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For a conformal manifold, we describe a new relation between the ambient obstruction tensor of Fefferman and Graham and the holonomy of the normal conformal Cartan connection. This relation allows us to prove several results on the vanishing and the rank of the obstruction tensor, for example for conformal structures admitting twistor spinors or normal conformal Killing forms. As our main tool we introduce the notion of a conformal holonomy distribution and show that its integrability is closely related to the exceptional conformal structures in dimensions five and six that were found by Nurowski and Bryant.

1. Introduction

A conformal structure of signature (p, q) on a smooth manifold M is an equivalence class c of semi-Riemannian metrics on M of signature (p, q), where two metrics g and $\hat{g}$ are equivalent if $\hat{g} = \mathrm{e}^{2f} g$ for a smooth function f. For conformal structures the construction of local invariants is more complicated than for semi-Riemannian structures, where all local invariants can be derived from the Levi-Civita connection and its curvature. For conformal geometry, essentially there are two invariant constructions: the conformal ambient metric of Fefferman and Graham [1985; 2012] and the normal conformal Cartan [1924] connection with the induced tractor calculus [Bailey et al. 1994]. We investigate a new relationship between two essential ingredients of these invariant constructions, the *ambient obstruction tensor* on one hand, and the *conformal holonomy* on the other. We briefly introduce these notions:

The ambient metric construction assigns to any conformal manifold $(M, [g])$ of signature (p, q) and dimension n a pseudo-Riemannian metric $\tilde{g}$ on an open neighborhood $\tilde{M}$ of $Q = M \times \mathbb{R}^{>0}$ in $\mathbb{R} \times Q$, of signature $(p+1, q+1)$ and with specific properties that link $[g]$ and $\tilde{g}$ as closely as possible. More precisely, denoting

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the coordinates on $\mathbb{R}^{>0}$ and $\mathbb{R}$ by t and ρ, respectively, $\tilde{g}$ is required to restrict to $t^2 g$ along Q and moreover its Ricci tensor vanishes along Q to infinite order in ρ when n is odd and to order $\rho^{(n/2)-1}$ when n is even. The seminal result in [Fefferman and Graham 1985; 2012] is that for smooth conformal structures, such an ambient metric always exists and is unique to all orders for n odd or $n = 2$ and up to order $\rho^{(n/2)-1}$ when $n \geq 4$ is even. Moreover, in even dimensions the existence of an ambient metric whose Ricci tensor vanishes along Q to all orders is closely related to the vanishing of a certain symmetric, divergence-free and conformally covariant (0, 2)-tensor $\mathcal{O}$ on M, the *Fefferman–Graham obstruction tensor* or *ambient obstruction tensor*. In four dimensions the obstruction tensor is given by the well-known Bach tensor, but in general even dimension no general explicit formula for $\mathcal{O}$ exists. The obstruction tensor will be the focus of the present article.

The other invariant construction in conformal geometry is the *normal conformal Cartan connection*. This is an $\mathfrak{so}(p+1, q+1)$-valued Cartan connection defined on a P-bundle, where P is the parabolic subgroup defined by the stabilizer in $\mathrm{O}(p+1, q+1)$ of a lightlike line in $\mathbb{R}^{p+1,q+1}$, and it satisfies a certain normalization condition that defines it uniquely. The normal conformal Cartan connection defines a covariant derivative ∇^{nc} on a vector bundle $\mathcal{T}$, the *conformal tractor connection* on the *standard tractor bundle*. To $(\mathcal{T}, \nabla^{\mathrm{nc}})$ one can associate the holonomy group of ∇^{nc}-parallel transports along loops based at $x \in M$. As this group only depends on the conformal structure, it is denoted by $\mathrm{Hol}_x(M, c)$ and called the *conformal holonomy*. It is contained in $\mathrm{O}(p+1, q+1)$ and its Lie algebra is denoted by

$$\mathfrak{hol}_x(M, c) \subset \mathfrak{so}(p+1, q+1).$$

Many interesting conformal structures are related to conformal *holonomy reductions*, i.e., conformal structures for which the conformal holonomy algebra is a proper subalgebra of $\mathfrak{so}(p+1, q+1)$. Examples are manifolds admitting twistor spinors, for which the spin representation of the conformal holonomy group admits an invariant spinor. This includes conformal Fefferman [1976] spaces that are closely related to CR-geometry, and for which the conformal holonomy reduces to the special unitary group. Other fascinating examples are the conformal structures that are determined by generic distributions of rank 2 in dimension 5. Such distributions played an important role in the history of the simple Lie algebra with exceptional root system G_2: Cartan [1893] discovered that for some of these distributions the Lie algebra of symmetries is given by the noncompact exceptional Lie algebra $\mathfrak{g}_2$ of type G_2. Related to the equivalence problem for such distributions, Cartan [1910] constructed the corresponding $\mathfrak{g}_2$-valued Cartan connection. It was then realized by Nurowski [2005] that to any such distribution one can associate a conformal structure of signature (2, 3) whose conformal holonomy is reduced from $\mathfrak{so}(3, 4)$ to $\mathfrak{g}_2$. Similarly, Bryant associated to any generic rank 3 distribution in dimension 6 a conformal structure of signature $(3, 3)$ whose holonomy reduces to $\mathfrak{spin}(3, 4) \subset \mathfrak{so}(4, 4)$. Both, and in particular the latter will be relevant to us.

The ambient metric construction and the normal conformal Cartan connection turn out to be closely related. Indeed, in [Čap and Gover 2003] tractor data are formulated entirely in terms of ambient data, and in [Gover and Peterson 2006] the ambient curvature tensors are rewritten in terms of tractor curvature and derivatives thereof. The main result in our paper reveals another interesting correspondence, now between the ambient obstruction tensor $\mathcal{O}$ and the conformal holonomy. We show that the image of $\mathcal{O}$, when considered as a $(1, 1)$-tensor, can be identified with a distinguished subspace in the conformal holonomy algebra $\mathfrak{hol}_x(M, c)$. To be more precise, recall that the Lie algebra $\mathfrak{so}(p + 1, q + 1)$ is $|1|$-graded as $\mathfrak{so}(p+1, q+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $g_0 \simeq \mathfrak{co}(p, q)$ is the conformal Lie algebra and $\mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{p}$ is the Lie algebra of the parabolic subgroup P. It is important to note that $\mathfrak{g}_1$ can be identified with $\mathbb{R}^{p,q}$ and hence with the tangent space T_xM. This allows us to prove the following theorem:

Theorem 1.1. *Let $(M^{p,q}, c)$ be a smooth conformal manifold of even dimension $n \geq 4$ and with ambient obstruction tensor $\mathcal{O}$. Then the image of $\mathcal{O}$ at $x \in M$ is contained in $\mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1$. In particular, the rank of $\mathcal{O}$ at each point is limited by the dimension of $\mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1$. Moreover, if $\mathfrak{hol}(M, c)$ is a proper subalgebra of $\mathfrak{so}(p+1, q+1)$, then the image of $\mathcal{O}$ is totally lightlike. In particular,* $\mathrm{rk}(\mathcal{O}) \leq \min(p, q)$.

The implications of this result are evident. On the one hand it shows that if the obstruction tensor has maximal rank n at some point, then the holonomy is generic. Hence, $\mathcal{O}$ can be interpreted as a universal obstruction to the existence of parallel tractors on (M, c) of any type. Namely for such a tractor to exist, $\mathcal{O}$ needs to have a nontrivial kernel everywhere. On the other hand, conformal holonomy reductions can be used to restrict the rank of the obstruction tensor. For example, it is well known that the existence of a parallel standard tractor (and hence of a local Einstein metric in c) forces the obstruction tensor to vanish, however no substantially more general conditions on the conformal class are known to have a similar effect on $\mathcal{O}$. Our results provide such conditions. For example, we obtain:

Corollary 1.2. *Under the assumptions of Theorem 1.1, $\mathcal{O} = 0$ for each of the following cases*:

(1) *the conformal structure is Riemannian and $\mathfrak{hol}(M, c) \subsetneq \mathfrak{so}(1, n+1)$*;

(2) *the conformal structure is Lorentzian and $\mathfrak{hol}(M, c) \subsetneq \mathfrak{su}(1, n/2)$*;

(3) *the conformal class contains an almost Einstein metric or special Einstein product* (*in the sense of* [Gover and Leitner 2009]);

(4) *there is a normal conformal vector field V of nonzero length or the dimension of the space of normal conformal vector fields is ≥ 2. In particular, this is the case for Fefferman spaces over quaternionic contact structures in signature* $(4k+3, 4l+3)$ (*characterized by* $\mathfrak{hol}(M, c) \subset \mathfrak{sp}(k+1, l+1)$);

(5) (M, c) *is spin and for $g \in c$ with spinor bundle S^g there are twistor spinors $\varphi_{i=1,2} \in \Gamma(M, S^g)$ such that the spaces $\{X \in TM \mid X \cdot \varphi_i = 0\}$ are complementary at each point.*

Corollary 1.3. *Under the assumptions of Theorem 1.1*, $\mathrm{rk}(\mathcal{O}) \leq 1$ *for each of the following cases*:

(1) $(p, q) = (3, 3)$ *and* $\mathfrak{hol}(M, c) \subsetneq \mathfrak{spin}(3, 4)$;

(2) $(p, q) = (n, n)$ *and* $\mathfrak{hol}(M, c) \subset \mathfrak{gl}(n+1)$;

(3) $\mathrm{Hol}(M, c)$ *fixes a nontrivial* 2*-form*, *i.e.*, (M, c) *admits a normal conformal vector field. In particular*, *this applies to Fefferman conformal structures*, *i.e.*, *to* $(p, q) = (2r+1, 2s+1)$ *and* $\mathfrak{hol}(M, c) \subset \mathfrak{su}(r+1, s+1)$;

(4) *the action of* $\mathrm{Hol}(M, c)$ *on the light cone* $\mathcal{N} \subset \mathbb{R}^{p+1,q+1}$ *does not have an open orbit.*

For each of these geometries one can give an explicit subspace $V \subset TM$ with $\mathrm{Im}(\mathcal{O}) \subset V$ *at each point.*

Two results in these corollaries can be found in the literature — the statement about almost Einstein [Fefferman and Graham 1985] and special Einstein products [Gover and Leitner 2009] in Corollary 1.2 and the statement about Fefferman conformal structures [Graham and Hirachi 2008] in Corollary 1.3 — but the general theory as developed here allows alternative proofs of these facts. Note also that the last two conditions in Corollary 1.2 are conformally invariant and do not refer to a distinguished metric in the conformal class.

As the main tool in proving these results, we introduce what we call *the conformal holonomy distribution*. At each point $x \in M$ it is defined as

$$\mathcal{E}_x := \mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1.$$

The vector space $\mathcal{E}_x$ can be canonically identified with a subspace in T_xM. When varying x, its dimension however may not be constant. Instead, varying x provides a stratification of the manifold into sets over which the dimension of $\mathcal{E}_x$ is constant. We will see in Theorem 4.1 that these strata are unions of the *curved orbits* defined by conformal holonomy reductions, introduced recently in [Čap et al. 2014] in the context of Cartan geometries. Moreover we will show that an open and dense set in M can be covered by open sets over which the dimension of $\mathcal{E}_x$ is constant. Very surprisingly, we find that, when considered over such an open set, $\mathcal{E}$ is closely related to the aforementioned generic distributions:

Theorem 1.4. *Let $(M^{p,q}, c)$ be a smooth conformal manifold. Then there is an open and dense set in M that is covered by open sets U over which $\mathcal{E}|_U$ is a vector distribution. Over each such U, $\mathcal{E}|_U$ is either integrable, or*

- *$(p,q) = (2,3)$ and $\mathcal{E}|_U$ is a generic rank 2 distribution, or*
- *$(p,q) = (3,3)$ and $\mathcal{E}|_U$ is a generic rank 3 distribution.*

In both cases, $\mathcal{E}|_U$ defines the conformal class c on U in the sense of [Nurowski 2005; Bryant 2006].

We should also point out that the statements in Theorem 1.1 remain valid when $\mathrm{rk}(\mathcal{O})$ at x is replaced by the dimension of $\mathcal{E}_x$. We believe that the conformal holonomy distribution will turn out to be a powerful tool that allows us to obtain not only results about the obstruction tensor but also about other aspects of special conformal structures.

This article is organized as follows: Section 2 reviews the relevant tractor calculus and the ambient metric construction in conformal geometry. Moreover, we discuss special conformal structures that will be important in the sequel from the point of view of holonomy reductions. Section 3 is then devoted to the proof of the first part of Theorem 1.1. The key ingredient is a recently established relation between conformal and ambient holonomy [Čap et al. 2016]. In Section 4A we introduce the conformal holonomy distribution $\mathcal{E}$ and study its basic properties. These results are then applied in Section 5 to derive constraints on the obstruction tensor for many families of special conformal structures, in particular those in signature (3, 3) discovered by Bryant.

2. Conformal structures, tractors and ambient metrics

2A. ***Conventions.*** Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇^g. denote by $\Lambda^k := \Lambda^k T^*M$ the k-forms and by $\mathfrak{so}(TM)$ the endomorphisms of TM that are skew with respect to g. By $\mathrm{R} = \mathrm{R}^g \in \Lambda^2 \otimes \mathfrak{so}(TM)$ we will denote the curvature endomorphism of ∇^g, i.e., one has for all vector fields $X, Y \in \mathfrak{X}(M)$

$$\mathrm{R}^g(X,Y) = [\nabla^g_X, \nabla^g_Y] - \nabla^g_{[X,Y]}.$$

By contraction one obtains the Ricci tensor and scalar curvature,

$$\mathrm{Ric}^g(X,Y) := \mathrm{tr}(Z \mapsto \mathrm{R}^g(Z,X)Y), \qquad \mathrm{scal}^g := \mathrm{tr}_g \mathrm{Ric}^g,$$

and we denote by P^g the Schouten tensor

$$\mathsf{P}^g := \frac{1}{n-2}\Big(\mathrm{Ric}^g - \frac{1}{2(n-1)}\,\mathrm{scal}^g\, g\Big). \tag{1}$$

Using g to raise and lower indices, we will also consider P^g and Ric^g as g-symmetric endomorphisms of TM denoted with the same symbol. The metric dual 1-form of a vector $V \in TM$ is $V^\flat = g(V, \cdot)$ and from a 1-form $\alpha \in T^*M$ we obtain a tangent vector $\alpha^\sharp$ via $g(\alpha^\sharp, \cdot) = \alpha$. From the Schouten tensor we obtain the Cotton tensor $C \in \Lambda^2 \otimes TM$,

$$C^g(X, Y) := (\nabla^g_X \mathsf{P}^g)(Y) - (\nabla^g_Y \mathsf{P}^g)(X),$$

and the Weyl tensor $W \in \Lambda^2 \otimes \mathfrak{so}(TM)$, considered as skew-symmetric bilinear map from $TM \times TM$ to $\mathfrak{so}(TM)$,

$$W^g(X, Y) := \mathrm{R}^g(X, Y) + X^\flat \otimes \mathsf{P}^g(Y) + \mathsf{P}^g(X) \otimes Y - \mathsf{P}^g(Y) \otimes X - Y^\flat \otimes \mathsf{P}^g(X).$$

We will also write $C^g(Z; X, Y) := g(Z, C^g(X, Y))$ for the metric dual of C^g, drop the g and use the index convention $C_{kij} = C(\partial_k; \partial_i, \partial_j)$.

