of the set Ω.

Let $\widehat{M}$ be the Stone–Čech compactification of M. Then every bounded continuous function on M can be continuously extended to $\widehat{M}$. Let Ω be an f-massive subset of M, we then define the set

$$S := \bigcap_{\substack{h:\, f\text{-potential} \\ \text{functions of } \Omega}} \{\hat{x} \in \widehat{M} \mid h(\hat{x}) = \sup h\}.$$

By the maximum principle for f-subharmonic functions, we know $S \subset \widehat{M} \setminus M$.

Then, by the same arguments as in [Li and Wang 1998, Theorem 2.1], we can prove $S \neq \varnothing$. Furthermore, for any bounded f-subharmonic function v, we have $S \subset \{\hat{x} \in \widehat{M} \mid v(\hat{x}) = \sup v\}$.

Let us take a point $Q_0 \in \overline{u(M)}$. If $u(M) = \{Q_0\}$, then we complete the proof. Otherwise, we have $u(M) \setminus \{Q_0\} \neq \varnothing$. Since u is an f-harmonic map, by Theorem 4.7, the function $h_1(x) := d(u(x), Q_0)$ is an f-subharmonic function, which is bounded and nonconstant. Hence h_1 attains its maximum at every point of S. For a point $\hat{x} \in S$, there is a sequence $\{x_n\}$ in M converging to $\hat{x}$ in $\widehat{M}$. Note that u has bounded image. Thus by local compactness of the target Y, there exists a subsequence of $\{u(x_n)\}$ converging to $Q_1 \in Y$. Now again, if $u(M) = \{Q_1\}$, the proof is complete. Therefore, we can assume $u(M) \setminus \{Q_1\} \neq \varnothing$. By Theorem 4.7, the function $h_2(x) := d(u(x), Q_1)$ is a bounded f-subharmonic function. Thus h_2 achieves its maximum on S, in particular at $\hat{x}$. That is,

$$\sup h_2(x) = h_2(\hat{x}) = d(Q_1, Q_1) = 0.$$

This contradicts our assumption. Therefore $u(M) = \{Q_1\}$ is a constant map. □

Remark 4.9. As pointed out to us by K. Kuwae, one can prove Kendall's theorem by combining the methods of Li and Wang [1998] and Kuwae and Sturm [2008] for harmonic maps into Hadamard spaces if the weak topology on the target (see [Jost 1994, Definition 2.7]) coincides with the strong one, or equivalently any distance function $d(x, \cdot)$ on the target is weakly continuous for any $x \in Y$.

5. Liouville-type theorems

In this section, we shall prove our main theorem. First, we review the classical classification theory of Riemannian manifolds in the framework of weighted manifolds. For more details we refer to [Glasner and Nakai 1972] and [Sario and Nakai 1970].

We recall some function spaces of $(M, g, e^{-f} dV_g)$. Let $D^f(M)$ be the set of Tonelli functions[1] on M with finite Dirichlet f-energy. The Royden algebra $BD^f(M)$ is the set of bounded functions in $D^f(M)$. Under the norm $\|u\| = \sup_M |u| + \sqrt{D^f(u)}$, $BD^f(M)$ becomes a Banach algebra. For a sequence $\{u_n\}$ in $D^f(M)$, we say $u = C - \lim u_n$ if u_n converges to u uniformly on compact subsets and $u = B - \lim u_n$ if in addition $\{u_n\}$ is uniformly bounded. We say $u = D^f - \lim u_n$ if $\lim D^f(u_n - u) = 0$. We also write $u = CD^f - \lim u_n$ or $u = BD^f - \lim u_n$ to indicate two types of convergence.

Let $C_0^\infty(M)$ be the set of smooth functions with compact support and $D_0^f(M)$ be its closure under the CD^f-topology. We also denote by $HD^f(M)$ and $HBD^f(M)$ the sets of f-harmonic functions in $D^f(M)$ and $BD^f(M)$ respectively.

