Pacific Journal of Mathematics

Volume 290 No. 2 October 2017

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.257

NONCONTRACTIBLE HAMILTONIAN LOOPS IN THE KERNEL OF SEIDEL'S REPRESENTATION

SÍLVIA ANJOS AND RÉMI LECLERCQ

The main purpose of this note is to exhibit a Hamiltonian diffeomorphism loop undetected by the Seidel morphism of a 1-parameter family of 2-point blow-ups of $S^2 \times S^2$, exactly one of which is monotone. As side remarks, we show that Seidel's morphism is injective on all Hirzebruch surfaces, and discuss how to adapt the monotone example to the Lagrangian setting.

1. Introduction

The motivation for this work is the search for homotopy classes of loops of Hamiltonian diffeomorphisms which are not detected by Seidel's morphism. Given a symplectic manifold (M, ω) and its Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$, recall that Seidel's morphism

$$\mathcal{S} : \pi_1(\mathrm{Ham}(M, \omega)) \to \mathrm{QH}_*(M, \omega)^\times$$

was defined on a covering of $\pi_1(\mathrm{Ham}(M, \omega))$ by Seidel [1997] for strongly semipositive symplectic manifolds and then on the fundamental group itself and for any closed symplectic manifold by Lalonde, McDuff and Polterovich [1999].

The target space, $\mathrm{QH}_*(M, \omega)^\times$, is the group of invertible elements of the quantum homology of (M, ω). More precisely, the small quantum homology of (M, ω) is $\mathrm{QH}_*(M, \omega) = H_*(M; \mathbb{Z}) \otimes \Pi$, where Π is equal to $\Pi^{\mathrm{univ}}[q, q^{-1}]$, with q a variable of degree 2 and the ring Π^{univ} consisting of generalized Laurent series in a variable t of degree 0:

$$\Pi^{\mathrm{univ}} := \Big\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \;\Big|\; r_\kappa \in \mathbb{Q} \text{ and } \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \text{ for all } c \in \mathbb{R} \Big\}. \tag{1}$$

Since its construction, Seidel's morphism has been successfully used to detect many Hamiltonian loops (see, e.g., [McDuff 2010]), and was extended or generalized

The authors would like to thank Dusa McDuff for her interest and useful discussions. Anjos is partially funded by FCT/Portugal through project UID/MAT/04459/2013 and project EXCL/MAT-GEO/0222/2012. Leclercq is partially supported by ANR Grant ANR-13-JS01-0008-01.
MSC2010: primary 53D45; secondary 53D05, 57S05.
Keywords: symplectic geometry, Seidel morphism, toric symplectic manifolds, Hirzebruch surfaces.

to various situations (see, e.g., [Hutchings 2008; Savelyev 2008; Hu and Lalonde 2010; Hu et al. 2011; Fukaya et al. 2017]). One particular extension consists of secondary-type invariants, whose construction is based on Seidel's construction after enriching Floer homology by considering Leray–Serre spectral sequences introduced by Barraud and Cornea [2007], and which should detect loops undetected by Seidel's morphism [Barraud and Cornea $\geq$ 2017]. However, there were no Hamiltonian loops with nontrivial homotopy class known to be undetected by Seidel's morphism (as far as we know). This short note intends to provide the first example of such a loop on a family of symplectic manifolds. Moreover, the example is explicit and thus can easily be used to test other constructions. Notice finally that this example can also be used to construct other examples (e.g., by products, see [Leclercq 2009]).

First try: symplectically aspherical manifolds. Looking for elements in the kernel of the Seidel morphism, one might first consider symplectically aspherical manifolds, by which we mean that both the symplectic form and the first Chern class vanish on the second homotopy group of the manifold. Indeed, such manifolds have trivial Seidel morphism.

The geometric reason for this is that, by construction, the Seidel morphism of (M, ω) counts pseudo-holomorphic section classes of a fibration over S^2 with fiber (M, ω). The difference between two such classes is thus given by elements of $\pi_2(M)$ admitting a pseudo-holomorphic representative, whose existence is prevented by symplectic asphericity.

Alternatively, this can be proved via purely algebraic methods, using the equivalent description of Seidel's morphism, as a representation of $\pi_1(\mathrm{Ham}(M, \omega))$ into the Floer homology of (M, ω). Given a loop of Hamiltonian diffeomorphisms, one gets an automorphism of $\mathrm{HF}_*(M, \omega)$ which can be shown to act trivially by using the following facts:

(i) Morse homology (the quantum homology of symplectically aspherical manifolds) is a ring over which Floer homology is a module.

(ii) All involved morphisms (PSS, Seidel, continuation) are module morphisms.

(iii) Any automorphism of Morse homology preserves the fundamental class, since it generates the top degree homology group.

(iv) The fundamental class is the unit of the Morse homology ring.

This line of ideas, which goes back to Seidel, has been used by McDuff and Salamon [2004] to simplify Schwarz's original proof of invariance of spectral invariants. It has been adapted by Leclercq [2008] to Lagrangian spectral invariants and used to prove the triviality of the relative (i.e., Lagrangian) Seidel morphism by Hu, Lalonde and Leclercq [2011] (see Lemma 5.5).

Now, even though aspherical manifolds seem to be ideal candidates, there are no homotopically nontrivial loops of Hamiltonian diffeomorphisms known to the authors in such manifolds.

Second try: symplectic toric manifolds. Symplectic toric geometry provides a large class of natural examples of symplectic manifolds which are complicated enough to be interesting while simple enough that many rather involved constructions can be explicitly performed. In [Anjos and Leclercq 2015], we computed the Seidel morphism on NEF toric 4-manifolds following work of McDuff and Tolman [2006]. Recall that by definition (M, J) is an NEF pair if there are no J-pseudo-holomorphic spheres in M with negative first Chern number. This gave, in the particular case of 4-dimensional toric manifolds, an elementary and somehow purely symplectic way to perform these computations previously obtained by Chan, Lau, Leung, and Tseng [2017] (and using works by Fukaya, Oh, Ohta, and Ono [2016], and González and Iritani [2012]). We also showed that one could then deduce the Seidel morphism of some non-NEF symplectic manifolds and, as an example, we made explicit computations for some Hirzebruch surface.

The easiest symplectic toric 4-manifolds for which we can exhibit a nontrivial element in the kernel of the Seidel morphism are 2-point blow-ups of $S^2 \times S^2$. More precisely, start with the monotone product $(S^2 \times S^2, \omega_1)$[1] on which we perform two blow-ups. Notice that the resulting symplectic manifold is monotone only when the respective sizes of the blow-ups coincide and are equal to $\frac{1}{2}$.

In Section 4, we exhibit a specific loop of Hamiltonian diffeomorphisms whose homotopy class is in the kernel of Seidel's morphism if and only if the size of the two blow-ups coincide. Since this loop, obtained from two circle actions, can easily be seen to be nontrivial (Anjos and Pinsonnault [2013] computed the rational homotopy of symplectomorphism groups of these manifolds), this obviously yields a family of symplectic manifolds, only one of which is monotone, with noninjective Seidel morphism.

Theorem 1.1. *The Seidel morphism of the* 2*-point blow-ups of* $(S^2 \times S^2, \omega_1)$ *with blow-ups of equal* (*arbitrary*) *sizes is not injective.*

In our search for undetected Hamiltonian loops, we realized the following:

Theorem 1.2. *Seidel's morphism is injective on all Hirzebruch surfaces.*

While this is not hard to prove and might be well known to experts, we did not find it in the literature and therefore include a proof in Section 3.

[1] Traditionally, ω_μ denotes the product symplectic form with total area $\mu \geq 1$ on the first factor and area 1 on the second one.

Discussion on the adaptation to the Lagrangian setting. As mentioned above, there is a relative (i.e., Lagrangian) version of the Seidel morphism defined by Hu and Lalonde [2010] and further studied by Hu, Lalonde and Leclercq [2011]. There are two ways to adapt the example of Theorem 1.1 to the Lagrangian setting which we discuss here. (However, in order to keep this note short, and to avoid too many technical details on the standard tools involved, we will not investigate these ideas further here.)

First, let us remark that to get the Lagrangian version of the Seidel morphism, we need to consider a monotone Lagrangian of minimal Maslov at least 2. So, in what follows, we have in mind the only monotone symplectic manifold of the family mentioned above, i.e., the monotone product $S^2 \times S^2$ with the area of each factor equal to 1 on which we perform two blow-ups of size $\frac{1}{2}$.

The first way to relate absolute and relative settings is to consider the diagonal of the symplectic product. More precisely, let (M, ω) be a monotone symplectic manifold. The diagonal $\Delta \simeq M$ is a monotone Lagrangian of the product $(M \times M, \omega \oplus (-\omega))$, which we denote $(\widehat{M}, \widehat{\omega})$ for short, with minimal Maslov number equal to twice the minimal first Chern number of (M, ω) and thus greater than or equal to 2. This allows us to consider the Lagrangian Seidel morphism:

$$\mathcal{S}_\Delta : \pi_1\big(\mathrm{Ham}(\widehat{M}, \widehat{\omega}), \mathrm{Ham}_\Delta(\widehat{M}, \widehat{\omega})\big) \to \mathrm{QH}_*(\Delta)^\times,$$

where Ham_Δ denotes the subgroup of Ham formed by Hamiltonian diffeomorphisms which preserve Δ, and $\mathrm{QH}_*(\Delta)$ denotes the Lagrangian quantum homology of Δ.

An element $\phi \in \pi_1(\mathrm{Ham}(M, \omega))$ generated by the Hamiltonian $H : M \times [0, 1] \to \mathbb{R}$, induces $\widehat{\phi} \in \pi_1(\mathrm{Ham}(\widehat{M}, \widehat{\omega}), \mathrm{Ham}_\Delta(\widehat{M}, \widehat{\omega}))$, generated by $\widehat{F} = F \oplus 0 : \widehat{M} \times [0, 1] \to \mathbb{R}$. To get an element in the kernel of the Lagrangian Seidel morphism, it only remains to prove that

(i) $\mathcal{S}(\phi) = \mathcal{S}_\Delta(\widehat{\phi})$ in $\mathrm{QH}_*(M, \omega) \simeq \mathrm{QH}_*(\Delta)$, and

(ii) $\widehat{\phi}$ is nonzero.

Note that in (i), not only are the quantum homologies canonically identified but the chain complexes themselves coincide and this identification agrees with the PSS morphisms in the following sense:

$$\begin{tikzcd}
\mathrm{QH}_*(M, \omega) \arrow[r, equal] \arrow[d, "\mathrm{PSS}"'] & \mathrm{QH}_*(\Delta) \arrow[d, "\mathrm{PSS}"] \\
\mathrm{HF}_*(H, J) \arrow[r, equal] & \mathrm{HF}_*(\widehat{H}, \widehat{J} : \Delta)
\end{tikzcd}$$

as proved in the monotone setting by Leclercq and Zapolsky [2017] (J denotes an almost complex structure on M, compatible with and tamed by ω, while $\widehat{J}$

denotes an almost complex structure on $\widehat{M}$ adapted to J). This suggests that it is straightforward to show that (i) holds.

On the other hand, proving (ii) will require a different technique.

The second way to the Lagrangian setting is to use Albers's comparison map [2008] between Hamiltonian and Lagrangian Floer homologies, denoted below by $\mathcal{A}$, which relates the absolute and relative Seidel morphisms via the following commutative diagram (see [Hu and Lalonde 2010]):

$$\begin{tikzcd}
\pi_1(\mathrm{Ham}(M,\omega)) \arrow[r] \arrow[d, "\mathcal{S}"'] & \pi_1(\mathrm{Ham}(M,\omega), \mathrm{Ham}_L(M,\omega)) \arrow[r] \arrow[d, "\mathcal{S}_L"] & \pi_0(\mathrm{Ham}_L(M,\omega)) \\
\mathrm{HF}_*(M,\omega) \arrow[r, "\mathcal{A}"'] & \mathrm{HF}_*(M,\omega;L) &
\end{tikzcd}$$

where L is a closed monotone Lagrangian of (M,ω) with minimal Maslov number at least 2.

To get an interesting example via this method, one must choose L such that $\mathrm{HF}_*(M,\omega;L) \neq 0$ and prove (again) that the image of $\phi \in \pi_1(\mathrm{Ham}(M,\omega))$ in $\pi_1(\mathrm{Ham}(M,\omega), \mathrm{Ham}_L(M,\omega))$ is nontrivial.

2. Background and user manual for Sections 3 and 4

In order to prove Theorems 1.1 and 1.2 in the following sections, we need to describe the setting and give some information whose nature we now explain. We also give some details about previous works on which it relies.

Step A: Geometric setting. We will first introduce the symplectic toric 4-manifold (M,ω) in which we are interested and describe the associated circle actions, moment map, and polytope. Then we will give topological information which will be useful:

- the fundamental group of $\mathrm{Ham}(M,\omega)$, on which the Seidel morphism is defined, and
- the second homology group of M, which consists of generators of the quantum homology of (M,ω) (as a module over the Novikov ring).

Background for Step A. (See [Cannas da Silva 2001] for more details.) First, consider a Hamiltonian circle action on (M,ω). It is generated by a function $\phi : M \to \mathbb{R}$, called the *moment map*, which is assumed to be normalized, that is, satisfying

$$\int_M \phi\, \omega^n = 0.$$

Now (M,ω) is called *toric* if it admits an effective action by a Hamiltonian torus $\mathbb{T}^2 \subset \mathrm{Ham}(M,\omega)$. We will denote by Φ the corresponding moment map and by $P = \Phi(M)$ the moment polytope. If η is an outward primitive normal to the facet

D_η of P, we consider the associated Hamiltonian circle action, Γ_η, whose moment map is $\phi := \langle \eta, \Phi(\cdot) \rangle$.[2]

Note that $\phi^{-1}(D_\eta)$ is a *semifree* maximum component for Γ_η, as the action is semifree (i.e., the stabilizer of every point is trivial or the whole circle) on some neighborhood of $\phi^{-1}(D_\eta)$.

Step B: The Seidel morphism. In this step, we will give the expression of the image of the aforementioned circle actions Γ_η via the Seidel morphism, $\mathcal{S}$.

Background for Step B. (See [McDuff and Tolman 2006; Anjos and Leclercq 2015].)

We consider a toric 4-manifold (M, ω, Φ) as above. To compute the image of a Hamiltonian circle action via the Seidel morphism, we pick a ω-compatible, S^1-invariant almost complex structure, J. The main case we are concerned with here is the Fano case. Recall that (M, J) is said to be *Fano* if any J-pseudo holomorphic sphere in M has positive first Chern number.

When this is the case, [McDuff and Tolman 2006, Theorem 1.10] or [Anjos and Leclercq 2015, Theorem 4.5] tells us that the associated Seidel element consists of only one term (the one of highest order). More precisely:

Theorem 2.1 [McDuff and Tolman 2006, Theorem 1.10]. *Let (M, ω, J, Φ) be a compact Fano toric symplectic 4-manifold. Let η be an outward primitive normal to the facet D_η of the moment polytope P and let Γ_η be the associated Hamiltonian circle action. Then*

$$\mathcal{S}(\Gamma_\eta) = [F_{\max}] \otimes q t^{\phi_{\max}},$$

where ϕ is the moment map associated to Γ_η, and $F_{\max} = \phi^{-1}(D_\eta)$ is the maximal fixed point component of ϕ and $\phi_{\max} = \phi(F_{\max})$.

Step C: The quantum homology of (M, ω). The computation of the Seidel elements $\mathcal{S}(\Gamma_\eta)$ in Step B also gives us explicit relations involving the quantum product. This allows us to complete the description of the quantum homology as an algebra. Since the generators of $\pi_1(\mathrm{Ham}(M, \omega))$ can be expressed in terms of the Γ_η, this also gives us the image of the Seidel morphism so that, by understanding $\mathrm{im}(\mathcal{S}) \subset \mathrm{QH}_*(M, \omega)^\times$, we can prove Theorems 1.1 and 1.2.

Background for Step C. (See [McDuff and Tolman 2006, Section 5.1] for the general setting.) Let us recall how to obtain the quantum homology algebra in our specific setting. Let $D_1, \ldots, D_n$ be the facets of P and $\eta_1, \ldots, \eta_n \in \mathbb{R}^2$ the respective outward primitive integral normal vectors. Let C be the set of *primitive sets*, i.e., subsets $I = \{i_1, i_2\} \subset \{1, \ldots, n\}$ such that $D_{i_1} \cap D_{i_2} = \varnothing$. Let $u_i = [D_i] \otimes q$.

[2]To lighten the notation, we will actually denote by D_i and Γ_i, respectively, the facet and the circle action associated to the normal η_i (instead of D_{η_i} and Γ_{η_i}).

There are two linear relations,

$$\sum_{i=1}^{n}\langle(1,0),\eta_i\rangle u_i = 0 \quad\text{and}\quad \sum_{i=1}^{n}\langle(0,1),\eta_i\rangle u_i = 0,$$

which generate the ideal of linear relations $\mathrm{Lin}(P)$ in $\mathbb{Q}[u_1,\ldots,u_n]$. Moreover, relations between the normal vectors η_i yield equations satisfied by the corresponding Seidel elements $\mathcal{S}(\Gamma_i)$. Using these, it is then possible to exhibit the quantum product $u_{i_1} * u_{i_2}$, for every primitive set $\{i_1,i_2\}$, as a linear combination of the classes p (the class of a point), $\mathbb{1}$ (the fundamental class), and u_i: $f_{i_1i_2} = (\alpha p\otimes q^2 + \beta\mathbb{1} + \sum\alpha_i u_i)t^{\gamma}$ for some $\alpha,\beta,\alpha_i\in\mathbb{Z}$ and $\gamma\in\mathbb{R}$. Then, the Stanley–Reisner ideal is defined by

$$\mathrm{SR}_Y(P) = \langle u_{i_1}\cdot u_{i_2} - f_{i_1i_2} \mid \{i_1,i_2\}\in C\rangle.$$

Finally, there is an isomorphism of Π^{univ}-algebras

$$\mathrm{QH}_*(M,\omega) \simeq \mathbb{Q}\,[u_1,\ldots,u_n]\otimes\Pi^{\mathrm{univ}}/(\mathrm{Lin}(P)+\mathrm{SR}_Y(P)). \tag{2}$$

3. Hirzebruch surfaces

We proceed in two steps as the "even" and "odd" Hirzebruch surfaces have to be dealt with separately. Throughout the section, we follow the notation and conventions used in [Anjos and Leclercq 2015] (in particular in Section 5.3), most of them having been recalled in Section 2 above.

3.1. ***Even Hirzebruch surfaces.*** Recall that the toric "even" Hirzebruch surfaces $(\mathbb{F}_{2k},\omega_\mu)$, $0\le k\le\ell$ with $\ell\in\mathbb{N}$ and $\ell<\mu\le\ell+1$, can be identified with the symplectic manifolds $M_\mu=(S^2\times S^2,\omega_\mu)$ where ω_μ is the split symplectic form with area $\mu\ge 1$ for the first S^2-factor, and with area 1 for the second factor. The moment polytope of $\mathbb{F}_{2k}$ is

$$P_{2k} = \{(x_1,x_2)\in\mathbb{R}^2 \mid 0\le x_1\le 1,\ x_2+kx_1\ge 0,\ x_2-kx_1\le\mu-k\}.$$

Let $\Lambda^{2k}_{e_1}$ and $\Lambda^{2k}_{e_2}$ represent the circle actions whose moment maps are, respectively, the first and second components of the moment map associated to the torus action T_{2k} acting on $\mathbb{F}_{2k}$. We will also denote by $\Lambda^{2k}_{e_1}$ and $\Lambda^{2k}_{e_2}$ the corresponding generators in $\pi_1(T_{2k})$.

It is well known (see, e.g., [Abreu and McDuff 2000, Theorem 1.1 or Corollary 2.7]) that for $k=0$, $\pi_1(\mathrm{Ham}(\mathbb{F}_0,\omega_\mu))=\mathbb{Z}/2\oplus\mathbb{Z}/2$ and that for $k\ge 1$, $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k},\omega_\mu))=\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}$. Moreover, the authors explain in [Abreu and McDuff 2000] (see Section 2.5 and in particular Lemma 2.10) that the $\mathbb{Z}/2$ terms of the fundamental groups are respectively generated by $\Lambda^0_{e_1}$ and $\Lambda^0_{e_2}$, while the generator of the additional $\mathbb{Z}$ term is $\Lambda^2_{e_1}$.

Let $B = [S^2 \times \{p\}]$ and $F = [\{p\} \times S^2] \in H_2(S^2 \times S^2; \mathbb{Z})$ and denote $u = B \otimes q$ and $v = F \otimes q$ where q is the degree 2 variable entering into play in the definition of $\Pi = \Pi^{\mathrm{univ}}[q, q^{-1}]$ and Π^{univ} is the ring of generalized Laurent series defined by (1).

We now gather from [Anjos and Leclercq 2015] the results we will need for the proof of Theorem 1.2 in this case. First, in Section 5.3 of that paper, we computed the image of the generators $\Lambda^0_{e_1}$, $\Lambda^0_{e_2}$, and $\Lambda^2_{e_1}$ by the Seidel morphism, $\mathcal{S}$. Namely, we obtained:

$$\begin{aligned} &\mathcal{S}(\Lambda^0_{e_1}) = B \otimes q t^{\frac{1}{2}} = u t^{\frac{1}{2}}, \quad \mathcal{S}(\Lambda^0_{e_2}) = F \otimes q t^{\frac{\mu}{2}} = v t^{\frac{\mu}{2}}, \text{ and} \\ &\mathcal{S}(\Lambda^2_{e_1}) = (B+F) \otimes q t^{\frac{1}{2}-\epsilon} = (u+v) t^{\frac{1}{2}-\epsilon}, \text{ with } \epsilon = \frac{1}{6\mu}. \end{aligned} \tag{3}$$

Note that the circle action $\Lambda^2_{e_1}$ acts on the second Hirzebruch surface $\mathbb{F}_2$ and the almost complex structure in this case is not Fano, because the class $B - F$ is represented by a pseudo-holomorphic sphere and its first Chern number vanishes. Nevertheless, by Theorem 4.4 in [Anjos and Leclercq 2015], the Seidel element of this action still does not contain any lower order terms.

The computation of the Seidel elements associated to each one of the facets of the polytope yields the quantum product identities

$$F * F = \mathbb{1} \otimes q^{-2} t^{-\mu}, \quad B * B = \mathbb{1} \otimes q^{-2} t^{-1}, \quad \text{and} \quad F * B = p, \tag{4}$$

so $\mathcal{S}(\Lambda^0_{e_1})^2 = \mathcal{S}(\Lambda^0_{e_2})^2 = \mathbb{1}$. Finally recall that, thanks to [Anjos and Leclercq 2015, Proposition 5.1] (see (2) in our setting), we were able to express the (small) quantum homology algebra as

$$\mathrm{QH}_*(\mathbb{F}_{2k}, \omega_\mu) \simeq \Pi^{\mathrm{univ}}[u, v] \,/\, \langle u^2 = t^{-1}, v^2 = t^{-\mu} \rangle.$$

From (3) and (4), it is now easy to check that the inverse of $\mathcal{S}(\Lambda^2_{e_1})$ is given by

$$\mathcal{S}(\Lambda^2_{e_1})^{-1} = (B - F) \otimes q \frac{t^{\frac{1}{2}+\epsilon}}{1 - t^{1-\mu}} = (u - v) \frac{t^{\frac{1}{2}+\epsilon}}{1 - t^{1-\mu}}. \tag{5}$$

Let us now prove the theorem.

Proof of Theorem 1.2 for even Hirzebruch surfaces. Since $\Lambda^0_{e_1}$ and $\Lambda^0_{e_2}$ are of order 2, any element in $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k}, \omega_\mu))$ is of the form $\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2} + \ell \Lambda^2_{e_1}$, with ε_1 and ε_2 in $\{0, 1\}$ and $\ell \in \mathbb{Z}$. Moreover, it is in the kernel of $\mathcal{S}$ if and only if $\mathcal{S}(\Lambda^2_{e_1})^{-\ell} = \mathcal{S}(\Lambda^0_{e_1})^{\varepsilon_1} \mathcal{S}(\Lambda^0_{e_2})^{\varepsilon_2}$, which is equivalent to the fact that $\mathcal{S}(\Lambda^2_{e_1})^{-\ell}$ is either u, v, or uv, up to a power of t.

Let $\ell' \in \mathbb{N} \setminus \{0\}$, and expand the ℓ'-th power of $\mathcal{S}(\Lambda^2_{e_1})$ (whose expression is recalled in (3) above) using the binomial theorem to get

$$\mathcal{S}(\Lambda^2_{e_1})^{\ell'} = \sum_{k=0}^{\ell'} \binom{\ell'}{k} u^k v^{\ell'-k} t^{(\frac{1}{2}-\epsilon)\ell'}.$$

The identities $u^2 = t^{-1}$ and $v^2 = t^{-\mu}$ ensure $\mathcal{S}(\Lambda^2_{e_1})^{\ell'}$ is of the form $C_1 \cdot u + C_2 \cdot v$ if ℓ' is odd, or $C_1 + C_2 \cdot uv$ otherwise, where (in both cases) C_1 and C_2 are linear combinations of powers of t with positive rational coefficients (hence nonzero), so

$$\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2} + \ell \Lambda^2_{e_1} \notin \ker(\mathcal{S})$$

for any $\varepsilon_1$and ε_2 in $\{0, 1\}$ and $\ell < 0$.

We proceed along the same lines for a positive ℓ: $\mathcal{S}(\Lambda^2_{e_1})^{-\ell}$ is, by the binomial theorem together with (5), of the form

$$\frac{C'_1 \cdot u - C'_2 \cdot v}{(1-t^{1-\mu})^\ell} \quad \text{or} \quad \frac{C'_1 - C'_2 \cdot uv}{(1-t^{1-\mu})^\ell},$$

which shows that $\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2} + \ell \Lambda^2_{e_1}$ is not in $\ker(\mathcal{S})$ for any $\ell > 0$ either.

This implies that the only elements of $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k}, \omega_\mu))$ which could be in $\ker(\mathcal{S})$ are of the form $\varepsilon_1 \Lambda^0_{e_1} + \varepsilon_2 \Lambda^0_{e_2}$ so that in the end $\ker(\mathcal{S}) = \{0\}$. $\square$

3.2. ***Odd Hirzebruch surfaces.*** Similarly, "odd" Hirzebruch surfaces $(\mathbb{F}_{2k-1}, \omega'_\mu)$, $1 \le k \le \ell$ with $\ell \in \mathbb{N}$ and $\ell < \mu \le \ell + 1$, can be identified with the symplectic manifolds $(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2, \omega'_\mu)$ where the symplectic area of the exceptional divisor is $\mu > 0$ and the area of the projective line is $\mu + 1$. Its moment polytope is

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le x_1 + x_2 \le 1,\ x_2(k-1) + kx_1 \ge 0, \\ kx_2 + (k-1)x_1 \ge k - \mu - 1 \end{array} \right\}.$$

Let $\Lambda^{2k-1}_{e_1}$ and $\Lambda^{2k-1}_{e_2}$ represent the circle actions whose moment maps are, respectively, the first and the second component of the moment map associated to the torus action T_{2k-1} acting on $\mathbb{F}_{2k-1}$. As before, we will also denote by $\Lambda^{2k-1}_{e_1}$ and $\Lambda^{2k-1}_{e_2}$ the generators of $\pi_1(T_{2k-1})$.

Similarly to the even case the fundamental group of $(\mathbb{F}_{2k-1}, \omega'_\mu)$ is computed in [Abreu and McDuff 2000, Theorem 1.4 or Corollary 2.7]. More precisely, $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k-1}, \omega'_\mu)) = \mathbb{Z}\langle \Lambda^1_{e_1} \rangle$ for all $k \ge 1$, that is, $\Lambda^1_{e_1}$ is the generator of the fundamental group as explained in [Abreu and McDuff 2000, Section 2.5 (in particular Lemma 2.11)]. So, in order to prove that the Seidel morphism is injective, we only need to show that the order of $\mathcal{S}(\Lambda^1_{e_1})$ in $\mathrm{QH}_*(\mathbb{F}_{2k+1}, \omega'_\mu)$ is infinite.

We now need to expand Remark 5.6 of [Anjos and Leclercq 2015] (which quickly dealt with the odd case), along the lines of [Anjos and Leclercq 2015, Section 5.3] (where we focused in more detail on the even case). Let $B \in H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2; \mathbb{Z})$ denote the homology class of the exceptional divisor with self intersection -1 and F the class of the fiber of the fibration $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \to S^2$. If we set $u_1 = (B + F) \otimes q$, $u_2 = u_4 = F \otimes q$, and $u_3 = B \otimes q$, clearly the additive relations are given by

$$u_2 = u_4 \quad \text{and} \quad u_1 = u_2 + u_3. \tag{6}$$

The normal vectors to the moment polytope of $\mathbb{F}_1$ are given by $\eta_1 = (1,1)$, $\eta_2 = (0,-1)$, $\eta_3 = (-1,-1)$, and $\eta_4 = (-1,0)$. We denote by Γ_i the actions associated to η_i.

As explained in Section 2, since $\mathbb{F}_1$ is Fano, it follows from [McDuff and Tolman 2006, Theorem 1.10] that the Seidel elements associated to the Γ_i are given by

$$\begin{aligned}
\mathcal{S}(\Gamma_1) &= (B+F)\otimes q t^{1+\mu-2\varepsilon} = u_1 t^{1+\mu-2\varepsilon},\\
\mathcal{S}(\Gamma_2) &= \mathcal{S}(\Gamma_4) = F\otimes q t^{\varepsilon} = u_2 t^{\varepsilon},\\
\mathcal{S}(\Gamma_3) &= B\otimes q t^{2\varepsilon-\mu} = u_3 t^{2\varepsilon-\mu},
\end{aligned}$$

with $\varepsilon = (3\mu^2+3\mu+1)/(3(1+2\mu))$.

The relation $\eta_1+\eta_3=0$ yields $\mathcal{S}(\Gamma_1)*\mathcal{S}(\Gamma_3)=\mathbb{1}$, that is, $B*(B+F)\otimes q^2 t=\mathbb{1}$. Similarly, since $\eta_2+\eta_4=\eta_3$ it follows that $\mathcal{S}(\Gamma_2)*\mathcal{S}(\Gamma_4)=\mathcal{S}(\Gamma_3)$, which is equivalent to $F*F = B\otimes q^{-1}t^{-\mu}$. Therefore the primitive relations are given by

$$u_1u_3 = t^{-1} \quad \text{and} \quad u_2u_4 = u_3 t^{-\mu}. \tag{7}$$

Now, following Step C of Section 2 above, we set $u = F\otimes q$ and deduce from the relations (6) and (7) that

$$\mathrm{QH}_*(\mathbb{F}_{2k+1},\omega'_\mu) = \Pi^{\mathrm{univ}}[u]/(u^4t^{2\mu}+u^3t^{\mu}-t^{-1}). \tag{8}$$

Note that $\Lambda^1_{e_1}$, the generator of $\pi_1(\mathrm{Ham}(\mathbb{F}_{2k-1},\omega'_\mu))$, is the action associated to the vector $(1,0)$. We thus get that $\mathcal{S}(\Lambda^1_{e_1}) = \mathcal{S}(\Gamma_4)^{-1}$.

Now we can proceed with the proof of the theorem.

Proof of Theorem 1.2 for odd Hirzebruch surfaces. From the discussion above, we see that $\mathcal{S}(\Lambda^1_{e_1})^{-1} = \mathcal{S}(\Gamma_4) = ut^{\varepsilon}$. So, in order to show that Seidel's morphism is injective we only need to show that

$$\mathcal{S}(\ell\Lambda^1_{e_1})^{-1} = u^{\ell}t^{\ell\varepsilon} \neq 1$$

for any $\ell\in\mathbb{N}\setminus\{0\}$.

First, note the polynomial $M(u)=u^4t^{2\mu}+u^3t^{\mu}-t^{-1}\in\Pi^{\mathrm{univ}}[u]$ in (8) above has invertible main coefficient, so that for any positive integer ℓ, there exist uniquely determined polynomials Q_ℓ and R_ℓ such that $u^\ell t^{\ell\varepsilon}-1 = M(u)Q_\ell(u)+R_\ell(u)$ and the degree of R_ℓ is less than the degree of M.

Assume Seidel's morphism is not injective: then there exists $\ell_0\in\mathbb{N}\setminus\{0\}$ such that $R_{\ell_0}=0$. To find the polynomial Q_{ℓ_0}, we proceed to the long division of $u^{\ell_0}t^{\ell_0\varepsilon}-1$ by M which consists of a finite number (at most ℓ_0-3) of steps. This ensures that the coefficients of Q_{ℓ_0} are *finite* linear combinations of powers of t (with rational coefficients). Therefore Q_{ℓ_0} induces a polynomial $Q^1_{\ell_0}$ in $\mathbb{Q}[u]$ when t is set to 1, satisfying $u^{\ell_0}-1=(u^4+u^3-1)Q^1_{\ell_0}(u)$ in $\mathbb{Q}[u]$. Since the roots of u^4+u^3-1 are not roots of unity, we get a contradiction. So, there is no positive integer ℓ_0 such that $u^{\ell_0}t^{\ell_0\varepsilon}=1$, which concludes the proof. □

4. 2-point blow-ups of $S^2 \times S^2$

We now consider the manifold obtained from

$$(M_\mu, \omega_\mu) = (S^2 \times S^2, \omega_\mu)$$

(see Section 3.1) by performing two successive symplectic blow-ups of capacities c_1 and c_2 with $0 < c_2 \le c_1 < c_1 + c_2 \le 1 \le \mu$, which we denote by $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$. Let B, $F \in H_2(M_{\mu,c_1,c_2}; \mathbb{Z})$ be the homology classes defined by $B = [S^2 \times \{p\}]$ and $F = [\{p\} \times S^2]$ and let $E_i \in H_2(M_{\mu,c_1,c_2}; \mathbb{Z})$ be the exceptional class corresponding to the blow-up of capacity c_i.

Remark 4.1. There is an alternative description of this manifold as the 3-point blow-up of $\mathbb{CP}^2$. Indeed, consider $\mathbb{X}_3 = \mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$ equipped with the symplectic form $\omega_{\nu;\delta_1,\delta_2,\delta_3}$ obtained from the symplectic blow-up of $(\mathbb{CP}^2, \omega_\nu)$ at three disjoint balls of capacities δ_1, δ_2 and δ_3, where ω_ν is the standard Fubini–Study form on $\mathbb{CP}^2$ rescaled so that $\omega_\nu(\mathbb{CP}^1) = \nu$. Let $\{L, V_1, V_2, V_3\}$ be the standard basis of $H_2(\mathbb{X}_3; \mathbb{Z})$ consisting of the class L of a line together with the classes V_i of the exceptional divisors. It is well known that $\mathbb{X}_3$ is diffeomorphic to M_{μ,c_1,c_2}. The diffeomorphism $\mathbb{X}_3 \to M_{\mu,c_1,c_2}$ can be chosen to map the ordered basis $\{L, V_1, V_2, V_3\}$ to $\{B + F - E_1, B - E_1, F - E_1, E_2\}$. When one considers this birational equivalence in the symplectic category, uniqueness of symplectic blow-ups implies that $(\mathbb{X}_3, \omega_{\nu;\delta_1,\delta_2,\delta_3})$ is symplectomorphic, after rescaling, to M_μ blown-up with capacities c_1 and c_2, where $\mu = (\nu - \delta_2)/(\nu - \delta_1)$, $c_1 = (\nu - \delta_1 - \delta_2)/(\nu - \delta_1)$, and $c_2 = \delta_3/(\nu - \delta_1)$. In Section 2.1 of [Anjos and Pinsonnault 2013], it is explained why it is sufficient to consider values of c_1 and c_2 in the range above: $0 < c_2 \le c_1 < c_1 + c_2 \le 1 \le \mu$. ◄

The quantum algebra of $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ was computed by Entov and Polterovich [2008] (as $(\mathbb{X}_3, \omega_{\nu;\delta_1,\delta_2,\delta_3})$, see their proof of Proposition 4.3). More precisely, setting $u = (F - E_2) \otimes q$ and $v = (B - E_2) \otimes q$, they proved that:

Lemma 4.2. *As a Π^{univ}-algebra we have*

$$\mathrm{QH}_*(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2}) \cong \Pi^{\text{univ}}[u, v]/I_{\mu,c_1,c_2}$$

where I_{μ,c_1,c_2} is the ideal generated by

$$u^2v^2 + u^2vt^{-c_2} = vt^{-\mu-c_2} + t^{c_1-\mu-1-c_2} \text{ and}$$
$$u^2v^2 + uv^2t^{-c_2} = ut^{-1-c_2} + t^{c_1-\mu-1-c_2}.$$

We recall here parts of this computation, using the formalism of [Anjos and Leclercq 2015], as they will be needed below to understand the proof of the noninjectivity result stated as Theorem 1.1. These parts correspond to Steps B and C of Section 2.

Sketch of proof. Consider $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ endowed with the standard action of the torus $T = S^1 \times S^1$ for which the moment polytope is given by

$$P = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_2 \le \mu,\ -1 \le x_1 \le 0,\ c_1 \le x_2 - x_1 \le \mu + 1 - c_2\right\} \tag{9}$$

so the primitive outward normals to P are as follows:

$$\begin{aligned} \eta_1 &= (0,1), & \eta_2 &= (1,0), & \eta_3 &= (1,-1),\\ \eta_4 &= (0,-1), & \eta_5 &= (-1,0), & \eta_6 &= (-1,1). \end{aligned}$$

The Delzant construction gives a method to obtain, from the polytope P, the symplectic manifold $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ with the toric action T: first consider the standard action of the torus $\mathbb{T}^6$ on $\mathbb{C}^6$ and then perform a symplectic reduction at a regular level of that action (for more details, see, for example, [Cannas da Silva 2001, Section 29]). Then the normalized moment map $\Phi : M_{\mu,c_1,c_2} \to \mathbb{R}^2$ of the remaining T action, obtained through the Delzant construction, is given by

$$\Phi(z_1, \dots, z_6) = \left(-\tfrac{1}{2}|z_2|^2 + \epsilon_1, -\tfrac{1}{2}|z_1|^2 + \mu - \epsilon_2\right), \quad z_i \in \mathbb{C},$$

where ϵ_1 and ϵ_2 are given by the symplectic parameters μ, c_1, and c_2 as

$$\epsilon_1 = \frac{c_1^3 + 3c_2^2 - c_2^3 - 3\mu}{3(c_1^2 + c_2^2 - 2\mu)} \quad \text{and} \quad \epsilon_2 = \frac{c_1^3 - c_2^3 + 3c_2^2\mu - 3\mu^2}{3(c_1^2 + c_2^2 - 2\mu)}. \tag{10}$$

Moreover, the homology classes $A_i = [\Phi^{-1}(D_i)]$ of the pre-images of the corresponding facets D_i are: $A_1 = F - E_2$, $A_2 = B - E_1$, $A_3 = E_1$, $A_4 = F - E_1$, $A_5 = B - E_2$, and $A_6 = E_2$.

For $1 \le i \le 6$, let Γ_i be the circle action associated to the primitive outward normal η_i. Since the toric complex structure on M_{μ,c_1,c_2} is Fano and T-invariant, it follows from [McDuff and Tolman 2006, Theorem 1.10] or [Anjos and Leclercq 2015, Theorem 4.5] (recalled as Theorem 2.1 in Section 2) that the Seidel elements associated to the Γ_i are given by the expressions

$$\begin{aligned} \mathcal{S}(\Gamma_1) &= (F - E_2) \otimes q t^{\mu - \epsilon_2}, & \mathcal{S}(\Gamma_2) &= (B - E_1) \otimes q t^{\epsilon_1},\\ \mathcal{S}(\Gamma_3) &= E_1 \otimes q t^{\epsilon_1 + \epsilon_2 - c_1}, & \mathcal{S}(\Gamma_4) &= (F - E_1) \otimes q t^{\epsilon_2},\\ \mathcal{S}(\Gamma_5) &= (B - E_2) \otimes q t^{1 - \epsilon_1}, & \mathcal{S}(\Gamma_6) &= E_2 \otimes q t^{\mu + 1 - c_2 - \epsilon_1 - \epsilon_2}. \end{aligned} \tag{11}$$

There are nine primitive sets: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$, and $\{4,6\}$ which yield nine multiplicative relations (which form the Stanley–Reisner ideal) that, combined with the two linear relations ($A_5 = A_1 + A_2 - A_4$ and $A_6 = A_3 + A_4 - A_1$), give the desired result as explained in Step C of Section 2 above. □

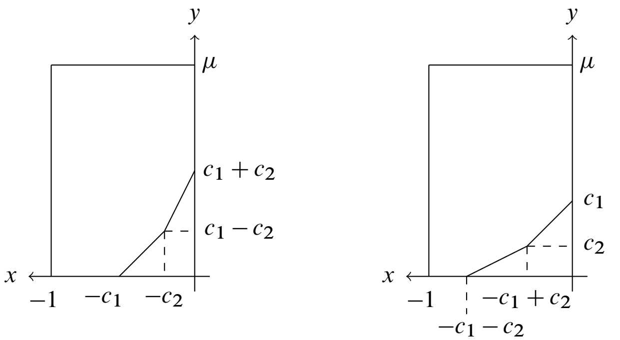


Figure 1. $(M_{\mu,c_1,c_2}, \omega_{\mu,c_1,c_2})$ with toric actions T_1 and T_2.

Assume from now on that $\mu = 1$. Recall from [Anjos and Pinsonnault 2013, Theorem 1.1] that if $c_2 < c_1$ then

$$\pi_1(\mathrm{Ham}(M_{1,c_1,c_2}, \omega_{1,c_1,c_2})) \simeq \mathbb{Z}\langle x_0, x_1, y_0, y_1, z\rangle \simeq \mathbb{Z}^5,$$

where the generators x_0, x_1, y_0, y_1, z correspond to circle actions contained in maximal tori of the Hamiltonian group. In particular, the generators in which we will be most interested are $x_0 = \Gamma_2$ and $y_0 = \Gamma_1$ where the Γ_i are the circle actions associated to the primitive outward normals η_i to the polytope P defined in (9).

Remark 4.3. In order to understand the remaining generators, consider the two toric manifolds given by the polytopes in Figure 1. We denote by $\{x_{0,i}, y_{0,i}\}$ the generators in $\pi_1(T_i)$, where T_i, $i = 1, 2$, represent the two torus actions in this figure and the generators $\{x_{0,i}, y_{0,i}\}$ correspond to the circle actions whose moment maps are, respectively, the first and second components of the moment map associated to each one of the toric actions. It was shown in [Anjos and Pinsonnault 2013, Lemma 4.5] that $x_1 = x_{0,1}$, $z = y_{0,2}$, and $y_1 = y_{0,1} - x_1 = z - x_{0,2}$.

Note that the case $c_1 = c_2$ is an interesting limit case in terms of the topology of the Hamiltonian group since y_1 disappears. For more details see [Anjos and Pinsonnault 2013, Section 5.1]. ◀

To prove Theorem 1.1, we will prove Proposition 4.4.

Proposition 4.4. *The class of* $2(x_0 + y_0)$ *belongs to* $\ker(\mathcal{S})$ *if and only if* $\mu = 1$ *and* $c_1 = c_2$.

Proof. From the computation of the Seidel elements in (11) one gets that in the general case (by which we mean for all $\mu \geq 1$), $\mathcal{S}(\Gamma_1) = ut^{\mu-\epsilon_2}$ and $\mathcal{S}(\Gamma_5) = vt^{1-\epsilon_1}$. As the Seidel elements are *invertible* quantum classes, this yields invertibility of u and v. Note that

$$\mathcal{S}(x_0) = \mathcal{S}(\Gamma_2) = \mathcal{S}(\Gamma_5)^{-1} = v^{-1}t^{\varepsilon_1 - 1} \quad \text{and} \quad \mathcal{S}(y_0) = \mathcal{S}(\Gamma_1) = ut^{\mu-\varepsilon_2}.$$

Since $\mu \geq 1 > c_2^2$, it is straightforward to deduce from (10) that $\varepsilon_1 = \varepsilon_2$ if and only if $\mu = 1$: we now restrict our attention to this case and denote by ε the common value of $\varepsilon_1 = \varepsilon_2$. By invertibility of u and v, the fact that $2(x_0 + y_0)$ belongs to $\ker(\mathcal{S})$ is equivalent to $u^2 = v^2$, since

$$\mathcal{S}(2(x_0 + y_0)) = \mathcal{S}(x_0)^2 * \mathcal{S}(y_0)^2 = v^{-2} t^{\epsilon-1} u^2 t^{1-\epsilon} = v^{-2} u^2.$$

On the other hand, note that multiplying the first and second relations in I_{1,c_1,c_2} by $v^{-1} t^{c_2}$ and $u^{-1} t^{c_2}$, respectively, these become equivalent to

$$u^2 = t^{-1} + v^{-1} t^{c_1-2} - u^2 v t^{c_2} \quad \text{and} \quad v^2 = t^{-1} + u^{-1} t^{c_1-2} - u v^2 t^{c_2}$$

so that $u^2 = v^2$ is equivalent to $v^{-1} t^{c_1-2} - u^2 v t^{c_2} = u^{-1} t^{c_1-2} - u v^2 t^{c_2}$.

Multiplying both relations in I_{1,c_1,c_2} by t^{2c_2}, we see that

$$\begin{aligned} -u^2 v t^{c_2} &= (u^2 v^2 t^{2c_2} - t^{c_1+c_2-2}) - v t^{c_2-1}, \text{ and} \\ -u v^2 t^{c_2} &= (u^2 v^2 t^{2c_2} - t^{c_1+c_2-2}) - u t^{c_2-1} \end{aligned} \tag{12}$$

so we can replace $u^2 v t^{c_2}$ and $u v^2 t^{c_2}$ in the previous equation to obtain

$$u^2 = v^2 \iff v^{-1} t^{c_1-1} + u t^{c_2} = u^{-1} t^{c_1-1} + v t^{c_2}. \tag{13}$$

Finally, (12) also induces, by subtracting one from the other, the equation

$$(u^2 v - u v^2) t^{-c_2} = (v - u) t^{-1-c_2},$$

which is equivalent to $(v^{-1} - u^{-1}) t^{-1} = v - u$. Using these together with (13) we conclude that $u^2 = v^2$ if and only if $(u - v)(t^{c_1} - t^{c_2}) = 0$ which is equivalent to $c_1 = c_2$ since otherwise $t^{c_1} - t^{c_2}$ would be invertible. □

References

[Abreu and McDuff 2000] M. Abreu and D. McDuff, "Topology of symplectomorphism groups of rational ruled surfaces", *J. Amer. Math. Soc.* **13**:4 (2000), 971–1009. MR Zbl

[Albers 2008] P. Albers, "A Lagrangian Piunikhin–Salamon–Schwarz morphism and two comparison homomorphisms in Floer homology", *Int. Math. Res. Not.* **2008**:4 (2008), art. id. rnm134. MR Zbl

[Anjos and Leclercq 2015] S. Anjos and R. Leclercq, "Seidel's morphism of toric 4-manifolds", 2015. To appear in *J. Symplectic Geom.* arXiv

[Anjos and Pinsonnault 2013] S. Anjos and M. Pinsonnault, "The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of the projective plane", *Math. Z.* **275**:1-2 (2013), 245–292. MR Zbl

[Barraud and Cornea 2007] J.-F. Barraud and O. Cornea, "Lagrangian intersections and the Serre spectral sequence", *Ann. of Math.* (2) **166**:3 (2007), 657–722. MR Zbl

[Barraud and Cornea ≥ 2017] J.-F. Barraud and O. Cornea, "Higher order Seidel invariants for loops of Hamiltonian isotopies", in preparation.

[Cannas da Silva 2001] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics **1764**, Springer, Berlin, 2001. MR Zbl

[Chan et al. 2017] K. Chan, S.-C. Lau, N. C. Leung, and H.-H. Tseng, "Open Gromov–Witten invariants, mirror maps, and Seidel representations for toric manifolds", *Duke Math. J.* (online publication February 2017).

[Entov and Polterovich 2008] M. Entov and L. Polterovich, "Symplectic quasi-states and semi-simplicity of quantum homology", pp. 47–70 in *Toric topology*, edited by M. Harada et al., Contemp. Math. **460**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Fukaya et al. 2016] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds*, Astérisque **376**, Société Mathématique de France, Paris, 2016. MR Zbl

[Fukaya et al. 2017] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, "Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory", 2017. To appear in *Memoirs Amer. Math. Soc.* arXiv

[González and Iritani 2012] E. González and H. Iritani, "Seidel elements and mirror transformations", *Selecta Math.* (*N.S.*) **18**:3 (2012), 557–590. MR Zbl

[Hu and Lalonde 2010] S. Hu and F. Lalonde, "A relative Seidel morphism and the Albers map", *Trans. Amer. Math. Soc.* **362**:3 (2010), 1135–1168. MR Zbl

[Hu et al. 2011] S. Hu, F. Lalonde, and R. Leclercq, "Homological Lagrangian monodromy", *Geom. Topol.* **15**:3 (2011), 1617–1650. MR Zbl

[Hutchings 2008] M. Hutchings, "Floer homology of families, I", *Algebr. Geom. Topol.* **8**:1 (2008), 435–492. MR Zbl

[Lalonde et al. 1999] F. Lalonde, D. McDuff, and L. Polterovich, "Topological rigidity of Hamiltonian loops and quantum homology", *Invent. Math.* **135**:2 (1999), 369–385. MR Zbl

[Leclercq 2008] R. Leclercq, "Spectral invariants in Lagrangian Floer theory", *J. Mod. Dyn.* **2**:2 (2008), 249–286. MR Zbl

[Leclercq 2009] R. Leclercq, "The Seidel morphism of Cartesian products", *Algebr. Geom. Topol.* **9**:4 (2009), 1951–1969. MR Zbl

[Leclercq and Zapolsky 2017] R. Leclercq and F. Zapolsky, "Spectral invariants for monotone Lagrangians", *Journal of Topology and Analysis* (online publication May 2017).

[McDuff 2010] D. McDuff, "Loops in the Hamiltonian group: a survey", pp. 127–148 in *Symplectic topology and measure preserving dynamical systems*, edited by A. Fathi et al., Contemp. Math. **512**, American Mathematical Society, Providence, RI, 2010. MR Zbl

[McDuff and Salamon 2004] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications **52**, American Mathematical Society, Providence, RI, 2004. MR Zbl

[McDuff and Tolman 2006] D. McDuff and S. Tolman, "Topological properties of Hamiltonian circle actions", *Int. Math. Res. Pap.* (2006), art. id. 72826. MR Zbl

[Savelyev 2008] Y. Savelyev, "Quantum characteristic classes and the Hofer metric", *Geom. Topol.* **12**:4 (2008), 2277–2326. MR Zbl

[Seidel 1997] P. Seidel, "π_1 of symplectic automorphism groups and invertibles in quantum homology rings", *Geom. Funct. Anal.* **7**:6 (1997), 1046–1095. MR Zbl

Received April 6, 2016. Revised March 9, 2017.

SÍLVIA ANJOS
CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS
MATHEMATICS DEPARTMENT
INSTITUTO SUPERIOR TÉCNICO
AV. ROVISCO PAIS
1049-001 LISBOA
PORTUGAL

sanjos@math.ist.utl.pt

RÉMI LECLERCQ
LABORATOIRE DE MATHÉMATIQUES D'ORSAY
UNIVERSITÉ PARIS-SUD, CNRS, UNIVERSITÉ PARIS-SACLAY
91405 ORSAY
FRANCE

remi.leclercq@math.u-psud.fr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.273

DIFFERENTIAL HARNACK ESTIMATES FOR FISHER'S EQUATION

XIAODONG CAO, BOWEI LIU, IAN PENDLETON AND ABIGAIL WARD

We derive several differential Harnack estimates (also known as Li–Yau–Hamilton-type estimates) for positive solutions of Fisher's equation. We use the estimates to obtain lower bounds on the speed of traveling wave solutions and to construct classical Harnack inequalities.

1. Introduction

Fisher's equation, or the Fisher–KPP partial differential equation, is given by

$$f_t = \Delta f + cf(1-f), \tag{1}$$

where f is a real-valued function on an n-dimensional Riemannian manifold M^n, and c is a positive constant. The equation was proposed by R. A. Fisher [1937] to describe the propagation of an evolutionarily advantageous gene in a population, and was also independently described in a seminal paper by A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov [1937] in the same year; for this reason, it is often referred to in the literature as the Fisher–KPP equation. The density of the gene evolves according to diffusion (the term Δf) and reaction (the term $cf(1-f)$). Since the two papers in 1937, the equation has found many applications including in the description of the branching Brownian motion process [McKean 1975], in neuropsychology [Tuckwell 1988], and in describing certain chemical reactions [Ó Náraigh and Kamhawi 2013]. Because a solution f often describes a concentration or density, it is natural to study solutions to the equation for which $0 < f < 1$; our main theorems will simply assume positive solutions.

It is clear that $f = 0$ and $f = 1$ are stationary solutions to this equation on any manifold; it is also known that when $M^n = \mathbb{R}^n$ the equation admits traveling wave solutions, i.e., solutions $f(x,t)$ that we can express as a function of $z = x + \eta t$ for some vector $\eta \in \mathbb{R}^n$. Under a broad range of conditions, general solutions to the equation in $\mathbb{R}^1$ approach a traveling wave solution with a unique minimal speed (see for example, [Kolmogorov et al. 1937, Theorem 17] or [Fisher 1937; Sherratt 1998]).

MSC2010: 58J35, 35K59.
Keywords: differential Harnack, classical Harnack, Fisher's equation, Fisher–KPP, traveling wave.

A bound on the minimum speed of such a traveling wave solution on $\mathbb{R}^1$ was known to Kolmogorov, Petrovskii and Piskunov [1937]; our work results in bounds for the minimum speed of a solution on $\mathbb{R}^n$ for $n = 1, 2, 3$. While our bound in dimension 1 is weaker than the previously known bounds, the bounds in higher dimensions are new and suggest that the study of Harnack inequalities may be used to bound the minimal speed of traveling waves in higher dimensions as well.

Our work introduces and proves three Li–Yau–Hamilton-type Harnack inequalities which constrain positive functions satisfying the Fisher–KPP equation on an arbitrary Riemannian manifold M^n. Depending on the setting we obtain different inequalities. The study of differential Harnack inequalities was first initiated by P. Li and S.-T. Yau [1986] (also see [Aronson and Bénilan 1979]). Harnack inequalities have since played an important role in the study of geometric analysis and geometric flows (for example, see [Hamilton 1993; Perelman 2002]). Applications have also been found to the study of nonlinear parabolic equations, e.g., in [Hamilton 2011]. One of these is a recent reproof of the classical result of H. Fujita [1966], which states that any positive solution to the endangered species equation in dimension n,

$$f_t = \Delta f + f^p,$$

blows up in finite time provided $0 < n(p-1) < 2$; see [Cao et al. 2015].

When the dimension falls into a certain range we can integrate our differential Harnack inequality along any spacetime curve to obtain a classical Harnack inequality which allows us to compare the values of positive solutions at any two points in spacetime when time is large.

The organization for the paper is as follows: In Section 2 we present the precise formulations and the proofs of our two inequalities governing closed manifolds. In Section 3 we state and prove a similar Harnack inequality for complete noncompact manifolds. In Section 4, we end the paper with the aforementioned results on the minimum speed of traveling wave solutions and classical Harnack inequalities.

2. On closed manifolds

In this section, we will deal with the case when the Riemannian manifold M is closed, and we also assume that its Ricci curvature is nonnegative.

In what follows, the time derivative will always be taken to mean the derivative from the left if the two-sided derivative does not exist.

Theorem 1. *Let (M^n, g) be an n-dimensional closed Riemannian manifold with nonnegative Ricci curvature and let $f(x,t) : M \times [0,\infty) \to \mathbb{R}$ be a positive solution of the Fisher–KPP equation $f_t = \Delta f + cf(1-f)$, where f is C^2 in x and C^1 in t, and $c > 0$.*

(A) *Let $u = \log f$ and define*

$$\phi_0^<(t) = \frac{\left(\frac{\beta cn}{cn+8\beta(1-\alpha)}\right)e^{-ct} - \beta}{1-e^{-ct}}.$$

Then we have

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0^<(t) \geq 0 \tag{2}$$

for all x and t, provided that

(i) $0 < \alpha < 1$, (ii) $\beta \leq \dfrac{-cn(1+\alpha)}{4\alpha^2 - 4\alpha + 2n} < 0$ and (iii) $\dfrac{8\beta(1-\alpha)}{n} + c < 0$.

(B) *Now set*

$$\phi_0^>(t) = \begin{cases} \dfrac{n}{2(1-\alpha)t} & \text{if } t \leq T_2, \\ \dfrac{-\beta c(e^{c(t-T_2)}+1)}{c + \frac{8\beta(1-\alpha)}{n} + ce^{c(t-T_2)}} & \text{otherwise}, \end{cases}$$

where

$$T_2 := \frac{n}{2(1-\alpha)(-\beta c)}\left(\frac{4\beta(1-\alpha)}{n} + c\right).$$

If instead of (iii) we have

(iv) $\dfrac{8\beta(1-\alpha)}{n} + c \geq 0,$

in addition to (i) and (ii), then

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0^>(t) \geq 0. \tag{3}$$

In summary, our theorem is that $\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0(t) \geq 0$, where

$$\phi_0(t) = \begin{cases} \dfrac{(\beta cn/(cn+8\beta(1-\alpha)))e^{-ct} - \beta}{1-e^{-ct}} & \text{if (iii) holds}, \\ \dfrac{n}{2(1-\alpha)t} & \text{if (iv) holds and } t \leq T_2, \\ \dfrac{-\beta c(e^{c(t-T_2)}+1)}{c + \frac{8\beta(1-\alpha)}{n} + ce^{c(t-T_2)}} & \text{if (iv) holds and } t > T_2. \end{cases}$$

We briefly describe the main idea of our proof here, which uses the parabolic maximum principle and an argument by contradiction. We first define a quantity

$$h(x,t) : M \times (0,\infty) \to \mathbb{R},$$

which will depend on a given solution to Fisher's equation. We start with $h(x,\varepsilon) > 0$ for any sufficiently small $\varepsilon > 0$, and our goal is to prove this quantity $h(x,t)$ remains positive for all points in $M \times \mathbb{R}^+$. As suggested in [Cao 2008; Cao and Hamilton 2009], we then compute what we call the time evolution of h, namely $\partial h/\partial t$, in the following form:

$$\frac{\partial h}{\partial t}(x,t) = \Delta h(x,t) + A_1(x,t)\cdot\nabla h(x,t) + A_2(x,t),$$

for some $A_1 : M \times (0,\infty) \to \mathbb{R}^n$, and $A_2 : M \times (0,\infty) \to \mathbb{R}$. We then assume for the sake of a contradiction that there exists a first (with respect to t) point (x_1, t_1) where $h(x,t) \le 0$; it follows that $(\partial h/\partial t)(x_1,t_1) \le 0$. Since $h(x_1,t_1)$ must be a local minimum in M of the function $h(x,t_1) : M \to \mathbb{R}$, it also follows that $\Delta h(x_1,t_1) \ge 0$, and $\nabla h(x_1,t_1) = (0,\dots,0)$. Thus our time evolution simplifies to

$$\frac{\partial h}{\partial t}(x_1,t_1) \ge A_2(x_1,t_1).$$

By our construction of $h(x,t)$ we will force $A_2(x_1,t_1) > 0$, and so we will have

$$0 \ge \frac{\partial h}{\partial t}(x_1,t_1) \ge A_2(x_1,t_1) > 0,$$

which is a contradiction. Thereby we conclude that $h(x,t) > 0$ for all $(x,t) \in M \times (0,\infty)$.

Technical lemmas. In this section we prove the technical lemmas needed in the case that M is a closed manifold.

Lemma 2 gives us the time evolution of h in terms of 4 quantities P_1, P_2, P_3, P_4 (which sum to A_2 above). Lemma 3 gives a lower bound for P_2 which also applies in the noncompact case. Lemma 4 introduces quantities P_5, $P_{5.1}$, $P_{5.2}$ which depend only on ϕ and which give a lower bound for P_3. Lemma 5 puts a lower bound on P_5. Lemma 6, used for our second Harnack inequality, bounds P_3 when Lemma 5 is inapplicable. Finally, P_1 and P_4 are bounded in the proof of the main theorem.

Lemma 2. *Let (M^n, g) be a complete Riemannian manifold with Ricci curvature bounded from below by* $\mathrm{Ric} \ge -K$. *Let $f(x,t) : M^n \to \mathbb{R}$ be a positive solution to $f_t = \Delta f + cf(1-f)$ which is C^2 in x and C^1 in t. Let $u(x,t) = \log f(x,t)$, and let α, β, c be any constants. Define $h(x,t)$ as follows*:

$$h(x,t) := \Delta u + \alpha|\nabla u|^2 + \beta e^u + \varphi,$$
$$\varphi = \varphi(x,t) = \phi(t) + \psi(x),$$

where $\phi(t)$ is any C^1 function and $\psi(x)$ is any C^2 function. Then the following inequality holds:

$$h_t - \Delta h - 2\nabla u\cdot\nabla h \ge P_1 h + P_2 + P_3 + P_4,$$

where

$$P_1 = \frac{2(1-\alpha)}{n}h - \frac{4(1-\alpha)}{n}(\alpha|\nabla u|^2 + \beta e^u + \phi + \psi) - ce^u,$$

$$P_2 = \frac{2(1-\alpha)}{n}(\alpha^2|\nabla u|^4 + 2\phi\psi) - 2K(1-\alpha)|\nabla u|^2 + \frac{4\alpha(1-\alpha)}{n}\phi|\nabla u|^2 + |\nabla u|^2 e^u\Big(\frac{4\alpha\beta(1-\alpha)}{n} - 2\beta - \alpha c - c\Big),$$

$$P_3 = e^{2u}\frac{2\beta^2(1-\alpha)}{n} + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t,$$

$$P_4 = \frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi + e^u\psi\Big(c + \frac{4\beta(1-\alpha)}{n}\Big) + \frac{2(1-\alpha)}{n}\psi^2 - \Delta\psi.$$

Lemma 2 will be used in the proofs of both Theorem 1 and Theorem 7, with different choices of α, β, c, ϕ and ψ. The statement of Lemma 2 is independent of these choices.

Proof. The proof is based on a straightforward but fairly long calculation. Let $f : M \times [0, \infty) \to \mathbb{R}$ satisfy (1); hence u must satisfy

$$u_t = \Delta u + |\nabla u|^2 + c(1 - e^u).$$

We then compute

$$(\partial_t - \Delta)u = c - ce^u + |\nabla u|^2,$$

$$(\partial_t - \Delta)(\Delta u) = \Delta|\nabla u|^2 - c(\Delta u)e^u - c|\nabla u|^2 e^u,$$

$$(\partial_t - \Delta)(\alpha|\nabla u|^2) = 2\alpha\nabla u\cdot\nabla(\Delta u) + 2\alpha\nabla u\cdot\nabla|\nabla u|^2 - 2\alpha c|\nabla u|^2 e^u - \alpha\Delta|\nabla u|^2,$$

$$(\partial_t - \Delta)(\beta e^u) = \beta ce^u - \beta ce^{2u},$$

$$(\partial_t - \Delta)\varphi(t) = \phi_t - \Delta\psi,$$

$$2\nabla u\cdot\nabla h = 2\nabla u\cdot\nabla(\Delta u) + 2\alpha\nabla u\cdot\nabla|\nabla u|^2 + 2\beta|\nabla u|^2 e^u + 2\nabla u\cdot\nabla\psi.$$

Here we use the Weitzenböck-Bochner formula,

$$\Delta|\nabla u|^2 = 2|\nabla\nabla u|^2 + 2\nabla u\cdot\nabla(\Delta u) + 2\,\mathrm{Ric}(\nabla u, \nabla u),$$

where $\nabla\nabla u$ is the Hessian of $u(x,t)$.

This leads to the equality

$$\begin{aligned}(\partial_t - \Delta)h - 2\nabla u\cdot\nabla h &= 2(1-\alpha)|\nabla\nabla u|^2 - ce^u(\Delta u) - |\nabla u|^2 e^u(2\alpha c + 2\beta + c)\\ &\quad + 2(1-\alpha)\,\mathrm{Ric}(\nabla u, \nabla u) + \beta ce^u - \beta ce^{2u} + \phi_t - \Delta\psi - 2\nabla u\cdot\nabla\psi.\end{aligned}$$

Using Cauchy–Schwarz $|\nabla\nabla u|^2 \geq (1/n)(\Delta u)^2$ and $\mathrm{Ric} \geq -K$ yields that

$$(\partial_t - \Delta)h - 2\nabla u \cdot \nabla h \geq 2\frac{(1-\alpha)}{n}(\Delta u)^2 - ce^u(\Delta u) - |\nabla u|^2 e^u(2\alpha c + 2\beta + c)$$
$$-2(1-\alpha)K|\nabla u|^2 + \beta c e^u - \beta c e^{2u} + \phi_t - \Delta\psi - 2\nabla u \cdot \nabla\psi.$$

Finally, we substitute for Δu:

$$\Delta u = h - \alpha|\nabla u|^2 - \beta e^u - \phi - \psi,$$

to expand and conclude that

$$\begin{aligned}
&h_t - \Delta h - 2\nabla u \cdot \nabla h \\
&\quad\geq h\Big(\frac{2(1-\alpha)}{n}h - \frac{4(1-\alpha)}{n}(\alpha|\nabla u|^2 + \beta e^u + \phi + \psi) - ce^u\Big) \\
&\qquad + \Big[\frac{2(1-\alpha)}{n}(\alpha^2|\nabla u|^4 + 2\phi\psi) - 2K(1-\alpha)|\nabla u|^2 + \frac{4\alpha(1-\alpha)}{n}\phi|\nabla u|^2 \\
&\qquad\qquad + |\nabla u|^2 e^u\Big(\frac{4\alpha\beta(1-\alpha)}{n} - 2\beta - \alpha c - c\Big)\Big] \\
&\qquad + \Big[e^{2u}\Big(\frac{2\beta^2(1-\alpha)}{n}\Big) + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t\Big] \\
&\qquad + \Big[\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u \cdot \nabla\psi + e^u\psi\Big(c + \frac{4\beta(1-\alpha)}{n}\Big) + \frac{2(1-\alpha)}{n}\psi^2 - \Delta\psi\Big] \\
&\quad= P_1 h + P_2 + P_3 + P_4,
\end{aligned}$$

as desired. □

We now show that P_2 is nonnegative under the assumptions of Theorem 1.

