Pacific Journal of Mathematics

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Volume 290 No. 2 October 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.467

LENGTH-PRESERVING EVOLUTION OF IMMERSED CLOSED CURVES AND THE ISOPERIMETRIC INEQUALITY

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It is shown that all immersed closed, locally convex curves with total curvature of $2m\pi$ and n-fold rotational symmetry ($m/n \le 1$) finally evolve into m-fold circles under the length-preserving curvature flow. Sufficient conditions for the occurrence of the finite-time singularities in the flow are also established. As a byproduct, an isoperimetric inequality for rotationally symmetric, locally convex curves is proved via the flow method.

1. Introduction

In this paper we investigate the evolution of immersed closed curves $X(p,t)$ parametrized by p and driven by the inner normal speed

$$V(p,t) = \left(-\int_{X(\,\cdot\,,t)} k^2\,ds \Big/ \int_{X(\,\cdot\,,t)} k\,ds + k(p,t)\right)\boldsymbol{n}(p,t), \tag{1-1}$$

where $k(p,t)$ denotes the curvature of $X(p,t)$ with respect to inner normal $\boldsymbol{n}(p,t)$. Denote by X_0 the given smooth closed initial curve. When X_0 is a simple convex closed curve ($m = 1$), this flow has been studied by Ma and Zhu [2012]. It is shown that the flow preserves convexity and length while it increases the enclosed area, finally converging to a round circle in the C^∞ metric.

When X_0 is an immersed, locally convex closed curve, it is not difficult to show that the convexity and length of evolving curves are still preserved under the flow, and the enclosed algebraic area is increasing. Moreover, in [Wang and Wo 2014], two special classes of rotationally symmetric, locally convex closed initial curves, which both enclose a positive algebraic area, are found to guarantee the convergence of the flow (1-1) to m-fold circles. One class consists of highly symmetric convex curves. Specifically, they are locally convex closed curves with total curvature $2m\pi$ and n-fold rotational symmetry where $n > 2m$. The other is Abresch–Langer type convex curves, which still have total curvature of $2m\pi$ and n-fold rotational symmetry but with $n < 2m$ and some additional conditions on the curvature (see

MSC2010: 35B40, 35K59, 53C44.
Keywords: curvature flow, nonlocal, blow-up, convergence, isoperimetric inequality.

its definition in [Wang and Wo 2014]). Note that Abresch–Langer curves [1986] belong to the later class.

One may naturally ask about the behavior for a general rotationally symmetric curve under the flow (1-1). Furthermore, is there any possibility of the occurrence of singularity in the flow (1-1)? We devote this short paper to answering these questions. For the convenience of the reader, we use the following notation:

ds	the differential element of arclength,
θ	the normal angle of $X(\cdot,t)$,
$L(t)$	the length of $X(\cdot,t)$,
$A(t)$	the algebraic area of $X(\cdot,t)$ defined by $-\dfrac{1}{2}\displaystyle\int_X \langle X, \boldsymbol{n}\rangle\, ds$,
$k(\cdot,t)$	the curvature of $X(\cdot,t)$ with respect to $\boldsymbol{n}$.

Here, we always take the orientation of $X(\cdot,t)$ to be counterclockwise.

Define

$$\bar{k} = \frac{\int_X k^2\, ds}{\int_X k\, ds} = \frac{\int_X k^2\, ds}{2m\pi}.$$

We write down the evolution of various geometric quantities along the flow (1-1). They can be deduced from the general formulas in [Chou and Zhu 2001].

$$\frac{\partial k}{\partial t} = k_{ss} + k^2(k-\bar{k}), \quad \frac{dL}{dt} = -\int_X k(k-\bar{k})\, ds = 0, \quad \frac{dA}{dt} = -\int_X (k-\bar{k})\, ds \ge 0.$$

Here, it can be seen that the length of the evolving curves is preserved while the enclosed algebraic area is increasing.