Proposition 5.1. *Let $(M, g, e^{-f} dV_g)$ be an f-parabolic weighted Riemannian manifold. Then any f-subharmonic function with finite Dirichlet f-energy is constant. In particular, any function in $HD^f(M)$ is constant.*

[1] A Tonelli function is a continuous function with locally L^2-integrable weak derivatives.

Proof. Let $u \in D^f(M)$ be f-subharmonic. we may assume $u \geq 0$ since $\max\{u, 0\}$ is also f-subharmonic. Let $\{M_n\}$ be an exhaustion of M and take $w_k \in BD^f(M)$ with $w_k|_{M_0} = 1$, $w_k|_{M \setminus M_k} = 0$ and f-harmonic in $M_k \setminus \overline{M_0}$. It follows from the f-parabolicity of M that $BD^f - \lim w_k = 1$. On the other hand, set $v_k \in BD^f(M)$ with $v_k|_{M_0} = u$, $v_k|_{M \setminus M_k} = 0$ and f-harmonic in $M_k \setminus \overline{M_0}$, one can verify that $v = BD^f - \lim v_k$ exists. Set now $\tilde{u} = u - v$, and $\tilde{u}_m = \min\{\tilde{u}, m\}$. Then $\tilde{u} = D^f - \lim \tilde{u}_m$. Since $\tilde{u}$ is nonnegative and f-subharmonic, we can compute

$$0 \geq -\int_{M_k \setminus M_0} \tilde{u}_m w_k \Delta_f \tilde{u} e^{-f}\, dV_g = \int_M \langle \nabla(\tilde{u}_m w_k), \nabla \tilde{u} \rangle e^{-f}\, dV_g. \tag{10}$$

As $w_k \to 1$ in D^f-topology, we deduce from (10) by letting $k \to \infty$ that

$$\int_M \langle \nabla \tilde{u}_m, \nabla \tilde{u} \rangle e^{-f}\, dV_g = 0,$$

which yields $D^f(\tilde{u}) = 0$ by letting $m \to \infty$. Since $\tilde{u}|_{M_0} = 0$, we see $u = v$. Finally,

$$D^f(u) = \int_M \langle \nabla u, \nabla v \rangle e^{-f}\, dV_g = \lim_{k \to \infty} \int_M \langle \nabla u, \nabla v_k \rangle e^{-f}\, dV_g \leq 0,$$

and hence u is a constant. □

The following are the weighted version of the Royden–Nakai decomposition theorem and the Virtanen theorem. The proofs are almost the same as the unweighted case. For the convenience of the reader, we shall give proofs here.

Theorem 5.2 (Royden–Nakai decomposition theorem). *Let $(M, g, e^{-f} dV_g)$ be a non-f-parabolic weighted Riemannian manifold. Then any function $u \in D^f(M)$ has a unique decomposition $u = h + g$, where $h \in HD^f(M)$ and $g \in D_0^f(M)$. Moreover, if u is f-subharmonic, then $u \leq h$.*

Proof. Let $u \in D^f(M)$. Assume first $u \geq 0$. Let $\{M_k\}$ be an exhaustion of M. Take $h_k \in D^f(M)$ such that h_k is f-harmonic in M_k and $h_k|_{M \setminus M_k} = u$. Denote $g_k = u - h_k$. It follows from the maximum principle that $h_k \geq 0$. One can check

$$D^f(u) = \int_M (|\nabla h_k|^2 + |\nabla g_k|^2 + 2\langle \nabla h_k, \nabla g_k \rangle) e^{-f}\, dV_g = D^f(h_k) + D^f(g_k),$$

where in the second equality we used integration by parts and the facts $g_k|_{M \setminus M_k} = 0$ and h_k is f-harmonic in M_k. Similarly we have for $m \leq k$

$$D^f(h_k - h_m) = D^f(h_k) - D^f(h_m).$$

Thus $\{h_k\}$ is a D^f-Cauchy sequence, i.e., $D^f(h_k - h_m)$ is small enough when m and k are large enough.