2B. ***Conformal tractor calculus.*** Let (M, c) be a smooth conformal manifold of signature (p, q), dimension $n = p+q \geq 3$ and let $\mathcal{T} \to M$ denote the standard tractor bundle for (M, c) with normal conformal Cartan connection ∇^{nc} and tractor metric h as introduced in [Bailey et al. 1994]. The tractor bundle $\mathcal{T}$ is equipped with a canonical filtration $\mathcal{I} \subset \mathcal{I}^\perp \subset \mathcal{T}$, where $\mathcal{I}$ is a distinguished lightlike line. For each metric $g \in c$, one finds distinguished lightlike tractors $s_\pm$ which lead to an identification

$$\mathcal{T} \to \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}, \quad T \mapsto \alpha s_- + V + \beta s_+ \mapsto (\alpha, V, \beta)^\top, \tag{2}$$

under which the tractor metric becomes

$$h((\alpha_1, V_1, \beta_1), (\alpha_2, V_2, \beta_2)) = \alpha_1\beta_2 + \alpha_2\beta_1 + g(V_1, V_2),$$

and in this identification, s_- generates $\mathcal{I}$. Under a conformal change $\tilde{g} = e^{2\sigma} g$, the transformation of the metric identification (2) of a standard tractor is given by

$$\begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\alpha} \\ \tilde{Y} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} e^{-\sigma}(\alpha - Y(\sigma) - \frac{1}{2}\beta \cdot \|\operatorname{grad}^g \sigma\|^2_g) \\ e^{-\sigma}(Y + \beta \cdot \operatorname{grad}^g \sigma) \\ e^\sigma \beta \end{pmatrix}. \tag{3}$$

From this one observes the image of a linear subspace $H \subset \mathcal{I}^\perp \subset \mathcal{T}$ under the map

$$\mathcal{I}^\perp \to \mathcal{I}^\perp/\mathcal{I} \to TM, \quad \alpha s_- + V \mapsto [\alpha s_- + V] \mapsto V$$

is conformally invariant, i.e., independent of the choice of $g \in c$. For ∇^{nc} expressed in terms of the splitting (2) we find

$$\nabla^{\mathrm{nc}}_X \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} X(\alpha) - \mathsf{P}^g(X, Y) \\ \nabla^g_X Y + \alpha X + \beta \mathsf{P}^g(X) \\ X(\beta) - g(X, Y) \end{pmatrix}. \tag{4}$$

The curvature of ∇^{nc} is given by $\mathrm{R}^{\mathrm{nc}}(X,Y) = C^g(X,Y)\wedge s_-^\flat + W^g(X,Y)$, where we identified the bundles $\mathfrak{so}(\mathcal{T},h)$ and $\Lambda^2\mathcal{T}^*$ by means of h in the usual way by the musical isomorphisms $^\flat$ and $^\sharp$.

Turning to adjoint tractors, it follows from identification (2) that for fixed $g\in c$, each fiber of the bundle $\mathfrak{so}(\mathcal{T},h)$ of skew-symmetric endomorphisms of the tractor bundle can be identified with skew-symmetric matrices of the form

$$\Phi(\mu,(a,A),Z) := \begin{pmatrix} -a & \mu & 0 \\ Z & A & -\mu^\sharp \\ 0 & -Z^\flat & a \end{pmatrix},$$

where Z is a vector, μ a 1-form, $a\in\mathbb{R}$, and A is skew-symmetric for g. For example, the curvature of ∇^{nc} is identified with

$$\mathrm{R}^{\mathrm{nc}}(X,Y) = \begin{pmatrix} 0 & C^g(X,Y)^\flat & 0 \\ 0 & W^g(X,Y) & -C^g(X,Y) \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

In particular, each choice of g yields an obvious pointwise $|1|$-grading of $\mathfrak{so}(\mathcal{T},h)$ according to the splitting

$$\mathfrak{g}_{-1} = \{\Phi(0,0,Z)\}, \quad \mathfrak{g}_0 = \{\Phi(0,(a,A),0)\}, \quad \mathfrak{g}_1 = \{\Phi(\mu,0,0)\}, \tag{6}$$

with brackets given by

$$[(a,A),Z] = (a+A)Z, \quad [(a,A),\mu] = -\mu\circ(A+a\mathrm{Id}), \quad [Z,\mu] = (\mu(Z),\mu\wedge Z^\flat).$$

In particular, $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$. It follows that the induced derivative ∇^{nc} on a section $\Phi=\Phi(\mu,(a,A),Z)$ of $\mathfrak{so}(\mathcal{T},h)$ is given by

$$\begin{aligned} &\nabla^{\mathrm{nc}}_X\Phi \\ &= \begin{pmatrix} -X(a)-\mathsf{P}^g(X,Z)-\mu(X) & \nabla^g_X\mu-\mathsf{P}^g(X,(A+a\mathrm{Id})\cdot) & 0 \\ \nabla^g_X Z-(A+a)X & \nabla^g_X A+\mu\wedge X^\flat - Z^\flat\wedge\mathsf{P}^g(X,\cdot) & -\nabla^g_X\mu^\sharp+(a-A)\mathsf{P}^g(X) \\ 0 & -\nabla^g_X Z^\flat+(AX)^\flat+aX^\flat & X(a)+\mathsf{P}^g(X,Z)+\mu(X) \end{pmatrix}. \end{aligned} \tag{7}$$

2C. *Holonomy reductions of conformal structures.* In this section we list definitions and properties of the conformal structures which have appeared in the introduction and to which the main Theorem 1.1 can be applied. They all turn out to be characterized in terms of a conformal holonomy reduction. Here, for $(M^{p,q},c)$ a smooth conformal manifold, its conformal holonomy at $x\in M$ is defined as

$$\mathrm{Hol}_x(M,c) := \mathrm{Hol}_x(\mathcal{T},\nabla^{\mathrm{nc}})$$

and gives a class of conjugated subgroups in $\mathrm{O}(p+1,q+1)$. The interplay between *conformal holonomy reductions*, i.e., when $\mathrm{Hol}^0_x(M,c)$ is a proper subgroup of $\mathrm{SO}(p+1,q+1)$, and distinguished metrics in the conformal class has been the focus of active research. We will review the most important ones relevant here.

2C.1. *Geometries with reducible holonomy representation.* One initial result is that holonomy invariant lines $L \subset \mathbb{R}^{p+1,q+1}$ are in one-to-one correspondence to *almost Einstein scales* in c [Gauduchon 1990; Bailey et al. 1994; Gover 2005; Gover and Nurowski 2006; Leitner 2005; Leistner 2006] by which we mean that on an open, dense subset of M there exists around each point locally an Einstein metric $g \in c$. If L is lightlike, g is Ricci flat and otherwise one has $\mathrm{sgn}(\mathrm{scal}^g) = -\,\mathrm{sgn}\langle L, L\rangle_{p+1,q+1}$.

A holonomy-invariant nondegenerate subspace $H \subset \mathbb{R}^{p+1,q+1}$ of dimension $k \geq 2$ corresponds locally and off a singular set to the existence of a *special Einstein product* in the conformal class [Leitner 2004; Armstrong 2007; Armstrong and Leitner 2012]. Here, we say that a pseudo-Riemannian manifold (M, g) is a special Einstein product if (M, g) is isometric to a product $(M_1, g_1) \times (M_2, g_2)$, where (M_i, g_i) are Einstein manifolds of dimensions $k-1$ and $n-k-1$ for $k \geq 2$ and in case $k \neq 2, n$ we additionally require that

$$\mathrm{scal}^{g_1} = -\frac{(k-1)(n-2)}{(n-k+1)(n-k)}\,\mathrm{scal}^{g_2} \neq 0.$$

Finally, if $H \subset \mathbb{R}^{p+1,q+1}$ is totally degenerate, of dimension $k+1 \geq 2$ and holonomy invariant, there exists — again locally and off a singular set — a metric $g \in c$ admitting a ∇^g-invariant and totally degenerate distribution $\mathcal{L} \subset TM$ of rank k which additionally satisfies $\mathrm{Im}(\mathrm{Ric}^g) \subset \mathcal{L}$, as has been shown in [Leistner 2006; Leistner and Nurowski 2012; Lischewski 2015].

2C.2. *Geometries defined via normal conformal Killing forms.* Suppose next that $\mathrm{Hol}(M, c)$ lies in the isotropy subgroup of a $(k+1)$-form, i.e., there exists a ∇^{nc}-parallel tractor $k+1$-form $\hat{\alpha} \in \Gamma(M, \Lambda^{k+1}\mathcal{T}^*)$. Such holonomy reductions have been studied in [Leitner 2005]. For fixed $g \in c$, consider the splitting of $\mathcal{T}$ with respect to g and write $\hat{\alpha}$ as

$$\hat{\alpha} = s_+^\flat \wedge \alpha + \alpha_0 + s_-^\flat \wedge s_+^\flat \wedge \alpha_\pm + s_-^\flat \wedge \alpha_- \tag{8}$$

for uniquely determined differential forms $\alpha, \alpha_0, \alpha_\pm, \alpha_-$ on M. The k-form $\alpha \in \Omega^k(M)$ turns out to be *normal conformal (nc)*, that is α is a conformal Killing form subject to additional conformally covariant differential normalization conditions that can be found in [Leitner 2005]. Moreover, $\alpha_0, \alpha_\pm, \alpha_-$ can be expressed in terms of α and ∇^g. Conversely, every normal conformal Killing form determines a parallel tractor form. The situation simplifies considerably if $k = 1$, i.e., there is a parallel adjoint tractor. In this case it is convenient to consider the metric dual $V = \alpha^\sharp \in \mathfrak{X}(M)$ of the associated normal conformal Killing form α, which is a *normal conformal vector field*. By this, we mean that V is a conformal vector field which additionally satisfies $C^g(V, \cdot) = W^g(V, \cdot) = 0$.

Examples of manifolds admitting normal conformal vector fields are so-called *Fefferman spaces* [Fefferman 1976]. They yield conformal structures (M, c) of signature $(2r+1, 2s+1)$ defined on the total spaces of S^1-bundles over strictly pseudoconvex CR manifolds. From the holonomy point of view they are (at least locally) equivalently characterized by the existence of a parallel adjoint tractor [Leitner 2007; Čap and Gover 2010], which is an almost complex structure for the tractor metric, i.e., $\mathrm{Hol}(M, c) \subset \mathrm{SU}(r+1, s+1)$. Here, we used a result from [Leitner 2008; Čap and Gover 2010] which asserts that unitary conformal holonomy is automatically special unitary.

Other geometries that are characterized by the existence of distinguished normal conformal vector fields include pseudo-Riemannian manifolds (M, g) of signature $(4r+3, 4m+3)$ with conformal holonomy group in the symplectic group

$$\mathrm{Sp}(r+1, m+1) \subset \mathrm{SO}(4r+4, 4m+4),$$

see [Alt 2008]. The models of such manifolds are S^3-bundles over a quaternionic contact manifold equipped with a canonical conformal structure, introduced in [Biquard 2000].