Lemma 3. *If* $K = 0$ *and assuming that* (i) *and* (ii) *hold*, *then for any* x, t *where* $\phi(t)$, $\psi(x) \geq 0$ *we have*

$$P_2 \geq 0.$$

Proof. We have assumed that α, $1-\alpha$, ϕ, ψ, $K \geq 0$. Note that

$$\frac{4\alpha\beta(1-\alpha)}{n} - 2\beta - \alpha c - c \geq 0$$

is equivalent to

$$(4\alpha(1-\alpha) - 2n)\beta - cn(\alpha + 1) \geq 0,$$

or

$$(-4\alpha(1-\alpha) + 2n)\beta \leq -cn(1+\alpha),$$

which is exactly condition (ii) since $2n \geq 1 \geq 4\alpha(1-\alpha)$. □

Next, we find quantities depending only on ϕ which we will eventually use to guarantee that P_3 is strictly positive.

Lemma 4. *Assume* $\alpha < 1$. *Define*

$$\mu_1 := \frac{1}{2} c \sqrt{\frac{n}{2(1-\alpha)}}, \qquad \nu_1 = \frac{c + \frac{4\beta(1-\alpha)}{n}}{2\beta} \sqrt{\frac{n}{2(1-\alpha)}},$$

$$\omega_1 = \sqrt{\frac{2(1-\alpha)}{n}},$$

$$P_5(\phi) := -(\mu_1 + \nu_1 \phi)^2 + (\omega_1 \phi)^2 + \phi_t,$$

$$P_{5.1}(\phi) := \Big(\frac{4\beta(1-\alpha)}{n} + c\Big)\phi + \beta c, \qquad P_{5.2}(\phi) := \frac{2(1-\alpha)}{n}\phi^2 + \phi_t.$$

Then for any (x, t), $P_5 > 0$ *implies that* $P_3 > 0$. *Alternatively, if* $P_{5.1} \geq 0$ *and* $P_{5.2} > 0$, *then* $P_3 > 0$.

Proof. Recall that

$$P_3(\phi) = e^{2u}\Big(\frac{2\beta^2(1-\alpha)}{n}\Big) + e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t.$$

If $P_5 > 0$, then by using $x^2 + 2xy \geq -y^2$, where $x^2 = e^{2u}(2\beta^2(1-a)/n)$, we get

$$\begin{aligned} P_3(\phi) &\geq -\frac{n}{8(1-\alpha)\beta^2}\Big[\beta c + \Big(c + \frac{4(1-\alpha)\beta}{n}\Big)\phi\Big]^2 + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t \\ &= -(\mu_1 + \nu_1\phi)^2 + (\omega_1\phi)^2 + \phi_t = P_5(\phi) > 0. \end{aligned}$$

Alternatively, if $P_{5.1} \geq 0$ and $P_{5.2} > 0$, then since $(1-\alpha) > 0$ we can ignore the first term of P_3 and get

$$\begin{aligned} P_3(\phi) &\geq e^u\Big(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Big) + \frac{2(1-\alpha)}{n}\phi^2 + \phi_t \\ &= e^u P_{5.1} + P_{5.2} > 0. \end{aligned}$$

□

We now find functions $\phi(t)$ such that $P_3(\phi) > 0$. In Lemma 5 we construct $\phi(t)$ in the case that (iii) is true, and in Lemma 6 we construct $\phi(t)$ when (iv) is true.

Lemma 5. *Let* μ, ν, ω *be any constants such that* $\mu \neq 0$, $\nu^2 < \omega^2$ *and* $\omega > 0$. *If for sufficiently small* $\varepsilon > 0$ *we define*

$$\phi(t) := \frac{\mu\Big(\frac{1}{\nu-(\omega-\varepsilon)} e^{2\mu(\omega-\varepsilon)t} - \frac{1}{\nu+(\omega-\varepsilon)}\Big)}{1 - e^{2\mu(\omega-\varepsilon)t}},$$

then

$$-(\mu + \nu\phi)^2 + (\omega\phi)^2 + \phi_t > 0,$$

where $\lim_{t\to 0^+} \phi(t) = \infty$ *and* $\phi(t) \geq 0$ *for all* t.

Proof. Choose ε small enough so that $\nu^2 < (\omega-\varepsilon)^2$. We claim that $\phi(t)$ satisfies the following equation:

$$-(\mu+\nu\phi)^2 + [(\omega-\varepsilon)\phi]^2 + \phi_t(t) = 0$$

for all time. This follows from the direct computation below. On the one hand we get that

$$\begin{aligned}
-(\mu+\nu\phi)^2 + [(\omega-\varepsilon)\phi]^2 &= \frac{\mu^2(\omega-\varepsilon)^2\Big(\dfrac{e^{2\mu(\omega-\varepsilon)t}}{\nu-(\omega-\varepsilon)} - \dfrac{1}{\nu+(\omega-\varepsilon)}\Big)^2}{(1-e^{2\mu(\omega-\varepsilon)t})^2} \\
&\qquad - \left(\mu + \frac{\mu\nu\Big(\dfrac{e^{2\mu(\omega-\varepsilon)t}}{\nu-(\omega-\varepsilon)} - \dfrac{1}{\nu+(\omega-\varepsilon)}\Big)}{1-e^{2\mu(\omega-\varepsilon)t}}\right)^2 \\
&= \frac{\mu^2[2(\omega-\varepsilon)(\omega-\varepsilon-\nu)][2(\omega-\varepsilon)(\omega-\varepsilon+\nu)e^{2\mu(\omega-\varepsilon)t}]}{(1-e^{2\mu(\omega-\varepsilon)t})^2(\nu-(\omega-\varepsilon))^2(\nu+(\omega-\varepsilon))^2} \\
&= -\frac{4\mu^2(\omega-\varepsilon)^2 e^{2\mu(\omega-\varepsilon)t}}{(\nu+(\omega-\varepsilon))(\nu-(\omega-\varepsilon))(e^{2\mu(\omega-\varphi)t}-1)^2}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\phi_t(t) &= \frac{2\mu^2(\omega-\varepsilon)e^{2\mu(\omega-\varepsilon)t}}{(1-e^{2\mu(\omega-\varepsilon)t})(\nu-(\omega-\varepsilon))} + \frac{2\mu^2(\omega-\varepsilon)e^{2\mu(\omega-\varepsilon)t}\Big(\dfrac{e^{2\mu(\omega-\varepsilon)t}}{\nu-(\omega-\varepsilon)} - \dfrac{1}{\nu+(\omega-\varepsilon)}\Big)}{(1-e^{2\mu(\omega-\varepsilon)t})^2} \\
&= \frac{4\mu^2(\omega-\varepsilon)^2 e^{2\mu(\omega-\varepsilon)t}}{(\nu+(\omega-\varepsilon))(\nu-(\omega-\varepsilon))(1-e^{2\mu(\omega-\varepsilon)t})^2}.
\end{aligned}$$

Therefore it follows that

$$-(\mu+\nu\phi)^2 + [(\omega-\varepsilon)\phi]^2 + \phi_t = 0,$$

and hence

$$-(\mu+\nu\phi)^2 + (\omega\phi)^2 + \phi_t = 2\varepsilon\omega\phi^2 - \varepsilon^2\phi^2 = \phi^2(2\varepsilon\omega - \varepsilon^2).$$

Note that $\nu-(\omega-\varepsilon)$ and $\nu+(\omega-\varepsilon)$ must have different signs since their product is $\nu^2-(\omega-\varepsilon)^2 < 0$; hence $\phi(t) \neq 0$ for all time. It then follows that for sufficiently small ε,

$$-(\mu+\nu\phi)^2 + (\omega\phi)^2 + \phi_t = \phi^2(2\varepsilon\omega - \varepsilon^2) > 0.$$

To show that $\lim_{t\to 0^+}\phi(t) = \infty$, we split $\phi(t)$ into two parts. First, note that

$$\begin{aligned}
\lim_{t\to 0^+}\Big(\frac{1}{\nu-(\omega-\varepsilon)}e^{2\mu(\omega-\varepsilon)t} - \frac{1}{\nu+(\omega-\varepsilon)}\Big) &= \frac{1}{\nu-(\omega-\varepsilon)} - \frac{1}{\nu+(\omega-\varepsilon)} \\
&= \frac{2(\omega-\varepsilon)}{\nu^2-(\omega-\varepsilon)^2} < 0.
\end{aligned}$$

Further, it is clear that

$$\lim_{t\to 0^+} \frac{\mu}{1-e^{2\mu(\omega-\varepsilon)t}} = -\infty.$$

Combining these two calculations lets us conclude that

$$\lim_{t\to 0^+} \phi(t) = \infty.$$

Finally, since $\phi(t)$ is continuous and starts out positive and $\phi(t) \neq 0$ for any $t > 0$, it follows that $\phi(t) > 0$ for all $t > 0$. □

Remark. We can also compute $\lim_{t\to\infty} \phi(t)$.

If $\mu > 0$ then $e^{2\mu(\omega-\varepsilon)t} \to \infty$ as $t \to \infty$; hence we find that

$$\lim_{t\to\infty} \phi(t) = \frac{\dfrac{\mu}{\nu-(\omega-\varepsilon)}}{-1} = \frac{\mu}{-\nu+(\omega-\varepsilon)}.$$

If $\mu < 0$ then $e^{2\mu(\omega-\varepsilon)t} \to 0$ as $t \to \infty$, which gives us

$$\lim_{t\to\infty} \phi(t) = \frac{-\mu}{\nu+(\omega-\varepsilon)}.$$

Next we deal with the other case.

Lemma 6. *Let μ_1, ν_1, ω_1 be defined as in Lemma 4, and suppose (iv) is true (i.e., (iii) becomes false). Let*

$$T_2 = T_2(\varepsilon) := \frac{n}{2(1-\alpha)(1-\varepsilon)(-\beta c)} \cdot \left(\frac{4\beta(1-\alpha)}{n} + c\right).$$

If for some sufficiently small $\varepsilon > 0$ we define

$$\phi(t) := \begin{cases} \dfrac{n}{2(1-\alpha)(1-\varepsilon)t} & \text{if } t \le T_2, \\ \dfrac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}} & \text{if } t > T_2, \end{cases}$$

then for $t \le T_2$ we get $P_{5.1} \ge 0$ and $P_{5.2} > 0$, and for $t > T_2$ we get $P_5 > 0$. Therefore $P_3(\phi) > 0$ for all t.

In addition, $\lim_{t\to 0^+} \phi(t) = \infty$ and $\phi(t) > 0$ for all t.

Proof. For $\varepsilon < 1$, we have

$$\lim_{t\to 0^+} \phi(t) = \lim_{t\to 0^+} \frac{n}{2(1-\alpha)(1-\varepsilon)t} = \infty.$$

To show that $\phi(t)$ is continuous at T_2, we check its limits from the left and right. The limit from the left is

$$\lim_{t\to T_2^-} \phi(t) = \frac{n}{2(1-\alpha)(1-\varepsilon)T_2} = \frac{-\beta cn}{4\beta(1-\alpha)+cn}.$$

And the limit from the right is

$$\lim_{t\to T_2^+}\phi(t)=\frac{-\mu_1(1+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))}=\frac{-2\mu_1}{2\nu_1}=-\frac{c}{2}\cdot\frac{2\beta n}{(cn+4\beta(1-\alpha))}$$
$$=\frac{-\beta cn}{4\beta(1-\alpha)+cn}.$$

Therefore $\phi(t)$ is continuous.

Next we check that $\phi(t)>0$ for all $t>0$. Note that $\phi(t)$ is continuous, and clearly is positive between 0 and T_2. For $t\geq T_2$, since $\mu_1\neq 0$, it follows that

$$-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)\neq 0,$$

and therefore $\phi(t)\neq 0$ for any $t\geq T_2$. By continuity, it follows that $\phi(t)>0$ for all $t>0$.

Next we show that for $t\leq T_2$ we have $P_{5.1}\geq 0$. That is, we need

$$P_{5.1}=\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi(t)+\beta c\geq 0.$$

First we note that condition (iv) states that $4\beta(1-\alpha)/n+c\geq 0$. Since $\phi(t)$ is decreasing in $t<T_2$, it suffices to check that $P_{5.1}\geq 0$ holds for $t=T_2$:

$$\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi(t)+\beta c\geq\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi(T_2)+\beta c$$
$$=\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\left(\frac{-\beta c}{\frac{4\beta(1-\alpha)}{n}+c}\right)+\beta c=0.$$

Therefore $P_{5.1}\geq 0$ for all $t\leq T_2$.

Now we show that $P_{5.2}>0$ for all $t\leq T_2$. That is, we need

$$P_{5.2}=\frac{2(1-\alpha)}{n}\phi(t)^2+\phi_t(t)>0.$$

We have

$$P_{5.2}=\frac{2(1-\alpha)}{n}\Big[\frac{n}{2(1-\alpha)(1-\varepsilon)t}\Big]^2+\frac{-n}{2(1-\alpha)(1-\varepsilon)t^2}$$
$$=\frac{n}{2(1-\alpha)(1-\varepsilon)^2t^2}-\frac{n}{2(1-\alpha)(1-\varepsilon)t^2}=\frac{\varepsilon n}{2(1-\alpha)(1-\varepsilon)^2t^2}>0.$$

This implies that $P_3(\phi)>0$ for $t\leq T_2$. Next we show that $P_5>0$ for all $t>T_2$. That is, we need that

$$P_5=-(\mu_1+\nu_1\phi)^2+(\omega_1\phi)^2+\phi_t>0$$

for

$$\phi(t)=\frac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}.$$

We first show that for $t > T_2$, $\phi(t)$ satisfies

$$-(\mu_1+\nu_1\phi)^2+[(\omega_1-\varepsilon)\phi]^2+\phi_t=0.$$

Plugging in $\phi(t)$ for $t > T_2$ gives us that

$$\begin{aligned}
&-(\mu_1+\nu_1\phi)^2+[(\omega_1-\varepsilon)\phi]^2\\
&\qquad=-\Big[\mu_1-\frac{\mu_1\nu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}\Big]^2\\
&\qquad\quad+\Big[(\omega_1-\varepsilon)\frac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}\Big]^2\\
&\qquad=\frac{\mu_1^2(\omega_1-\varepsilon)^2[-(1-e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)})^2+(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)^2]}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}\\
&\qquad=\frac{4\mu_1^2(\omega_1-\varepsilon)^2e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\phi_t(t)&=\frac{-2\mu_1^2(\omega_1-\varepsilon)e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}\\
&\quad-\frac{(\nu_1-(\omega_1-\varepsilon))(2\mu_1(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}[-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)]}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}\\
&=-\frac{4\mu_1^2(\omega_1-\varepsilon)^2e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}{[(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}]^2}.
\end{aligned}$$

Therefore

$$-(\mu_1+\nu_1\phi)^2+[(\omega_1-\varepsilon)\phi]^2+\phi_t=0,$$

and it follows that

$$P_5=-(\mu_1+\nu_1\phi)^2+(\omega_1\phi)^2+\phi_t=(2\varepsilon\omega_1-\varepsilon^2)\phi^2>0$$

for small enough ε. Therefore $P_3(\phi)>0$ for $t > T_2$. □

Remark. Here we observe that

$$\begin{aligned}
\lim_{t\to\infty}\phi(t)&=\lim_{t\to\infty}\frac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}}\\
&=\frac{-\mu_1}{\nu_1-(\omega_1-\varepsilon)}=\frac{\mu_1}{-\nu_1+(\omega_1-\varepsilon)},
\end{aligned}$$

which is the same limit as $\phi(t)$ from Lemma 5 since $\mu_1>0$.

Now we are ready to finish the proof of Theorem 1.

Proof of Theorem 1. Let $f : M \times [0, \infty) \to \mathbb{R}$ be a positive solution of $f_t = \Delta f + cf(1-f)$ for $c > 0$, and assume that the following hold:

(i) $0 < \alpha < 1$,

(ii) $\beta \le \dfrac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n} < 0$.

Let $u = \log f$, and define

$$h(x,t) := \Delta u + \alpha|\nabla u|^2 + \beta e^u + \varphi,$$

where

$$\varphi = \varphi(x,t) = \phi(t) + \psi(x),$$

and since we are in the closed case we set $\psi(x) = 0$.

With μ_1, ν_1, ω_1, and T_2 as defined in Lemma 4 and Lemma 6, and $\varepsilon > 0$ small enough to satisfy Lemmas 5 and 6, we let

$$\phi(t) = \begin{cases} \dfrac{\mu_1\Big(\dfrac{1}{\nu_1-(\omega_1-\varepsilon)}e^{2\mu_1(\omega_1-\varepsilon)t} - \dfrac{1}{\nu_1+(\omega_1-\varepsilon)}\Big)}{1-e^{2\mu_1(\omega_1-\varepsilon)t}} & \text{if (iii)}, \\ \dfrac{n}{2(1-\alpha)(1-\varepsilon)t} & \text{if (iv) and } t \le T_2, \\ \dfrac{-\mu_1(e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}+1)}{(\nu_1+(\omega_1-\varepsilon))+(\nu_1-(\omega_1-\varepsilon))e^{2\mu_1(\omega_1-\varepsilon)(t-T_2)}} & \text{if (iv) and } t > T_2. \end{cases}$$

We first show that $h(x,t) > 0$ for all t. Suppose for the sake of a contradiction that $h \le 0$ somewhere; let t_1 be the first time such that $\min_x h(x,t) = 0$. Since M is closed the minimum is attained, say at the point (x_1, t_1). By Lemmas 5 and 6, $\lim_{t\to 0^+} \phi(t) = \infty$ so it follows that t_1 exists.

By applying Lemma 2, we get that

$$h_t - \Delta h - 2\nabla u \cdot \nabla h \ge P_1 h + P_2 + P_3 + P_4, \tag{4}$$

where $P_1, \dots, P_4$ are defined as in Lemma 2. Note that in the case (iv), the derivative ϕ_t at $t = T_2$ is considered to be the derivative from the left.

We have $P_1 h = 0$ since $h(x_1,t_1) = 0$. Lemma 3 yields that $P_2 \ge 0$ since $K = 0$, and $P_4 = 0$ since $\psi(x) \equiv 0$.

Since (x_1, t_1) is the first spacetime where $h(x,t) = 0$, the maximum principle yields that $h_t(x_1,t_1) \le 0$ (where this is a derivative as $t \to t_1^-$), $\Delta h(x_1,t_1) \ge 0$ and $\nabla h(x_1,t_1) = 0$.

Hence (4) yields that

$$0 \ge h_t - \Delta h - 2\nabla u \cdot \nabla h \ge P_1 h + P_2 + P_3 + P_4 \ge P_3. \tag{5}$$

Now we split into cases based on whether (iii) or (iv) holds.

If (iii) is true, since $c > 0$ we have the following inequalities:

$$\begin{aligned}\frac{4\beta(1-\alpha)}{n} < c + \frac{4\beta(1-\alpha)}{n} &< -\frac{4\beta(1-\alpha)}{n},\\ \left|c+\frac{4\beta(1-\alpha)}{n}\right| &< \left|\frac{4\beta(1-\alpha)}{n}\right|,\\ \left(\frac{c+4\beta(1-\alpha)/n}{2\beta}\right)^2 &< \left(\frac{2(1-\alpha)}{n}\right)^2,\\ \nu_1^2 = \left(\frac{c+4\beta(1-\alpha)/n}{2\beta}\right)^2 \frac{n}{2(1-\alpha)} &< \omega_1^2 = \frac{2(1-\alpha)}{n}.\end{aligned}$$

Therefore by Lemmas 4 and 5 it follows that $P_3 > 0$, which contradicts (5).

Otherwise, if (iv) is true, it follows from Lemmas 4 and 6 that $P_3 > 0$ again, which still contradicts (5).

This proves that $h(x,t) > 0$ for all x, t. Finally, letting $\varepsilon \to 0$ with

$$T_2\big|_{\varepsilon=0} = \frac{n}{2(1-\alpha)(-\beta c)}\left(\frac{4\beta(1-\alpha)}{n}+c\right),$$

we get that $\phi(t) \to \phi_0(t)$, where

$$\phi_0(t) = \begin{cases} \dfrac{\left(\dfrac{\beta cn}{cn+8\beta(1-\alpha)}\right)e^{-ct}-\beta}{1-e^{-ct}} & \text{if (iii) holds,}\\ \dfrac{n}{2(1-\alpha)t} & \text{if (iv) holds and } t \le T_2\big|_{\varepsilon=0},\\ \dfrac{-\beta c(e^{c(t-T_2)}+1)}{c+\dfrac{8\beta(1-\alpha)}{n}+ce^{c(t-T_2)}} & \text{if (iv) holds and } t > T_2\big|_{\varepsilon=0}.\end{cases}$$

Therefore $\lim_{\varepsilon\to 0} h(x,t) = \Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_0(t) \ge 0$ as desired. □

3. On complete noncompact manifolds

In this section, we study the case in which the manifold is complete but noncompact. The idea is similar to the case when the manifold is compact without boundary. The main technical difficulty here is to ensure that the minimum of the Harnack quantity is attained in a compact region. We first state our main theorem of this section.

Theorem 7. *Let (M^n, g) be an n-dimensional complete (noncompact) Riemannian manifold with nonnegative Ricci curvature. Let $f(x,t) : M \times [0,\infty) \to \mathbb{R}$ be a positive solution of the Fisher–KPP equation $f_t = \Delta f + cf(1-f)$, where f is C^2 in x and C^1 in t, and $c > 0$ is a constant. Let $u = \log f$. Then we have*

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi_1(t) \ge 0, \tag{6}$$

provided the following constraints are satisfied:

(i) $0<\alpha<1$,

(ii) $\beta < \dfrac{-cn(1+\alpha)}{2(2\alpha^2-2\alpha+n)} < 0$,

(iii) $\dfrac{-cn(2+\sqrt{2})}{4(1-\alpha)} < \beta < \dfrac{-cn(2-\sqrt{2})}{4(1-\alpha)}$,

where

$$\phi_1(t) = \frac{\mu_2\Bigl(\dfrac{1}{\nu_2-\omega_2}e^{2\mu_2\omega_2 t} - \dfrac{1}{\mu_2+\omega_2}\Bigr)}{1-e^{2\mu_2\omega_2 t}},$$

with

$$\mu_2 = \beta c\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}},$$

$$\nu_2 = \Bigl(\frac{4\beta(1-\alpha)}{n}+c\Bigr)\cdot\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}}, \qquad \omega_2 = \sqrt{\frac{2(1-\alpha)}{n}}.$$

Technical lemmas. In this subsection, we state and prove some additional lemmas which will be needed in the proof of Theorem 7. Lemma 8 allows us to substitute the sum P_6+P_7 for P_3+P_4; then Lemma 9 bounds P_6 using a new quantity P_8. Lemma 10 allows us to apply Lemma 5 to control P_8. Lemma 11 gives sufficient conditions for bounding P_7. After bounding P_1, we are in a position to prove our theorem.

For any given $\varepsilon'>0$, let

$$A = A(\varepsilon') := \frac{2\beta^2(1-\alpha)}{n} - \frac{n\Bigl(c+\dfrac{4\beta(1-\alpha)}{n}\Bigr)^2}{8(1-\alpha-\varepsilon')}.$$

Lemma 8. *Let P_3 and P_4 be as defined in Lemma 2. Define*

$$P_6 := Ae^{2u} + e^u\Bigl(\frac{4\beta(1-\alpha)\phi}{n}+c\beta+c\phi\Bigr) + \frac{2(1-\alpha)}{n}\phi^2+\phi_t,$$

$$P_7 := \frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi + \frac{2\varepsilon'}{n}\psi^2 - \Delta\psi.$$

For any $\varepsilon'>0$ and any (x,t) we have

$$P_3+P_4 \ge P_6+P_7.$$

Proof of Lemma 8. Recall that,

$$P_3+P_4 = \frac{2\beta^2(1-\alpha)}{n}e^{2u} + e^u\Bigl(\frac{4\beta(1-\alpha)}{n}\phi + c\phi + c\beta\Bigr) + \frac{2(1-\alpha)}{n}\phi^2+\phi_t$$
$$+\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2 - 2\nabla u\cdot\nabla\psi - \Delta\psi + e^u\psi\Bigl(c+\frac{4\beta(1-\alpha)}{n}\Bigr) + \frac{2(1-\alpha)}{n}\psi^2.$$

We write the last two terms as

$$e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha)}{n}\psi^2=e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha-\varepsilon')}{n}\psi^2+\frac{2\varepsilon'}{n}\psi^2.$$

Using $2xy+x^2\geq -y^2$ in the form

$$e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha-\varepsilon')}{n}\psi^2\geq -\frac{n\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}e^{2u},$$

gives us

$$e^u\psi\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)+\frac{2(1-\alpha)}{n}\psi^2\geq \frac{2\varepsilon'}{n}\psi^2-\frac{n\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}e^{2u}.$$

Applying this inequality then gives

$$\begin{aligned}P_3+P_4\geq e^{2u}&\left(\frac{2\beta^2(1-\alpha)}{n}-\frac{n\Big(c+\frac{4\beta(1-\alpha)}{n}\Big)^2}{8(1-\alpha-\varepsilon')}\right)+e^u\Big(\frac{4\beta(1-\alpha)\phi}{n}+c\beta+c\phi\Big)\\&+\frac{2(1-\alpha)}{n}\phi^2+\phi_t+\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2-2\nabla u\cdot\nabla\psi+\frac{2\varepsilon'}{n}\psi^2-\Delta\psi,\\=P_6+P_7,&\end{aligned}$$

which finishes the proof. □

Lemma 9. *For* $\mu_1=\beta c/(2\sqrt{A})$, $\nu_1=(4\beta(1-\alpha)/n+c)/(2\sqrt{A})$, *and* $\omega_1=\sqrt{2(1-\alpha)/n}$, *define*

$$P_8(\phi):=-(\mu_1+\nu_1\phi)^2+(\omega_1\phi)^2+\phi_t.$$

If $A>0$, *then* $P_8>0$ *implies* $P_6>0$ *for any* (x,t).

Proof of Lemma 9. Recall that

$$P_6=Ae^{2u}+\Big[\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi+\beta c\Big]e^u+\frac{2(1-\alpha)}{n}\phi^2+\phi_t.$$

Since $A>0$, we use the fact that $x^2+xy\geq -\frac{1}{4}y^2$ in the form

$$Ae^{2u}+\Big[\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi+\beta c\Big]e^u\geq -\frac{\Big[\Big(\frac{4\beta(1-\alpha)}{n}+c\Big)\phi+\beta c\Big]^2}{4A}.$$

This gives

$$P_6 \geq -\frac{\left[\left(\frac{4\beta(1-\alpha)}{n}+c\right)\phi+\beta c\right]^2}{4A}+\frac{2(1-\alpha)}{n}\phi^2+\phi_t$$
$$=-\left[\frac{\beta c}{2\sqrt{A}}+\frac{1}{2\sqrt{A}}\left(\frac{4\beta(1-\alpha)}{n}+c\right)\phi\right]^2+\left(\phi\sqrt{\frac{2(1-\alpha)}{n}}\right)^2+\phi_t$$

as desired. □

Lemma 10. *If condition (iii) of Theorem 7 holds, then there always exists some $\varepsilon' > 0$ such that $A > 0$ and $\nu_1^2 < \omega_1^2$.*

Proof of Lemma 10. We first want to show that $A(\varepsilon') > 0$ for some $\varepsilon' > 0$. We will show that $A(0) > 0$, and since A is a continuous function of ε', this implies that $A(\varepsilon') > 0$ for some $\varepsilon' > 0$.

We have

$$A(0) = \frac{2\beta^2(1-\alpha)}{n} - \frac{n\left(c+\frac{4\beta(1-\alpha)}{n}\right)^2}{8(1-\alpha-0)}$$
$$= \frac{16\beta^2(1-\alpha)^2-(cn+4\beta(1-\alpha))^2}{8n(1-\alpha)}$$
$$= \frac{-c^2n^2-8\beta cn(1-\alpha)}{8n(1-\alpha)}.$$

It follows from (iii) that

$$-8 < -4-2\sqrt{2} < \frac{cn}{\beta(1-\alpha)},$$

which rearranges to give $c^2n^2+8\beta cn(1-\alpha) < 0$. Thus $A(0) > 0$, and so there exists some $\varepsilon' > 0$ such that $A(\varepsilon') > 0$.

Next we want to show that $\nu_1^2 < \omega_1^2$ for some $\varepsilon' > 0$, where

$$\nu_1 = \frac{4\beta(1-\alpha)/n+c}{2\sqrt{A}} \quad \text{and} \quad \omega_1 = \sqrt{\frac{2(1-\alpha)}{n}}.$$

Since ν_1 and ω_1 are continuous functions of ε', if we can show that $\nu_1^2 < \omega_1^2$ for $\varepsilon' = 0$, then it must be that $\nu_1^2 < \omega_1^2$ for some $\varepsilon' > 0$.

When $\varepsilon' = 0$, $\nu_1^2 < \omega_1^2$ is equivalent to

$$\frac{c^2n^2+8\beta cn(1-\alpha)+16(1-\alpha)^2\beta^2}{-c^2n^2-8\beta cn(1-\alpha)} < 1.$$

Restriction (iii) implies

$$-4-2\sqrt{2} < \frac{cn}{\beta(1-\alpha)} < -4+2\sqrt{2},$$

which leads to

$$\frac{c^2n^2}{\beta^2(1-\alpha)^2}+\frac{8cn}{\beta(1-\alpha)}+8<0.$$

This is equivalent to

$$c^2n^2+8\beta cn(1-\alpha)+16(1-\alpha)^2\beta^2<-(c^2n^2+8\beta cn(1-\alpha)),$$

and therefore $\nu_1^2<\omega_1^2$ for $\varepsilon'=0$. □

Lemma 11. *Suppose $R\geq 1$ is a constant and $\rho : M^n\to\mathbb{R}$ is a function that satisfies*

$$\rho(x)\geq 0,\qquad |\nabla\rho(x)|\leq 1,\qquad \Delta\rho\leq\frac{c_1}{\rho},$$

for some constant $c_1>0$. Define

$$\psi(x):=k\frac{R^2+\rho^2}{(R^2-\rho^2)^2}. \tag{7}$$

Then for k sufficiently large, $\psi(x)$ satisfies $P_7>0$.

Proof of Lemma 11. Let

$$\Psi(x):=\frac{R^2+\rho^2}{(R^2-\rho^2)^2},$$

so that $\psi=k\Psi$. We claim that Ψ satisfies

$$|\nabla\Psi|^2\leq 18\Psi^3\quad\text{and}\quad\Delta\Psi\leq c_2\Psi^2, \tag{8}$$

where c_2 depends only on c_1.

Indeed, we can compute

$$\nabla\Psi=\nabla\rho\Big(\frac{6\rho R^2+2\rho^3}{(R^2-\rho^2)^3}\Big),$$
$$|\nabla\Psi|^2\leq 4\rho^2\frac{(3R^2+\rho^2)^2}{(R^2-\rho^2)^6}\leq 18\Psi^3,$$

and

$$\begin{aligned}\Delta\Psi&=\Delta\rho\Big(\frac{6\rho R^2+2\rho^3}{(R^2-\rho^2)^3}\Big)+|\nabla\rho|^2\Big(\frac{6R^4+36\rho^2R^2+6\rho^4}{(R^2-\rho^2)^4}\Big)\\&\leq 6c_1\frac{R^2+\rho^2}{(R^2-\rho^2)^3}+18\frac{(R^2+\rho^2)^2}{(R^2-\rho^2)^4}\\&\leq(6c_1+18)\Psi^2.\end{aligned}$$

Recall that

$$P_7=\frac{4\alpha(1-\alpha)}{n}\psi|\nabla u|^2-2\nabla u\cdot\nabla\psi+\frac{2\varepsilon'}{n}\psi^2-\Delta\psi.$$

Completing the square gives us

$$P_7 \geq \frac{2\varepsilon'}{n}\psi^2 - \Delta\psi - \frac{n}{4\alpha(1-\alpha)\psi}|\nabla\psi|^2.$$

By (8), we know that

$$\frac{\varepsilon'}{n}k^2\Psi^2 \geq \frac{\varepsilon' k}{c_2 n}k\Delta\Psi \quad \text{and} \quad \frac{\varepsilon'}{n}k^2\Psi^2 \geq \frac{\varepsilon' k}{18n}\cdot\frac{k^2|\nabla\Psi|^2}{k\Psi},$$

so if

$$k > \max\Big(\frac{c_2 n}{\varepsilon'}, \frac{18n^2}{4\alpha(1-\alpha)\varepsilon'}\Big),$$

we immediately obtain $P_7 > 0$. □

Proof of Theorem 7. Fix a point $p \in M$, let $r = r(x) := d(x, p)$, where $d(\cdot, \cdot)$ denotes the geodesic distance in M. We define the Harnack quantity h on the geodesic ball $B_R(p) := \{x \in M \mid d(x, p) < R\}$. The quantity h depends on the positive constants ε, ε', k, R and is defined as follows:

$$h(x,t) = \Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi(t) + \psi(x),$$

$$\phi = \phi(t) := \frac{\mu_2\Big(\frac{1}{\nu_2-(\omega_2-\varepsilon)}e^{2\mu_2(\omega_2-\varepsilon)t} - \frac{1}{\nu_2+(\omega_2-\varepsilon)}\Big)}{1-e^{2\mu_2(\omega_2-\varepsilon)t}},$$

$$\psi = \psi(x) := k\frac{R^2+r^2}{(R^2-r^2)^2},$$

with μ_2, ν_2, ω_2, and A defined as in Lemma 9 and the paragraph following Theorem 7. Fix $R > 1$. Let ε, ε' and k be positive constants to be chosen later. Note that h is C^1 in t and C^2 in x, except possibly for those x in the cut locus $\mathcal{C}(p)$. We will show that we can choose ε, ε', and k so that $h(x, t) > 0$ for all x, t. Assume for the sake of a contradiction that $h(x, t) \leq 0$ for some x, t.

Let t_1 be the first time t such that $\inf_{x\in B_R(p)} h(x, t) = 0$. Since $\lim_{t\to 0^+} h(t) = \infty$ by Lemma 5, it follows that t_1 exists. Note also that $\psi(x) \to \infty$ as $r = d(x, p)$ approaches R, so the infimum of h is attained inside $B_R(p)$; let (x_1, t_1) be such a point, so that $h(x_1, t_1) = 0$. Now we split into cases based on whether or not x_1 is in the cut locus $\mathcal{C}(p)$.

Case 1: Suppose that $x_1 \notin \mathcal{C}(p)$, so that $\psi(x)$ is twice differentiable at x_1. Then by Lemmas 2 and 3 and 8 we have

$$0 > h_t - \Delta h - 2\nabla h\cdot\nabla u - P_1 h \geq P_2 + P_3 + P_4 \geq P_6 + P_7.$$

By Lemma 10, we can choose $\varepsilon' > 0$ small enough such that $A > 0$ and $\nu^2 < \omega^2$; then, since ϕ is the same as the one defined as in Lemma 5, it follows by Lemmas 5 and 9 that we can choose ε small enough so that $P_6 > 0$.

Note that ψ takes the form of (7), with the distance function $\rho(x)=r(x)=d(x, p)$. We have $r \geq 0$ and $|\nabla r|^2 = 1$; furthermore, by the Laplacian comparison theorem we have $\Delta r \leq (n-1)/r$. Thus we can apply Lemma 11 and choose k sufficiently large such that $P_7 > 0$ as well, which leads to a contradiction.

Case 2: Suppose that $x_1 \in \mathcal{C}(p)$. We apply Calabi's trick. Let $\delta \in (0, d(x_1, p)/2)$ be a positive constant, and let $\gamma(t)$ be any length-minimizing geodesic from p to x_1. Define $p_\delta := \gamma(\delta)$, so that $x_1 \notin \mathcal{C}(p_\delta)$, and define

$$r_\delta(x) := d(x, p_\delta) + \delta, \qquad \psi_\delta(x) := k\frac{R^2 + r_\delta^2}{(R^2 - r_\delta)^2},$$
$$h_\delta := \Delta u + \alpha|\nabla u|^2 + \beta e^u + \phi + \psi_\delta.$$

Note that by the triangle inequality, $r_\delta(x)=d(x, p_\delta)+d(p_\delta, p)\geq r(x)$, with equality at $x = x_1$. Since ψ is an increasing function of r, it follows that $\psi_\delta(x) \geq \psi(x)$ with equality at $x = x_1$. This implies that (x_1, t_1) is still the first time and place where $h_\delta(x, t) = 0$. Furthermore, h_δ is now C^2 at (x_1, t_1) so applying Lemmas 2, 3, 8, 5, and 9 gives that $0 > P_7$.

Note that clearly $r_\delta \geq 0$ and $|\nabla r_\delta| \leq 1$, and at x_1 we get

$$\Delta r_\delta = \Delta(d(x_1, p_\delta)) \leq \frac{n-1}{d(x_1, p_\delta)} = \frac{n-1}{r(x_1)-\delta} \leq \frac{2(n-1)}{r(x_1)},$$

since we assumed that $\delta \leq \frac{1}{2}r(x_1)$. Therefore applying Lemma 11 gets us a contradiction in this case as well.

This shows that $h(x, t) > 0$ for all x, t. Since h varies continuously as a function of $R, \varepsilon, \varepsilon'$, we can take the limit $R \to \infty$ to get $\psi \to 0$. Then by taking $\varepsilon, \varepsilon' \to 0$, we get that $\phi \to \phi_1$ and so

$$\Delta u + \alpha|\nabla u|^2 + \beta e^u + \frac{\mu_2\left(\dfrac{1}{\nu_2-\omega_2}e^{2\mu_2\omega_2 t} - \dfrac{1}{\mu_2+\omega_2}\right)}{1 - e^{2\mu_2\omega_2 t}} \geq 0,$$

with

$$\mu_2 = \beta c\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}},$$
$$\nu_2 = \left(\frac{4\beta(1-\alpha)}{n} + c\right)\cdot\sqrt{\frac{2(1-\alpha)}{c(-cn-8\beta(1-\alpha))}}, \qquad \omega_2 = \sqrt{\frac{2(1-\alpha)}{n}},$$

which finishes the proof. □

4. Applications

In this section, we derive two applications of our differential Harnack estimates.

Bounds on the wave speed of traveling wave solutions. The first such application shows that our Harnack inequality can be used to prove an interesting fact about traveling wave solutions to Fisher's equation. In particular we look at traveling plane waves, i.e., solutions to (1) of the form

$$f(x,t) = v(z) := v(x + \eta t\hat{a}),$$

for some function $v : \mathbb{R}^n \to \mathbb{R}$ and some wave direction $\hat{a} \in \mathbb{R}^n$, $|\hat{a}| = 1$ and wave speed $\eta > 0$. For $n = 1$, these solutions were first studied by Fisher [1937] (also see [Kolmogorov et al. 1937; Sherratt 1998]) and were considered by him to be a natural model for propagation of mutations. He was able to show that if $n = 1$ and $\lim_{t\to-\infty} f(x,t) = 0$, then it must be that $\eta \ge 2\sqrt{c}$.

We will show a weaker bound that generalizes to higher dimensions.

Theorem 12. *Let $f(x,t) = v(x + \eta t\hat{a})$ be a traveling plane wave solution to (1), with wave speed η and wave direction $\hat{a}$. Suppose that*

$$\lim_{\substack{x=k\hat{b},\\ k\to\infty}} v(x) = 0 \quad \text{for some direction } \hat{b} \in \mathbb{R}^n,\ |\hat{b}| \ne 0. \tag{9}$$

Then

$$\eta \ge \begin{cases} \sqrt{(3-\sqrt{3})c} & \text{if } n = 1,\\ \sqrt{2c} & \text{if } n = 2,\\ \sqrt{(7-3\sqrt{3})c} & \text{if } n = 3. \end{cases}$$

When $n = 1$, $\eta \ge 2\sqrt{c}$ is both a necessary and sufficient condition for the existence of traveling wave solutions. The same condition is sufficient in any higher dimension, but it is not known (at least to us) if it is necessary as well. Our bounds above give a weaker necessary wave speed in dimensions two and three.

Remark. In the proof below we have not used the fact that the traveling wave v approaches 1 in some direction. Although we were ourselves unsuccessful, the authors would like to encourage an attempt to use this additional restriction to obtain a better bound on the wave speed η.

Lemma 13. *For any $v(z)$ and any η that satisfy the conditions of Theorem 12, and for any α, β that satisfy (i), (ii), and (iii) as in Theorem 7, we have*

$$\eta^2 \ge M' := 4(1-\alpha)[(c - \phi(t)) - (\beta + c)v(z)],$$

for all x, t, where

$$\phi(t) = \frac{\mu\left(\frac{1}{\nu-\omega}e^{2\mu\omega t} - \frac{1}{\nu+\omega}\right)}{(1 - e^{2\mu\omega t})}$$

(*which appears as $\phi_1(t)$ in the statement of Theorem 7*).

Proof. Since Fisher's equation is spherically symmetric, we may assume without loss of generality that $\hat{a} = \hat{x}_1 = (1, 0, 0, \dots, 0)$. Therefore

$$f(x,t) = v(x_1 + \eta t, x_2, \dots, x_n) = v(z_1, z_2, \dots, z_n) = v(\hat{z}).$$

It then follows from (1) that (where $\partial_i := \partial/\partial z_i$)

$$\eta \partial_1 v = \Delta v + cv(1-v).$$

Combining this with Theorem 7 gives

$$\begin{aligned}
\Delta(\log v) + \alpha|\nabla(\log v)|^2 + \beta v + \phi &\geq 0,\\
\frac{\Delta v}{v} - (1-\alpha)\frac{|\nabla v|^2}{v^2} + \beta v + \phi &\geq 0,\\
\frac{\eta\partial_1 v - cv(1-v)}{v} - (1-\alpha)\frac{|\nabla v|^2}{v^2} + \beta v + \phi &\geq 0,\\
(1-\alpha)\frac{\sum_{i=2}^n (\partial_i v)^2}{v^2} + (1-\alpha)\frac{(\partial_1 v)^2}{v^2} - \eta\frac{\partial_1 v}{v} - (\beta + c)v + (c - \phi) &\leq 0.
\end{aligned}$$

It follows from standard Cauchy–Schwarz that

$$-\frac{\eta^2}{4(1-\alpha)} - (\beta + c)v + (c - \phi) \leq 0,$$

hence $\eta^2 \geq 4(1-\alpha)[(c-\phi) - (\beta + c)v]$, as desired. □

Lemma 14. *Assume that $v(x) \to 0$ along some path, as in (9). Then for any $\varepsilon_3 > 0$ there exists (x_3, t_3), possibly depending on n, α, β, and c, such that at (x_3, t_3)*

$$M' > M'' - \tfrac{1}{3}\varepsilon_3,$$

where

$$M'' := 4(1-\alpha)\Big(c - \frac{-\mu}{\nu + \omega}\Big).$$

Proof. Fix $\varepsilon_3 > 0$. Note that

$$\lim_{t\to\infty} \phi(t) = \frac{-\mu}{\nu + \omega}.$$

Choosing $t \geq t_3$ large enough gives

$$\Big|\phi(t_3) - \frac{-\mu}{\nu + \omega}\Big| < \frac{\varepsilon_3}{24(1-\alpha)},$$

so that

$$4(1-\alpha)(c - \phi) > 4(1-\alpha)\Big(c - \frac{-\mu}{\nu + \omega}\Big) - \frac{\varepsilon_3}{6}.$$

Having fixed t_3, we then set $x_3 := -\eta t_3 \hat{a} + \lambda \hat{b}$ with λ sufficiently large. Then by (9) it follows that

$$|v-0| < \frac{\varepsilon_3}{24(1-\alpha)} \frac{1}{|\beta+c|} \quad \text{and} \quad -4(1-\alpha)(\beta+c)v > 0 - \tfrac{1}{6}\varepsilon_3.$$

Therefore

$$M' = 4(1-\alpha)[(c-\phi) - (\beta+c)\phi] > M'' - \tfrac{1}{3}\varepsilon_3. \qquad \square$$

Remark. Condition (9) can be weakened; it suffices to have $\lim_{z\to\infty} v(z) = 0$ along some path that goes to infinity.

Lemma 15. *If* $n \leq 3$, *and* $\beta = -cn(1+\alpha)/(4\alpha^2 - 4\alpha + 2n)$, *and* $0 < \alpha < \alpha_0(\varepsilon_3)$ *is sufficiently close to* 0, *then conditions* (i), (ii), *and* (iii) *are satisfied*, *and* $M'' > M''' - \frac{1}{3}\varepsilon_3$, *where*

$$M''' := M'''(n) = 2c\left(\frac{n-4+2\sqrt{4n-n^2}}{n-2+\sqrt{4n-n^2}}\right).$$

Proof. Conditions (i) and (ii) are clearly satisfied by construction. And note that (iii) is equivalent to

$$-\frac{2+\sqrt{2}}{4} < \frac{\beta(1-\alpha)}{cn} < -\frac{2-\sqrt{2}}{4}.$$

But the quantity in the middle varies continuously with α near $\alpha = 0$, so it suffices to check it at $\alpha = 0$, where we indeed have

$$-\frac{2+\sqrt{2}}{4} < -\frac{1}{2n} < -\frac{2-\sqrt{2}}{4},$$

which holds for all $n \leq 3$, so there must exist some α_0 sufficiently small such that (iii) holds for all $\alpha < \alpha_0$.

Next, we compute M'':

$$\begin{aligned}
M'' &= 4(1-\alpha)\left(c - \frac{-\mu}{\nu+\omega}\right) \\
&= 4(1-\alpha)\left(c + \frac{\frac{\beta c}{2\sqrt{A}}}{\frac{1}{2\sqrt{A}}\left(\frac{4\beta(1-\alpha)}{n}+c\right) + \sqrt{\frac{2(1-\alpha)}{n}}}\right) \\
&= 4(1-\alpha)\left(c + \frac{\beta c}{\left(c+\frac{4\beta(1-\alpha)}{n}\right) + \sqrt{\frac{8A(1-\alpha)}{n}}}\right).
\end{aligned}$$

Here $A = A(\varepsilon' = 0)$, so that

$$\frac{8A(1-\alpha)}{n} = \frac{16\beta^2(1-\alpha)^2}{n^2} - \left(c + \frac{4\beta(1-\alpha)}{n}\right)^2 = c^2\left(-1 - \frac{8\beta(1-\alpha)}{cn}\right).$$

This gives

$$M'' = 4(1-\alpha)c\left(1 + \frac{\beta/c}{1+\frac{4\beta(1-\alpha)}{cn} + \sqrt{-1-\frac{8\beta(1-\alpha)}{cn}}}\right).$$

Again, this involves only $(1-\alpha)$ and β, both of which are continuous at $\alpha = 0$, where we have $\beta = -c/2$, so

$$M'' = 4c\left(1 + \frac{-1/2}{1-\frac{2}{n}+\sqrt{-1+\frac{4}{n}}}\right) = 2c\left(2 - \frac{n}{n-2+\sqrt{4n-n^2}}\right) = M'''.$$

Hence for α sufficiently close to 0 we can get $|M'' - M'''| < \varepsilon_3/3$, which gives us the desired conclusion. □

Proof of Theorem 12. Fix a solution $f(x,t) = v(x + \eta t\hat{a})$ of (1) which also satisfies (9), and fix a $\varepsilon_3 > 0$.

Let $\alpha < \alpha_0$ and $\beta = -c/(2(1-\alpha))$, so that (i), (ii), (iii) are satisfied (by Lemma 15). Applying Lemma 13 then gives $\eta^2 \geq M$ for all x, t.

Applying Lemma 14, we find a pair (x_3, t_3) such that $M' > M'' - \varepsilon_3/3$. Then applying Lemma 15 again, we have $M'' > M''' - \varepsilon_3/3$ so that

$$\eta^2 > M''' - \varepsilon_3.$$

However, note that M''' depends only on n. Hence we send $\varepsilon_3 \to 0$, to get that

$$\eta^2 \geq M'''(n) = \begin{cases} c(3-\sqrt{3}), & n = 1, \\ 2c, & n = 2, \\ c(7-3\sqrt{3}), & n = 3, \end{cases}$$

as desired. □

Classical Harnack inequality. In this subsection, we integrate our differential Harnack estimates along a spacetime curve to derive classical Harnack inequalities. We further assume that M is closed, and that $f(x,t) < 1$ for all x, t.

Theorem 16. *Let M be a closed Riemannian manifold with nonnegative Ricci curvature, and $0 < f < 1$ be a bounded positive solution to Fisher's equation. Let α and β satisfy the conditions of Theorem 1. Furthermore, if $\alpha \leq n/4$, then there will always exist β such that $\beta + c \geq 0$ in addition to the constraints of Theorem 1. For such an α and β,*

(i) *if* $8\beta(1-\alpha)+cn<0$, *then we have*

$$\frac{f(x_2,t_2)}{f(x_1,t_1)} \ge \left(\frac{1-e^{-ct_2}}{1-e^{-ct_1}}\right)^{\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}} \exp\left(-\frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}\right);$$

(ii) *if* $8\beta(1-\alpha)+cn>0,\ t_2>t_1>T_2$, *then we have*

$$\frac{f(x_2,t_2)}{f(x_1,t_1)} \ge \left[\frac{\left(1+\dfrac{8\beta(1-\alpha)}{cn}\right)e^{-c(t_2-T_2)}+1}{\left(1+\dfrac{8\beta(1-\alpha)}{cn}\right)e^{-c(t_1-T_2)}+1}\right]^{\frac{8\beta^2(1-\alpha)}{c(cn+8\beta(1-\alpha))}} \exp\left(-\frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}\right);$$

(iii) *if* $8\beta(1-\alpha)+cn=0,\ t_2>t_1>T_2$, *then we have*

$$\frac{f(x_2,t_2)}{f(x_1,t_1)} \ge \exp\left[-\frac{\beta}{c}\left(e^{-c(t_2-T_2)}-e^{-c(t_1-T_2)}\right)\right]\exp\left(-\frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}\right).$$

Proof of Theorem 16. Let $f(x,t)$ solve $f_t=\Delta f+cf(1-f)$, and $u=\log f$. Fix points (x_1,t_1), (x_2,t_2) and let $\gamma:[t_1,t_2]\to M^n$ be an arbitrary spacetime path connecting them, i.e., $\gamma(t_1)=x_1$, $\gamma(t_2)=x_2$.

Let $v(t):=u(\gamma(t),t)$ be the value of u along γ. We compute

$$v'(t)=u_t+\nabla u\cdot\frac{d\gamma}{dt}.$$

Using the time evolution for $u_t=(\log f)_t=f_t/f$, this is equal to

$$v'(t)=\Delta u+|\nabla u|^2+c(1-e^u)+\nabla u\cdot\frac{d\gamma}{dt}.$$

Applying the Harnack inequality gives

$$v'(t)\ge(1-\alpha)|\nabla u|^2+(c-\phi)-(\beta+c)e^u+\nabla u\cdot\frac{d\gamma}{dt}.$$

By assumption, $f<1$ and $\beta+c\ge 0$. This implies $-(\beta+c)e^u\ge-(\beta+c)$, so defining $\tilde{\phi}(t)=-\beta-\phi(t)$, we then get

$$\begin{aligned}
v'(t)&\ge(1-\alpha)|\nabla u|^2+(c-\phi)-(\beta+c)+\nabla u\cdot\frac{d\gamma}{dt}\\
&=-\beta-\phi+(1-\alpha)|\nabla u|^2+\nabla u\cdot\frac{d\gamma}{dt}\\
&=\tilde{\phi}(t)+(1-\alpha)|\nabla u|^2+\nabla u\cdot\frac{d\gamma}{dt},\\
v'(t)&\ge\tilde{\phi}(t)-\frac{1}{4(1-\alpha)}\left|\frac{d\gamma}{dt}\right|^2.
\end{aligned}$$

Integrating in time, we get

$$u(x_2,t_2)-u(x_1,t_1)=v(t_2)-v(t_1)=\int_{t_1}^{t_2} v'(t)\,dt \geq \int_{t_1}^{t_2} \tilde{\phi}(t)\,dt - \frac{1}{4(1-\alpha)}\int_{t_1}^{t_2} \left|\frac{d\gamma}{dt}\right|^2 dt.$$

Since γ was chosen to be an arbitrary path, we can choose it to be the path minimizing $\int |\gamma'|^2$, which is the minimizing geodesic between the two endpoints. The integral thus becomes

$$\int_{t_1}^{t_2} |\gamma'|^2\,dt = \frac{d(x_1,x_2)^2}{t_2-t_1}.$$

Thus the spacetime Harnack is given by

$$\log\left(\frac{f(x_2,t_2)}{f(x_1,t_1)}\right) = u(x_2,t_2)-u(x_1,t_1) \geq \int_{t_1}^{t_2} \tilde{\phi}(t)\,dt - \frac{d(x_1,x_2)^2}{4(1-\alpha)(t_2-t_1)}.$$

We compute the definite integral, dividing into three cases. First we deal with the case $8\beta(1-\alpha)+cn<0$. In this case we have

$$\phi(t) = \frac{\left(\frac{\beta cn}{cn+8\beta(1-\alpha)}\right)e^{-ct}-\beta}{1-e^{-ct}},$$

and

$$\tilde{\phi}(t) = \left(\beta e^{-ct} - \frac{\beta cne^{-ct}}{cn+8\beta(1-\alpha)}\right)\frac{1}{1-e^{-ct}} = \beta\cdot\frac{8\beta(1-\alpha)}{cn+8\beta(1-\alpha)}\cdot\frac{e^{-ct}}{1-e^{-ct}}.$$

Then we can explicitly integrate

$$\int_{t_1}^{t_2} \tilde{\phi}(t)\,dt = \frac{\beta}{c}\left(\frac{8\beta(1-\alpha)}{cn+8\beta(1-\alpha)}\right)\log\left[\frac{1-e^{-ct_2}}{1-e^{-ct_1}}\right].$$

Therefore we get that

$$\exp\left(\int_{t_1}^{t_2} \tilde{\phi}(t)\,dt\right) = \left(\frac{1-e^{-ct_2}}{1-e^{-ct_1}}\right)^{\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}},$$

and the claim follows.

Second, we deal with the case $8\beta(1-\alpha)+cn>0$. Then for $t>T_2$ (recall that T_2 is a constant) we have

$$\phi(t) = \frac{-\beta cne^{c(t-T_2)}-\beta cn}{cn+8\beta(1-\alpha)+cne^{c(t-T_2)}},$$

and so

$$\tilde{\phi}(t) = -\beta-\phi(t) = \frac{-8\beta^2(1-\alpha)e^{-c(t-T_2)}}{(8\beta(1-\alpha)+cn)e^{-c(t-T_2)}+cn}.$$

If we let $B = -8\beta^2(1-\alpha)$ and $D = cn + 8\beta(1-\alpha)$, then we get that

$$\tilde{\phi}(t) = \frac{Be^{-c(t-T_2)}}{De^{-c(t-T_2)}+cn}.$$

We can integrate

$$\int_{t_1}^{t_2} \tilde{\phi}(t)\,dt = \Big(\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}\Big)\log\Big(\frac{(8\beta(1-\alpha)+cn)e^{-c(t_2-T_2)}+cn}{(8\beta(1-\alpha)+cn)e^{-c(t_1-T_2)}+cn}\Big).$$

Therefore

$$\exp\Big(\int_{t_1}^{t_2} \tilde{\phi}(t)\,dt\Big) = \left[\frac{\Big(1+8\beta(1-\alpha)/(cn)\Big)e^{-c(t_2-T_2)}+1}{\Big(1+8\beta(1-\alpha)/(cn)\Big)e^{-c(t_1-T_2)}+1}\right]^{\frac{8\beta^2(1-\alpha)}{c^2n+8\beta c(1-\alpha)}}$$

as claimed in the statement of Theorem 16.

In the last case that $8\beta(1-\alpha)+cn = 0$, we have

$$\phi(t) = \frac{-\beta e^{c(t-T_2)}-\beta}{e^{c(t-T_2)}},$$

and so

$$\tilde{\phi}(t) = -\beta - \phi(t) = \frac{\beta}{e^{c(t-T_2)}}.$$

Therefore

$$\exp\Big(\int_{t_1}^{t_2} \tilde{\phi}(t)\,dt\Big) = \exp\Big[-\frac{\beta}{c}\big(e^{-c(t_2-T_2)} - e^{-c(t_1-T_2)}\big)\Big]$$

as desired.

To finish the proof of our theorem we need to show that we can choose $\beta + c \geq 0$, i.e., $\beta \geq -c$. We have the constraint (ii):

$$\beta \leq \frac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n},$$

so we need to have

$$-c \leq \beta \leq \frac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n}.$$

Note that since $0 < \alpha < 1$, we have $4\alpha^2 - 4\alpha + 2n \geq -1 + 2n \geq 1$; thus it remains to choose α so that

$$-(4\alpha^2 - 4\alpha + 2n) \leq -n(1+\alpha),$$

which simplifies to

$$\alpha \leq \tfrac{1}{4}n.$$

This is automatically true if $n \geq 4$, which means we can choose any α we wish, and there will be at least one β satisfying all the constraints including $\beta + c \geq 0$. $\square$

Note that $\lim_{t\to\infty}\phi(t) = -\beta$, and $\lim_{t\to\infty}\widetilde{\phi}(t) = 0$. Thus, as $t_1, t_2 \to \infty$, the estimate approaches the classical Li–Yau–Harnack [Li and Yau 1986].

Remark. In the compact case we obtain a good bound as t_1 and t_2 get large. In the complete noncompact case, one can still integrate along spacetime curves to obtain an inequality, but the estimate degenerates when time becomes large.

Acknowledgements

Cao's research was partially supported by NSF grant DMS 0904432. Liu, Pendleton, and Ward's research was supported by NSF grant DMS 1156350 through the Research Experience for Undergraduates Program at Cornell University during the summer of 2012. The authors would like to thank Professor Robert Strichartz for his encouragement and Benjamin Fayyazuddin-Ljungberg and Hung Tran for helpful discussions. They also thank the referee for many detailed suggestions which improved the quality of this paper.

References

[Aronson and Bénilan 1979] D. G. Aronson and P. Bénilan, "Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$", *C. R. Acad. Sci. Paris Sér. A-B* **288**:2 (1979), A103–A105. MR Zbl

[Cao 2008] X. Cao, "Differential Harnack estimates for backward heat equations with potentials under the Ricci flow", *J. Funct. Anal.* **255**:4 (2008), 1024–1038. MR Zbl

[Cao and Hamilton 2009] X. Cao and R. S. Hamilton, "Differential Harnack estimates for time-dependent heat equations with potentials", *Geom. Funct. Anal.* **19**:4 (2009), 989–1000. MR Zbl

[Cao et al. 2015] X. Cao, M. Cerenzia, and D. Kazaras, "Harnack estimate for the endangered species equation", *Proc. Amer. Math. Soc.* **143**:10 (2015), 4537–4545. MR Zbl

[Fisher 1937] R. A. Fisher, "The wave of advance of advantageous genes", *Ann. Eugenics* **7**:4 (1937), 355–369. Zbl

[Fujita 1966] H. Fujita, "On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$", *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124. MR Zbl

[Hamilton 1993] R. S. Hamilton, "The Harnack estimate for the Ricci flow", *J. Differential Geom.* **37**:1 (1993), 225–243. MR Zbl

[Hamilton 2011] R. S. Hamilton, "Li–Yau estimates and their Harnack inequalities", pp. 329–362 in *Geometry and analysis, I*, edited by L. Ji, Adv. Lect. Math. **17**, International Press, Somerville, MA, 2011. MR Zbl

[Kolmogorov et al. 1937] A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov, "A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem", *Bull. Univ. Moscou, Sér. Int. A* **1**:6 (1937), 1–26. In Russian; translated in *Selected works of A. N. Kolmogorov*, edited by V. M. Tikhomirov, Math. Appl. (Soviet Series) **25**, Springer, 1991, pp. 242–270. Zbl

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR Zbl

[McKean 1975] H. P. McKean, "Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov", *Comm. Pure Appl. Math.* **28**:3 (1975), 323–331. MR Zbl

[Ó Náraigh and Kamhawi 2013] L. Ó Náraigh and K. Kamhawi, "Multiscale methods and modelling for chemical reactions on oscillating surfaces", *IMA J. Appl. Math.* **78**:3 (2013), 537–565. MR Zbl

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. Zbl arXiv

[Sherratt 1998] J. A. Sherratt, "On the transition from initial data to travelling waves in the Fisher–KPP equation", *Dynam. Stability Systems* **13**:2 (1998), 167–174. MR Zbl

[Tuckwell 1988] H. C. Tuckwell, *Introduction to theoretical neurobiology, II: Nonlinear and stochastic theories*, Cambridge Studies in Mathematical Biology **8**, Cambridge University Press, 1988. MR Zbl

Received October 26, 2015. Revised August 30, 2016.

XIAODONG CAO
DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NY 14853
UNITED STATES
cao@math.cornell.edu

BOWEI LIU
DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CA 94305
UNITED STATES
bowei@math.stanford.edu

IAN PENDLETON
DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NY 14853
UNITED STATES
iap26@cornell.edu

ABIGAIL WARD
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF CHICAGO
CHICAGO, IL 60637
UNITED STATES
abigailward@uchicago.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.301

A DIRECT METHOD OF MOVING PLANES FOR THE SYSTEM OF THE FRACTIONAL LAPLACIAN

CHUNXIA CHENG, ZHONGXUE LÜ AND YINGSHU LÜ

We establish a direct method of moving planes for systems of fractional Laplacian equations. By using this direct method of moving planes, we obtain symmetry and nonexistence of positive solutions for the following system of fractional Laplacian equations:

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n. \end{cases}$$

1. Introduction

In this paper, we consider the following system of fractional Laplacian equations:

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n. \end{cases} \tag{1-1}$$

When $\alpha = 2$, system (1-1) is an important model, the Lane–Emden system. It is conjectured that if $1/(p+1) + 1/(q+1) > (n-2)/n$, then there are no nontrivial classical solutions of (1-1) in $\mathbb{R}^N$ with $N \geq 3$. The conjecture has been proved to be true for radial solutions in all dimensions in [Mitidieri 1996]. The cases of $N = 3, 4$ for the conjecture in general have also been solved recently in [Poláčik et al. 2007] and [Souplet 2009], respectively. The interested reader can refer to the above papers for detailed descriptions (see also the works [Busca and Manásevich 2002; Serrin and Zou 1998], etc.).

More generally, Troy [1981] used the *maximum principle* and the *method of moving parallel planes* to investigate symmetry properties of solutions of systems of semilinear elliptic equations $\Delta u_i + f_i(u_1, \dots, u_n) = 0$, $i = 1, \dots, n$, in a domain of $\mathbb{R}^n$.

The work was partially supported by NSFC(No.11271166), sponsored by Qing Lan Project.
MSC2010: 35B09, 35B50, 35B53, 35J61.
Keywords: the fractional Laplacian, maximum principles for antisymmetric functions, narrow region principle, decay at infinity, method of moving planes, radial symmetry, nonexistence of positive solutions.

In the special case $p = q,\ u = v$, (1-1) changes to

$$(-\Delta)^{\alpha/2} u(x) = u^p(x), \quad x \in \mathbb{R}^n. \tag{1-2}$$

Here the fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudodifferential operator assuming the form

$$(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x-z|^{n+\alpha}}\, dz, \tag{1-3}$$

where α is any real number between 0 and 2. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^n$. In this space, it can also be equivalently defined in terms of the Fourier transform

$$\widehat{(-\Delta)^{\alpha/2}} u(\xi) = |\xi|^\alpha \hat{u}(\xi),$$

where $\hat{u}$ is the Fourier transform of u. One can extend this operator to a wider space of functions.

Let

$$L_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}}\, dx < \infty \right\}.$$

Then it is easy to verify that for $u \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}$, the integral on the right-hand side of (1-3) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

The nonlocality of the fractional Laplacian makes it difficult to study. To circumvent this difficulty, Caffarelli and Silvestre [2007] introduced the *extension method*, which reduced this nonlocal problem into a local one in higher dimensions. For a function $u : \mathbb{R}^n \to \mathbb{R}$, consider the extension $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases} \operatorname{div}(y^{1-\alpha} \nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x). \end{cases}$$

Then

$$(-\Delta)^{\alpha/2} u = -C_{n,\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}.$$

This extension method has been applied successfully to study equations involving the fractional Laplacian, and a series of fruitful results have been obtained (see the references in that work).

In [Busca and Manásevich 2002], among many interesting results, when the authors considered the properties of the positive solutions for (1-2), they first used the above extension method to reduce the nonlocal problem into a local one for $U(x, y)$ in one higher dimensional half space $\mathbb{R}^n \times [0, \infty)$, then applied the method of moving planes to show the symmetry of $U(x, y)$ in x, and hence derived the nonexistence in the subcritical case.

Proposition 1.1. *Let* $1 \le \alpha < 2$. *Then the problem*

$$\begin{cases} \operatorname{div}(y^{1-\alpha}\nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ -\lim\limits_{y\to 0^+} y^{1-\alpha}\frac{\partial U}{\partial y} = U^p(x, 0), & x \in \mathbb{R}^n, \end{cases}$$

has no positive bounded solution provided $p < (n+\alpha)/(n-\alpha)$.

They then took trace to obtain:

Corollary 1.2. *Assume that* $1 \le \alpha < 2$ *and* $1 < p < (n-\alpha)/(n-\alpha)$. *Then equation* (1-2) *possesses no bounded positive solution.*

A similar extension method was adapted in [Chen and Zhu 2016] to obtain the nonexistence of positive solutions for an indefinite fractional problem.

Proposition 1.3. *Let* $1 \le \alpha < 2$ *and* $1 < p < \infty$. *Then the equation*

$$(-\Delta)^{\alpha/2} = x_1 u^p, \quad x \in \mathbb{R}^n,$$

possesses no positive bounded solutions.