Let $w_k \in BD^f(M)$ with $w_k|_{M_0} = 1,\ w_k|_{M\setminus M_k} = 0$ and harmonic in $M_k \setminus \overline{M_0}$. It follows from the non-f-parabolicity of M that $w = BD^f - \lim w_k$ satisfies $D^f(w) > 0$.

We can compute

$$\int_M \langle \nabla g_k, \nabla w_k\rangle e^{-f}\, dV_g = \int_{M_k\setminus\overline{M_0}} \langle \nabla g_k, \nabla w_k\rangle e^{-f}\, dV_g = \int_{\partial M_0} g_k \frac{\partial w_k}{\partial \nu} e^{-f}\, dA_g,$$

where ν is the unit inward normal of ∂M_0. Since w_k is f-harmonic in $M_k \setminus \overline{M_0}$, it follows from the Hopf lemma that $\partial w_k/\partial \nu > 0$ along M_0. It follows that

$$\begin{aligned}(\inf_{\partial M_0} h_k - \sup_{\partial M_0} u) \int_{\partial M_0} \frac{\partial w_k}{\partial \nu} e^{-f}\, dA_g &\le \int_{\partial M_0} -g_k \frac{\partial w_k}{\partial \nu} e^{-f}\, dA_g \\ &= -\int_M \langle \nabla g_k, \nabla w_k\rangle e^{-f}\, dV_g \\ &\le \left[D^f(g_k) D^f(w_k)\right]^{1/2} \le \left[D^f(u) D^f(w_k)\right]^{1/2}.\end{aligned}$$

Combining this with the fact that $\int_{\partial M_0} (\partial w_k/\partial \nu) e^{-f}\, dV_g = D^f(w_k)$, we find

$$\inf_{M_0} h_k \le \inf_{\partial M_0} h_k \le \sup_{M_0} u + \left[\frac{D^f(u)}{D^f(w_k)}\right]^{1/2}.$$

Since $w = BD^f - \lim w_k$ satisfies $D^f(w) > 0$, we see $\inf_{M_0} h_k$ is bounded. Consequently, by the Harnack inequality for f-harmonic functions, $\sup_{M_0} h_k$ is bounded. Hence there exists a subsequence of h_k, still denoted by h_k, such that $\{h_k\}$ is a C^f-Cauchy sequence.

Together with the fact $\{h_k\}$ is a D^f-Cauchy sequence, we conclude that h_k converges to some h in the CD^f-topology and $h \in HD^f(M)$. One may directly check that g_k converges to $g = u - h$ in the CD^f-topology and thus $g \in D_0^f(M)$.

Furthermore, if u is f-subharmonic, from the construction of h_k we see $u - h_k$ is f-subharmonic and vanishes on ∂M_k and in turn by the maximum principle that $h \ge u$.

If u is not nonnegative, we can run the same process for $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ as before and get the same result.

The uniqueness follows from the fact that any $h \in HD^f(M)$ and $g \in D_0^f(M)$ satisfy $\int_M \langle \nabla h, \nabla g\rangle e^{-f}\, dV_g = 0$. □

Theorem 5.3 (Virtanen's theorem). *For every $u \in HD^f(M)$ there exists a sequence $h_k \in HBD^f(M)$ such that $u = CD^f - \lim h_k$. In particular, M admits no nonconstant f-harmonic function on M with finite Dirichlet f-energy if and only if M admits no nonconstant bounded f-harmonic function on M with finite Dirichlet f-energy.*

Proof. We may assume M is non-f-parabolic, since otherwise, any $u \in HD^f(M)$ is constant, due to Proposition 5.1, whence the statement is trivial. We may also assume $u \geq 0$, since otherwise we do the same analysis on u^+ and u^-. Set for any $k \in \mathbb{N}$, $u_k = \min\{u, k\}$. Then u_k is f-superharmonic and $u = D^f - \lim u_k$. By Royden–Nakai decomposition, $u_k = h_k + g_k$, where $h_k \in HD^f(M)$ and $g_k \in D_0^f(M)$. Moreover, $g_k \geq 0$. One can verify

$$D^f(u - u_k) = D^f(u - h_k) + D^f(g_k).$$

Hence $D^f(u - h_k) \to 0$ and $D^f(g_k) \to 0$. Since $0 \leq g_k \leq u_k \leq u$ is bounded in any compact set of M, we conclude that g_k converges to some constant function c in the CD^f-topology. It follows from the non-f-parabolicity of M that $c = 0$. Therefore h_k converges to u in the CD^f-topology.