2C.3. *Conformal holonomy and twistor spinors.* If (M, c) is actually spin for one, and hence all, $g \in c$, the presence of conformal Killing spinors always leads to reductions of $\mathrm{Hol}(M, c)$. To formulate these, let $S^g \to M$ denote the real or complex spinor bundle over M which possesses a spinor covariant derivative ∇^{S^g} and vectors act on spinors by Clifford multiplication $\mathrm{cl} = \cdot$, see [Baum 1981]. Given these data, the spin Dirac operator is given as $D^g = \mathrm{cl} \circ \nabla^{S^g}$. Now assume that (M, g) admits a twistor spinor, i.e., a section $\varphi \in \Gamma(M, S^g)$ solving

$$\nabla_X^{S^g} \varphi + \frac{1}{n} X \cdot D^g \varphi = 0. \tag{9}$$

Equation (9) is conformally invariant, see [Baum et al. 1991], and to φ we can associate the union of subspaces

$$\mathcal{L}_\varphi := \{X \in TM \mid X \cdot \varphi = 0\} \subset TM,$$

which does not depend on the choice of $g \in c$. Equation (9) can be prolonged, see [Baum et al. 1991], and using this prolonged system it becomes immediately clear that a twistor spinor φ is equivalently described as a parallel section ψ of the spin tractor bundle associated to (M, c). Its construction can be found in [Leitner 2007], for instance. As ψ is parallel, it is at each point annihilated by $\mathfrak{hol}_x(M, c)$ under Clifford multiplication, i.e.,

$$\mathfrak{hol}_x(M, c) \cdot \psi_x = 0 \quad \text{for all } x \in M. \tag{10}$$

2C.4. *Exceptional cases.* Finally we describe conformal structures in dimension 5 and 6 with holonomy algebra contained in $\mathfrak{g}_2 \subset \mathfrak{so}(3, 4)$, the noncompact simple

Lie algebra of dimension 14, or in $\mathfrak{spin}(3,4) \subset \mathfrak{so}(4,4)$, respectively. They turn out to be closely related to generic distributions:

Recall that a distribution $\mathcal{D}$ of rank 2 on a 5-manifold M is *generic* if

$$[\mathcal{D},[\mathcal{D},\mathcal{D}]]+[\mathcal{D},\mathcal{D}]+\mathcal{D}=TM.$$

It is known by work of Nurowski [2005] that $\mathcal{D}$ canonically defines a conformal structure $c_{\mathcal{D}}$ of signature $(2,3)$ on M^5 whose conformal holonomy is reduced to $\mathfrak{g}_2 \subset \mathfrak{so}(3,4)$, see also [Čap and Sagerschnig 2009]. Analogously, a distribution $\mathcal{D}$ of rank 3 on a 6-manifold M is *generic* if $[\mathcal{D},\mathcal{D}]+\mathcal{D}=TM$, and Bryant [2006] showed that $\mathcal{D}$ canonically defines a conformal structure $c_{\mathcal{D}}$ of signature $(3,3)$ on M whose conformal holonomy is reduced to $\mathfrak{spin}(4,3) \subset \mathfrak{so}(4,4)$. In both cases, the holonomy characterization implies that $(M, c_{\mathcal{D}})$ admits a parallel tractor 3- or 4-form, respectively. Moreover, [Hammerl and Sagerschnig 2011b] shows that there is in both cases a distinguished twistor spinor φ which encodes $\mathcal{D}$ in the sense that

(11) $$\mathcal{L}_\varphi=\mathcal{D} \quad \text{at each point.}$$

2D. ***Conformal ambient metrics.*** Let (M,c) be a smooth conformal manifold of dimension ≥ 3. For our purposes we do not need the general theory of ambient metrics as presented in [Fefferman and Graham 2012], which can be consulted for more details, but it suffices to deal with ambient metrics which are in normal form with respect to some $g \in c$. A (straight) preambient metric in normal form with respect to $g \in c$ is a pseudo-Riemannian metric $\tilde{g}$ on an open neighborhood $\tilde{M}$ of $\{1\} \times M \times \{0\}$ in $\mathbb{R}^+ \times M \times \mathbb{R}$ such that for $(t,x,\rho) \in \tilde{M}$

(12) $$\tilde{g}=2t\,dt\,d\rho+2\rho\,dt^2+t^2 g_\rho(x),$$

with $g_0=g$. We call $(\tilde{M},\tilde{g})$ an *ambient metric for* $(M,[g])$ *in normal form* with respect to g if

- $\widetilde{\mathrm{Ric}}=O(\rho^\infty)$ if n is odd, and
- $\widetilde{\mathrm{Ric}}=O(\rho^{(n/2)-1})$ and $\mathrm{tr}_g(\rho^{1-(n/2)}\widetilde{\mathrm{Ric}}_{|TM\otimes TM})=0$ along $\rho=0$, if n is even.

The existence and uniqueness assertion for ambient metrics [Fefferman and Graham 1985; 2012] states that for each choice of g there is an ambient metric in normal form with respect to g. In all dimensions $n \geq 3$, g_ρ has an expansion of the form $g_\rho=\sum_{k\geq 0} g^{(k)}\rho^k$ starting with

$$g_\rho=g+2\rho\mathsf{P}^g+O(\rho^2),$$

and in odd dimensions the Ricci flatness condition determines $g^{(k)}$ for all k, whereas in even dimensions only the $g^{(k<n/2)}$ and the trace of $g^{(n/2)}$ are determined.

We shall sometimes work with ambient indices $I \in \{0, i, \infty\}$, where the i are indices for coordinates on M, 0 refers to ∂_t and ∞ to ∂_ρ, i.e.,

$$T\tilde{M} \ni V = V^0 \partial_t + V^i \partial_i + V^\infty \partial_\rho.$$

For the Levi-Civita connection of any metric of the form (12) one computes [Fefferman and Graham 2012, Lemma 3.2],

$$\begin{aligned} &\widetilde{\nabla}_{\partial_i} \partial_j = -\frac{1}{2} t \dot{g}_{ij} \partial_t + \Gamma^k_{ij} \partial_k + (\rho \dot{g}_{ij} - g_{ij}) \partial_\rho, && \widetilde{\nabla}_{\partial_t} \partial_t = \widetilde{\nabla}_{\partial_\rho} \partial_\rho = 0, \\ &\widetilde{\nabla}_{\partial_i} \partial_t = \frac{1}{t} \partial_i, \qquad \widetilde{\nabla}_{\partial_i} \partial_\rho = \frac{1}{2} g^{kl} \dot{g}_{il} \partial_k, && \widetilde{\nabla}_{\partial_\rho} \partial_t = \frac{1}{t} \partial_\rho, \end{aligned} \tag{13}$$

where, abusing notation, g_{ij} denotes the components of g_ρ and Γ^k_{ij} the Christoffel symbols of g_ρ. In particular, $T := t\partial_t$ is an Euler vector field for $(\tilde{M}, \tilde{g})$, i.e.,

$$\widetilde{\nabla} T = \mathrm{Id}. \tag{14}$$

For n even a conformally invariant $(0, 2)$-tensor on M, the *ambient obstruction tensor* $\mathcal{O}$, obstructs the existence of smooth solutions to $\widetilde{\mathrm{Ric}} = O(\rho^{n/2})$. For $\tilde{g}$ in normal form with respect to g it is given by

$$\mathcal{O} = c_n (\rho^{1-(n/2)} (\widetilde{\mathrm{Ric}}_{|TM \otimes TM}))_{\rho=0}, \tag{15}$$

where c_n is some known nonzero constant; see [Fefferman and Graham 2012]. From this one can deduce that $\mathcal{O}$ is trace- and divergence-free.

Tractor data can be recovered from ambient data as shown in [Čap and Gover 2003]. For ambient metrics in normal form with respect to $g \in c$, this reduces to the following observation, see [Graham and Willse 2012] for more details: Identify M with the level set $\{\rho = 0, t = 1\}$ in $\tilde{M}$. Then $T\tilde{M}_{|M}$ splits into $\mathbb{R}\partial_t \oplus TM \oplus \mathbb{R}\partial_\rho$, which is isomorphic to the g-metric identification of the tractor bundle $\mathcal{T}$ under the map

$$\partial_t \mapsto s_-, \qquad TM \overset{\mathrm{Id}}{\longmapsto} TM, \qquad \partial_\rho \mapsto s_+. \tag{16}$$

The map (16) is an isometry of bundles over M with respect to $\tilde{g}$ and h and the pullback of $\widetilde{\nabla}$, the Levi-Civita connection of $\tilde{g}$, to $T\tilde{M}_{|M}$ coincides with (4). This also follows directly from an inspection of (3) and (13). With these identifications, for fixed $g \in c$ we view the tractor data as restrictions of ambient data for an ambient metric which is in normal form with respect to g.

3. The ambient obstruction tensor and conformal holonomy

We outline how the image of the obstruction tensor can be identified with a distinguished subspace of the infinitesimal conformal holonomy algebra at each point. This requires some preparation:

Let V be a vector space. The standard action # of $\mathrm{End}(V)$ on V extends to an action on the space $T^{r,s}V$ of (r, s) tensors over V. This action will be denoted by the same symbol. Thus, $\mathrm{End}(V) \otimes \mathrm{End}(V)$ acts on $T^{r,s}V$ with a double #-action, explicitly given by

$$(A \otimes B)\,\#\#\,(\eta) = A\,\#\,(B\,\#\,\eta). \tag{17}$$

Given a pseudo-Riemannian manifold (N, h), we can view its curvature tensor R^h as section of the bundle $\mathfrak{so}(N, h) \otimes \mathfrak{so}(N, h)$, and applying (17) pointwise yields an action $\mathrm{R}^h\#\#$ of the curvature on arbitrary tensor bundles of N.

Returning to the original setting, let (M, g) be a pseudo-Riemannian manifold of even dimension and let $(\tilde{M}, \tilde{g})$ be an associated ambient metric which is in normal form with respect to g. Let $\widetilde{\Delta} = \widetilde{\nabla}_A \widetilde{\nabla}^A$ denote the usual connection Laplacian on the ambient manifold. In [Gover and Peterson 2006] a modified Laplace-type operator

$$\not{\Delta} = \widetilde{\Delta} + \tfrac{1}{2}\widetilde{\mathrm{R}}\#\# \tag{18}$$

is introduced and will be used in the subsequent calculations.

The previous observations enable us to prove the main result of this section:

Theorem 3.1. *Let $(M, c = [g])$ be of even dimension > 2. For every $g \in c$ one has*

$$s_-^\flat \wedge (X \lrcorner \mathcal{O}) \in \mathfrak{hol}_x(M, [g]) \quad \textit{for all } x \in M \textit{ and } X \in T_xM.$$

Proof. The proof uses the notion of *infinitesimal holonomy*: within in the Lie algebra $\mathfrak{hol}_x(M, c)$ of $\mathrm{Hol}_x(M, c)$ at a point $x \in M$, we consider the *infinitesimal holonomy algebra at x*, i.e., the Lie algebra of iterated derivatives of the tractor curvature evaluated at x,

$$\mathfrak{hol}'_x(M,c) := \mathrm{span}_{\mathbb{R}}\{\nabla^{\mathrm{nc}}_{X_1}(\cdots(\nabla^{\mathrm{nc}}_{X_{l-1}}(\mathrm{R}^{\mathrm{nc}}(X_{l-1}, X_l))))(x) \mid l \geq 2, X_1, \dots, X_l \in \mathfrak{X}(M)\}.$$

For more details on the infinitesimal holonomy refer to [Kobayashi and Nomizu 1963, Chap. II.10] or [Nijenhuis 1953a; 1953b; 1954]. We will in fact show that $s_-^\flat \wedge (X \lrcorner \mathcal{O}) \in \mathfrak{hol}'_x(M, [g])$ for all $x \in M$ and $X \in T_xM$.

Assume first that $n > 4$. Let $(\tilde{M}, \tilde{g})$ be an associated ambient manifold for $(M, [g])$ which is in normal form with respect to some fixed g in the conformal class. For $x \in M$ let

$$\mathfrak{hol}_x(\tilde{M}, \tilde{g}) := \mathrm{span}_{\mathbb{R}}\{\widetilde{\nabla}_{X_1}(\cdots \widetilde{\nabla}_{X_{l-2}}(\widetilde{\mathrm{R}}(X_{l-1}, X_l)))(x) \mid l \geq 2,\, X_i \in \mathfrak{X}(\tilde{M})\}$$

denote the infinitesimal holonomy algebra of $(\tilde{M}, \tilde{g})$ at x and for $k \geq 0$ let $\mathfrak{hol}^k_x(\tilde{M}, \tilde{g})$ denote the subspace of elements for which at most k of the X_i have a not identically zero ∂_ρ-component. Then [Čap et al. 2016, Theorem 3.1] asserts that under the identifications from Section 2D,

$$\mathfrak{hol}'_x(M, c) = \mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g}). \tag{19}$$

Indeed, for $(\tilde{M}, \tilde{g})$ which is in normal form with respect to g, equality (19) can be verified as follows:

From the identifications from Section 2D one obtains immediately the inclusion $\subset$ in (19). In order to prove the converse, we obtain with [Graham and Willse 2012, Lemma 3.1] and [Fefferman and Graham 2012, Propositon 6.1] that

$$(20)\qquad \begin{aligned} &\widetilde{\mathrm{R}}(\partial_i, \partial_j)(x) = \mathrm{R}^{\mathrm{nc}}(\partial_i, \partial_j)(x), && \widetilde{\mathrm{R}}(\partial_t, \partial_I)(x) = 0,\\ &\widetilde{\mathrm{R}}(\partial_\rho, \partial_i)(x) = 3g^{kl}(\nabla^{\mathrm{nc}}_{\partial_k}\mathrm{R}^{\mathrm{nc}})(\partial_l, \partial_i)(x). \end{aligned}$$

The right sides of these expressions clearly lie in $\mathfrak{hol}'_x(M, c)$. To proceed, using linearity and commuting covariant derivatives, it suffices to prove that

$$(21)\qquad (\widetilde{\nabla}^k_{X_i}\widetilde{\nabla}^l_{\partial_\rho}\widetilde{\nabla}^j_{\partial_t}\widetilde{\mathrm{R}})(Y, Z)(x) \in \mathfrak{hol}'_x(M, c),$$

where $k, j, l \geq 0$, $X_i \in T_xM$, $Y, Z \in T_x\tilde{M}$ and $l \leq \frac{1}{2}n - 3$ or $\frac{1}{2}n - 2$ (depending on whether one of Y, Z has a ∂_ρ-component): given an element of the form (21) one first applies Proposition 6.1 from [Fefferman and Graham 2012], which rewrites ∂_t derivatives of $\widetilde{\mathrm{R}}$, and obtains a linear combination of elements of the form (21) with $j = 0$ and Y, Z have no ∂_t-component. Thus, it suffices to prove (21) for $j = 0$. This is then achieved by induction over l. Indeed, for $l = 0$ the statement follows from the last equation in (20). Furthermore, we may assume that $Y = \partial_\rho$ (otherwise all differentiations are tangent to M or we use the second Bianchi identity) and $Z \in T_xM$. However, Lemma 3.1 from [Graham and Willse 2012] allows us to rewrite ∂_ρ-derivatives $(\widetilde{\nabla}^l_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, Z)$ up to $l \leq \frac{1}{2}n - 3$ in terms of $(\widetilde{\nabla}^l_{\partial_\rho}\widetilde{\mathrm{R}})|_{TM\times TM}$. Then applying the second Bianchi identity and the induction hypothesis shows the claim (21). This proves the equality (19).