Each point on the locally convex solution $X(\cdot,t)$ has a unique tangent and one can use the tangent angle $\theta \in S^1_m := \mathbb{R}/2m\pi\mathbb{Z}$ to parametrize it. Generally speaking, θ is a function depending on t. One can make θ independent of time t by adding a tangential component to the velocity vector $\partial X/\partial t$, which does not affect the geometric shape of the evolving curve (see, for instance, [Gage 1986]). Then the evolution equations can be expressed in the coordinates of θ and t. If we denote by $k(\theta,t)$ the curvature function of $X(\theta,t)$, the evolution problem of (1-1) can be reformulated equivalently into equations of the curvature k:

$$\begin{cases} k_t = k^2(k_{\theta\theta} + k - \bar{k}), & (\theta,t) \in I \times (0,T), \\ k(\theta,0) = k_0(\theta), & \theta \in I, \end{cases} \tag{1-2}$$

where k_0 is the curvature of X_0 and T is the maximal existence time of the flow. Here and after, I always denotes the circle S^1_m. In terms of the new coordinates, we have

$$\bar{k} = \frac{\int_I k(\theta,t)\, d\theta}{2m\pi}.$$

The first main theorem is:

Theorem 1. *If the initial curve is locally convex, closed and its curvature $k_0(\theta)$ satisfies*

$$\int_I (k_0 - \bar{k}_0)^2 \, d\theta \geq \int_I (k_{0\theta})^2 \, d\theta, \tag{1-3}$$

and $k_0(\theta)$ is nonconstant in I, then the solution $k(\theta, t)$ to problem (1-2) blows up in some finite time and a singularity appears during the evolution of the flow (1-1).

We note that the condition (1-3) is not void since by the Poincaré inequality,

$$\int_I (k_0 - \bar{k}_0)^2 \, d\theta \leq m^2 \int_I (k_{0\theta})^2 \, d\theta.$$

If the curvature of initial curve does not satisfy (1-3), how about the behavior of the flow? In fact, we find a large class of initial curves which do not satisfy the condition (1-3) and can evolve into m-fold circles under the flow. This is our second main theorem.

Theorem 2. *If the initial curve is locally convex, closed and has total curvature of $2m\pi$ and n-fold rotational symmetry with $m/n \leq 1$, then the flow (1-1) exists globally and converges to an m-fold circle in the C^∞-metric as time goes to infinity.*

When the initial curve is simple closed and convex, it can be regarded as the case of $m = n = 1$ in Theorem 2. In addition, its curvature cannot satisfy the condition (1-3) except by being a constant, in view of the Poincaré inequality.

The third theorem gives an isoperimetric condition such that the singularity appears.

Theorem 3. *Assume the initial curve X_0 is locally convex, closed and has total curvature of $2m\pi$. If X_0 satisfies*

$$L_0^2 < 4m\pi A_0, \tag{1-4}$$

where L_0 and A_0 denote its length and enclosed algebraic area respectively, then the solution $k(\theta, t)$ to problem (1-2) blows up in some finite time and a singularity appears during the evolution of the flow (1-1)

As a result, we can present a new proof of the following isoperimetric inequality for the rotationally symmetric and locally convex curves, which was proven in [Chou 2003] and [Süssmann 2011]:

Proposition 4. *For the rotationally symmetric and locally convex curves, with total curvature of $2m\pi$ and n-fold symmetry ($m/n < 1$), the length L and the enclosed algebraic area A satisfy*

$$L^2 \geq 4m\pi A. \tag{1-5}$$

We give some remarks on the above theorems and the nonlocal flow. As an interesting variant of the popular curve shortening flow [Gage and Hamilton 1986; Angenent 1991; Andrews 1998; Chou and Zhu 2001], the nonlocal curvature flow, arising in many application fields [Sapiro and Tannenbaum 1995; Capuzzo Dolcetta et al. 2002; Xu and Yang 2014], such as phase transitions, image processing, etc., has received much attention in recent years. Before the work of Ma and Zhu [2012], there was an original study by Gage [1986], where an area-preserving flow was investigated with its inner normal velocity given by

$$V = \left(-\int_{X(\,\cdot\,,t)} k\,ds \Big/ \int_{X(\,\cdot\,,t)} ds + k\right)\boldsymbol{n}. \tag{1-6}$$

After that, there are a lot of papers on the nonlocal flow for simple convex curves, including [Jiang and Pan 2008; Lin and Tsai 2012]. In the higher dimensional case, people also consider nonlocal flows. For example, there are volume-preserving mean curvature flows; see [Huisken 1987; McCoy 2005; Cabezas-Rivas and Sinestrari 2010]. And also there are surface area-preserving mean curvature flows, see [McCoy 2003]. Recently, the study of nonlocal flow extends to the case of Riemannian manifolds; see [Xu et al. 2014].