The second assertion follows easily from this approximation. □

The following lemma was first proved by Cheng, Tam and Wan [Cheng et al. 1996, Theorem 1.2].

Lemma 5.4. *Let $(M, g, e^{-f} dV_g)$ be a weighted Riemannian manifold. Then the following two statements are equivalent*:

(i) *any $u \in HD^f(M)$ is bounded*;

(ii) *any nonnegative f-subharmonic function on M with finite Dirichlet f-energy is bounded.*

Proof. (ii) ⇒(i). This is quite simple by observing the fact that if $u \in HD^f(M)$, then $\sqrt{u^2 + 1}$ is a nonnegative f-subharmonic function on M with finite Dirichlet f-energy.

(i) ⇒(ii). Assume u is a nonnegative f-subharmonic function on M with finite Dirichlet f-energy. If M is f-parabolic, then the two statements are both true by virtue of Proposition 5.1 and hence equivalent. If M is non-f-parabolic, then by Theorem 5.2, $u = h + g$ for $h \in HD^f(M)$ and $g \in D_0^f(M)$. Moreover, since u is f-subharmonic, we know $u \leq h$. By the assumption (i), h is bounded. Thus u is also bounded. This proves the lemma. □

Using Lemma 5.4, we can prove the main Theorem 1.2.

Proof of Theorem 1.2. Let u be an f-harmonic map from M to Y with finite f-energy. It follows from Theorem 4.7 that the function $v : M \to \mathbb{R}$, $v(x) = \sqrt{d^2(u(x), Q) + 1}$ is subharmonic, where $Q \in Y$. Also, the finiteness of the f-energy of u implies the finiteness of the Dirichlet f-energy of v (recall (6)). Using the assumption and the equivalence in Lemma 5.4, we know that any nonnegative f-subharmonic function on M with finite Dirichlet f-energy is bounded. Hence v is bounded, and in turn, u has bounded image. This proves the theorem. □

For harmonic maps from f-parabolic weighted manifolds, we don't need the local compactness assumption of the targets to obtain the Liouville theorem.

Corollary 5.5. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact f-parabolic weighted Riemannian manifold and (Y, d) be an Hadamard space. Then any f-harmonic map from M to Y with finite f-energy is a constant map.*

Proof. Let u be an f-harmonic map from M to Y with finite f-energy. By Proposition 5.1 and Theorem 1.2, the image of u is bounded. Hence for any $Q \in Y$, the f-subharmonic function $d(u(x), Q)$ is bounded. By the f-parabolicity of M and Proposition 2.2, the function $d(u(x), Q)$ is constant for any $Q \in Y$. This yields that u is a constant map. The corollary follows. □

Combining Theorems 1.1 and 1.2, we obtain Theorem 1.3 by the potential theory.

Proof of Theorem 1.3.. By assumption, any bounded f-harmonic function on M is constant. By Theorem 5.3, we know that any f-harmonic function on M with finite Dirichlet f-energy is constant. Using Theorem 1.2, we see that any f-harmonic map from M to Y with finite f-energy must have bounded image.

On the other hand, by Theorem 1.1, we know that any f-harmonic map from M to Y having bounded image is constant. Hence any f-harmonic map from M to Y with finite f-energy is a constant map. This proves the theorem. □

Proof of Theorem 1.4.. By a theorem of Brighton [2013], the weighted manifold $(M, g, e^{-f} dV_g)$ satisfying $\mathrm{Ric}_f \geq 0$ admits no nonconstant bounded f-harmonic functions. The assertion follows from Theorem 1.3 immediately. □

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