Using again the identifications from Section 2D, we will now show that for $x \in M$ and $X \in T_xM$ we have

$$(22)\qquad \partial_t^\flat(x) \wedge (X \lrcorner \mathcal{O})(x) \in \mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g}).$$

With this, equality (19) and the inclusion $\mathfrak{hol}'_x(M, c) \subset \mathfrak{hol}(M, c)$ will imply Theorem 3.1. In order to verify property (22), note that, as observed in [Gover and Peterson 2006], on any pseudo-Riemannian manifold one has (in abstract indices)

$$(23)\qquad 4\widetilde{\nabla}_{A_1}\widetilde{\nabla}_{B_1}\widetilde{\mathrm{Ric}}_{A_2B_2} = \tilde{\Delta}\widetilde{\mathrm{R}}_{A_1A_2B_1B_2} - \widetilde{\mathrm{Ric}}_{CA_1}\widetilde{\mathrm{R}}^C{}_{A_2B_1B_2} + \widetilde{\mathrm{Ric}}_{CB_1}\widetilde{\mathrm{R}}^C{}_{B_2A_1A_2},$$

where here A_1, A_2 and B_1, B_2 are pairwise skew-symmetrized. Indeed, (23) is a straightforward consequence of the second Bianchi identity. As in our situation, $\widetilde{\mathrm{Ric}} = O(\rho^{(n/2)-1})$, it follows that

$$(24)\qquad 4\widetilde{\nabla}_{A_1}\widetilde{\nabla}_{B_1}\widetilde{\mathrm{Ric}}_{A_2B_2} = \tilde{\Delta}\widetilde{\mathrm{R}}_{A_1A_2B_1B_2} + O(\rho^{(n/2)-1}).$$

In the next steps we use the general fact that if B is a tensor field on $\tilde{M}$, for example a $(0,2)$-tensor field, such that $B = O(\rho^m)$ for an $m \geq 1$, then, for all $X, Y \in TM$

$$\rho^{-m} B(X,Y)|_{\rho=0} = (\widetilde{\nabla}^m_{\partial_\rho} B)(X,Y)|_{\rho=0}, \qquad (\widetilde{\nabla}^k_{\partial_\rho} B)(X,Y)|_{\rho=0} = 0 \quad \text{for } k = 0, \dots, m-1. \tag{25}$$

Indeed, $B = O(\rho^m)$ implies for $k = 0, \dots, m-1$ that

$$0 = \partial^k_\rho (B(X,Y))|_{\rho=0} = (\widetilde{\nabla}^k_{\partial_\rho} B)(X,Y)|_{\rho=0},$$

where the second equality holds because of $\widetilde{\nabla}_{\partial_\rho}\partial_\rho = 0$ and $\widetilde{\nabla}_{\partial_\rho} X \in \mathfrak{X}(M)$ for $X \in \mathfrak{X}(M)$. This also implies that

$$\rho^{-m} B(X,Y)|_{\rho=0} = \partial^m_\rho (B(X,Y))|_{\rho=0} = (\widetilde{\nabla}^m_{\partial_\rho} B)(X,Y)|_{\rho=0},$$

proving both relations in (25).

Now we return to equation (24) and see, using (25), that it implies

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \tilde{\Delta} \widetilde{\mathrm{R}})(Y_1, Y_2, Z_1, Z_2)(x) = 4(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \widetilde{\nabla}_{Y_1} \widetilde{\nabla}_{Z_1} \widetilde{\mathrm{Ric}})(Y_2, Z_2)(x), \tag{26}$$

where now $x \in M$, Y_i, Z_i are ambient vector fields and Y_1, Y_2 as well as Z_1, Z_2 are skew-symmetrized. Now let $Y_1 = \partial_\rho$ and $Y_2 = X \in \mathfrak{X}(M)$ and insert $\widetilde{\mathrm{Ric}} = O(\rho^{(n/2)-1})$ into the right side of (26). It follows that, with $x \in M$, the resulting expression is zero unless one of the Z_i is proportional to ∂_ρ and the other one is a tangent vector $Y \in T_x M$. For this choice of vectors we have

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \tilde{\Delta} \widetilde{\mathrm{R}})(\partial_\rho, X, \partial_\rho, Y)(x) = (\widetilde{\nabla}^{(n/2)-1}_{\partial_\rho} \widetilde{\mathrm{Ric}})(X,Y)(x), \tag{27}$$

for $X, Y \in TM$. Hence, by definition (15) and the observation (25), one obtains a multiple of $\mathcal{O}(X,Y)$,

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \tilde{\Delta} \widetilde{\mathrm{R}})(\partial_\rho, X)(x) = k(n) \cdot \partial_t^\flat(x) \wedge (X \lrcorner \mathcal{O})(x), \tag{28}$$

for some nonzero numerical constant $k(n)$ which depends only on the dimension n. Note that along $M = \{\rho = 0, t = 1\}$ we have $\partial_t^\flat = d\rho$. To proceed, we analyze the left side in (28). Equations (13) show that the ambient Laplacian applied to some tensor field η has an expansion of the form

$$\widetilde{\Delta}\eta = \tilde{g}^{IJ} \widetilde{\nabla}_I \widetilde{\nabla}_J \eta = \frac{1}{t} \widetilde{\nabla}_{\partial_\rho}(\widetilde{\nabla}_{\partial_t}\eta) + \frac{1}{t} \widetilde{\nabla}_{\partial_t}(\widetilde{\nabla}_{\partial_\rho}\eta) - \frac{2\rho}{t^2} \widetilde{\nabla}_{\partial_\rho}(\widetilde{\nabla}_{\partial_\rho}\eta) + f \widetilde{\nabla}_{\partial_\rho}\eta + \tilde{D}\eta, \tag{29}$$

where f is a certain known function on $\tilde{M}$ and $\tilde{D}$ is an operator of the form

$$\tilde{D}\eta = \sum_{i,j} a_{ij} \widetilde{\nabla}_i (\widetilde{\nabla}_j \eta) + \sum_{K \in \{k,0\}} b_K \widetilde{\nabla}_K \eta. \tag{30}$$

We conclude inductively that for an arbitrary ambient tensor field η and an element $Z \in \mathfrak{so}(T\tilde{M})$ one has

$$\begin{gathered}\eta = O(\rho^l) \Rightarrow \widetilde{\Delta}^k \eta = O(\rho^{l-k}),\\ Z(x) \in \mathfrak{hol}^l_x(\tilde{M}, \tilde{g}) \Rightarrow (\widetilde{\Delta}^k Z)(x) \in \mathfrak{hol}^{k+l}_x(\tilde{M}, \tilde{g}).\end{gathered} \tag{31}$$

Moreover, a straightforward linear algebra calculation using the algebraic Bianchi identity for the ambient curvature reveals that (in abstract ambient indices and with brackets denoting skew symmetrization)

$$\begin{aligned}&(\widetilde{\mathrm{R}} \,\#\#\, \widetilde{\mathrm{R}})_{ABCD}\\ &\quad = 2\widetilde{\mathrm{R}}_{ABVW}\widetilde{\mathrm{R}}^{VW}{}_{CD} + 8\widetilde{\mathrm{R}}_C{}^P{}_{[A}{}^Q\widetilde{\mathrm{R}}_{B]PDQ} - 2(\widetilde{\mathrm{Ric}}^V{}_{[A}\widetilde{\mathrm{R}}_{B]VCD} + \widetilde{\mathrm{Ric}}^V{}_{[C}\widetilde{\mathrm{R}}_{D]VAB}).\end{aligned}$$

For each A, B the first term on the right hand side is contained in the holonomy algebra as it is a linear combination of curvature tensors. Similarly, the second term is a linear combination of commutators of curvature tensors and hence also in the holonomy algebra. Differentiating this $\frac{1}{2}n - 3$ times in ∂_ρ direction and using that $\widetilde{\mathrm{Ric}}$ vanishes to order $\frac{1}{2}n - 1$ shows via induction that

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho}(\widetilde{\mathrm{R}} \,\#\#\, \widetilde{\mathrm{R}}))(\partial_\rho, X)(x) \in \mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g}).$$

Next, we focus on the ρ-derivatives of $\widetilde{\Delta}$ in (28). Using the form of $\widetilde{\Delta}$ in (29) and (30) and calculating mod $\mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g})$, we find they are given by

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho}\widetilde{\Delta}\widetilde{\mathrm{R}})(\partial_\rho, X)(x) = \widetilde{\nabla}^{(n/2)-3}_{\partial_\rho}(\widetilde{\Delta}\widetilde{\mathrm{R}}(\partial_\rho, X))(x) = l(n)(\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, X)(x)$$

for some numerical constant $l(n)$. Thus, we have found that for $x \in M$, $X \in T_xM$,

$$k(n)\partial_t^\flat(x) \wedge (X \lrcorner \mathcal{O})(x) - l(n) \cdot (\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, X) = E_X \tag{32}$$

for some $E_X \in \mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g})$. Now insert $Y \in T_xM$ and ∂_ρ into the 2-forms in (32). One obtains

$$k(n)\mathcal{O}(X, Y) - l(n)(\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, X, Y, \partial_\rho) = E_X(Y, \partial\rho). \tag{33}$$

By [Fefferman and Graham 2012, Proposition 6.6] we have

$$2(\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, \partial_i, \partial_j, \partial_\rho) = \mathrm{tf}(\partial^{(n/2)}_\rho g_{ij}) + K_{ij},$$

where K_{ij} can be expressed algebraically in terms of $(\partial^k_\rho g_{ij})_{|\rho=0}$, $k < \frac{1}{2}n$, as well as $g^{ij}_{|\rho=0}$. Moreover, as follows from reviewing the above argument, E can be expressed algebraically in terms of derivatives of g_ρ and its inverse in M-directions and at most $\frac{1}{2}n - 1$ derivatives in ρ-direction and $\mathcal{O}$ is a natural tensor invariant. But then, as the ambiguity, i.e., the term $\mathrm{tf}(\partial^{n/2}_\rho g_{ij})$, can be arbitrary, equation (33) can only be true if $l(n) = 0$ from which the theorem follows if $n > 4$.

In general, it holds in every dimension that for $X \in TM$ one has

$$\mathrm{tr}_g \nabla^{\mathrm{nc}}_{\cdot} \mathrm{R}^{\mathrm{nc}}(X, \cdot) = (n-4)C(X; \cdot, \cdot) + B(X) \wedge s_-^\flat \in \mathfrak{hol}(M, c),$$

where

$$B_{ij} = \nabla^k C_{ijk} - P^{kl} W_{kijl}$$

is the Bach tensor, and where for each pair j, k we understand $\mathrm{R}^{\mathrm{nc}}_{jk}$ as an element in $\Lambda^2 \mathcal{T}^*$. From this observation the theorem follows in case $n = 4$, as here $\mathcal{O}$ is a multiple of the Bach tensor □

Remark. Consider the case $n = 6$. It is an entirely mechanical process to turn the formulas in [Gover and Peterson 2006], section 4B into an explicit formula for derivatives of the tractor curvature, which gives a more explicit proof of Theorem 3.1 for this dimension. In order to make this more explicit, assume that there is a metric $g \in c$ and a totally lightlike subspace $\mathcal{L} \subset TM$ such that $\mathrm{Im}(\mathrm{Ric}^g) \subset \mathcal{L}$ and $\mathcal{L}$ is ∇^g invariant. Such geometries correspond to invariant null subspaces which are invariant under $\mathrm{Hol}(M, c)$ and are of importance in Section 5B. Let ∇ denote the tractor derivative ∇^{nc} coupled to ∇^g. One can explicitly compute for this case that

$$g^{ij} s_-^\flat \wedge \mathcal{O}_{mi} \partial_j^\flat = g^{ij} g^{kl} \nabla_i \nabla_j \nabla_k \mathrm{R}^{\mathrm{nc}}_{ml} + 4\mathrm{P}^{ij} \nabla_i \mathrm{R}^{\mathrm{nc}}_{mj} + 2[\mathrm{R}^{\mathrm{nc}}_{mi}, \nabla^{\mathrm{nc}}_j \mathrm{R}^{\mathrm{nc}\, ij}] + 2C_m{}^{kl} \mathrm{R}^{\mathrm{nc}}_{kl}.$$

4. The conformal holonomy distribution

In this section we will introduce and study the fundamental object that provides us with the link between conformal holonomy and the ambient obstruction tensor.

4A. ***The conformal holonomy distribution.*** Let $(M, c = [g])$ be a smooth conformal manifold of arbitrary signature (p, q) and dimension $n = p + q$. For $x \in M$ consider the conformal holonomy algebra $\mathfrak{hol}_x(M, c) \subset \mathfrak{so}(\mathcal{T}_x, h_x)$. Fix $g \in c$. Theorem 3.1 motivates us to study the following subspaces of $T_x M$,

$$\mathcal{E}_x^g := \{\mathrm{pr}_{T_x M} \mathrm{Im}(A) \mid A \in \mathfrak{hol}_x(M, c),\, A\mathcal{I} = 0,\, h(A\mathcal{I}^\perp, \mathcal{I}^\perp) = 0\} \subset T_x M. \tag{34}$$

It follows immediately from the transformation formulas that $\mathcal{E}_x^g$ does not depend on the choice of $g \in c$, so that we can write $\mathcal{E}_x$. With respect to $g \in c$, however, $\mathcal{E}_x$ is identified with the space of elements of the holonomy algebra that are of the form $s_-^\flat \wedge X^\flat$ for some $X \in T_x M$. Equivalently and more invariantly, the space $\mathcal{E}_x$ can be identified with the space $\mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1$. We call the subset of TM defined by

$$\mathcal{E} := \bigcup_{x \in M} \mathcal{E}_x \subset TM$$

the *conformal holonomy distribution*. This is a slight abuse of terminology, as the dimension of $\mathcal{E}_x$ may vary with x, so that $\mathcal{E}$ is not a vector distribution in the usual sense. Indeed, the holonomy algebras with respect to different base points are related by the adjoint action of elements in $\mathrm{O}(p+1, q+1)$ that generically do not lie in the stabilizer of s_-. Instead, define a function on M by

$$r^{\mathcal{E}}(x) := \dim \mathcal{E}_x.$$

The function $r^{\mathcal{E}}$ need not be constant over M but leads to an obvious stratification

$$M = \bigcup_{k=0}^{n} M_k,$$

where $M_k = \{x \in M \mid r^{\mathcal{E}}(x) = k\}$.