The common restriction $\alpha \ge 1$ is due to the approach that they need to carry out the method of moving planes on the solutions U of the extended problem

$$\operatorname{div}(y^{1-\alpha}\nabla U) = 0, \quad (x, y) \in \mathbb{R}^n \times [0, \infty). \tag{1-4}$$

Because of the monotonicity requirement, they have to assume that $\alpha \ge 1$.

Jarohs and Weth [2016] without going through the extended equation (1-4), introduced *antisymmetric maximum principles* and applied them to carry on the method of moving planes directly on nonlocal problems to show the symmetry of solutions. The operators they considered are quite general; however, their maximum principles only apply to bounded regions.

Chen, Li and Li [Chen et al. 2017] developed a systematic approach to carry out the method of moving planes for nonlocal problems, either on bounded or unbounded domains, corresponding to approaches for local elliptic operators that were introduced more than twenty years ago in the paper [Chen and Li 1991] and then summarized in the book [Chen and Li 2010].

In this paper, we will establish the direct method of moving planes for the system of the fractional Laplacian equations. This will be accomplished in Section 2, in which the main results are the following:

Theorem 2.1 (maximum principle for antisymmetric functions). *Let* T *be a hyperplane in* $\mathbb{R}^n$. *Without loss of generality, we may assume that*

$$T = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \quad \textit{for some } \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \quad \text{and} \quad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded domain in H. Assume that $u \in L_\alpha \cap C^{1,1}_{\rm loc}(\Omega)$ and is lower semicontinuous on $\overline{\Omega}$. If

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ v(\tilde{x}) = -v(x) & \text{in } H, \end{cases}$$

then

$$u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 \quad \text{in } \ \Omega.$$

This conclusion holds for unbounded region Ω if we further assume that

$$\varliminf_{|x|\to\infty} u(x) \geq 0 \quad \text{and} \quad \varliminf_{|x|\to\infty} v(x) \geq 0.$$

If $u = 0$ and $v = 0$ at some point in Ω, then

$$u(x) = 0 \quad \text{and} \quad v(x) = 0 \quad \text{almost everywhere in } \mathbb{R}^n.$$

Theorem 2.2 (narrow region principle). *Let T be a hyperplane in $\mathbb{R}^n$. Without loss of generality,we may assume that*

$$T = \{x = (x^1, x') \in \mathbb{R}^n \mid x_1 = \lambda \quad \text{for some } \ \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \ldots, x_n),$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \quad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded narrow region in H such that it is contained in $\{x \mid \lambda - l < x_1 < \lambda\}$ with small l. Suppose that $u, v \in L_\alpha \cap C^{1,1}_{\rm loc}(\Omega)$ and both are lower semicontinuous on $\overline{\Omega}$. If $c_1(x)$ and $c_2(x)$ are both bounded from below in Ω, $c_1(x) \leq 0$ and $c_2(x) \leq 0$ and

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x) v(x) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x) u(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ u(\tilde{x}) = -u(x) & \text{in } H, \end{cases}$$

then for sufficiently small l, we have

$$u(x) \geq 0 \quad \textit{and} \quad v(x) \geq 0 \quad \textit{in} \quad \Omega.$$

This conclusion holds for unbounded regions Ω if we further assume that

$$\varliminf_{|x|\to\infty} u(x) \geq 0 \quad \textit{and} \quad \varliminf_{|x|\to\infty} v(x) \geq 0.$$

Theorem 2.3 (decay at infinity). *Let $H = \{x \in \mathbb{R}^n \mid x_1 < \lambda \quad \text{for some} \quad \lambda \in \mathbb{R}\}$ and let Ω be an unbounded region in H. Assume*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x) v(x) \geq 0 & \textit{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x) u(x) \geq 0 & \textit{in } \Omega, \\ u(x) \geq 0 \quad \textit{and} \quad v(x) \geq 0 & \textit{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \textit{in } H, \\ v(\tilde{x}) = -v(x) & \textit{in } H, \end{cases}$$

with

$$\varliminf_{|x|\to\infty} |x|^\alpha c_1(x) = 0, \quad c_1(x) \leq 0,$$

and

$$\varliminf_{|x|\to\infty} |x|^\alpha c_2(x) = 0, \quad c_2(x) \leq 0,$$

then there exists a constant R_0 such that if

$$u(x^0) = \min_\Omega u(x) < 0 \quad \textit{or} \quad v(x^0) = \min_\Omega v(x) < 0,$$

then

$$|x^0| \leq R_0.$$

As a simple application, we consider system (1-1).

Theorem 3.1. *Assume that $0 < \alpha < 2$ and $u, v \in L_\alpha \cup C^{1,1}_{\mathrm{loc}}$ is a nonnegative solution of equation (1-1). Then*

(i) *in the subcritical case $1 < p, q < (n+\alpha)/(n-\alpha)$, $(u, v) \equiv (0, 0)$;*

(ii) *in the critical case $p = q = (n+\alpha)/(n-\alpha)$, (u, v) is radially symmetric about some point.*

2. Various maximum principles

Maximum principle for antisymmetric functions.

Theorem 2.1. *Let T be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that*

$$T = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for some } \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \quad \text{and} \quad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded domain in H. Assume that $u \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ is lower semicontinuous on $\bar{\Omega}$. If

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ v(\tilde{x}) = -v(x) & \text{in } H, \end{cases} \tag{2-1}$$

then

$$u(x) \geq 0 \quad \text{and} \quad v(x) \geq 0 \quad \text{in } \Omega. \tag{2-2}$$

This conclusion holds for unbounded region Ω if we further assume that

$$\varliminf_{|x| \to \infty} u(x) \geq 0 \quad \text{and} \quad \varliminf_{|x| \to \infty} v(x) \geq 0.$$

If $u = 0$ and $v = 0$ at some point in Ω, then

$$u(x) = 0 \quad \text{and} \quad v(x) = 0 \quad \text{almost everywhere in } \mathbb{R}^n.$$

Proof. If (2-2) does not hold, then the lower semicontinuity of u and v on $\bar{\Omega}$ indicates that there exists a $x^0 \in \bar{\Omega}$ such that

$$u(x^0) = \min_{\bar{\Omega}} u < 0$$

or

$$v(x^0) = \min_{\bar{\Omega}} v < 0,$$

and one can further deduce from condition (2-1) that x^0 is in the interior of Ω.

If $u(x^0) < 0$, it follows that

$$\begin{aligned}
(-\Delta)^{\alpha/2} u(x^0) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy + \int_{\tilde{H}} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy + \int_H \frac{u(x^0) - u(\tilde{y})}{|x^0 - \tilde{y}|^{n+\alpha}} \, dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy + \int_H \frac{u(x^0) + u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} \, dy \right\} \\
&\leq C_{n,\alpha} \int_H \left\{ \frac{u(x^0) - u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} + \frac{u(x^0) + u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} \right\} dy \\
&= C_{n,\alpha} \int_H \frac{2u(x^0)}{|x^0 - \tilde{y}|^{n+\alpha}} \, dy \\
&< 0,
\end{aligned}$$

which contradicts inequality (2-1).

Similarly, if $v(x^0) < 0$, we also get a contradiction with (2-1). This verifies (2-2).

Now we show that $u \geq 0$ and $v \geq 0$ in H. If there is some point $x^0 \in \Omega$, such that $u(x^0) = 0$ and $v(x^0) = 0$, then from

$$\begin{aligned}
0 \leq (-\Delta)^{\alpha/2} u(x^0) &= C_{n,\alpha} PV \int_H \frac{-u(y)}{|x^0 - y|^{n+\alpha}} \, dy, \\
0 \leq (-\Delta)^{\alpha/2} v(x^0) &= C_{n,\alpha} PV \int_H \frac{-v(y)}{|x^0 - y|^{n+\alpha}} \, dy,
\end{aligned}$$

we derive immediately that

$$u(x) = 0 \quad \text{and} \quad v(x) = 0 \quad \text{almost everywhere in} \quad \mathbb{R}^n.$$

This completes the proof. □

Narrow region principle.

Theorem 2.2. *Let T be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that*

$$T = \{x = (x^1, x') \in \mathbb{R}^n \mid x_1 = \lambda \quad \textit{for some} \quad \lambda \in \mathbb{R}\}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \ldots, x_n),$$

be the reflection of x about the plane T. Denote

$$H = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \qquad \tilde{H} = \{x \mid \tilde{x} \in H\}.$$

Let Ω be a bounded narrow region in H such that it is contained in $\{x \mid \lambda - l < x_1 < \lambda\}$ with small l. Suppose that $u, v \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ and both are lower semicontinuous on $\bar{\Omega}$. If $c_1(x)$ and $c_2(x)$ are both bounded from below in Ω, $c_1(x) \le 0$ and $c_2(x) \le 0$ and

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x) v(x) \ge 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x) u(x) \ge 0 & \text{in } \Omega, \\ u(x) \ge 0 \quad \text{and} \quad v(x) \ge 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H, \\ v(\tilde{x}) = -v(x) & \text{in } H, \end{cases} \tag{2-3}$$

then for sufficiently small l, we have

$$u(x) \ge 0 \quad \text{and} \quad v(x) \ge 0 \quad \text{in } \Omega. \tag{2-4}$$

This conclusion holds for unbounded regions Ω if we further assume that

$$\varliminf_{|x| \to \infty} u(x) \ge 0 \quad \text{and} \quad \varliminf_{|x| \to \infty} v(x) \ge 0.$$

Proof. If (2-4) does not hold, then the lower semicontinuity of u and v on $\bar{\Omega}$ indicates that there exists an $x^0 \in \bar{\Omega}$ such that

$$u(x^0) = \min_{\bar{\Omega}} u < 0 \quad \text{or} \quad v(x^0) = \min_{\bar{\Omega}} v < 0,$$

and one can further deduce from condition (2-3) that x^0 is in the interior of Ω.

Next we discuss the problem in three different cases.

Case i. ($u(x^0) = \min_\Omega u < 0$ and $v(x^0) \ge 0$).

It follows that

$$\begin{aligned} (-\Delta)^{\alpha/2} u(x^0) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy \\ &= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy + \int_{\tilde{H}} \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy \right\} \\ &= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy + \int_H \frac{u(x^0) - u(\tilde{y})}{|x^0 - \tilde{y}|^{n+\alpha}} \, dy \right\} \\ &= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0) - u(y)}{|x^0 - y|^{n+\alpha}} \, dy + \int_H \frac{u(x^0) + u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} \, dy \right\} \\ &\le C_{n,\alpha} \int_H \left\{ \frac{u(x^0) - u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} + \frac{u(x^0) + u(y)}{|x^0 - \tilde{y}|^{n+\alpha}} \right\} dy \\ &= C_{n,\alpha} \int_H \frac{2u(x^0)}{|x^0 - \tilde{y}|^{n+\alpha}} \, dy. \end{aligned} \tag{2-5}$$

Let $D = \{y \mid l < y_1 - x_1^0 < 1,\ |y' - (x^0)'| < 1\}$, $s = y_1 - x_1^0$, $\tau = y' - (x^0)'$ and $\omega_{n-2} = |B_1(0)|$ in $\mathbb{R}^{n-2}$. Now we have

$$\begin{aligned}
\int_H \frac{1}{|x^0 - \tilde{y}|^{n+\alpha}}\, dy &\geq \int_D \frac{1}{|x^0 - y|^{n+\alpha}}\, dy \\
&= \int_l^1 \int_0^1 \frac{\omega_{n-2}\tau^{n-2}}{(s^2+\tau^2)^{\frac{n+\alpha}{2}}}\, d\tau\, ds \\
&= \int_l^1 \int_0^{\frac{1}{s}} \frac{\omega_{n-2}(st)^{n-2}s}{s^{n+\alpha}(1+t^2)^{\frac{n+\alpha}{2}}}\, dt\, ds \qquad \text{(2-6)} \\
&= \int_l^1 \frac{1}{s^{1+\alpha}} \int_0^{\frac{1}{s}} \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{\frac{n+\alpha}{2}}}\, dt\, ds \\
&\geq \int_l^1 \frac{1}{s^{1+\alpha}} \int_0^{1} \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{\frac{n+\alpha}{2}}}\, dt\, ds \\
&\geq C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds \to \infty, \qquad \text{(2-7)}
\end{aligned}$$

where (2-6) follows from the substitution $\tau = st$ and (2-7) is true when $l \to 0$.

Hence $c_1(x) \leq 0$ leads to

$$\begin{aligned}
(-\Delta)^{\alpha/2}u(x^0) + c_1(x)v(x^0) &\leq C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds\, u(x^0) + c_1(x^0)v(x^0) \\
&= u(x^0)\Big[C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0)\frac{v(x^0)}{u(x^0)}\Big] \\
&< 0,
\end{aligned}$$

when l sufficiently small. This is a contradiction with condition (2-3).

Case ii ($v(x^0) = \min_\Omega v < 0$ and $u(x^0) \geq 0$). Similarly to Case i, $c_2(x) \leq 0$ leads to

$$(-\Delta)^{\alpha/2}v(x^0) + c_2(x^0)u(x^0) \leq v(x^0)\Big[C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^0)\frac{u(x^0)}{v(x^0)}\Big] < 0,$$

when l sufficiently small. This is a contradiction with condition (2-3).

Case iii ($u(x^0) = \min_\Omega u < 0$ and $v(x^0) < 0$). Similarly to Case i, by (2-3), we have

$$0 \leq (-\Delta)^{\alpha/2}u(x^0) + c_1(x^0)v(x^0) \leq C\int_l^1 \frac{1}{s^{1+\alpha}}\, ds\, u(x^0) + c_1(x^0)v(x^0). \qquad \text{(2-8)}$$

By $v(x^0) < 0$, there exists $x^1 \in \Omega$ such that $v(x^1) = \min_\Omega v < 0$. Similarly to Case ii, by (2-3) and $c_2(x) \le 0$, we have

$$0 \le (-\Delta)^{\alpha/2} v(x^1) + c_2(x^1)u(x^1) \le C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds\, v(x^0) + c_2(x^1)u(x^0). \tag{2-9}$$

Adding (2-8) to (2-9), we get

$$\left[C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1)\right] u(x^0) + \left[C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0)\right] v(x^0) \ge 0. \tag{2-10}$$

As $u(x^0) < 0$ and $v(x^0) < 0$, if (2-10) holds, then at least one of

$$C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1) \le 0 \quad \text{or} \quad C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0) \le 0$$

holds.

Equivalently,

$$C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1) \le 0 \quad \text{or} \quad C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0) \le 0. \tag{2-11}$$

However, when l sufficiently small, from the fact that $c_1(x)$ and $c_2(x)$ are both bounded from below in Ω, we have

$$C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_2(x^1) > 0 \quad \text{and} \quad C \int_l^1 \frac{1}{s^{1+\alpha}}\, ds + c_1(x^0) > 0.$$

which is a contradiction with (2-11).

Similarly, we can prove the case $v(x^0) = \min_\Omega v < 0$ and $u(x^0) < 0$.

Therefore, (2-4) must be true. This completes the proof. □

Decay at infinity.

Theorem 2.3. *Let $H = \{x \in \mathbb{R}^n \mid x_1 < \lambda \ \text{ for some } \lambda \in \mathbb{R}\}$ and let Ω be an unbounded region in H. Assume*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + c_1(x)v(x) \ge 0 & \text{in } \Omega, \\ (-\Delta)^{\alpha/2} v(x) + c_2(x)u(x) \ge 0 & \text{in } \Omega, \\ u(x) \ge 0 \quad \text{and} \quad v(x) \ge 0 & \text{in } H \setminus \Omega, \\ u(\tilde{x}) = -u(x) & \text{in } H \\ v(\tilde{x}) = -v(x) & \text{in } H \end{cases} \tag{2-12}$$

with

$$\varliminf_{|x|\to\infty} |x|^\alpha c_1(x) = 0, \quad c_1(x) \le 0, \tag{2-13}$$

and

$$\lim_{|x|\to\infty} |x|^\alpha c_2(x) = 0, \quad c_2(x) \le 0. \tag{2-14}$$

Then there exists a constant R_0 such that if

$$u(x^0) = \min_\Omega u(x) < 0 \quad or \quad v(x^0) = \min_\Omega v(x) < 0, \tag{2-15}$$

then

$$|x^0| \le R_0. \tag{2-16}$$

Proof. Following from (2-15), there are three different cases for this proof.

Case i ($u(x^0) < 0$ and $v(x^0) \ge 0$). It follows that

$$\begin{aligned}
(-\Delta)^{\alpha/2}u(x^0) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy + \int_{\tilde H} \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy + \int_{H} \frac{u(x^0)-u(\tilde y)}{|x^0-\tilde y|^{n+\alpha}}\,dy \right\} \\
&= C_{n,\alpha} PV \left\{ \int_H \frac{u(x^0)-u(y)}{|x^0-y|^{n+\alpha}}\,dy + \int_{H} \frac{u(x^0)+u(y)}{|x^0-\tilde y|^{n+\alpha}}\,dy \right\} \\
&\le C_{n,\alpha} \int_H \left\{ \frac{u(x^0)-u(y)}{|x^0-\tilde y|^{n+\alpha}} + \frac{u(x^0)+u(y)}{|x^0-\tilde y|^{n+\alpha}} \right\} dy \\
&= C_{n,\alpha} \int_H \frac{2u(x^0)}{|x^0-\tilde y^{n+\alpha}|}\,dy.
\end{aligned}$$

For each fixed λ, when $|x^0| \ge \lambda$, we have $B_{|x^0|}(x^1) \subset \tilde H$ with $x^1 = (3|x^0| + x^0_1, (x^0)')$, and it follows that

$$\begin{aligned}
\int_H \frac{1}{|x^0-\tilde y|^{n+\alpha}}\,dy &= \int_{\tilde H} \frac{1}{|x^0-y|^{n+\alpha}}\,dy \\
&\ge \int_{B_{|x^0|}(x^1)} \frac{1}{|x^0-y|^{n+\alpha}}\,dy \\
&\ge \int_{B_{|x^0|}(x^1)} \frac{1}{4^{n+\alpha}|x^0|^{n+\alpha}}\,dy = \frac{\omega_n}{4^{n+\alpha}|x^0|^{n+\alpha}},
\end{aligned} \tag{2-17}$$

where (2-17) follows from $|x^0-y| \le |x^0 - x_1| + |x^0| = 4|x^0|$ for all $y \in B_{|x^0|}(x^1)$. Then we have

$$0 \le (-\Delta)^{\alpha/2}u(x^0) + c_1(x^0)v(x^0) \le \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} u(x^0) + c_1(x^0)v(x^0). \tag{2-18}$$

Following from (2-13), $c_1(x^0) \le 0$ for all $x^0 \in H$, we have

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} u(x^0) + c_1(x^0)v(x^0) < 0.$$

This contradicts (2-18).

Case ii ($v(x^0) < 0$ and $u(x^0) \ge 0$). Using the same method as Case i, we have

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} v(x^0) + c_2(x^0)u(x^0) \ge 0,$$

which is a contradiction with

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} v(x^0) + c_2(x^0)u(x^0) < 0,$$

for $c_2(x^0) \le 0$ for all $x^0 \in H$.

Case iii ($u(x^0) < 0$ and $v(x^0) < 0$). We have

$$0 \le (-\Delta)^{\alpha/2} u(x^0) + c_1(x^0)v(x^0) \le \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} u(x^0) + c_1(x^0)v(x^0), \tag{2-19}$$

$$0 \le (-\Delta)^{\alpha/2} v(x^0) + c_2(x^0)u(x^0) \le \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} v(x^0) + c_2(x^0)u(x^0). \tag{2-20}$$

Adding (2-19) to (2-20), we get

$$\left[\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0)\right] u(x^0) + \left[\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0)\right] v(x^0) \ge 0. \tag{2-21}$$

As $u(x^0) < 0$ and $v(x^0) < 0$, if (2-21) holds, at least one of

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0) \le 0 \quad \text{or} \quad \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0) \le 0$$

holds. Equivalently,

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0) \le 0 \quad \text{or} \quad \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0) \le 0. \tag{2-22}$$

However, if $|x^0|$ is sufficiently large, following from (2-13) and (2-14), we have

$$\frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_2(x^0) > 0 \quad \text{and} \quad \frac{2\omega_n C_{n,\alpha}}{4^{n+\alpha}|x^0|^\alpha} + c_1(x^0) > 0.$$

which is a contradiction with (2-22).

Therefore, (2-16) holds. This completes the proof. □

3. Method of moving planes and its applications

Theorem 3.1. *Assume that $u, v \in C^{1,1}_{\mathrm{loc}} \cap L_\alpha$ and*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = v^q(x), & x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2} v(x) = u^p(x), & x \in \mathbb{R}^n. \end{cases} \tag{3-1}$$

Then

(i) *in the subcritical case $1 < p, q < (n+\alpha)/(n-\alpha)$, (3-1) has no positive solution*;

(ii) *in the critical case $p = q = (n+\alpha)/(n-\alpha)$, the positive solutions must be radially symmetric about some point in $\mathbb{R}^n$.*

Proof. Because no decay condition on u near infinity is assumed, we are not able to carry out the method of moving planes on u directly. To circumvent this difficulty, we make a Kelvin transform.

Let x^0 be a point in $\mathbb{R}^n$, and let

$$\bar{u}(x^0) = \frac{1}{|x-x^0|^{n-\alpha}} u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right), \qquad \bar{v}(x^0) = \frac{1}{|x-x^0|^{n-\alpha}} v\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right)$$

be the Kelvin transform of (u, v) centered at x^0. Then it follows that

$$\begin{aligned} \bar{u}(x) &= \frac{1}{|x-x^0|^{n-\alpha}} u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right) \\ &= \frac{1}{|x-x^0|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{v^q(y)}{|y - \frac{x-x^0}{|x-x^0|^2} - x^0|^{n-\alpha}} \, dy \\ &= \frac{1}{|x-x^0|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{\left(\frac{1}{|y-x^0|^{n-\alpha}}\right)^q \bar{v}^q\left(\frac{y-x^0}{|y-x^0|^2} + x^0\right)}{\left|y - x^0 - \frac{x-x^0}{|x-x^0|^2}\right|^{n-\alpha}} \, dy \\ &= \frac{1}{|x-x^0|^{n-\alpha}} \int_{\mathbb{R}^n} \frac{|z-x^0|^{q(n-\alpha)} \bar{v}^q(z)}{\left|\frac{z-x^0}{|z-x^0|^2} - \frac{x-x^0}{|x-x^0|^2}\right|^{n-\alpha}} \frac{1}{|z-x^0|^{2n}} \, dz \qquad (3\text{-}2) \\ &= \int_{\mathbb{R}^n} \frac{\bar{v}^q(z)}{|z-x^0|^\tau |x-z|^{n-\alpha}} \, dz, \end{aligned}$$

where the step (3-2) follows from the substitution $z = (y-x^0)/|y-x^0|^2 + x^0$ and $\tau = n + \alpha - q(n-\alpha)$.

This means

$$(-\Delta)^{\alpha/2} \bar{u}(x) = \frac{\bar{v}^q(x)}{|x-x^0|^\tau}. \tag{3-3}$$

Similarly, we have

$$(-\Delta)^{\alpha/2}\bar{v}(x) = \frac{\bar{u}^p(x)}{|x-x^0|^\gamma}, \tag{3-4}$$

with $\gamma = n+\alpha-p(n-\alpha)$. Obviously, $\tau=\gamma=0$ in the critical case.

Choose any direction to be the x_1 direction. For $\lambda < x_1^0$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}, \qquad x^\lambda = (2\lambda - x_1, x'), \quad \bar{u}_\lambda(x) = \bar{u}(x^\lambda),$$
$$w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x), \qquad \bar{v}_\lambda(x) = \bar{v}(x^\lambda), \qquad \varphi_\lambda(x) = \bar{v}_\lambda(x) - \bar{v}(x),$$

and

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \quad \tilde{\Sigma}_\lambda = \{x^\lambda \mid x \in \Sigma_\lambda\}.$$

First, notice that, by the definition of w_λ and φ_λ, we have

$$\lim_{|x|\to\infty} w_\lambda(x) = 0, \qquad \lim_{|x|\to\infty} \varphi_\lambda(x) = 0.$$

Hence, if w_λ or φ_λ is negative somewhere in Σ_λ, then the negative minima of w_λ or φ_λ was attained in the interior of Σ_λ.

From (3-3), at points where φ_λ is negative, we have

$$\begin{aligned}(-\Delta)^{\alpha/2}w_\lambda(x) &= \frac{\bar{v}_\lambda^q(x)}{|x^\lambda - x^0|^\tau} - \frac{\bar{v}^q(x)}{|x-x^0|^\tau} \\ &\geq \frac{\bar{v}_\lambda^q(x) - \bar{v}^q(x)}{|x-x^0|^\tau} \\ &\geq \frac{q\bar{v}^{q-1}(x)\varphi_{\lambda(x)}}{|x-x^0|^\tau}, \end{aligned} \tag{3-5}$$

where (3-5) follows from the *mean value theorem*, that is,

$$(-\Delta)^{\alpha/2}w_\lambda(x) + c_1(x)\varphi_\lambda(x) \geq 0$$

with

$$c_1(x) = -\frac{q\bar{v}^{q-1}(x)}{|x-x^0|^\tau}. \tag{3-6}$$

From (3-4), at points where w_λ is negative, we similarly have

$$(-\Delta)^{\alpha/2}\varphi_\lambda(x) + c_2(x)w_\lambda(x) \geq 0 \tag{3-7}$$

with

$$c_2(x) = -\frac{p\bar{u}^{p-1}(x)}{|x-x^0|^\gamma}. \tag{3-8}$$

The subcritical case. For $1 < p, q < (n+\alpha)/(n-\alpha)$, we show that (3-1) admits no positive solution.

Step 1. We show that, for λ sufficiently negative,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{in } \Sigma_\lambda. \tag{3-9}$$

This is done by using Theorem 2.3 (*decay at infinity*).

It follows from (3-6) that,

$$\begin{aligned} c_1(x) &= -\frac{q\left(\frac{1}{|x-x^0|^{n-\alpha}}\right)^{q-1} v^{q-1}\left(\frac{x-x^0}{|x-x^0|^2}+x^0\right)}{|x-x^0|^{n+\alpha-q(n-\alpha)}} \\ &= -\frac{q v^{q-1}\left(\frac{x-x^0}{|x-x^0|^2}+x^0\right)}{|x-x^0|^{2\alpha}}. \end{aligned}$$

It is easy to verify that, for $|x|$ sufficiently large,

$$c_1(x) \sim \frac{1}{|x|^{2\alpha}}. \tag{3-10}$$

In the same way,

$$c_2(x) \sim \frac{1}{|x|^{2\alpha}}. \tag{3-11}$$

In addition, following from (3-6) and (3-8), we have $c_1(x) \leq 0$ and $c_2(x) \leq 0$. Hence, $c_1(x)$ and $c_2(x)$ satisfy conditions (2-13) and (2-14) respectively in Theorem 2.3. Applying Theorem 2.3 to w_λ and φ_λ with $\Omega = H = \Sigma_\lambda$, we conclude that, there exists an $R_0 > 0$ (independent of λ), such that if $\bar{x}$ is a negative minimum of w_λ or φ_λ in Σ_λ, then

$$|\bar{x}| \leq R_0. \tag{3-12}$$

Now for $\lambda \leq -R_0$, we must have

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda.$$

This verifies (3-9).

Step 2. Step 1 provides a starting point, from which we can now move the plane T_λ to the right as long as (3-9) holds to its limiting position.

Let

$$\lambda_0 = \sup\{\lambda \leq x_1^0 \mid w_\mu(x) \geq 0 \quad \text{and} \quad \varphi_\mu(x) \geq 0, \quad \text{for all } x \in \Sigma_\mu,\ \mu \leq \lambda\}.$$

In this part, we show that

$$\lambda_0 = x_1^0$$

and

$$w_{\lambda_0}(x) \equiv 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \equiv 0, \quad \text{for all } x \in \Sigma_{\lambda_0}. \tag{3-13}$$

Suppose that $\lambda_0 < x_1^0$. We show that the plane T_λ can be moved further right. To be more rigorous, there exists some $\epsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0, \quad \text{for all } x \in \Sigma_\lambda. \tag{3-14}$$

This is a contradiction with the definition of λ_0. Hence we must have

$$\lambda_0 = x_1^0. \tag{3-15}$$

Now we prove (3-14) by combining the use of the *narrow region principle* and *decay at infinity*.

Again by (3-12), the negative minimum of w_λ cannot be attained outside of $B_{R_0}(0)$. Next we argue that it can neither be attained inside of $B_{R_0}(0)$. Actually, we will show that for λ sufficiently close to λ_0,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0, \quad \text{for all } x \in \Sigma_\lambda \cap B_{R_0}(0). \tag{3-16}$$

From the *narrow region principle*, there is a small $\delta > 0$, such that for $\lambda \in [\lambda_0, \lambda_0+\delta)$, if

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0-\delta}, \tag{3-17}$$

then

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda \setminus \Sigma_{\lambda_0-\delta}. \tag{3-18}$$

To see this, in Theorem 2.2, we let $H = \Sigma_\lambda$ and the narrow region $\Omega = \Sigma_\lambda \setminus \Sigma_{\lambda_0-\delta}$, while the lower bound of $c_1(x)$, $c_2(x)$ can be seen from (3-10) and (3-11).

Then what is left to show is (3-17), and actually we only need

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0-\delta} \cap B_{R_0}(0). \tag{3-19}$$

In fact, when $\lambda_0 < x_1^0$, we have

$$w_{\lambda_0}(x) > 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) > 0 \quad \text{for all } x \in \Sigma_{\lambda_0}. \tag{3-20}$$

If not, there exists some $\hat{x}$ such that

$$w_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0 \quad \text{or} \quad \varphi_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} \varphi_{\lambda_0}(x) = 0.$$

Case i ($w_{\lambda_0}(\hat{x}) = 0$ and $\varphi_{\lambda_0}(\hat{x}) > 0$). It follows that

$$
\begin{aligned}
(-\Delta)^{\alpha/2} w_{\lambda_0}(\hat{x}) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy \\
&= C_{n,\alpha} PV \left[\int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy + \int_{\tilde{\Sigma}_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy \right] \\
&= C_{n,\alpha} PV \left[\int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy + \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(\tilde{y})}{|\hat{x}-\tilde{y}|^{n+\alpha}}\,dy \right] \\
&= C_{n,\alpha} PV \left[\int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}}\,dy + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y)}{|\hat{x}-\tilde{y}|^{n+\alpha}}\,dy \right] \\
&\le C_{n,\alpha} \int_{\Sigma_{\lambda_0}} \left[\frac{-w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} + \frac{w_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} \right] dy \\
&= 0. \qquad (3\text{-}21)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} w_{\lambda_0}(\hat{x}) &= \frac{\bar{v}^q_{\lambda_0}(\hat{x})}{|\hat{x}^{\lambda_0}-x^0|^\tau} - \frac{\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^\tau} \\
&> \frac{\bar{v}^q_{\lambda_0}(\hat{x}) - \bar{v}^q(\hat{x})}{|\hat{x}-x^0|^\tau} \\
&> \frac{q\bar{v}^{q-1}(\hat{x})\varphi_{\lambda_0}(\hat{x})}{|\hat{x}-x^0|^\tau} \\
&> 0,
\end{aligned}
$$

which is a contradiction with (3-21).

Case ii ($\varphi_{\lambda_0}(\hat{x}) = 0$ and $w_{\lambda_0}(\hat{x}) > 0$). As in Case i, there will be a contradiction.

Case iii ($w_{\lambda_0}(\hat{x}) = 0$ and $\varphi_{\lambda_0}(\hat{x}) = 0$). We have

$$
(-\Delta)^{\frac{\alpha}{2}} w_{\lambda_0}(\hat{x}) = \frac{\bar{v}^q_{\lambda_0}(\hat{x})}{|\hat{x}^{\lambda_0}-x^0|^\tau} - \frac{\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^\tau} = \frac{\bar{v}^q(\hat{x})}{|\hat{x}^{\lambda_0}-x^0|^\tau} - \frac{\bar{v}^q(\hat{x})}{|\hat{x}-x^0|^\tau} > 0,
$$

a contradiction with (3-21).

These three cases prove (3-20). It follows from (3-20) that there exists a constant $c_0 > 0$, such that

$$
w_{\lambda_0}(x) \ge c_0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \ge c_0 \quad \text{for all } x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}.
$$

Since w_λ and φ_λ both depend on λ continuously, there exist $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0+\epsilon)$, we have

$$
w_{\lambda_0}(x) \ge 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \ge 0 \quad \text{for all } x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_{R_0}(0)}. \qquad (3\text{-}22)
$$

Combining (3-18), (3-12) and (3-22), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda. \tag{3-23}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = x_1^0 \quad \text{and} \quad w_{\lambda_0} \geq 0,\ \varphi_{\lambda_0} \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0}.$$

Similarly, one can move the plane T_λ from $+\infty$ to the left and show that

$$w_{\lambda_0}(x) \geq 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_0}. \tag{3-24}$$

Now we have shown that

$$\lambda_0 = x_1^0 \quad \text{and} \quad w_{\lambda_0}(x) \equiv 0,\ \varphi_{\lambda_0}(x) \equiv 0 \quad \text{for all } x \in \Sigma_{\lambda_0}.$$

This completes Step 2.

So far, we have proved that $(\bar{u}, \bar{v})$ is symmetric about the plane $T_{x_1^0}$. Since the x_1 direction can be chosen arbitrarily, we have actually shown that $(\bar{u}, \bar{v})$ is radially symmetric about x^0.

For any two points $X^i \in \mathbb{R}^n$, $i = 1, 2$. Choose x_0 to be the midpoint, i.e., $x^0 = (X^1 + X^2)/2$. Since $(\bar{u}, \bar{v})$ is radially symmetric about x^0, so is (u, v), hence $(u(X^1), v(X^1)) = (u(X^2), v(X^2))$. This implies that u is constant. A positive constant function does not satisfy (3-1). This proves the nonexistence of positive solutions for (3-1) when $1 < p, q < (n+\alpha)/(n-\alpha)$.

The critical case. Let $(\bar{u}, \bar{v})$ be the Kelvin transform of (u, v) centered at the origin. Then

$$(-\Delta)^{\alpha/2}\bar{u}(x) = \bar{v}^q(x), \quad (-\Delta)^{\alpha/2}\bar{v}(x) = \bar{u}^p(x). \tag{3-25}$$

We will show that either $(\bar{u}, \bar{v})$ is symmetric about the origin or (u, v) is symmetric about some point.

We still use the notation as in the subcritical case. **Step 1** is entirely the same as that in the subcritical case, that is, we can show that for λ sufficiently negative,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad \varphi_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda.$$

Let

$$\lambda_0 = \sup\{\lambda \geq 0 \mid w_\mu(x) \geq 0 \quad \text{and} \quad \varphi_\mu(x) \geq 0 \quad \text{for all } x \in \Sigma_\mu,\ \mu \leq \lambda\}.$$

Case i. $\lambda_0 < 0$. Similarly to the subcritical case, one can show that

$$w_{\lambda_0}(x) \equiv 0 \quad \text{and} \quad \varphi_{\lambda_0}(x) \equiv 0 \quad \text{for all } x \in \Sigma_{\lambda_0}.$$

It follows that 0 is not a singular point of $\bar{u}$ or $\bar{v}$, and hence following from Kelvin transform of $\bar{u}$ centered at the origin

$$u(x)=\frac{1}{|x|^{n-\alpha}}\bar{u}\Big(\frac{x}{|x|^2}\Big),$$

we have

$$\lim_{|x|\to\infty}|x|^{n-\alpha}u(x)=\lim_{|x|\to\infty}\bar{u}\Big(\frac{x}{|x|^2}\Big)=\bar{u}(0)>0,$$

that is,

$$u(x)=O\Big(\frac{1}{|x|^{n-\alpha}}\Big)\quad\text{when } |x|\to\infty.$$

Similarly for v,

$$v(x)=O\Big(\frac{1}{|x|^{n-\alpha}}\Big)\quad\text{when } |x|\to\infty.$$

This enables us to apply the method of moving planes to (u, v) directly and show that (u, v) is symmetric about some point in $\mathbb{R}^n$.

Case ii. $\lambda_0=0$. Then by moving planes from near $x_1=+\infty$, we derive that $(\bar{u}, \bar{v})$ is symmetric about the origin, and so is (u, v).

In any case, (u, v) is symmetric about some point in $\mathbb{R}^n$. $\square$

References

[Busca and Manásevich 2002] J. Busca and R. Manásevich, "A Liouville-type theorem for Lane–Emden systems", *Indiana Univ. Math. J.* **51**:1 (2002), 37–51. MR Zbl

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:8 (2007), 1245–1260. MR Zbl

[Chen and Li 1991] W. X. Chen and C. Li, "Classification of solutions of some nonlinear elliptic equations", *Duke Math. J.* **63**:3 (1991), 615–622. MR Zbl

[Chen and Li 2010] W. Chen and C. Li, *Methods on nonlinear elliptic equations*, AIMS Series Diff. Equations Dyn. Syst. **4**, Amer. Inst. Math. Sci., Springfield, MO, 2010. MR Zbl

[Chen and Zhu 2016] W. Chen and J. Zhu, "Indefinite fractional elliptic problem and Liouville theorems", *J. Differential Equations* **260**:5 (2016), 4758–4785. MR Zbl

[Chen et al. 2017] W. Chen, C. Li, and Y. Li, "A direct method of moving planes for the fractional Laplacian", *Adv. Math.* **308** (2017), 404–437. MR Zbl

[Jarohs and Weth 2016] S. Jarohs and T. Weth, "Symmetry via antisymmetric maximum principles in nonlocal problems of variable order", *Ann. Mat. Pura Appl.* (4) **195**:1 (2016), 273–291. MR Zbl

[Mitidieri 1996] E. Mitidieri, "Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$", *Differential Integral Equations* **9**:3 (1996), 465–479. MR Zbl

[Poláčik et al. 2007] P. Poláčik, P. Quittner, and P. Souplet, "Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems", *Duke Math. J.* **139**:3 (2007), 555–579. MR Zbl

[Serrin and Zou 1998] J. Serrin and H. Zou, "Existence of positive solutions of the Lane–Emden system", *Atti Sem. Mat. Fis. Univ. Modena* **46**:Suppl. (1998), 369–380. MR Zbl

[Souplet 2009] P. Souplet, "The proof of the Lane–Emden conjecture in four space dimensions", *Adv. Math.* **221**:5 (2009), 1409–1427. MR Zbl

[Troy 1981] W. C. Troy, "Symmetry properties in systems of semilinear elliptic equations", *J. Differential Equations* **42**:3 (1981), 400–413. MR Zbl

Received November 29, 2015. Revised October 13, 2016.

CHUNXIA CHENG
SCHOOL OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, 221116
CHINA
860119947@qq.com

ZHONGXUE LÜ
SCHOOL OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, 221116
CHINA
6020010017@jsnu.edu.cn

YINGSHU LÜ
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAO TONG UNIVERSITY
SHANGHAI, 200240
CHINA
lvyingshu@126.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.321

A VECTOR-VALUED BANACH–STONE THEOREM WITH DISTORTION $\sqrt{2}$

ELÓI MEDINA GALEGO AND ANDRÉ LUIS PORTO DA SILVA

Let K and S be locally compact Hausdorff spaces and H a real Hilbert space of finite dimension at least two. We prove that if T is an isomorphism from $C_0(K, H)$ onto $C_0(S, H)$ whose distortion $\|T\|\|T^{-1}\|$ is exactly $\sqrt{2}$, then K and S are homeomorphic. This is the vector-valued Banach–Stone theorem via isomorphisms with the largest distortion that is known. It improves a 1976 classical result due to Cambern.

1. Introduction

If K is a locally compact Hausdorff space and X is a Banach space, we denote by $C_0(K, X)$ the Banach space of continuous functions vanishing at infinity on K, taking values in X, and provided with the usual supremum norm. If K is compact, we use the notation $C(K, X)$ to represent this space. Moreover, if $X = \mathbb{R}$ we will denote these spaces by $C_0(K)$ and $C(K)$ respectively. In the present paper, the word "isomorphism" means "linear homeomorphism".

The well-known Banach–Stone theorem states that if K and S are locally compact Hausdorff spaces, then the existence of an isometric isomorphism T of $C_0(K)$ onto $C_0(S)$ implies that K and S are homeomorphic [Banach 1932; Behrends 1979; Stone 1937]. Cambern [1966; 1967] strengthened this theorem by showing that the conclusion holds if the requirement that T be an isometric isomorphism is replaced by the requirement that T be an isomorphism satisfying $\|T\|\|T^{-1}\| < 2$. Amir [1965] established the same result independently for K and S compact. Cambern [1970] showed that 2 is indeed the greatest number for which the formulation of the Banach–Stone theorem given in [Cambern 1967] is valid, by exhibiting a pair of locally compact Hausdorff spaces K and S, with K compact, S noncompact, and an isomorphism T of $C(K)$ onto $C_0(S)$ with $\|T\|\|T^{-1}\| = 2$. Cohen [1975] showed there was such an example where both K and S are compact.

Cambern [1976] was also the first to get a vector-valued Banach–Stone theorem via isomorphisms with distortion $\lambda > 1$. He proved:

MSC2010: primary 46B03, 46E15; secondary 46B25, 46E40.
Keywords: vector-valued Banach–Stone theorem, $C_0(K, X)$ spaces, finite-dimensional Hilbert space.

Theorem 1.1. *Let K and S be locally compact Hausdorff spaces and H a finite-dimensional Hilbert space. If there exists an isomorphism T from $C_0(K, H)$ onto $C_0(S, H)$ satisfying $\|T\| \|T^{-1}\| < \sqrt{2}$, then K and S are homeomorphic*

It is still an open question whether the bound $\sqrt{2}$ can be improved. Moreover, after Cambern's theorem, all vector-valued Banach–Stone theorems have been obtained via isomorphisms with distortion $1 \le \lambda < \sqrt{2}$; see [Cidral et al. 2015].

Thus, in view of the above mentioned isomorphisms with distortion 2 between $C_0(K, H)$ spaces constructed independently by Cambern and Cohen in the case where H is the scalar field, it is natural to turn our attention to the isomorphisms with distortion $\sqrt{2}$ between $C_0(K, H)$ spaces in the case where H is an n-dimensional Hilbert space with $n \ge 2$. In other words, the following question arises naturally.

Problem 1.2. *Let K and S be locally compact Hausdorff spaces and H a Hilbert space of finite dimension greater than or equal to 2. Suppose that there exists an isomorphism T from $C_0(K, H)$ onto $C_0(S, H)$ satisfying $\|T\| \|T^{-1}\| = \sqrt{2}$. Does it follow that K and S are homeomorphic*?

The principal purpose of this paper is to show that Problem 1.2 has a positive solution when the scalar field is the real numbers $\mathbb{R}$.

So, henceforward $H = \mathbb{R}^n_2$ the space of n tuples of real numbers with the usual 2 norm and $n \ge 2$. Our main theorem is as follows.

Theorem 1.3. *Let K and S be locally compact Hausdorff spaces. Suppose that there exists an isomorphism T from $C_0(K, H)$ onto $C_0(S, H)$ satisfying*

$$\frac{\|f\|}{\sqrt[4]{2}} \le \|T(f)\| \le \sqrt[4]{2}\|f\|, \tag{1-1}$$

for every $f \in C_0(K, H)$. Then K and S are homeomorphic.

Then, the solution of Problem 1.2 follows immediately from Theorem 1.3 by considering $\tau = T\|T^{-1}\|2^{-1/4}$ and noticing that (1-1) holds for the isomorphism τ. Moreover, Theorem 1.1 in the real case is also a direct consequence of Theorem 1.3. Indeed, put $\|T\| \|T^{-1}\| = \lambda < \sqrt{2}$ and $\tau = T\|T^{-1}\|\lambda^{-1/2}$. Therefore, it suffices to observe that (1-1) again holds for the isomorphism τ.

It is worth mentioning that Theorem 1.3 cannot be extended to infinite dimensional Hilbert spaces. Indeed, let I be an infinite set and write $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \varnothing$ and the cardinalities of I_1 and I_2 equal to the cardinality of I. Let $K_1 = \{1\}$ and $K_2 = \{1, 2\}$ be two discrete compact Hausdorff spaces. Consider the natural isometries

$$\Theta : C(K_2, l_2(I)) \to l_2(I_1) \oplus_\infty l_2(I_2) \quad \text{and} \quad \Upsilon : l_2(I) \to C(K_1, l_2(I)).$$

Now, define $T : l_2(I_1) \oplus_\infty l_2(I_2) \to l_2(I)$ by

$$T((a_i)_{i \in I_1}, (b_i)_{i \in I_2}) = (c_i)_{i \in I},$$

where $c_i = a_i$ if $i \in I_1$ and $c_i = b_i$ if $i \in I_2$. Then, it is easy to check that

$$\|\Upsilon T \Theta\| = \sqrt{2} \quad \text{and} \quad \|(\Upsilon T \Theta)^{-1}\| = 1.$$

But, of course K_1 and K_2 are not homeomorphic.

As we will see, the proof of Theorem 1.3 depends not only on the fact that H has finite dimension but the intrinsic geometry of H as a real Hilbert space. It is divided into five sections.

2. Special sets associated to isomorphisms between $C_0(K, H)$ spaces

We begin by recalling that a bijective map $T : C_0(K, H) \to C_0(S, H)$ is said to be a bijective coarse quasi-isometry if for some constants $M > 0$ and $L \geq 0$ the inequalities

$$\frac{1}{M}\|f - g\| - L \leq \|T(f) - T(g)\| \leq M\|f - g\| + L,$$

are satisfied for all $f, g \in C_0(K, H)$.

In our recent study of these maps ([Galego and Porto da Silva 2016]; henceforth abbreaviated [GPS]) we introduced some classes of subsets $\Gamma_w(k, v)$ and $\Gamma_v(s, w)$ of S and K respectively, where $k \in K$, $s \in S$ and v and w are suitable elements of $\mathbb{R}$. We shall define these sets for $v, w \in H$ instead of $\mathbb{R}$.

In order to prove Theorem 1.3, we will need to state some new properties of these sets in the particular case where T is linear, $M = \sqrt[4]{2}$ and $L = 0$. So, in this short preliminary section we will remember some definitions and results already adapted to the context of Theorem 1.3.

From now on $M = \sqrt[4]{2}$ and T will be an isomorphism of $C_0(K, H)$ onto $C_0(S, H)$ satisfying

$$\frac{\|f\|}{M} \leq \|T(f)\| \leq M\|f\|, \tag{2-1}$$

for every $f \in C_0(K, H)$.

Let $k \in K$, $f \in C_0(K, H)$ and $v \in H$. Following [GPS, Definition 2.2] we set

$$\omega(k, f, v) = \max\{\|f\|, \|f(k) - v\|\}.$$

Moreover, if $v, w \in H$ with $v \neq 0$ satisfy $\|w\| = \|v\|/M$, following [GPS, Definition 3.1], we also set

$$\Gamma_w(k, v) = \{s \in S : \|Tf(s) - w\| \leq M\omega(k, f, v), \forall f \in C_0(K, H)\}.$$

Analogously, for $s \in S$, w and $v \in H$ with $w \neq 0$ and $\|v\| = \|w\|/M$, we set

$$\Gamma_v(s, w) = \{k \in K : \|T^{-1}g(k) - v\| \leq M\omega(s, g, w), \forall g \in C_0(S, H)\}.$$

Let us summarize the results concerning these sets which will be used in the present paper. We will denote by $\langle \cdot, \cdot \rangle$ the usual inner product on H. When the vectors v and w of H are orthogonal we will write $v \perp w$.

Proposition 2.1. *Let $k \in K$ and $v \in H$ with $v \neq 0$.*

(1) *There exists $w \in H$ such that $\Gamma_w(k, v) \neq \varnothing$.*

(2) *For all $t \in \mathbb{R}$ with $t \neq 0$ and $w \in H$ we have $\Gamma_w(k, v) = \Gamma_{tw}(k, tv)$.*

(3) *Let $v', w, w' \in H$ and $k' \in K$ with $k \neq k'$. Suppose that*

$$\Gamma_w(k, v) \cap \Gamma_{w'}(k', v') \neq \varnothing,$$

then $w \perp w'$.

(4) *Let $w \in H$ and suppose that $s \in \Gamma_w(k, v)$. If $\Gamma_z(s, w) \neq \varnothing$ for some $z \in H$ then $\Gamma_z(s, w) = \{k\}$.*

Proof. (1) The proof is essentially the same proof of [GPS, Proposition 3.2]. We leave it to the reader to transpose to the Hilbert context.

(2) It suffices to prove that $\Gamma_w(k, v) \subset \Gamma_{tw}(k, tv)$ for all $t \neq 0$. Let $s \in \Gamma_w(k, v)$. Given $f \in C_0(K, H)$ put $f' = t^{-1} f$. By the definition of $\Gamma_w(k, v)$ it follows that

$$\|Tf'(s) - w\| \leq M\omega(k, f', v),$$

and hence

$$\|Tf(s) - tw\| = |t|\|Tf'(s) - w\| \leq M|t|\omega(k, f', v) = M\omega(k, f, tv).$$

Consequently $s \in \Gamma_{tw}(k, tv)$.

(3) By item (2) of the proposition we may assume that $\|v\| = \|v'\| = 1$. By Urysohn's lemma pick $f \in C_0(K, H)$ such that $\|f\| = \frac{1}{2}$, $f(k) = \frac{v}{2}$ and $f(k') = \frac{v'}{2}$. It is easy to check that $\omega(k, f, v) = \omega(k', f, v') = \frac{1}{2}$. Pick $s \in \Gamma_w(k, v) \cap \Gamma_{w'}(k', v')$. Then, by the definitions of these sets we have

$$\|w - w'\| \leq \|Tf(s) - w\| + \|Tf(s) - w'\| \leq \frac{M}{2} + \frac{M}{2} = M.$$

Now, by applying the law of cosines we see that

$$\langle w, w' \rangle \geq \tfrac{1}{2}(\|w\|^2 + \|w'\|^2 - M^2),$$

Since $\|w\| = \|w'\| = 1/M$ and $M = \sqrt[4]{2}$, it follows that

$$\langle w, w' \rangle \geq \frac{1}{2}\left(\frac{2}{M^2} - M^2\right) = 0.$$

On the other hand, by item (2) of the proposition we have

$$s \in \Gamma_w(k, v) \cap \Gamma_{-w'}(k', -v').$$

So, proceeding as above we obtain that $\langle w, -w' \rangle \geq 0$. Hence $\langle w, w' \rangle = 0$.

(4) According to item (2) of the proposition we may assume that $\|v\| = 1$. By item (1) of the proposition there is $z \in H$ such that $\Gamma_z(s, w) \neq \varnothing$. Fix $m \in \Gamma_z(s, w)$; we need to show that $m = k$. Assume then that $m \neq k$ and choose $h \in C_0(K)$ satisfying

$$\|h\| = \frac{1}{2}, \quad h(k) = \frac{v}{2} \quad \text{and} \quad h(m) = -\frac{1}{2}\frac{z}{\|z\|}.$$

Since $\Gamma_w(k, v)$ and $\Gamma_z(s, w)$ are well defined, we have $\|z\| = 1/M^2 = 1/\sqrt{2}$. Moreover, observe that z is negatively proportional to $h(m)$. Thus, we have

$$\|h(m) - z\| = \|h(m)\| + \|z\| = \frac{1}{2} + \frac{1}{\sqrt{2}}. \tag{2-2}$$

On the other hand, $\omega(k, h, v) = \frac{1}{2}$ and $s \in \Gamma_w(k, v)$ imply that

$$\|Th(s) - w\| \leq \frac{M}{2}.$$

Since $\|Th\| \leq M/2$ it follows that $\omega(s, Th, w) \leq M/2$ and by the definition of $\Gamma_z(s, w)$ (using the function Th and the map T^{-1})

$$\|h(m) - z\| \leq M\omega(s, Th, w) \leq \frac{M^2}{2} = \frac{1}{\sqrt{2}},$$

which by (2-2) lead us to a contradiction. □

Note that since the definitions of $\Gamma_w(k, v)$ and $\Gamma_v(s, w)$ are symmetric the properties proved in Proposition 2.1 on $k \in K$ and $\Gamma_w(k, v)$ are also valid for $s \in S$ and $\Gamma_v(s, w)$.

Henceforth our task will be to construct a homeomorphism $\varphi : K \to S$ using the subsets $\Gamma_w(k, v)$, for every $k \in K$. In fact, we will see that these subsets contain the candidates to be the image of k by φ.

3. On the subsets $\Gamma_w(k, v)$ of K containing irregular points

The purpose of this section is to establish Proposition 3.1. It allows us to relate the vectors v and w involved in the construction of certain special sets $\Gamma_w(k, v)$. For convenience, we introduce the following definition.

A point $s \in S$ is said to be irregular if there exist two different points k and $k' \in K$ such that $s \in \Gamma_w(k, v) \cap \Gamma_{w'}(k', v')$ for some $v, w, v', w' \in H$. Symmetrically, we will say that a point $k \in K$ is irregular if $k \in \Gamma_v(s, w) \cap \Gamma_{v'}(s', w')$ for some different points $s, s' \in S$ and $v, w, v', w' \in H$.

Proposition 3.1. *Suppose that $k \in K$ and s is an irregular point of S.*

(1) *If $s \in \Gamma_{w_1}(k, v_1) \cap \Gamma_{w_2}(k, v_2)$ for some $v_1, v_2, w_1, w_2 \in H$ then*

$$\langle v_1, v_2 \rangle = M^2 \langle w_1, w_2 \rangle.$$

(2) *If* $(v_i)_{1\le i\le l}$ *is a linearly independent set of* H *and* $s\in\Gamma_{w_i}(k,v_i)$, *for some* $w_i\in H$, $1\le i\le l$, *then* $(w_i)_{1\le i\le l}$ *is a linearly independent set.*

Proof. In virtue of Proposition 2.1(2) we can assume that $\|v_1\|=\|v_2\|=1$. Hence $\|w_1\|=\|w_2\|=1/M$. Since s is irregular, there exists $k'\in K$, $k'\ne k$ and vectors $v', w'\in H$ with $\|v'\|=1$ and $\|w'\|=1/M$ such that $s\in\Gamma_{w'}(k',v')$. According to Proposition 2.1(3) we have

$$w'\perp w_1 \quad\text{and}\quad w'\perp w_2. \tag{3-1}$$

Since $k\ne k'$ by Urysohn's lemma there exist $f, f'\in C_0(K)$ satisfying:

(i) $f(K), f'(K)\subset[0,1]$.

(ii) $f(k)=f'(k')=1$.

(iii) $\operatorname{supp} f\cap\operatorname{supp} f'=\varnothing$.

Put $h_1=f\cdot(v_1/2)$, $h_2=f\cdot(v_2/2)$, $h_3=f'\cdot(v'/2)$ and

$$h=h_1+h_2+\|v_1+v_2\|h_3. \tag{3-2}$$

According to (iii)

$$\|h\|=\tfrac12\|v_1+v_2\|. \tag{3-3}$$

Next we will calculate $\|Th(s)\|$. In order to do this consider the function h_1+h_3. It is easy to see that

$$\omega(k,h_1+h_3,v_1)=\omega(k',h_1+h_3,v')=\tfrac12.$$

Thus, since $s\in\Gamma_{w_1}(k,v_1)\cap\Gamma_{w'}(k',v')$ it follows by the definition of these sets that

$$\|T(h_1+h_3)(s)-w_1\|\le\frac{M}{2}\quad\text{and}\quad\|T(h_1+h_3)(s)-w'\|\le\frac{M}{2}. \tag{3-4}$$

On the other hand, (3-1) gives us that

$$\|w_1-w'\|=\sqrt{\|w_1\|^2+\|w'\|^2}=\sqrt{\frac{2}{M^2}}=M. \tag{3-5}$$

By (3-4) and (3-5) we deduce that

$$T(h_1+h_3)(s)=\frac{w_1+w'}{2}. \tag{3-6}$$

In the same way we obtain

$$T(h_1 - h_3)(s) = \frac{w_1 - w'}{2}, \tag{3-7}$$

and

$$T(h_2 + h_3)(s) = \frac{w_2 + w'}{2}. \tag{3-8}$$

By combining (3-6), (3-7) and (3-8) we infer that

$$Th_1(s) = \frac{w_1}{2}, \quad Th_2(s) = \frac{w_2}{2} \quad \text{and} \quad Th_3(s) = \frac{w'}{2}.$$

Thus, taking in mind (3-1) and (3-2) we get

$$\|Th(s)\|^2 = \frac{\|w_1 + w_2\|^2}{4} + \|v_1 + v_2\|^2 \frac{\|w'\|^2}{4}.$$

Since that $\|Th\| \le M\|h\|$ and (3-3) holds, it follows that

$$\frac{\|w_1 + w_2\|^2}{4} + \|v_1 + v_2\|^2 \frac{\|w'\|^2}{4} \le M^2 \frac{\|v_1 + v_2\|^2}{4}.$$

Recalling that $\|w'\| = 1/M$, we have

$$\|w_1 + w_2\|^2 \le \left(M^2 - \frac{1}{M^2}\right) \|v_1 + v_2\|^2.$$

But $\|w_1 + w_2\|^2 = 2/M^2 + 2\langle w_1, w_2\rangle$ and $\|v_1 + v_2\|^2 = 2 + 2\langle v_1, v_2\rangle$. Hence

$$\frac{2}{M^2} + 2\langle w_1, w_2\rangle \le \left(M^2 - \frac{1}{M^2}\right)(2 + 2\langle v_1, v_2\rangle).$$

By using that $M^2 = \sqrt{2}$ we conclude

$$M^2 \langle w_1, w_2\rangle \le \langle v_1, v_2\rangle.$$

Similarly working with $-v_2$ and $-w_2$ instead of v_2 and w_2 we derive that

$$M^2 \langle w_1, -w_2\rangle \le \langle v_1, -v_2\rangle,$$

so the equality holds.

(2) It suffices to notice that item (1) of the proposition implies the following identity of matrices:

$$[\langle v_i, v_j\rangle]_{1 \le i, j \le l} = M^2 [\langle w_i, w_j\rangle]_{1 \le i, j \le l}. \qquad \square$$

4. The functions $\Phi : K \to \mathcal{P}(S)$ and $\Psi : S \to \mathcal{P}(K)$

Here it is convenient to introduce two functions $\Phi : K \to \mathcal{P}(S)$ and $\Psi : S \to \mathcal{P}(K)$ given by

$$\Phi(k) = \bigcup \left\{ \Gamma_w(k, v) : v \neq 0 \quad \text{and} \quad \|w\| = \frac{\|v\|}{M} \right\},$$

and

$$\Psi(s) = \bigcup \left\{ \Gamma_v(s, w) : w \neq 0 \quad \text{and} \quad \|v\| = \frac{\|w\|}{M} \right\}.$$

Our next step is to prove that the sets $\Phi(k)$ and $\Psi(s)$ are singletons, see Proposition 5.1. The next proposition works on the assumption that $\Phi(k)$ is not a singleton set. Later, in the proof of Proposition 4.1, we will use it to derive a contradiction.

Proposition 4.1. *Let $k \in K$. Suppose that $\Phi(k)$ is not a singleton set. Then*:

(1) *k is an irregular point of K.*

(2) *$\Phi(k)$ contains only irregular points of S.*

Proof. (1) Pick two different points $s, s' \in \Phi(k)$. So, there are $v, v', w, w' \in H$ such that

$$s \in \Gamma_w(k, v) \quad \text{and} \quad s' \in \Gamma_{w'}(k, v').$$

By Proposition 2.1.4 there exist z and $z' \in H$ satisfying

$$k \in \Gamma_z(s, w) \cap \Gamma_{z'}(s', w'),$$

hence k is an irregular point of K.

(2) First of all notice that by item (1) of the proposition applied to $\Psi(s)$, it suffices to prove that for all $s \in \Phi(k)$, $\Psi(s)$ is not a singleton set.

Assume by contradiction that $\Psi(s)$ is a singleton set for some $s \in \Phi(k)$. Since $s \in \Phi(k)$, there exist $v, w \in H$ such that $s \in \Gamma_w(k, v)$. By Proposition 2.1(4) there exists $z \in H$ satisfying $\Gamma_z(s, w) = \{k\}$. Then $k \in \Psi(s)$ and therefore

$$\Psi(s) = \{k\}. \tag{4-1}$$

Now fix $(w_i)_{1 \leq i \leq n}$, a basis of H with $\|w_i\| = 1$ for every $1 \leq i \leq n$. There exist, by Proposition 2.1(1), $(v_i)_{1 \leq i \leq n}$ in H such that $\Gamma_{v_i}(s, w_i) \neq \varnothing$ for every $1 \leq i \leq n$. Thus (4-1) implies that

$$\Gamma_{v_i}(s, w_i) = \{k\}, \tag{4-2}$$

for every $1 \leq i \leq n$.

On the other hand, since by item (1) of the proposition k is an irregular point of K, it follows from (4-2) and Proposition 3.1(2) that $(v_i)_{1 \leq i \leq n}$ is linearly independent.

Next, since k is an irregular point of K, there exist $s' \in S$, $s' \neq s$ and $w', v' \in H$ such that $k \in \Gamma_{v'}(s', w')$. So, by (4-2) and Proposition 2.1(3) we conclude that

$$v' \perp v_i,$$

for every $1 \leq i \leq n$, a contradiction because the dimension of H is n. □

5. The cardinality of $\Phi(k)$ for every $k \in K$

We are now in position to state the key proposition for proving Theorem 1.3. The span of a subset V of E will be denoted by $[V]$.

Proposition 5.1. *$\Phi(k)$ is a singleton set for every $k \in K$.*

Proof. Assume that there exists $k \in K$ such that $\Phi(k) = \{s_i : i \in I\}$ with cardinality of I greater than or equal two. For all $i \in I$ put

$$V_i = \{v \in H, v \neq 0 : s_i \in \Gamma_w(k, v) \text{ for some } w \in H\}.$$

It follows from the definition of $\Phi(k)$ that $V_i \neq \varnothing$ for every $i \in I$, and according to Proposition 2.1(1)

$$\bigcup_{i \in I} V_i = H \setminus \{0\},$$

and therefore

$$\bigcup_{i \in I} [V_i] = H. \tag{5-1}$$

On the other hand, for all $i \in I$ set

$$Z_i = \{z \in H, z \neq 0 : k \in \Gamma_z(s_i, w) \text{ for some } w \in H\}.$$

Pick $i \in I$. Since $V_i \neq \varnothing$ there exists $v \in H$ such that $s_i \in \Gamma_w(k, v)$ for some $w \in H$. By Proposition 2.1(4), $\Gamma_z(s_i, w) = \{k\}$ for some $z \in H$. Hence $Z_i \neq \varnothing$.

According to Proposition 2.1(2) we can assume that $\|z_i\| = \|z_j\|$ and by the definition of $(Z_i)_{i \in I}$ there are w_i and $w_j \in H$ such that

$$k \in \Gamma_{z_i}(s_i, w_i) \cap \Gamma_{z_j}(s_j, w_j).$$

So by Proposition 2.1(3), $z_i \perp z_j$. Consequently

$$[Z_i] \perp [Z_j]. \tag{5-2}$$

Now we will prove that for all $i \in I$

$$[Z_i] = [V_i]. \tag{5-3}$$

First we will show that $Z_i \subset V_i$. Indeed, let $z \in Z_i$ and take $w \in H$ such that $k \in \Gamma_z(s_i, w)$. By Proposition 2.1(4) there exists $w' \in H$ satisfying $\Gamma_{w'}(k, z) = \{s_i\}$. So $z \in V_i$.

Next we will complete the proof of (5-3) by showing that the dimension of $[V_i]$ is less than or equal to the dimension of $[Z_i]$. Let $\{v_1, \dots, v_l\} \subset V_i$ be a basis of $[V_i]$. Thus, by the definition of V_i there are $\{w_1, \dots, w_l\} \subset H$ such that

$$s_i \in \Gamma_{w_j}(k, v_j), \tag{5-4}$$

for every $1 \le j \le l$. Since the cardinality of I is greater than or equal to two, k is an irregular element of K. Thus, according to Proposition 4.1(2), s_i is an irregular element of S. Then, by (5-4) and Proposition 3.1(2) we see that $\{w_1, \dots, w_l\}$ is linearly independent.

In view of (5-4), Proposition 2.1(4) implies that there are $\{z_1, \dots, z_l\} \subset H$ such that for all $1 \le j \le l$,

$$\Gamma_{z_j}(s_i, w_j) = \{k\}. \tag{5-5}$$

So, for all $1 \le j \le l$, $z_j \in Z_i$ and by (5-5) and Proposition 3.1(2) we deduce that $\{z_1, \dots, z_l\}$ is linearly independent. Then, we are done.

Finally, by combining (5-2) and (5-3) it follows that for all $i, j \in I$ with $i \neq j$

$$[V_i] \perp [V_j],$$

a contradiction with (5-1), because H would be a union of nontrivial mutually perpendicular subspaces. $\square$

6. The isomorphisms between $C_0(K, H)$ spaces with distortion $\sqrt{2}$

Proposition 5.1 allows us to define two functions $\varphi : K \to S$ and $\psi : S \to K$ by

$$\Phi(k) = \{\varphi(k)\} \quad \text{and} \quad \Psi(s) = \{\psi(s)\}.$$

Thus, to complete the proof of Theorem 1.3 it remains to prove the following proposition.

Proposition 6.1. *The functions $\varphi : K \to S$ and $\psi : S \to K$ are continuous and $\psi = \varphi^{-1}$.*

Proof. First we will show that $\psi = \varphi^{-1}$. Fix $k \in K$. By the definition of $\Phi(k)$ there are $v, w \in H$ such that

$$\varphi(k) \in \Gamma_w(k, v).$$

Thus, applying the items (1) and (3) of Proposition 2.1, there exists $z \in H$ satisfying

$$\Gamma_z(\varphi(k), w) = \{k\}.$$

Therefore $k \in \Psi(\varphi(k)) = \{\psi(\varphi(k))\}$. That is, $k = \psi(\varphi(k))$. Hence $\psi \circ \varphi = \mathrm{Id}_K$. Analogously we deduce that $\varphi \circ \psi = \mathrm{Id}_S$.

We now prove that φ is continuous. The proof that ψ is continuous is analogous.

Observe that it suffices to prove that each net $(k_j)_{j\in J}$ of K converging to $k \in K$ admits a subnet $(k_{j_p})_{p\in P}$ such that $(\varphi(k_{j_p}))_{p\in P}$ converges to $\varphi(k)$.