In all of the papers mentioned above, the main concern is the global existence and convergence of the flow. For a study of the singularity, one can refer to [Escher and Ito 2005], or to [Wang and Kong 2014], where the area-preserving flow of immersed curves is studied and some geometric initial conditions are given to guarantee the occurrence of singularity. This urges us to carry the present work on the length-preserving flow of immersed curves.

One interesting aspect of this paper is that we have obtained the sufficient conditions for the flow (1-1) to yield the singularity. Moreover, the geometric condition (1-4) given in Theorem 3 can be interpreted as

$$\int_I (h_0 - \bar{h}_0)^2\,d\theta > \int_I (h_{0\theta})^2\,d\theta, \tag{1-7}$$

where $h_0(\theta)$ is the support function of the initial curve X_0, defined by $h_0(\theta) = -\langle X_0(\theta), \boldsymbol{n}_0(\theta)\rangle$ with $\boldsymbol{n}_0$ being the inner normal of X_0, and $\bar{h}_0 = \int_I h_0\,d\theta/(2m\pi)$. Indeed, we can deduce (1-7) from the following observations:

$$k_0 = (h_0 + h_{0\theta\theta})^{-1}, \quad L_0 = \int_I \frac{d\theta}{k_0} = \int_I h_0\,d\theta,$$

and

$$A_0 = \frac{1}{2}\int_{X_0} h_0\,ds = \frac{1}{2}\int_I h_0(h_0 + h_{0\theta\theta})\,d\theta,$$

where k_0 is the curvature function of X_0.

Another interesting aspect is that we have refined the results of [Wang and Wo 2014] in Theorem 2 and showed that the convergence result holds for all rotationally symmetric, locally convex immersed curves whether the enclosed algebraic area A_0 of the initial curve X_0 is negative or not. This differs with the flow (1-6), since a singularity must happen in the flow (1-6) if $A_0 < 0$, see [Escher and Ito 2005] for reference. One may also compare it with the different evolution of rotationally symmetric curves in the curve shortening flow, see [Au 2010].

We organize this paper in the following way. Some basic and useful lemmas are prepared in Section 2. Then we prove Theorems 1 and 2 in Section 3, and prove Theorem 3 in Section 4.

2. Lemmas

In this section, we present some lemmas for later use. The first lemma shows the flow exists as long as its curvature is bounded.

Lemma 2.1. *When the initial curve is immersed closed, locally convex and smooth, problem (1-1) has a unique smooth, locally convex solution in a time interval* $[0, T)$ *for some* $T > 0$, *which can be continued as long as the curvature of evolving curves is finite.*

Proof. The unique existence of the flow can be proven by applying the classical Leray–Schauder fixed point theorem to problem (1-2). See details in [Mao et al. 2013], where a general area-preserving flow is studied. One can also find the relative references in [McCoy 2003; 2005; Cabezas-Rivas and Sinestrari 2010], where the nonlocal flows in higher dimensions are discussed. The preserved convexity will be proved in the next lemma. □

By the maximum principle, we can show that the local convexity of the initial curve is preserved by the flow (1-1).

Lemma 2.2. *If the initial curve* X_0 *is locally convex, then* $X(\cdot, t)$ *is locally convex as long as the flow exists.*

Proof. By the continuity, $\min_{\theta\in I} k(\theta, t)$ remains positive on a small time interval. Assume that the time span of the flow is T. Suppose to the contrary that the conclusion is not true. Then there must be a first time, say $t_1 < T$, such that

$$\min_{\theta\in I} k(\theta, t_1) = 0. \tag{2-1}$$

We will deduce a contradiction. Consider the quantity

$$\Phi(\theta, t) = \frac{1}{k(\theta, t)} - \frac{L(t)}{2m\pi} - \frac{1}{2m\pi}\int_0^t \int_0^{2m\pi} k(\theta, \tau)\, d\theta\, d\tau,$$

with $(\theta, t) \in I \times [0, t_1)$. By (1-2), we have

$$\Phi_t(\theta, t) = -k_{\theta\theta} - k \le k^2(\theta, t)\Phi_{\theta\theta}(\theta, t).$$

Hence by the maximum principle,

$$\frac{1}{k(\theta, t)} \le \max_{\theta\in I}\Bigl(\frac{1}{k_0(\theta)}\Bigr) + \frac{L(t)-L(0)}{2m\pi} + \frac{1}{2m\pi}\int_0^t \int_0^{2m\pi} k(\theta, \tau)\, d\theta\, d\tau$$

for all $(\theta, t) \in I \times [0, t_1)$, where we note that $L(t) = L(0)$ for all time t and

$$\sup_{(\theta,t)\in I\times[0,t_1)} k(\theta, t) \le C_1(t_1) < \infty$$

for some constant $C_1(t_1)$. Therefore,

$$\inf_{\theta\in I} k(\theta, t) \ge C_2(t_1) > 0 \quad \text{for all } \ t \in [0, t_1)$$

for some constant $C_2(t_1)$. This is a contraction with (2-1)! The proof is done. □

The following lemma is the gradient estimate.