4B. ***Relation to the curved orbit decomposition.*** We now proceed to establish a relation between the stratification defined by $\mathcal{E}$ and the curved orbit decomposition for holonomy reductions of arbitrary Cartan geometries in [Čap et al. 2014]. When doing this, we restrict to the case that $\mathfrak{hol}(M, c)$ equals the stabilizer of some tensor:

Starting with the tractor data $(\mathcal{T} \to M, h, \nabla^{\mathrm{nc}})$, one recovers an underlying Cartan geometry as follows [Čap and Gover 2003]: Fix a lightlike line $L \subset \mathbb{R}^{p+1,q+1}$ and at each point $x \in M$ consider the set of all linear, orthogonal maps $\mathbb{R}^{p+1,q+1} \to \mathcal{T}_x$ which additionally map L to $\mathcal{I}_x$. This defines a principal P-bundle $\mathcal{G} \to M$, where $P \subset G = \mathrm{O}(p+1, q+1)$ is the stabilizer subgroup of L. Then the tractor connection ∇^{nc} induces a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ of type (G, P), i.e., ω is equivariant with respect to the P-right action, reproduces the generators of fundamental vector fields, and provides a global parallelism $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$. In this way, $(\mathcal{G} \to M, \omega)$ is a Cartan geometry of type (G, P). Conversely, one obtains the standard tractor bundle from these data as $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{p+1,q+1} = \widehat{\mathcal{G}} \times_G \mathbb{R}^{p+1,q+1}$, where $\widehat{\mathcal{G}} = \mathcal{G} \times_P G$ denotes the enlarged G-bundle. The Cartan connection ω lifts to a principal bundle connection $\widehat{\omega}$ on $\widehat{\mathcal{G}}$ and ∇^{nc} is then the induced covariant derivative on the associated bundle $\mathcal{T}$.

Now assume that there is a faithful representation ρ of G on some vector space V with associated vector bundle $\mathcal{H} = \widehat{\mathcal{G}} \times_G V$ and induced covariant derivative $\nabla^{\mathcal{H}}$ such that $\mathrm{Hol}(M, c)$ equals pointwise the stabilizer of a $\nabla^{\mathcal{H}}$-parallel section $\psi \in \Gamma(M, \mathcal{H})$ (if actually (M, c) is spin, the same discussion is possible for spin coverings of the groups and bundles under consideration). Such a ψ is equivalently encoded in a G-equivariant map $s : \widehat{\mathcal{G}} \to V$ which is constant along $\widehat{\omega}$-horizontal curves. To this situation the general machinery developed in [Čap et al. 2014] applies and one defines for $x \in M$ the P-type of x (with respect to ψ) to be the P-orbit $s(\mathcal{G}_x) \subset V$. Then M decomposes into a union of initial submanifolds M_α of elements with the same P-type, where α runs over all possible P-types, which in turn can be found by looking at the homogeneous model $G \to G/P$. In that work, the M_α are called *curved orbits* and it was shown that they carry a naturally induced Cartan geometry of type $(H, P \cap H)$.

Theorem 4.1. *If* $\mathrm{Hol}(M, c)$ *is equal to the stabilizer of a tensor, then the subsets of* M *on which* $r^{\mathcal{E}}$ *is constant are unions of curved orbits in the sense of* [Čap et al. 2014]. *In particular, they are unions of initial submanifolds.*

Proof. We fix a curved orbit M_α with element x_1. By definition, $x_2 \in M_\alpha$ if and only if

$$s(\mathcal{G}_{x_1}) = s(\mathcal{G}_{x_2}). \tag{35}$$

We unwind the condition (35) as follows: Let $u_{x_i} \in \mathcal{G}_{x_i}$ and let

$$[u_{x_i}] : V \ni v \mapsto [u_{x_i}, v] \in \mathcal{H}_{x_i}$$

denote the associated fiber isomorphism. As ρ is faithful the holonomy group $\mathrm{Hol}_{u_{x_i}}(\widehat{\omega}) \subset G$ will coincide with the stabilizer of $[u_{x_i}]^{-1}\psi_{x_i} \in V$ under the (ρ, G)-action. Moreover (35) is equivalent to the existence of $p \in P$ such that

$$\rho(p)([u_{x_1}]^{-1}\psi_{x_1}) = [u_{x_2}]^{-1}\psi_{x_2},$$

from which one deduces that

$$\mathrm{Ad}(p^{-1})(\mathfrak{hol}_{u_{x_1}}(\widehat{\omega})) = \mathfrak{hol}_{u_{x_2}}(\widehat{\omega}). \tag{36}$$

Using that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, one sees that (36) restricts to a map between the $\mathfrak{g}_1$-components of $\mathfrak{hol}_{u_{x_i}}(\widehat{\omega})$ which therefore have the same dimension. As

$$\mathfrak{hol}_x = [u_x] \circ \mathfrak{hol}_{u_x}(\widehat{\omega}) \circ [u_x]^{-1} \subset \mathfrak{so}(\mathcal{T}_x, h_x)$$

and $[u_x]$ preserves the lightlike line by definition of $\mathcal{G}$, we obtain that the dimensions of $\mathfrak{hol}_{x_i} \cap \mathfrak{g}_1$ also agree. Consequently, $r^{\mathcal{E}}$ is constant on the curved orbit M_α. □

Theorem 4.1 shows that, in general, the holonomy distribution $\mathcal{E}$ as studied here will induce a stratification of M that is *coarser* than the curved orbit decomposition in [Čap et al. 2014]. The following example shows that in some cases it induces the same stratification.

Example. Suppose (M, c) is of Riemannian signature and $\mathrm{Hol}_x(M, c)$ equals the stabilizer of some tractor $\zeta_x \in \mathcal{T}_x$. For any metric $g \in c$ write $\zeta = (\alpha, Y, \beta)^\top$ for smooth functions α, β and a vector field Y on M. Evaluating $\nabla^{\mathrm{nc}}\zeta = 0$ using (4) yields

$$Y = \mathrm{grad}^g \beta, \quad \alpha g = \beta \mathrm{P}^g - \mathrm{Hess}^g(\beta).$$

An element $V^\flat \wedge s_-^\flat$ lies in $\mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1$ if and only if $d\beta(V) = 0$ as well as $\beta \cdot V = 0$ at x. If $h(\zeta, \zeta) \neq 0$, we conclude that

$$M = M_0 \cup M_{n-1}, \quad \text{with } M_0 = \{\beta \neq 0\} \text{ and } M_{n-1} = \{\beta = 0\}.$$

For $x \in M_{n-1}$ we have $\mathcal{E}_x = \ker d\beta \neq T_xM$. In particular, M_{n-1} is a smooth embedded submanifold of M. Similarly, if $h(\zeta,\zeta)=0$, we have

$$M = M_0 \cup M_n = \{\beta \neq 0\} \cup \{\beta = 0\}.$$

Here $\{\beta=0\}$ consists only of isolated points because $\beta(x)=0$ implies that $d\beta(x)=0$ and $\mathrm{Hess}^g(\beta)(x)$ is proportional to g_x.

4C. ***Open sets adapted to the holonomy distribution.*** We analyze the function $r^{\mathcal{E}}$ in more detail. Obviously, if $\mathfrak{hol}_x(M,c)$ is generic at some point of M, i.e., if $\mathfrak{hol}_x(M,c) = \mathfrak{so}(p+1,q+1)$, then $r^{\mathcal{E}} \equiv n$. Conversely, one finds:

Proposition 4.2. *Suppose that there is a curve γ in M with $g(\dot\gamma,\dot\gamma) \neq 0$ and $r^{\mathcal{E}} \circ \gamma \equiv n$. Then $\mathfrak{hol}(M,c)$ is generic. In particular, $r^{\mathcal{E}} \equiv n$.*

Proof. All calculations are carried out with respect to some fixed $g \in c$. By assumption, $s_-^\flat \wedge V^\flat \in \mathfrak{hol}_{\gamma(t)}(M,c)$ for every vector field V along γ. Applying $\nabla^{\mathrm{nc}}_{\dot\gamma}$ to this expression using (7) reveals that

$$-g(V,\dot\gamma)s_-^\flat \wedge s_+^\flat + \dot\gamma^\flat \wedge V^\flat \in \mathfrak{hol}_{\gamma(t)}(M,c). \tag{37}$$

Letting $V=\dot\gamma$ shows that $s_-^\flat \wedge s_+^\flat \in \mathfrak{hol}_{\gamma(t)}(M,c)$. Moreover, letting $(V_1, V_2, \dot\gamma)$ be mutually orthogonal to each other and taking the Lie brackets of the expressions (37) with $V=V_1$ and $V=V_2$, respectively, shows that

$$\|\dot\gamma\|^2 V_1^\flat \wedge V_2^\flat \in \mathfrak{hol}_{\gamma(t)}(M,c).$$

But this establishes that $\mathfrak{g}_0 \in \mathfrak{hol}_{\gamma(t)}(M,c)$. Thus, $\mathfrak{g}_1 \oplus \mathfrak{g}_0 \in \mathfrak{hol}_{\gamma(t)}(M,c)$. Differentiating elements $\dot\gamma^\flat \wedge V^\flat \in \mathfrak{hol}_{\gamma(t)}(M,c)$ in the direction of γ, where V is again a vector field along γ shows using (7) that also $\mathfrak{g}_{-1} \cap \dot\gamma^\perp$ is contained in the infinitesimal holonomy along γ and differentiating $s_-^\flat \wedge s_+^\flat$ along γ shows that all of $\mathfrak{g}_{-1}$ is contained in the holonomy. Thus, $\mathfrak{hol}_{\gamma(t)}(M,c)$ is generic along γ, and thus generic everywhere. □

In order to continue with our analysis, we need to show that there are *sufficiently many* open sets U on which $r^{\mathcal{E}}$ is constant, i.e., such that $\mathcal{E}|_U$ is a vector bundle, and on which there is a basis of local smooth sections of $U \to \mathcal{E}$. For this purpose we define: An open set $U \subset M$ is an *$\mathcal{E}$-adapted open set* if

(1) $r^{\mathcal{E}} \equiv k$ constant on U,

(2) there are smooth and pointwise linearly independent sections $V_1,\dots,V_k : U \to \mathcal{E}$.

Then:

Theorem 4.3. *For each open set $U \subset M$ there exists an $\mathcal{E}$-adapted open subset $V \subset U$. In particular, there is an open dense subset of M which is the union of $\mathcal{E}$-adapted open sets.*

Proof. After restricting U if necessary, we may assume that U is contained in a coordinate neighborhood for M. It is then possible to choose a local basis of $\mathfrak{hol}_x(M,c)$ over U which depends smoothly on x. Write such a basis as

$$U \ni x \mapsto (v_i^\flat(x) \wedge s_-^\flat + A^i(x)), \tag{38}$$

where $i = 1, \dots, m := \dim \mathfrak{hol}(M,c)$, for certain $v_i \in T_xM$ and $A^i \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. With respect to the fixed coordinates we may think of the $A^i = (A^i_{jk})_{j,k}$ as $\mathfrak{so}(p+1,q+1)$-matrices. Let

$$\tilde{A}^i := (A^i_{11}, A^i_{12}, \dots, A^i_{n+1,n+2}, A^i_{n+2,n+2})^\top$$

and introduce the $(n+2)^2 \times m$-matrix $A := (\tilde{A}^1 \cdots \tilde{A}^m)$. By elementary linear algebra,

$$r^{\mathcal{E}}(x) = k \iff k = \dim \ker A = \dim \mathfrak{hol}_x(M,c) - \operatorname{rk} A_x. \tag{39}$$

The set of matrices with rank greater or equal to some fixed integer is open in the set of all matrices. Thus, it follows from (39) that $\{x \mid r^{\mathcal{E}}(x) \le k\}$ is open in M. In particular, $(r^{\mathcal{E}})^{-1}(0) = \{x \mid r^{\mathcal{E}}(x) \le 0\}$ is open and $r^{\mathcal{E}} < n$ is an open condition.

Assume now that there is $x \in U$ with $r^{\mathcal{E}}(x) = 0$. It follows that $r^{\mathcal{E}} = 0$ on some open subset $V \subset U$. Thus the claim follows for this case. Otherwise, we have $r^{\mathcal{E}} \ge 1$ everywhere. If there is $x \in U$ with $r^{\mathcal{E}}(x) = 1$, it follows that there is an open neighborhood V in U with $r^{\mathcal{E}} \le 1$ of x in U. Thus, $r^{\mathcal{E}} = 1$ on V. Otherwise we have $r^{\mathcal{E}} \ge 2$ on U etc. So the statement regarding the existence of V with $r^{\mathcal{E}}|_V =: l =$ constant follows inductively. The above proof starts with a smooth local basis (38) and constructs (on an open subset of V) via smooth linear algebra operations a basis on V of the form $(\tilde{v}^\flat_{i=1,\dots,l} \wedge s_-^\flat, \dots)$. It is thus clear that the $\tilde{v}_i$ depend smoothly on $x \in V$ and yield local sections.

Finally, if every open set in M contains an $\mathcal{E}$-adapted open subset, the union of all $\mathcal{E}$-adapted open sets is open and dense in M. □

By virtue of this theorem, after restricting to an open and dense subset of M if necessary, we may from now on always assume that M is the union of $\mathcal{E}$-adapted open sets. In particular, the level sets of $r^{\mathcal{E}}$ are then (possibly empty) unions of $\mathcal{E}$-adapted open sets. From this point of view, we may restrict ourselves to such open sets in the following local analysis. Note that restricting to an open and dense subset in the context of Cartan holonomy reductions is a basic feature of the curved orbit decomposition as revealed in [Čap et al. 2014].