Assume then that $(k_j)_{j\in J}$ is a net of K converging to k. By Propositions 2.1(1) and 5.1, for all $j \in J$ take v_j and $w_j \in H$ with $\|v_j\| = 1$ such that

$$\varphi(k_j) \in \Gamma_{w_j}(k_j, v_j). \tag{6-1}$$

Since the nets $(v_j)_{j\in J}$ and $(w_j)_{j\in J}$ are contained in compact sets, we can assume that there are $v, w \in H$ such that $v_j \to v$ and $w_j \to w$.

For each $f \in C_0(K, H)$ we have

$$\omega(k_j, f, v_j) \to \omega(k, f, v), \tag{6-2}$$

and according to (6-1),

$$\|Tf(\varphi(k_j)) - w_j\| \le M\omega(k_j, f, v_j), \quad \forall j \in J. \tag{6-3}$$

Fix $f_1 \in C_0(K, H)$ satisfying $\|f_1\| = \frac{1}{2}$ and $f_1(x) = \frac{v}{2}$. Then (6-2) and (6-3) imply that

$$\|Tf_1(\varphi(k_j))\| \ge \|w_j\| - \|Tf_1(\varphi(k_j)) - w_j\| \ge \frac{1}{M} - M\omega(k_j, f_1, v_j),$$

for every $j \in J$. Notice that $\omega(k, f_1, v) = \frac{\|v\|}{2} = \frac{1}{2}$, so by (6-2) we have

$$\liminf_{j\in J} \|Tf_1(\varphi(k_j))\| \ge \frac{1}{M} - \frac{M}{2} > 0.$$

Since Tf_1 vanishes at infinity, this implies that $(\varphi(k_j))_{j\in J}$ admits a subnet converging to some $s \in S$, so we assume that $\varphi(k_j) \to s$. Hence, by (6-2) and (6-3),

$$\|Tf(s) - w\| \le M\omega(k, f, v), \quad \forall f \in C_0(K, H),$$

which means that $s \in \Gamma_w(k, v) \subset \Phi(k) = \{\varphi(k)\}$, and consequently $s = \varphi(k)$. □

7. Open questions

In view of Theorem 1.3, the following questions arise naturally:

Problem 7.1. *Is Theorem 1.3 optimal, in the sense that $\sqrt[4]{2}$ is the best number for formalizing it?*

Problem 7.2. *What are the Banach spaces X satisfying the following property: whenever K and S are locally compact Hausdorff spaces and there exists an isomorphism T from $C_0(K, X)$ onto $C_0(S, X)$ with $\|T\|\|T^{-1}\| = \sqrt{2}$, then K and S are homeomorphic?*

References

[Amir 1965] D. Amir, "On isomorphisms of continuous function spaces", *Israel J. Math.* **3** (1965), 205–210. MR Zbl

[Banach 1932] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne **1**, Seminarium Matematyczne Uniwersytetu Warszawskiego, Warsaw, 1932. Zbl JFM

[Behrends 1979] E. Behrends, *M-structure and the Banach–Stone theorem*, Lecture Notes in Mathematics **736**, Springer, Berlin, 1979. MR Zbl

[Cambern 1966] M. Cambern, "A generalized Banach–Stone theorem", *Proc. Amer. Math. Soc.* **17** (1966), 396–400. MR Zbl

[Cambern 1967] M. Cambern, "On isomorphisms with small bound", *Proc. Amer. Math. Soc.* **18** (1967), 1062–1066. MR Zbl

[Cambern 1970] M. Cambern, "Isomorphisms of $C_0(Y)$ onto $C(X)$", *Pacific J. Math.* **35** (1970), 307–312. MR Zbl

[Cambern 1976] M. Cambern, "Isomorphisms of spaces of continuous vector-valued functions", *Illinois J. Math.* **20**:1 (1976), 1–11. MR Zbl

[Cidral et al. 2015] F. C. Cidral, E. M. Galego, and M. A. Rincón-Villamizar, "Optimal extensions of the Banach–Stone theorem", *J. Math. Anal. Appl.* **430**:1 (2015), 193–204. MR Zbl

[Cohen 1975] H. B. Cohen, "A bound-two isomorphism between $C(X)$ Banach spaces", *Proc. Amer. Math. Soc.* **50** (1975), 215–217. MR Zbl

[Galego and Porto da Silva 2016] E. M. Galego and A. L. Porto da Silva, "An optimal nonlinear extension of Banach–Stone theorem", *J. Funct. Anal.* **271**:8 (2016), 2166–2176. MR Zbl

[Stone 1937] M. H. Stone, "Applications of the theory of Boolean rings to general topology", *Trans. Amer. Math. Soc.* **41**:3 (1937), 375–481. MR Zbl

Received September 19, 2016. Revised March 31, 2017.

ELÓI MEDINA GALEGO
DEPARTMENT OF MATHEMATICS, IME
UNIVERSITY OF SÃO PAULO
05508-090 SÃO PAULO
BRAZIL

eloi@ime.usp.br

ANDRÉ LUIS PORTO DA SILVA
DEPARTMENT OF MATHEMATICS, IME
UNIVERSITY OF SÃO PAULO
05508-090 SÃO PAULO
BRAZIL

porto@ime.usp.br

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.333

DISTINGUISHED THETA REPRESENTATIONS FOR CERTAIN COVERING GROUPS

FAN GAO

To Professor Freydoon Shahidi on his 70th birthday

For Brylinski–Deligne covering groups of an arbitrary split reductive group, we consider theta representations attached to certain exceptional genuine characters. The goal of the paper is to study the dimension of the space of Whittaker functionals of a theta representation. In particular, we investigate when the dimension is exactly one, in which case the theta representation is called distinguished. For this purpose, we first give effective lower and upper bounds for the dimension of Whittaker functionals for general theta representations. Consequently, the dimension in many cases can be reduced to simple combinatorial computations, e.g., the Kazhdan–Patterson covering groups of the general linear groups, or covering groups whose complex dual groups (à la Finkelberg, Lysenko, McNamara and Reich) are of adjoint type. In the second part of the paper, we consider coverings of certain semisimple simply connected groups and give necessary and sufficient conditions for the theta representation to be distinguished. There are subtleties arising from the relation between the rank and the degree of the covering group. However, in each case we will determine the exceptional character whose associated theta representation is distinguished.

1. Introduction and main results

1A. ***Introduction.*** Let F be a nonarchimedean local field of characteristic 0 and residue characteristic p. Let $\mathbb{G}$ be a connected split reductive group over F, and let $G := \mathbb{G}(F)$ be its rational points. One of the central ingredients in the study of irreducible admissible representation of G is the uniqueness of Whittaker functionals (see [Rodier 1973; Shalika 1974]). For instance, this uniqueness property is crucial in the Langlands–Shahidi theory of L-functions [Shahidi 2010] for the so-called generic representations of G, i.e., those with nontrivial Whittaker functionals.

MSC2010: primary 11F70; secondary 22E50.
Keywords: Brylinski–Deligne covering groups, theta representations, Whittaker functionals, distinguished characters, dual groups.

For a natural number $n \geq 1$, we assume that $F^\times$ contains the full subgroup of the n-th roots of unity, which is then denoted by μ_n. In this paper, we work with the Brylinski–Deligne n-fold covering groups $\overline{G}^{(n)}$ of G, see Section 2A for a description on such covering groups. We may write $\overline{G}^{(n)}$ and $\overline{G}$ interchangeably if no confusion arises. For simplicity, the phrase *covering groups* in this paper is used to refer to the Brylinski–Deligne covering groups. For this purpose, it is noteworthy to mention that the Brylinski–Deligne framework is quite encompassing and contains almost all classically interesting covering groups [Steinberg 1962; Moore 1968; Matsumoto 1969], in particular the Matsumoto covering groups of semisimple simply connected groups [Moore 1968] and the Kazhdan–Patterson covering groups $\overline{\mathrm{GL}}_r^{(n)}$ of GL_r [Kazhdan and Patterson 1984].

For covering groups, the uniqueness of Whittaker functionals for genuine representations of $\overline{G}^{(n)}$ holds rarely and one nontrivial example is the classical double cover $\overline{\mathrm{Sp}}_{2r}^{(2)}$ of the symplectic group Sp_{2r}, see [Szpruch 2007]. This uniqueness plays a pivotal role in the work of Szpruch [2009b; 2013] generalizing the method of Langlands and Shahidi to $\overline{\mathrm{Sp}}_{2r}^{(2)}$. Besides this special family of examples, the uniqueness of Whittaker functionals fails widely, and one almost never expects such a uniform property for all genuine representations of a general covering group. For example, it is well known that certain theta representations for the Kazhdan–Patterson coverings $\overline{\mathrm{GL}}_r^{(n)}$ of GL_r could have high dimensional space of Whittaker functionals [Kazhdan and Patterson 1984]. In fact, such theta representations show that the analogous standard module conjecture (which is a theorem for linear algebraic groups from [Casselman and Shahidi 1998]) does not hold for covering groups.

The failure of the uniqueness of Whittaker functionals for general genuine representations of covering groups, however, has been the source of both obstacles and inspirations to some advancement of the representation theory of such groups. On the one hand, for instance, it is not a priori clear how to generalize the Langlands–Shahidi theory of L-functions to covering groups because of the nonuniqueness of Whittaker functionals for unramified principal series representations. Equivalently, the difficulty for such generalization is essentially due to the fact that the analogous Casselman–Shalika formula for covering groups as in [Chinta and Offen 2013; McNamara 2016] is vector-valued, whereas for linear algebraic groups it is scalar-valued; see [Casselman and Shalika 1980].

On the other hand, there are various streams of rich theories stemming from the nonexistence or multidimensionality of Whittaker functionals. For instance, for genuine representations of covering groups without Whittaker functionals, one may consider semi-Whittaker functionals as in [Takeda 2014] or degenerate Whittaker-functionals [Mœglin and Waldspurger 1987], which interact fruitfully with the arithmetic and character theory of the representations. Meanwhile, the theory of unipotent orbit as discussed in [Ginzburg 2006; Friedberg and Ginzburg 2014;

Friedberg and Ginzburg 2016a] for instance also rectify the situation in the absence of Whittaker functionals. In the latter case where multidimensionality holds, the theory of multiple Weyl Dirichlet series makes deep and fascinating connections between representation theory of covering groups, quantum physics and statistical mechanics etc, see [Brubaker et al. 2011; Bump et al. 1990; 2012] for some of the ideas involved. In particular, the book [Bump et al. 2012] contains several excellent expository articles on multiple Dirichlet series.

Nevertheless, in this paper we consider only the so-called theta representations $\Theta(\overline{G}^{(n)}, \overline{\chi})$ which appear as the local representations for the residue of the Borel Eisenstein series (see Definition 2.1). Moreover, we are mostly interested in determining when the space of Whittaker functionals for $\Theta(\overline{G}^{(n)}, \overline{\chi})$ has dimension one, in which case $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is called distinguished following Suzuki [1998]. Here $\overline{\chi}$ is an exceptional genuine character (see Definition 2.1) of the center $Z(\overline{T})$ of the covering torus $\overline{T} \subseteq \overline{G}$. The reason for considering this problem is two-fold.

First, $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is in a certain sense the simplest family of genuine representations of a general covering group $\overline{G}^{(n)}$. Indeed, if $n = 1$, then it follows from definition that $\Theta(\overline{G}^{(n)}, \overline{\chi})$ could be the trivial representation of the linear group $\overline{G} = \overline{G}^{(1)}$, depending on a proper choice of the exceptional character $\overline{\chi}$. Therefore, for the genericity question regarding Whittaker functionals of genuine representations, it is reasonable to consider this family first. Moreover, theta representations for the Kazhdan–Patterson covering groups of GL_r, to which we have just alluded, are already studied in depth in the seminal paper [Kazhdan and Patterson 1984]. Despite the fact that the idea therein could be applicable for general covering groups, to the best of our knowledge, it seems that there is no systematic treatment on theta representations for general covering groups in the literature. Perhaps this gap is caused by the tedious cocycle computation to be carried out by any potential author. However, the Brylinski–Deligne framework enables us to compute by invoking some neat structural fact of the covering groups of interest, and to handle only a minimized usage of a cocycle on the torus. In brief, we wish to fill in the gap by generalizing the relevant work of Kazhdan and Patterson to Brylinski–Deligne covering groups.

Second, distinguished theta representations have important and emergingly wider applications. Theta representations are the representation-theoretic analogues of theta functions, one of the early applications of which was given by Riemann in his seminal paper to prove the functional equation of the Riemann zeta function. In the language of modern theory of representations, theta representations for $\overline{\mathrm{Sp}}_{2r}^{(2)}$ gain deep applications in the Shimura correspondence [Shimura 1973; Gelbart 1976]. On the other hand, following the work of Kazhdan and Patterson, theta representations for $\overline{\mathrm{GL}}_r^{(n)}$ are also studied extensively in [Bump and Hoffstein 1987; Suzuki 1998; 2012], to mention a few. In particular, these authors made some deep conjectures and also provided evidence for a generalized Shimura correspondence regarding $\overline{\mathrm{GL}}_r^{(n)}$, and the distinguishedness property is exploited to achieve the goals in their work. Another significant direction of applications is the Rankin–Selberg integral representation for the symmetric square and cube L-functions [Bump and Ginzburg 1992; Bump et al. 1996; Takeda 2014; Kaplan 2016]. Evidently, it should be mentioned that for distinguished theta representations, the theory of L-functions could be developed as in the linear algebraic case, since the Casselman–Shalika formula is then scalar-valued. More recently, the work of E. Kaplan [2015a; 2015b], and S. Friedberg and D. Ginzburg [2014; 2016a] also relies heavily on the local and global theta representations in their consideration of Fourier coefficient, Rankin–Selberg L-function and descent integral etc. Notably in their work, distinguishedness is responsible for proving that a global integral admits an Euler factorization into local factors. Besides these, the problem on global cuspidal theta representations is important and many problems are open (see [Friedberg and Ginzburg 2016a; Suzuki 1998]). In any case, we believe that distinguished theta representations are objects of great interest and significance, and we hope that our paper could shed some light on the relevant questions.

1B. ***Main results.*** We consider a Brylinski–Deligne n-fold covering group $\overline{G}^{(n)}$. Let $\overline{\chi}$ be an exceptional character for $\overline{G}^{(n)}$. Fix an unramified additive character ψ of F and consider the space $\mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ of ψ-Whittaker functionals of the theta representation $\Theta(\overline{G}^{(n)}, \overline{\chi})$. The pair $(\overline{G}^{(n)}, \overline{\chi})$ such that

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$$

is quite unique, and the goal is to investigate when $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is distinguished. We remark that for fixed $\overline{G}^{(n)}$, the set of unramified exceptional characters $\overline{\chi}$ is a torsor over $Z(\overline{G}^\vee)$, the center of the complex dual group $\overline{G}^\vee$ of $\overline{G}$. For details on $\overline{G}^\vee$, see [Finkelberg and Lysenko 2010; McNamara 2012; Reich 2012; Weissman 2015].

We outline the structure of the paper and state the main results.

In Section 2, we recall the basic structural facts on a Brylinski–Deligne covering group $\overline{G}^{(n)}$ which will be crucial for our computations. In this paper, we consider exclusively unramified covering group $\overline{G}^{(n)}$ and unramified exceptional character $\overline{\chi}$. In Section 3, the space $\mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}), \overline{\chi})$ is analyzed following the strategy in [Kazhdan and Patterson 1984] closely. In particular, it relies crucially on the Shahidi local coefficient matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ for covering groups. Note that $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ is also referred to as the scattering matrix in [Brubaker et al. 2016] and transition matrix in [Chinta and Offen 2013]. Since the matrix is an analogue (and in fact the reciprocal) of Shahidi's local coefficient in the linear algebraic case [Shahidi 2010, Chapter 5], we call it the Shahidi local coefficient matrix in this paper. See also [Budden 2006; Szpruch 2016]. In the unramified setting, the matrix is computed in [McNamara 2016]; it is also computed for ramified places in [Goldberg and Szpruch 2015].

The first main result is Theorem 3.14 from Section 3:

Theorem 1.1. *Let $\overline{G}^{(n)}$ be an arbitrary unramified Brylinski–Deligne covering group. Let $\overline{\chi}$ be an unramified exceptional genuine character of $\overline{G}^{(n)}$ with associated theta representation $\Theta(\overline{G}^{(n)}, \overline{\chi})$. Then,*

$$|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \le \dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) \le |\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})|.$$

These two bounds are combinatorial quantities involving certain Weyl-action on lattices. The readers are referred to Section 2 for details. We highlight here some consequences from the above theorem.

Firstly, Theorem 1.1 recovers the results of Kazhdan and Patterson. More precisely, for covering groups $\overline{\mathrm{GL}}_r^{(n)}$ studied in [Kazhdan and Patterson 1984], the authors determine that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{GL}}_r^{(n)}, \overline{\chi})) = 1$ if and only if

(1) $n = r$ and $\overline{\mathrm{GL}}_r^{(n)}$ is any Kazhdan–Patterson covering group, or

(2) $n = r + 1$ and $\overline{\mathrm{GL}}_r^{(n)}$ belongs to a special type of degree n Kazhdan–Patterson covering groups.

In fact, for any covering group $\overline{\mathrm{GL}}_r^{(n)}$ studied in [Kazhdan and Patterson 1984], one has $\mathcal{O}^F_{Q,n} = \mathcal{O}^F_{Q,n,\mathrm{sc}}$. Therefore $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{GL}}_r^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})|$. In particular, the dimension does not depend on the choice of the exceptional character $\overline{\chi}$ and can be computed effectively. For details, see Example 3.16.

In general, for cases where the two bounds in Theorem 1.1 actually agree, the computation of the dimension is reduced to a purely combinatorial problem, and thus amenable to a straightforward calculation. This includes the case where $Y_{Q,n} = Y^{\mathrm{sc}}_{Q,n}$, or equivalently $Z(\overline{G}^\vee) = 1$. For example, odd degree coverings of simply connected groups of type B_r, C_r have this property. See Sections 5 and 6.

Secondly in contrast, when the two bounds in Theorem 1.1 do not agree, $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ becomes sensitive to the choice of the exceptional character $\overline{\chi}$. The second half of this paper is devoted to investigating this. This phenomenon already occurs for the degree two metaplectic covering $\overline{\mathrm{SL}}_2^{(2)}$, see Example 4.7. In this case $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi})$ is the even Weil representation. Consider $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$, where $\overline{\chi}_{\psi_a}$ is an exceptional character defined by using the twisted additive character ψ_a, where $a \in F^\times$. It is well known that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})) \le 1$ and the equality holds if and only if $a \in (F^\times)^2$. Our analysis shows that similar phenomenon occurs for higher rank groups, see Section 4B, in particular Corollary 4.5.

In any case, we summarize our results for certain coverings of simply connected groups as follows. We write for instance $\overline{A}_r^{(n)}$ for the degree n covering of the simply connected group of type A_r of rank r. Here the covering group arises from a quadratic form Q on the coroot lattice $Y = Y^{\mathrm{sc}}$ such that $Q(\alpha^\vee) = 1$ for any short coroot $\alpha^\vee$. The following theorem is an amalgam of Theorems 4.10, 5.3, 6.2 and 7.1. Only for $\overline{A}_r^{(n)}$, we impose the condition $n \le r+2$ for technical reasons.

Theorem 1.2. *Let $\overline{G}^{(n)}$ be an unramified Brylinski–Deligne degree n covering of a simply connected semisimple group of type A_r, B_r, C_r or G_2. If $\overline{G}^{(n)} = \overline{A}_r^{(n)}$, we further assume $n \le r+2$. Let $\overline{\chi}$ be an unramified exceptional character for $\overline{G}^{(n)}$. In each case for $\overline{G}^{(n)}$ below, if* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$, *then the following relations between r and n must hold*:

$$\begin{cases} \overline{A}_r^{(n)}, r \ge 1, n \le r+2, & n = r+2 \ \textit{or} \ r+1; \\ \overline{C}_r^{(n)}, r \ge 2, & n = 4r-2,\ 4r,\ 4r+2 \ \textit{or} \ 2r+1; \\ \overline{B}_r^{(n)}, r \ge 3, & n = 2r+1 \ \textit{or} \ 2r+2; \\ \overline{G}_2^{(n)}, & n = 7 \ \textit{or} \ 12. \end{cases}$$

Conversely, suppose that r and n satisfy the above relations; *then for every case above except $\overline{C}_r^{(4r)}$, there exists a unique exceptional character $\overline{\chi}$ such that* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$ *for above $\overline{G}^{(n)}$.*

We actually determine the unique exceptional character specified in Theorem 1.2, see Theorems 4.10, 5.3, 6.2 and 7.1. In the $\overline{A}_r^{(r+1)}$ case, our result generalizes the result for the even Weil representation of $\overline{\mathrm{SL}}_2^{(2)}$ mentioned above. As noted, the collection of unramified exceptional characters is a torsor over $Z(\overline{G}^\vee)$. Moreover, for covering groups of simply connected groups, the choice of ψ actually gives a base point for this torsor. Thus, any exceptional character $\overline{\chi}$ gives rise to an element in $Z(\overline{G}^\vee)$, depending on the choice of ψ. That is, the explicit requirement given in those theorems could be viewed as determining the corresponding element in $Z(\overline{G}^\vee)$.

We note that for classical groups and similitude groups, an extensive study is included in [Friedberg et al. $\ge$ 2017]. Our result from Theorem 1.2 also agrees with the pertinent discussion in [Friedberg and Ginzburg 2016b] for symplectic groups. For example, the local statement for the second part of Conjecture 1 in Friedberg and Ginzburg's paper follows from our Proposition 5.1 here. Moreover, the factorizability property of the Whittaker function in that paper for $\overline{\mathrm{Sp}}_{2n}^{(4n-2)}$ also agrees with our result for the $\overline{C}_r^{(n)}$ case in Theorem 1.2.

Finally, we remark that groups of type D_r, E_6, E_7, E_8, F_4 could be analyzed by the same procedure. In principle, Theorem 1.1 coupled with the analogous argument for Theorem 1.2 enable one to determine completely $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ for arbitrary $(\overline{G}^{(n)}, \overline{\chi})$.

2. Basic setup

2A. ***Structural facts on $\overline{G}$.*** For ease of reading, we first recall some structural facts on $\overline{G}$. The main references are [Brylinski and Deligne 2001; Finkelberg and Lysenko 2010; Reich 2012; McNamara 2012; 2016; Weissman 2015; Gan and Gao 2016].

In this paper, we concentrate exclusively on unramified Brylinski–Deligne covering groups $\overline{G}$ (to be explained below). We follow the notations in [Gan and Gao 2016].

Let F be a nonarchimedean field of characteristic 0, with residual characteristic p. Fix a uniformizer ϖ of F. Let $\mathbb{G}$ be a split linear algebraic group over F with maximal split torus $\mathbb{T}$. Write $(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee)$ for the root data of $\mathbb{G}$. Here X (respectively, Y) is the character lattice (respectively, cocharacter lattice) for $(\mathbb{G}, \mathbb{T})$. Choose a set $\Delta \subseteq \Phi$ of simple roots from the set of roots Φ, and $\Delta^\vee$ the corresponding simple coroots from $\Phi^\vee$. Let $\mathbb{B}$ be the Borel subgroup associated with Δ. Write $Y^{\mathrm{sc}} \subseteq Y$ for the lattice generated by $\Phi^\vee$.

Fix a Chevalley system of pinnings for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$. That is, fix an isomorphism $e_\alpha : \mathbb{G}_{\mathrm{a}} \to \mathbb{U}_\alpha$ for each $\alpha \in \Phi$, where $\mathbb{U}_\alpha \subseteq \mathbb{G}$ is the root subgroup associated with α. Moreover, for each $\alpha \in \Phi$, there is a unique morphism $\phi_\alpha : \mathrm{SL}_2 \to \mathbb{G}$ which restricts to $e_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices of SL_2.

Consider the algebro-geometric covering $\overline{\mathbb{G}}$ of $\mathbb{G}$ by $\mathbb{K}_2$, which is categorically equivalent to the pairs $\{(D, \eta)\}$ (see [Gan and Gao 2016]). Here $\eta : Y^{\mathrm{sc}} \to F^\times$ is a homomorphism. On the other hand, D is a bisector associated to a Weyl-invariant quadratic form $Q : Y \to \mathbb{Z}$. That is, let B_Q be the Weyl-invariant bilinear form associated to Q such that $B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$, then D is a bilinear form on Y satisfying

$$D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2).$$

The bisector D is not necessarily symmetric. Any $\overline{\mathbb{G}}$ is, up to isomorphism, incarnated by (i.e., categorically associated to) (D, η) for a bisector D and some η.

Let $n \geq 1$ be a natural number. Assume that $F^\times$ contains the full group μ_n of n-th roots of unity and $p \nmid n$. Let $\overline{\mathbb{G}}$ be incarnated by (D, η). One naturally obtains degree n topological covering groups $\overline{G}, \overline{T}, \overline{B}$ of the rational points $G := \mathbb{G}(F), T := \mathbb{T}(F), B := \mathbb{B}(F)$, such as

$$\mu_n \hookrightarrow \overline{G} \twoheadrightarrow G.$$

We may write $\overline{G}^{(n)}$ for $\overline{G}$ to emphasize the degree of covering. For any set $H \subseteq G$, we write $\overline{H} \subseteq \overline{G}$ for the preimage of H with respect to the quotient map $\overline{G} \to G$. The Bruhat–Tits theory gives a maximal compact subgroup $K \subseteq G$, which depends on the fixed pinnings. We assume that $\overline{G}$ splits over K and fixes such a splitting; call $\overline{G}$ an unramified Brylinski–Deligne covering group in this case. We remark that if the derived group of $\mathbb{G}$ is simply connected, then $\overline{G}$ splits over K (see [Gan and Gao 2016, Theorem 4.2]). On the other hand, we refer the reader to [Gan and Gao 2016, § 4.6] for a counterexample from a certain double cover of PGL_2 where the splitting does not exist.

The data (D, η) play the following role for the structural fact on $\overline{G}$:

- The group $\overline{G}$ splits canonically over any unipotent element of G. In particular, we write $\bar{e}_\alpha(u) \in \overline{G}$, $\alpha \in \Phi$, $u \in F$ for the canonical lifting of $e_\alpha(u) \in G$. For any $\alpha \in \Phi$, there is a natural representative $w_\alpha := e_\alpha(1)e_{-\alpha}(-1)e_\alpha(1) \in K$ (and therefore $\bar{w}_\alpha \in \overline{G}$ by the splitting of K) of the Weyl element $\mathbb{w}_\alpha \in W$. Moreover, for $h_\alpha(a) := \alpha^\vee(a) \in G$, $\alpha \in \Phi$, $a \in F^\times$, there is a natural lifting $\bar{h}_\alpha(a) \in \overline{G}$ of $h_\alpha(a)$, which depends only on the pinning and the canonical unipotent splitting. For details, see [Gan and Gao 2016].
- There is a section $\boldsymbol{s}$ of $\overline{T}$ over T such that the group law on $\overline{T}$ is given by

$$\boldsymbol{s}(y_1(a)) \cdot \boldsymbol{s}(y_2(b)) = (a, b)_n^{D(y_1,y_2)} \cdot \boldsymbol{s}(y_1(a) \cdot y_2(b)). \tag{1}$$

Moreover, for the natural lifting $\bar{h}_\alpha(a)$, one has

$$\bar{h}_\alpha(a) = (\eta(\alpha^\vee), a)_n \cdot \boldsymbol{s}(h_\alpha(a)) \in \overline{T}. \tag{2}$$

- Let $w_\alpha \in G$ be the natural representative of $\mathbb{w}_\alpha \in W$. For any $\overline{y(a)} \in \overline{T}$,

$$w_\alpha \cdot \overline{y(a)} \cdot w_\alpha^{-1} = \overline{y(a)} \cdot \bar{h}_\alpha(a^{-\langle y,\alpha\rangle}), \tag{3}$$

where $\langle -, - \rangle$ is the pairing between Y and X.

Consider the sublattice $Y_{Q,n} := \{y \in Y : B_Q(y, y') \in n\mathbb{Z}\}$ of Y. For every $\alpha^\vee \in \Phi^\vee$, define $n_\alpha := n/\gcd(n, Q(\alpha^\vee))$. Write $\alpha_{Q,n}^\vee := n_\alpha \alpha^\vee$ and $\alpha_{Q,n} := n_\alpha^{-1}\alpha$. Let $Y_{Q,n}^{\mathrm{sc}} \subseteq Y$ be the sublattice generated by $\{\alpha_{Q,n}^\vee\}_{\alpha\in\Phi}$. The complex dual group $\overline{G}^\vee$ for $\overline{G}$ as given in [Finkelberg and Lysenko 2010; McNamara 2012; Reich 2012] has root data $(Y_{Q,n}, \{\alpha_{Q,n}^\vee\}, \mathrm{Hom}(Y_{Q,n}, \mathbb{Z}), \{\alpha_{Q,n}\})$. In particular, $Y_{Q,n}^{\mathrm{sc}}$ is the root lattice for $\overline{G}^\vee$. What is most pertinent to our paper is that the center $Z(\overline{G}^\vee)$ could be identified as

$$Z(\overline{G}^\vee) := \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times).$$

2B. ***Theta representations*** $\boldsymbol{\Theta(\overline{G}, \overline{\chi})}$**.** Fix an embedding $\iota : \mu_n \hookrightarrow \mathbb{C}^\times$. A representation of $\overline{G}$ is called ι-genuine if μ_n acts via ι. We consider throughout the paper ι-genuine (or simply genuine) representations of $\overline{G}$.

Let U be the unipotent subgroup of $B = TU$. As U splits canonically in $\overline{G}$, we have $\overline{B} = \overline{T}U$. The covering torus $\overline{T}$ is a Heisenberg group with center $Z(\overline{T})$. The image of $Z(\overline{T})$ in T is equal to the image of the isogeny $Y_{Q,n} \otimes F^\times \to T$ induced from $Y_{Q,n} \to Y$.

Let $\overline{\chi} \in \mathrm{Hom}_\iota(Z(\overline{T}), \mathbb{C}^\times)$ be a genuine character of $Z(\overline{T})$, write $i(\overline{\chi}) := \mathrm{Ind}_{\overline{A}}^{\overline{T}} \overline{\chi}'$ for the induced representation on $\overline{T}$, where A is any maximal abelian subgroup of $\overline{T}$, and $\overline{\chi}'$ is any extension of $\overline{\chi}$. By the Stone–von Neumann theorem (see [Weissman 2009, Theorem 3.1; McNamara 2012, Theorem 3]), the construction $\overline{\chi} \mapsto i(\overline{\chi})$ gives a bijection between isomorphism classes of genuine representations of $Z(\overline{T})$ and $\overline{T}$. Since we consider an unramified covering group $\overline{G}$ in this paper, we take $\overline{A}$ to be $Z(\overline{T}) \cdot (K \cap T)$ from now.

View $i(\bar{\chi})$ as a genuine representation of $\bar{B}$ by inflation from the quotient map $\bar{B} \to \bar{T}$. Write $I(i(\bar{\chi})) := \mathrm{Ind}_{\bar{B}}^{\bar{G}}\ i(\bar{\chi})$ for the normalized induced principal series representation of $\bar{G}$. For simplicity, we may also write $I(\bar{\chi})$ for $I(i(\bar{\chi}))$. One knows that $I(\bar{\chi})$ is unramified (i.e., $I(\bar{\chi})^K \neq 0$) if and only if $\bar{\chi}$ is unramified, i.e., $\bar{\chi}$ is trivial on $Z(\bar{T}) \cap K$. We consider in this paper only unramified genuine representations (and characters). In fact, one has the naturally arising abelian extension

$$\mu_n \hookrightarrow \overline{Y}_{Q,n} \twoheadrightarrow Y_{Q,n} \tag{4}$$

such that unramified genuine characters of $\bar{\chi}$ of $Z(\bar{T})$ correspond to genuine characters of $\overline{Y}_{Q,n}$. Here $\overline{Y}_{Q,n} := Z(\bar{T})/Z(\bar{T}) \cap K$. Since $\bar{A}/(T \cap K) \simeq \overline{Y}_{Q,n}$ as well, there is a canonical extension (also denoted by $\bar{\chi}$) of an unramified character $\bar{\chi}$ of $Z(\bar{T})$ to $\bar{A}$, by composing $\bar{\chi}$ with $\bar{A} \twoheadrightarrow \overline{Y}_{Q,n}$. Therefore, we will identify $i(\bar{\chi})$ as $\mathrm{Ind}_{\bar{A}}^{\bar{T}}\ \bar{\chi}$ for this $\bar{\chi}$.

For any $\mathbb{w} \in W$, the intertwining operator $T_{\mathbb{w},\chi} : I(\bar{\chi}) \to I({}^{\mathbb{w}}\bar{\chi})$ is defined by

$$(T_{\mathbb{w},\bar{\chi}} f)(\bar{g}) = \int_{U_w} f(w^{-1} u \bar{g})\, du$$

whenever it is absolutely convergent. Moreover, it can be meromorphically continued for all $\bar{\chi}$ (see [McNamara 2012, § 7]). For $I(\bar{\chi})$ unramified and $\mathbb{w} = \mathbb{w}_\alpha$ with $\alpha \in \Delta$, $T_{\mathbb{w}_\alpha,\chi}$ is determined by

$$T_{\mathbb{w}_\alpha,\chi}(f_0) = c(\mathbb{w}_\alpha, \bar{\chi}) \cdot f_0' \text{ with } c(\mathbb{w}_\alpha, \bar{\chi}) = \frac{1 - q^{-1}\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))}{1 - \bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha})},$$

where $f_0 \in I(\bar{\chi})$ and $f_0' \in I({}^{\mathbb{w}_\alpha}\bar{\chi})$ are the unramified vectors. Moreover, $T_{\mathbb{w},\bar{\chi}}$ satisfies the cocycle condition as in the linear case. The coefficient $c(\mathbb{w}_\alpha, \bar{\chi})$ was determined in [McNamara 2016, Theorem 12.1] and later reformulated in [Gao $\geq$ 2017]. We use the latter formalism which is more suitable for our needs in this paper.

The following definition mimics that in [Kazhdan and Patterson 1984, § I.2].

Definition 2.1. An unramified genuine character $\bar{\chi}$ of $Z(\bar{T})$ is called exceptional if $\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha})) = q^{-1}$ for all $\alpha \in \Delta$. The theta representation $\Theta(\bar{G}, \bar{\chi})$ associated to an exceptional character $\bar{\chi}$ is the unique Langlands quotient (see [Ban and Jantzen 2013]) of $I(\bar{\chi})$, which is also equal to the image of the intertwining operator $T_{\mathbb{w}_0,\bar{\chi}} : I(\bar{\chi}) \to I({}^{\mathbb{w}_0}\bar{\chi})$, where $\mathbb{w}_0 \in W$ is the longest Weyl element.

The extension $\overline{Y}_{Q,n}$ gives rise to an extension $\overline{Y}_{Q,n}^{\mathrm{sc}}$ of $Y_{Q,n}^{\mathrm{sc}}$ by restriction. All exceptional characters agree on $\overline{Y}_{Q,n}^{\mathrm{sc}}$, and therefore the set of exceptional characters is a torsor over $Z(\bar{G}^\vee)$.

2C. ***Unitary distinguished characters.*** Depending on the choice of a nontrivial additive character ψ' of F, a special class of the so-called distinguished genuine characters of $Z(\overline{T})$ is singled out in [Gan and Gao 2016] for the consideration of the L-group extension for $\overline{G}$. Distinguished characters, in the sense of [Gan and Gao 2016], may not exist for general Brylinski–Deligne covering groups. However, if $\mathbb{G}$ has a simply connected derived group or if the composition

$$\eta : Y^{\mathrm{sc}} \to F^{\times} \to F/(F^{\times})^n$$

is trivial, such characters exist. One special property of a distinguished character is its Weyl-invariance, and thus it could serve as a *distinguished* base point in the set of genuine characters of $Z(\overline{T})$.

For the purpose of Sections 4 to 7, we recall the explicit construction in [Gan and Gao 2016] when a distinguished character exists. In particular, we make the above assumption on $\overline{G}$, which is clearly satisfied in the simply connected case in Sections 4 to 7.

First, let $\{y_i\}$ be a basis of $Y_{Q,n}$ such that $\{k_i y_i\}$ is a basis for the lattice $J = nY + Y_{Q,n}^{\mathrm{sc}}$ for some $k_i \in \mathbb{Z}$. Let ψ' be a nontrivial additive character of F. Let $\boldsymbol{\gamma}_{\psi'}$ be the Weil index valued in μ_4 satisfying

$$\boldsymbol{\gamma}_{\psi'}(b^2) = 1, \quad \boldsymbol{\gamma}_{\psi'}(b)^2 = (b, b)_2, \quad \boldsymbol{\gamma}_{\psi'}(bc) = \boldsymbol{\gamma}_{\psi'}(b)\boldsymbol{\gamma}_{\psi'}(c) \cdot (b, c)_2.$$

For any $a \in F^{\times}$, let $\psi'_a : x \mapsto \psi'(ax)$ be the twisted additive character. Then

$$\boldsymbol{\gamma}_{\psi'_a}(b) = \boldsymbol{\gamma}_{\psi'}(b) \cdot (a, b)_2.$$

By definition, a unitary distinguished character $\overline{\chi}_{\psi'}^{\,0}$ of $Z(\overline{T})$ is given by

$$\overline{\chi}_{\psi'}^{\,0}(y_i(a)) = \boldsymbol{\gamma}_{\psi'}(a)^{2(k_i-1)Q(y_i)/n},$$

and for $y = \sum_i n_i y_i$ and $a \in F^{\times}$,

$$\overline{\chi}_{\psi'}^{\,0}(y(a)) = (a, a)_n^{\sum_{i<j} n_i n_j D(y_i, y_j)} \cdot \prod_i \overline{\chi}_{\psi'}^{\,0}(y_i(a^{n_i}))^{2(k_i-1)Q(y_i)/n}. \tag{5}$$

Note that in [Gan and Gao 2016], the exponent of $\boldsymbol{\gamma}_{\psi'}(a)$ in the formula of $\overline{\chi}_{\psi'}^{\,0}(y_i(a))$ is the negative of what we use here. However, both give rise to distinguished characters.

2D. ***Conventions and notations.*** Let $2\rho := \sum_{\alpha^\vee > 0} \alpha^\vee$ be the sum of all positive coroots of $\mathbb{G}$. Consider the affine translation $\ell_\rho : Y \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$ given by $y \mapsto y - \rho$. Write $\mathbb{w}(y)$ for the natural Weyl group action on Y and $Y \otimes \mathbb{Q}$. Endow the codomain of ℓ_ρ with this action. By transport of structure, one has an induced action of W on the domain of ℓ_ρ (i.e., the first $Y \otimes \mathbb{Q}$), which we denote by $\mathbb{w}[y]$. That is,

$$\mathbb{w}[y] := \mathbb{w}(y - \rho) + \rho.$$

Clearly Y is stable under this action. Write $y_\rho := y - \rho$ for any $y \in Y$, then $\mathbb{w}[y] - y = \mathbb{w}(y_\rho) - y_\rho$. From now, by Weyl orbits in Y or $Y \otimes \mathbb{Q}$ we always refer to the ones with respect to the action $\mathbb{w}[y]$. Write $\mathcal{O}$ (respectively $\mathcal{O}^F$) for the set of W-orbits (respectively, free W-orbits) in Y.

We remark that for $\mathbb{GL}_r$, the Weyl-action considered by Kazhdan and Patterson [1984, page 78] is actually $\mathbb{w}(y+\rho)-\rho$. However, the indexing of Whittaker functionals also differs from ours by taking an "inverse", thus our terminology is different but equivalent to that of [Kazhdan and Patterson 1984].

Definition 2.2. For any subgroup $\Lambda \subseteq Y$, a free orbit $\mathcal{O}_y \in \mathcal{O}^F$ is called Λ-free if the quotient map $Y \to Y/\Lambda$ is injective on $\mathcal{O}_y$. We write $\mathcal{O}^F_\Lambda \subseteq \mathcal{O}^F$ for the set of Λ-free orbits of Y.

Note that Λ-free orbits are assumed to be free by definition. For simplicity, we write $\mathcal{O}^F_{Q,n,\mathrm{sc}}$ and $\mathcal{O}^F_{Q,n}$ for the set of $Y^{\mathrm{sc}}_{Q,n}$ and $Y_{Q,n}$-free orbits of Y, respectively. Clearly, the inclusions $\mathcal{O} \supseteq \mathcal{O}^F \supseteq \mathcal{O}^F_{Q,n,\mathrm{sc}} \supseteq \mathcal{O}^F_{Q,n}$ hold.

Generally, notations will be either self-explanatory or explained the first time they occur. For convenience, we list some notations which appear frequently in the text:

ε: the element $\iota((-1,\varpi)_n) \in \mathbb{C}^\times$. In particular, for n odd, $\varepsilon = 1$. We use the following identity freely in the paper:

$$\varepsilon^{D(y,y')} = \varepsilon^{D(y',y)} \ \text{ for any } \ y \in Y_{Q,n},\ y' \in Y.$$

$\wp_{Q,n}$: the projection $Y \to Y/Y_{Q,n}$.

$\wp^{\mathrm{sc}}_{Q,n}$: the projection $Y \to Y/Y^{\mathrm{sc}}_{Q,n}$.

ψ: a fixed additive character of F into $\mathbb{C}^\times$ with conductor O_F. For any $a \in F^\times$, the twisted character ψ_a is given by $\psi_a : x \mapsto \psi(ax)$.

$\boldsymbol{s}_y$: for any $y \in Y$, we write $\boldsymbol{s}_y := \boldsymbol{s}(\varpi^y) \in \overline{T}$.

$\lceil x \rceil$: the minimum integer such that $\lceil x \rceil \geq x$ for a real number x.

3. Bounds for $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$

3A. ***Whittaker functionals.*** We follow the notations in Section 2B. Consider, in particular, the principal series $I(\overline{\chi}) := I(i(\overline{\chi}))$ for an unramified character $\overline{\chi} \in \mathrm{Hom}_\iota(Z(\overline{T}), \mathbb{C}^\times)$.

Let $\mathrm{Ftn}(i(\overline{\chi}))$ be the vector space of functions $\boldsymbol{c}$ on $\overline{T}$ satisfying

$$\boldsymbol{c}(\bar{t}\cdot\bar{z}) = \boldsymbol{c}(\bar{t})\cdot\overline{\chi}(\bar{z}), \qquad \bar{t} \in \overline{T} \text{ and } \bar{z} \in \overline{A}.$$

The support of any $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ is a disjoint union of cosets in $\overline{T}/\overline{A}$. Moreover, $\dim(\mathrm{Ftn}(i(\overline{\chi}))) = |Y/Y_{Q,n}|$ since $\overline{T}/\overline{A}$ has the same size as $Y/Y_{Q,n}$.

There is a natural isomorphism of vector spaces $\mathrm{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee$, where $i(\overline{\chi})^\vee$ is the complex dual space of functionals of $i(\overline{\chi})$. More explicitly, letting $\{\gamma_i\} \subseteq \overline{T}$ be a chosen set of representatives of $\overline{T}/\overline{A}$, consider $\boldsymbol{c}_{\gamma_i} \in \mathrm{Ftn}(i(\overline{\chi}))$ which has support $\gamma_i \cdot \overline{A}$ and $\boldsymbol{c}_{\gamma_i}(\gamma_i) = 1$. It gives rise to a linear functional $\lambda^{\overline{\chi}}_{\gamma_i} \in i(\overline{\chi})^\vee$ such that $\lambda^{\overline{\chi}}_{\gamma_i}(f_{\gamma_j}) = \delta_{ij}$, where $f_{\gamma_j} \in i(\overline{\chi})$ is the unique element such that $\operatorname{supp}(f_{\gamma_j}) = \overline{A} \cdot \gamma_j^{-1}$ and $f_{\gamma_j}(\gamma_j^{-1}) = 1$. That is, $f_{\gamma_j} = i(\overline{\chi})(\gamma_j)\phi_0$, where $\phi_0 \in i(\overline{\chi})$ is the normalized unramified vector of $i(\overline{\chi})$ such that $\phi_0(1_{\overline{T}}) = 1$. Thus, the isomorphism $\operatorname{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee$ is given explicitly by

$$\boldsymbol{c} \mapsto \lambda_{\boldsymbol{c}}^{\overline{\chi}} := \sum_{\gamma_i \in \overline{T}/\overline{A}} \boldsymbol{c}(\gamma_i)\lambda_{\gamma_i}^{\overline{\chi}}.$$

It can be checked easily that the isomorphism does not depend on the choice of representatives for $\overline{T}/\overline{A}$.

Let $\psi_U : U \to \mathbb{C}^\times$ be the character on U such that its restriction to every U_α, $\alpha \in \Delta$ is given by $\psi \circ e_\alpha^{-1}$. We may write ψ for ψ_U if no confusion arises.

Definition 3.1. For any genuine representation $(\overline{\sigma}, V_{\overline{\sigma}})$ of $\overline{G}$, a linear functional $\ell : V_{\overline{\sigma}} \to \mathbb{C}$ is called a ψ-Whittaker functional if $\ell(\overline{\sigma}(u)v) = \psi(u) \cdot v$ for all $u \in U$ and $v \in V_{\overline{\sigma}}$. Write $\mathrm{Wh}_\psi(\overline{\sigma})$ for the space of ψ-Whittaker functionals for $\overline{\sigma}$.

An isomorphism exists between $i(\overline{\chi})^\vee$ and the space $\mathrm{Wh}_\psi(I(\overline{\chi}))$ of ψ-Whittaker functionals on $I(\overline{\chi})$ (see [McNamara 2016, § 6]), given by $\lambda \mapsto W_\lambda$ with

$$W_\lambda : I(\overline{\chi}) \to \mathbb{C}, \quad f \mapsto \lambda\biggl(\int_U f(w_0^{-1}u)\psi(u)^{-1}\mu(u)\biggr),$$

where $f \in I(\overline{\chi})$ is an $i(\overline{\chi})$-valued function on $\overline{G}$. Here U^- is the unipotent subgroup opposite to U; also, $w_0 = w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_k} \in K$ is a representative of w_0, where $\mathsf{w}_0 = \mathsf{w}_{\alpha_1}\mathsf{w}_{\alpha_2}\cdots\mathsf{w}_{\alpha_k}$ is a minimum decomposition of w_0. For any $\boldsymbol{c} \in \operatorname{Ftn}(i(\overline{\chi}))$, by abuse of notation, we will write $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(I(\overline{\chi}))$ for the resulting ψ-Whittaker functional of $I(\overline{\chi})$ from the isomorphism $\operatorname{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee \simeq \mathrm{Wh}_\psi(I(\overline{\chi}))$. An easy consequence is

$$\dim \mathrm{Wh}_\psi(I(\overline{\chi})) = |Y/Y_{Q,n}|.$$

Let $J(\mathsf{w}, \overline{\chi})$ be the image of $T_{\mathsf{w},\overline{\chi}}$. The operator $T_{\mathsf{w},\overline{\chi}}$ induces a homomorphism $T_{\mathsf{w},\overline{\chi}}^*$ of vectors spaces with image $\mathrm{Wh}_\psi(J(\mathsf{w}, \overline{\chi}))$:

$$\begin{tikzcd}
T_{\mathsf{w},\overline{\chi}}^* : \mathrm{Wh}_\psi(I({}^{\mathsf{w}}\overline{\chi})) \arrow[r] \arrow[dr, two heads] & \mathrm{Wh}_\psi(I(\overline{\chi})) \\
& \mathrm{Wh}_\psi(J(\mathsf{w}, \overline{\chi})) \arrow[u, hook]
\end{tikzcd}$$

given by $\langle \lambda_{\boldsymbol{c}}^{{}^{\mathsf{w}}\overline{\chi}}, - \rangle \mapsto \langle \lambda_{\boldsymbol{c}}^{{}^{\mathsf{w}}\overline{\chi}}, T_{\mathsf{w},\overline{\chi}}(-) \rangle$ for any $\boldsymbol{c} \in \operatorname{Ftn}(i({}^{\mathsf{w}}\overline{\chi}))$. Letting $\{\lambda_\gamma^{{}^{\mathsf{w}}\overline{\chi}}\}_{\gamma \in \overline{T}/\overline{A}}$ be a basis for $\mathrm{Wh}_\psi(I({}^{\mathsf{w}}\overline{\chi}))$, and $\{\lambda_{\gamma'}^{\overline{\chi}}\}$ a basis for $\mathrm{Wh}_\psi(I(\overline{\chi}))$, the map $T_{\mathsf{w},\overline{\chi}}^*$ is then determined by the square matrix $[\tau(\overline{\chi}, \mathsf{w}, \gamma, \gamma')]_{\gamma,\gamma' \in \overline{T}/\overline{A}}$ of size $|Y/Y_{Q,n}|$ such that

$$T_{\mathsf{w},\overline{\chi}}^*(\lambda_\gamma^{{}^{\mathsf{w}}\overline{\chi}}) = \sum_{\gamma' \in \overline{T}/\overline{A}} \tau(\overline{\chi}, \mathsf{w}, \gamma, \gamma') \cdot \lambda_{\gamma'}^{\overline{\chi}}.$$

Some immediate properties are as follows.

Lemma 3.2. *For* $\mathsf{w} \in W$ *and* $\bar{z}, \bar{z}' \in \bar{A}$, *the following identity holds*:

$$\tau(\bar{\chi}, \mathsf{w}, \gamma \cdot \bar{z}, \gamma' \cdot \bar{z}') = ({}^{\mathsf{w}}\bar{\chi})^{-1}(\bar{z}) \cdot \tau(\bar{\chi}, \mathsf{w}, \gamma, \gamma') \cdot \bar{\chi}(\bar{z}').$$

Moreover, *for* $\mathsf{w}_1, \mathsf{w}_2 \in W$ *such that* $l(\mathsf{w}_2\mathsf{w}_1) = l(\mathsf{w}_2) + l(\mathsf{w}_1)$, *one has*

$$\tau(\bar{\chi}, \mathsf{w}_2\mathsf{w}_1, \gamma, \gamma') = \sum_{\gamma'' \in \bar{T}/\bar{A}} \tau({}^{\mathsf{w}_1}\bar{\chi}, \mathsf{w}_2, \gamma, \gamma'') \cdot \tau(\bar{\chi}, \mathsf{w}_1, \gamma'', \gamma'),$$

which is referred to as the cocycle relation.

Proof. The first equality follows from a change of basis formula from a different choice of representations for $\bar{T}/\bar{A}$. The second equality follows from the cocycle relation of intertwining operators. □

3B. *Reduction of* $\mathbf{Wh}_\psi(\Theta(\bar{G}, \bar{\chi}))$. Let w_0 be the longest Weyl element of $\mathbb{G}$. Consider the theta representation $\Theta(\bar{G}, \bar{\chi}) = T_{\mathsf{w}_0, \bar{\chi}}(I(\bar{\chi}))$ attached to an unramified exceptional character $\bar{\chi}$ (see Definition 2.1).

Definition 3.3. A theta representation $\Theta(\bar{G}, \bar{\chi})$ attached to an unramified exceptional genuine character $\bar{\chi}$ is called distinguished if

$$\dim \mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) = 1.$$

The distinguishedness of a theta representation here is not to be confused with that of a distinguished genuine character as given in Section 2C.

Proposition 3.4. *Let* $\bar{\chi}$ *be an unramified exceptional character of* $\bar{G}$, *and* Δ *the set of simple roots. Then*

$$\mathrm{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) = \bigcap_{\alpha \in \Delta} \mathrm{Ker}(T^*_{\mathsf{w}_\alpha, {}^{\mathsf{w}_\alpha}\bar{\chi}} : \mathrm{Wh}_\psi(I(\bar{\chi})) \to \mathrm{Wh}_\psi(I({}^{\mathsf{w}_\alpha}\bar{\chi}))),$$

where $T_{\mathsf{w}_\alpha, {}^{\mathsf{w}_\alpha}\bar{\chi}}$ *is the intertwining operator from* $I({}^{\mathsf{w}_\alpha}\bar{\chi})$ *to* $I(\bar{\chi})$.

Proof. The same proof for [Kazhdan and Patterson 1984, Theorem I.2.9] applies here mutatis mutandis. □

Let $\lambda_\gamma^{\bar{\chi}} \in \mathrm{Wh}_\psi(I(\bar{\chi}))$ and $\alpha \in \Delta$, then

$$T^*_{\mathsf{w}_\alpha, {}^{\mathsf{w}_\alpha}\bar{\chi}}(\lambda_\gamma^{\bar{\chi}}) = \sum_{\gamma'} \tau\big({}^{\mathsf{w}_\alpha}\bar{\chi}, \mathsf{w}_\alpha, \gamma, \gamma'\big) \cdot \lambda_{\gamma'}^{{}^{\mathsf{w}_\alpha}\bar{\chi}}.$$

In general, let $\boldsymbol{c} \in \mathrm{Ftn}(i(\bar{\chi}))$, and write

$$\lambda_{\boldsymbol{c}}^{\bar{\chi}} = \sum_{\gamma \in \bar{T}/\bar{A}} \boldsymbol{c}(\gamma)\lambda_\gamma^{\bar{\chi}}.$$

Then,

$$T^*_{\mathbb{w}_\alpha, {}^{\mathbb{w}_\alpha}\overline{\chi}}(\lambda_{\boldsymbol{c}}^{\overline{\chi}}) = \sum_{\gamma} \boldsymbol{c}(\gamma)\Big(\sum_{\gamma'} \tau\big({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\big) \cdot \lambda_{\gamma'}^{{}^{\mathbb{w}_\alpha}\overline{\chi}}\Big)$$
$$= \sum_{\gamma'}\Big(\sum_{\gamma} \boldsymbol{c}(\gamma)\tau\big({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\big)\Big)\lambda_{\gamma'}^{{}^{\mathbb{w}_\alpha}\overline{\chi}}.$$

As an immediate consequence of Proposition 3.4, one has (see also [Kazhdan and Patterson 1984, page 76]):

Corollary 3.5. *A function* $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ *gives rise to a functional in* $\mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ (*i.e.*, $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$) *if and only if for all* $\alpha \in \Delta$,

$$\sum_{\gamma \in \overline{T}/\overline{A}} \boldsymbol{c}(\gamma)\tau\big({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma'\big) = 0 \text{ for all } \gamma'.$$

The left-hand side is independent of the choice of representatives for $\overline{T}/\overline{A}$ *by Lemma 3.2.*

3C. ***The Shahidi local coefficient matrix.*** We would like to compute the matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ for any unramified character $\overline{\chi}$ (not necessarily exceptional) and simple reflection $\mathbb{w}_\alpha$, $\alpha \in \Delta$.

For Kazhdan–Patterson coverings $\overline{\mathrm{GL}}_r^{(n)}$, the matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ is first studied in [Kazhdan and Patterson 1984]. It also appears in the work of Suzuki [1998], Chinta and Offen [2013] among others. For a subclass of Brylinski–Deligne covering groups, the study of matrix $[\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')]_{\gamma,\gamma'}$ is conducted in [McNamara 2016] for unramified characters χ, generalizing that of Kazhdan and Patterson. Meanwhile, for ramified characters, it is included in the work of [Goldberg and Szpruch 2015]. However, in order to work with the full class of Brylinski–Deligne covering groups and also remove the assumption $\mu_{2n} \subseteq F^\times$ in [McNamara 2016], we refine the computation in [McNamara 2016] slightly. This is achieved by invoking the structural facts of Brylinski–Deligne covering groups, in particular those from Section 2A. We also note that interesting phenomena dissipate when the assumption $\mu_{2n} \subseteq F^\times$ is imposed, for example for the type A_r case in Section 4. There are subtleties arising from the fact that -1 is not a square root. For this purpose, it is important to rigidify the formula for the matrix and express its entries in terms of naturally defined elements of the group.

Consider the Haar measure μ of F such that $\mu(O_F) = 1$. Thus,

$$\mu(O_F^\times) = 1 - 1/q.$$

The Gauss sum is given by

$$G_\psi(a, b) = \int_{O_F^\times} (u, \varpi)_n^a \cdot \psi(\varpi^b u)\mu(u), \quad a, b \in \mathbb{Z}.$$

It is known that

$$G_\psi(a,b)=\begin{cases}0 & \text{if } b<-1,\\ 1-1/q & \text{if } n|a,\, b\geq 0,\\ 0 & \text{if } n\nmid a,\, b\geq 0,\\ -1/q & \text{if } n|a,\, b=-1,\\ G_\psi(a,-1) \text{ with } |G_\psi(a,-1)|=q^{-1/2} & \text{if } n\nmid a,\, b=-1.\end{cases}$$

Recall $\varepsilon := \iota((-1,\varpi)_n)\in\mathbb{C}^\times$. One has $\overline{G_\psi(a,b)}=\varepsilon^a\cdot G_\psi(-a,b)$. For any $k\in\mathbb{Z}$, we write

$$\boldsymbol{g}_\psi(k) := G_\psi(k,-1).$$

As in [McNamara 2016, § 9], let $f_{\gamma'}\in I(\overline{\chi})$ be the function with $\mathrm{supp}(f_{\gamma'})=\overline{B}w_0K_1$, and $f_{\gamma'}(w_0^{-1})=i(\overline{\chi})(\gamma')\phi_0$ for a certain compact open subgroup K_1. Here $\phi_0\in i(\overline{\chi})^{T\cap K}$ is the unramified vector in $i(\overline{\chi})$. From [McNamara 2016, Corollary 9.2], one has $\tau(\overline{\chi},\mathbb{w}_\alpha,\gamma,\gamma')=\langle\lambda_\gamma^{{}^{\mathbb{w}_\alpha}\overline{\chi}}, T_{\mathbb{w}_\alpha,\overline{\chi}}(f_{\gamma'})\rangle/|U^-\cap K_1|$. More precisely, from equality (9.3) of [McNamara 2016] one could evaluate $\tau(\overline{\chi},\mathbb{w}_\alpha,\gamma,\gamma')$ by applying $\lambda_\gamma^{{}^{\mathbb{w}_\alpha}\overline{\chi}}\in i\big({}^{\mathbb{w}_\alpha}\overline{\chi}\big)^\vee$ to the integral

$$\int_F f_{\gamma'}\big(\overline{h}_\alpha(x^{-1})\cdot\overline{e}_\alpha(-x)\cdot w_0^{-1}\big)\cdot\psi^{-1}\big(\overline{e}_\alpha(x^{-1})\big)\mu(x)\in i\big({}^{\mathbb{w}_\alpha}\overline{\chi}\big). \tag{6}$$

Note that the integrand of (6) takes values in $i(\overline{\chi})$. However, on the one hand, as vector spaces of functions on $\overline{T}$, the underlying space $i(\overline{\chi})$ is identical to that of ${}^{w_\alpha}i(\overline{\chi})$ (see [Gao ≥ 2017]); on the other hand, it follows from the Stone–von Neumann theorem that ${}^{w_\alpha}i(\overline{\chi})\simeq i({}^{\mathbb{w}_\alpha}\overline{\chi})$ as representations of $\overline{T}$. Therefore, there is a canonical vector space isomorphism $i(\overline{\chi})\simeq i({}^{\mathbb{w}_\alpha}\overline{\chi})$. For the computation below, we will follow [McNamara 2016] closely and adopt this viewpoint implicitly.

To ease notations, write $\pi=i(\overline{\chi})$. Use the partition $F=\bigcup_{m\in\mathbb{Z}}\varpi^{-m}O_F^\times$ and write $x=\varpi^{-m}u^{-1}$, where $u\in O_F^\times$. Then $\mu(x)=|\varpi|^{-m}\mu(u)$ and the integral in (6) is equal to

$$\begin{aligned}\sum_{m\in\mathbb{Z}}|\varpi|^{-m}&\int_{O_F^\times} f_{\gamma'}(\overline{h}_\alpha(\varpi^m\cdot u)\cdot\overline{e}_\alpha(-\varpi^{-m}u^{-1})\cdot w_0^{-1})\cdot\psi^{-1}(\overline{e}_\alpha(\varpi^m\cdot u))\mu(u)\\ &=\sum_{m\in\mathbb{Z}}\int_{O_F^\times}(u,\varpi)_n^{mQ(\alpha^\vee)}\cdot\pi(\overline{h}_\alpha(\varpi^m))\cdot\pi(\overline{h}_\alpha(u))\cdot\pi(\gamma')\phi_0\cdot\psi^{-1}(\varpi^m\cdot u)\mu(u).\end{aligned}$$

Suppose $\gamma'=\boldsymbol{s}_y\in\overline{T}$ for some $y\in Y$. (We write $\boldsymbol{s}_y := \boldsymbol{s}(\varpi^y)\in\overline{T}$ for $y\in Y$, see Section 2 for notations.) Then the above is equal to

$$\sum_{m\in\mathbb{Z}}\int_{O_F^\times}(u,\varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee,y)}\cdot\pi(\overline{h}_\alpha(\varpi^m))\cdot\pi(\boldsymbol{s}_y)\phi_0\cdot\psi^{-1}(\varpi^m\cdot u)\mu(u). \tag{7}$$

From now, we write $\Gamma(m, y, \alpha^\vee) := \varepsilon^{(m+\langle y,\alpha\rangle)D(y,\alpha^\vee)}$ and $\mathbf{\Gamma}(y, \alpha^\vee) := \Gamma(-1, y, \alpha^\vee)$, which lie in $\{\pm 1\}$. Following (3), $\bar{h}_\alpha(\varpi^m)\cdot \boldsymbol{s}_y = w_\alpha \cdot (\Gamma(m, y, \alpha^\vee)\cdot \boldsymbol{s}_{y+m\alpha^\vee})\cdot w_\alpha^{-1}$. Therefore (7) is equal to

$$\sum_{m\in\mathbb{Z}} \Gamma(m, y, \alpha^\vee)\cdot {}^{w_\alpha}\pi\left(\boldsymbol{s}_{\mathbb{w}_\alpha(y+m\alpha^\vee)}\right)\phi_0 \cdot \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee,y)}\psi^{-1}(\varpi^m\cdot u)\mu(u).$$

There are three cases for each term in the sum:

- For $m \le -2$, the integral over $O_F^\times$ vanishes, and thus the contribution to $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ is 0.
- For $m = -1$, the contribution $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ is nonzero only when $\mathbb{w}_\alpha(y_1) \equiv y - \alpha^\vee \bmod Y_{Q,n}$ where $\gamma = \boldsymbol{s}_{y_1}$, $\gamma' = \boldsymbol{s}_y$. When $\mathbb{w}_\alpha(y_1) = y - \alpha^\vee$, the contribution to $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ is
$$\mathbf{\Gamma}(y, \alpha^\vee)\cdot \boldsymbol{g}_{\psi^{-1}}(B(\alpha^\vee, y) - Q(\alpha^\vee)) = \mathbf{\Gamma}(y, \alpha^\vee)\cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee)).$$
- For any $x \in \mathbb{R}$, recall that we denote by $\lceil x\rceil$ the minimum integer such that $\lceil x\rceil \ge x$. The sum for $m \ge 0$ is equal to

$$\begin{aligned}
&\sum_{m\ge 0} \Gamma(m, y, \alpha^\vee)\cdot {}^{w_\alpha}\pi(\boldsymbol{s}_{\mathbb{w}_\alpha(y+m\alpha^\vee)})\phi_0 \cdot \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee,y)}\mu(u)\\
&\qquad = \sum_{\substack{m=k\cdot n_\alpha - B(\alpha^\vee,y)/Q(\alpha^\vee)\\ k\ge\lceil B(\alpha^\vee,y)/n_\alpha Q(\alpha^\vee)\rceil}} \begin{aligned}&\Gamma(m, y, \alpha^\vee)\cdot \varepsilon^{(m+\langle y,\alpha,\rangle)D(\alpha^\vee,y)}\\ &\cdot {}^{w_\alpha}\pi(\boldsymbol{s}_{(-m-\langle y,\alpha\rangle)\alpha^\vee})^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0\cdot(1-q^{-1})\end{aligned}\\
&\qquad = (1-q^{-1})\sum_{k\ge\lceil\langle y,\alpha^\vee\rangle/n_\alpha\rceil} \varepsilon^{kn_\alpha B(\alpha^\vee,y)}\cdot {}^{w_\alpha}\pi(\bar{h}_\alpha(\varpi^{-kn_\alpha}))\cdot {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0\\
&\qquad = (1-q^{-1})\sum_{k\ge\lceil\langle y,\alpha^\vee\rangle/n_\alpha\rceil} \bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^k\cdot {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0\\
&\qquad = (1-q^{-1})\frac{\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1-\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))}\cdot {}^{w_\alpha}\pi(\boldsymbol{s}_y)\phi_0,\quad \text{where } k_{y,\alpha} = \lceil\langle y,\alpha\rangle/n_\alpha\rceil.
\end{aligned}$$

The contribution is nonzero only for $\gamma = \boldsymbol{s}_{y_1}$ with $y_1 \equiv y \bmod Y_{Q,n}$. In particular, if $y_1 = y$, then the contribution to $\tau(\bar{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ (for $\gamma = \gamma' = \boldsymbol{s}_y$) is

$$(1-q^{-1})\frac{\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1-\bar{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))},\quad \text{where } k_{y,\alpha} = \left\lceil\frac{\langle y,\alpha\rangle}{n_\alpha}\right\rceil.$$

To summarize, we state the following theorem by McNamara which generalizes [Kazhdan and Patterson 1984, Lemma I.3.3]:

Theorem 3.6 [McNamara 2016, Theorem 13.1]. *Suppose that $\gamma = \boldsymbol{s}_{y_1}$ is represented by y_1 and $\gamma' = \boldsymbol{s}_y$ by y. Then we can write $\tau(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = \tau^1(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') + \tau^2(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma')$ with the following properties*:

- $\tau^i(\overline{\chi}, \mathbb{w}_\alpha, \gamma \cdot \overline{z}, \gamma' \cdot \overline{z}') = ({}^{\mathbb{w}_\alpha}\overline{\chi})^{-1}(\overline{z}) \cdot \tau^i(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') \cdot \overline{\chi}(\overline{z}'), \qquad \overline{z}, \overline{z}' \in \overline{A}$;
- $\tau^1(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = 0$ *unless* $y_1 \equiv y \bmod Y_{Q,n}$;
- $\tau^2(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = 0$ *unless* $y_1 \equiv \mathbb{w}_\alpha[y] \bmod Y_{Q,n}$.