Lemma 2.3. *Along the flow* (1-1), *we have*

$$\int_I (k_\theta)^2\, d\theta \le \int_I k^2\, d\theta + C$$

for some constant C independent of time.

Proof. From (1-2), we have

$$\frac{1}{2}\frac{d}{dt}\int_I [(k_\theta)^2 - k^2 + 2\bar{k}k]\, d\theta = -\int_I k^2(k_{\theta\theta} + k - \bar{k})^2 + \frac{d\bar{k}}{dt}\int_I k\, d\theta \le \frac{d\bar{k}}{dt}\int_I k\, d\theta.$$

Hence,

$$\frac{d}{dt}\int_I (k_\theta)^2\, d\theta \le \frac{d}{dt}\int_I (k^2 - 2\bar{k}k)\, d\theta + 2\frac{d\bar{k}}{dt}\int_I k\, d\theta,$$

and the integration yields

$$\begin{aligned}
\int_I (k_\theta)^2 &\le \int_I (k^2 - 2\bar{k}k)\, d\theta + \frac{1}{2m\pi}\int_0^t \frac{d}{d\tau}\Bigl(\int_I k\, d\theta\Bigr)^2 d\tau + C_1 \\
&= \int_I (k^2 - 2\bar{k}k)\, d\theta + \frac{1}{2m\pi}\Bigl(\int_I k\, d\theta\Bigr)^2 + C_2 \\
&= \int_I k^2\, d\theta - \bar{k}\int_I k\, d\theta + C_2 \\
&\le \int_I k^2\, d\theta + C_2,
\end{aligned}$$

where C_1, C_2 only depend on the initial data. The proof is done. □

By the obtained gradient estimate, if the curvature k blows up, we can show that the blow-up set for k must contain at least some open interval.

Denote

$$k_{\max}(t) = \max_{\theta \in I} k(\theta, t), \quad t \in [0, T).$$

Lemma 2.4. *Assume that $k_{\max}(t) = k(\theta_t, t)$ for some $\theta_t \in [0, 2m\pi]$. Then for any small $\varepsilon > 0$, there exists a number $\delta > 0$, depending only on ε, such that*

$$(1-\varepsilon) k_{\max}(t) \le k(\theta, t) + \sqrt{2m\pi |C|}$$

for all $\theta \in (\theta_t - \delta^2, \theta_t + \delta^2)$ and all $t \in (0, T)$, where C is the constant in Lemma 2.3.

Proof. An easy integration combined with the Hölder inequality shows that

$$k_{\max}(t) = k(\theta, t) + \int_\theta^{\theta_t} k_\theta(\theta, t)\, d\theta \le k(\theta, t) + |\theta_t - \theta|^{1/2} \left(\int_\theta^{\theta_t} k_\theta^2 \, d\theta \right)^{1/2}.$$

Then from Lemma 2.3 we have

$$\begin{aligned} k_{\max}(t) &\le k(\theta, t) + |\theta_t - \theta|^{1/2} \left(\int_I k^2 \, d\theta + |C| \right)^{1/2} \\ &\le k(\theta, t) + |\theta_t - \theta|^{1/2} (2m\pi k_{\max}^2(t) + |C|)^{1/2} \\ &\le k(\theta, t) + |\theta_t - \theta|^{1/2} \sqrt{2m\pi}\, k_{\max}(t) + |\theta_t - \theta|^{1/2} |C|^{1/2} \\ &\le k(\theta, t) + |\theta_t - \theta|^{1/2} \sqrt{2m\pi}\, k_{\max}(t) + \sqrt{2m\pi |C|}. \end{aligned}$$

Take δ such that $|\theta_t - \theta|^{1/2} \le \delta := \varepsilon / \sqrt{2m\pi}$ and the lemma is proved. □

We need the following lemma, proven in [Wang and Wo 2014], to conclude the convergence of the flow after we obtain the a priori estimate for the curvature.