Proposition 4.4. *Let $U \subset M$ be a $\mathcal{E}$-adapted open set. Then $\mathcal{E}_x$ is a totally lightlike subspace of T_xM for every $x \in U$ or $\mathfrak{hol}(M,c)$ is generic.*

Proof. Let V be a vector field defined on U such that $s_-^\flat \wedge V^\flat(x) \in \mathfrak{hol}_x(M,c)$ for $x \in U$. Differentiating in the direction of some $X \in TM$ using (7) reveals that

$$-\nabla^{\mathrm{nc}}_X(s_-^\flat \wedge V^\flat)(x) = g(V,X) s_-^\flat \wedge s_+^\flat + X^\flat \wedge V^\flat + (\nabla_X V)^\flat \wedge s_-^\flat \in \mathfrak{hol}_x(M,c). \tag{40}$$

Suppose that there is $x \in U$ with $g(V, V)(x) \neq 0$. It follows that $g(V, V) \neq 0$ on some open neighborhood $x \in W \subset U$. Let X be orthogonal to V on W. As $\mathfrak{hol}_x(M, c)$ is a Lie algebra with the usual commutator Lie bracket, it follows that on W also

$$[X^\flat \wedge V^\flat + (\nabla_X V)^\flat \wedge s_-^\flat, s_-^\flat \wedge V^\flat] = -g(V, V) X^\flat \wedge s_-^\flat \in \mathfrak{hol}(M, c). \tag{41}$$

Thus, $r^{\mathcal{E}}|_W = n$ and the statement follows from Proposition 4.2. □

4D. ***Rank and integrability of the holonomy distribution.*** Interestingly, it turns out that, at least locally, $\mathcal{E}$ is always integrable or it is maximally nonintegrable and one of the exceptional holonomy reductions occurs. More precisely, we will see that if $\mathcal{E}$ is not integrable, M is of dimension 5 or 6, $\mathcal{E}$ is generic and of rank 2 or 3, respectively, and $\mathfrak{hol}(M, c)$ is $\mathfrak{g}_2$ or $\mathfrak{spin}(4, 3)$, respectively.

In order to analyze the integrability of $\mathcal{E}$, we need some preparations.

Proposition 4.5. *Let (M^n, c) be a conformal manifold of even dimension. Either there is an open dense subset of M on which $r^{\mathcal{E}} \leq 1$ or $\mathrm{Hol}^0(M, c)$ acts on the lightcone $\mathcal{N} \subset \mathbb{R}^{p+1,q+1}$ with an open orbit.*

Proof. Suppose first that $r^{\mathcal{E}} \geq 2$ on some open set $U \subset M$. After restricting to an open, dense subset of U, if necessary, we may assume that U is an $\mathcal{E}$-adapted open set. We may also assume that the holonomy is not generic and hence that $\mathcal{E}$ is lightlike. Let V be a local section of $\mathcal{E}$ and let V' be a lightlike vector field with $g(V, V') = 1$. Moreover, let $X \in (V, V')^\perp$. We have on U

$$\nabla^{\mathrm{nc}}_{V'}(s_-^\flat \wedge V^\flat) = s_-^\flat \wedge s_+^\flat + A_1 \in \mathfrak{hol}(M, c), \tag{42}$$

$$\nabla^{\mathrm{nc}}_X(\nabla^{\mathrm{nc}}_{V'}(s_-^\flat \wedge V^\flat)) = -X^\flat \wedge s_+^\flat + A_2 \in \mathfrak{hol}(M, c), \tag{43}$$

where $A_{1,2} \in \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{p}$. As $r^{\mathcal{E}} \geq 2$ on U and $\mathcal{E}$ is totally lightlike, linear algebra shows that at $x \in U$, equation (43) implies

$$\mathfrak{so}(p+1, q+1) = \mathfrak{hol}_x(M, c) + \mathfrak{p}. \tag{44}$$

This, together with equation (42) shows that the orbit of $\mathrm{Hol}^0(M, c)$ through $s_- \in \mathcal{N}$ has dimension $n+1$, i.e., it is open. Otherwise, the subset of M on which $r^{\mathcal{E}} \leq 1$ is dense. It is also open as follows from the proof of Theorem 4.3. □

In relation to this proposition, we point out that conformal structures for which the holonomy group acts not only with an open orbit on $\mathcal{N}$, but transitively and irreducibly on the homogeneous model were classified in [Alt 2012].

Proposition 4.6. *Suppose that $(M, [g])$ admits an nc-Killing form $\alpha \in \Omega^k(M)$. Then $V^\flat \wedge \alpha = 0$ for every $V \in \mathcal{E}$.*

Proof. Following the discussion in Section 2C, every nc-Killing k-form α uniquely determines a parallel tractor $(k+1)$-form $\hat{\alpha}$. With respect to a metric g in the conformal class, decompose $\hat{\alpha}$ as in (8). Pointwise, $\hat{\alpha}$ is annihilated by the action # of $\mathfrak{hol}(M, c)$ on forms. In particular, one has for every $V \in \mathcal{E}_x$ that

$$(s_-^\flat \wedge V^\flat) \# \hat{\alpha}_x = 0.$$

Inserting (8), one immediately obtains that $V^\flat \wedge \alpha = 0$. □

Proposition 4.7. *Suppose M is orientable and the action of $\mathfrak{hol}(M, c)$ leaves invariant a nontrivial nondegenerate subspace of $\mathbb{R}^{p+1,q+1}$. Then $\mathcal{E} = 0$ on an open, dense subset of M.*

Proof. As the holonomy invariant space (of dimension $k+1$) is nondegenerate and M is orientable, there is actually a decomposable parallel tractor form in $\Omega^k \mathcal{T}^*$. The associated nc-Killing form α is of the form $\alpha = t_1 \wedge \cdots \wedge t_k$, defining a k-dimensional nondegenerate subspace $H \subset TM$ on an open, dense subset of M as follows from the discussion in [Leitner 2005], Thus, Proposition 4.6 implies that $\mathcal{E} \subset H$ on an open dense subset M' of M. On the other hand, by Proposition 4.2, $\mathcal{E}$ is over M' contained in a totally degenerate subspace. We conclude $\mathcal{E}_{|M'} = 0$. □

Proposition 4.8. *Suppose that* $\mathrm{Hol}(M, c)$ *fixes a totally lightlike* (*with respect to h*) *subbundle $\mathcal{H} \subset \mathcal{T}$. Then there is an open and dense subset of M and at least locally a metric $g \in c$ such that with respect to g*

$$\mathcal{H} = \mathbb{R}s_+ \oplus \mathcal{L}, \tag{45}$$

with $\mathcal{L} \subset TM$ a ∇^g-parallel distribution containing $\mathcal{E}$ and the image of Ric^g.

Proof. The existence of a parallel distribution $\mathcal{L} \subset TM$ containing the image of Ric^g was proven in [Lischewski 2015]. To see that at each $x \in M$, the fiber $\mathcal{L}_x$ contains $\mathcal{E}_x$, consider $V \in \mathcal{E}_x$ such that $s_-^\flat \wedge V^\flat \in \mathfrak{hol}_x(M, c)$. Then $(s_-^\flat \wedge V^\flat)(s_+) = V$ lies in $\mathcal{H}$, which shows that $\mathcal{E} \subset \mathcal{L}$. □

These results enable us to prove the main result of this section:

Theorem 4.9. *Let $U \subset M$ be a $\mathcal{E}$-adapted open set. Then exactly one of the following cases occurs on U*:

(1) *$\mathcal{E}$ is integrable.*

(2) *The dimension of M is* 5 *and $\mathcal{E}$ is a generic rank* 2 *distribution. Moreover,* $\mathfrak{hol}(M, c) = \mathfrak{g}_2$ *and hence the conformal structure $c = c_{\mathcal{E}}$ is defined by the generic distribution $\mathcal{E}$.*

(3) *The dimension of M is* 6 *and $\mathcal{E}$ is a generic rank* 3 *distribution. Moreover,* $\mathfrak{hol}(M, c) = \mathfrak{spin}(3, 4)$ *and the conformal structure $c = c_{\mathcal{E}}$ is defined by the generic distribution $\mathcal{E}$.*

Proof. If $\mathfrak{hol}(M, c)$ is generic the statement is trivial as $\mathcal{E} = TM$ in this case. Thus, we may assume that the holonomy algebra is reduced and by the previous Proposition, $\mathcal{E}_x$ is a totally lightlike subspace of T_xM for $x \in U$.

Let V_1, V_2 be vector fields on U such that $s_-^\flat \wedge V_{i=1,2}^\flat \in \mathfrak{hol}_x(M, c)$ for $x \in U$. It follows that

$$\nabla^{\mathrm{nc}}_{V_1}(s_-^\flat \wedge V_2^\flat) - \nabla^{\mathrm{nc}}_{V_2}(s_-^\flat \wedge V_1^\flat) = -V_1^\flat \wedge V_2^\flat + s_-^\flat \wedge ([V_1, V_2])^\flat \in \mathfrak{hol}(M, c). \tag{46}$$

Moreover, let X be a vector field on U which is orthogonal to V_i for $i = 1, 2$. It follows from evaluating $[\nabla^{\mathrm{nc}}_X(s_-^\flat \wedge V_1^\flat), \nabla^{\mathrm{nc}}_X(s_-^\flat \wedge V_2^\flat)]$ that

$$2g(\nabla_X V_1, V_2)X^\flat \wedge s_-^\flat + g(X, X)V_1^\flat \wedge V_2^\flat \in \mathfrak{hol}(M, c). \tag{47}$$

Combining (46) and (47) it follows for X orthogonal to (V_1, V_2) that

$$X \cdot g(\nabla_X V_1, V_2) \in \mathcal{E} \quad \text{for } g(X, X) = 0, \tag{48}$$

$$[V_1, V_2] - \frac{2g(\nabla_X V_1, V_2)}{g(X, X)} \cdot X \in \mathcal{E} \quad \text{for } g(X, X) \neq 0. \tag{49}$$

Now we distinguish several cases: Obviously the statement is trivial in case $r^{\mathcal{E}} \leq 1$. Thus, we may assume that V_1, V_2 are linearly independent. Fix a local g-pseudoorthonormal basis $(s_1, \ldots, s_n)$ over U such that

$$\mathcal{E} = \operatorname{span}(V_i := s_{2i-1} + s_{2i} \mid i = 1, \ldots, r^{\mathcal{E}}). \tag{50}$$

Moreover, let $V_i' := s_{2i-1} - s_{2i}$ for $i = 1, \ldots, e$. That is, $g(V_i, V_j') = 2\delta_{ij}$.

Case 1: $r^{\mathcal{E}} \geq 3$ and $n > 6$. In (48) let $X = V_3'$. It follows that $g(\nabla_{s_5} V_1, V_2) = g(\nabla_{s_6} V_1, V_2)$. But then letting $X = s_5, s_6$, (49) can only be true if $[V_1, V_2] - f \cdot V_3' \in \mathcal{E}$ for some function f. On the other hand, applying (49) to $X = s_n$ reveals that $[V_1, V_2] - h \cdot s_n \in \mathcal{E}$ for some function h. But this can only be true if $f = h = 0$, i.e., $[V_1, V_2] \in \mathcal{E}$.

Case 2: $r^{\mathcal{E}} = 2$ and $n > 5$. In complete analogy to the previous case, we obtain that $[V_1, V_2] - f s_5 \in \mathcal{E}$ for some function f as well as $[V_1, V_2] - h s_6 \in \mathcal{E}$ for some function h from which one has to conclude that $f = h = 0$.

Case 3: $r^{\mathcal{E}} = 2$ and $n = 4$. Necessarily, M is of signature $(2, 2)$. It follows from (48) that for $i, j, k \in \{1, 2\}$ we have $g(\nabla_{V_i}, V_j, V_k) = 0$. But this implies that $g(\nabla_{V_1} V_2 - \nabla_{V_2} V_1, V_k) = 0$, i.e., $[V_1, V_2] \in \mathcal{E}^\perp = \mathcal{E}$.

It remains to show that in signatures $(3, 2)$ with $\mathcal{E}$ of dimension 2 and in signature $(3, 3)$ with $\mathcal{E}$ being of dimension 3 and not integrable, $\mathcal{E}$ is generic.

First, let us consider signature $(3, 2)$ and assume that $\mathcal{E}$ is not integrable. In particular, $\mathcal{E}$ is of rank 2 on an open and dense set. One could proceed with the proof for this case analogously as with the $(3, 3)$ case below. However, as we are considering a conformal structure in *odd* dimension, one of the main results in [Čap et al. 2016] yields that $\mathfrak{hol}(M, c)$ is the holonomy algebra of a Ricci flat pseudo-Riemannian manifold of signature $(4, 3)$. If the standard action of $\mathfrak{hol}(M, c)$ was reducible, then by Propositions 4.7 and 4.8, $\mathcal{E}$ would be either zero or contained in an integrable totally lightlike distribution, both contradicting the assumptions in the current case. Thus, the action of the holonomy algebra is irreducible and from $\mathcal{E} \neq TM$ and the pseudo-Riemannian version of the Berger list it follows that $\mathfrak{hol}(M, c) = \mathfrak{g}_2$, where $\mathfrak{g}_2$ denotes the noncompact simple Lie algebra of dimension 14. For this case, however, $\mathcal{E}$ is generic. This follows from the discussion of $\mathfrak{g}_2$-conformal structures in Section 2C in complete analogy to the proof of Corollary 5.11 in Section 5B.

Let us now treat the 6-dimensional case. Fix a local basis $(V_1, V_2, V_3, V_1', V_2', V_3')$ for TM over U as specified in (50) such that $g(V_i, V_j') = 2\delta_{ij}$. Moreover, without loss of generality, we may assume that

$$[V_1, V_2] \notin \mathcal{E}. \tag{51}$$

From (48) we obtain $g(\nabla_{V_3'} V_1, V_2) = 0$ and (49) applied to $X = V_3 + V_3'$ then yields

$$[V_1, V_2] - g(\nabla_{V_3} V_1, V_2) V_3' \in \mathcal{E}. \tag{52}$$

We conclude from (51) that $g(\nabla_{V_3} V_1, V_2) \neq 0$. Moreover, it follows from subtracting $\nabla_{V_2}(s_-^\flat \wedge V_1^\flat) \in \mathfrak{hol}(M, c)$ from (46) that

$$\nabla_{V_2} V_1 + [V_1, V_2] \in \mathcal{E}. \tag{53}$$

In complete analogy to the derivation of (52) we obtain $[V_1, V_3] - g(\nabla_{V_2} V_1, V_3) V_2' \in \mathcal{E}$. Inserting (53) and then using (51) and (52) reveals that the coefficient $g(\nabla_{V_2} V_1, V_3)$ is nonzero. The same argument applies to $[V_2, V_3]$ and we conclude that there are nowhere vanishing functions f_k for $k = 1, 2, 3$ such that

$$[V_i, V_j] = \epsilon_{ijk} f_k V_k' \mod \mathcal{E}.$$

In particular, $[\mathcal{E}, \mathcal{E}] = TM$.