Moreover,

- *If $y_1 = y$, then*
$$\tau^1(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = (1-q^{-1}) \frac{\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1-\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))}, \quad \text{where } k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil.$$
- *If $y_1 = \mathbb{w}_\alpha[y]$, then*
$$\tau^2(\overline{\chi}, \mathbb{w}_\alpha, \gamma, \gamma') = \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee)).$$

As an analogue of [Kazhdan and Patterson 1984, Corollary I.3.4], we have the following result.

Corollary 3.7. *Let $\overline{\chi}$ be an unramified exceptional character. Let $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(I(\overline{\chi}))$ be the ψ-Whittaker functional of $I(\overline{\chi})$ associated to some $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$. Then, $\lambda_{\boldsymbol{c}}^{\overline{\chi}}$ lies in $\mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ if and only if for any simple root $\alpha \in \Delta$ one has*

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]})) = q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \text{ for all } y. \tag{8}$$

Proof. By Corollary 3.5, for all $\alpha \in \Delta$, we have

$$\boldsymbol{c}(\boldsymbol{s}_y) \cdot \tau({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \boldsymbol{s}_y, \boldsymbol{s}_y) + \boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) \cdot \tau({}^{\mathbb{w}_\alpha}\overline{\chi}, \mathbb{w}_\alpha, \boldsymbol{s}_{\mathbb{w}_\alpha[y]}, \boldsymbol{s}_y) = 0,$$

where $y \in Y$ is any element. The preceding theorem gives

$$\begin{aligned}\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) &= -(1-q^{-1}) \frac{(\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha})))^{-k_{y,\alpha}}}{1-\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))^{-1}} \cdot \boldsymbol{\Gamma}(y,\alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \\ &= q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y,\alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y).\end{aligned}$$
□

From now on, for $y \in Y$ and $\alpha \in \Delta$, we write

$$\boldsymbol{t}(\mathbb{w}_\alpha, y) := q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1}, \tag{9}$$

where

$$k_{y,\alpha} = \left\lceil \frac{\langle y_\rho, \alpha \rangle + 1}{n_\alpha} \right\rceil \text{ and } \boldsymbol{\Gamma}(y, \alpha^\vee) = \varepsilon^{\langle y_\rho, \alpha \rangle \cdot D(y, \alpha^\vee)}.$$

It is clear $\boldsymbol{t}(\mathbb{w}_\alpha, y) \neq 0$.

Definition 3.8. For $\boldsymbol{c} \in \mathrm{Ftn}(i(\bar{\chi}))$, we say that $\boldsymbol{c}$ vanishes on $y \in Y$ if and only if $\boldsymbol{c}(\boldsymbol{s}_y) = 0$. It is said to vanish on the orbit $\mathcal{O}_{y_0} \subset Y$ if and only if it vanishes on all $y \in \mathcal{O}_{y_0}$, in which case we write $\boldsymbol{c}(\mathcal{O}_{y_0}) = 0$.

Assume that $\boldsymbol{c}$ gives rise to $\lambda_{\boldsymbol{c}}^{\bar{\chi}} \in \mathrm{Wh}_\psi(\Theta(\overline{G}, \bar{\chi}))$. Since $\boldsymbol{t}(\mathbb{w}_\alpha, y) \neq 0$ for all y and $\alpha \in \Delta$, it follows from Corollary 3.7 that $\boldsymbol{c}$ vanishes on $\mathcal{O}_{y_0}$ if and only if it vanishes on any $y \in \mathcal{O}_{y_0}$. It is therefore easy to see that

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \bar{\chi})) = \left| \left\{ \wp_{Q,n}(\mathcal{O}_{y_0}) : \begin{array}{l} \mathcal{O}_{y_0} \in \mathcal{O} \text{ is a } W\text{-orbit in } Y\text{, and there} \\ \text{exists } \boldsymbol{c} \in \mathrm{Ftn}(i(\bar{\chi})) \text{ satisfying (8) for} \\ \text{all } \alpha \in \Delta,\ y \in \mathcal{O}_{y_0}\text{. Also } \boldsymbol{c}(\mathcal{O}_{y_0}) \neq 0. \end{array} \right\} \right| \tag{10}$$

In the remaining part of this section we will prove an effective lower and upper bound for $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \bar{\chi}))$.

3D. ***A lower bound for* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \bar{\chi}))$.** The Weyl group W of $\mathbb{G}$ has the presentation

$$W = \langle \mathbb{w}_\alpha : (\mathbb{w}_\alpha \mathbb{w}_\beta)^{m_{\alpha\beta}} = 1 \text{ for } \alpha, \beta \in \Delta \rangle.$$

Lemma 3.9. *Let $\mathcal{O}_y \in \mathcal{O}^F_{Q,n,\mathrm{sc}}$ be a $Y^{\mathrm{sc}}_{Q,n}$-free orbit in Y. Then the following holds*:

$$\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) = 1 \text{ for all } \alpha \in \Delta.$$

Proof. Note that $\mathbb{w}_\alpha[y] = \mathbb{w}_\alpha(y) + \alpha^\vee = y + (1 - \langle y, \alpha \rangle)\alpha^\vee$. It follows that $\langle \mathbb{w}_\alpha[y], \alpha \rangle = 2 - \langle y, \alpha \rangle$. Therefore

$$\begin{aligned} &\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \\ &\quad = q^{\lceil \langle \mathbb{w}_\alpha[y], \alpha \rangle / n_\alpha \rceil - 1} \cdot \boldsymbol{\Gamma}(\mathbb{w}_\alpha[y], \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha^\vee)(\langle \mathbb{w}_\alpha[y], \alpha \rangle - 1))^{-1} \\ &\quad = q^{\lceil (2 - \langle y, \alpha \rangle)/n_\alpha \rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle \cdot (D(y, \alpha^\vee) - \langle y_\rho, \alpha^\vee \rangle Q(\alpha^\vee))} \cdot \boldsymbol{g}_{\psi^{-1}}(-Q(\alpha^\vee)\langle y_\rho, \alpha \rangle)^{-1} \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) \\ &\qquad = q^{\lceil (2 - \langle y, \alpha \rangle)/n_\alpha \rceil + \lceil \langle y, \alpha \rangle / n_\alpha \rceil - 2} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle^2 \cdot Q(\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha^\vee) \cdot \langle y_\rho, \alpha \rangle)^{-1} \\ &\qquad\qquad \cdot \boldsymbol{g}_{\psi^{-1}}(-Q(\alpha^\vee) \cdot \langle y_\rho, \alpha \rangle)^{-1}. \end{aligned}$$

However, it follows from $\boldsymbol{g}_{\psi^{-1}}(k) = \varepsilon^k \cdot \overline{\boldsymbol{g}_{\psi^{-1}}(-k)}$ that $|\boldsymbol{g}_{\psi^{-1}}(k)| = q^{-1/2}$. Moreover, since $\mathcal{O}_y$ is a $Y^{\mathrm{sc}}_{Q,n}$-free orbit, $\mathbb{w}_\alpha[y] - y \notin Y^{\mathrm{sc}}_{Q,n}$. Therefore, $n_\alpha \nmid (1 - \langle y, \alpha \rangle)$ and so

$$\left\lceil \frac{2 - \langle y, \alpha \rangle}{n_\alpha} \right\rceil + \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil = 1.$$

Now it can be checked easily that $\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) = 1$. $\square$

Consider adjacent $\alpha, \beta \in \Delta$ from the Dynkin diagram. We would like to show that for the $Y^{\mathrm{sc}}_{Q,n}$-free orbit $\mathcal{O}_y$ the equality

$$\prod_{i=1}^{m_{\alpha\beta}} \boldsymbol{t}(\mathbb{w}_\alpha \mathbb{w}_\beta, (\mathbb{w}_\alpha \mathbb{w}_\beta)^i[y]) = 1$$

holds, where $\boldsymbol{t}(\mathbb{w}_\alpha \mathbb{w}_\beta, y) := \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y)$. This will follow from a case by case discussion. We will give the details for $m_{\alpha\beta} = 3, 4$ below and leave the case for $m_{\alpha\beta} = 6$ to the reader.

<u>Case $m_{\alpha\beta} = 3$</u>: The relation $(\mathbb{w}_\alpha \mathbb{w}_\beta)^{m_{\alpha\beta}} = 1$ is equivalent to $\mathbb{w}_\alpha \mathbb{w}_\beta \mathbb{w}_\alpha = \mathbb{w}_\beta \mathbb{w}_\alpha \mathbb{w}_\beta$. By Lemma 3.9, it suffices to show

$$\boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) = \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y). \tag{11}$$

We first note that

$$\boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{\left\lceil \frac{\langle y_\rho, \alpha\rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle \cdot D(y, \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(B_Q(y_\rho, \alpha^\vee))^{-1}.$$

We also have $\langle \mathbb{w}_\beta \mathbb{w}_\alpha(y_\rho), \alpha\rangle = \langle y_\rho, \beta\rangle$ since the pairing $\langle -, -\rangle$ is W-equivariant and $\mathbb{w}_\alpha \mathbb{w}_\beta(\alpha) = \beta$. Similarly, $\langle \mathbb{w}_\alpha \mathbb{w}_\beta(y_\rho), \beta\rangle = \langle y_\rho, \alpha\rangle$. A simple computation gives

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{\left\lceil \frac{\langle y_\rho, \alpha\rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle D(y, \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) = q^{\left\lceil \frac{\langle y_\rho, \alpha+\beta\rangle + 1}{n_\beta} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha+\beta\rangle D(\mathbb{w}_\alpha[y], \beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha[y]) = q^{\left\lceil \frac{\langle y_\rho, \beta\rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \beta\rangle D(\mathbb{w}_\beta \mathbb{w}_\alpha[y], \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \beta\rangle Q(\alpha^\vee))^{-1}. \end{cases}$$

Meanwhile,

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\beta, y) = q^{\left\lceil \frac{\langle y_\rho, \beta\rangle + 1}{n_\beta} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \beta\rangle D(y, \beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) = q^{\left\lceil \frac{\langle y_\rho, \alpha+\beta\rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha+\beta\rangle D(\mathbb{w}_\beta[y], \alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+\beta\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta \mathbb{w}_\alpha[y]) = q^{\left\lceil \frac{\langle y_\rho, \alpha\rangle + 1}{n_\beta} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle D(\mathbb{w}_\alpha \mathbb{w}_\beta[y], \beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\beta^\vee))^{-1}. \end{cases}$$

Since $Q(\alpha^\vee) = Q(\beta^\vee)$ and thus $n_\alpha = n_\beta$, to show that (11) holds, it suffices to check that the powers of ε on the two sides of (11) are equal. However, a straightforward computation shows that this is indeed the case, and we may omit the details.

<u>Case $m_{\alpha\beta} = 4$</u>: Let $\alpha, \beta \in \Delta$ be two adjacent roots such that $m_{\alpha\beta} = 4$. We assume that α is the longer one. Thus, $\langle \alpha^\vee, \beta\rangle = -1$, $\langle \beta^\vee, \alpha\rangle = -2$, and $Q(\beta^\vee) = 2Q(\alpha^\vee)$. As in the preceding case, we want to show

$$\begin{aligned} \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, y) \\ = \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta \mathbb{w}_\alpha \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta, y). \end{aligned} \tag{12}$$

A simple computation yields

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle+1}{n_\alpha}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \alpha\rangle D(y,\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \alpha+\beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \alpha+\beta\rangle D(\mathbb{w}_\alpha[y],\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta\mathbb{w}_\alpha[y]) \\ \qquad = q^{\lceil \frac{\langle y_\rho, \alpha+2\beta\rangle+1}{n_\alpha}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \alpha+2\beta\rangle D(\mathbb{w}_\beta\mathbb{w}_\alpha[y],\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+2\beta\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha\mathbb{w}_\beta\mathbb{w}_\alpha[y]) = q^{\lceil \frac{\langle y_\rho, \beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \beta\rangle D(\mathbb{w}_\alpha\mathbb{w}_\beta\mathbb{w}_\alpha[y],\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \beta\rangle Q(\beta^\vee))^{-1}. \end{cases}$$

On the other hand, for the right-hand side of (12), one has

$$\begin{cases} \boldsymbol{t}(\mathbb{w}_\beta, y) = q^{\lceil \frac{\langle y_\rho, \beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \beta\rangle D(y,\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta[y]) = q^{\lceil \frac{\langle y_\rho, \alpha+2\beta\rangle+1}{n_\alpha}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \alpha+2\beta\rangle D(\mathbb{w}_\beta[y],\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+2\beta\rangle Q(\alpha^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\beta, \mathbb{w}_\alpha\mathbb{w}_\beta\mathbb{w}_\alpha[y]) \\ \qquad = q^{\lceil \frac{\langle y_\rho, \alpha+\beta\rangle+1}{n_\beta}\rceil -1} \cdot \varepsilon^{\langle y_\rho, \alpha+\beta\rangle D(\mathbb{w}_\alpha\mathbb{w}_\beta[y],\beta^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha+\beta\rangle Q(\beta^\vee))^{-1}, \\ \boldsymbol{t}(\mathbb{w}_\alpha, \mathbb{w}_\beta\mathbb{w}_\alpha\mathbb{w}_\beta[y]) = q^{\lceil \frac{\langle y_\rho, \alpha\rangle+1}{n_\alpha}\rceil -1} \varepsilon^{\langle y_\rho, \alpha\rangle D(\mathbb{w}_\beta\mathbb{w}_\alpha\mathbb{w}_\beta[y],\alpha^\vee)} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee))^{-1}. \end{cases}$$

To show equality (12), again it suffices to show that the powers of ε of the two sides have the same parities, which is achieved from a straightforward check.

Analogous argument for $m_{\alpha\beta} = 6$ works, and we give a summary.

Proposition 3.10. *Let $\mathcal{O}_y$ be a $Y_{Q,n}^{\mathrm{sc}}$-free orbit. For all adjacent $\alpha, \beta \in \Delta$, one has*

$$\prod_{i=1}^{m_{\alpha\beta}} \boldsymbol{t}(\mathbb{w}_\alpha\mathbb{w}_\beta,\, (\mathbb{w}_\alpha\mathbb{w}_\beta)^i[y]) = 1,$$

where $\boldsymbol{t}(\mathbb{w}_\alpha\mathbb{w}_\beta,\, y) := \boldsymbol{t}(\mathbb{w}_\alpha,\, \mathbb{w}_\beta[y]) \cdot \boldsymbol{t}(\mathbb{w}_\beta,\, y)$.

Definition 3.11. Let $\mathcal{O}_y \in \mathcal{O}_{Q,n,\mathrm{sc}}^{F}$ be a $Y_{Q,n}^{\mathrm{sc}}$-free orbit. For any

$$\mathbb{w} = \mathbb{w}_k\mathbb{w}_{k-1}\cdots\mathbb{w}_2\mathbb{w}_1 \in W$$

written as a minimum expansion, write

$$\boldsymbol{T}(\mathbb{w},\, y) := \prod_{i=1}^{k} \boldsymbol{t}(\mathbb{w}_i,\, \mathbb{w}_{i-1}\cdots\mathbb{w}_1[y]),$$

which, by Lemma 3.9 and Proposition 3.10, is independent of the choice of minimum expansion of $\mathbb{w}$.

Let $\mathcal{O}_y \in \mathcal{O}_{Q,n}^{F}$ be a $Y_{Q,n}$-free orbit (and therefore $Y_{Q,n}^{\mathrm{sc}}$-free). We define a nonzero $\boldsymbol{c}$ with support $\mathcal{O}_y$ as follows. First, let $\boldsymbol{c}(\boldsymbol{s}_y) = 1$, and for any $\alpha \in \Delta$, let

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) := \boldsymbol{t}(\mathbb{w}_\alpha,\, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y).$$

Inductively, one can define $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]}) := \boldsymbol{T}(\mathbb{w}, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$ for any $\mathbb{w} \in W$. It is well defined and independent of the minimum decomposition of $\mathbb{w}$. Second, extend $\boldsymbol{c}$ by

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]} \cdot \bar{z}) = \boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]}) \cdot \overline{\chi}(\bar{z}), \quad \bar{z} \in \bar{A},$$

and

$$\boldsymbol{c}(\bar{t}) = 0 \text{ if } \bar{t} \notin \bigcup_{\mathbb{w} \in W} \boldsymbol{s}_{\mathbb{w}[y]} \cdot \bar{A}.$$

By using the property that $\boldsymbol{T}(\mathbb{w}, y)$ and $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}[y]})$ are independent of the minimum decomposition of $\mathbb{w}$, we see that equality (8) is satisfied. It follows that $\wp_{Q,n}(\mathcal{O}_y)$ belongs to the right-hand side of (10). Therefore,

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) \geq |\wp_{Q,n}(\mathcal{O}^F_{Q,n})|. \tag{13}$$

3E. ***An upper bound for*** $\mathbf{dim\,Wh}_{\boldsymbol{\psi}}(\boldsymbol{\Theta}(\overline{\boldsymbol{G}}, \overline{\boldsymbol{\chi}}))$. First we show a result in the general setting regarding the usual Weyl action. Let Ψ be a root system and Ψ_s be a fixed choice of simple roots. Write $\mathrm{L} := \langle \Psi \rangle$ for the lattice generated by Ψ and $V = \mathrm{L} \otimes \mathbb{R}$. The Weyl group W associated to Ψ acts on V naturally by the usual linear transformation generated by simple reflections. Recall that we write $\mathbb{w}(v)$, $\mathbb{w} \in W$, $v \in V$ for this action.

Lemma 3.12. *Let $v \in V$ be any vector such that $\mathbb{w}(v) \equiv v \bmod \mathrm{L}$. Then there exist $\mathbb{w}' \in W$ and $\alpha \in \Psi_s$ such that $\mathbb{w}_\alpha(\mathbb{w}'(v)) \equiv \mathbb{w}'(v) \bmod \mathrm{L}$.*

Proof. Let $W_{\mathrm{aff}} = \mathrm{L} \rtimes W$ be the affine Weyl group, and denote any element of W_{aff} by $\mathbb{w}_{\mathrm{a}} = (y, \mathbb{w})$. We call $\mathbb{w}$ the Weyl component of $\mathbb{w}_{\mathrm{a}}$. The congruence $\mathbb{w}(v) \equiv v \bmod \mathrm{L}$ is equivalent to $\mathbb{w}_{\mathrm{a}}(v) = v$ for some $\mathbb{w}_{\mathrm{a}}$ which projects to $\mathbb{w} \in W$.

If $\mathbb{w}_{\mathrm{a}}(v) = v$, it then follows that $v \in V$ lies on the boundary of $\bar{C}$, where C is an alcove (i.e., a fundamental domain) of the action of W_{aff} on V, see [Bourbaki 2002]. Note that $\bar{C}$ is a simplicial complex whose boundary consists of $|\Psi_s| + 1$ walls $\{\mathrm{E}_i\}$. Moreover, we may assume that for $1 \leq i \leq |\Psi_s|$, the wall E_i lies in the hyperplane fixed by $\mathbb{w}_{\mathrm{a}}$ whose Weyl component is $\mathbb{w}_{\alpha_i}$ for some $\alpha_i \in \Psi_s$. In this case, one also knows that $\mathrm{E}_{|\Psi_s|+1}$ is fixed by $(y, \mathbb{w}_\beta) \in W_{\mathrm{aff}}$ for some $\beta \in \Psi - \Psi_s$.

Since $v \in \bigcup_i \mathrm{E}_i$, there are two cases. First, suppose $v \in \mathrm{E}_i$ for some $1 \leq i \leq |\Psi_s|$; then clearly $\mathbb{w}_{\alpha_i}(v) \equiv v \bmod \mathrm{L}$ for some $\alpha_i \in \Psi_s$. Otherwise, suppose $v \in \mathrm{E}_{|\Psi_s|+1}$. Let $\mathbb{w}' \in W$ be such that $\mathbb{w}'(\beta) \in \Psi_s$. It follows that $\mathbb{w}'(\mathrm{E}_{|\Psi_s|+1})$ is fixed by some $\mathbb{w}_{\mathrm{a}} = (y, \mathbb{w}_\alpha)$ with $\alpha \in \Psi_s$. That is, $\mathbb{w}_\alpha(\mathbb{w}'(v)) \equiv \mathbb{w}'(v) \bmod \mathrm{L}$. The proof is completed. □

Proposition 3.13. *Consider $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ such that $\lambda_{\boldsymbol{c}}^{\overline{\chi}}$ is a ψ-Whittaker functional on $\Theta(\overline{G}, \overline{\chi})$. If $\mathcal{O}_{y^0}$ is not $Y_{Q,n}^{\mathrm{sc}}$-free, then $\boldsymbol{c}$ is zero on $\mathcal{O}_{y^0}$. It follows that* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) \leq |\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})|$.

Proof. Write $V = Y \otimes \mathbb{R}$. One has $V = (Y^{\mathrm{sc}} \otimes \mathbb{R}) \oplus V_0$ where $V_0 \subseteq V$ is fixed by W pointwise with respect to the usual action, i.e., the action $\mathbb{w}(v)$ of W. In general

$y_\rho^0 \in V$; however, without loss of generality, we may assume $y_\rho^0 \in Y^{\mathrm{sc}} \otimes \mathbb{R}$ now. There is a canonical W-equivariant isomorphism $Y_{Q,n}^{\mathrm{sc}} \otimes \mathbb{R} \simeq Y^{\mathrm{sc}} \otimes \mathbb{R}$ with respect to that usual action. Moreover, $\{\alpha_{Q,n}^\vee\}_{\alpha\in\Phi}$ forms a root system.

If $\mathcal{O}_{y^0}$ is not $Y_{Q,n}^{\mathrm{sc}}$-free, there exists $\mathbb{w} \in W$ such that $\mathbb{w}[y^0] \equiv y^0 \bmod Y_{Q,n}^{\mathrm{sc}}$, i.e., $\mathbb{w}(y_\rho^0) \equiv y_\rho^0 \bmod Y_{Q,n}^{\mathrm{sc}}$. By the preceding Lemma, there exist $y \in \mathcal{O}_{y^0}$ and $\alpha \in \Delta$ such that $\mathbb{w}_\alpha(y_\rho) \equiv y_\rho \bmod Y_{Q,n}^{\mathrm{sc}}$. Now it suffices to show that $\boldsymbol{c}$ vanishes on y.

By Corollary 3.7, $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) = \boldsymbol{t}(\mathbb{w}_\alpha, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$. Since $\mathbb{w}_\alpha(y_\rho) \equiv y_\rho \bmod Y_{Q,n}^{\mathrm{sc}}$, it follows that $n_\alpha | \langle y_\rho, \alpha\rangle$. Write $\langle y_\rho, \alpha\rangle = k \cdot n_\alpha$. Since

$$\boldsymbol{s}_{\mathbb{w}_\alpha[y]} = \boldsymbol{s}_y \cdot \boldsymbol{s}_{-\langle y_\rho,\alpha\rangle)\alpha^\vee} \cdot \varepsilon^{\langle y_\rho,\alpha\rangle\cdot D(\alpha^\vee,y)},$$

one has

$$\begin{aligned}\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) &= \overline{\chi}(\boldsymbol{s}_{-kn_\alpha\alpha^\vee}) \cdot \boldsymbol{c}(\boldsymbol{s}_y) \cdot \varepsilon^{\langle y_\rho,\alpha\rangle\cdot D(\alpha^\vee,y)} \\ &= q^k \cdot \varepsilon^{kn_\alpha \cdot D(\alpha^\vee,y)} \cdot \boldsymbol{c}(\boldsymbol{s}_y).\end{aligned}$$

On the other hand,

$$\begin{aligned}\boldsymbol{t}(\mathbb{w}_\alpha, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y) &= q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha\rangle Q(\alpha^\vee))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \\ &= q^k \cdot (-1, \varpi)_n^{kn_\alpha\cdot D(y,\alpha^\vee)} \cdot (-q^{-1}) \cdot \boldsymbol{c}(\boldsymbol{s}_y).\end{aligned}$$

It follows that $\boldsymbol{c}(\boldsymbol{s}_y) = -q^{-1} \cdot \varepsilon^{kn_\alpha B(y,\alpha^\vee)} \cdot \boldsymbol{c}(\boldsymbol{s}_y) = (-q^{-1}) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$. Therefore $\boldsymbol{c}(\boldsymbol{s}_y) = 0$. The proof is completed. □

Theorem 3.14. *Let $\overline{G}$ be an unramified Brylinski–Deligne covering group incarnated by (D, η). Let $\overline{\chi}$ be an unramified exceptional character and $\Theta(\overline{G}, \overline{\chi})$ the theta representation associated with $\overline{\chi}$. Then*

$$|\wp_{Q,n}(\mathcal{O}_{Q,n}^{\digamma})| \le \dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) \le |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^{\digamma})|.$$

The group $\mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times)$ is identified with $Z(\overline{G}^\vee)$, the center of the dual group $\overline{G}^\vee$ of $\overline{G}$, so $Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n}$ if and only if $Z(\overline{G}^\vee) = \{1\}$. Immediately it follows that:

Corollary 3.15. *If the dual group $\overline{G}^\vee$ of $\overline{G}$ is of adjoint type, i.e., $Z(\overline{G}^\vee) = 1$, then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^{\digamma})|$.

For groups of type E_8, F_4 and G_2, the complex dual group of their covering group has trivial center and thus Corollary 3.15 applies.

More generally, if $\mathcal{O}_{Q,n}^{\digamma} = \mathcal{O}_{Q,n,\mathrm{sc}}^{\digamma}$, then the dimension of $\mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ can be uniquely determined. We will illustrate below that Theorem 3.14 recovers the result of Kazhdan and Patterson in this case.

Example 3.16. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for the cocharacter lattice Y of GL_r. The simple coroots $\Delta^\vee$ of GL_r are $\Delta^\vee = \{\alpha_i^\vee := e_i - e_{i+1}\}_{1\le i\le r-1}$. The isomorphism class of (D, η) in the incarnation category corresponds to a Weyl-invariant quadratic form Q, or equivalently, to the bilinear form B_Q. Let $B_Q(e_i, e_j)$ be the Weyl-invariant bilinear form determined by

$$B_Q(e_i, e_i) = 2\boldsymbol{p}, \qquad B_Q(e_i, e_j) = \boldsymbol{q} \qquad \text{if } i \neq j.$$

For any root α, one has $Q(\alpha^\vee) = 2\boldsymbol{p} - \boldsymbol{q}$. We assume $2\boldsymbol{p} - \boldsymbol{q} = -1$ and therefore $n_\alpha = n$. The covering groups $\overline{\mathrm{GL}}_r^{(n)}$ arising from such B_Q are exactly those studied by Kazhdan and Patterson. The parameter $\boldsymbol{p}$ corresponds to the twisting parameter c in [Kazhdan and Patterson 1984].

From B_Q, the lattice $Y_{Q,n}$ is given by

$$\left\{ \sum_i x_i e_i \in \bigoplus_{i=1}^r \mathbb{Z} e_i : x_1 \equiv x_2 \equiv \cdots \equiv x_r \bmod n, \ \text{ and } \ n | (\boldsymbol{q} r - 1) x_i \right\}.$$

The lattice $Y_{Q,n}^{\mathrm{sc}}$ is generated by $\{\alpha_{Q,n}^\vee\}_{\alpha \in \Phi}$. It is easy to check $Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n} \cap Y^{\mathrm{sc}}$, and this has the following implications:

Suppose that $\mathcal{O}_y$ is not $Y_{Q,n}$-free, i.e., $\mathbb{w}[y] - y \in Y_{Q,n}$ for some $\mathbb{w} \neq 1 \in W$. Clearly $\mathbb{w}[y] - y \in Y^{\mathrm{sc}}$ as well. It follows that $\mathbb{w}[y] - y \in Y_{Q,n}^{\mathrm{sc}}$, that is, $\mathcal{O}_y$ is not $Y_{Q,n}^{\mathrm{sc}}$-free. Therefore, for the Kazhdan–Patterson covering group $\overline{\mathrm{GL}}_r^{(n)}$, one has that $\mathcal{O}_{Q,n}^F$ is equal to $\mathcal{O}_{Q,n,\mathrm{sc}}^F$. Consequently, for the covering group $\overline{\mathrm{GL}}_r^{(n)}$ with parameter $(\boldsymbol{p}, \boldsymbol{q})$ such that $2\boldsymbol{p} - \boldsymbol{q} = -1$, Theorem 3.14 yields

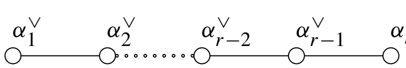
$$\dim \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)|,$$

which is the content of [Kazhdan and Patterson 1984, Theorem I.3.5]. Moreover, distinguished theta representations (see Definition 3.3) for $\overline{\mathrm{GL}}_r^{(n)}$ are completely determined in [Kazhdan and Patterson 1984, Corollary I.3.6].

In the remaining part of the paper, we will determine the distinguished theta representations for coverings of simply connected groups of type A_r, B_r, C_r and G_2. To ease the computations, we will use the standard coordinates for the coroot system of each type as in [Bourbaki 2002, pages 265–290].

4. The A_r, $r \geq 1$ case

Consider the Dynkin diagram for the simple coroots of A_r:



The cocharacter lattice is $Y = Y^{\mathrm{sc}} = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$. As in [Bourbaki 2002, page 265], consider the embedding $\boldsymbol{i}_A : \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i$, which is given by

$$\boldsymbol{i}_A : y = (x_1, x_2, \ldots, x_r) \mapsto \boldsymbol{i}_A(y) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_r - x_{r-1}, -x_r).$$

In particular, we can identify the image of $\boldsymbol{i}_A$: any $(y_1, y_2, \ldots, y_r, y_{r+1}) \in \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i$ is equal to $\boldsymbol{i}_A(y)$ for some y if and only if $\sum_{i=1}^{r+1} y_i = 0$.

Meanwhile, $\rho = \sum_{i=1}^{r} \frac{i}{2}(r-i+1)\alpha_i^\vee$. We use $\boldsymbol{i}_A : \bigoplus_{i=1}^{r} \mathbb{Q}\alpha_i^\vee \to \bigoplus_{i=1}^{r+1} \mathbb{Q}e_i$ to denote the canonical extension of $\boldsymbol{i}_A$. Then,

$$\boldsymbol{i}_A(\rho) = \Big(\frac{r}{2}, \frac{r-2}{2}, \dots, \frac{-(r-2)}{2}, \frac{-r}{2}\Big) \in \bigoplus_{i=1}^{r+1} \mathbb{Q}e_i.$$

It follows that for any $y \in Y$,

$$\begin{aligned}\boldsymbol{i}_A(y_\rho) &= \Big(x_1 - \frac{r}{2}, \dots, x_i - x_{i-1} + (i-1) - \frac{r}{2}, \dots, -x_r + r - \frac{r}{2}\Big), \qquad 1 \le i \le r \\ &= \big(x_1, x_2 - x_1 + 1, \dots, x_i - x_{i-1} + (i-1), \dots, -x_r + r\big) + \Big(\frac{-r}{2}, \frac{-r}{2}, \dots, \frac{-r}{2}\Big).\end{aligned}$$

From now, we write $\boldsymbol{i}_A^*(y_\rho) := (x_1^*, x_2^*, \dots, x_r^*, x_{r+1}^*)$ for

$$\big(x_1, x_2 - x_1 + 1, \dots, x_i - x_{i-1} + (i-1), \dots, -x_r + r\big) \in \bigoplus_i \mathbb{Z}e_i.$$

Thus,

$$\boldsymbol{i}_A(y_\rho) = \boldsymbol{i}_A^*(y_\rho) + \Big(\frac{-r}{2}, \frac{-r}{2}, \dots, \frac{-r}{2}\Big).$$

Meanwhile, any $(x_1^*, x_2^*, \dots, x_r^*, x_{r+1}^*) \in \bigoplus_i \mathbb{Z}e_i$ is equal to $\boldsymbol{i}_A^*(y_\rho)$ for some y if and only if $\sum_{i=1}^{r+1} x_i^* = r(r+1)/2$.

Consider the quadratic form Q on $Y = \langle \alpha_i^\vee, 1 \le i \le r\rangle$ with $Q(\alpha_i^\vee) = 1$ for all i. Then

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } j = i+1, \\ 0, & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

This gives rise to the degree n covering group $\overline{\mathrm{SL}}_{r+1}^{(n)}$. Any element $\sum_{i=1}^{r} x_i\alpha_i^\vee \in Y$ lies in $Y_{Q,n}$ if and only if

$$2x_1 - x_2, \;\; -x_1 + 2x_2 - x_3, \;\; -x_2 + 2x_3 - x_4, \;\; \dots \;\; -x_{r-2} + 2x_{r-1} - x_r, \;\; -x_{r-1} + 2x_r$$

are in $n\mathbb{Z}$.

By using $\boldsymbol{i}_A$, we see

$$Y_{Q,n} = \Big\{(y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z}e_i \; : \; \sum_{i=1}^{r+1} y_i = 0, \;\text{ and }\; y_1 \equiv \cdots \equiv y_r \equiv y_{r+1} \bmod n\Big\}$$

and

$$Y_{Q,n}^{\mathrm{sc}} = \Big\{(y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z}e_i \; : \; \sum_{i=1}^{r+1} y_i = 0, \;\text{ and }\; n | y_i \text{ for all } i.\Big\}$$

The Weyl group $W = S_{r+1}$ acts as permutations on $\bigoplus_{i=1}^{r+1} \mathbb{Z}e_i$. In particular, $\mathbb{w}_{\alpha_i}$ for $\alpha_i \in \Delta$ acts by exchanging e_i and e_{i+1}.

4A. *Case I:* $\overline{\mathrm{SL}}^{(n)}_{r+1}$, $n \le r$. Suppose $n \le r$, then for any $y \in Y$ with $\boldsymbol{i}_A^*(y_\rho) = (x_1^*, x_2^*, \dots, x_{r+1}^*)$, there exists $x_i^*, x_j^*, i \neq j$ such that $n | (x_i^* - x_j^*)$. Then clearly $\mathbb{w}(y_\rho) - y_\rho \in Y_{Q,n}^{\mathrm{sc}}$ for some $\mathbb{w} \in W$. That is, $\mathcal{O}_y \notin \mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}$ and one has in this case

$$\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}} = \varnothing.$$

Therefore, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_{r+1}, \overline{\chi})) = 0$ for $n \le r$.

4B. *Case II:* $\overline{\mathrm{SL}}^{(n)}_{r+1}$, $n = r+1$. In this case, the dual group for $\overline{\mathrm{SL}}^{(n)}_{n}$ is SL_n, see [Weissman 2015]. Consider $\mathcal{O}_y \in \mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}$ such that

$$\boldsymbol{i}_A^*(y_\rho) = (0, 1, 2, \dots, r-1, r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i.$$

It is easy to check $\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \{\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}_y)\}$, and this implies $|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}})| = 1$. However, $\mathcal{O}_y \notin \mathcal{O}^{\digamma}_{Q,n}$. For example, let $\mathbb{w}_\natural$ be such that $\boldsymbol{i}_A^*(\mathbb{w}_\natural(y_\rho)) = (1, 2, \dots, r, 0)$, then $\boldsymbol{i}_A(\mathbb{w}_\natural(y_\rho)) - \boldsymbol{i}_A(y_\rho) = (1, 1, \dots, 1, -r) \in Y_{Q,n}$. That is, $\mathbb{w}_\natural[y] - y \in Y_{Q,n}$. Therefore,

$$|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})| = 0.$$

It follows that $0 \le \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_{n}, \overline{\chi})) \le 1$. In this case, determining $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_{n}, \overline{\chi}))$ is delicate, and there are additional constraints on the exceptional character $\overline{\chi}$ such that $\Theta(\overline{\mathrm{SL}}^{(n)}_{n}, \overline{\chi})$ is distinguished. The analysis below is devoted to this.

4B1. *The reduction step.* It is clear that $\boldsymbol{i}_A^*(y_\rho) = (0, 1, 2, \dots, r-1, r)$ if and only if $y = 0$. Moreover, $\boldsymbol{i}_A^*(\mathbb{w}_\natural(y_\rho)) = (1, 2, 3, \dots, r, 0)$ for $\mathbb{w}_\natural = \mathbb{w}_{\alpha_r}\mathbb{w}_{\alpha_{r-1}} \cdots \mathbb{w}_{\alpha_2}\mathbb{w}_{\alpha_1}$. As above,

$$\mathbb{w}_\natural[0] - 0 = \sum_{i=1}^{r} i \cdot \alpha_i^\vee \in Y_{Q,n}.$$

Write $y_{Q,n} := \sum_{i=1}^r i \cdot \alpha_i^\vee$. In fact, the set $\{n\alpha_i^\vee : 2 \le i \le r\} \cup \{y_{Q,n}\}$ forms a basis for $Y_{Q,n}$, whereas $\{n\alpha_i^\vee : 2 \le i \le r\} \cup \{n \cdot y_{Q,n}\}$ is a basis for $Y_{Q,n}^{\mathrm{sc}}$. It follows that any exceptional character $\overline{\chi}$ is determined by its value at $\boldsymbol{s}_{y_{Q,n}}$.

We choose the bisector D on Y^{sc} such that $D(\alpha_i^\vee, \alpha_j^\vee)$ is given by

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} Q(\alpha_i^\vee) & \text{if } i = j, \\ 0 & \text{if } i < j, \\ B_Q(\alpha_i^\vee, \alpha_j^\vee) & \text{if } i > j. \end{cases}$$

Recall from Corollary 3.7 that $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ gives rise to a ψ-Whittaker functional of $\Theta(\overline{\mathrm{SL}}^{(n)}_{n}, \overline{\chi})$ if and only if for all y and $\alpha \in \Delta$,

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_\alpha[y]}) = q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(B(\alpha^\vee, y_\rho))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y).$$

For $1 \le i \le r$, write $y_{\langle i \rangle} = \mathbb{w}_{\alpha_i}\mathbb{w}_{\alpha_{i-1}} \cdots \mathbb{w}_{\alpha_1}[0]$ and we set $y_{\langle 0 \rangle} = 0$. Recall that $\boldsymbol{t}(\mathbb{w}_\alpha, y)$ is the coefficient in the above formula. In this case, it reads $\boldsymbol{t}(\mathbb{w}_\alpha, y) = q^{k_{y,\alpha}-1} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle)^{-1}$ since $Q(\alpha^\vee) = 1$ (and therefore $n_\alpha = n$) for all $\alpha \in \Delta$. In order to have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})) = 1$, we must have the equality

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) = \boldsymbol{T}(\mathbb{w}_\natural, 0) \ \text{ where } \ \boldsymbol{T}(\mathbb{w}_\natural, 0) = \prod_{i=1}^{r} \boldsymbol{t}(\mathbb{w}_{\alpha_i}, y_{\langle i-1 \rangle}). \tag{14}$$

We would like to show that the equality (14) is also sufficient. Consider any $\mathbb{w}' \in W$, $y \in \mathcal{O}_0$, one has $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}'[y]}) = \boldsymbol{T}(\mathbb{w}', y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$. Now assume $\mathbb{w}'[y] - y \in Y_{Q,n}$, we have

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}'[y]-y+y}) = \overline{\chi}(\boldsymbol{s}_{\mathbb{w}'[y]-y}) \cdot \boldsymbol{c}(\boldsymbol{s}_y) \cdot \varepsilon^{D(\mathbb{w}'[y]-y, y)}.$$

To show $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})) = 1$, it suffices to show $\boldsymbol{c}(\boldsymbol{s}_y)$ to be nonzero for all $y \in \mathcal{O}_0$ such that $\mathbb{w}'[y] - y \in Y_{Q,n}$. That is, it requires

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}'[y]-y}) = \varepsilon^{D(\mathbb{w}'[y]-y, y)} \cdot \boldsymbol{T}(\mathbb{w}', y). \tag{15}$$

Write $\mathbb{w}'[y]-y = \sum_{i=2}^{r} k_i \cdot \alpha_{i,Q,n}^\vee + k_1 \cdot y_{Q,n}$. Note that $\mathcal{O}_0$ is $Y_{Q,n}^{\mathrm{sc}}$-free, thus $k_1 \neq 0$. We may reduce the negative case to the positive case by a simple computation, and therefore we can assume that $k_1 \ge 1$. Furthermore, we may apply induction on k_1, and thus it suffices to: i) prove the inductive step, ii) check the equality (15) when $\mathbb{w}'[y] - y = \sum_{i=2}^{r} k_i \alpha_{i,Q,n}^\vee + y_{Q,n}$. The assertion i) can be checked easily, and thus we will only outline the proof of ii).

For ii), if $\mathbb{w}'[y] - y = \sum_{i=2}^{r} k_i \alpha_{i,Q,n}^\vee + y_{Q,n}$, then it is not hard to see that $\mathbb{w}'[y] - y = \mathbb{w}(y_{Q,n})$, i.e., $\mathbb{w}^{-1}\mathbb{w}'[y] - \mathbb{w}^{-1}[y] = y_{Q,n}$ for some $\mathbb{w} \in W$. We may change $\mathbb{w}$ if necessary such that $\mathbb{w}^{-1}[y] = 0$. With this assumption, $\mathbb{w}^{-1}\mathbb{w}'\mathbb{w} = \mathbb{w}_\natural$, i.e., $\mathbb{w}' = \mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1}$. Therefore, we need only show that for any $\mathbb{w} \in W$,

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}[0]}) = \varepsilon^{D(\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}[0], \mathbb{w}[0])} \cdot \boldsymbol{T}(\mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1}, \mathbb{w}[0]). \tag{16}$$

To show (16), we would like to apply induction on the length of $\mathbb{w}$. When $\mathbb{w} = 1$, it is just the equality (14). For the induction step, assuming the equality (16), we would like to prove that for $\alpha \in \Delta$ the following equality holds:

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}_\alpha\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}[0]}) = \varepsilon^{D(\mathbb{w}_\alpha\mathbb{w}\mathbb{w}_\natural[0]-\mathbb{w}_\alpha\mathbb{w}[0], \mathbb{w}_\alpha\mathbb{w}[0])} \cdot \boldsymbol{T}(\mathbb{w}_\alpha\mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1}\mathbb{w}_\alpha^{-1}, \mathbb{w}_\alpha\mathbb{w}[0]). \tag{17}$$

For this purpose, write $x := \mathbb{w}\mathbb{w}_\natural[0] - \mathbb{w}[0] \in Y_{Q,n}$. We have $n_\alpha | \langle x, \alpha \rangle$. Write $\langle x, \alpha \rangle = k \cdot n_\alpha$.

The left-hand side of (17) is

$$\begin{aligned} \overline{\chi}(\boldsymbol{s}_{x-\langle x,\alpha\rangle\alpha^\vee}) &= \overline{\chi}(\boldsymbol{s}_x) \cdot \overline{\chi}(\boldsymbol{s}_{-kn_\alpha\alpha^\vee}) \cdot \varepsilon^{D(x, -kn_\alpha\alpha^\vee)} \\ &= \overline{\chi}(\boldsymbol{s}_x) \cdot \overline{\chi}(\bar{h}_\alpha(\varpi^{n_\alpha}))^{-k} = q^k \cdot \overline{\chi}(\boldsymbol{s}_x). \end{aligned}$$

The right-hand side of (17) is

$$\begin{aligned}
&\varepsilon^{D(\mathbb{w}_\alpha(x),\mathbb{w}_\alpha\mathbb{w}[0])}\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}\mathbb{w}_\natural[0])\cdot \boldsymbol{T}(\mathbb{w}\mathbb{w}_\natural\mathbb{w}^{-1},\mathbb{w}[0])\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}_\alpha\mathbb{w}[0])\\
&\qquad=\varepsilon^{D(x,\mathbb{w}_\alpha\mathbb{w}[0])-\mathbb{w}[0])}\cdot\overline{\chi}(\boldsymbol{s}_x)\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}\mathbb{w}_\natural[0])\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}_\alpha\mathbb{w}[0]) \qquad \text{by (16)}\\
&\qquad=\varepsilon^{D(x,\mathbb{w}_\alpha\mathbb{w}[0])-\mathbb{w}[0])}\cdot\overline{\chi}(\boldsymbol{s}_x)\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}\mathbb{w}_\natural[0])\cdot \boldsymbol{t}(\mathbb{w}_\alpha,\mathbb{w}[0])^{-1}\\
&\qquad=\varepsilon^{D(x,\mathbb{w}_\alpha\mathbb{w}[0])-\mathbb{w}[0])}\cdot q^{\lceil\frac{\langle\mathbb{w}[0],\alpha\rangle}{n_\alpha}\rceil-1+k}\cdot \boldsymbol{g}_{\psi^{-1}}(\langle\mathbb{w}(0_\rho),\alpha\rangle Q(\alpha^\vee))^{-1}\\
&\qquad\qquad\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle\cdot D(\mathbb{w}\mathbb{w}_\natural[0],\alpha^\vee)}\cdot\overline{\chi}(\boldsymbol{s}_x)\cdot q^{-\lceil\frac{\langle\mathbb{w}[0],\alpha\rangle}{n_\alpha}\rceil+1}\\
&\qquad\qquad\qquad\cdot \boldsymbol{g}_{\psi^{-1}}(\langle\mathbb{w}(0_\rho),\alpha\rangle Q(\alpha^\vee))\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle\cdot D(\mathbb{w}[0],\alpha^\vee)}\\
&\qquad=\overline{\chi}(\boldsymbol{s}_x)\cdot q^k\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle D(x,\alpha^\vee)}\cdot\varepsilon^{\langle\mathbb{w}(0_\rho),\alpha\rangle D(x,\alpha^\vee)}\\
&\qquad=\overline{\chi}(\boldsymbol{s}_x)\cdot q^k,
\end{aligned}$$

which is clearly equal to the left-hand side. To summarize, we have:

Proposition 4.1. *Let* $\overline{\chi}\in \mathrm{Hom}_\iota(Z(\overline{T}),\mathbb{C}^\times)$ *be an exceptional character of* $\overline{\mathrm{SL}}_n^{(n)}$. *Then*

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)},\overline{\chi}))=1$$

if and only if $\overline{\chi}$ *is the unique exceptional character satisfying* (14).

We would like to explicate the condition given by (14).

Lemma 4.2. *One has*

$$\boldsymbol{T}(\mathbb{w}_\natural,0)=\begin{cases} q^{-r/2} & \text{if } n \text{ is odd},\\ \varepsilon^{n(n-2)/8}\cdot q^{-n/2}\cdot \boldsymbol{g}_{\psi^{-1}}(-n/2)^{-1} & \text{if } n \text{ is even}.\end{cases}$$

Proof. We compute each $\boldsymbol{t}(\mathbb{w}_{\alpha_i},y_{\langle i-1\rangle})$ for $1\le i\le r$. First, one can check easily that $y_{\langle i\rangle}=\sum_{j=1}^i i\cdot\alpha_i^\vee=\alpha_1^\vee+2\alpha_2^\vee+\cdots+i\cdot\alpha_i^\vee$. Thus, $\langle y_{\langle i-1\rangle},\alpha_i\rangle=-(i-1)$ and therefore

$$k_{y_{\langle i-1\rangle},\alpha_i}=0 \ \text{ for all } \ 1\le i\le r.$$

Second, $\boldsymbol{\Gamma}(y_{\langle i-1\rangle},\alpha_i^\vee)=\varepsilon^{-i\cdot D(y_{\langle i-1\rangle},\alpha_i^\vee)}$. Since $D(\alpha_j^\vee,\alpha_i^\vee)=0$ for all $j<i$, we see $\boldsymbol{\Gamma}(y_{\langle i-1\rangle},\alpha_i^\vee)=1$. Thus, $\boldsymbol{t}(\mathbb{w}_{\alpha_i},y_{\langle i-1\rangle})=q^{-1}\cdot \boldsymbol{g}_{\psi^{-1}}(-i)^{-1}$. Now, if $1\le i,j\le n$ and $i+j=n$, one has

$$\begin{aligned}
\boldsymbol{g}_{\psi^{-1}}(-i)^{-1}\cdot \boldsymbol{g}_{\psi^{-1}}(-j)^{-1}&=\boldsymbol{g}_{\psi^{-1}}(-i)^{-1}\cdot(\overline{\boldsymbol{g}_{\psi^{-1}}(j)}\cdot\varepsilon^j)^{-1}\\
&=|\boldsymbol{g}_{\psi^{-1}}(j)|^{-2}\cdot\varepsilon^i\\
&=q\cdot\varepsilon^i.
\end{aligned}$$

The result then follows from simply multiplying together each term. □

4B2. *Interlude: Weil-index.* Let $\boldsymbol{\gamma}_\psi$ be the Weil-index given in Section 2C.

Lemma 4.3. *Suppose $n = 2m$ is an even number. Then the following equality holds*:

$$\boldsymbol{g}_{\psi^{-1}}(m) = \frac{q^{-1/2}}{\boldsymbol{\gamma}_\psi(\varpi)}.$$

Proof. By definition, $\boldsymbol{g}_{\psi^{-1}}(m)$ is equal to

$$\begin{aligned}\int_{O_F^\times}(u,\varpi)_2\cdot\psi^{-1}(\varpi^{-1}u)\mu(u) &= \int_{O_F^\times}\boldsymbol{\gamma}_\psi(\varpi u)\boldsymbol{\gamma}_\psi(\varpi)^{-1}\boldsymbol{\gamma}_\psi(u)^{-1}\cdot\psi^{-1}(\varpi^{-1}u)\mu(u)\\ &= \boldsymbol{\gamma}_\psi(\varpi)^{-1}\cdot\int_{O_F^\times}\boldsymbol{\gamma}_\psi(\varpi u)\cdot\psi^{-1}(\varpi^{-1}u)\mu(u).\end{aligned}$$

However, by Equation (3.7) of [Szpruch 2009b, Lemma 3.2],

$$\boldsymbol{\gamma}_\psi(\varpi u) = q^{-1/2}\left(1 + q\int_{O_F^\times}\psi(\varpi^{-1}v^2u)\mu(v)\right).$$

Thus,

$$\begin{aligned}\boldsymbol{g}_{\psi^{-1}}(m) &= q^{-1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)^{-1}\cdot\int_{O_F^\times}\left(1+q\int_{O_F^\times}\psi(\varpi^{-1}v^2u)\mu(v)\right)\psi^{-1}(\varpi^{-1}u)\mu(u)\\ &= q^{-1/2}\cdot\boldsymbol{\gamma}_\psi(\varpi)^{-1}\cdot\left(-\frac{1}{q}+q\cdot\int_{O_F^\times}\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\mu(v)\right)\end{aligned}$$

Let $D = \{v \in O_F^\times : |1-v^2| = 1\}$ and $H = \{v \in O_F^\times : |1-v^2| \le q^{-1}\}$. We get

$$\begin{aligned}&\int_{O_F^\times}\left(\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\right)\mu(v)\\ &= \int_{v\in H}\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\mu(v)+\int_{v\in D}\int_{O_F^\times}\psi(\varpi^{-1}u(v^2-1))\mu(u)\mu(v)\\ &= \mu(H)\cdot(1-q^{-1})+\mu(D)\cdot(-q^{-1}) \text{ by (8.19) of [Szpruch 2009b, Lemma 8.6]}\\ &= 2q^{-1}\cdot(1-q^{-1})+(1-3q^{-1})\cdot(-q^{-1}) \text{ by [Szpruch 2009b, Lemma 8.9]}\\ &= q^{-1}+q^{-2}.\end{aligned}$$

The result follows easily by simplification. □

4B3. *An explicit criterion.* Consider the *unitary* distinguished character $\overline{\chi}^0_{\psi'}$ constructed in [Gan and Gao 2016], which we recalled and gave in (5). Then the character $\overline{\chi}_{\psi'} = \overline{\chi}^0_{\psi'}\cdot\delta_B(\cdot)^{1/2n}$ is an exceptional character. In the simply connected case, $J = Y^{\mathrm{sc}}_{Q,n}$. For the definition of $\overline{\chi}^0_{\psi'}$, we pick a basis $\{y_i\}$ for $Y_{Q,n}$ such that

$\{k_i y_i\}$ is a basis for $J = Y^{\mathrm{sc}}_{Q,n}$. Then by definition,

$$\overline{\chi}^{\,0}_{\psi'}(\boldsymbol{s}_{y_i}) = \boldsymbol{\gamma}_{\psi'}(\varpi)^{2(k_i-1)Q(y_i)/n}$$

and, for $y = \sum_i n_i y_i \in Y_{Q,n}$, one has

$$\overline{\chi}^{\,0}_{\psi'}(\boldsymbol{s}_y) = \prod_i \overline{\chi}^{\,0}_{\psi'}(\varpi^{n_i})^{2(k_i-1)Q(y_i)/n} \cdot \varepsilon^{\sum_{i<j} n_i n_j D(y_i,y_j)}.$$

For the covering group $\overline{\mathrm{SL}}^{(n)}_n$, we take $y_i = n\alpha_i^\vee$, $2 \le i \le r$ and $y_1 = y_{Q,n}$, with $k_i = 1$ for $2 \le i \le r$ and $k_1 = n$.

An easy computation shows $Q(y_{Q,n}) = r(r+1)/2$, and thus

$$\overline{\chi}_{\psi'}(\boldsymbol{s}_{y_{Q,n}}) = \overline{\chi}^{\,0}_{\psi'}(\boldsymbol{s}_{y_{Q,n}}) \cdot \delta_B(\boldsymbol{s}_{y_{Q,n}})^{\frac{1}{2n}} = \boldsymbol{\gamma}_{\psi'}(\varpi)^{(n-1)^2} \cdot q^{-(n-1)/2}. \tag{18}$$

Proposition 4.4. *For the exceptional character $\overline{\chi}_{\psi'} = \overline{\chi}^{\,0}_{\psi'} \cdot \delta_B(\cdot)^{\frac{1}{2n}}$ given above, one has that the dimension of* $\mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}^{(n)}_n, \overline{\chi}_{\psi'}))$ *equals* 1 *in the following cases, and* 0 *otherwise*:

$$\begin{cases} \text{any } \psi', & \text{if } n \text{ is odd};\\ \boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi), & \text{if } n \equiv 0, 2 \bmod 8;\\ \boldsymbol{\gamma}_{\psi'}(\varpi) = (-1,\varpi)_4 \cdot \boldsymbol{\gamma}_{\psi}(\varpi) & \text{if } n \equiv 4 \bmod 8;\\ \boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)^{-1} & \text{if } n \equiv 6 \bmod 8. \end{cases}$$

Proof. By the value of $\overline{\chi}_{\psi'}(\boldsymbol{s}_{y_{Q,n}})$ in (18), it follows from Lemma 4.2 that the equality (14) is equivalent to

$$\boldsymbol{\gamma}_{\psi'}(\varpi)^{(n-1)^2} \cdot q^{-\frac{(n-1)}{2}} = \begin{cases} q^{-r/2} & \text{if } n \text{ is odd};\\ (-1,\varpi)_n^{n(n-2)/8} \cdot q^{-\frac{n}{2}} \cdot \boldsymbol{g}_{\psi^{-1}}(-\frac{n}{2})^{-1} & \text{if } n \text{ is even.} \end{cases} \tag{19}$$

For n odd, the equality holds for any ψ'. Now we assume n even.

For $n = 4k+2$, by Lemma 4.3, the required equality in (19) becomes

$$\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)^{2k+1}.$$

In particular, if k is even, it is equivalent to $\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)$. If k is odd, it is equivalent to $\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)^{-1}$.

For $n = 4k$, applying Lemma 4.3 again, the equality in (19) reads

$$\boldsymbol{\gamma}_{\psi'}(\varpi) = (-1,\varpi)_n^k \cdot \boldsymbol{\gamma}_{\psi}(\varpi) = (-1,\varpi)_4 \cdot \boldsymbol{\gamma}_{\psi}(\varpi).$$

A special case is when k is even. In this case $(-1,\varpi)_4 = 1$ and therefore it is equivalent to $\boldsymbol{\gamma}_{\psi'}(\varpi) = \boldsymbol{\gamma}_{\psi}(\varpi)$. $\square$

Corollary 4.5. *Consider $\psi' = \psi_a$ for some $a \in F^\times$. Assume ψ_a has conductor O_F, i.e., $a \in O_F^\times$. Then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi}_{\psi_a})) = 1$ *if and only if the following hold*:

$$\begin{cases} a \in O_F^\times & \text{if } n \text{ is odd,} \\ a \in (O_F^\times)^2 & \text{if } n \equiv 0, 2 \bmod 8, \\ a^2 \in -(O_F^\times)^4 & \text{if } n \equiv 4 \bmod 8, \\ a \in -(O_F^\times)^2 & \text{if } n \equiv 6 \bmod 8. \end{cases}$$

Remark 4.6. The facts that for any exceptional representation $\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})$ there exists ψ such that it is ψ-generic, and that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_n^{(n)}, \overline{\chi})) \le 1$ for all ψ also follow from the work of [Kazhdan and Patterson 1984] on $\overline{\mathrm{GL}}_n^{(n)}$ combined with the relation between $\overline{\mathrm{SL}}_n^{(n)}$ and $\overline{\mathrm{GL}}_n^{(n)}$ in [Adams 2003]. (We thank the referee for pointing this out.) However, our Corollary 4.5 gives precise information for the matching between ψ and the distinguished theta representation in terms of the distinguished character.

Example 4.7. The first nontrivial example is the metaplectic covering $\overline{\mathrm{SL}}_2^{(2)}$. In this case, we have $Y_{Q,n} = Y = \mathbb{Z} \cdot \alpha^\vee$ and $Y_{Q,n}^{\mathrm{sc}} = \mathbb{Z} \cdot (2\alpha^\vee)$. As mentioned at the beginning of Section 4B, one has that the lower and upper bounds in Theorem 3.14 are 0 and 1 respectively and thus

$$0 \le \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi})) \le 1$$

for any exceptional $\overline{\chi}$. For the character ψ_a, the representation $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$ is the even Weil representation in the following exact sequence:

$$\mathrm{St}(\overline{\chi}_{\psi_a}) \hookrightarrow I(\overline{\chi}_{\psi_a}) \twoheadrightarrow \Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a}),$$

where $\mathrm{St}(\overline{\chi}_{\psi_a})$ is the metaplectic analogue of the Steinberg representation. From Corollary 4.5, we can recover the well-known fact, which follows from the work of Gelbart and Piatetski-Shapiro [1980], that for $\overline{\mathrm{SL}}_2^{(2)}$ the even Weil representation $\Theta(\overline{\mathrm{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$ (for unramified data) is ψ-generic if and only if $a \in (O_F^\times)^2$. We note that this also follows directly from the computation of the local coefficient for $\overline{\mathrm{SL}}_2^{(2)}$ in [Szpruch 2009a].

Example 4.8. We also discuss explicitly the example $\overline{\mathrm{SL}}_3^{(3)}$. Consider $\overline{\mathrm{SL}}_3^{(3)}$ with cocharacter lattice $Y = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$. Consider Q such that $Q(\alpha_i^\vee) = 1$. Then

$$Y_{Q,n} = \langle 2\alpha_1^\vee + \alpha_2^\vee, 3\alpha_1^\vee \rangle = \langle 2\alpha_2^\vee + \alpha_1^\vee, 3\alpha_2^\vee \rangle.$$

Note $Y = \langle 2\alpha_1^\vee + \alpha_2^\vee, \alpha_1^\vee \rangle = \langle 2\alpha_2^\vee + \alpha_1^\vee, \alpha_2^\vee \rangle$. We know $\rho = \alpha_1^\vee + \alpha_2^\vee$. For $y = 0$ one has

$$y_\rho = 0_\rho = -(\alpha_1^\vee + \alpha_2^\vee).$$

Consider $\mathbb{w}_\natural = \mathbb{w}_{\alpha_1}\mathbb{w}_{\alpha_2}$, then $\mathbb{w}_{\alpha_2}[y] = \alpha_2^\vee$ and moreover $\mathbb{w}_{\alpha_1}\mathbb{w}_{\alpha_2}[y] = 2\alpha_1^\vee + \alpha_2^\vee$. One has

$$\begin{aligned}
\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]}) &= q^{k_{\mathbb{w}_2[y],\alpha_1}-1} \cdot \boldsymbol{\Gamma}(\mathbb{w}_2[y], \alpha_1^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_1^\vee)(\langle \mathbb{w}_2[y], \alpha_1\rangle - 1))^{-1} \\
&\qquad \cdot q^{k_{y,\alpha_2}-1} \cdot \boldsymbol{\Gamma}(y, \alpha_2^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_2^\vee)(\langle y, \alpha_2\rangle - 1))^{-1} \cdot \boldsymbol{c}(\boldsymbol{s}_y) \\
&= q^{\lceil \frac{\langle \alpha_2^\vee, \alpha_1\rangle}{3} \rceil + \lceil \frac{\langle y, \alpha_2\rangle}{3}\rceil - 2} \cdot \boldsymbol{\Gamma}(\alpha_2^\vee, \alpha_1^\vee) \cdot \boldsymbol{\Gamma}(0, \alpha_2^\vee) \\
&\qquad \cdot \boldsymbol{g}_{\psi^{-1}}(-2)^{-1} \boldsymbol{g}_{\psi^{-1}}(-1)^{-1} \cdot \boldsymbol{c}(1_{\overline{\mathrm{SL}}_3^{(3)}}) \\
&= q^{-2} \cdot q \cdot \boldsymbol{c}(1_{\overline{\mathrm{SL}}_3^{(3)}}) = q^{-1},
\end{aligned}$$

where $\boldsymbol{c}$ is normalized to take value 1 at the $1 \in \overline{\mathrm{SL}}_3^{(3)}$. This implies that necessarily $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]}) = q^{-1}$, and thus

$$\overline{\chi}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]}) = q^{-1}.$$

Note, this is not a consequence of $\overline{\chi}$ being exceptional, although it is compatible. Clearly, an exceptional character $\overline{\chi}$ is such that

$$\begin{cases} \overline{\chi}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]})^3 = q^{-3}, \\ \overline{\chi}(\boldsymbol{s}_{3\alpha_1^\vee}) = q^{-1}. \end{cases}$$

In particular, if for some third root of unity $\zeta \neq 1$, $\overline{\chi}(\boldsymbol{s}_{\mathbb{w}_1\mathbb{w}_2[y]})$ is equal to $\zeta \cdot q^{-1}$, then $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_3^{(3)}, \overline{\chi})) = 0$ for such $\overline{\chi}$.

4C. *Case III:* $\overline{\mathbf{SL}}_{r+1}^{(n)}$, $n = r+2$. For $n = r+2$, we show $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ and therefore Corollary 3.15 applies. Picking any $(y_1, y_2, \ldots, y_{r+1}) \in Y_{Q,n}$, we have

$$a \equiv y_1 \equiv y_2 \equiv \cdots \equiv y_{r+1} \mod n,$$

where $a \in \{0, 1, 2, \ldots, r+1\}$. Write $y_i = k_i n + a$. Since $\sum_{i=1}^{r+1} y_i = 0$, one has

$$n \cdot \left(\sum_{i=1}^{r+1} k_i \right) + (r+1) \cdot a = 0.$$

In particular, $n | (r+1)a$. However, $\gcd(n, r+1) = 1$, so $n|a$ and $a = 0$. That is, $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ and therefore $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(r+2)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$. Note that, the equality $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ reflects the fact that the dual group for $\overline{\mathrm{SL}}_n^{(n+1)}$ is PGL_n (see [Weissman 2015, § 2.7.2]).

We claim that the dimension is equal to 1 in this case. Let $\mathcal{O}_y \in \mathcal{O}_{Q,n,\mathrm{sc}}^F$ be a $Y_{Q,n}^{\mathrm{sc}}$-free orbit with $\boldsymbol{i}_A^*(y_\rho) = (0, 1, \ldots, r-1, r) \in \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i$. We know that $\mathcal{O}_y$ is $Y_{Q,n}$-free (or equally, $Y_{Q,n}^{\mathrm{sc}}$-free). Moreover, one can check easily that $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Therefore $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(r+2)}, \overline{\chi})) = 1$ for the unique exceptional character $\overline{\chi}$ in this case.

4D. *Case IV:* $\overline{\mathrm{SL}}_{r+1}^{(n)}$, $n \geq r+3$.

Lemma 4.9. *Consider $y \in Y$ such that $\boldsymbol{i}_A^*(y_\rho) = (x_1^*, x_2^*, \ldots, x_r^*, x_{r+1}^*)$ with $x_i^* = i-1$. If $n \geq r+3$, the orbit $\mathcal{O}_y$ is $Y_{Q,n}$-free.*

Proof. Suppose not, then there exists $\mathbb{w} \neq 1$ such that $\mathbb{w}[y] - y \in Y_{Q,n}$. Identify $\mathbb{w}$ with a permutation, then we have

$$(x_1^*, x_2^*, \ldots, x_{r+1}^*) - (x_{\mathbb{w}(1)}^*, x_{\mathbb{w}(2)}^*, \ldots, x_{\mathbb{w}(r+1)}^*) \in Y_{Q,n}.$$

More precisely, $i - \mathbb{w}(i) \equiv j - \mathbb{w}(j) \bmod n$ for all i, j. Clearly, $n \nmid (i - \mathbb{w}(i))$ for all i, otherwise one can deduce $\mathbb{w}(i) = i$ for all i and therefore $\mathbb{w} = 1$. That is, $(i - \mathbb{w}(i))$ is either negative or positive. We reorder the terms $(i - \mathbb{w}(i))$ as

$$-r \leq (i_1 - \mathbb{w}(i_1)) \leq (i_2 - \mathbb{w}(i_2)) \leq \cdots < 0 < \cdots \leq (i_r - \mathbb{w}(i_r)) \leq (i_{r+1} - \mathbb{w}(i_{r+1})) \leq r.$$

Write $(i_1 - \mathbb{w}(i_1)) = -s$, $s \in \mathbb{N}$ and $(i_{r+1} - \mathbb{w}(i_{r+1})) = t$, $t \in \mathbb{N}$. It is easy to see that any negative $i - \mathbb{w}(i)$ must be equal to $-s$, and any positive $i - \mathbb{w}(i)$ must be equal to t.