Lemma 2.5. *If there is a constant C independent of time such that*

$$\max_{\theta \in I} k(\theta, t) \le C, \quad t \in [0, T),$$

with T being the maximal existence time, then the flow (1-1) must exist for all time and converge smoothly to an m-fold circle as time goes to infinity.

3. Proofs of Theorems 1 and 2

First, we deduce a sufficient condition for the occurrence of the singularity at some finite time. The following two lemmas are useful in the proof.

Lemma 3.1. *If the flow (1-1) exists for all time, then there exists a sequence $\{t_j\}_{j=1}^{\infty} \to \infty$ such that*

$$\int_I k(\theta, t_j)\, d\theta \le C$$

for some constant C independent of time.

Proof. We have

$$\frac{dA}{dt} = -\int_I (k - \bar{k})\, ds = \frac{L(t)}{2m\pi}\int_I k\, d\theta - 2m\pi \geq 0.$$

Since an isoperimetric inequality of Rado (see [Osserman 1978]) says that

$$L(t)^2 \geq 4\pi A(t)$$

and $L(t) = L_0$, we know that $A(t)$ is uniformly bounded from above. Notice that $A(t)$ is increasing in time. We have $\int_0^\infty (dA/d\tau)\, d\tau < \infty$. Thus for any small $\varepsilon > 0$, there exists a sequence $\{t_j\}_{j=1}^\infty \to \infty$, such that

$$\frac{dA}{dt}(t_j) < \varepsilon,$$

that is,

$$\int_I k(\theta, t_j)\, d\theta < \frac{2m\pi}{L_0}(\varepsilon + 2m\pi).$$

Then we can draw the conclusion by fixing an $\varepsilon > 0$. □

Denote

$$E(t) = \int_I (k_\theta)^2\, d\theta - \int_I k^2\, d\theta + \frac{1}{2m\pi}\left(\int_I k\, d\theta\right)^2.$$

That is,

$$E(t) = \int_I (k_\theta)^2\, d\theta - \int_I (k - \bar{k})^2\, d\theta.$$

Lemma 3.2. *For the energy $E(t)$ defined as above, we have*

$$\frac{dE(t)}{dt} \leq 0.$$

Proof. From the equation (1-2), we have

$$\int_I \frac{(k_t)^2}{k^2}\, d\theta = \int_I (k_{\theta\theta} + k - \bar{k}) k_t\, d\theta = -\frac{1}{2}\frac{d}{dt}\int_I [(k_\theta)^2 - k^2]\, d\theta - \bar{k}\int_I k_t\, d\theta,$$

where

$$\bar{k}\int_I k_t\, d\theta = \frac{d}{dt}\int_0^t \bar{k}(\tau)\int_I k_\tau\, d\theta d\tau = \frac{1}{4m\pi}\frac{d}{dt}\int_0^t \frac{d}{d\tau}\left(\int_I k\, d\theta\right)^2 d\tau.$$

Thus,

$$-\frac{1}{2}\frac{dE(t)}{dt} = \int_I \frac{(k_t)^2}{k^2}\, d\theta \geq 0,$$

and the proof is done. □

Proof of Theorem 1. Using the equation (1-2) and integrating by parts yield

$$\frac{d}{dt}\int_I \ln k \, d\theta = \int_I k(k_{\theta\theta} + k - \bar{k}) \, d\theta = -E(t).$$

From Lemma 3.2, we have

$$\frac{d}{dt}\int_I \ln k \, d\theta \geq -E(0) = -\int_I (k_{0\theta})^2 \, d\theta + \int_I (k_0 - \bar{k}_0)^2 \, d\theta.$$

First, we consider the case of $E(0) < 0$. If we suppose to the contrary the flow exists for all time, then $\lim_{t\to\infty} \int_I \ln k \, d\theta = \infty$. This implies that for any $t > 0$, we can find a $\theta_t \in I$, such that $\lim_{t\to\infty} k(\theta_t, t) = \infty$. Then by Lemma 2.4, we have $\lim_{t\to\infty} \int_I k(\theta, t) \, d\theta = \infty$, which is a contradiction to Lemma 3.1. Thus the flow must exist for some finite time.