It remains to show that in this case we have $\mathfrak{hol}(M, c) = \mathfrak{spin}(3, 4)$. Using (7), it is straightforward to compute that the 15 elements, $i, j = 1, 2, 3$,

$$s_-^\flat \wedge V_i^\flat, \quad \nabla^{\mathrm{nc}}_{V_i'}(s_-^\flat \wedge V_j^\flat) \quad \text{and} \quad \nabla^{\mathrm{nc}}_{V_i}(s_-^\flat \wedge V_j^\flat),\ i < j \tag{54}$$

are pointwise linearly independent in $\mathfrak{hol}(M, c) \cap \mathfrak{p}$. Then Proposition 4.5 comes into play, which ensures that $\mathfrak{so}(p+1, q+1) = \mathfrak{hol}(M, c) + \mathfrak{p}$ and hence that dim $\mathfrak{hol}(M, c) \geq 15 + 6 = 21$, which is the dimension of $\mathfrak{spin}(4, 3)$. Then the equality $\mathfrak{hol}(M, c) = \mathfrak{spin}(4, 3)$, and with it the last point in the theorem, follows from Lemma 4.10 below. □

Lemma 4.10. *Let* $\mathfrak{h} \subsetneq \mathfrak{so}(4,4)$ *be irreducible of dimension at least* 21. *Then* $\mathfrak{h} = \mathfrak{spin}(3,4)$.

Proof. Since $\mathfrak{h}$ acts irreducibly, it is reductive. Then either $\mathfrak{h}$ is semisimple and the complexified representation $\mathbb{C} \otimes \mathrm{R}^{4,4}$ is irreducible, or $\mathfrak{h} \subset \mathfrak{u}(2,2)$ and $\mathbb{C} \otimes \mathrm{R}^{4,4}$ is not irreducible (see for example [Di Scala and Leistner 2011, Section 2]). The second case however is excluded by the assumption $\dim(\mathfrak{h}) \geq 21$. Hence, we may consider $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{so}(8,\mathbb{C})$ semisimple acting irreducibly on $\mathbb{C}^8$. Inspecting the dimensions of simple complex Lie algebras below 28, it turns out that the only possibilities for $\mathfrak{h}$, apart from $\mathfrak{so}(7,\mathbb{C})$, are $\mathfrak{sl}_5\mathbb{C}$ and $\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_3\mathbb{C}$. Then $\mathfrak{sl}_5\mathbb{C}$ is excluded as it does not have an irreducible representation of dimension 8. On the other hand, any irreducible representation of $\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_3\mathbb{C}$ is a tensor product of irreducible representations, which is excluded as $\mathfrak{sl}_3\mathbb{C}$ does not have an irreducible representations of dimension 2 or 4. □

Finally, we want to derive universal integrability conditions for the Weyl and Cotton tensors for conformal manifolds with reduced holonomy.

Proposition 4.11. *Let* (M, c) *be a conformal manifold with nongeneric holonomy. Locally, and off a singular set there is a totally degenerate subspace* $\mathcal{L} \subset TM$, *which is integrable if* $(p,q) \notin \{(3,2),(3,3)\}$, *such that*

$$W(\mathcal{L}, \mathcal{L}^{\perp}) = 0, \tag{55}$$

$$(n-4)C(\mathcal{L}, \mathcal{L}^{\perp}) = 0. \tag{56}$$

In even dimensions, one has $\mathrm{Im}(\mathcal{O}) \subset \mathcal{L}$. *In particular, if a conformal manifold in even dimension* ≥ 4 *admits a parallel tractor (of any type) other than the tractor metric, then the conformally invariant system (55) – (56) either becomes a nontrivial integrability condition on the curvature (and it couples* $\mathcal{O}$ *to the curvature) or* $\mathcal{O} = 0$.

Proof. We restrict the local analysis to $\mathcal{E}$-adapted open sets and let $\mathcal{L} = \mathcal{E}$. The conditions (55) and (56) are easily seen to be an equivalent reformulation of

$$[\mathrm{R}^{\mathrm{nc}}(X,Y), s_-^{\flat} \wedge V^{\flat}] \in \mathfrak{hol}(M,c), \tag{57}$$

$$[\mathrm{tr}_g \nabla.\mathrm{R}^{\mathrm{nc}}(\,\cdot\,,X), s_-^{\flat} \wedge V^{\flat}] \in \mathfrak{hol}(M,c), \tag{58}$$

where $X, Y \in TM$ and $V \in \mathcal{E}$. The statement follows from the definition of $\mathcal{E}$ and Theorems 3.1 and 4.9. □

5. Applications to the obstruction tensor

Recall that according to Theorem 3.1 the image of the obstruction tensor $\mathcal{O}$ is contained in the holonomy distribution $\mathcal{E}$. In this section we apply the results about $\mathcal{E}$ to obtain the results in Corollaries 1.2 and 1.3. In the following we will always assume we have given a smooth conformal manifold (M, c) of even dimension n and with obstruction tensor $\mathcal{O}$. We view $\mathcal{O}$ as a $(1, 1)$-tensor by means of some $g \in c$ and define the rank of $\mathcal{O}$ at a point to be the rank of this $(1, 1)$-tensor. The holonomy reductions we will consider now were described in Section 2C.

5A. ***The obstruction tensor and holonomy reductions.*** We begin with a well-known case of a conformal holonomy reduction, the case of a parallel standard tractor. The existence of a parallel standard tractor is equivalent to the existence of an open dense subset in M, on which the conformal class contains local Einstein metrics. It is well known since [Fefferman and Graham 1985, Proposition 3.5], see also [Gover and Peterson 2006, Theorem 4.3] and [Fefferman and Graham 2012] that the existence of local Einstein metrics in the conformal class forces $\mathcal{O} = 0$. Our Theorem 3.1 provides us with an independent and alternative proof:

Corollary 5.1. *If locally on an open and dense subset of M there is an Einstein metric $g \in c$, then $\mathcal{O} = 0$.*

Proof. Given an Einstein metric on $U \subset M$ and splitting the tractor bundle over U with respect to g, there is on U a parallel standard tractor

$$T = -\frac{\mathrm{scal}^g}{2n(n-1)} s_- + s_+.$$

In particular, $\mathfrak{hol}_x(U, [g]) T_x = 0$. Theorem 3.1 yields $(s_-^\flat \wedge (X \lrcorner \mathcal{O}))(T) = \mathcal{O}(X) = 0$ on U for each $X \in TU$ which is equivalent to $\mathcal{O} = 0$ on U. $\square$

A weaker condition than admitting a parallel tractor is the existence of a subspace that is invariant under the conformal holonomy. In this situation Propositions 4.7 and 4.8 imply:

Corollary 5.2. *Suppose M is orientable and the action of* $\mathrm{Hol}(M, c)$ *leaves invariant a nontrivial subspace $\mathcal{H}$ of $\mathbb{R}^{p+1,q+1}$. Then we have the following alternatives (possibly replacing $\mathcal{H}$ with $\mathcal{H} \cap \mathcal{H}^\perp$ if it is degenerate):*

(1) *If $\mathcal{H}$ is nondegenerate, then $\mathcal{O} = 0$.*

(2) *If $\mathcal{H}$ is totally lightlike, then, locally on an open dense subset of M there is a metric $g \in c$ and a ∇^g-parallel distribution $\mathcal{L} \subset TM$ containing the image of* Ric^g *and of $\mathcal{O}$.*

Specializing the total lightlike case in this corollary further, in Section 5B we will consider Bryant's conformal structures as examples. Another example is the following:

Example. Suppose that M is of split signature (n, n) and that $\mathrm{Hol}(M, c)$ leaves invariant two complementary totally lightlike distributions $\mathcal{H} \oplus \mathcal{H}' = \mathcal{T}$, i.e., $\mathrm{Hol}(M, c) \subset \mathrm{GL}(n + 1, \mathbb{R}) \subset \mathrm{SO}(n + 1, n + 1)$. Such conformal structures arise from Fefferman type constructions starting with n-dimensional projective structures, see [Hammerl and Sagerschnig 2011a; Hammerl et al. 2015]. For $\mathcal{H}$ and $\mathcal{H}'$ define L and L' as above and fix a local metric g such that $\mathcal{H}$ is of the form (45) on some set $U \subset M$. Elementary linear algebra shows that on U the space $L \cap L'$ is at each point at most 1-dimensional. Moreover, we have from the conformal covariance of $\mathcal{O}$ and Corollary 5.2 that $\mathrm{Im}(\mathcal{O}) \subset L \cap L'$. It follows that the rank of $\mathcal{O}$ is less than or equal to one on an open, dense subset of M.

Proposition 5.3. *Let (M, c) be an even-dimensional conformal manifold admitting a twistor spinor φ. Then, at each point*

$$\mathrm{Im}(\mathcal{O}) \subset \mathcal{L}_\varphi. \tag{59}$$

In particular, $\mathcal{O}$ vanishes if there are twistor spinors whose associated subspaces L are transversal on an open and dense subset of M.

Proof. Combining Theorem 1.1 with relation (10) yields that

$$s_- \cdot \mathcal{O}(X) \cdot \psi = 0. \tag{60}$$

Filling in the technical details how ψ is related to φ by means of a metric in the conformal class as done in [Leitner 2007] reveals that (60) is equivalent to

$$\mathcal{O}(X) \cdot \varphi(x) = 0 \quad \text{for } \varphi(x) \neq 0, \tag{61}$$

which is clearly equivalent to (59). □

We continue by combining Theorem 3.1 with the results in Section 4C. In the nongeneric case, i.e., when $\mathfrak{hol}(M, c) \neq \mathfrak{so}(p+1, q+1)$, Proposition 4.4 shows that the image of $\mathcal{O}$ is lightlike over an open dense set in M, and hence everywhere:

Corollary 5.4. *If $\mathfrak{hol}(M, c)$ is not generic, then* $\mathrm{Im}(\mathcal{O})$ *is totally lightlike. In particular, if (M, c) is Riemannian and $\mathfrak{hol}(M, c)$ is not generic, then $\mathcal{O} = 0$.*

The statement in Corollary 5.4 about Riemannian conformal structure can be pieced together from several results in the literature: The decomposition theorem in [Armstrong 2007] states that a conformal structure with holonomy reduced from $\mathfrak{so}(1, n+1)$, locally over an open dense subset of M, contains an Einstein metric or a certain product of Einstein metrics. Corollary 5.1 and the results in [Gover and Leitner 2009] about products of Einstein metrics then ensure that (M, c) admits an ambient metric whose Ricci tensor vanishes to infinite order, and hence that the obstruction tensor vanishes. Our proof of $\mathcal{O} = 0$ for Riemannian nongeneric conformal classes in Corollary 5.4 is self-contained and does not make use of the results in the literature.

We consider now several options for the rank of $\mathcal{O}$. From Proposition 4.5 we get:

Corollary 5.5. *If* $\mathrm{Hol}^0(M,c)$ *has no open orbit on the lightcone* $\mathcal{N}\subset\mathbb{R}^{p+1,q+1}$, *then* $\mathrm{rk}(\mathcal{O})\le 1$.

Indeed, if $\mathrm{Hol}^0(M,c)$ has no open orbit on the lightcone $\mathcal{N}\subset\mathbb{R}^{p+1,q+1}$, then by Proposition 4.5 the rank of $\mathcal{O}$ is ≤ 1 on an open dense set. Hence, the rank is ≤ 1 everywhere.

Again we refer to [Alt 2012], where conformal structures with a transitive and irreducible action of the conformal holonomy are classified. Moreover, Proposition 4.2 implies:

Corollary 5.6. *If the rank of* $\mathcal{O}$ *is maximal at some point* $x\in M$, *then* $\mathfrak{hol}(M,c)=\mathfrak{so}(p+1,q+1)$ *is generic. In particular, all parallel tractors are obtained from the tractor metric* h *only.*

Corollary 5.6 demonstrates that the ambient obstruction tensor $\mathcal{O}$ can also be interpreted as an obstruction to the existence of parallel tractors on (M,c) of any type. Namely for such a tractor to exist, $\mathcal{O}$ needs to have a nontrivial kernel everywhere. We analyze this phenomenon in more detail by focusing on parallel tractor forms and the associated normal conformal Killing forms (see Section 2C). Proposition 4.6 implies:

Corollary 5.7. *If* (M,c) *admits a nc-Killing form* $\alpha\in\Omega^k(M)$, *then* $\mathrm{Im}(\mathcal{O})\wedge\alpha=0$.

Corollary 5.8. *If* V *is a normal conformal vector field for* (M,c), *then* $\mathrm{Im}(\mathcal{O})\subset\mathbb{R}V$ *whenever* $V\neq 0$. *In particular,* $\mathcal{O}$ *vanishes if there is a normal conformal vector field that is not lightlike, or if the space of normal conformal vector fields has dimension greater than* 1.

In particular, Corollary 5.8 applies to Fefferman conformal structures (M,c) of signature $(2k+1,2r+1)$, i.e., $\mathrm{Hol}(M,c)\subset\mathrm{SU}(k+1,r+1)$. They admit a distinguished normal conformal Killing vector field V_F. Thus,

$$\mathrm{Im}\,\mathcal{O}\subset\mathbb{R}V_F, \tag{62}$$

for which an independent proof can be found in [Graham and Hirachi 2008]. For the Lorentzian case, i.e., $k=0$, any additional holonomy reduction will force $\mathcal{O}$ to vanish.

Proposition 5.9. *Let* (M,c) *be a Lorentzian conformal manifold of even dimension* n *with* $\mathfrak{hol}(M,c)\subsetneq\mathfrak{su}(1,\frac{n}{2})$. *Then* $\mathcal{O}=0$.