We claim that $2 < t + s \leq r+1$ and therefore $n \nmid (t+s)$, i.e., $\mathbb{w}[y] - y \notin Y_{Q,n}$ for all $\mathbb{w} \neq 1$. Note $0 - \mathbb{w}(0) = -s$ and $r - \mathbb{w}(r) = t$. Suppose $t + s > r+1$, then there exists i_0 such that $r + 1 - t < i_0 < 1 + s$. However, there exists no i' such that $\mathbb{w}(i') = i_0$. This is a contradiction, and the claim follows.

Therefore $\mathcal{O}_y$ is $Y_{Q,n}$-free for the given y. □

It follows that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(n)}, \overline{\chi})) \geq 1$ for $n \geq r+3$. In principle, one could proceed as in Section 4B to analyze every element in $\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)$ and determine completely $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(n)}, \overline{\chi}))$ in this case. However, the level of complexity of the computation depends inevitably on (the center of) the dual group of $\overline{\mathrm{SL}}_r^{(n)}$ and could be quite involved for general $n \geq r+3$.

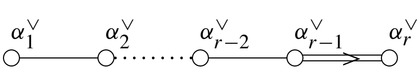
We summarize for the $n \leq r+2$ cases below.

Theorem 4.10. *Consider the Brylinski–Deligne covering $\overline{\mathrm{SL}}_{r+1}^{(n)}$, $n \leq r+2$ with $Q(\alpha^\vee) = 1$ for all coroots $\alpha^\vee$. Let $\overline{\chi}$ be an exceptional character of $\overline{\mathrm{SL}}_{r+1}^{(n)}$. Then $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{SL}}_{r+1}^{(n)}, \overline{\chi})) = 1$ if and only if*

- *$n = r+2$ and $\overline{\chi}$ is the only exceptional character, or*
- *$n = r+1$ and $\overline{\chi}$ is the unique exceptional character satisfying (14).*

5. The C_r, $r \geq 2$ case

Consider the Dynkin diagram for the simple coroots for C_r:



Let
$$Y = Y^{\mathrm{sc}} = \langle \alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$$
be the cocharacter lattice of Sp_{2r}, where $\alpha_r^\vee$ is the short coroot. Let Q be the Weyl-invariant quadratic on Y such that $Q(\alpha_r^\vee) = 1$. Then the bilinear form B_Q is given by
$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = r, \\ 4 & \text{if } 1 \le i = j \le r-1, \\ -2 & \text{if } j = i+1, \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$
A simple computation gives
$$Y_{Q,n} = \left\{ \sum_{i=1}^n x_i \alpha_i^\vee : n | (2x_i) \right\}.$$
We write $n_2 := n / \gcd(2, n)$. Then
$$Y_{Q,n} = \langle n_2\alpha_1^\vee, n_2\alpha_2^\vee, \ldots, n_2\alpha_{r-1}^\vee, n_2\alpha_r^\vee \rangle$$
and
$$Y_{Q,n}^{\mathrm{sc}} = \langle n_2\alpha_1^\vee, n_2\alpha_2^\vee, \ldots, n_2\alpha_{r-1}^\vee, n\alpha_r^\vee \rangle.$$
The map $\boldsymbol{i}_C : \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^r \mathbb{Z}e_i$ is given by
$$\boldsymbol{i}_C : (x_1, x_2, x_3, \ldots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{r-1} - x_{r-2}, x_r - x_{r-1}).$$
Here $\boldsymbol{i}_C$ is an isomorphism. The Weyl group is $W = S_r \rtimes (\mathbb{Z}/2\mathbb{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbb{Z}e_i$ and each $(\mathbb{Z}/2\mathbb{Z})_i$ acts by $e_i \mapsto \pm e_i$. In particular, w_{α_i}, $1 \le i \le r-1$, acts on $(y_1, y_2, \ldots, y_r) \in \bigoplus_i \mathbb{Z}e_i$ by exchanging y_i and y_{i+1}, while w_{α_r} acts by (-1) on $\mathbb{Z}e_r$.

Moreover, $y \in Y$ lies in $Y_{Q,n}$ if and only if all entries of $\boldsymbol{i}_C(y)$ are divisible by n_2. It is easy to obtain
$$Y_{Q,n} = \left\{ (y_1, y_2, \ldots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i \ : \ n_2 | y_i \text{ for all } i. \right\}$$
and
$$Y_{Q,n}^{\mathrm{sc}} = \left\{ (y_1, y_2, \ldots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i \ : \ n_2 | y_i \ \text{ for all } i, \ \text{ and } n \Big| \sum_i y_i. \right\}$$
We further note
$$2\rho = \sum_{i=1}^r (2r - 2i + 1)e_i = \sum_{i=1}^r i(2r - i)\alpha_i^\vee.$$
Assume $x_0 = 0$, then
$$\boldsymbol{i}_C(y_\rho) = (x_i - x_{i-1} - (r - i + 1/2))_{1 \le i \le r}.$$

Write $x_i^* := x_i - x_{i-1} - (r-i)$, and also $\boldsymbol{i}_C^*(y_\rho) := (x_1^*, x_2^*, \dots, x_{r-1}^*, x_r^*)$. Then

$$\boldsymbol{i}_C(y_\rho) = \boldsymbol{i}_C^*(y_\rho) - \Big(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}\Big).$$

We will discuss the two cases depending on the parity of n separately.

5A. ***The case where n is odd.*** Here, $n_2 = n$ and

$$nY = Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n} = \Big\{(y_1, \dots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i \;:\; n|y_i \text{ for all } i\Big\}.$$

The complex dual group for $\overline{\mathrm{Sp}}_{2r}^{(n)}$ for n odd is SO_{2r+1}.

Proposition 5.1. *Let n be an odd number, one has*

$$\begin{cases} |\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| \geq 2 & \textit{if } n \geq 2r+3, \\ |\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = 1 & \textit{if } n = 2r+1, \\ |\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = 0 & \textit{if } n \leq 2r-1. \end{cases}$$

So, we have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(n)}, \overline{\chi})) = 1$, *for n odd, if and only if* $n = 2r+1$ *for the only exceptional character of* $\overline{\mathrm{Sp}}_{2r}^{(2r+1)}$.

Proof. We have written

$$\boldsymbol{i}_C(y_\rho) = \boldsymbol{i}_C^*(y_\rho) - \Big(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}\Big).$$

Since $x_1, \dots, x_r$ are arbitrary, the associated x_i^* are also arbitrary.

First, when $n \geq 2r+3$, consider the orbits $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ where

$$\boldsymbol{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r) \quad \text{and} \quad \boldsymbol{i}_C^*(y'_\rho) = (1, 2, \dots, r-1, r+1).$$

If $r = 2$, consider $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ with $\boldsymbol{i}_C^*(y_\rho) = (1, 2)$ and $\boldsymbol{i}_C^*(y'_\rho) = (1, 3)$. Both $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are $Y_{Q,n}$-free orbits. For example, for $\mathcal{O}_y$, this follows from the fact that the entries of $\boldsymbol{i}_C(\mathbb{w}(y_\rho)) - \boldsymbol{i}_C(y_\rho)$ are either $j-i$ or $j+i-1$, for $0 \leq i, j \leq r-1$. One can check also that $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$, and therefore $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$.

Second, assume $n = 2r+1$. Consider $\mathcal{O}_y$ such that $\boldsymbol{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r)$. For $r = 2$, consider $\boldsymbol{i}_C^*(y_\rho) = (1, 2)$. It can be checked easily that $\wp_{Q,n}(\mathcal{O}^F_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Thus, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(2r+1)}, \overline{\chi})) = 1$.

Third, assume that $n \leq 2r-1$, we want to show that $\mathcal{O}^F_{Q,n,\mathrm{sc}} = \varnothing$. If $\boldsymbol{i}_C^*(y_\rho) = (x_1^*, x_2^*, \dots, x_i^*, \dots, x_r^*)$ is such that $x_i^* \equiv x_j^* \bmod n$ for some $i \neq j$, then clearly $\mathcal{O}_y \notin \mathcal{O}^F_{Q,n,\mathrm{sc}}$. Now if $n \nmid (x_i^* - x_j^*)$ for all $i \neq j$; since $n \leq 2r-1$, it is not hard to see that there always exist i, j such that $n|(x_j^* - 1/2) + (x_i^* - 1/2)$, i.e., $n|(x_j^* + x_i^* - 1)$. In this case, one also has $\mathcal{O}_y \notin \mathcal{O}^F_{Q,n,\mathrm{sc}}$. In any case, $\mathcal{O}^F_{Q,n,\mathrm{sc}} = \varnothing$ for $n \leq 2r-1$. The proof is completed. $\square$

5B. ***The case where n is even.*** Writing $n = 2m$,

$$Y_{Q,n} = \langle m\alpha_1^\vee, m\alpha_2^\vee, \dots, m\alpha_{r-1}^\vee, m\alpha_r^\vee \rangle, \qquad Y_{Q,n}^{\mathrm{sc}} = \langle m\alpha_1^\vee, m\alpha_2^\vee, \dots, m\alpha_{r-1}^\vee, n\alpha_r^\vee \rangle.$$

Equivalently, one has:

$$Y_{Q,n} = \Big\{ (y_1, y_2, \dots, y_r) \in \bigoplus\nolimits_{i=1}^{r} \mathbb{Z}e_i \ : \ m | y_i \text{ for all } i. \Big\}$$

and

$$Y_{Q,n}^{\mathrm{sc}} = \Big\{ (y_1, y_2, \dots, y_r) \in \bigoplus\nolimits_{i=1}^{r} \mathbb{Z}e_i : m | y_i \text{ for all } i, \text{ and } n \Big| \sum\nolimits_i y_i . \Big\}$$

The dual group for $\overline{\mathrm{Sp}}_{2r}^{(n)}$ with n even is Sp_{2r}.

5B1. *The case where* $m \geq 2r + 2$. Here, consider the orbits $\mathcal{O}_y$, $\mathcal{O}_{y'}$ given in the proof of Proposition 5.1. They are $Y_{Q,n}$-free; moreover, $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are distinct in the image of $\wp_{Q,n}$. Thus, we have $|\wp_{Q,n}(\mathcal{O}_{Q,n}^{\digamma})| \geq 2$.

5B2. *The case where* $m \leq 2r - 2$. Here, we can easily check $\mathcal{O}_{Q,n,\mathrm{sc}}^{\digamma} = \varnothing$.

5B3. *The case where* $m = 2r - 1$. Consider y with $\boldsymbol{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r)$, i.e.,

$$\boldsymbol{i}_C(y_\rho) = \Big(1 - \frac{1}{2}, 2 - \frac{1}{2}, \dots, (r-1) - \frac{1}{2}, r - \frac{1}{2} \Big).$$

Consider $\mathbb{w}_{\alpha_r} \in W$, then $\boldsymbol{i}_C(\mathbb{w}_{\alpha_r}(y_\rho)) = \big(1 - \tfrac{1}{2}, 2 - \tfrac{1}{2}, \dots, (r-1) - \tfrac{1}{2}, -(r - \tfrac{1}{2})\big)$.

Note $\mathcal{O}_y$ is $Y_{Q,n}^{\mathrm{sc}}$-free, and $\wp_{Q,n}^{\mathrm{sc}}(\mathcal{O}_{Q,n,\mathrm{sc}}^{\digamma}) = \{\wp_{Q,n}^{\mathrm{sc}}(\mathcal{O}_y)\} = \{\wp_{Q,n}^{\mathrm{sc}}(\mathcal{O}_0)\}$. However, it is not $Y_{Q,n}$-free, since $\boldsymbol{i}_C(y_\rho - \mathbb{w}_{\alpha_r}(y_\rho)) = (0, 0, \dots, m) \in Y_{Q,n}$. Remember that any $\boldsymbol{c} \in \mathrm{Ftn}(i(\overline{\chi}))$ which gives rise to $\lambda_{\boldsymbol{c}}^{\overline{\chi}} \in \mathrm{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ satisfies $\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_{\alpha_r}[y]}) = \boldsymbol{t}(\mathbb{w}_{\alpha_r}, y) \cdot \boldsymbol{c}(\boldsymbol{s}_y)$ where

$$\boldsymbol{t}(\mathbb{w}_{\alpha_r}, y) = q^{k_{y,\alpha_r} - 1} \cdot \boldsymbol{\Gamma}(y, \alpha_r^\vee) \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_r^\vee) \cdot \langle y_\rho, \alpha_r \rangle)^{-1}.$$

Meanwhile, in our case $\mathbb{w}_{\alpha_r}[y] - y = (-m)\alpha_r^\vee \in Y_{Q,n}$. It follows that

$$\boldsymbol{c}(\boldsymbol{s}_{\mathbb{w}_{\alpha_r}[y]}) = \varepsilon^{D(\mathbb{w}_{\alpha_r}(y_\rho) - y_\rho, y)} \cdot \overline{\chi}(\boldsymbol{s}_{\mathbb{w}_{\alpha_r}(y_\rho) - y_\rho}) \cdot \boldsymbol{c}(\boldsymbol{s}_y).$$

For $\boldsymbol{c}$ to be nonzero on $\mathcal{O}_y$, i.e., $\wp_{Q,n}(\mathcal{O}_y)$ contributes to the right-hand side of (10), one has

$$\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) = q^{k_{y,\alpha_r} - 1} \cdot \boldsymbol{g}_{\psi^{-1}}(Q(\alpha_r^\vee) \cdot \langle y_\rho, \alpha_r \rangle)^{-1}.$$

Moreover, we can argue as in Section 4B that this condition is also sufficient. One has $\langle y, \alpha_r \rangle = 2r$ and thus $k_{y,\alpha_r} = 1$. The equality is thus simplified to

$$\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^\vee}) = \boldsymbol{g}_{\psi^{-1}}(m)^{-1}. \tag{20}$$

Consider the exceptional character $\overline{\chi}_{\psi'} = \overline{\chi}_{\psi'}^{0} \cdot \delta_B^{1/2n}$, which relies on the distinguished unitary character $\overline{\chi}_{\psi'}^{0}$ depending on a nontrivial character $\psi' : F \to \mathbb{C}^\times$

(see Section 2C). Since $\overline{\chi}^{\,0}_{\psi'}(\boldsymbol{s}_{m\alpha_r^\vee}) = \boldsymbol{\gamma}_{\psi'}(\varpi)^{mQ(\alpha_r^\vee)}$, by Lemma 4.3, equality (20) becomes $\boldsymbol{\gamma}_{\psi}(\varpi) = (-1,\varpi)_n^{m^2}\cdot \boldsymbol{\gamma}_{\psi'}(\varpi)^{-m}$, which can be further reduced to

$$\boldsymbol{\gamma}_{\psi'}(\varpi) = (-1,\varpi)_n^{r+1}\cdot \boldsymbol{\gamma}_{\psi}(\varpi) = (-1,\varpi)_2^{r+1}\cdot \boldsymbol{\gamma}_{\psi}(\varpi).$$

In particular, if $\psi' = \psi_a$ with $a \in O_F^\times$, then the equality is equivalent to $(a(-1)^{r+1},\varpi)_2 = -1$, i.e., $a \in (-1)^{r+1}\cdot (O_F^\times)^2$.

5B4. *The case where $m = 2r$.* We claim that here $\mathcal{O}^{\digamma}_{Q,n} = \mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}$. Clearly it suffices to show that $\mathcal{O}^{\digamma}_{Q,n} \supseteq \mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}$. Equivalently, if $\mathcal{O}_y$ is not $Y_{Q,n}$-free, we would like to show that it is not $Y^{\mathrm{sc}}_{Q,n}$-free. Write $\boldsymbol{i}^*_C(y_\rho) = (x_1^*, x_2^*, \dots, x_r^*)$. By assumption,

$$\boldsymbol{i}_C(y - \mathbb{w}[y]) = \boldsymbol{i}^*_C(y_\rho - \mathbb{w}(y_\rho)) \in Y_{Q,n}$$

for some $\mathbb{w} \in W$. Entries of $\boldsymbol{i}_C(y-\mathbb{w}[y])$ cannot be of the form $2x_i^* - 1$ since m is even; thus they are of the form 0, $x_i^* - x_j^*$ or $x_i^* + x_j^* - 1$ for $i \neq j$. In this case, it is easy to see that $\boldsymbol{i}_C(y - \mathbb{w}'[y]) \in Y^{\mathrm{sc}}_{Q,n}$ for some $\mathbb{w}' \in W$, i.e., $\mathcal{O}_y$ is not $Y^{\mathrm{sc}}_{Q,n}$-free.

Consequently,

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}^{(4r)}_{2r}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})|.$$

On the other hand, consider $\mathcal{O}_y$ with $\boldsymbol{i}^*_C(y_\rho) = (1,\dots,r-1,r)$. It is easy to see $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Therefore, we always have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}^{(4r)}_{2r}, \overline{\chi})) = 1$ for any of the two exceptional characters of $\overline{\mathrm{Sp}}^{(4r)}_{2r}$.

5B5. *The case where $m = 2r+1$.* Consider $\mathcal{O}_y$ with $\boldsymbol{i}^*_C(y_\rho) = (1,2,\dots,r-1,r)$. One can check $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ with $\mathcal{O}_y \in \mathcal{O}^{\digamma}_{Q,n}$, i.e., $|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})| = 1$. On the other hand,

$$\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y)\} \cup \{\wp_{Q,n}(\mathcal{O}_{z_i}) : 1 \le i \le r\}$$

with z_i described as follows. Recall that we write $z_{i,\rho} := z_i - \rho$. For $1 \le i \le r-1$, z_i is such that $\boldsymbol{i}^*_C(z_{i,\rho}) = (0,2,3,\dots,\widehat{i+1},\dots,r,r+1)$, which denotes the r-tuple obtained from the $(r+1)$-tuple $(0,2,3,\dots,r-1,r,r+1)$ by removing the entry $i+1$. Meanwhile, z_r is such that $\boldsymbol{i}^*_C(z_{r,\rho}) = (2,3,\dots,r-1,r,r+1)$.

Note that $\mathcal{O}_{z_i} \in \mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}} \setminus \mathcal{O}^{\digamma}_{Q,n}$, since

$$\boldsymbol{i}_C(\mathbb{w}_{\alpha_r}[z_i] - z_i) = \boldsymbol{i}_C(\mathbb{w}_{\alpha_r}(z_{i,\rho}) - z_{i,\rho}) = -(0,0,\dots,0,m) = \boldsymbol{i}_C(-m\alpha_r^\vee) \in Y_{Q,n}.$$

The $r+1$ elements $\wp_{Q,n}(\mathcal{O}_y)$ and $\wp_{Q,n}(\mathcal{O}_{z_i})$ $(1 \le i \le r)$ are all distinct. It follows that $|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}})| = r+1$. Therefore,

$$1 \le \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}^{(4r+2)}_{2r}, \overline{\chi})) \le r+1.$$

However, because there are only two exceptional characters $\overline{\chi}$, the dimension $\mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}^{(4r+2)}_{2r}, \overline{\chi}))$ can take at most two values. In fact, we will determine completely the value and its dependence on $\overline{\chi}$.

Proposition 5.2. *Let $\overline{\chi}$ be an exceptional character of $\overline{\mathrm{Sp}}_{2r}^{(4r+2)}$. Then*

$$\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = \begin{cases} 1 & \text{if } \overline{\chi}(\boldsymbol{s}_{-m\alpha_r^{\vee}}) = -q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi), \\ r+1 & \text{if } \overline{\chi}(\boldsymbol{s}_{-m\alpha_r^{\vee}}) = q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi). \end{cases}$$

Proof. First, we show that $\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^{\vee}})$ is equal to $\pm q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi)$ if $\overline{\chi}$ is an exceptional character. Consider

$$\begin{aligned} \overline{\chi}(\boldsymbol{s}_{-m\alpha_r^{\vee}})^2 &= \overline{\chi}(\boldsymbol{s}_{-n\alpha_r^{\vee}}) \cdot \varepsilon^{m^2 Q(\alpha_r^{\vee})} \\ &= \overline{\chi}(\boldsymbol{s}_{n\alpha_r^{\vee}})^{-1} \cdot \varepsilon \\ &= q \cdot (-1, \varpi)_2, \end{aligned}$$

which has square roots exactly $\pm q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi)$. That is, an exceptional character $\overline{\chi}$ of $\overline{\mathrm{Sp}}_{2r}^{(4r+2)}$ is uniquely determined by the sign.

Second, arguing as in Section 4B , we see that $\wp_{Q,n}(\mathcal{O}_{z_i})$, $1 \le i \le r$ contributes to the right-hand side of equality (10) if and only if (as in equality (15))

$$\overline{\chi}(\boldsymbol{s}_{\mathrm{w}_{\alpha_r}[z_i]-z_i}) = \varepsilon^{D(\mathrm{w}_{\alpha_r}[z_i]-z_i, z_i)} \cdot \boldsymbol{t}(\mathrm{w}_{\alpha_r}, z_i). \tag{21}$$

That is, $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = 1 + |\{z_i : \text{ the equality (21) holds for } z_i\}|$. Note that, $\mathrm{w}_{\alpha_r}[z_i] - z_i = -m\alpha_r^{\vee}$ for all i. On the other hand, we claim that the right-hand side of (21) is independent of i. A simple computation gives $\langle z_{i,\rho}, \alpha_r \rangle = m$ and therefore

$$\begin{aligned} &\varepsilon^{D(\mathrm{w}_{\alpha_r}[z_i]-z_i, z_i)} \cdot \boldsymbol{t}(\mathrm{w}_{\alpha_r}, z_i) \\ &\quad = \varepsilon^{D(\alpha_r^{\vee}, z_i)} \cdot q^{\lceil \frac{\langle z_{i,\rho}, \alpha_r \rangle + 1}{n_{\alpha_r}} \rceil - 1} \cdot \varepsilon^{\langle z_{i,\rho}, \alpha_r \rangle \cdot D(z_i, \alpha_r^{\vee})} \cdot \boldsymbol{g}_{\psi^{-1}}(\langle z_{i,\rho}, \alpha_r \rangle \cdot Q(\alpha_r^{\vee}))^{-1} \\ &\quad = \varepsilon^{B_Q(z_i, \alpha_r^{\vee})} \cdot q^{\lceil \frac{m+1}{n} \rceil - 1} \cdot \boldsymbol{g}_{\psi^{-1}}(m)^{-1} \\ &\quad = \boldsymbol{g}_{\psi^{-1}}(m)^{-1}, \quad \text{by the evenness of } B_Q. \end{aligned}$$

Thus, it follows that $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = 1$ or $r+1$. Moreover, it is equal to 1 if and only if $\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^{\vee}}) \neq \boldsymbol{g}_{\psi^{-1}}(m)^{-1}$. By Lemma 4.3,

$$\boldsymbol{g}_{\psi^{-1}}(m)^{-1} = q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi).$$

Thus, $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi})) = 1$ (respectively, $r+1$) if and only if $\overline{\chi}(\boldsymbol{s}_{-m\alpha_r^{\vee}})$ is equal to $-q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi)$ (respectively $q^{1/2} \cdot \boldsymbol{\gamma}_{\psi}(\varpi)$). □

We summarize the results in this section as follows:

Theorem 5.3. *Consider the Brylinski–Deligne covering group $\overline{\mathrm{Sp}}_{2r}^{(n)}$, where $r \ge 2$, and $n \ge 1$. Let $\overline{\chi}$ be an unramified exceptional character, then $\dim \mathrm{Wh}_{\psi}(\Theta(\overline{\mathrm{Sp}}_{2r}^{(n)}, \overline{\chi}))$ is equal to* 1 *if and only if the following hold*:

- *$n = 4r-2$ and $\overline{\chi}$ is the unique exceptional character satisfying (20), or*
- *$n = 4r$ and $\overline{\chi}$ is any exceptional character of $\overline{\mathrm{Sp}}_{2r}^{(4r)}$, or*

- *$n = 4r+2$ and $\overline{\chi}$ is the unique exceptional character from Proposition 5.2, or*
- *$n = 2r+1$ and $\overline{\chi}$ is the only exceptional character of $\overline{\mathrm{Sp}}_{2r}^{(2r+1)}$.*

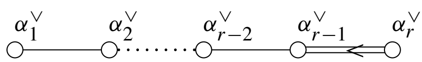
Further, consider the exceptional character $\overline{\chi}_{\psi_a} := \overline{\chi}^0_{\psi_a} \cdot \delta_B^{1/2n}$ associated with ψ_a. Assume ψ_a has conductor O_F, i.e., $a \in O_F^\times$. Then,

$$\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r-2)}, \overline{\chi}_{\psi_a})) = 1$$

if and only if $a \in (-1)^{r+1} \cdot (O_F^\times)^2$, and $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Sp}}_{2r}^{(4r+2)}, \overline{\chi}_{\psi_a})) = 1$ if and only if $a \in (-1)^r \cdot (O_F^\times)^2$.

6. The B_r, $r \geq 2$ case

Consider the Dynkin diagram for the simple coroots for the type B_r group Spin_{2r+1}:



Let $Y = \langle \alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$ be the cocharacter lattice of Spin_{2r+1}, where $\alpha_r^\vee$ is the long coroot. Let Q be the Weyl-invariant quadratic form on Y such that $Q(\alpha_r^\vee) = 2$, i.e., $Q(\alpha_i^\vee) = 1$ for $1 \leq i \leq r-1$. Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 4 & \text{if } i = j = r; \\ 2 & \text{if } 1 \leq i = j \leq r-1; \\ -1 & \text{if } 1 \leq i \leq r-2 \text{ and } j = i+1; \\ -2 & \text{if } i = r-1,\ j = r; \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

The map $\boldsymbol{i}_B : \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^r \mathbb{Z}e_i$ is given by

$$\boldsymbol{i}_B : (x_1, x_2, x_3, \ldots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{r-1} - x_{r-2}, 2x_r - x_{r-1}).$$

In particular, any $(y_1, \ldots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i$ is equal to $\boldsymbol{i}_B(y)$ for some y if and only if $2|(\sum_i y_i)$.

The Weyl group is $W = S_r \rtimes (\mathbb{Z}/2\mathbb{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbb{Z}e_i$ and $(\mathbb{Z}/2\mathbb{Z})_i : e_i \mapsto \pm e_i$. In particular, $\mathbb{w}_{\alpha_i}$, $1 \leq i \leq r-1$, acts on $(y_1, y_2, \ldots, y_r) \in \bigoplus_i \mathbb{Z}e_i$ by exchanging y_i and y_{i+1}. Also, $\mathbb{w}_{\alpha_r}$ acts by (-1) on $\mathbb{Z}e_r$.

A simple computation gives

$$Y_{Q,n} = \left\{(y_1, y_2, \ldots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i : 2 \left| \left(\sum_{i=1}^r y_i\right)\right., \ y_1 \equiv \cdots \equiv y_r \bmod n,\ n|2y_i \text{ for all } i.\right\},$$
$$Y_{Q,n}^{\mathrm{sc}} = \left\{(y_1, y_2, \ldots, y_r) \in \bigoplus_{i=1}^r \mathbb{Z}e_i : 2 \left| \left(\sum_{i=1}^r y_i\right)\right., \ n|y_i \text{ for all } i.\right\}$$

In particular, if n is odd, then $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$.

We note that $2\rho = \sum_{i=1}^{r} 2(r-i+1)e_i$, and therefore $\rho = \sum_{i=1}^{r}(r-i+1)e_i$. If $y = (x_1, x_2, \ldots, x_r) \in \bigoplus_i \mathbb{Z}\alpha_i^\vee$, then

$$\begin{aligned}\boldsymbol{i}_B(y_\rho) = (x_1 - (r-1+1), x_2 - x_1 - (r-2+1), \ldots, x_i - x_{i-1} - (r-i+1),\\ \ldots, x_{r-1} - x_{r-2} - (r-(r-1)+1), 2x_r - x_{r-1} - (r-r+1))\\ =: (x_1^*, x_2^*, \ldots, x_i^*, \ldots x_{r-1}^*, x_r^*).\end{aligned}$$

Any $(x_1^*, \ldots, x_r^*) \in \bigoplus_i \mathbb{Z}e_i$ such that $2 \mid \big(\sum_i x_i^* + r(r+1)/2\big)$ is equal to $\boldsymbol{i}_B(y_\rho)$ for some y.

6A. ***The case where n is odd.*** Here,

$$nY = Y_{Q,n}^{\mathrm{sc}} = Y_{Q,n}.$$

Therefore, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}_{2r+1}^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)|$, where $\overline{\chi}$ is the only exceptional character of $\overline{\mathrm{Spin}}_{2r+1}^{(n)}$. For n odd, the dual group for $\overline{\mathrm{Spin}}_{2r+1}^{(n)}$ is PGSp_{2r}.

Proposition 6.1. *Letting n be an odd number, one has*

$$\begin{cases} |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| \geq 2 & \textit{if } n \geq 2r+3,\\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| = 0 & \textit{if } n \leq 2r-1,\\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| = 1 & \textit{if } n = 2r+1. \end{cases}$$

Therefore, when n is odd, we have $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}_{2r+1}^{(n)}, \overline{\chi})) = 1$ *if and only if* $n = 2r+1$.

Proof. First, assume that $n \geq 2r+3$. We write

$$\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \ldots, x_i^*, \ldots, x_r^*), \quad \text{with } 2 \,\Big|\, \Big(\sum_{i=1}^{r} x_i^* + \frac{r(r+1)}{2}\Big).$$

For $r \geq 3$, let $y \in Y$ y' be such that $\boldsymbol{i}_B(y_\rho) = (1, 2, 3, \ldots, r-2, r-1, r)$ and y' be such that $\boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1)$). For $r = 2$, we take $(x_1^*, x_2^*) = (1, 2)$ or $(2, 3)$, and let y and y' be the associated element in Y respectively. In any case, the two orbits $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are $Y_{Q,n}$-free. Moreover, $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, for $n \geq 2r+3$, one has

$$|\wp_{Q,n}(\mathcal{O}_{Q,n,\mathrm{sc}}^F)| \geq 2.$$

Second, assuming that $n \leq 2r-1$, we want to show that $\mathcal{O}_{Q,n,\mathrm{sc}}^F = \varnothing$. If $\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \ldots, x_i^*, \ldots, x_r^*)$ is such that $x_i^* \equiv x_j^* \bmod n$ for some $i \neq j$, then clearly $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\mathrm{sc}}^F$. Suppose $n \nmid (x_i^* - x_j^*)$ for all $i \neq j$, then since $n \leq 2r-1$, it is not hard to see that there always exist i, j such that $n | (x_j^* + x_i^*)$. That is, $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\mathrm{sc}}^F$ for any $\mathcal{O}_y$.

Third, if $n = 2r+1$, consider the orbit $\mathcal{O}_y$ with

$$\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \ldots, x_{r-1}^*, x_r^*) = (1, 2, 3, \ldots, r-2, r-1, r).$$

(For $r = 2$, consider $\boldsymbol{i}_B(y_\rho) = (1, 2)$.) One has $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$, and therefore $|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}})| = 1$ for $n = 2r+1$. $\square$

6B. ***The case where n is even.*** Write $n = 2m$. Here,

$$Y = \bigl\{(y_1, y_2, \ldots, y_r) \in \bigoplus_i \mathbb{Z}e_i : 2 \mid \textstyle\sum_{i=1}^r y_i\bigr\}.$$

Moreover,

$$Y_{Q,n} = \Bigl\{(y_1, y_2, \ldots, y_r) \in \bigoplus_i \mathbb{Z}e_i : 2 \Bigm| \sum_{i=1}^r y_i, \text{ if } y_i = k_i n + m \text{ for all } i \text{ or } y_i = k_i n \text{ for all } i\Bigr\},$$

$$Y^{\mathrm{sc}}_{Q,n} = \bigl\{(y_1, y_2, \ldots, y_r) \in \bigoplus_i \mathbb{Z}e_i : n \mid y_i \text{ for all } i\bigr\}.$$

We see easily that for $y_i = k_i n + m$, one has $(y_1, y_2, \ldots, y_r) \in Y_{Q,n}$ if and only if $2 \mid (rm)$. In fact, for n even, the dual group for $\overline{\mathrm{Spin}}^{(n)}_{2r+1}$ is equal to SO_{2r+1} if m and r are both odd; otherwise, the dual group is Spin_{2r+1}, see [Weissman 2015]. We discuss case by case according to the parities of r and m.

6B1. *The case where m and r are odd.* In particular, one has $r \geq 3$. In this case, $Y_{Q,n} = Y^{\mathrm{sc}}_{Q,n}$, and $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n}) = \wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}})$. Consider the following situations:

- If $n > 2(r+1)$ (i.e., $m > r+1$ and therefore $m \geq r+2$), consider y such that $\boldsymbol{i}_B(y_\rho) = (x_1^*, x_2^*, \ldots, x_r^*)$ is equal to

  $$(1, 2, \ldots, r-2, r-1, r) \quad \text{or} \quad (1, 2, \ldots, r-2, r, r+1).$$

  We can check the two orbits $\mathcal{O}_y$ for these two choices of y are $Y_{Q,n}$-free, and moreover their images with respect to the map $\wp_{Q,n}$ are distinct in $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})$. Thus, $|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})| \geq 2$ in this case.
- If $n < 2r$ (i.e., $m < r$ and so $m \leq r-2$), one can check that $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2r$ (note $n \neq 2(r+1)$), i.e., $m = r$, one can also check $\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n,\mathrm{sc}}) = \varnothing$.

Therefore, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) \neq 1$ for both r and m odd.

6B2. *The case where m is odd and $r \geq 2$ is even.* Here, $Y_{Q,n} \neq Y^{\mathrm{sc}}_{Q,n}$. One has the following situations:

- Assume $n > 2(r+1)$ (i.e., $m > r+1$ and thus $m \geq r+3$).

  Case I: If $r \geq 3$, consider y and y' such that

  $$\boldsymbol{i}_B(y_\rho) = (1, 2, \ldots, r-2, r-1, r) \quad \text{and} \quad \boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1).$$

  We can check the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$-free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}^{\digamma}_{Q,n})| \geq 2$.

Case II: If $r = 2$ and $m \geq r+5$, consider $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ with $\boldsymbol{i}_B(y_\rho) = (1, 2)$ and $\boldsymbol{i}_B(y'_\rho) = (2, 3)$. Then as in the preceding case, they are $Y_{Q,n}$-free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$.

Case III: If $r = 2$ and $m = 5$, consider $\mathcal{O}_y$ with $\boldsymbol{i}_B(y_\rho) = (1, 2)$. It is easy to check $\wp_{Q,n}(\mathcal{O}^F_{Q,n}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. On the other hand, let z, z' be such that $\boldsymbol{i}_B(z_\rho) = (1, 4)$ and $\boldsymbol{i}_B(z'_\rho) = (2, 3)$. Then

$$\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y), \wp_{Q,n}(\mathcal{O}_z), \wp_{Q,n}(\mathcal{O}_{z'})\},$$

which is a set of size 3. Note, $\mathcal{O}_z, \mathcal{O}_{z'} \in \mathcal{O}^F_{Q,n,\mathrm{sc}} \backslash \mathcal{O}^F_{Q,n}$. That is, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| = 1$ and $|\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}})| = 3$ in this case.

Let $\mathbb{w}, \mathbb{w}' \in W$ be such that

$$\boldsymbol{i}_B(\mathbb{w}[z] - z) = \boldsymbol{i}_B(\mathbb{w}'[z'] - z') = -(5, 5) \in Y_{Q,n}.$$

Write $y_{Q,n} = \boldsymbol{i}_B(\mathbb{w}[z] - z) \in Y_{Q,n}$. Then, $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(10)}_5, \overline{\chi}))$ is equal to 1, as in Section 5B5, if and only if

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) \neq \varepsilon^{D(y_{Q,n},z)} \cdot \boldsymbol{T}(\mathbb{w}, z) \quad \text{and} \quad \overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) \neq \varepsilon^{D(y_{Q,n},z')} \cdot \boldsymbol{T}(\mathbb{w}', z'). \tag{22}$$

However, as in Proposition 5.2, that $\varepsilon^{D(y_{Q,n},z)} \cdot \boldsymbol{T}(\mathbb{w}, z) = \varepsilon^{D(y_{Q,n},z')} \cdot \boldsymbol{T}(\mathbb{w}', z')$ can be easily checked, and the condition (22) is equivalent to

$$\overline{\chi}(\boldsymbol{s}_{-5\alpha_r^\vee}) = -q^{1/2} \cdot \boldsymbol{\gamma}_\psi(\varpi). \tag{23}$$

This agrees with the result from Proposition 5.2 for the $\overline{C}^{(10)}_2$ case.

- If $n < 2r$ (i.e., $m \leq r$ and therefore $m \leq r-1$), one can check $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2(r+1)$ (note $n \neq 2r$), i.e., $r = m-1$, one can check $\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp^{\mathrm{sc}}_{Q,n}(\mathcal{O}_0)\}$ (and thus $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_0)\}$) is a singleton with

$$\boldsymbol{i}_B(0_\rho) = (-r, -(r-1), \ldots, -2, -1).$$

That is, $\mathcal{O}_0$ is $Y^{\mathrm{sc}}_{Q,n}$-free. However, it is not $Y_{Q,n}$-free, since there exists $\mathbb{w} \in W$ such that $\boldsymbol{i}_B(\mathbb{w}(0_\rho)) = (1, 2, \ldots, r-1, r)$. It follows that

$$\boldsymbol{i}_B(\mathbb{w}(0_\rho) - 0_\rho) = (m, m, \ldots, m, m) \in Y_{Q,n}.$$

Write $y_{Q,n} = \mathbb{w}(0_\rho) - 0_\rho = \mathbb{w}[0] - 0$. It follows from an analogous argument for Proposition 4.1 that $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(2r+2)}_{2r+1}, \overline{\chi})) = 1$ if and only if $\overline{\chi}$ is the unique exceptional character satisfying

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) = \boldsymbol{T}(\mathbb{w}, 0). \tag{24}$$

One can explicate the equality by computing the right-hand side as in Lemma 4.2. We omit the details here.

6B3. *The case where m is even and $r \geq 3$ is odd.* Here, $Y_{Q,n} \neq Y^{\mathrm{sc}}_{Q,n}$. We have:

- If $n > 2(r+1)$ (i.e., $m > r+1$ and therefore $m \geq r+3$), consider y and y' such that

$$\boldsymbol{i}_B(y_\rho) = (1, 2, \ldots, r-2, r-1, r) \quad \text{and} \quad \boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1).$$

  We can check the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$-free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$.
- If $n < 2r$ (i.e., $m < r$ and therefore $m \leq r-1$), one can check $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2(r+1)$ (note $n \neq 2r$), i.e., $r = m-1$, then $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ is a singleton with

$$\boldsymbol{i}_B(0_\rho) = (-r, -(r-1), \ldots, -2, -1).$$

  The situation is exactly as in the third case of Section 6B2, i.e., $\mathcal{O}_0$ is $Y^{\mathrm{sc}}_{Q,n}$-free but not $Y_{Q,n}$-free. Consider $\mathbb{w} \in W$ such that $\boldsymbol{i}_B(\mathbb{w}(0_\rho)) = (1, 2, \ldots, r-1, r)$ and

$$\boldsymbol{i}_B(\mathbb{w}(0_\rho) - 0_\rho) = (m, m, \ldots, m, m) \in Y_{Q,n}.$$

Write $y_{Q,n} = \mathbb{w}(0_\rho) - 0_\rho = \mathbb{w}[0] - 0$. Then $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(2r+2)}_{2r+1}, \overline{\chi})) = 1$ if and only if $\overline{\chi}$ is the unique exceptional character satisfying

$$\overline{\chi}(\boldsymbol{s}_{y_{Q,n}}) = \boldsymbol{T}(\mathbb{w}, 0). \tag{25}$$

6B4. *The case where m is even and $r \geq 2$ is even.* Here, $Y_{Q,n} \neq Y^{\mathrm{sc}}_{Q,n}$. One has the following situations:

- If $n > 2(r+1)$ (i.e., $m > r+1$ and therefore $m \geq r+2$), there are two cases to consider.

  <u>Case I</u>: $r \geq 4$. Consider y and y' such that

$$\boldsymbol{i}_B(y_\rho) = (1, 2, \ldots, r-2, r-1, r) \quad \text{and} \quad \boldsymbol{i}_B(y'_\rho) = (1, 2, \ldots, r-2, r, r+1).$$

  We can check easily that the orbits $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ for these two choices are $Y_{Q,n}$-free. Note that $|\wp_{Q,n}(\mathcal{O}^F_{Q,n})| \geq 2$, since $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$.

  <u>Case II</u>: $r = 2$. Consider y and y' such that $\boldsymbol{i}_B(y_\rho) = (1, 2)$ and $\boldsymbol{i}_B(y'_\rho) = (2, 3)$. For $m \geq 4$, $\mathcal{O}_y$ and $\mathcal{O}_{y'}$ are both $Y_{Q,n}$-free. Moreover, we can check that $\wp_{Q,n}(\mathcal{O}^F_{Q,n,sc}) \subseteq \{\wp_{Q,n}(\mathcal{O}_y), \wp_{Q,n}(\mathcal{O}_{y'})\}$. Now if $m \geq 6$, then $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. On the other hand, for $m = 4$, one has $\wp_{Q,n}(\mathcal{O}_y) = \wp_{Q,n}(\mathcal{O}_{y'})$ and so $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(8)}_5, \overline{\chi})) = 1$ for any exceptional character $\overline{\chi}$ in this case.

  To summarize for the case $m \geq r+2$:

$$\begin{cases} \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) = 1 & \text{if } m = 4, r = 2, \\ \dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) \geq 2 & \text{if } r \geq 4 \text{ and } m \geq r+2, \text{ or } r = 2 \text{ and } m \geq 6. \end{cases}$$

- If $n < 2r$ (i.e., $m < r$ and therefore $m \leq r-2$), one can check easily that $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.
- If $n = 2r$ (note $n \neq 2(r+1)$), i.e., $r = m$, one also has $\wp_{Q,n}(\mathcal{O}^F_{Q,n,\mathrm{sc}}) = \varnothing$.

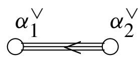
From the above discussion, we observe that for $r = 2$, the result agrees with that for covering groups of type C_2, as expected. Therefore, we just summarize our result for covering $\overline{\mathrm{Spin}}^{(n)}_{2r+1}$ with $r \geq 3$ as follows.

Theorem 6.2. *Consider Brylinski–Deligne covering* $\overline{\mathrm{Spin}}^{(n)}_{2r+1}$ *with* $r \geq 3$. *Let* $\overline{\chi}$ *be an exceptional character, then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{\mathrm{Spin}}^{(n)}_{2r+1}, \overline{\chi})) = 1$ *if and only if one of the following holds*:

- $n = 2(r+1)$ *and* $\overline{\chi}$ *is the unique exceptional character satisfying* (24) *or* (25),
- $n = 2r+1$ *and* $\overline{\chi}$ *is the only exceptional character of* $\overline{\mathrm{Spin}}^{(2r+1)}_{2r+1}$.

7. The G_2 case

Consider G_2 with Dynkin diagram for its simple coroots:



Let $Y = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$ be the cocharacter lattice of G_2, where $\alpha_1^\vee$ is the short coroot. Let Q be the Weyl-invariant quadratic on Y such that $Q(\alpha_1^\vee) = 1$ (thus $Q(\alpha_2^\vee) = 3$). Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = 1, \\ -3 & \text{if } i = 1, j = 2, \\ 6 & \text{if } i = j = 2. \end{cases}$$

A simple computation gives

$$Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}} = \mathbb{Z}(n_{\alpha_1}\alpha_1^\vee) \oplus \mathbb{Z}(n_{\alpha_2}\alpha_2^\vee),$$

where $n_{\alpha_2} = n/\gcd(n,3)$ and $n_{\alpha_1} = n$.

The map $\boldsymbol{i}_G : \bigoplus_{i=1}^2 \mathbb{Z}\alpha_i^\vee \to \bigoplus_{i=1}^3 \mathbb{Z}e_i$ is given by

$$\boldsymbol{i}_G : (x_1, x_2) \mapsto (x_1 - 2x_2, x_2 - x_1, x_2).$$

Any $(y_i)_i \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$ lies in the image of $\boldsymbol{i}_G$ if and only if $y_1 + y_2 + y_3 = 0$.

The Weyl group $W = \langle \mathbb{w}_{\alpha_1}, \mathbb{w}_{\alpha_2} \rangle$ generated by $\mathbb{w}_{\alpha_1}$ and $\mathbb{w}_{\alpha_2}$ is the dihedral group of order 12. In particular, $\mathbb{w}_{\alpha_1}(y_1, y_2, y_3) = (y_2, y_1, y_3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$, and $\mathbb{w}_{\alpha_2}(y_1, y_2, y_3) = (-y_1, -y_3, -y_2)$.

By using $\boldsymbol{i}_G$, we could identify

$$Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}} = \left\{ (y_1, y_2, y_3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i : y_1 + y_2 + y_3 = 0,\ y_1 \equiv y_2 \equiv y_3 \bmod n \right\}.$$

We have $\rho = 5\alpha_1^\vee + 3\alpha_2^\vee$ with $\boldsymbol{i}_G(\rho) = (-1, -2, 3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$. It follows that for any $y = (x_1, x_2) \in \bigoplus_{i=1}^2 \mathbb{Z}\alpha_i^\vee$,

$$\boldsymbol{i}_G(y_\rho) = (x_1 - 2x_2 - 1, x_2 - x_1 - 2, x_2 + 3) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i.$$

We may write $\boldsymbol{i}_G(y_\rho) = (x_1^*, x_2^*, x_3^*)$. In particular, $(x_1^*, x_2^*, x_3^*) \in \bigoplus_{i=1}^3 \mathbb{Z}e_i$ lies in the image of $\boldsymbol{i}_G$ if and only if $x_1^* + x_2^* + x_3^* = 0$.

Since $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$, it follows that $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$, where $\overline{\chi}$ is the only exceptional character of $\overline{G}_2^{(n)}$ as $Z(\overline{G}_2^\vee) = 1$.

To determine the n such that $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$, we only give an outline of the argument, the details of which consists of basic combinatorial computations:

- For $n = 7, 8$ or $n \geq 10$, the orbit $\mathcal{O}_y$ with $\boldsymbol{i}_G(y_\rho) = (-2, -1, 3)$ is $Y_{Q,n}$-free.
- For $n = 8, 10, 11$ or $n \geq 13$, the orbit $\mathcal{O}_{y'}$ with $\boldsymbol{i}_G(y'_\rho) = (-3, -1, 4)$ is $Y_{Q,n}$-free. Moreover, for $n = 8, 10, 11$ or $n \geq 13$, one has $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$ for $\boldsymbol{i}_G(y_\rho) = (-2, -1, 3)$ and $\boldsymbol{i}_G(y'_\rho) = (-3, -1, 4)$.
- If $\mathcal{O}_{Q,n,\mathrm{sc}}^F \neq \varnothing$, then necessarily $|Y/Y_{Q,n}^{\mathrm{sc}}| \geq |W|$, i.e., $n \cdot n_{\alpha_2} \geq 12$. Thus $n \geq 4$.
- One can also check by hand that $\mathcal{O}_{Q,n,\mathrm{sc}}^F = \varnothing$ for $n = 4, 5, 6, 9$.
- For $n = 7$ or 12, $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ with $\boldsymbol{i}_G(y_\rho) = (-2, -1, 3)$, i.e., $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$ for $n = 7$ or 12.

To summarize:

Theorem 7.1. *Consider the Brylinski–Deligne covering $\overline{G}_2^{(n)}$. Let $\overline{\chi}$ be the only exceptional character on $\overline{G}_2^{(n)}$, then* $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$ *if and only if $n = 7$ or* 12.

Acknowledgements. I would like to thank Solomon Friedberg, David Goldberg and Freydoon Shahidi for interesting discussions on various topics. Thanks are also due to Wee Teck Gan for helpful comments on the paper. Meanwhile, I would like to thank Boston College for its support and hospitality during a visit in the Fall semester. Finally, I would like to thank the referee for very helpful and detailed comments which greatly improve the exposition of the paper.

References

[Adams 2003] J. Adams, "Characters of covering groups of SL(n)", *J. Inst. Math. Jussieu* **2**:1 (2003), 1–21. MR Zbl

[Ban and Jantzen 2013] D. Ban and C. Jantzen, "The Langlands quotient theorem for finite central extensions of p-adic groups", *Glas. Mat. Ser. III* **48**:2 (2013), 313–334. MR Zbl

[Bourbaki 2002] N. Bourbaki, *Lie groups and Lie algebras, chapters 4–6*, Springer, Berlin, 2002. MR Zbl

[Brubaker et al. 2011] B. Brubaker, D. Bump, and S. Friedberg, "Weyl group multiple Dirichlet series, Eisenstein series and crystal bases", *Ann. of Math.* (2) **173**:2 (2011), 1081–1120. MR Zbl

[Brubaker et al. 2016] B. Brubaker, V. Buciumas, and D. Bump, "A Yang–Baxter equation for metaplectic ice", preprint, 2016. arXiv

[Brylinski and Deligne 2001] J.-L. Brylinski and P. Deligne, "Central extensions of reductive groups by $\mathbf{K_2}$", *Publ. Math. Inst. Hautes Études Sci.* 94 (2001), 5–85. MR Zbl

[Budden 2006] M. Budden, "Local coefficient matrices of metaplectic groups", *J. Lie Theory* **16**:2 (2006), 239–249. MR Zbl

[Bump and Ginzburg 1992] D. Bump and D. Ginzburg, "Symmetric square L-functions on $\mathrm{GL}(r)$", *Ann. of Math.* (2) **136**:1 (1992), 137–205. MR Zbl

[Bump and Hoffstein 1987] D. Bump and J. Hoffstein, "On Shimura's correspondence", *Duke Math. J.* **55**:3 (1987), 661–691. MR Zbl

[Bump et al. 1990] D. Bump, S. Friedberg, and J. Hoffstein, "Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives", *Ann. of Math.* (2) **131**:1 (1990), 53–127. MR Zbl

[Bump et al. 1996] D. Bump, D. Ginzburg, and J. Hoffstein, "The symmetric cube", *Invent. Math.* **125**:3 (1996), 413–449. MR Zbl

[Bump et al. 2012] D. Bump, S. Friedberg, and D. Goldfeld (editors), *Multiple Dirichlet series, L-functions and automorphic forms*, Progress in Mathematics **300**, Birkhäuser, New York, 2012. MR Zbl

[Casselman and Shahidi 1998] W. Casselman and F. Shahidi, "On irreducibility of standard modules for generic representations", *Ann. Sci. École Norm. Sup.* (4) **31**:4 (1998), 561–589. MR Zbl

[Casselman and Shalika 1980] W. Casselman and J. Shalika, "The unramified principal series of p-adic groups, II: The Whittaker function", *Compositio Math.* **41**:2 (1980), 207–231. MR Zbl

[Chinta and Offen 2013] G. Chinta and O. Offen, "A metaplectic Casselman–Shalika formula for GL_r", *Amer. J. Math.* **135**:2 (2013), 403–441. MR Zbl

[Finkelberg and Lysenko 2010] M. Finkelberg and S. Lysenko, "Twisted geometric Satake equivalence", *J. Inst. Math. Jussieu* **9**:4 (2010), 719–739. MR Zbl

[Friedberg and Ginzburg 2014] S. Friedberg and D. Ginzburg, "Descent and theta functions for metaplectic groups", 2014. To appear in *J. European Math. Soc.* arXiv

[Friedberg and Ginzburg 2016a] S. Friedberg and D. Ginzburg, "Criteria for the existence of cuspidal theta representations", *Res. Number Theory* **2** (2016), art. id. 16. MR Zbl

[Friedberg and Ginzburg 2016b] S. Friedberg and D. Ginzburg, "Theta functions on covers of symplectic groups", preprint, 2016. arXiv

[Friedberg et al. $\geq$ 2017] S. Friedberg, D. Goldberg, and D. Szpruch, "Exceptional representations of higher covers of classical and similitude groups", in preparation.

[Gan and Gao 2016] W. T. Gan and F. Gao, "The Langlands–Weissman program for Brylinski–Deligne extensions", preprint, 2016, Available at www.math.nus.edu.sg/~matgwt.

[Gao $\geq$ 2017] F. Gao, "The Langlands–Shahidi L-functions for Brylinski–Deligne extensions", Available at https://tinyurl.com/LangShahL-functionsGAO. To appear in *Amer. J. Math.*

[Gelbart 1976] S. S. Gelbart, *Weil's representation and the spectrum of the metaplectic group*, Lecture Notes in Mathematics **530**, Springer, Berlin, 1976. MR Zbl

[Gelbart and Piatetski-Shapiro 1980] S. Gelbart and I. I. Piatetski-Shapiro, "Distinguished representations and modular forms of half-integral weight", *Invent. Math.* **59**:2 (1980), 145–188. MR Zbl

[Ginzburg 2006] D. Ginzburg, "Certain conjectures relating unipotent orbits to automorphic representations", *Israel J. Math.* **151** (2006), 323–355. MR Zbl

[Goldberg and Szpruch 2015] D. Goldberg and D. Szpruch, "Plancherel measures for coverings of p-adic $SL_2(F)$", preprint, 2015. arXiv

[Kaplan 2015a] E. Kaplan, "The characterization of theta-distinguished representations of GL_n", preprint, 2015. arXiv

[Kaplan 2015b] E. Kaplan, "The theta period of a cuspidal automorphic representation of $\mathrm{GL}(n)$", *Int. Math. Res. Not.* **2015**:8 (2015), 2168–2209. MR Zbl

[Kaplan 2016] E. Kaplan, "Theta distinguished representations, inflation and the symmetric square L-function", *Math. Z.* **283**:3 (2016), 909–936. MR Zbl

[Kazhdan and Patterson 1984] D. A. Kazhdan and S. J. Patterson, "Metaplectic forms", *Inst. Hautes Études Sci. Publ. Math.* 59 (1984), 35–142. MR Zbl

[Matsumoto 1969] H. Matsumoto, "Sur les sous-groupes arithmétiques des groupes semi-simples déployés", *Ann. Sci. École Norm. Sup.* (4) **2**:1 (1969), 1–62. MR Zbl

[McNamara 2012] P. J. McNamara, "Principal series representations of metaplectic groups over local fields", pp. 299–327 in *Multiple Dirichlet series, L-functions and automorphic forms*, edited by D. Bump et al., Progr. Math. **300**, Birkhäuser, New York, 2012. MR Zbl

[McNamara 2016] P. J. McNamara, "The metaplectic Casselman–Shalika formula", *Trans. Amer. Math. Soc.* **368**:4 (2016), 2913–2937. MR Zbl

[Mœglin and Waldspurger 1987] C. Mœglin and J.-L. Waldspurger, "Modèles de Whittaker dégénérés pour des groupes p-adiques", *Math. Z.* **196**:3 (1987), 427–452. MR Zbl

[Moore 1968] C. C. Moore, "Group extensions of p-adic and adelic linear groups", *Inst. Hautes Études Sci. Publ. Math.* 35 (1968), 157–222. MR Zbl

[Reich 2012] R. C. Reich, "Twisted geometric Satake equivalence via gerbes on the factorizable Grassmannian", *Represent. Theory* **16** (2012), 345–449. MR Zbl

[Rodier 1973] F. Rodier, "Whittaker models for admissible representations of reductive p-adic split groups", pp. 425–430 in *Harmonic analysis on homogeneous spaces* (Williamstown, MA, 1972), edited by C. C. Moore, Proc. Sympos. Pure Math. **26**, American Mathematical Society, Providence, RI, 1973. MR Zbl

[Shahidi 2010] F. Shahidi, *Eisenstein series and automorphic L-functions*, American Mathematical Society Colloquium Publications **58**, American Mathematical Society, Providence, RI, 2010. MR Zbl

[Shalika 1974] J. A. Shalika, "The multiplicity one theorem for GL_n", *Ann. of Math.* (2) **100** (1974), 171–193. MR Zbl

[Shimura 1973] G. Shimura, "On modular forms of half integral weight", *Ann. of Math.* (2) **97** (1973), 440–481. MR Zbl

[Steinberg 1962] R. Steinberg, "Générateurs, relations et revêtements de groupes algébriques", pp. 113–127 in *Colloque sur la théorie des groupes algébriques* (Brussels, 1962), Librairie Universitaire, Louvain, Belgium, 1962. MR Zbl

[Suzuki 1998] T. Suzuki, "Distinguished representations of metaplectic groups", *Amer. J. Math.* **120**:4 (1998), 723–755. MR Zbl

[Suzuki 2012] T. Suzuki, "On the Fourier coefficients of metaplectic forms", *Ryukyu Math. J.* **25** (2012), 21–106. MR Zbl

[Szpruch 2007] D. Szpruch, "Uniqueness of Whittaker model for the metaplectic group", *Pacific J. Math.* **232**:2 (2007), 453–469. MR Zbl

[Szpruch 2009a] D. Szpruch, "Computation of the local coefficients for principal series representations of the metaplectic double cover of $\mathrm{SL}_2(\mathbb{F})$", *J. Number Theory* **129**:9 (2009), 2180–2213. MR Zbl

[Szpruch 2009b] D. Szpruch, *The Langlands–Shahidi method for the metaplectic group and applications*, Ph.D. thesis, Tel Aviv University, 2009. arXiv

[Szpruch 2013] D. Szpruch, "Some irreducibility theorems of parabolic induction on the metaplectic group via the Langlands–Shahidi method", *Israel J. Math.* **195**:2 (2013), 897–971. MR Zbl

[Szpruch 2016] D. Szpruch, "On Shahidi local coefficients matrix", preprint, 2016. arXiv

[Takeda 2014] S. Takeda, "The twisted symmetric square L-function of $\mathrm{GL}(r)$", *Duke Math. J.* **163**:1 (2014), 175–266. MR Zbl

[Weissman 2009] M. H. Weissman, "Metaplectic tori over local fields", *Pacific J. Math.* **241**:1 (2009), 169–200. MR Zbl

[Weissman 2015] M. H. Weissman, "L-groups and parameters for covering groups", 2015. To appear in *Astérisque*. arXiv

Received June 14, 2016. Revised March 21, 2017.

FAN GAO
DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
150 N. UNIVERSITY STREET
WEST LAFAYETTE, IN 47907
UNITED STATES
gaofan.math@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.381

LIOUVILLE THEOREMS FOR f-HARMONIC MAPS INTO HADAMARD SPACES

BOBO HUA, SHIPING LIU AND CHAO XIA

We study harmonic functions on weighted manifolds and harmonic maps from weighted manifolds into Hadamard spaces introduced by Korevaar and Schoen. We prove several Liouville-type theorems for these harmonic maps.

1. Introduction

Weighted Riemannian manifolds, also called manifolds with density or smooth metric measure spaces in the literature, are Riemannian manifolds equipped with weighted measures. Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, weighted manifolds have been proven to be nontrivial generalizations of Riemannian manifolds. There are many geometric investigations of weighted manifolds; see Morgan [2005], Wei and Wylie [2009] and many others. In this paper, we investigate various Liouville-type theorems for harmonic functions on weighted manifolds as well as harmonic maps from weighted manifolds into Hadamard spaces, i.e., globally nonpositively curved spaces in the sense of Alexandrov (also called CAT(0) spaces), see, e.g., [Jost 1997b; Burago et al. 2001].

A weighted Riemannian manifold is a triple $(M, g, e^{-f} dV_g)$, where (M, g) is an n-dimensional Riemannian manifold, dV_g is the Riemannian volume element induced by the metric g and f is a smooth positive function on M. The f-Laplacian

$$\Delta_f = \Delta - \nabla f \cdot \nabla$$

is a natural generalization of Laplace–Beltrami operator Δ as it is self-adjoint with

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 267087. B. H. is partially supported by NSFC (Grant No. 11401106). S. L. is partially supported by the EPSRC Grant EP/K016687/1. C. X. is partially supported by the Fundamental Research Funds for the Central Universities (Grant No. 20720150012) and NSFC (Grant No. 11501480).
MSC2010: primary 53C43; secondary 31C05.
Keywords: harmonic map, Liouville theorem, Hadamard space.

respect to the weighted measure $e^{-f}dV_g$, i.e.,

$$\int_M \Delta_f uve^{-f}\,dV_g = \int_M u\Delta_f ve^{-f}\,dV_g \quad \text{for } u, v \in C_0^\infty(M).$$

A function $u \in W^{1,2}_{\mathrm{loc}}(M)$ is called f-harmonic (f-subharmonic, f-superharmonic resp.) if it satisfies $\Delta_f u = 0$ (≥ 0, ≤ 0 resp.) in the weak sense, i.e.,

$$\int_M \langle \nabla u, \nabla\varphi\rangle e^{-f}\,dV_g = 0 \quad (\leq 0,\ \geq 0 \text{ resp.}) \text{ for any } 0 \leq \varphi \in C_0^\infty(M).$$

The Dirichlet f-energy of u is defined by

$$D^f(u) = \int_M |\nabla u|^2 e^{-f}\,dV_g.$$

On the other hand, f-harmonic maps from weighted manifolds $(M, g, e^{-f}dV_g)$ to (singular) metric spaces (Y, d) have wide geometric applications. Harmonic maps into metric spaces were initiated by Gromov and Schoen [1992] and then investigated independently by Korevaar and Schoen [1993] and Jost [1994]. In particular, when the domain is a Riemannian manifold, Korevaar and Schoen [1993; 1997] gave a complete exposition. In this paper we call a map $u : M \to Y$ f-harmonic if u locally minimizes the f-energy functional E^f in the sense of Korevaar and Schoen. For a detailed definition and its properties, we refer to [Korevaar and Schoen 1993] or Section 4 below. For the special case, f-harmonic maps from the Gaussian spaces, $(\mathbb{R}^n, |\cdot|, e^{-|x|^2/4}\,dx)$, to Riemannian manifolds are called quasiharmonic spheres, which emerge in the blowup analysis of harmonic heat flow [Lin and Wang 1999; Li and Tian 2000]. In this paper, we study Liouville theorems for f-harmonic maps into metric spaces, which generalize the previous results for harmonic maps in both aspects of domain manifolds and target spaces.

Analysis on weighted manifolds and the corresponding f-Laplacian have been extensively studied recently. We refer to [Munteanu and Wang 2011, 2012; Brighton 2013; Li 2005, 2016] for the f-harmonic functions on weighted manifolds, to [Li and Wang 2009; Zhu and Wang 2010; Li and Zhu 2010; Li and Yang 2012] for f-harmonic functions on the Gaussian spaces, to [Grigor'yan 2006, 2009] for heat kernel estimates, and to [Lin and Wang 1999; Wang and Xu 2012; Chen et al. 2012; Rimoldi and Veronelli 2013; Sinaei 2014, 2016] for f-harmonic maps.

In the first part of the paper we are concerned with Liouville-type theorems for f-harmonic functions on weighted manifolds. Several Liouville-type theorems for f-harmonic functions on the Gaussian spaces, also called quasiharmonic functions, have been proved in [Zhu and Wang 2010; Li and Wang 2009], in which the main techniques adopted are gradient estimates and separation of variables coupled with ODE results. In this paper, we propose another approach, which seems to be overlooked in the literature, to reprove many previous results. This method can be easily generalized, so that we may obtain Liouville theorems for f-harmonic functions for a large class of weighted manifolds; see Section 2.

Our observation is that the weighted version of L^p-Liouville theorem for weighted manifolds can be used to derive various Liouville theorems concerning the growth of f-harmonic functions. Yau [1976] first proved the L^p-Liouville theorem (for $1 < p < \infty$) for harmonic functions on any complete Riemannian manifold. Later, Karp [1982] obtained a quantitative version of this result. Li and Schoen [1984] proved other L^p-Liouville theorems (e.g., $0 < p < 1$) under the curvature assumption of manifolds. Karp's version of L^p-Liouville theorem has been generalized by Sturm [1994] to the setting of strongly local regular Dirichlet forms. In particular, our f-harmonic functions lie in this setting. By applying Sturm's L^p-Liouville theorem to f-harmonic functions, we immediately obtain several consequences which generalize previous results of [Zhu and Wang 2010; Li and Wang 2009; Li and Zhu 2010; Li and Yang 2012]. Although the proof of L^p-Liouville theorem is quite general and only involves integration by parts and the Caccioppoli inequality (thus it holds for all reasonable spaces), it is surprisingly powerful to obtain various Liouville theorems for weighted manifolds with slow volume growth, especially for the Gaussian spaces; see Corollaries 2.5 and 2.6 in Section 2. This does provide another approach to derive Liouville theorems without using any gradient estimate.

In the second part, we study Liouville-type theorems for harmonic maps from weighted manifolds to Hadamard spaces. For applications of f-harmonic maps with singular targets we refer to Gromov and Schoen [1992]. Our first result is an analogue to Kendall's theorem [1990, Theorem 3.2]. The essence of Kendall's theorem is that validity of a Liouville theorem for f-harmonic maps into Hadamard spaces, a priori a nonlinear problem, is reduced to that of a Liouville theorem of f-harmonic functions, a linear problem. Kendall [1990] proved this theorem for harmonic maps between Riemannian manifolds, by using probabilistic methods and potential theory. Kuwae and Sturm [2008] generalized Kendall's method to a class of harmonic maps between general metric spaces in the framework of Markov processes. Note that the harmonic maps they were concerned with are different from those of Korevaar and Schoen [1993] when targets are singular. In this paper, we consider harmonic maps into Hadamard spaces in the sense of Korevaar and Schoen. Following the argument by Li and Wang [1998], we are able to prove the following Kendall-type theorem by assuming local compactness of the targets. Recall that a geodesic space (Y, d) is called locally compact if every closed geodesic ball is compact.

Theorem 1.1. *Let $(M, g, e^{-f}\,dV_g)$ be a complete weighted Riemannian manifold satisfying that any bounded f-harmonic function is constant. Let (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y having bounded image is a constant map.*

In the same spirit as Kendall's theorem, Cheng, Tam and Wan [Cheng et al. 1996] proved a Liouville-type theorem for harmonic maps with finite energy. Our second result is a generalization of their theorem to f-harmonic maps into Hadamard spaces.

Theorem 1.2. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying that any f-harmonic function with finite Dirichlet f-energy is bounded. Let (Y, d) be an Hadamard space. Then any f-harmonic map from M to Y with finite f-energy has bounded image.*

We will follow the line of Cheng, Tam and Wan's reasoning, but using the techniques in potential theory, especially the theory of Royden and Nakai's decomposition on Riemannian manifolds [Royden 1952; Nakai 1960; Sario and Nakai 1970]. This possible approach of potential theory was implicitly suggested by Lyons in [Cheng et al. 1996, pp. 278]. We figure out the detailed arguments of this insight and apply them to Liouville theorems of f-harmonic maps. The Royden–Nakai decomposition theorem and Virtanen's theorem, see, e.g., Section 5 for weighted versions, play important roles in the classification theory of Riemannian manifolds developed by Royden, Nakai, Sario et al. many years ago. We shall dwell on these theories in the framework of weighted manifolds in Section 5 and utilize them to prove Theorem 1.2.