If $E(0) = 0$, we claim that $k_{0\theta\theta} + k_0 - \bar{k}_0 \neq 0$ must hold at some point of I and hence in some interval of I by the continuity. Indeed, if $k_{0\theta\theta} + k_0 - \bar{k}_0 = 0$ holds everywhere in I, we set $w = k_0 - \bar{k}_0$ and w satisfies

$$w_{\theta\theta} + w = 0 \quad \text{in } I,$$

which implies that w is a 2π-periodic function and so is k_0. Hence, $E(0)$=0 tells us that k_0 is a constant function in view of the Poincaré inequality, a contradiction with the assumption! Thus we have shown that $k_{0\theta\theta} + k_0 - \bar{k}_0 \neq 0$ must hold in some interval of I. Then by recalling the proof of Lemma 3.2, we have

$$\frac{dE(t)}{dt} = -2\int_I \frac{(k_t)^2}{k^2} \, d\theta < 0,$$

which implies that $E(t) < 0$ for $t > 0$. At last, we can still show the conclusion holds via a similar method to the one above. The proof is finished. □

One may naturally ask what happens if the condition (1-3) does not hold for the initial curve. A large class of rotationally symmetric curves belong to this case. In fact, the Poincaré inequality tells us the following lemma:

Lemma 3.3. *If a curve is locally convex, closed and has total curvature of* $2m\pi$ *and* n*-fold rotational symmetry with* $m/n \leq 1$, *then its curvature* $k(\theta)$ *satisfies*

$$\int_I (k - \bar{k})^2 \, d\theta \leq \left(\frac{m}{n}\right)^2 \int_I (k_\theta)^2 \, d\theta.$$

Proof. By the Poincaré inequality, we have

$$\int_0^{2m\pi/n} (k - \bar{k})^2 \, d\theta \leq \left(\frac{m}{n}\right)^2 \int_0^{2m\pi/n} (k_\theta)^2 \, d\theta,$$

and then the conclusion follows. □

Proof of Theorem 2. By equation (1-2) and integration by parts, we have

$$\frac{d}{dt}\int_I \ln k\, d\theta = \int_I k(k_{\theta\theta}+k-\bar{k})\, d\theta = -\int_I (k_\theta)^2\, d\theta + \int_I (k-\bar{k})^2\, d\theta.$$

From Lemma 3.3, we have $d\bigl(\int_I \ln k\, d\theta\bigr)/dt \le 0$. Thus there is a constant C_1 independent of time, such that $\int_I \ln k(\theta,t)\, d\theta \le C_1$ for all $t \in [0,T)$. This implies that there is a constant C_2 independent of time, such that

$$\max_{\theta\in I} k(\theta,t) \le C_2 \tag{3-1}$$

for all $t \in [0,T)$. Indeed, for $m/n < 1$, using Lemma 3.3 and the fact that $E(t) \le E(0)$, we can deduce an estimate of k_θ, which implies (3-1) holds. As a result of the a priori estimate (3-1), we can show the flow's global existence and its smooth convergence to an m-fold circle as time goes to infinity by using Lemma 2.5. □

4. Proof of Theorem 3

To prove Theorem 3, we need to show the following lemma holds, which states a subconvergence of the global flow without any a priori estimate on the curvature like that in Lemma 2.5.

Lemma 4.1. *If the flow (1-1) starts from a locally convex closed curve and exists for all time, then it subconverges to an m-fold circle in C^2 sense, that is, there exists a time sequence $\{t_j\}_{j=1}^{\infty} \to \infty$ such that $k(\theta,t_j)$ converges to a positive constant function in the L^∞ norm.*

Proof. Notice that a careful choice of $\{t_j\}_{j=1}^{\infty}$ in Lemma 3.1 can guarantee that $(dA/dt)(t_j) \to 0$ as $j \to \infty$, that is,

$$\frac{L_0}{2m\pi}\int_I k(\theta,t_j)\, d\theta \to 2m\pi, \quad j\to\infty. \tag{4-1}$$