Proof. From the classification of irreducibly acting subalgebras of $\mathfrak{so}(2,n)$ in [Di Scala and Leistner 2011] and the results in [Alt et al. 2014] it follows that $\mathfrak{hol}(M,c)$ has to act with an invariant subspace. If the holonomy representation fixes a nondegenerate subspace or a lightlike line in $\mathbb{R}^{2,n}$ the result follows with Corollaries 5.1 and 5.2. Otherwise, $\mathfrak{hol}(M,c)$ fixes a totally lightlike 2-plane in

$\mathbb{R}^{2,n}$ and again Corollary 5.2 applies. That is, there is (at least locally) a metric $g \in c$ admitting a recurrent and nowhere vanishing null vector field U, i.e., $\nabla^g U = \theta \otimes U$ for some 1-form θ and $\mathrm{Im}(\mathcal{O}) \subset \mathbb{R}U$. Assume now that $\mathcal{O}$ is nonzero at some point. It follows from (62) that there is an open subset of M on which $\mathbb{R}V_F = \mathbb{R}U$. However, this contradicts the fact that the twist[1] of U is given by $\omega_U = \theta \wedge U^\flat \wedge U^\flat = 0$ but $\omega_{V_F} \neq 0$; see [Baum and Leitner 2004]. Thus, $\mathcal{O} \equiv 0$. □

Remark. In similar fashion, Fefferman spaces over quaternionic contact structures, see [Alt 2008], admit 3 linearly independent $\mathrm{Hol}(M, c)$-invariant almost complex structures which descend to pointwise linear independent nc-vector fields (or 1-forms) on M. Thus $\mathcal{O} \equiv 0$ for this case by Corollary 5.8.

5B. ***The obstruction tensor for Bryant conformal structures.*** We now specialize to Bryant conformal structures in signature (3, 3) induced by a generic 3-distribution $\mathcal{D} \subset TM$ as in Section 2C, and deduce several new results about the relation of the generic distribution $\mathcal{D}$ and the image of $\mathcal{O}$.

Every Bryant conformal structure admits (and is equivalently characterized by) a parallel tractor 4-form $\hat{\alpha} \in \Gamma(M, \Lambda^4\mathcal{T})$ whose stabilizer under the SO(4, 4)-action at each point is isomorphic to $\mathrm{Spin}(4, 3) \subset \mathrm{SO}(4, 4)$. In particular, $\mathrm{Hol}(M, c) \subset \mathrm{Spin}(4, 3)$. For a fixed metric $g \in c$ and the corresponding splitting (8), i.e.,

$$\hat{\alpha} = s_+^\flat \wedge \alpha + \alpha_0 + \cdots , \tag{63}$$

one finds that $\alpha = l_1^\flat \wedge l_2^\flat \wedge l_3^\flat$ for $l_{i=1,2,3}$ some basis of $\mathcal{D}$ and α transforms conformally covariantly under a change of g. Using this, we can derive constraints on the obstruction tensor for Bryant conformal structures.

As an immediate consequence of Proposition 4.6 and Corollary 5.7 we obtain:

Corollary 5.10. *Let $(M, c_{\mathcal{D}})$ be a Bryant conformal structure induced by a generic 3-distribution $\mathcal{D} \subset TM$. Then $\mathcal{E} \subset \mathcal{D}$, and in particular,* $\mathrm{Im}(\mathcal{O}) \subset \mathcal{D}$.

Moreover:

Corollary 5.11. *If $\mathfrak{hol}(M, c) = \mathfrak{spin}(4, 3)$, then $\mathcal{D} = \mathcal{E}$ everywhere on M.*

Proof. The Lie algebra $\mathfrak{spin}(4, 3)$ equals the stabilizer algebra of a spinor ψ of nonzero length in signature (4, 4) which corresponds via some $g \in c$ to a twistor spinor φ with $L_\varphi = \mathcal{D}$ at every point (see Section 2C). Thus, $(s_-^\flat \wedge l^\flat) \cdot \psi = 0$ for every $l \in \mathcal{D}$, i.e., $\mathcal{D} \subset \mathcal{E}$. □

Remark. This agrees with the curved orbit decomposition from [Čap et al. 2014], cf., the discussion in Section 4B for this particular case. Indeed, as discussed in that work for the general case, the curved orbits correspond to the Spin(4, 3)-orbits

[1] Recall that for a vector field $X \in \mathfrak{X}(M)$, its twist is the 3-form $\omega_X := dX^\flat \wedge X^\flat$. Clearly, the condition $d\omega_X = 0$ depends on $\mathbb{R}X$ only.

on $\mathrm{SO}(4,4)/\operatorname{Stab}_{\mathrm{SO}(4,4)}(\mathcal{L})$, where $\mathcal{L}\subset\mathbb{R}^{4,4}$ is a null line. However, there is only one such orbit as $\mathrm{Spin}(4,3)$ acts transitively on the projectivized lightcone in $\mathbb{R}^{4,4}$.

Proposition 5.12. *Assume that* $\mathfrak{hol}(M,c)\subsetneq\mathfrak{spin}(4,3)\subset\mathfrak{so}(4,4)$. *Then* $\operatorname{rk}(\mathcal{O})\le 1$.

Proof. Suppose first that there is an open set $U\subset M$ on which $\mathcal{E}$ has dimension 3, i.e., by Corollary 5.10 we have $\mathcal{E}=\mathcal{D}$ over U. By passing to a subset of U if necessary, we may assume that U is a $\mathcal{E}$-adapted open set. Let $V_{i=1,2,3}$ be a pointwise basis of $\mathcal{E}$ over U depending smoothly on x. Let V_i' be lightlike vector fields on U such that $g(V_i,V_j')=\delta_{ij}$. We have seen that in this case the 15 elements in (54) are pointwise linearly independent in $\mathfrak{hol}(M,c)\cap\mathfrak{p}$. But then it follows immediately from Proposition 4.5 that $\dim\mathfrak{hol}(M,c)\ge 15+6=21$, which is the dimension of $\mathfrak{spin}(4,3)$. Thus $\mathfrak{hol}(M,c)$ is no proper subalgebra of $\mathfrak{so}(4,4)$.

We have to conclude that the set on which $r^{\mathcal{E}}\le 2$ is open and dense in M. In particular, $\operatorname{rk}(\mathcal{O})<3$ on an open and dense subset of M. However, the set on which $\operatorname{rk}(\mathcal{O})<3$ is also closed and since M is connected it follows that $\operatorname{rk}(\mathcal{O})<3$ on M. Assume next that there is $x\in M$ such that $\operatorname{rk}(\mathcal{O})=2$ at x. Since the subset on which $\operatorname{rk}(\mathcal{O})\ge 2$ is open in M it follows that $\operatorname{rk}(\mathcal{O})=2$ on some open set U of M. After restricting U we may assume that U is a $\mathcal{E}$-adapted open set and $r^{\mathcal{E}}=2$ on U. Thus, $\mathcal{E}$ is over U a 2-dimensional subbundle of $\mathcal{D}$. By Theorem 4.9, $\mathcal{E}$ is integrable over U which contradicts $\mathcal{D}$ being generic. Consequently, $\operatorname{rk}(\mathcal{O})\le 1$ everywhere. □

Example. Proposition 5.12 applies to the situation when $\mathrm{Hol}(M,c)$ lies in the intersection of $\mathrm{Spin}(4,3)$ with the stabilizer of a totally degenerate subspace $\mathcal{H}\subset\mathbb{R}^{4,4}$. For $\dim\mathcal{H}=4$, this intersection is isomorphic to

$$\mathfrak{spin}(3,4)_{\mathcal{H}}=\left\{\begin{pmatrix} Z & X\\ 0 & -Z^{\top}\end{pmatrix}\,\middle|\, Z\in\mathfrak{csp}_2\mathbb{R},\ \ X\in\mathfrak{so}(4),\,\operatorname{tr}(X\boldsymbol{J})=0\right\},$$

where

$$\boldsymbol{J}=\begin{pmatrix} 0 & \mathbf{1}_2\\ -\mathbf{1}_2 & 0\end{pmatrix},$$

and

$$\mathfrak{csp}_2\mathbb{R}=\left\{Z\in\mathfrak{gl}_4\mathrm{R}\mid Z^{\top}\boldsymbol{J}+\boldsymbol{J}Z-\tfrac{1}{2}\operatorname{tr}(Z)\boldsymbol{J}=0\right\}=\mathbb{R}\mathbf{1}_4\oplus\mathfrak{sp}_2\mathbb{R}.$$

Moreover, since the Lie group $\mathrm{Spin}(3,4)\subset\mathrm{SO}(4,4)$ corresponding to $\mathfrak{spin}(3,4)\subset\mathfrak{so}(4,4)$ acts transitively on triples

$$\{(\mathrm{s}_+,\mathcal{H},\mathrm{s}_-)\mid\mathcal{H}\text{ a totally null 4-plane, }\mathrm{s}_+\in\mathcal{H},\ \mathrm{s}_-\in\mathbb{R}^8\text{ null, }g(\mathrm{s}_+,\mathrm{s}_-)=1\},$$

we can express the stabilizer of $\mathcal{H}$ in conjunction with the $|1|$-grading $\mathfrak{spin}(3,4) = \boldsymbol{g}_{-1} \oplus \boldsymbol{g}_0 \oplus \boldsymbol{g}_1$ in a basis $(\mathrm{s}_+, \mathrm{e}_a, \mathrm{s}_-, \mathrm{e}_{\bar{a}})$ for $a = 1, 2, 3$ and $\bar{a} = a + 3$, as

$$\mathfrak{spin}(3,4)_{\mathcal{H}} = \left\{ \left(\begin{array}{cc|cc} r & w^\top & 0 & \overline{w}^\top \\ v^\top & Z & -\overline{w} & X \\ \hline 0 & 0 & -r & -v \\ 0 & 0 & -w & -Z^\top \end{array} \right) \;\middle|\; \begin{array}{c} w = (w^a) \in \mathrm{R}^3,\ \overline{w} = (\overline{w}^{\bar{a}}) \in \mathbb{R}^3,\ v = (v_{\bar{b}}) \in (\mathbb{R}^3)^*, \\ X = (X^{\bar{b}}{}_a) \in \mathfrak{so}(3),\ Z = (Z^b{}_a) \in \mathfrak{gl}_3\mathbb{R}, \\ w^3 = Z^2{}_1,\ w^1 = -Z^2{}_3,\ v_1 = -Z^3{}_2,\ v_3 = Z^1{}_2, \\ r = Z^1{}_1 - Z^2{}_2 + Z^3{}_3,\ \overline{w}^2 = -X^1{}_3. \end{array} \right\}.$$

Here (r, Z, X) corresponds to the $\boldsymbol{g}_0$ part whereas $(w, \overline{w})$ correspond tot he $\boldsymbol{g}_{-1}$ and v to the $\boldsymbol{g}_1$-part. In particular, the intersection $\mathfrak{p}_{\mathcal{H}}$ of $\mathfrak{spin}(3,4)_{\mathcal{H}}$ with the parabolic $\mathfrak{p}$ is given by setting w and $\overline{w}$ to zero, and the intersection $\mathcal{E}$ of $\mathfrak{spin}(3,4)_{\mathcal{H}}$ with $\boldsymbol{g}_1$ by requiring in addition that $X = Z = r = 0$. Note that $\mathcal{E}$ is one dimensional.

In regards to examples of this situation, we recall that in [Anderson et al. 2015] a certain class of Bryant's conformal structures was studied. They are defined by a rank 3 distribution $\mathcal{D}_f$ on $\mathbb{R}^6$ with coordinates $(x^1, x^2, x^3, y^1, y^2, y^3)$ given by the annihilator of three 1-forms

$$\theta_1 = dy^1 + x^2 dx^3, \quad \theta_2 = dy^2 + f dx^1, \quad \theta_3 = dy^3 + x^1 dx^2,$$

where $f = f(x^1, x^2, x^3)$ is a differentiable function of the variables (x^1, x^2, x^3) only. It was shown that, whenever f depends only on x^3 and x^1, the corresponding conformal class contains a metric for which the image of the Schouten tensor lies in a parallel rank 3 distribution, which implies [Lischewski 2015] that the conformal holonomy is contained in $\mathfrak{spin}(3,4)_{\mathcal{H}}$. In addition, these conformal structures turned out to have vanishing obstruction tensor, and therefore they admit ambient metrics. For the conformal class defined by $\mathcal{D}_f$ with $f = x^1(x^3)^2$, an ambient metric with holonomy equal to $\mathfrak{spin}(3,4)_{\mathcal{H}}$ was found, and for this example also the conformal holonomy is equal to $\mathfrak{spin}(3,4)_{\mathcal{H}}$.

Remark. We point out that there is a large class of examples of Bryant conformal structures with f depending on three variables x^1, x^2, x^3 for which the obstruction tensor has rank 3, e.g., the one with $f = x^3 + x^1x^2 + (x^2)^2 + (x^3)^2$ in [Anderson et al. 2015]. From our Proposition 5.12 it follows that these examples have $\mathfrak{hol}(M, c_{\mathcal{D}_f}) = \mathfrak{spin}(4,3)$.

More difficult is the question of finding examples with $\mathrm{rk}(\mathcal{O}) = 1$. Of course, a general conformal structure with holonomy $\mathfrak{su}(2,2) \subset \mathfrak{spin}(4,3)$ has $\mathrm{rk}(\mathcal{O}) = 1$, but we are not aware of an explicit example with $\mathrm{rk}(\mathcal{O}) = 1$ and $\mathfrak{hol}(M, c_{\mathcal{D}}) \subset \mathfrak{spin}(4,3)_{\mathcal{H}}$. Other examples with $\mathrm{rk}(\mathcal{O}) = 1$, not necessarily with $\mathfrak{hol}(M, c_{\mathcal{D}}) \subset \mathfrak{spin}(4,3)$, are given by pp-waves and their generalization to arbitrary signature [Leistner and Nurowski 2010; Anderson et al. 2017].

Finally, Theorem 4.9 implies:

Corollary 5.13. *Suppose* (M, c) *is of signature* $(3, 3)$ *and* $\mathrm{rk}(\mathcal{O}) \leq 3$ *on some open set and* $\mathrm{Im}(\mathcal{O})$ *is not integrable. Then* $\mathfrak{hol}(M, c)$ *is either equal to* $\mathfrak{so}(4, 4)$ *or to* $\mathfrak{spin}(4, 3)$.

Proof. From the assumptions, $\mathrm{rk}(\mathcal{O}) \geq 2$ on an open set. If $\mathrm{rk}(\mathcal{O}) = 2$ on an open set, it follows from Theorem 4.9 that $\mathcal{E}$ must have dimension at least 3 on this set. Otherwise the image of $\mathcal{O}$ would be integrable. But then the statement follows from Theorem 4.9. Otherwise the set on which $\mathrm{rk}(\mathcal{O}) \geq 3$ is open and dense and the statement is an immediate consequence of Theorem 4.9. □

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