The following theorem is, more or less, a consequence of the combination of Theorems 1.1 and 1.2.

Theorem 1.3. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying that any bounded f-harmonic functions is constant. Let (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y with finite f-energy is a constant map.*

This theorem has an interesting application which motivates our studies in some sense. Bakry and Émery [1985] introduced weighted Ricci curvature for weighted manifolds. In particular, the so-called ∞-Bakry–Émery Ricci curvature

$$\mathrm{Ric}_f := \mathrm{Ric} + \nabla^2 f$$

turns out to be a suitable and important curvature quantity for weighted manifolds. The nonnegativity of Ric_f corresponds to the curvature-dimension condition $CD(0, \infty)$ on metric measure spaces via optimal transport, in the sense of Lott and Villani [2009] and Sturm [2006a; 2006b]. By a theorem of Brighton [2013], see also [Li 2016], the weighted manifold $(M, g, e^{-f} dV_g)$ satisfying $\mathrm{Ric}_f \geq 0$ admits no nonconstant bounded f-harmonic functions. Hence by Theorem 1.3 we immediately have:

Theorem 1.4. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact weighted Riemannian manifold satisfying $\mathrm{Ric}_f \geq 0$ and (Y, d) be a locally compact Hadamard space. Then any f-harmonic map from M to Y with finite f-energy is a constant map.*

The novelty of the result lies in the generality of targets, i.e., including singular metric spaces. In the smooth setting, Hadamard spaces are in fact Cartan–Hadamard manifolds, i.e., simply connected Riemannian manifolds with nonpositive sectional curvature. On Riemannian manifolds, Theorem 1.4 has been proved by Wang and Xu [2012] and Rimoldi and Veronelli [2013] independently under an additional assumption of $\int_M e^{-f}\, dV_g = \infty$ for domain manifolds, while simply-connectedness of the targets is not needed. Note that the weighted volume assumption here cannot be derived from the curvature condition $\mathrm{Ric}_f \geq 0$ in general. In addition, there is a nontrivial f-harmonic map from a domain manifold with $\mathrm{Ric}_f \geq 0$ and $\int_M e^{-f}\, dV_g < \infty$ to a nonpositively curved target manifold, constructed by Rimoldi and Veronelli [2013, Remark 3.7]. Our contribution is to drop the weighted volume assumption by assuming simply-connectedness of the targets and to extend the result to singular spaces.

For harmonic maps into singular Hadamard spaces, the arguments in [Wang and Xu 2012; Rimoldi and Veronelli 2013], both following Schoen and Yau [1976], do not work any more since we cannot apply Bochner techniques as in those works due to the singularity of targets. Although a weak Bochner formula can also be derived following Korevaar and Schoen [1993], it is insufficient for our purpose. Fortunately, we can circumvent these technical problems by proving Theorem 1.3, which follows from Kendall-type theorems. This does provide another approach to Liouville theorems for f-harmonic maps without using Bochner techniques. This is one of the main points of the paper.

The rest of the paper is organized as follows. In Section 2, we study L^p Liouville theorem for f-harmonic functions and give some applications. In Section 3, we consider harmonic maps with smooth targets. In Section 4, we define f-harmonic maps into Hadamard spaces and prove Theorem 1.1. In Section 5, we dwell on the Royden-Nakai theory and prove Theorems 1.2 and 1.3.

2. f-harmonic functions

In this section, we study L^p-Liouville theorems for f-harmonic functions and their applications. We will show that L^p-Liouville theorems are quite powerful for weighted manifolds with finite volume.

The L^p-Liouville theorem, $1 < p < \infty$, for harmonic functions (or nonnegative subharmonic functions) was initiated by Yau [1976] on complete Riemannian manifolds. Karp [1982] obtained a quantitative version of this Liouville theorem. Later, Sturm [1994] proved an L^p-Liouville theorem for strongly local regular Dirichlet forms. The following theorem is a special case of Sturm's result for f-harmonic functions. We denote by $B_r := B_r(x_0)$ the closed geodesic ball of radius r centered at a fixed point $x_0 \in M$.

Theorem 2.1 [Sturm 1994, Theorem 1]. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold and u be a nonnegative f-subharmonic function (or an f-harmonic function). For $1 < p < \infty$, set $v(r) := \int_{B_r} |u|^p e^{-f}\, dV_g$. Then either*

$$\inf_{a>0} \int_a^\infty \frac{r}{v(r)}\, dr < \infty,$$

or u is a constant.

We state several consequences of Theorem 2.1.

A quite useful consequence is about f-parabolicity of M. Recall that a weighted manifold $(M, g, e^{-f} dV_g)$ is called *f-parabolic* if there are no nonconstant nonnegative f-superharmonic functions on M. For a compact set $K \subset M$, the *f-capacity* of K is defined as

$$\operatorname{cap}^f(K) := \inf_{\substack{\varphi \in \operatorname{Lip}_0(M)\\ \varphi|_K = 1}} \int_M |\nabla \varphi|^2 e^{-f}\, dV_g,$$

where $\operatorname{Lip}_0(M)$ is the space of compactly supported Lipschitz functions on M.

Proposition 2.2 (f-parabolicity). *Let $(M, g, e^{-f} dV_g)$ be a complete weighted manifold. Then the following are equivalent:*

(i) *M is f-parabolic;*

(ii) $\operatorname{cap}^f(K) = 0$ *for some (then any) compact set $K \subset M$;*

(iii) *any bounded f-superharmonic function on M is constant.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from [Grigor'yan 1985, Proposition 3]; see also Proposition 2.1 of [Grigor'yan 1999].

(i) $\Leftrightarrow$ (iii). This follows from the fact that any nonnegative f-superharmonic function u can be approximated by bounded f-superharmonic functions $u_n = \min\{u, n\}$, $n \in \mathbb{N}$. □

We say a weighted manifold $(M, g, e^{-f} dV_g)$ has the *moderate volume growth property* if

$$\int_1^\infty \frac{r}{V_f(B_r)}\, dr = \infty, \tag{1}$$

where $V_f(B_r) := \int_{B_r} e^{-f}\, dV_g$.

Corollary 2.3. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold satisfying the moderate volume growth property. Then M is f-parabolic.*

Proof. Let u be a bounded f-superharmonic function on M. Then for any $a > 0$,

$$\int_a^\infty \frac{r}{v(r)}\, dr \geq C \int_a^\infty \frac{r}{V_f(B_r)}\, dr = \infty.$$

Theorem 2.1 yields that u is a constant. This proves the corollary. □

Remark 2.4. Corollary 2.3 slightly generalizes [Wang and Xu 2012, Theorem 1.4]. In particular, this corollary implies [Zhu and Wang 2010, Theorem 2].

We can also derive several Liouville-type theorems for f-harmonic functions from Theorem 2.1.

Corollary 2.5. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold and u be a nonnegative f-subharmonic function (or f-harmonic function). Assume one of the following holds:*

(i) *$u = O(w^\alpha)$ for some nonnegative function w with $\int_M w d^{-2}(\cdot, x_0) e^{-f} dV_g < \infty$ and some $\alpha \in (0, 1)$;*

(ii) *$\int_M d^k(\cdot, x_0) e^{-f} dV_g < \infty$ for some $k > -2$ and $u = O(d^\beta(\cdot, x_0))$ for some $\beta \in (0, k+2)$;*

(iii) *$\int_M e^{-f} dV_g < \infty$ and $u = O(d^\beta(\cdot, x_0))$ for $\beta \in (0, 2)$;*

(iv) *$f \geq C d(\cdot, x_0)^\beta$ for some $C > 0$, $\beta > 0$ and $\int_M e^{-\delta f} dV_g < \infty$ for some $0 < \delta < 1$ and u has polynomial growth;*

(v) *$f \geq C d(\cdot, x_0)^\beta$ for some $C > 0$, $\beta > 0$ and the Riemannian volume has polynomial volume growth and $u = O(e^{\alpha C d(\cdot, x_0)^\beta})$, $\alpha \in (0, 1)$.*

Then u is a constant.

Proof. For (i), we see that there exists $p \in (1, \infty)$ such that $|u|^p = O(w)$. Hence

$$\frac{1}{r^2 \log r} v(r) = \frac{1}{r^2 \log r} \int_{B_r} |u|^p e^{-f} dV_g \leq \frac{C}{\log r} \int_{B_r} \frac{w(x)}{d^2(x, x_0)} e^{-f(x)} dV_g(x) = o(1).$$

It follows from Theorem 2.1 that u is a constant. The case (ii) follows from (i) by letting $w = d^{k+2}(\cdot, x_0)$. The case (iii) follows from (ii) by letting $k = 0$.

For (iv), let us observe for any $1 < p < \infty$,

$$\int_M |u|^p e^{-f} dV_g \leq C \int_M d^{sp}(x, x_0) e^{-f(x)} dV_g(x) \leq C \int_M e^{-\delta f} dV_g < \infty,$$

where $s > 0$. Then the statement also follows from Theorem 2.1. The case (v) can be proved in a similar way. □

The following result is a direct corollary of the above (v).

Corollary 2.6. *Let u be an f-harmonic function on the Gaussian space, i.e.,*

$$\Delta u - \tfrac{1}{2} \langle x, \nabla u \rangle = 0.$$

If $u = O(e^{\alpha |x|^2/4})$ as $x \to \infty$, for some $0 < \alpha < 1$, then u is a constant.

Remark 2.7. Corollary 2.6 implies that there are no nonconstant polynomial growth f-harmonic functions on the Gaussian space. This improves the result in [Li and Wang 2009, Theorem 4.2]. By Caccioppoli's inequality, Corollary 2.6 can be also derived from Li and Yang [2012, Corollary 1.2] .

In the remaining part of this section, we study the L^p-Liouville theorem introduced by Zhu and Wang [2010] using a different measure from ours. We shall explain why the critical exponent of the L^p-Liouville theorem in [Zhu and Wang 2010, Theorem 3] is $p = n/(n-2)$ $(n \geq 3)$ by applying our result. Let $(M, g, e^{-f} dV_g)$ be an n-dimensional $(n \geq 3)$ complete weighted manifold. In fact, they consider the L^p space with respect to the Riemannian volume in a modified Riemannian manifold $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$, denoted by $L^p(\widetilde{M}, dV_{\tilde{g}})$, where $\tilde{g}$ is a conformal change of g given by $\tilde{g} = e^{-2f/(n-2)} g$. Since this new manifold $\widetilde{M}$ may be incomplete, e.g., Gaussian space, Yau's L^p-Liouville theorem fails in this setting. In the following, we use the L^p-Liouville theorem on weighted manifolds to show the one on modified Riemannian manifolds.

Theorem 2.8. *Let $(M, g, e^{-f} dV_g)$ be an n-dimensional $(n \geq 3)$ complete weighted manifold, $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ be the modified Riemannian manifold and u be a nonnegative f-subharmonic function (or f-harmonic function) on M. For any $p > n/(n-2)$, there exists a constant $\delta = \delta(p, n) \in (0, 1)$ such that if $\int_M e^{-\delta f} dV_g < \infty$ and $u \in L^p(\widetilde{M}, dV_{\tilde{g}})$, then u is a constant.*

Proof. For any $p > n/(n-2)$, let $q = 2p/(p+n/(n-2)) > 1$, $\alpha = p/q > n/(n-2)$ and $\alpha^* = \alpha/(\alpha - 1) \in (1, n/2)$. Set $\delta = (n - 2\alpha^*)/(n-2) \in (0, 1)$. By Hölder's inequality, we can verify that

$$\begin{aligned}\int_M u^q e^{-f} dV_g = \int_M u^q e^{\frac{2f}{n-2}} dV_{\tilde{g}} &\leq \left(\int_M u^{q\alpha} dV_{\tilde{g}}\right)^{\frac{1}{\alpha}} \left(\int_M e^{\frac{2\alpha^* f}{n-2}} dV_{\tilde{g}}\right)^{\frac{1}{\alpha^*}} \\ &= \left(\int_M u^p dV_{\tilde{g}}\right)^{\frac{1}{\alpha}} \left(\int_M e^{-\delta f} dV_g\right)^{\frac{1}{\alpha^*}} < \infty.\end{aligned}$$

The statement follows from Theorem 2.1. □

This yields a direct corollary which generalizes [Zhu and Wang 2010, Theorem 3], which is restricted to the Gaussian spaces, to general weighted manifolds. The Riemannian manifold (M, g, dV_g) is said to be *of subexponential volume growth* if $V_g(r) := V_g(B_r(x_0)) = e^{o(r)}$ for some (then all) $x_0 \in M$.

Corollary 2.9. *Let $(M, g, e^{-f} dV_g)$ be an n-dimensional $(n \geq 3)$ complete weighted manifold satisfying that $f \geq C d^\beta(\cdot, x_0)$ for some $C > 0$, $\beta > 0$ and $V_g(r) = e^{o(r^\beta)}$. Let $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ be the modified Riemannian manifold. Then for any $p > n/(n-2)$, the f-harmonic function in $L^p(\widetilde{M}, dV_{\tilde{g}})$ is constant. In particular, for $\beta = 1$, it suffices to assume (M, g, dV_g) has subexponential volume growth.*

Proof. By virtue of Theorem 2.8, it is sufficient to prove $\int_M e^{-\delta f}\,dV_g < \infty$ where δ is the constant in Theorem 2.8. We see by the coarea formula that

$$\begin{aligned}\int_M e^{-\delta f}\,dV_g &= \int_0^1\int_{S_r(x_0)} e^{-\delta f}\,dA_r\,dr + \int_1^\infty\int_{S_r(x_0)} e^{-\delta f}\,dA_r\,dr\\ &\le C_0 + \int_1^\infty\int_{S_r(x_0)} e^{-\delta C r^\beta}\\ &= C_0 + \int_1^\infty e^{-\delta C r^\beta}\frac{d}{dr}V_g(r)\,dr\\ &= C_0 + e^{-\delta C r^\beta}V_g(r)\big|_1^\infty + \delta C\int_1^\infty \beta r^{\beta-1}e^{-\delta C r^\beta}V_g(r)\,dr.\end{aligned}$$

Since $V_g(r) = e^{o(r^\beta)}$, there exists R large such that

$$V_g(r) \le e^{\frac{1}{2}\delta C r^\beta} \quad \text{for } r > R.$$

It follows that $\lim_{r\to\infty} e^{-\delta C r^\beta}V_g(r) = 0$ and $\int_1^\infty \beta r^{\beta-1}e^{-\delta C r^\beta}V_g(r)\,dr < \infty$. It follows that $\int_M e^{-\delta f}\,dV_g < \infty$. This completes the proof. □

3. f-harmonic maps into Cartan–Hadamard manifolds

In this section, we prove Theorem 1.4 in the case that the target $Y = N$ is a Cartan–Hadamard manifold.

Theorem 3.1. *Let $(M, g, e^{-f}dV_g)$ be a complete weighted Riemannian manifold which is f-parabolic and N be a Cartan–Hadamard manifold. Then any f-harmonic map with finite f-energy, i.e., $E^f(u) := \int_M |\nabla u|^2 e^{-f}\,dV_g < \infty$, is a constant map.*

Proof. We use a construction by Rimoldi and Veronelli [2013] which associates an f-harmonic map with a harmonic map on some higher dimensional warped product manifold.

Precisely, let $\overline{M} := M \times_{e^{-f}} \mathbb{S}^1$ denote a warped product, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with $\mathrm{Vol}(\mathbb{S}^1) = 1$, with the metric on $\overline{M}$ given by $\bar g(x, t) = g(x) + e^{-2f(x)}dt^2$. Note that $\overline{M}$ is complete. It follows from [Rimoldi and Veronelli 2013, Proposition 2.5 and Lemma 2.6] that $\overline{M}$ is parabolic and the map $\bar u : \overline{M} \to N$, defined by $\bar u(x, t) = u(x)$ is a harmonic map. Moreover, $E_{\overline{M}}(\bar u) = E^f_M(u) < \infty$.

Now by applying [Cheng et al. 1996, Proposition 2.1 and Theorem 3.1] to $\bar u$ and $\overline{M}$, we know that the image of $\bar u$, $\bar u(\overline{M}) = u(M)$, is bounded in N. Since N is a Cartan–Hadamard manifold, $d^2(\bar u(\cdot), Q)$ is a subharmonic function for any $Q \in N$, which is also bounded. By the parabolicity of $\overline{M}$, we know that $d^2(\bar u(\cdot), Q)$ is constant for any $Q \in N$. This proves the theorem. □

Theorem 3.2. *Let $(M, g, e^{-f} dV_g)$ be a complete weighted Riemannian manifold satisfying* $\mathrm{Ric}_f \geq 0$ *and N be a Cartan–Hadamard manifold. Then any f-harmonic map with finite f-energy $E^f(u) < \infty$ is a constant map.*

Proof. We divide the theorem into two cases:

(a) $\int_M e^{-f}\, dV_g = \infty$,

(b) $\int_M e^{-f}\, dV_g < \infty$.

For case (a), it was already proved in [Wang and Xu 2012, Theorem 1.2] or [Rimoldi and Veronelli 2013, Theorem 3.3] for general Riemannian target of nonpositive curvature (without the assumption of simply-connectedness). For case (b), we observe that M satisfies the moderate volume growth property (1). By Corollary 2.3, M is f-parabolic. Then the statement follows from Theorem 3.1. □

Remark 3.3. Comparing Theorem 3.2 with [Wang and Xu 2012, Theorem 1.2] or [Rimoldi and Veronelli 2013, Theorem 3.3], we remove the condition of the infinity of f-volume for M but add the assumption that N is simply connected.

4. f-harmonic maps into Hadamard spaces

In this section, we define f-harmonic maps from an n-dimensional complete weighted Riemannian manifold $(M, g, e^{-f} dV_g)$ to a general metric space (Y, d). For that purpose we investigate an f-energy functional E^f whose definition given here follows Korevaar and Schoen [1993], where a Sobolev space theory for maps from Riemannian domains to metric spaces was developed. Note that the energy functional has been further extended to maps from complete noncompact Riemannian manifolds, and even more generally the so-called admissible Riemannian polyhedrons with simplexwise smooth Riemannian metric, in Eells and Fuglede [2001] (see Chapter 9 therein).

We consider Borel-measurable (equivalently, measurable with respect to $e^{-f} dV_g$) maps $u : M \to Y$ (u then has separable range since M is a separable metric space; see [Dudley 2002, Problem 10 in Section 4.2]). The space $L^2_{\mathrm{loc}}(M_f, Y)$ is defined as the set of Borel-measurable maps u for which $d(u(\cdot), Q) \in L^2_{\mathrm{loc}}(M, e^{-f} dV_g)$ for some point Q (and hence for any Q by the triangle inequality) in Y. Since this space is unchanged if we use the unweighted measure dV_g instead of $e^{-f} dV_g$ in its definition, we will write $L^2_{\mathrm{loc}}(M, Y)$ for simplicity in the following. When M is compact, $L^2_{\mathrm{loc}}(M, Y)$ is a complete metric space, with distance function $\hat{d}$ defined by

$$\hat{d}^2(u, v) := \int_M d^2(u(x), v(x)) e^{-f(x)}\, dV_g(x),$$

provided that (Y, d) is complete.

The approximate energy density for a map $u \in L^2_{\mathrm{loc}}(M, Y)$ is defined for $\varepsilon > 0$ as

$$e_\varepsilon(u) := \frac{1}{\omega_n} \int_{S(x,\varepsilon)} \frac{d^2(u(x), u(y))}{\varepsilon^2} \frac{d\sigma_{x,\varepsilon}(y)}{\varepsilon^{n-1}}, \tag{2}$$

where $d\sigma_{x,\varepsilon}(y)$ is the $(n-1)$-dimensional surface measure on the sphere $S(x, \varepsilon)$ of radius ε centered at x induced by the Riemannian metric g, and ω_n is the volume of the n-dimensional unit Euclidean ball. One can check that the function $e_\varepsilon(u) \in L^1_{\mathrm{loc}}(M)$ (see [Korevaar and Schoen 1993]). Then we can define the f-energy functional E^f by

$$E^f(u) := \sup_{\substack{\eta \in C_0(M) \\ 0 \le \eta \le 1}} \left(\limsup_{\varepsilon \to 0} \int_M \eta e_\varepsilon(u) e^{-f} \, dV_g \right).$$

We say a map $u \in L^2_{\mathrm{loc}}(M, Y)$ is locally of finite energy, denoted by $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$, if $E^f(u_{|\Omega}) < \infty$ for any relatively compact domain $\Omega \subset M$.

Theorem 4.1. *If $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$, then there exists a function $e(u) \in L^1_{\mathrm{loc}}(M)$, such that for any $\eta \in C_0(M)$, the following limit exists*

$$\lim_{\varepsilon \to 0} \int_M \eta e_\varepsilon(u) e^{-f} \, dV_g =: \int_M \eta e(u) e^{-f} \, dV_g, \tag{3}$$

which serves as the definition of $e(u)$.

Proof. By definition, $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ implies that for any connected, open and relatively compact subset $\Omega \subset M$, $u_{|\Omega} \in L^2(\Omega, Y)$ and

$$\sup_{\substack{\zeta \in C_0(\Omega) \\ 0 \le \zeta \le 1}} \left(\limsup_{\varepsilon \to 0} \int_\Omega \zeta e_\varepsilon(u_{|\Omega}) \, dV_g \right) < \infty,$$

that is, $u_{|\Omega} \in W^{1,2}(\Omega, Y)$ in Korevaar and Schoen's notation [1993].

Now by their Theorem 1.5.1 and Theorem 1.10, we know that there exists a function $e(u_{|\Omega}) \in L^1(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \int_\Omega \zeta e_\varepsilon(u) \, dV_g = \int_\Omega \zeta e(u_{|\Omega}) \, dV_g \quad \text{for all } \zeta \in C_0(\Omega). \tag{4}$$

In particular, one has

$$\lim_{\varepsilon \to 0} \int_\Omega \eta e_\varepsilon(u) e^{-f} \, dV_g = \int_\Omega \eta e(u_{|\Omega}) e^{-f} \, dV_g \quad \text{for all } \eta \in C_0(\Omega). \tag{5}$$

We then define a function $e(u)$ on M by $e(u)_{|\Omega} := e(u_{|\Omega})$ for any $\Omega \subset M$ with smooth boundary. One can show that $e(u)$ is well defined. For that purpose, one only needs to check $e(u_{|\Omega}) = e(u_{|\Omega_1})$ on $\Omega_1 \subset \Omega$ where both Ω_1 and $\Omega \setminus \Omega_1$ have

Lipschitz boundary. This is true since by the trace theory [Korevaar and Schoen 1993, Theorem 1.12.3], one has

$$\int_{\Omega} e(u_{|\Omega})\,dV_g = \int_{\Omega_1} e(u_{|\Omega_1})\,dV_g + \int_{\Omega\setminus\Omega_1} e(u_{|\Omega\setminus\Omega_1})\,dV_g.$$

Then (3) follows from (5) which proves this theorem. □

Remark 4.2. By the definition of $e(u)$ and (4), we know

$$e(u)(x) = |\nabla u|^2(x),$$

where $|\nabla u|^2(x)$ is the energy density function in [Korevaar and Schoen 1993]. This function is consistent with the usual way of defining $|du|^2$ for maps between Riemannian manifolds. Therefore we use $|\nabla u|^2(x)$ instead of $e(u)(x)$ in the following.

Remark 4.3. By a polarization argument, we can check that for any two functions $h_1, h_2 \in W^{1,2}_{\mathrm{loc}}(M, e^{-f}\,dV_g)$,

$$\lim_{\varepsilon\to 0}\int_M \eta(x)\frac{1}{\omega_n}\int_{S(x,\varepsilon)} \frac{(h_1(x)-h_1(y))(h_2(x)-h_2(y))}{\varepsilon^2}\frac{d\sigma_{x,\varepsilon}(y)}{\varepsilon^{n-1}} e^{-f(x)}\,dV_g(x)$$
$$= \int_M \eta(x)\langle\nabla h_1(x), \nabla h_2(x)\rangle e^{-f(x)}\,dV_g(x) \quad \text{for all } \eta\in C_0(M).$$

Remark 4.4. With (3) in hand, by the definition of E^f, we can derive (see [Eells and Fuglede 2001, Theorem 9.1]),

$$E^f(u) = \int_M |\nabla u|^2 e^{-f}\,dV_g \quad \text{for all } u\in W^{1,2}_{\mathrm{loc}}(M, Y).$$

In particular, we define $D^f(u) = E^f(u)$ when u is a scalar function.

Remark 4.5. As in [Korevaar and Schoen 1993], the definition of E^f is unchanged if we replace $e_\varepsilon(x)$ by ${}_\nu e_\varepsilon(x) := \int_0^2 e_{\lambda\varepsilon}(x)\,d\nu(\lambda)$, where ν is any Borel measure on the interval $(0, 2)$ satisfying $\nu \geq 0$, $\nu((0, 2)) = 1$, $\int_0^2 \lambda^{-2}\,d\nu(\lambda) < \infty$. For example, the approximate energy density function can be chosen as follows.

(1) When $n \geq 3$, for the measure $d\nu_1(\lambda) = n\lambda^{n-1}d\lambda,\ 0 < \lambda < 1$,

$${}_{\nu_1}e_\varepsilon(x) = \frac{n}{\omega_n}\int_{B(x,\varepsilon)} \frac{d^2(u(x), u(y))}{d^2(x, y)}\frac{dV_g(y)}{\varepsilon^n};$$

(2) For the measure $d\nu_2(\lambda) = (n+2)\lambda^{n+1}d\lambda,\ 0 < \lambda < 1$,

$${}_{\nu_2}e_\varepsilon(x) = \frac{n+2}{\omega_n}\int_{B(x,\varepsilon)} \frac{d^2(u(x), u(y))}{\varepsilon^2}\frac{dV_g(y)}{\varepsilon^n}.$$

Remark 4.6. For $n \geq 3$, by introducing a conformal change of the metric $\widetilde{M} = (M, \tilde{g}, dV_{\tilde{g}})$ where $\tilde{g} = e^{-2f/(n-2)}g$ and employing the energy density ${}_{\nu_1}e_\varepsilon$, many problems for weighted manifolds can be reduced to those on (possibly incomplete) unweighted manifolds. However, we prefer to write the proofs in a unified way which includes the case $n = 2$.

We call a map $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ *f-harmonic* if it is a local minimizer of the energy functional E^f, i.e., for any connected, open and relatively compact domain $\Omega \subset M$, $E^f(u) \le E^f(v)$ for every map $v \in W^{1,2}_{\mathrm{loc}}(M, Y)$ such that $u = v$ in $M \setminus \Omega$.

We now investigate the properties of the function $d(u(\cdot), Q)$ on M, where $u : M \to Y$ is an f-harmonic map and $Q \in Y$. The first observation is that

$$E^f(d(u, Q)) \le E^f(u). \tag{6}$$

This can be derived from the triangle inequality

$$(d(u(x), Q) - d(u(y), Q))^2 \le d^2(u(x), u(y)).$$

Recall that an *Hadamard space* (also called global NPC space) is a complete geodesic space which is globally nonpositively curved in the sense of Alexandrov, i.e., Toponogov's triangle comparison for nonpositive curvature holds for any geodesic triangle. The class of Hadamard spaces, natural generalizations of Cartan–Hadamard manifolds, includes all simply connected local NPC spaces (see, e.g., [Burago et al. 2001]). When the target space (Y, d) is an Hadamard space, we have the following theorem.

Theorem 4.7. *If $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ is an f-harmonic map into an Hadamard space Y, then for any $Q \in Y$,*

$$-\int_M \langle \nabla\eta(x), \nabla d(u(x), Q)\rangle e^{-f}\, dV_g \ge 0 \quad \textit{for all } 0 \le \eta \in \mathrm{Lip}_0(M), \tag{7}$$

i.e., $d(u(x), Q) \in W^{1,2}_{\mathrm{loc}}(M)$ is an f-subharmonic function.

This theorem is a consequence of Jost [1997a, Lemma 5]. The subharmonicity of $d(u(\cdot), Q)$ for harmonic maps from an admissible Riemannian polyhedron with simplexwise smooth Riemannian metric to an Hadamard space was obtained by Eells and Fuglede [2001, Lemma 10.2]. Their argument essentially also works in our setting. Using Remark 4.3, Jost's lemma can be reformulated in our setting as follows.

Lemma 4.8 [Jost 1997a, Lemma 5]. *If $u \in W^{1,2}_{\mathrm{loc}}(M, Y)$ is an f-harmonic map into an Hadamard space Y, then for any $Q \in Y$ and $\eta \in \mathrm{Lip}_0(M)$, $0 \le \eta \le 1$,*

$$-\int_M \langle \nabla\eta(x), \nabla d^2(u(x), Q)\rangle e^{-f(x)}\, dV_g(x) \ge 2\int_M \eta(x)|\nabla u|^2(x) e^{-f(x)}\, dV_g(x). \tag{8}$$

In fact, (8) still holds for nonnegative functions $\eta \in W^{1,2}(M)$ with compact support. (When $E^f(u)$ is finite, (8) even holds for $0 \le \eta \in W^{1,2}_0(M)$.) Now we can prove Theorem 4.7 concerning the f-subharmonicity of $d(u(\cdot), Q)$.

Proof of Theorem 4.7. Denote $\varphi(x) := \sqrt{x^2+\varepsilon}$ for $\varepsilon > 0$. For any $0 \le \eta \in \mathrm{Lip}_0(M)$, we choose a compactly supported function

$$\eta_1(x) := \frac{\eta(x)}{2\varphi(d(u(x),\,Q))} \in W^{1,2}(M).$$

Then we calculate (we suppress the measure $e^{-f}dV_g$ in the notation)

$$\begin{aligned}
-\int_M \Big\langle \nabla\eta(x), \nabla\sqrt{d^2(u(x),Q)+\varepsilon}\Big\rangle &= -\int_M \Big\langle \nabla\eta(x), \frac{\nabla d^2(u(x),Q)}{2\varphi(d(u(x),Q))}\Big\rangle \\
&= -\int_M \langle \nabla\eta_1(x), \nabla d^2(u(x),Q)\rangle \\
&\quad -\int_M 2\eta_1 \frac{d(u(x),Q)\varphi'(d(u(x),Q))}{\varphi(d(u(x),Q))}|\nabla d(u(x),Q)|^2.
\end{aligned}$$

Note that

$$\frac{d(u(x),\,Q)\varphi'(d(u(x),\,Q))}{\varphi(d(u(x),\,Q))} = \frac{d^2(u(x),\,Q)}{d^2(u(x),\,Q)+\varepsilon} \le 1,$$

and by (6), $|\nabla d(u(x),\,Q)|^2 \le |\nabla u(x)|^2$, we obtain

$$-\int_M \Big\langle \nabla\eta(x), \nabla\sqrt{d^2(u(x),Q)+\varepsilon}\Big\rangle \ge -\int_M \langle \nabla\eta_1(x), \nabla d^2(u(x),Q)\rangle - 2\int_M \eta_1 |\nabla u(x)|^2. \tag{9}$$

Applying Lemma 4.8, and letting $\varepsilon \to 0$, we complete the proof. □

Now we adopt the method of Li and Wang [1998], a geometric analysis method, to prove Kendall's theorem when the target is a locally compact Hadamard space.

Proof of Theorem 1.1. By assumption, the space of bounded f-harmonic functions is of dimension one. Then by the arguments of Grigor'yan [1990], every two f-massive subsets of M have a nonempty intersection. Here by a f-massive subset, we mean an open proper subset of $\Omega \subset M$ on which there is a bounded, nonnegative, nontrivial, f-subharmonic function h such that $h_{|\partial\Omega} = 0$. Such function h is called an f-potential of the set Ω.

Let $\widehat{M}$ be the Stone–Čech compactification of M. Then every bounded continuous function on M can be continuously extended to $\widehat{M}$. Let Ω be an f-massive subset of M, we then define the set

$$S := \bigcap_{\substack{h:\, f\text{-potential} \\ \text{functions of } \Omega}} \{\hat{x} \in \widehat{M} \mid h(\hat{x}) = \sup h\}.$$

By the maximum principle for f-subharmonic functions, we know $S \subset \widehat{M} \setminus M$.

Then, by the same arguments as in [Li and Wang 1998, Theorem 2.1], we can prove $S \ne \varnothing$. Furthermore, for any bounded f-subharmonic function v, we have $S \subset \{\hat{x} \in \widehat{M} \mid v(\hat{x}) = \sup v\}$.

Let us take a point $Q_0 \in \overline{u(M)}$. If $u(M) = \{Q_0\}$, then we complete the proof. Otherwise, we have $u(M) \setminus \{Q_0\} \neq \varnothing$. Since u is an f-harmonic map, by Theorem 4.7, the function $h_1(x) := d(u(x), Q_0)$ is an f-subharmonic function, which is bounded and nonconstant. Hence h_1 attains its maximum at every point of S. For a point $\hat{x} \in S$, there is a sequence $\{x_n\}$ in M converging to $\hat{x}$ in $\widehat{M}$. Note that u has bounded image. Thus by local compactness of the target Y, there exists a subsequence of $\{u(x_n)\}$ converging to $Q_1 \in Y$. Now again, if $u(M) = \{Q_1\}$, the proof is complete. Therefore, we can assume $u(M) \setminus \{Q_1\} \neq \varnothing$. By Theorem 4.7, the function $h_2(x) := d(u(x), Q_1)$ is a bounded f-subharmonic function. Thus h_2 achieves its maximum on S, in particular at $\hat{x}$. That is,

$$\sup h_2(x) = h_2(\hat{x}) = d(Q_1, Q_1) = 0.$$

This contradicts our assumption. Therefore $u(M) = \{Q_1\}$ is a constant map. □

Remark 4.9. As pointed out to us by K. Kuwae, one can prove Kendall's theorem by combining the methods of Li and Wang [1998] and Kuwae and Sturm [2008] for harmonic maps into Hadamard spaces if the weak topology on the target (see [Jost 1994, Definition 2.7]) coincides with the strong one, or equivalently any distance function $d(x, \cdot)$ on the target is weakly continuous for any $x \in Y$.

5. Liouville-type theorems

In this section, we shall prove our main theorem. First, we review the classical classification theory of Riemannian manifolds in the framework of weighted manifolds. For more details we refer to [Glasner and Nakai 1972] and [Sario and Nakai 1970].

We recall some function spaces of $(M, g, e^{-f} dV_g)$. Let $D^f(M)$ be the set of Tonelli functions[1] on M with finite Dirichlet f-energy. The Royden algebra $BD^f(M)$ is the set of bounded functions in $D^f(M)$. Under the norm $\|u\| = \sup_M |u| + \sqrt{D^f(u)}$, $BD^f(M)$ becomes a Banach algebra. For a sequence $\{u_n\}$ in $D^f(M)$, we say $u = C - \lim u_n$ if u_n converges to u uniformly on compact subsets and $u = B - \lim u_n$ if in addition $\{u_n\}$ is uniformly bounded. We say $u = D^f - \lim u_n$ if $\lim D^f(u_n - u) = 0$. We also write $u = CD^f - \lim u_n$ or $u = BD^f - \lim u_n$ to indicate two types of convergence.

Let $C_0^\infty(M)$ be the set of smooth functions with compact support and $D_0^f(M)$ be its closure under the CD^f-topology. We also denote by $HD^f(M)$ and $HBD^f(M)$ the sets of f-harmonic functions in $D^f(M)$ and $BD^f(M)$ respectively.

Proposition 5.1. *Let $(M, g, e^{-f} dV_g)$ be an f-parabolic weighted Riemannian manifold. Then any f-subharmonic function with finite Dirichlet f-energy is constant. In particular, any function in $HD^f(M)$ is constant.*

[1] A Tonelli function is a continuous function with locally L^2-integrable weak derivatives.

Proof. Let $u \in D^f(M)$ be f-subharmonic. we may assume $u \geq 0$ since $\max\{u, 0\}$ is also f-subharmonic. Let $\{M_n\}$ be an exhaustion of M and take $w_k \in BD^f(M)$ with $w_k|_{M_0} = 1$, $w_k|_{M \setminus M_k} = 0$ and f-harmonic in $M_k \setminus \overline{M_0}$. It follows from the f-parabolicity of M that $BD^f - \lim w_k = 1$. On the other hand, set $v_k \in BD^f(M)$ with $v_k|_{M_0} = u$, $v_k|_{M \setminus M_k} = 0$ and f-harmonic in $M_k \setminus \overline{M_0}$, one can verify that $v = BD^f - \lim v_k$ exists. Set now $\tilde{u} = u - v$, and $\tilde{u}_m = \min\{\tilde{u}, m\}$. Then $\tilde{u} = D^f - \lim \tilde{u}_m$. Since $\tilde{u}$ is nonnegative and f-subharmonic, we can compute

$$0 \geq -\int_{M_k \setminus M_0} \tilde{u}_m w_k \Delta_f \tilde{u} e^{-f} \, dV_g = \int_M \langle \nabla(\tilde{u}_m w_k), \nabla \tilde{u} \rangle e^{-f} \, dV_g. \tag{10}$$

As $w_k \to 1$ in D^f-topology, we deduce from (10) by letting $k \to \infty$ that

$$\int_M \langle \nabla \tilde{u}_m, \nabla \tilde{u} \rangle e^{-f} \, dV_g = 0,$$

which yields $D^f(\tilde{u}) = 0$ by letting $m \to \infty$. Since $\tilde{u}|_{M_0} = 0$, we see $u = v$. Finally,

$$D^f(u) = \int_M \langle \nabla u, \nabla v \rangle e^{-f} \, dV_g = \lim_{k \to \infty} \int_M \langle \nabla u, \nabla v_k \rangle e^{-f} \, dV_g \leq 0,$$

and hence u is a constant. □

The following are the weighted version of the Royden–Nakai decomposition theorem and the Virtanen theorem. The proofs are almost the same as the unweighted case. For the convenience of the reader, we shall give proofs here.

Theorem 5.2 (Royden–Nakai decomposition theorem). *Let $(M, g, e^{-f} dV_g)$ be a non-f-parabolic weighted Riemannian manifold. Then any function $u \in D^f(M)$ has a unique decomposition $u = h + g$, where $h \in HD^f(M)$ and $g \in D_0^f(M)$. Moreover, if u is f-subharmonic, then $u \leq h$.*

Proof. Let $u \in D^f(M)$. Assume first $u \geq 0$. Let $\{M_k\}$ be an exhaustion of M. Take $h_k \in D^f(M)$ such that h_k is f-harmonic in M_k and $h_k|_{M \setminus M_k} = u$. Denote $g_k = u - h_k$. It follows from the maximum principle that $h_k \geq 0$. One can check

$$D^f(u) = \int_M (|\nabla h_k|^2 + |\nabla g_k|^2 + 2\langle \nabla h_k, \nabla g_k \rangle) e^{-f} \, dV_g = D^f(h_k) + D^f(g_k),$$

where in the second equality we used integration by parts and the facts $g_k|_{M \setminus M_k} = 0$ and h_k is f-harmonic in M_k. Similarly we have for $m \leq k$

$$D^f(h_k - h_m) = D^f(h_k) - D^f(h_m).$$

Thus $\{h_k\}$ is a D^f-Cauchy sequence, i.e., $D^f(h_k - h_m)$ is small enough when m and k are large enough.

Let $w_k \in BD^f(M)$ with $w_k|_{M_0} = 1,\ w_k|_{M\setminus M_k} = 0$ and harmonic in $M_k \setminus \overline{M_0}$. It follows from the non-f-parabolicity of M that $w = BD^f - \lim w_k$ satisfies $D^f(w) > 0$.

We can compute

$$\int_M \langle \nabla g_k, \nabla w_k \rangle e^{-f}\, dV_g = \int_{M_k\setminus\overline{M_0}} \langle \nabla g_k, \nabla w_k \rangle e^{-f}\, dV_g = \int_{\partial M_0} g_k \frac{\partial w_k}{\partial \nu} e^{-f}\, dA_g,$$

where ν is the unit inward normal of ∂M_0. Since w_k is f-harmonic in $M_k \setminus \overline{M_0}$, it follows from the Hopf lemma that $\partial w_k/\partial\nu > 0$ along M_0. It follows that

$$\begin{aligned}(\inf_{\partial M_0} h_k - \sup_{\partial M_0} u) \int_{\partial M_0} \frac{\partial w_k}{\partial \nu} e^{-f}\, dA_g &\le \int_{\partial M_0} -g_k \frac{\partial w_k}{\partial \nu} e^{-f}\, dA_g \\ &= -\int_M \langle \nabla g_k, \nabla w_k \rangle e^{-f}\, dV_g \\ &\le \big[D^f(g_k) D^f(w_k)\big]^{1/2} \le \big[D^f(u) D^f(w_k)\big]^{1/2}.\end{aligned}$$

Combining this with the fact that $\int_{\partial M_0} (\partial w_k/\partial\nu) e^{-f}\, dV_g = D^f(w_k)$, we find

$$\inf_{M_0} h_k \le \inf_{\partial M_0} h_k \le \sup_{M_0} u + \left[\frac{D^f(u)}{D^f(w_k)}\right]^{1/2}.$$

Since $w = BD^f - \lim w_k$ satisfies $D^f(w) > 0$, we see $\inf_{M_0} h_k$ is bounded. Consequently, by the Harnack inequality for f-harmonic functions, $\sup_{M_0} h_k$ is bounded. Hence there exists a subsequence of h_k, still denoted by h_k, such that $\{h_k\}$ is a C^f-Cauchy sequence.

Together with the fact $\{h_k\}$ is a D^f-Cauchy sequence, we conclude that h_k converges to some h in the CD^f-topology and $h \in HD^f(M)$. One may directly check that g_k converges to $g = u - h$ in the CD^f-topology and thus $g \in D_0^f(M)$.

Furthermore, if u is f-subharmonic, from the construction of h_k we see $u - h_k$ is f-subharmonic and vanishes on ∂M_k and in turn by the maximum principle that $h \ge u$.

If u is not nonnegative, we can run the same process for $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ as before and get the same result.

The uniqueness follows from the fact that any $h \in HD^f(M)$ and $g \in D_0^f(M)$ satisfy $\int_M \langle \nabla h, \nabla g \rangle e^{-f}\, dV_g = 0$. □

Theorem 5.3 (Virtanen's theorem). *For every $u \in HD^f(M)$ there exists a sequence $h_k \in HBD^f(M)$ such that $u = CD^f - \lim h_k$. In particular, M admits no nonconstant f-harmonic function on M with finite Dirichlet f-energy if and only if M admits no nonconstant bounded f-harmonic function on M with finite Dirichlet f-energy.*

Proof. We may assume M is non-f-parabolic, since otherwise, any $u \in HD^f(M)$ is constant, due to Proposition 5.1, whence the statement is trivial. We may also assume $u \geq 0$, since otherwise we do the same analysis on u^+ and u^-. Set for any $k \in \mathbb{N}$, $u_k = \min\{u, k\}$. Then u_k is f-superharmonic and $u = D^f - \lim u_k$. By Royden–Nakai decomposition, $u_k = h_k + g_k$, where $h_k \in HD^f(M)$ and $g_k \in D_0^f(M)$. Moreover, $g_k \geq 0$. One can verify

$$D^f(u - u_k) = D^f(u - h_k) + D^f(g_k).$$

Hence $D^f(u - h_k) \to 0$ and $D^f(g_k) \to 0$. Since $0 \leq g_k \leq u_k \leq u$ is bounded in any compact set of M, we conclude that g_k converges to some constant function c in the CD^f-topology. It follows from the non-f-parabolicity of M that $c = 0$. Therefore h_k converges to u in the CD^f-topology.

The second assertion follows easily from this approximation. $\square$

The following lemma was first proved by Cheng, Tam and Wan [Cheng et al. 1996, Theorem 1.2].

Lemma 5.4. *Let $(M, g, e^{-f} dV_g)$ be a weighted Riemannian manifold. Then the following two statements are equivalent*:

(i) *any $u \in HD^f(M)$ is bounded*;

(ii) *any nonnegative f-subharmonic function on M with finite Dirichlet f-energy is bounded.*

Proof. (ii) $\Rightarrow$(i). This is quite simple by observing the fact that if $u \in HD^f(M)$, then $\sqrt{u^2+1}$ is a nonnegative f-subharmonic function on M with finite Dirichlet f-energy.

(i) $\Rightarrow$(ii). Assume u is a nonnegative f-subharmonic function on M with finite Dirichlet f-energy. If M is f-parabolic, then the two statements are both true by virtue of Proposition 5.1 and hence equivalent. If M is non-f-parabolic, then by Theorem 5.2, $u = h + g$ for $h \in HD^f(M)$ and $g \in D_0^f(M)$. Moreover, since u is f-subharmonic, we know $u \leq h$. By the assumption (i), h is bounded. Thus u is also bounded. This proves the lemma. $\square$

Using Lemma 5.4, we can prove the main Theorem 1.2.

Proof of Theorem 1.2. Let u be an f-harmonic map from M to Y with finite f-energy. It follows from Theorem 4.7 that the function $v : M \to \mathbb{R}$, $v(x) = \sqrt{d^2(u(x), Q) + 1}$ is subharmonic, where $Q \in Y$. Also, the finiteness of the f-energy of u implies the finiteness of the Dirichlet f-energy of v (recall (6)). Using the assumption and the equivalence in Lemma 5.4, we know that any nonnegative f-subharmonic function on M with finite Dirichlet f-energy is bounded. Hence v is bounded, and in turn, u has bounded image. This proves the theorem. $\square$

For harmonic maps from f-parabolic weighted manifolds, we don't need the local compactness assumption of the targets to obtain the Liouville theorem.

Corollary 5.5. *Let $(M, g, e^{-f} dV_g)$ be a complete noncompact f-parabolic weighted Riemannian manifold and (Y, d) be an Hadamard space. Then any f-harmonic map from M to Y with finite f-energy is a constant map.*

Proof. Let u be an f-harmonic map from M to Y with finite f-energy. By Proposition 5.1 and Theorem 1.2, the image of u is bounded. Hence for any $Q \in Y$, the f-subharmonic function $d(u(x), Q)$ is bounded. By the f-parabolicity of M and Proposition 2.2, the function $d(u(x), Q)$ is constant for any $Q \in Y$. This yields that u is a constant map. The corollary follows. □

Combining Theorems 1.1 and 1.2, we obtain Theorem 1.3 by the potential theory.

Proof of Theorem 1.3.. By assumption, any bounded f-harmonic function on M is constant. By Theorem 5.3, we know that any f-harmonic function on M with finite Dirichlet f-energy is constant. Using Theorem 1.2, we see that any f-harmonic map from M to Y with finite f-energy must have bounded image.

On the other hand, by Theorem 1.1, we know that any f-harmonic map from M to Y having bounded image is constant. Hence any f-harmonic map from M to Y with finite f-energy is a constant map. This proves the theorem. □

Proof of Theorem 1.4.. By a theorem of Brighton [2013], the weighted manifold $(M, g, e^{-f} dV_g)$ satisfying $\mathrm{Ric}_f \geq 0$ admits no nonconstant bounded f-harmonic functions. The assertion follows from Theorem 1.3 immediately. □

Acknowledgements

The authors thank Prof. Jürgen Jost for numerous inspiring discussions on harmonic maps into singular spaces. The authors thank Prof. Kazuhiro Kuwae for the discussions on Kendall's theorem and suggestions on the conditions of the targets (see Remark 4.9). We would also like to thank the referee for a critical reading and useful suggestions.

References

[Bakry and Émery 1985] D. Bakry and M. Émery, "Diffusions hypercontractives", pp. 177–206 in *Séminaire de probabilités, XIX, 1983/84*, edited by J. Azéma and M. Yor, Lecture Notes in Math. **1123**, Springer, Berlin, 1985. MR Zbl

[Brighton 2013] K. Brighton, "A Liouville-type theorem for smooth metric measure spaces", *J. Geom. Anal.* **23**:2 (2013), 562–570. MR Zbl

[Burago et al. 2001] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics **33**, American Mathematical Society, Providence, RI, 2001. MR Zbl

[Chen et al. 2012] Q. Chen, J. Jost, and H. Qiu, "Existence and Liouville theorems for V-harmonic maps from complete manifolds", *Ann. Global Anal. Geom.* **42**:4 (2012), 565–584. MR Zbl

[Cheng et al. 1996] S. Y. Cheng, L.-F. Tam, and T. Y.-H. Wan, "Harmonic maps with finite total energy", *Proc. Amer. Math. Soc.* **124**:1 (1996), 275–284. MR Zbl

[Dudley 2002] R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics **74**, Cambridge University Press, 2002. MR Zbl

[Eells and Fuglede 2001] J. Eells and B. Fuglede, *Harmonic maps between Riemannian polyhedra*, Cambridge Tracts in Mathematics **142**, Cambridge University Press, 2001. MR Zbl

[Glasner and Nakai 1972] M. Glasner and M. Nakai, "Riemannian manifolds with discontinuous metrics and the Dirichlet integral", *Nagoya Math. J.* **46** (1972), 1–48. MR Zbl

[Grigor'yan 1985] A. A. Grigor'yan, "The existence of positive fundamental solutions of the Laplace equation on Riemannian manifolds", *Mat. Sb.* (*N.S.*) **128(170)**:3 (1985), 354–363. In Russian; translated in *Math. USSR-Sb.* **56**:2 (1987), 349–358. MR Zbl

[Grigor'yan 1990] A. A. Grigor'yan, "Dimensions of spaces of harmonic functions", *Mat. Zametki* **48**:5 (1990), 55–61. In Russian; translated in *Math. Notes* **48**:5 (1990), 1114–1118. MR

[Grigor'yan 1999] A. Grigor'yan, "Escape rate of Brownian motion on Riemannian manifolds", *Appl. Anal.* **71**:1-4 (1999), 63–89. MR Zbl

[Grigor'yan 2006] A. Grigor'yan, "Heat kernels on weighted manifolds and applications", pp. 93–191 in *The ubiquitous heat kernel*, edited by J. Jorgenson and L. Walling, Contemp. Math. **398**, American Mathematical Society, Providence, RI, 2006. MR Zbl

[Grigor'yan 2009] A. Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics **47**, American Mathematical Society, Providence, RI, 2009. MR Zbl

[Gromov and Schoen 1992] M. Gromov and R. Schoen, "Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one", *Inst. Hautes Études Sci. Publ. Math.* 76 (1992), 165–246. MR Zbl

[Jost 1994] J. Jost, "Equilibrium maps between metric spaces", *Calc. Var. Partial Differential Equations* **2**:2 (1994), 173–204. MR Zbl

[Jost 1997a] J. Jost, "Generalized Dirichlet forms and harmonic maps", *Calc. Var. Partial Differential Equations* **5**:1 (1997), 1–19. MR Zbl

[Jost 1997b] J. Jost, *Nonpositive curvature: geometric and analytic aspects*, Birkhäuser, Basel, 1997. MR Zbl

[Karp 1982] L. Karp, "Subharmonic functions on real and complex manifolds", *Math. Z.* **179**:4 (1982), 535–554. MR Zbl

[Kendall 1990] W. S. Kendall, "Probability, convexity, and harmonic maps with small image, I: Uniqueness and fine existence", *Proc. London Math. Soc.* (3) **61**:2 (1990), 371–406. MR Zbl

[Korevaar and Schoen 1993] N. J. Korevaar and R. M. Schoen, "Sobolev spaces and harmonic maps for metric space targets", *Comm. Anal. Geom.* **1**:3-4 (1993), 561–659. MR Zbl

[Korevaar and Schoen 1997] N. J. Korevaar and R. M. Schoen, "Global existence theorems for harmonic maps to non-locally compact spaces", *Comm. Anal. Geom.* **5**:2 (1997), 333–387. MR

[Kuwae and Sturm 2008] K. Kuwae and K.-T. Sturm, "On a Liouville type theorem for harmonic maps to convex spaces via Markov chains", pp. 177–191 in *Proceedings of RIMS Workshop on Stochastic Analysis and Applications* (Kyoto, 2006), edited by M. Fukushima and I. Shigekawa, RIMS Kôkyûroku Bessatsu **B6**, Res. Inst. Math. Sci., Kyoto, 2008. MR Zbl

[Li 2005] X.-D. Li, "Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds", *J. Math. Pures Appl.* (9) **84**:10 (2005), 1295–1361. MR Zbl

[Li 2016] X.-D. Li, "Hamilton's Harnack inequality and the W-entropy formula on complete Riemannian manifolds", *Stochastic Process. Appl.* **126**:4 (2016), 1264–1283. MR Zbl

[Li and Schoen 1984] P. Li and R. Schoen, "L^p and mean value properties of subharmonic functions on Riemannian manifolds", *Acta Math.* **153**:3-4 (1984), 279–301. MR Zbl

[Li and Tian 2000] J. Li and G. Tian, "The blow-up locus of heat flows for harmonic maps", *Acta Math. Sin.* (*Engl. Ser.*) **16**:1 (2000), 29–62. MR Zbl

[Li and Wang 1998] P. Li and J. Wang, "Convex hull properties of harmonic maps", *J. Differential Geom.* **48**:3 (1998), 497–530. MR Zbl

[Li and Wang 2009] J. Li and M. Wang, "Liouville theorems for self-similar solutions of heat flows", *J. Eur. Math. Soc.* (*JEMS*) **11**:1 (2009), 207–221. MR Zbl

[Li and Yang 2012] J. Li and Y. Yang, "Nonexistence of quasi-harmonic spheres with large energy", *Manuscripta Math.* **138**:1-2 (2012), 161–169. MR Zbl

[Li and Zhu 2010] J. Li and X. Zhu, "Non existence of quasi-harmonic spheres", *Calc. Var. Partial Differential Equations* **37**:3-4 (2010), 441–460. MR Zbl

[Lin and Wang 1999] F. Lin and C. Wang, "Harmonic and quasi-harmonic spheres", *Comm. Anal. Geom.* **7**:2 (1999), 397–429. MR Zbl

[Lott and Villani 2009] J. Lott and C. Villani, "Ricci curvature for metric-measure spaces via optimal transport", *Ann. of Math.* (2) **169**:3 (2009), 903–991. MR Zbl

[Morgan 2005] F. Morgan, "Manifolds with density", *Notices Amer. Math. Soc.* **52**:8 (2005), 853–858. MR Zbl

[Munteanu and Wang 2011] O. Munteanu and J. Wang, "Smooth metric measure spaces with non-negative curvature", *Comm. Anal. Geom.* **19**:3 (2011), 451–486. MR Zbl

[Munteanu and Wang 2012] O. Munteanu and J. Wang, "Analysis of weighted Laplacian and applications to Ricci solitons", *Comm. Anal. Geom.* **20**:1 (2012), 55–94. MR Zbl

[Nakai 1960] M. Nakai, "A measure on the harmonic boundary of a Riemann surface", *Nagoya Math. J.* **17** (1960), 181–218. MR Zbl

[Rimoldi and Veronelli 2013] M. Rimoldi and G. Veronelli, "Topology of steady and expanding gradient Ricci solitons via f-harmonic maps", *Differential Geom. Appl.* **31**:5 (2013), 623–638. MR Zbl

[Royden 1952] H. L. Royden, "Harmonic functions on open Riemann surfaces", *Trans. Amer. Math. Soc.* **73** (1952), 40–94. MR Zbl

[Sario and Nakai 1970] L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Die Grundlehren der mathematischen Wissenschaften **164**, Springer, New York, 1970. MR Zbl

[Schoen and Yau 1976] R. Schoen and S. T. Yau, "Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature", *Comment. Math. Helv.* **51**:3 (1976), 333–341. MR Zbl

[Sinaei 2014] Z. Sinaei, "Riemannian polyhedra and Liouville-type theorems for harmonic maps", *Anal. Geom. Metr. Spaces* **2**:1 (2014), 294–318. MR Zbl

[Sinaei 2016] Z. Sinaei, "Convergence of harmonic maps", *J. Geom. Anal.* **26**:1 (2016), 529–556. MR Zbl

[Sturm 1994] K.-T. Sturm, "Analysis on local Dirichlet spaces, I: Recurrence, conservativeness and L^p-Liouville properties", *J. Reine Angew. Math.* **456** (1994), 173–196. MR Zbl

[Sturm 2006a] K.-T. Sturm, "On the geometry of metric measure spaces, I", *Acta Math.* **196**:1 (2006), 65–131. MR Zbl

[Sturm 2006b] K.-T. Sturm, "On the geometry of metric measure spaces, II", *Acta Math.* **196**:1 (2006), 133–177. MR Zbl

[Wang and Xu 2012] G. Wang and D. Xu, "Harmonic maps from smooth metric measure spaces", *Internat. J. Math.* **23**:9 (2012), art. id. 1250095, 21 pp. MR Zbl

[Wei and Wylie 2009] G. Wei and W. Wylie, "Comparison geometry for the Bakry–Emery Ricci tensor", *J. Differential Geom.* **83**:2 (2009), 377–405. MR Zbl

[Yau 1976] S. T. Yau, "Some function-theoretic properties of complete Riemannian manifold and their applications to geometry", *Indiana Univ. Math. J.* **25**:7 (1976), 659–670. MR Zbl

[Zhu and Wang 2010] X. Zhu and M. Wang, "Liouville theorems for quasi-harmonic functions", *Nonlinear Anal.* **73**:9 (2010), 2890–2896. MR Zbl

Received July 3, 2015. Revised July 25, 2016.

BOBO HUA
SCHOOL OF MATHEMATICAL SCIENCES, LMNS
FUDAN UNIVERSITY
200433 SHANGHAI
CHINA
bobohua@fudan.edu.cn

SHIPING LIU
DEPARTMENT OF MATHEMATICAL SCIENCES
DURHAM UNIVERSITY
DH13LE DURHAM
UNITED KINGDOM
Current address:
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
230026 HEFEI
CHINA
spliu@ustc.edu.cn

CHAO XIA
SCHOOL OF MATHEMATICAL SCIENCES
XIAMEN UNIVERSITY
361005 XIAMEN
CHINA
chaoxia@xmu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.403

THE AMBIENT OBSTRUCTION TENSOR AND CONFORMAL HOLONOMY

THOMAS LEISTNER AND ANDREE LISCHEWSKI

For a conformal manifold, we describe a new relation between the ambient obstruction tensor of Fefferman and Graham and the holonomy of the normal conformal Cartan connection. This relation allows us to prove several results on the vanishing and the rank of the obstruction tensor, for example for conformal structures admitting twistor spinors or normal conformal Killing forms. As our main tool we introduce the notion of a conformal holonomy distribution and show that its integrability is closely related to the exceptional conformal structures in dimensions five and six that were found by Nurowski and Bryant.

1. Introduction

A conformal structure of signature (p, q) on a smooth manifold M is an equivalence class c of semi-Riemannian metrics on M of signature (p, q), where two metrics g and $\hat{g}$ are equivalent if $\hat{g} = e^{2f} g$ for a smooth function f. For conformal structures the construction of local invariants is more complicated than for semi-Riemannian structures, where all local invariants can be derived from the Levi-Civita connection and its curvature. For conformal geometry, essentially there are two invariant constructions: the conformal ambient metric of Fefferman and Graham [1985; 2012] and the normal conformal Cartan [1924] connection with the induced tractor calculus [Bailey et al. 1994]. We investigate a new relationship between two essential ingredients of these invariant constructions, the *ambient obstruction tensor* on one hand, and the *conformal holonomy* on the other. We briefly introduce these notions:

The ambient metric construction assigns to any conformal manifold $(M, [g])$ of signature (p, q) and dimension n a pseudo-Riemannian metric $\tilde{g}$ on an open neighborhood $\tilde{M}$ of $Q = M \times \mathbb{R}^{>0}$ in $\mathbb{R} \times Q$, of signature $(p+1, q+1)$ and with specific properties that link $[g]$ and $\tilde{g}$ as closely as possible. More precisely, denoting

This research was supported by the Australian Research Council via the grants FT110100429 and DP120104582.
MSC2010: primary 53A30; secondary 53C29.
Keywords: Fefferman–Graham ambient metric, obstruction tensor, conformal holonomy, exceptional conformal structures, normal conformal Killing forms.

the coordinates on $\mathbb{R}^{>0}$ and $\mathbb{R}$ by t and ρ, respectively, $\tilde{g}$ is required to restrict to $t^2 g$ along Q and moreover its Ricci tensor vanishes along Q to infinite order in ρ when n is odd and to order $\rho^{(n/2)-1}$ when n is even. The seminal result in [Fefferman and Graham 1985; 2012] is that for smooth conformal structures, such an ambient metric always exists and is unique to all orders for n odd or $n = 2$ and up to order $\rho^{(n/2)-1}$ when $n \geq 4$ is even. Moreover, in even dimensions the existence of an ambient metric whose Ricci tensor vanishes along Q to all orders is closely related to the vanishing of a certain symmetric, divergence-free and conformally covariant (0, 2)-tensor $\mathcal{O}$ on M, the *Fefferman–Graham obstruction tensor* or *ambient obstruction tensor*. In four dimensions the obstruction tensor is given by the well-known Bach tensor, but in general even dimension no general explicit formula for $\mathcal{O}$ exists. The obstruction tensor will be the focus of the present article.

The other invariant construction in conformal geometry is the *normal conformal Cartan connection*. This is an $\mathfrak{so}(p+1, q+1)$-valued Cartan connection defined on a P-bundle, where P is the parabolic subgroup defined by the stabilizer in $\mathrm{O}(p+1, q+1)$ of a lightlike line in $\mathbb{R}^{p+1,q+1}$, and it satisfies a certain normalization condition that defines it uniquely. The normal conformal Cartan connection defines a covariant derivative ∇^{nc} on a vector bundle $\mathcal{T}$, the *conformal tractor connection* on the *standard tractor bundle*. To $(\mathcal{T}, \nabla^{\mathrm{nc}})$ one can associate the holonomy group of ∇^{nc}-parallel transports along loops based at $x \in M$. As this group only depends on the conformal structure, it is denoted by $\mathrm{Hol}_x(M, c)$ and called the *conformal holonomy*. It is contained in $\mathrm{O}(p+1, q+1)$ and its Lie algebra is denoted by

$$\mathfrak{hol}_x(M, c) \subset \mathfrak{so}(p+1, q+1).$$

Many interesting conformal structures are related to conformal *holonomy reductions*, i.e., conformal structures for which the conformal holonomy algebra is a proper subalgebra of $\mathfrak{so}(p+1, q+1)$. Examples are manifolds admitting twistor spinors, for which the spin representation of the conformal holonomy group admits an invariant spinor. This includes conformal Fefferman [1976] spaces that are closely related to CR-geometry, and for which the conformal holonomy reduces to the special unitary group. Other fascinating examples are the conformal structures that are determined by generic distributions of rank 2 in dimension 5. Such distributions played an important role in the history of the simple Lie algebra with exceptional root system G_2: Cartan [1893] discovered that for some of these distributions the Lie algebra of symmetries is given by the noncompact exceptional Lie algebra $\mathfrak{g}_2$ of type G_2. Related to the equivalence problem for such distributions, Cartan [1910] constructed the corresponding $\mathfrak{g}_2$-valued Cartan connection. It was then realized by Nurowski [2005] that to any such distribution one can associate a conformal structure of signature (2, 3) whose conformal holonomy is reduced from $\mathfrak{so}(3, 4)$ to $\mathfrak{g}_2$. Similarly, Bryant associated to any generic rank 3 distribution in dimension 6 a conformal structure of signature $(3, 3)$ whose holonomy reduces to $\mathfrak{spin}(3, 4) \subset \mathfrak{so}(4, 4)$. Both, and in particular the latter will be relevant to us.

The ambient metric construction and the normal conformal Cartan connection turn out to be closely related. Indeed, in [Čap and Gover 2003] tractor data are formulated entirely in terms of ambient data, and in [Gover and Peterson 2006] the ambient curvature tensors are rewritten in terms of tractor curvature and derivatives thereof. The main result in our paper reveals another interesting correspondence, now between the ambient obstruction tensor $\mathcal{O}$ and the conformal holonomy. We show that the image of $\mathcal{O}$, when considered as a $(1, 1)$-tensor, can be identified with a distinguished subspace in the conformal holonomy algebra $\mathfrak{hol}_x(M, c)$. To be more precise, recall that the Lie algebra $\mathfrak{so}(p+1, q+1)$ is $|1|$-graded as $\mathfrak{so}(p+1, q+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 \simeq \mathfrak{co}(p, q)$ is the conformal Lie algebra and $\mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{p}$ is the Lie algebra of the parabolic subgroup P. It is important to note that $\mathfrak{g}_1$ can be identified with $\mathbb{R}^{p,q}$ and hence with the tangent space T_xM. This allows us to prove the following theorem:

Theorem 1.1. *Let $(M^{p,q}, c)$ be a smooth conformal manifold of even dimension $n \geq 4$ and with ambient obstruction tensor $\mathcal{O}$. Then the image of $\mathcal{O}$ at $x \in M$ is contained in $\mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1$. In particular, the rank of $\mathcal{O}$ at each point is limited by the dimension of $\mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1$. Moreover, if $\mathfrak{hol}(M, c)$ is a proper subalgebra of $\mathfrak{so}(p+1, q+1)$, then the image of $\mathcal{O}$ is totally lightlike. In particular,* $\mathrm{rk}(\mathcal{O}) \leq \min(p, q)$.