We claim that along the sequence $\{t_j\}_{j=1}^{\infty}$ we have

$$\max_{\theta\in I} k(\theta,t_j) \le C_1 \tag{4-2}$$

for some constant C_1 independent of time. Suppose $\limsup_{j\to\infty} \max_{\theta\in I} k(\theta,t_j) = \infty$. Then we can find a subsequence, still denoted by $\{t_j\}_{j=1}^{\infty}$, and a sequence $\{\theta_j\}_{j=1}^{\infty} \subset I$, such that $t_j \to \infty$ and $k(\theta_j,t_j) \to \infty$. By Lemma 2.4, $\int_I k(\theta,t_j)\, d\theta \to \infty$, contradicting Lemma 3.1! Thus we have (4-2). Furthermore, by Lemma 2.3,

$$\int_I (k_\theta)^2(\theta,t_j)\, d\theta \le C_2, \tag{4-3}$$

for some constant C_2 independent of time. Combining (4-2) with (4-3) we obtain

$$\|k(\,\cdot\,,t_j)\|_{W^{1,2}(I)} \le C_3$$

for some constant C_3 independent of time. The compactness yields a subsequence of $\{k(\theta, t_j)\}_{j=1}^{\infty}$, still denoted by $\{k(\theta, t_j)\}_{j=1}^{\infty}$, which converges to a continuous function $k_\infty(\theta)$ in the L^∞ norm as $j \to \infty$. Taking the limit in (4-1) along the time sequence $\{t_j\}_{j=1}^{\infty}$, we have

$$\frac{L_0}{2m\pi}\int_I k_\infty(\theta)\,d\theta = 2m\pi. \tag{4-4}$$

By Fatou's lemma,

$$\int_I \frac{d\theta}{k_\infty(\theta)} \le \int_I \frac{d\theta}{k(\theta, t_j)} = L_0. \tag{4-5}$$

Thus, substituting (4-5) into (4-4) yields

$$\int_I k_\infty\,d\theta \int_I \frac{d\theta}{k_\infty(\theta)} \le (2m\pi)^2.$$

We notice that

$$(2m\pi)^2 = \left(\int_I 1\,d\theta\right)^2 \le \int_I k_\infty\,d\theta \int_I \frac{d\theta}{k_\infty}.$$

Thus k_∞ must be a constant function, i.e., the sequence $\{k(\theta, t_j)\}_{j=1}^{\infty}$ converges to a constant function in L^∞ norm as $j \to \infty$. □

Proof of Theorem 3. Assume the initial curve satisfies

$$L_0^2 < 4m\pi A_0.$$

Since $dL(t)/dt \equiv 0$ and $dA(t)/dt \ge 0$, we have $L_0 = L(\infty) := \lim_{t\to\infty} L(t)$ and $A_0 \le A(\infty) := \lim_{t\to\infty} A(t)$. Thus,

$$L^2(\infty) < 4m\pi A(\infty). \tag{4-6}$$

Suppose to the contrary that the flow exists for all time. Then by Lemma 4.1 the flow converges to an m-fold circle along some time sequence $\{t_j\}_{j=1}^{\infty} \to \infty$, implying

$$L^2(\infty) = 4m\pi A(\infty).$$

This contradicts (4-6)! Thus, the singularity must happen at some finite time during the evolution of the flow. □

As a result of Theorem 3, we can give a proof for Proposition 4.

Proof of Proposition 4. On one hand, by Theorem 2 the flow (1-1) starting from such rotationally symmetric curves must converge to m-fold circles at $t \to \infty$. However, on the other hand, if (1-5) does not hold, then by Theorem 3 there is a finite-time singularity during the evolution. This contradiction shows (1-5) holds. □

Acknowledgments

We thank the referees for their careful reading and useful comments which have helped improve the presentation of this paper. Our work is partially supported by the Fundamental Research Funds for the Central Universities 2242015R30012, the NSF of China 11101078, the NSF of China 11171064 and the Natural Science Foundation of Jiangsu Province BK20161412.

References

[Abresch and Langer 1986] U. Abresch and J. Langer, "The normalized curve shortening flow and homothetic solutions", *J. Differential Geom.* **23**:2 (1986), 175–196. MR Zbl

[Andrews 1998] B. Andrews, "Evolving convex curves", *Calc. Var. Partial Differential Equations* **7**:4 (1998), 315–371. MR Zbl

[Angenent 1991] S. Angenent, "On the formation of singularities in the curve shortening flow", *J. Differential Geom.* **33**:3 (1991), 601–633. MR Zbl

[Au 2010] T. K.-K. Au, "On the saddle point property of Abresch–Langer curves under the curve shortening flow", *Comm. Anal. Geom.* **18**:1 (2010), 1–21. MR Zbl

[Cabezas-Rivas and Sinestrari 2010] E. Cabezas-Rivas and C. Sinestrari, "Volume-preserving flow by powers of the mth mean curvature", *Calc. Var. Partial Differential Equations* **38**:3-4 (2010), 441–469. MR Zbl

[Capuzzo Dolcetta et al. 2002] I. Capuzzo Dolcetta, S. Finzi Vita, and R. March, "Area-preserving curve-shortening flows: from phase separation to image processing", *Interfaces Free Bound.* **4**:4 (2002), 325–343. MR Zbl

[Chou 2003] K.-S. Chou, "A blow-up criterion for the curve shortening flow by surface diffusion", *Hokkaido Math. J.* **32**:1 (2003), 1–19. MR Zbl

[Chou and Zhu 2001] K.-S. Chou and X.-P. Zhu, *The curve shortening problem*, Chapman & Hall, Boca Raton, FL, 2001. MR Zbl

[Escher and Ito 2005] J. Escher and K. Ito, "Some dynamic properties of volume preserving curvature driven flows", *Math. Ann.* **333**:1 (2005), 213–230. MR Zbl

[Gage 1986] M. Gage, "On an area-preserving evolution equation for plane curves", pp. 51–62 in *Nonlinear problems in geometry* (Mobile, AL, 1985), edited by D. M. DeTurck, Contemp. Math. **51**, American Mathematical Society, Providence, RI, 1986. MR Zbl

[Gage and Hamilton 1986] M. Gage and R. S. Hamilton, "The heat equation shrinking convex plane curves", *J. Differential Geom.* **23**:1 (1986), 69–96. MR Zbl

[Huisken 1987] G. Huisken, "The volume preserving mean curvature flow", *J. Reine Angew. Math.* **382** (1987), 35–48. MR Zbl

[Jiang and Pan 2008] L. Jiang and S. Pan, "On a non-local curve evolution problem in the plane", *Comm. Anal. Geom.* **16**:1 (2008), 1–26. MR Zbl

[Lin and Tsai 2012] Y.-C. Lin and D.-H. Tsai, "Application of Andrews and Green–Osher inequalities to nonlocal flow of convex plane curves", *J. Evol. Equ.* **12**:4 (2012), 833–854. MR Zbl

[Ma and Zhu 2012] L. Ma and A. Zhu, "On a length preserving curve flow", *Monatsh. Math.* **165**:1 (2012), 57–78. MR Zbl

[Mao et al. 2013] Y. Mao, S. Pan, and Y. Wang, "An area-preserving flow for closed convex plane curves", *Internat. J. Math.* **24**:4 (2013), art. id. 1350029, 31 pp. MR Zbl

[McCoy 2003] J. McCoy, "The surface area preserving mean curvature flow", *Asian J. Math.* **7**:1 (2003), 7–30. MR Zbl

[McCoy 2005] J. A. McCoy, "Mixed volume preserving curvature flows", *Calc. Var. Partial Differential Equations* **24**:2 (2005), 131–154. MR Zbl

[Osserman 1978] R. Osserman, "The isoperimetric inequality", *Bull. Amer. Math. Soc.* **84**:6 (1978), 1182–1238. MR Zbl

[Sapiro and Tannenbaum 1995] G. Sapiro and A. Tannenbaum, "Area and length preserving geometric invariant scale-spaces", *IEEE Trans. Pattern Anal. Machine Intel.* **17**:1 (1995), 67–72.

[Süssmann 2011] B. Süssmann, "Isoperimetric inequalities for special classes of curves", *Differential Geom. Appl.* **29**:1 (2011), 1–6. MR Zbl

[Wang and Kong 2014] X.-L. Wang and L.-H. Kong, "Area-preserving evolution of nonsimple symmetric plane curves", *J. Evol. Equ.* **14**:2 (2014), 387–401. MR Zbl

[Wang and Wo 2014] X. Wang and W. Wo, "Length-preserving evolution of non-simple symmetric plane curves", *Math. Methods Appl. Sci.* **37**:6 (2014), 808–816. MR Zbl

[Xu and Yang 2014] G. Xu and X. Yang, "Construction of several second- and fourth-order geometric partial differential equations for space curves", *Comput. Aided Geom. Design* **31**:2 (2014), 63–80. MR Zbl

[Xu et al. 2014] H. Xu, Y. Leng, and E. Zhao, "Volume-preserving mean curvature flow of hypersurfaces in space forms", *Internat. J. Math.* **25**:3 (2014), art. id. 1450021, 19 pp. MR Zbl

Received January 21, 2016. Revised February 8, 2017.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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