The implications of this result are evident. On the one hand it shows that if the obstruction tensor has maximal rank n at some point, then the holonomy is generic. Hence, $\mathcal{O}$ can be interpreted as a universal obstruction to the existence of parallel tractors on (M, c) of any type. Namely for such a tractor to exist, $\mathcal{O}$ needs to have a nontrivial kernel everywhere. On the other hand, conformal holonomy reductions can be used to restrict the rank of the obstruction tensor. For example, it is well known that the existence of a parallel standard tractor (and hence of a local Einstein metric in c) forces the obstruction tensor to vanish, however no substantially more general conditions on the conformal class are known to have a similar effect on $\mathcal{O}$. Our results provide such conditions. For example, we obtain:

Corollary 1.2. *Under the assumptions of Theorem 1.1, $\mathcal{O} = 0$ for each of the following cases:*

(1) *the conformal structure is Riemannian and $\mathfrak{hol}(M, c) \subsetneq \mathfrak{so}(1, n+1)$;*

(2) *the conformal structure is Lorentzian and $\mathfrak{hol}(M, c) \subsetneq \mathfrak{su}(1, n/2)$;*

(3) *the conformal class contains an almost Einstein metric or special Einstein product (in the sense of [Gover and Leitner 2009]);*

(4) *there is a normal conformal vector field V of nonzero length or the dimension of the space of normal conformal vector fields is ≥ 2. In particular, this is the case for Fefferman spaces over quaternionic contact structures in signature* $(4k+3, 4l+3)$ (*characterized by* $\mathfrak{hol}(M, c) \subset \mathfrak{sp}(k+1, l+1)$);

(5) (M, c) *is spin and for $g \in c$ with spinor bundle S^g there are twistor spinors $\varphi_{i=1,2} \in \Gamma(M, S^g)$ such that the spaces $\{X \in TM \mid X \cdot \varphi_i = 0\}$ are complementary at each point.*

Corollary 1.3. *Under the assumptions of Theorem 1.1*, $\mathrm{rk}(\mathcal{O}) \leq 1$ *for each of the following cases*:

(1) $(p, q) = (3, 3)$ *and* $\mathfrak{hol}(M, c) \subsetneq \mathfrak{spin}(3, 4)$;

(2) $(p, q) = (n, n)$ *and* $\mathfrak{hol}(M, c) \subset \mathfrak{gl}(n+1)$;

(3) $\mathrm{Hol}(M, c)$ *fixes a nontrivial* 2*-form, i.e.*, (M, c) *admits a normal conformal vector field. In particular, this applies to Fefferman conformal structures, i.e., to* $(p, q) = (2r+1, 2s+1)$ *and* $\mathfrak{hol}(M, c) \subset \mathfrak{su}(r+1, s+1)$;

(4) *the action of* $\mathrm{Hol}(M, c)$ *on the light cone* $\mathcal{N} \subset \mathbb{R}^{p+1,q+1}$ *does not have an open orbit.*

For each of these geometries one can give an explicit subspace $V \subset TM$ with $\mathrm{Im}(\mathcal{O}) \subset V$ *at each point.*

Two results in these corollaries can be found in the literature — the statement about almost Einstein [Fefferman and Graham 1985] and special Einstein products [Gover and Leitner 2009] in Corollary 1.2 and the statement about Fefferman conformal structures [Graham and Hirachi 2008] in Corollary 1.3 — but the general theory as developed here allows alternative proofs of these facts. Note also that the last two conditions in Corollary 1.2 are conformally invariant and do not refer to a distinguished metric in the conformal class.

As the main tool in proving these results, we introduce what we call *the conformal holonomy distribution*. At each point $x \in M$ it is defined as

$$\mathcal{E}_x := \mathfrak{hol}_x(M, c) \cap \mathfrak{g}_1.$$

The vector space $\mathcal{E}_x$ can be canonically identified with a subspace in T_xM. When varying x, its dimension however may not be constant. Instead, varying x provides a stratification of the manifold into sets over which the dimension of $\mathcal{E}_x$ is constant. We will see in Theorem 4.1 that these strata are unions of the *curved orbits* defined by conformal holonomy reductions, introduced recently in [Čap et al. 2014] in the context of Cartan geometries. Moreover we will show that an open and dense set in M can be covered by open sets over which the dimension of $\mathcal{E}_x$ is constant. Very surprisingly, we find that, when considered over such an open set, $\mathcal{E}$ is closely related to the aforementioned generic distributions:

Theorem 1.4. *Let $(M^{p,q}, c)$ be a smooth conformal manifold. Then there is an open and dense set in M that is covered by open sets U over which $\mathcal{E}|_U$ is a vector distribution. Over each such U, $\mathcal{E}|_U$ is either integrable, or*

- *$(p, q) = (2, 3)$ and $\mathcal{E}|_U$ is a generic rank 2 distribution, or*
- *$(p, q) = (3, 3)$ and $\mathcal{E}|_U$ is a generic rank 3 distribution.*

In both cases, $\mathcal{E}|_U$ defines the conformal class c on U in the sense of [Nurowski 2005; Bryant 2006].

We should also point out that the statements in Theorem 1.1 remain valid when $\mathrm{rk}(\mathcal{O})$ at x is replaced by the dimension of $\mathcal{E}_x$. We believe that the conformal holonomy distribution will turn out to be a powerful tool that allows us to obtain not only results about the obstruction tensor but also about other aspects of special conformal structures.

This article is organized as follows: Section 2 reviews the relevant tractor calculus and the ambient metric construction in conformal geometry. Moreover, we discuss special conformal structures that will be important in the sequel from the point of view of holonomy reductions. Section 3 is then devoted to the proof of the first part of Theorem 1.1. The key ingredient is a recently established relation between conformal and ambient holonomy [Čap et al. 2016]. In Section 4A we introduce the conformal holonomy distribution $\mathcal{E}$ and study its basic properties. These results are then applied in Section 5 to derive constraints on the obstruction tensor for many families of special conformal structures, in particular those in signature (3, 3) discovered by Bryant.

2. Conformal structures, tractors and ambient metrics

2A. ***Conventions.*** Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇^g. denote by $\Lambda^k := \Lambda^k T^*M$ the k-forms and by $\mathfrak{so}(TM)$ the endomorphisms of TM that are skew with respect to g. By $\mathrm{R} = \mathrm{R}^g \in \Lambda^2 \otimes \mathfrak{so}(TM)$ we will denote the curvature endomorphism of ∇^g, i.e., one has for all vector fields $X, Y \in \mathfrak{X}(M)$

$$\mathrm{R}^g(X, Y) = [\nabla^g_X, \nabla^g_Y] - \nabla^g_{[X,Y]}.$$

By contraction one obtains the Ricci tensor and scalar curvature,

$$\mathrm{Ric}^g(X, Y) := \mathrm{tr}(Z \mapsto \mathrm{R}^g(Z, X)Y), \qquad \mathrm{scal}^g := \mathrm{tr}_g \mathrm{Ric}^g,$$

and we denote by P^g the Schouten tensor

$$\mathsf{P}^g := \frac{1}{n-2}\Big(\mathrm{Ric}^g - \frac{1}{2(n-1)}\,\mathrm{scal}^g\, g\Big). \tag{1}$$

Using g to raise and lower indices, we will also consider P^g and Ric^g as g-symmetric endomorphisms of TM denoted with the same symbol. The metric dual 1-form of a vector $V \in TM$ is $V^\flat = g(V, \cdot)$ and from a 1-form $\alpha \in T^*M$ we obtain a tangent vector $\alpha^\sharp$ via $g(\alpha^\sharp, \cdot) = \alpha$. From the Schouten tensor we obtain the Cotton tensor $C \in \Lambda^2 \otimes TM$,

$$C^g(X, Y) := (\nabla^g_X \mathsf{P}^g)(Y) - (\nabla^g_Y \mathsf{P}^g)(X),$$

and the Weyl tensor $W \in \Lambda^2 \otimes \mathfrak{so}(TM)$, considered as skew-symmetric bilinear map from $TM \times TM$ to $\mathfrak{so}(TM)$,

$$W^g(X, Y) := \mathrm{R}^g(X, Y) + X^\flat \otimes \mathsf{P}^g(Y) + \mathsf{P}^g(X) \otimes Y - \mathsf{P}^g(Y) \otimes X - Y^\flat \otimes \mathsf{P}^g(X).$$

We will also write $C^g(Z; X, Y) := g(Z, C^g(X, Y))$ for the metric dual of C^g, drop the g and use the index convention $C_{kij} = C(\partial_k; \partial_i, \partial_j)$.

2B. ***Conformal tractor calculus.*** Let (M, c) be a smooth conformal manifold of signature (p, q), dimension $n = p+q \geq 3$ and let $\mathcal{T} \to M$ denote the standard tractor bundle for (M, c) with normal conformal Cartan connection ∇^{nc} and tractor metric h as introduced in [Bailey et al. 1994]. The tractor bundle $\mathcal{T}$ is equipped with a canonical filtration $\mathcal{I} \subset \mathcal{I}^\perp \subset \mathcal{T}$, where $\mathcal{I}$ is a distinguished lightlike line. For each metric $g \in c$, one finds distinguished lightlike tractors $s_\pm$ which lead to an identification

$$\mathcal{T} \to \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}, \quad T \mapsto \alpha s_- + V + \beta s_+ \mapsto (\alpha, V, \beta)^\top, \tag{2}$$

under which the tractor metric becomes

$$h((\alpha_1, V_1, \beta_1), (\alpha_2, V_2, \beta_2)) = \alpha_1\beta_2 + \alpha_2\beta_1 + g(V_1, V_2),$$

and in this identification, s_- generates $\mathcal{I}$. Under a conformal change $\tilde{g} = e^{2\sigma} g$, the transformation of the metric identification (2) of a standard tractor is given by

$$\begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\alpha} \\ \tilde{Y} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} e^{-\sigma}(\alpha - Y(\sigma) - \frac{1}{2}\beta \cdot \| \operatorname{grad}^g \sigma \|_g^2) \\ e^{-\sigma}(Y + \beta \cdot \operatorname{grad}^g \sigma) \\ e^\sigma \beta \end{pmatrix}. \tag{3}$$

From this one observes the image of a linear subspace $H \subset \mathcal{I}^\perp \subset \mathcal{T}$ under the map

$$\mathcal{I}^\perp \to \mathcal{I}^\perp/\mathcal{I} \to TM, \quad \alpha s_- + V \mapsto [\alpha s_- + V] \mapsto V$$

is conformally invariant, i.e., independent of the choice of $g \in c$. For ∇^{nc} expressed in terms of the splitting (2) we find

$$\nabla^{\mathrm{nc}}_X \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} X(\alpha) - \mathsf{P}^g(X, Y) \\ \nabla^g_X Y + \alpha X + \beta \mathsf{P}^g(X) \\ X(\beta) - g(X, Y) \end{pmatrix}. \tag{4}$$

The curvature of ∇^{nc} is given by $\mathrm{R}^{\mathrm{nc}}(X,Y) = C^g(X,Y) \wedge s_-^\flat + W^g(X,Y)$, where we identified the bundles $\mathfrak{so}(\mathcal{T},h)$ and $\Lambda^2\mathcal{T}^*$ by means of h in the usual way by the musical isomorphisms $^\flat$ and $^\sharp$.

Turning to adjoint tractors, it follows from identification (2) that for fixed $g \in c$, each fiber of the bundle $\mathfrak{so}(\mathcal{T},h)$ of skew-symmetric endomorphisms of the tractor bundle can be identified with skew-symmetric matrices of the form

$$\Phi(\mu,(a,A),Z) := \begin{pmatrix} -a & \mu & 0 \\ Z & A & -\mu^\sharp \\ 0 & -Z^\flat & a \end{pmatrix},$$

where Z is a vector, μ a 1-form, $a \in \mathbb{R}$, and A is skew-symmetric for g. For example, the curvature of ∇^{nc} is identified with

$$\mathrm{R}^{\mathrm{nc}}(X,Y) = \begin{pmatrix} 0 & C^g(X,Y)^\flat & 0 \\ 0 & W^g(X,Y) & -C^g(X,Y) \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

In particular, each choice of g yields an obvious pointwise $|1|$-grading of $\mathfrak{so}(\mathcal{T},h)$ according to the splitting

$$\mathfrak{g}_{-1} = \{\Phi(0,0,Z)\}, \quad \mathfrak{g}_0 = \{\Phi(0,(a,A),0)\}, \quad \mathfrak{g}_1 = \{\Phi(\mu,0,0)\}, \tag{6}$$

with brackets given by

$$[(a,A),Z] = (a+A)Z, \quad [(a,A),\mu] = -\mu\circ(A+a\mathrm{Id}), \quad [Z,\mu] = (\mu(Z),\mu\wedge Z^\flat).$$

In particular, $[\mathfrak{g}_i,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. It follows that the induced derivative ∇^{nc} on a section $\Phi = \Phi(\mu,(a,A),Z)$ of $\mathfrak{so}(\mathcal{T},h)$ is given by

$$\begin{aligned} &\nabla^{\mathrm{nc}}_X\Phi \\ &= \begin{pmatrix} -X(a)-\mathsf{P}^g(X,Z)-\mu(X) & \nabla^g_X\mu-\mathsf{P}^g(X,(A+a\mathrm{Id})\cdot) & 0 \\ \nabla^g_X Z-(A+a)X & \nabla^g_X A+\mu\wedge X^\flat-Z^\flat\wedge\mathsf{P}^g(X,\cdot) & -\nabla^g_X\mu^\sharp+(a-A)\mathsf{P}^g(X) \\ 0 & -\nabla^g_X Z^\flat+(AX)^\flat+aX^\flat & X(a)+\mathsf{P}^g(X,Z)+\mu(X) \end{pmatrix}. \end{aligned} \tag{7}$$

2C. *Holonomy reductions of conformal structures.* In this section we list definitions and properties of the conformal structures which have appeared in the introduction and to which the main Theorem 1.1 can be applied. They all turn out to be characterized in terms of a conformal holonomy reduction. Here, for $(M^{p,q},c)$ a smooth conformal manifold, its conformal holonomy at $x \in M$ is defined as

$$\mathrm{Hol}_x(M,c) := \mathrm{Hol}_x(\mathcal{T},\nabla^{\mathrm{nc}})$$

and gives a class of conjugated subgroups in $\mathrm{O}(p+1,q+1)$. The interplay between *conformal holonomy reductions*, i.e., when $\mathrm{Hol}^0_x(M,c)$ is a proper subgroup of $\mathrm{SO}(p+1,q+1)$, and distinguished metrics in the conformal class has been the focus of active research. We will review the most important ones relevant here.

2C.1. *Geometries with reducible holonomy representation.* One initial result is that holonomy invariant lines $L \subset \mathbb{R}^{p+1,q+1}$ are in one-to-one correspondence to *almost Einstein scales* in c [Gauduchon 1990; Bailey et al. 1994; Gover 2005; Gover and Nurowski 2006; Leitner 2005; Leistner 2006] by which we mean that on an open, dense subset of M there exists around each point locally an Einstein metric $g \in c$. If L is lightlike, g is Ricci flat and otherwise one has $\operatorname{sgn}(\mathrm{scal}^g) = -\operatorname{sgn}\langle L, L\rangle_{p+1,q+1}$.

A holonomy-invariant nondegenerate subspace $H \subset \mathbb{R}^{p+1,q+1}$ of dimension $k \geq 2$ corresponds locally and off a singular set to the existence of a *special Einstein product* in the conformal class [Leitner 2004; Armstrong 2007; Armstrong and Leitner 2012]. Here, we say that a pseudo-Riemannian manifold (M, g) is a special Einstein product if (M, g) is isometric to a product $(M_1, g_1) \times (M_2, g_2)$, where (M_i, g_i) are Einstein manifolds of dimensions $k-1$ and $n-k-1$ for $k \geq 2$ and in case $k \neq 2, n$ we additionally require that

$$\mathrm{scal}^{g_1} = -\frac{(k-1)(n-2)}{(n-k+1)(n-k)}\,\mathrm{scal}^{g_2} \neq 0.$$

Finally, if $H \subset \mathbb{R}^{p+1,q+1}$ is totally degenerate, of dimension $k+1 \geq 2$ and holonomy invariant, there exists — again locally and off a singular set — a metric $g \in c$ admitting a ∇^g-invariant and totally degenerate distribution $\mathcal{L} \subset TM$ of rank k which additionally satisfies $\mathrm{Im}(\mathrm{Ric}^g) \subset \mathcal{L}$, as has been shown in [Leistner 2006; Leistner and Nurowski 2012; Lischewski 2015].

2C.2. *Geometries defined via normal conformal Killing forms.* Suppose next that $\mathrm{Hol}(M, c)$ lies in the isotropy subgroup of a $(k+1)$-form, i.e., there exists a ∇^{nc}-parallel tractor $k+1$-form $\hat{\alpha} \in \Gamma(M, \Lambda^{k+1}\mathcal{T}^*)$. Such holonomy reductions have been studied in [Leitner 2005]. For fixed $g \in c$, consider the splitting of $\mathcal{T}$ with respect to g and write $\hat{\alpha}$ as

$$\hat{\alpha} = s_+^\flat \wedge \alpha + \alpha_0 + s_-^\flat \wedge s_+^\flat \wedge \alpha_\pm + s_-^\flat \wedge \alpha_- \tag{8}$$

for uniquely determined differential forms $\alpha, \alpha_0, \alpha_\pm, \alpha_-$ on M. The k-form $\alpha \in \Omega^k(M)$ turns out to be *normal conformal (nc)*, that is α is a conformal Killing form subject to additional conformally covariant differential normalization conditions that can be found in [Leitner 2005]. Moreover, $\alpha_0, \alpha_\pm, \alpha_-$ can be expressed in terms of α and ∇^g. Conversely, every normal conformal Killing form determines a parallel tractor form. The situation simplifies considerably if $k = 1$, i.e., there is a parallel adjoint tractor. In this case it is convenient to consider the metric dual $V = \alpha^\sharp \in \mathfrak{X}(M)$ of the associated normal conformal Killing form α, which is a *normal conformal vector field*. By this, we mean that V is a conformal vector field which additionally satisfies $C^g(V, \cdot) = W^g(V, \cdot) = 0$.

Examples of manifolds admitting normal conformal vector fields are so-called *Fefferman spaces* [Fefferman 1976]. They yield conformal structures (M, c) of signature $(2r+1, 2s+1)$ defined on the total spaces of S^1-bundles over strictly pseudoconvex CR manifolds. From the holonomy point of view they are (at least locally) equivalently characterized by the existence of a parallel adjoint tractor [Leitner 2007; Čap and Gover 2010], which is an almost complex structure for the tractor metric, i.e., $\mathrm{Hol}(M, c) \subset \mathrm{SU}(r+1, s+1)$. Here, we used a result from [Leitner 2008; Čap and Gover 2010] which asserts that unitary conformal holonomy is automatically special unitary.

Other geometries that are characterized by the existence of distinguished normal conformal vector fields include pseudo-Riemannian manifolds (M, g) of signature $(4r+3, 4m+3)$ with conformal holonomy group in the symplectic group

$$\mathrm{Sp}(r+1, m+1) \subset \mathrm{SO}(4r+4, 4m+4),$$

see [Alt 2008]. The models of such manifolds are S^3-bundles over a quaternionic contact manifold equipped with a canonical conformal structure, introduced in [Biquard 2000].

2C.3. *Conformal holonomy and twistor spinors.* If (M, c) is actually spin for one, and hence all, $g \in c$, the presence of conformal Killing spinors always leads to reductions of $\mathrm{Hol}(M, c)$. To formulate these, let $S^g \to M$ denote the real or complex spinor bundle over M which possesses a spinor covariant derivative ∇^{S^g} and vectors act on spinors by Clifford multiplication $\mathrm{cl} = \cdot$, see [Baum 1981]. Given these data, the spin Dirac operator is given as $D^g = \mathrm{cl} \circ \nabla^{S^g}$. Now assume that (M, g) admits a twistor spinor, i.e., a section $\varphi \in \Gamma(M, S^g)$ solving

$$\nabla_X^{S^g} \varphi + \frac{1}{n} X \cdot D^g \varphi = 0. \tag{9}$$

Equation (9) is conformally invariant, see [Baum et al. 1991], and to φ we can associate the union of subspaces

$$\mathcal{L}_\varphi := \{X \in TM \mid X \cdot \varphi = 0\} \subset TM,$$

which does not depend on the choice of $g \in c$. Equation (9) can be prolonged, see [Baum et al. 1991], and using this prolonged system it becomes immediately clear that a twistor spinor φ is equivalently described as a parallel section ψ of the spin tractor bundle associated to (M, c). Its construction can be found in [Leitner 2007], for instance. As ψ is parallel, it is at each point annihilated by $\mathfrak{hol}_x(M, c)$ under Clifford multiplication, i.e.,

$$\mathfrak{hol}_x(M, c) \cdot \psi_x = 0 \quad \text{for all } x \in M. \tag{10}$$

2C.4. *Exceptional cases.* Finally we describe conformal structures in dimension 5 and 6 with holonomy algebra contained in $\mathfrak{g}_2 \subset \mathfrak{so}(3, 4)$, the noncompact simple

Lie algebra of dimension 14, or in $\mathfrak{spin}(3,4) \subset \mathfrak{so}(4,4)$, respectively. They turn out to be closely related to generic distributions:

Recall that a distribution $\mathcal{D}$ of rank 2 on a 5-manifold M is *generic* if

$$[\mathcal{D}, [\mathcal{D}, \mathcal{D}]] + [\mathcal{D}, \mathcal{D}] + \mathcal{D} = TM.$$

It is known by work of Nurowski [2005] that $\mathcal{D}$ canonically defines a conformal structure $c_{\mathcal{D}}$ of signature $(2,3)$ on M^5 whose conformal holonomy is reduced to $\mathfrak{g}_2 \subset \mathfrak{so}(3,4)$, see also [Čap and Sagerschnig 2009]. Analogously, a distribution $\mathcal{D}$ of rank 3 on a 6-manifold M is *generic* if $[\mathcal{D}, \mathcal{D}] + \mathcal{D} = TM$, and Bryant [2006] showed that $\mathcal{D}$ canonically defines a conformal structure $c_{\mathcal{D}}$ of signature $(3,3)$ on M whose conformal holonomy is reduced to $\mathfrak{spin}(4,3) \subset \mathfrak{so}(4,4)$. In both cases, the holonomy characterization implies that $(M, c_{\mathcal{D}})$ admits a parallel tractor 3- or 4-form, respectively. Moreover, [Hammerl and Sagerschnig 2011b] shows that there is in both cases a distinguished twistor spinor φ which encodes $\mathcal{D}$ in the sense that

$$\mathcal{L}_\varphi = \mathcal{D} \quad \text{at each point.} \tag{11}$$

2D. *Conformal ambient metrics.* Let (M, c) be a smooth conformal manifold of dimension ≥ 3. For our purposes we do not need the general theory of ambient metrics as presented in [Fefferman and Graham 2012], which can be consulted for more details, but it suffices to deal with ambient metrics which are in normal form with respect to some $g \in c$. A (straight) preambient metric in normal form with respect to $g \in c$ is a pseudo-Riemannian metric $\tilde{g}$ on an open neighborhood $\tilde{M}$ of $\{1\} \times M \times \{0\}$ in $\mathbb{R}^+ \times M \times \mathbb{R}$ such that for $(t, x, \rho) \in \tilde{M}$

$$\tilde{g} = 2t\,dt\,d\rho + 2\rho\,dt^2 + t^2 g_\rho(x), \tag{12}$$

with $g_0 = g$. We call $(\tilde{M}, \tilde{g})$ an *ambient metric for* $(M, [g])$ *in normal form* with respect to g if

- $\widetilde{\mathrm{Ric}} = O(\rho^\infty)$ if n is odd, and
- $\widetilde{\mathrm{Ric}} = O(\rho^{(n/2)-1})$ and $\mathrm{tr}_g(\rho^{1-(n/2)}\widetilde{\mathrm{Ric}}_{|TM\otimes TM}) = 0$ along $\rho = 0$, if n is even.

The existence and uniqueness assertion for ambient metrics [Fefferman and Graham 1985; 2012] states that for each choice of g there is an ambient metric in normal form with respect to g. In all dimensions $n \geq 3$, g_ρ has an expansion of the form $g_\rho = \sum_{k\geq 0} g^{(k)} \rho^k$ starting with

$$g_\rho = g + 2\rho \mathsf{P}^g + O(\rho^2),$$

and in odd dimensions the Ricci flatness condition determines $g^{(k)}$ for all k, whereas in even dimensions only the $g^{(k<n/2)}$ and the trace of $g^{(n/2)}$ are determined.

We shall sometimes work with ambient indices $I \in \{0, i, \infty\}$, where the i are indices for coordinates on M, 0 refers to ∂_t and ∞ to ∂_ρ, i.e.,

$$T\tilde{M} \ni V = V^0\partial_t + V^i\partial_i + V^\infty\partial_\rho.$$

For the Levi-Civita connection of any metric of the form (12) one computes [Fefferman and Graham 2012, Lemma 3.2],

$$\begin{aligned} &\widetilde{\nabla}_{\partial_i}\partial_j = -\frac{1}{2}t\dot{g}_{ij}\partial_t + \Gamma^k_{ij}\partial_k + (\rho\dot{g}_{ij} - g_{ij})\partial_\rho, && \widetilde{\nabla}_{\partial_t}\partial_t = \widetilde{\nabla}_{\partial_\rho}\partial_\rho = 0,\\ &\widetilde{\nabla}_{\partial_i}\partial_t = \frac{1}{t}\partial_i, \qquad \widetilde{\nabla}_{\partial_i}\partial_\rho = \frac{1}{2}g^{kl}\dot{g}_{il}\partial_k, && \widetilde{\nabla}_{\partial_\rho}\partial_t = \frac{1}{t}\partial_\rho, \end{aligned} \tag{13}$$

where, abusing notation, g_{ij} denotes the components of g_ρ and Γ^k_{ij} the Christoffel symbols of g_ρ. In particular, $T := t\partial_t$ is an Euler vector field for $(\tilde{M}, \tilde{g})$, i.e.,

$$\widetilde{\nabla}T = \mathrm{Id}. \tag{14}$$

For n even a conformally invariant $(0, 2)$-tensor on M, the *ambient obstruction tensor* $\mathcal{O}$, obstructs the existence of smooth solutions to $\widetilde{\mathrm{Ric}} = O(\rho^{n/2})$. For $\tilde{g}$ in normal form with respect to g it is given by

$$\mathcal{O} = c_n(\rho^{1-(n/2)}(\widetilde{\mathrm{Ric}}_{|TM\otimes TM}))_{\rho=0}, \tag{15}$$

where c_n is some known nonzero constant; see [Fefferman and Graham 2012]. From this one can deduce that $\mathcal{O}$ is trace- and divergence-free.

Tractor data can be recovered from ambient data as shown in [Čap and Gover 2003]. For ambient metrics in normal form with respect to $g \in c$, this reduces to the following observation, see [Graham and Willse 2012] for more details: Identify M with the level set $\{\rho = 0, t = 1\}$ in $\tilde{M}$. Then $T\tilde{M}_{|M}$ splits into $\mathbb{R}\partial_t \oplus TM \oplus \mathbb{R}\partial_\rho$, which is isomorphic to the g-metric identification of the tractor bundle $\mathcal{T}$ under the map

$$\partial_t \mapsto s_-, \quad TM \xmapsto{\mathrm{Id}} TM, \quad \partial_\rho \mapsto s_+. \tag{16}$$

The map (16) is an isometry of bundles over M with respect to $\tilde{g}$ and h and the pullback of $\widetilde{\nabla}$, the Levi-Civita connection of $\tilde{g}$, to $T\tilde{M}_{|M}$ coincides with (4). This also follows directly from an inspection of (3) and (13). With these identifications, for fixed $g \in c$ we view the tractor data as restrictions of ambient data for an ambient metric which is in normal form with respect to g.

3. The ambient obstruction tensor and conformal holonomy

We outline how the image of the obstruction tensor can be identified with a distinguished subspace of the infinitesimal conformal holonomy algebra at each point. This requires some preparation:

Let V be a vector space. The standard action # of $\mathrm{End}(V)$ on V extends to an action on the space $T^{r,s}V$ of (r,s) tensors over V. This action will be denoted by the same symbol. Thus, $\mathrm{End}(V)\otimes\mathrm{End}(V)$ acts on $T^{r,s}V$ with a double #-action, explicitly given by

$$(A\otimes B)\,\#\#\,(\eta)=A\,\#\,(B\,\#\,\eta). \tag{17}$$

Given a pseudo-Riemannian manifold (N,h), we can view its curvature tensor R^h as section of the bundle $\mathfrak{so}(N,h)\otimes\mathfrak{so}(N,h)$, and applying (17) pointwise yields an action $\mathrm{R}^h\#\#$ of the curvature on arbitrary tensor bundles of N.

Returning to the original setting, let (M,g) be a pseudo-Riemannian manifold of even dimension and let $(\tilde{M},\tilde{g})$ be an associated ambient metric which is in normal form with respect to g. Let $\widetilde{\Delta}=\widetilde{\nabla}_A\widetilde{\nabla}^A$ denote the usual connection Laplacian on the ambient manifold. In [Gover and Peterson 2006] a modified Laplace-type operator

$$\not{\Delta}=\widetilde{\Delta}+\tfrac{1}{2}\widetilde{\mathrm{R}}\#\# \tag{18}$$

is introduced and will be used in the subsequent calculations.

The previous observations enable us to prove the main result of this section:

Theorem 3.1. *Let $(M,c=[g])$ be of even dimension >2. For every $g\in c$ one has*

$$s_-^\flat\wedge(X\lrcorner\,\mathcal{O})\in\mathfrak{hol}_x(M,[g])\quad \textit{for all } x\in M \textit{ and } X\in T_xM.$$

Proof. The proof uses the notion of *infinitesimal holonomy*: within in the Lie algebra $\mathfrak{hol}_x(M,c)$ of $\mathrm{Hol}_x(M,c)$ at a point $x\in M$, we consider the *infinitesimal holonomy algebra at x*, i.e., the Lie algebra of iterated derivatives of the tractor curvature evaluated at x,

$$\mathfrak{hol}'_x(M,c):=\mathrm{span}_{\mathbb{R}}\{\nabla^{\mathrm{nc}}_{X_1}(\cdots(\nabla^{\mathrm{nc}}_{X_{l-1}}(\mathrm{R}^{\mathrm{nc}}(X_{l-1},X_l))))(x)\mid l\geq 2, X_1,\ldots,X_l\in\mathfrak{X}(M)\}.$$

For more details on the infinitesimal holonomy refer to [Kobayashi and Nomizu 1963, Chap. II.10] or [Nijenhuis 1953a; 1953b; 1954]. We will in fact show that $s_-^\flat\wedge(X\lrcorner\,\mathcal{O})\in\mathfrak{hol}'_x(M,[g])$ for all $x\in M$ and $X\in T_xM$.

Assume first that $n>4$. Let $(\tilde{M},\tilde{g})$ be an associated ambient manifold for $(M,[g])$ which is in normal form with respect to some fixed g in the conformal class. For $x\in M$ let

$$\mathfrak{hol}_x(\tilde{M},\tilde{g}):=\mathrm{span}_{\mathbb{R}}\{\widetilde{\nabla}_{X_1}(\cdots\widetilde{\nabla}_{X_{l-2}}(\widetilde{\mathrm{R}}(X_{l-1},X_l)))(x)\mid l\geq 2,\ X_i\in\mathfrak{X}(\tilde{M})\}$$

denote the infinitesimal holonomy algebra of $(\tilde{M},\tilde{g})$ at x and for $k\geq 0$ let $\mathfrak{hol}^k_x(\tilde{M},\tilde{g})$ denote the subspace of elements for which at most k of the X_i have a not identically zero ∂_ρ-component. Then [Čap et al. 2016, Theorem 3.1] asserts that under the identifications from Section 2D,

$$\mathfrak{hol}'_x(M,c)=\mathfrak{hol}^{(n/2)-2}_x(\tilde{M},\tilde{g}). \tag{19}$$

Indeed, for $(\tilde{M}, \tilde{g})$ which is in normal form with respect to g, equality (19) can be verified as follows:

From the identifications from Section 2D one obtains immediately the inclusion $\subset$ in (19). In order to prove the converse, we obtain with [Graham and Willse 2012, Lemma 3.1] and [Fefferman and Graham 2012, Propositon 6.1] that

$$\begin{aligned} &\widetilde{\mathrm{R}}(\partial_i, \partial_j)(x) = \mathrm{R}^{\mathrm{nc}}(\partial_i, \partial_j)(x), \qquad\qquad \widetilde{\mathrm{R}}(\partial_t, \partial_I)(x) = 0, \\ &\widetilde{\mathrm{R}}(\partial_\rho, \partial_i)(x) = 3g^{kl}(\nabla^{\mathrm{nc}}_{\partial_k}\mathrm{R}^{\mathrm{nc}})(\partial_l, \partial_i)(x). \end{aligned} \tag{20}$$

The right sides of these expressions clearly lie in $\mathfrak{hol}'_x(M, c)$. To proceed, using linearity and commuting covariant derivatives, it suffices to prove that

$$(\widetilde{\nabla}^k_{X_i}\widetilde{\nabla}^l_{\partial_\rho}\widetilde{\nabla}^j_{\partial_t}\widetilde{\mathrm{R}})(Y, Z)(x) \in \mathfrak{hol}'_x(M, c), \tag{21}$$

where $k, j, l \geq 0$, $X_i \in T_xM$, $Y, Z \in T_x\tilde{M}$ and $l \leq \frac{1}{2}n - 3$ or $\frac{1}{2}n - 2$ (depending on whether one of Y, Z has a ∂_ρ-component): given an element of the form (21) one first applies Proposition 6.1 from [Fefferman and Graham 2012], which rewrites ∂_t derivatives of $\widetilde{\mathrm{R}}$, and obtains a linear combination of elements of the form (21) with $j = 0$ and Y, Z have no ∂_t-component. Thus, it suffices to prove (21) for $j = 0$. This is then achieved by induction over l. Indeed, for $l = 0$ the statement follows from the last equation in (20). Furthermore, we may assume that $Y = \partial_\rho$ (otherwise all differentiations are tangent to M or we use the second Bianchi identity) and $Z \in T_xM$. However, Lemma 3.1 from [Graham and Willse 2012] allows us to rewrite ∂_ρ-derivatives $(\widetilde{\nabla}^l_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, Z)$ up to $l \leq \frac{1}{2}n - 3$ in terms of $(\widetilde{\nabla}^l_{\partial_\rho}\widetilde{\mathrm{R}})|_{TM\times TM}$. Then applying the second Bianchi identity and the induction hypothesis shows the claim (21). This proves the equality (19).

Using again the identifications from Section 2D, we will now show that for $x \in M$ and $X \in T_xM$ we have

$$\partial_t^\flat(x) \wedge (X \lrcorner \mathcal{O})(x) \in \mathfrak{hol}_x^{(n/2)-2}(\tilde{M}, \tilde{g}). \tag{22}$$

With this, equality (19) and the inclusion $\mathfrak{hol}'_x(M, c) \subset \mathfrak{hol}(M, c)$ will imply Theorem 3.1. In order to verify property (22), note that, as observed in [Gover and Peterson 2006], on any pseudo-Riemannian manifold one has (in abstract indices)

$$4\widetilde{\nabla}_{A_1}\widetilde{\nabla}_{B_1}\widetilde{\mathrm{Ric}}_{A_2B_2} = \tilde{\Delta}\widetilde{\mathrm{R}}_{A_1A_2B_1B_2} - \widetilde{\mathrm{Ric}}_{CA_1}\widetilde{\mathrm{R}}^C{}_{A_2B_1B_2} + \widetilde{\mathrm{Ric}}_{CB_1}\widetilde{\mathrm{R}}^C{}_{B_2A_1A_2}, \tag{23}$$

where here A_1, A_2 and B_1, B_2 are pairwise skew-symmetrized. Indeed, (23) is a straightforward consequence of the second Bianchi identity. As in our situation, $\widetilde{\mathrm{Ric}} = O(\rho^{(n/2)-1})$, it follows that

$$4\widetilde{\nabla}_{A_1}\widetilde{\nabla}_{B_1}\widetilde{\mathrm{Ric}}_{A_2B_2} = \tilde{\Delta}\widetilde{\mathrm{R}}_{A_1A_2B_1B_2} + O(\rho^{(n/2)-1}). \tag{24}$$

In the next steps we use the general fact that if B is a tensor field on $\tilde{M}$, for example a $(0,2)$-tensor field, such that $B = O(\rho^m)$ for an $m \geq 1$, then, for all $X, Y \in TM$

$$\begin{aligned} \rho^{-m} B(X,Y)|_{\rho=0} &= (\widetilde{\nabla}^m_{\partial_\rho} B)(X,Y)|_{\rho=0}, \\ (\widetilde{\nabla}^k_{\partial_\rho} B)(X,Y)|_{\rho=0} &= 0 \quad \text{for } k = 0, \dots, m-1. \end{aligned} \tag{25}$$

Indeed, $B = O(\rho^m)$ implies for $k = 0, \dots, m-1$ that

$$0 = \partial^k_\rho (B(X,Y))|_{\rho=0} = (\widetilde{\nabla}^k_{\partial_\rho} B)(X,Y)|_{\rho=0},$$

where the second equality holds because of $\widetilde{\nabla}_{\partial_\rho}\partial_\rho = 0$ and $\widetilde{\nabla}_{\partial_\rho} X \in \mathfrak{X}(M)$ for $X \in \mathfrak{X}(M)$. This also implies that

$$\rho^{-m} B(X,Y)|_{\rho=0} = \partial^m_\rho (B(X,Y))|_{\rho=0} = (\widetilde{\nabla}^m_{\partial_\rho} B)(X,Y)|_{\rho=0},$$

proving both relations in (25).

Now we return to equation (24) and see, using (25), that it implies

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \tilde{\Delta} \widetilde{\mathrm{R}})(Y_1, Y_2, Z_1, Z_2)(x) = 4(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \widetilde{\nabla}_{Y_1} \widetilde{\nabla}_{Z_1} \widetilde{\mathrm{Ric}})(Y_2, Z_2)(x), \tag{26}$$

where now $x \in M$, Y_i, Z_i are ambient vector fields and Y_1, Y_2 as well as Z_1, Z_2 are skew-symmetrized. Now let $Y_1 = \partial_\rho$ and $Y_2 = X \in \mathfrak{X}(M)$ and insert $\widetilde{\mathrm{Ric}} = O(\rho^{(n/2)-1})$ into the right side of (26). It follows that, with $x \in M$, the resulting expression is zero unless one of the Z_i is proportional to ∂_ρ and the other one is a tangent vector $Y \in T_x M$. For this choice of vectors we have

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \tilde{\Delta} \widetilde{\mathrm{R}})(\partial_\rho, X, \partial_\rho, Y)(x) = (\widetilde{\nabla}^{(n/2)-1}_{\partial_\rho} \widetilde{\mathrm{Ric}})(X,Y)(x), \tag{27}$$

for $X, Y \in TM$. Hence, by definition (15) and the observation (25), one obtains a multiple of $\mathcal{O}(X,Y)$,

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho} \tilde{\Delta} \widetilde{\mathrm{R}})(\partial_\rho, X)(x) = k(n) \cdot \partial_t^\flat(x) \wedge (X \lrcorner \mathcal{O})(x), \tag{28}$$

for some nonzero numerical constant $k(n)$ which depends only on the dimension n. Note that along $M = \{\rho = 0, t = 1\}$ we have $\partial_t^\flat = d\rho$. To proceed, we analyze the left side in (28). Equations (13) show that the ambient Laplacian applied to some tensor field η has an expansion of the form

$$\widetilde{\Delta}\eta = \tilde{g}^{IJ}\widetilde{\nabla}_I \widetilde{\nabla}_J \eta = \frac{1}{t}\widetilde{\nabla}_{\partial_\rho}(\widetilde{\nabla}_{\partial_t}\eta) + \frac{1}{t}\widetilde{\nabla}_{\partial_t}(\widetilde{\nabla}_{\partial_\rho}\eta) - \frac{2\rho}{t^2}\widetilde{\nabla}_{\partial_\rho}(\widetilde{\nabla}_{\partial_\rho}\eta) + f\widetilde{\nabla}_{\partial_\rho}\eta + \tilde{D}\eta, \tag{29}$$

where f is a certain known function on $\tilde{M}$ and $\tilde{D}$ is an operator of the form

$$\tilde{D}\eta = \sum_{i,j} a_{ij} \widetilde{\nabla}_i (\widetilde{\nabla}_j \eta) + \sum_{K \in \{k,0\}} b_K \widetilde{\nabla}_K \eta. \tag{30}$$

We conclude inductively that for an arbitrary ambient tensor field η and an element $Z \in \mathfrak{so}(T\tilde{M})$ one has

$$\begin{aligned} \eta = O(\rho^l) &\Rightarrow \widetilde{\Delta}^k \eta = O(\rho^{l-k}), \\ Z(x) \in \mathfrak{hol}^l_x(\tilde{M}, \tilde{g}) &\Rightarrow (\widetilde{\Delta}^k Z)(x) \in \mathfrak{hol}^{k+l}_x(\tilde{M}, \tilde{g}). \end{aligned} \tag{31}$$

Moreover, a straightforward linear algebra calculation using the algebraic Bianchi identity for the ambient curvature reveals that (in abstract ambient indices and with brackets denoting skew symmetrization)

$$\begin{aligned} &(\widetilde{\mathrm{R}} \,\#\#\, \widetilde{\mathrm{R}})_{ABCD} \\ &\quad = 2\widetilde{\mathrm{R}}_{ABVW}\widetilde{\mathrm{R}}^{VW}{}_{CD} + 8\widetilde{\mathrm{R}}_C{}^P{}_{[A}{}^Q\widetilde{\mathrm{R}}_{B]PDQ} - 2(\widetilde{\mathrm{Ric}}^V{}_{[A}\widetilde{\mathrm{R}}_{B]VCD} + \widetilde{\mathrm{Ric}}^V{}_{[C}\widetilde{\mathrm{R}}_{D]VAB}). \end{aligned}$$

For each A, B the first term on the right hand side is contained in the holonomy algebra as it is a linear combination of curvature tensors. Similarly, the second term is a linear combination of commutators of curvature tensors and hence also in the holonomy algebra. Differentiating this $\frac{1}{2}n - 3$ times in ∂_ρ direction and using that $\widetilde{\mathrm{Ric}}$ vanishes to order $\frac{1}{2}n - 1$ shows via induction that

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho}(\widetilde{\mathrm{R}} \,\#\#\, \widetilde{\mathrm{R}}))(\partial_\rho, X)(x) \in \mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g}).$$

Next, we focus on the ρ-derivatives of $\widetilde{\Delta}$ in (28). Using the form of $\widetilde{\Delta}$ in (29) and (30) and calculating mod $\mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g})$, we find they are given by

$$(\widetilde{\nabla}^{(n/2)-3}_{\partial_\rho}\widetilde{\Delta}\widetilde{\mathrm{R}})(\partial_\rho, X)(x) = \widetilde{\nabla}^{(n/2)-3}_{\partial_\rho}(\widetilde{\Delta}\widetilde{\mathrm{R}}(\partial_\rho, X))(x) = l(n)(\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, X)(x)$$

for some numerical constant $l(n)$. Thus, we have found that for $x \in M$, $X \in T_xM$,

$$k(n)\partial_t^\flat(x) \wedge (X \lrcorner \mathcal{O})(x) - l(n) \cdot (\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, X) = E_X \tag{32}$$

for some $E_X \in \mathfrak{hol}^{(n/2)-2}_x(\tilde{M}, \tilde{g})$. Now insert $Y \in T_xM$ and ∂_ρ into the 2-forms in (32). One obtains

$$k(n)\mathcal{O}(X, Y) - l(n)(\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, X, Y, \partial_\rho) = E_X(Y, \partial\rho). \tag{33}$$

By [Fefferman and Graham 2012, Proposition 6.6] we have

$$2(\widetilde{\nabla}^{(n/2)-2}_{\partial_\rho}\widetilde{\mathrm{R}})(\partial_\rho, \partial_i, \partial_j, \partial_\rho) = \mathrm{tf}(\partial_\rho^{(n/2)} g_{ij}) + K_{ij},$$

where K_{ij} can be expressed algebraically in terms of $(\partial^k_\rho g_{ij})_{|\rho=0}$, $k < \frac{1}{2}n$, as well as $g^{ij}_{|\rho=0}$. Moreover, as follows from reviewing the above argument, E can be expressed algebraically in terms of derivatives of g_ρ and its inverse in M-directions and at most $\frac{1}{2}n - 1$ derivatives in ρ-direction and $\mathcal{O}$ is a natural tensor invariant. But then, as the ambiguity, i.e., the term $\mathrm{tf}(\partial_\rho^{n/2} g_{ij})$, can be arbitrary, equation (33) can only be true if $l(n) = 0$ from which the theorem follows if $n > 4$.

In general, it holds in every dimension that for $X \in TM$ one has

$$\mathrm{tr}_g \nabla^{\mathrm{nc}}_{\cdot}\mathrm{R}^{\mathrm{nc}}(X,\cdot) = (n-4)C(X;\cdot,\cdot) + B(X) \wedge s_-^\flat \in \mathfrak{hol}(M,c),$$

where

$$B_{ij} = \nabla^k C_{ijk} - P^{kl} W_{kijl}$$

is the Bach tensor, and where for each pair j,k we understand $\mathrm{R}^{\mathrm{nc}}_{jk}$ as an element in $\Lambda^2\mathcal{T}^*$. From this observation the theorem follows in case $n=4$, as here $\mathcal{O}$ is a multiple of the Bach tensor □

Remark. Consider the case $n=6$. It is an entirely mechanical process to turn the formulas in [Gover and Peterson 2006], section 4B into an explicit formula for derivatives of the tractor curvature, which gives a more explicit proof of Theorem 3.1 for this dimension. In order to make this more explicit, assume that there is a metric $g \in c$ and a totally lightlike subspace $\mathcal{L} \subset TM$ such that $\mathrm{Im}(\mathrm{Ric}^g) \subset \mathcal{L}$ and $\mathcal{L}$ is ∇^g invariant. Such geometries correspond to invariant null subspaces which are invariant under $\mathrm{Hol}(M,c)$ and are of importance in Section 5B. Let ∇ denote the tractor derivative ∇^{nc} coupled to ∇^g. One can explicitly compute for this case that

$$g^{ij} s_-^\flat \wedge \mathcal{O}_{mi}\partial_j^\flat = g^{ij}g^{kl}\nabla_i\nabla_j\nabla_k \mathrm{R}^{\mathrm{nc}}_{ml} + 4\mathrm{P}^{ij}\nabla_i \mathrm{R}^{\mathrm{nc}}_{mj} + 2[\mathrm{R}^{\mathrm{nc}}_{mi}, \nabla^{\mathrm{nc}}_j \mathrm{R}^{\mathrm{nc}ij}] + 2C_m{}^{kl}\mathrm{R}^{\mathrm{nc}}_{kl}.$$

4. The conformal holonomy distribution

In this section we will introduce and study the fundamental object that provides us with the link between conformal holonomy and the ambient obstruction tensor.

4A. ***The conformal holonomy distribution.*** Let $(M, c=[g])$ be a smooth conformal manifold of arbitrary signature (p,q) and dimension $n=p+q$. For $x \in M$ consider the conformal holonomy algebra $\mathfrak{hol}_x(M,c) \subset \mathfrak{so}(\mathcal{T}_x, h_x)$. Fix $g \in c$. Theorem 3.1 motivates us to study the following subspaces of T_xM,

$$\mathcal{E}^g_x := \{\mathrm{pr}_{T_xM}\,\mathrm{Im}(A) \mid A \in \mathfrak{hol}_x(M,c),\, A\mathcal{I}=0,\, h(A\mathcal{I}^\perp,\mathcal{I}^\perp)=0\} \subset T_xM. \tag{34}$$

It follows immediately from the transformation formulas that $\mathcal{E}^g_x$ does not depend on the choice of $g \in c$, so that we can write $\mathcal{E}_x$. With respect to $g \in c$, however, $\mathcal{E}_x$ is identified with the space of elements of the holonomy algebra that are of the form $s_-^\flat \wedge X^\flat$ for some $X \in T_xM$. Equivalently and more invariantly, the space $\mathcal{E}_x$ can be identified with the space $\mathfrak{hol}_x(M,c) \cap \mathfrak{g}_1$. We call the subset of TM defined by

$$\mathcal{E} := \bigcup_{x\in M} \mathcal{E}_x \subset TM$$

the *conformal holonomy distribution*. This is a slight abuse of terminology, as the dimension of $\mathcal{E}_x$ may vary with x, so that $\mathcal{E}$ is not a vector distribution in the usual sense. Indeed, the holonomy algebras with respect to different base points are related by the adjoint action of elements in $\mathrm{O}(p+1, q+1)$ that generically do not lie in the stabilizer of s_-. Instead, define a function on M by

$$r^{\mathcal{E}}(x) := \dim \mathcal{E}_x.$$

The function $r^{\mathcal{E}}$ need not be constant over M but leads to an obvious stratification

$$M = \bigcup_{k=0}^{n} M_k,$$

where $M_k = \{x \in M \mid r^{\mathcal{E}}(x) = k\}$.

4B. ***Relation to the curved orbit decomposition.*** We now proceed to establish a relation between the stratification defined by $\mathcal{E}$ and the curved orbit decomposition for holonomy reductions of arbitrary Cartan geometries in [Čap et al. 2014]. When doing this, we restrict to the case that $\mathfrak{hol}(M, c)$ equals the stabilizer of some tensor:

Starting with the tractor data $(\mathcal{T} \to M, h, \nabla^{\mathrm{nc}})$, one recovers an underlying Cartan geometry as follows [Čap and Gover 2003]: Fix a lightlike line $L \subset \mathbb{R}^{p+1,q+1}$ and at each point $x \in M$ consider the set of all linear, orthogonal maps $\mathbb{R}^{p+1,q+1} \to \mathcal{T}_x$ which additionally map L to $\mathcal{I}_x$. This defines a principal P-bundle $\mathcal{G} \to M$, where $P \subset G = \mathrm{O}(p+1, q+1)$ is the stabilizer subgroup of L. Then the tractor connection ∇^{nc} induces a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ of type (G, P), i.e., ω is equivariant with respect to the P-right action, reproduces the generators of fundamental vector fields, and provides a global parallelism $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$. In this way, $(\mathcal{G} \to M, \omega)$ is a Cartan geometry of type (G, P). Conversely, one obtains the standard tractor bundle from these data as $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{p+1,q+1} = \widehat{\mathcal{G}} \times_G \mathbb{R}^{p+1,q+1}$, where $\widehat{\mathcal{G}} = \mathcal{G} \times_P G$ denotes the enlarged G-bundle. The Cartan connection ω lifts to a principal bundle connection $\widehat{\omega}$ on $\widehat{\mathcal{G}}$ and ∇^{nc} is then the induced covariant derivative on the associated bundle $\mathcal{T}$.

Now assume that there is a faithful representation ρ of G on some vector space V with associated vector bundle $\mathcal{H} = \widehat{\mathcal{G}} \times_G V$ and induced covariant derivative $\nabla^{\mathcal{H}}$ such that $\mathrm{Hol}(M, c)$ equals pointwise the stabilizer of a $\nabla^{\mathcal{H}}$-parallel section $\psi \in \Gamma(M, \mathcal{H})$ (if actually (M, c) is spin, the same discussion is possible for spin coverings of the groups and bundles under consideration). Such a ψ is equivalently encoded in a G-equivariant map $s : \widehat{\mathcal{G}} \to V$ which is constant along $\widehat{\omega}$-horizontal curves. To this situation the general machinery developed in [Čap et al. 2014] applies and one defines for $x \in M$ the P-type of x (with respect to ψ) to be the P-orbit $s(\mathcal{G}_x) \subset V$. Then M decomposes into a union of initial submanifolds M_α of elements with the same P-type, where α runs over all possible P-types, which in turn can be found by looking at the homogeneous model $G \to G/P$. In that work, the M_α are called *curved orbits* and it was shown that they carry a naturally induced Cartan geometry of type $(H, P \cap H)$.

Theorem 4.1. *If* $\mathrm{Hol}(M,c)$ *is equal to the stabilizer of a tensor, then the subsets of* M *on which* $r^{\mathcal{E}}$ *is constant are unions of curved orbits in the sense of* [Čap et al. 2014]. *In particular, they are unions of initial submanifolds.*

Proof. We fix a curved orbit M_α with element x_1. By definition, $x_2 \in M_\alpha$ if and only if

$$s(\mathcal{G}_{x_1}) = s(\mathcal{G}_{x_2}). \tag{35}$$

We unwind the condition (35) as follows: Let $u_{x_i} \in \mathcal{G}_{x_i}$ and let

$$[u_{x_i}] : V \ni v \mapsto [u_{x_i}, v] \in \mathcal{H}_{x_i}$$

denote the associated fiber isomorphism. As ρ is faithful the holonomy group $\mathrm{Hol}_{u_{x_i}}(\widehat{\omega}) \subset G$ will coincide with the stabilizer of $[u_{x_i}]^{-1}\psi_{x_i} \in V$ under the (ρ, G)-action. Moreover (35) is equivalent to the existence of $p \in P$ such that

$$\rho(p)([u_{x_1}]^{-1}\psi_{x_1}) = [u_{x_2}]^{-1}\psi_{x_2},$$

from which one deduces that

$$\mathrm{Ad}(p^{-1})(\mathfrak{hol}_{u_{x_1}}(\widehat{\omega})) = \mathfrak{hol}_{u_{x_2}}(\widehat{\omega}). \tag{36}$$

Using that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, one sees that (36) restricts to a map between the $\mathfrak{g}_1$-components of $\mathfrak{hol}_{u_{x_i}}(\widehat{\omega})$ which therefore have the same dimension. As

$$\mathfrak{hol}_x = [u_x] \circ \mathfrak{hol}_{u_x}(\widehat{\omega}) \circ [u_x]^{-1} \subset \mathfrak{so}(\mathcal{T}_x, h_x)$$

and $[u_x]$ preserves the lightlike line by definition of $\mathcal{G}$, we obtain that the dimensions of $\mathfrak{hol}_{x_i} \cap \mathfrak{g}_1$ also agree. Consequently, $r^{\mathcal{E}}$ is constant on the curved orbit M_α. □

Theorem 4.1 shows that, in general, the holonomy distribution $\mathcal{E}$ as studied here will induce a stratification of M that is *coarser* than the curved orbit decomposition in [Čap et al. 2014]. The following example shows that in some cases it induces the same stratification.

Example. Suppose (M,c) is of Riemannian signature and $\mathrm{Hol}_x(M,c)$ equals the stabilizer of some tractor $\zeta_x \in \mathcal{T}_x$. For any metric $g \in c$ write $\zeta = (\alpha, Y, \beta)^\top$ for smooth functions α, β and a vector field Y on M. Evaluating $\nabla^{\mathrm{nc}}\zeta = 0$ using (4) yields

$$Y = \mathrm{grad}^g \beta, \quad \alpha g = \beta \mathrm{P}^g - \mathrm{Hess}^g(\beta).$$

An element $V^\flat \wedge s_-^\flat$ lies in $\mathfrak{hol}_x(M,c) \cap \mathfrak{g}_1$ if and only if $d\beta(V) = 0$ as well as $\beta \cdot V = 0$ at x. If $h(\zeta,\zeta) \neq 0$, we conclude that

$$M = M_0 \cup M_{n-1}, \quad \text{with } M_0 = \{\beta \neq 0\} \text{ and } M_{n-1} = \{\beta = 0\}.$$

For $x \in M_{n-1}$ we have $\mathcal{E}_x = \ker d\beta \neq T_x M$. In particular, M_{n-1} is a smooth embedded submanifold of M. Similarly, if $h(\zeta, \zeta) = 0$, we have

$$M = M_0 \cup M_n = \{\beta \neq 0\} \cup \{\beta = 0\}.$$

Here $\{\beta = 0\}$ consists only of isolated points because $\beta(x) = 0$ implies that $d\beta(x) = 0$ and $\mathrm{Hess}^g(\beta)(x)$ is proportional to g_x.

4C. ***Open sets adapted to the holonomy distribution.*** We analyze the function $r^{\mathcal{E}}$ in more detail. Obviously, if $\mathfrak{hol}_x(M, c)$ is generic at some point of M, i.e., if $\mathfrak{hol}_x(M, c) = \mathfrak{so}(p+1, q+1)$, then $r^{\mathcal{E}} \equiv n$. Conversely, one finds:

Proposition 4.2. *Suppose that there is a curve γ in M with $g(\dot\gamma, \dot\gamma) \neq 0$ and $r^{\mathcal{E}} \circ \gamma \equiv n$. Then $\mathfrak{hol}(M, c)$ is generic. In particular, $r^{\mathcal{E}} \equiv n$.*

Proof. All calculations are carried out with respect to some fixed $g \in c$. By assumption, $s_-^\flat \wedge V^\flat \in \mathfrak{hol}_{\gamma(t)}(M, c)$ for every vector field V along γ. Applying $\nabla^{\mathrm{nc}}_{\dot\gamma}$ to this expression using (7) reveals that

$$-g(V, \dot\gamma) s_-^\flat \wedge s_+^\flat + \dot\gamma^\flat \wedge V^\flat \in \mathfrak{hol}_{\gamma(t)}(M, c). \tag{37}$$

Letting $V = \dot\gamma$ shows that $s_-^\flat \wedge s_+^\flat \in \mathfrak{hol}_{\gamma(t)}(M, c)$. Moreover, letting $(V_1, V_2, \dot\gamma)$ be mutually orthogonal to each other and taking the Lie brackets of the expressions (37) with $V = V_1$ and $V = V_2$, respectively, shows that

$$\|\dot\gamma\|^2 V_1^\flat \wedge V_2^\flat \in \mathfrak{hol}_{\gamma(t)}(M, c).$$

But this establishes that $\mathfrak{g}_0 \in \mathfrak{hol}_{\gamma(t)}(M, c)$. Thus, $\mathfrak{g}_1 \oplus \mathfrak{g}_0 \in \mathfrak{hol}_{\gamma(t)}(M, c)$. Differentiating elements $\dot\gamma^\flat \wedge V^\flat \in \mathfrak{hol}_{\gamma(t)}(M, c)$ in the direction of γ, where V is again a vector field along γ shows using (7) that also $\mathfrak{g}_{-1} \cap \dot\gamma^\perp$ is contained in the infinitesimal holonomy along γ and differentiating $s_-^\flat \wedge s_+^\flat$ along γ shows that all of $\mathfrak{g}_{-1}$ is contained in the holonomy. Thus, $\mathfrak{hol}_{\gamma(t)}(M, c)$ is generic along γ, and thus generic everywhere. $\square$

In order to continue with our analysis, we need to show that there are *sufficiently many* open sets U on which $r^{\mathcal{E}}$ is constant, i.e., such that $\mathcal{E}|_U$ is a vector bundle, and on which there is a basis of local smooth sections of $U \to \mathcal{E}$. For this purpose we define: An open set $U \subset M$ is an *$\mathcal{E}$-adapted open set* if

(1) $r^{\mathcal{E}} \equiv k$ constant on U,

(2) there are smooth and pointwise linearly independent sections $V_1, \ldots, V_k : U \to \mathcal{E}$.

Then:

Theorem 4.3. *For each open set $U \subset M$ there exists an $\mathcal{E}$-adapted open subset $V \subset U$. In particular, there is an open dense subset of M which is the union of $\mathcal{E}$-adapted open sets.*

Proof. After restricting U if necessary, we may assume that U is contained in a coordinate neighborhood for M. It is then possible to choose a local basis of $\mathfrak{hol}_x(M,c)$ over U which depends smoothly on x. Write such a basis as

$$U \ni x \mapsto (v_i^\flat(x) \wedge s_-^\flat + A^i(x)), \tag{38}$$

where $i = 1, \dots, m := \dim \mathfrak{hol}(M,c)$, for certain $v_i \in T_xM$ and $A^i \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. With respect to the fixed coordinates we may think of the $A^i = (A^i_{jk})_{j,k}$ as $\mathfrak{so}(p+1, q+1)$-matrices. Let

$$\tilde{A}^i := (A^i_{11}, A^i_{12}, \dots, A^i_{n+1,n+2}, A^i_{n+2,n+2})^\top$$

and introduce the $(n+2)^2 \times m$-matrix $A := (\tilde{A}^1 \cdots \tilde{A}^m)$. By elementary linear algebra,

$$r^{\mathcal{E}}(x) = k \iff k = \dim\ker A = \dim \mathfrak{hol}_x(M,c) - \operatorname{rk} A_x. \tag{39}$$

The set of matrices with rank greater or equal to some fixed integer is open in the set of all matrices. Thus, it follows from (39) that $\{x \mid r^{\mathcal{E}}(x) \le k\}$ is open in M. In particular, $(r^{\mathcal{E}})^{-1}(0) = \{x \mid r^{\mathcal{E}}(x) \le 0\}$ is open and $r^{\mathcal{E}} < n$ is an open condition.

Assume now that there is $x \in U$ with $r^{\mathcal{E}}(x) = 0$. It follows that $r^{\mathcal{E}} = 0$ on some open subset $V \subset U$. Thus the claim follows for this case. Otherwise, we have $r^{\mathcal{E}} \ge 1$ everywhere. If there is $x \in U$ with $r^{\mathcal{E}}(x) = 1$, it follows that there is an open neighborhood V in U with $r^{\mathcal{E}} \le 1$ of x in U. Thus, $r^{\mathcal{E}} = 1$ on V. Otherwise we have $r^{\mathcal{E}} \ge 2$ on U etc. So the statement regarding the existence of V with $r^{\mathcal{E}}|_V =: l =$ constant follows inductively. The above proof starts with a smooth local basis (38) and constructs (on an open subset of V) via smooth linear algebra operations a basis on V of the form $(\tilde{v}^\flat_{i=1,\dots,l} \wedge s^\flat_-, \dots)$. It is thus clear that the $\tilde{v}_i$ depend smoothly on $x \in V$ and yield local sections.

Finally, if every open set in M contains an $\mathcal{E}$-adapted open subset, the union of all $\mathcal{E}$-adapted open sets is open and dense in M. □

By virtue of this theorem, after restricting to an open and dense subset of M if necessary, we may from now on always assume that M is the union of $\mathcal{E}$-adapted open sets. In particular, the level sets of $r^{\mathcal{E}}$ are then (possibly empty) unions of $\mathcal{E}$-adapted open sets. From this point of view, we may restrict ourselves to such open sets in the following local analysis. Note that restricting to an open and dense subset in the context of Cartan holonomy reductions is a basic feature of the curved orbit decomposition as revealed in [Čap et al. 2014].

Proposition 4.4. *Let $U \subset M$ be a $\mathcal{E}$-adapted open set. Then $\mathcal{E}_x$ is a totally lightlike subspace of T_xM for every $x \in U$ or $\mathfrak{hol}(M,c)$ is generic.*

Proof. Let V be a vector field defined on U such that $s_-^\flat \wedge V^\flat(x) \in \mathfrak{hol}_x(M,c)$ for $x \in U$. Differentiating in the direction of some $X \in TM$ using (7) reveals that

$$-\nabla^{\mathrm{nc}}_X(s_-^\flat \wedge V^\flat)(x) = g(V,X) s_-^\flat \wedge s_+^\flat + X^\flat \wedge V^\flat + (\nabla_X V)^\flat \wedge s_-^\flat \in \mathfrak{hol}_x(M,c). \tag{40}$$

Suppose that there is $x \in U$ with $g(V, V)(x) \neq 0$. It follows that $g(V, V) \neq 0$ on some open neighborhood $x \in W \subset U$. Let X be orthogonal to V on W. As $\mathfrak{hol}_x(M, c)$ is a Lie algebra with the usual commutator Lie bracket, it follows that on W also

$$[X^\flat \wedge V^\flat + (\nabla_X V)^\flat \wedge s_-^\flat, s_-^\flat \wedge V^\flat] = -g(V, V) X^\flat \wedge s_-^\flat \in \mathfrak{hol}(M, c). \tag{41}$$

Thus, $r^{\mathcal{E}}|_W = n$ and the statement follows from Proposition 4.2. □

4D. ***Rank and integrability of the holonomy distribution.*** Interestingly, it turns out that, at least locally, $\mathcal{E}$ is always integrable or it is maximally nonintegrable and one of the exceptional holonomy reductions occurs. More precisely, we will see that if $\mathcal{E}$ is not integrable, M is of dimension 5 or 6, $\mathcal{E}$ is generic and of rank 2 or 3, respectively, and $\mathfrak{hol}(M, c)$ is $\mathfrak{g}_2$ or $\mathfrak{spin}(4, 3)$, respectively.

In order to analyze the integrability of $\mathcal{E}$, we need some preparations.

Proposition 4.5. *Let (M^n, c) be a conformal manifold of even dimension. Either there is an open dense subset of M on which $r^{\mathcal{E}} \leq 1$ or $\mathrm{Hol}^0(M, c)$ acts on the lightcone $\mathcal{N} \subset \mathbb{R}^{p+1,q+1}$ with an open orbit.*

Proof. Suppose first that $r^{\mathcal{E}} \geq 2$ on some open set $U \subset M$. After restricting to an open, dense subset of U, if necessary, we may assume that U is an $\mathcal{E}$-adapted open set. We may also assume that the holonomy is not generic and hence that $\mathcal{E}$ is lightlike. Let V be a local section of $\mathcal{E}$ and let V' be a lightlike vector field with $g(V, V') = 1$. Moreover, let $X \in (V, V')^\perp$. We have on U

$$\nabla^{\mathrm{nc}}_{V'}(s_-^\flat \wedge V^\flat) = s_-^\flat \wedge s_+^\flat + A_1 \in \mathfrak{hol}(M, c), \tag{42}$$

$$\nabla^{\mathrm{nc}}_X(\nabla^{\mathrm{nc}}_{V'}(s_-^\flat \wedge V^\flat)) = -X^\flat \wedge s_+^\flat + A_2 \in \mathfrak{hol}(M, c), \tag{43}$$

where $A_{1,2} \in \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{p}$. As $r^{\mathcal{E}} \geq 2$ on U and $\mathcal{E}$ is totally lightlike, linear algebra shows that at $x \in U$, equation (43) implies

$$\mathfrak{so}(p+1, q+1) = \mathfrak{hol}_x(M, c) + \mathfrak{p}. \tag{44}$$

This, together with equation (42) shows that the orbit of $\mathrm{Hol}^0(M, c)$ through $s_- \in \mathcal{N}$ has dimension $n+1$, i.e., it is open. Otherwise, the subset of M on which $r^{\mathcal{E}} \leq 1$ is dense. It is also open as follows from the proof of Theorem 4.3. □

In relation to this proposition, we point out that conformal structures for which the holonomy group acts not only with an open orbit on $\mathcal{N}$, but transitively and irreducibly on the homogeneous model were classified in [Alt 2012].

Proposition 4.6. *Suppose that $(M, [g])$ admits an nc-Killing form $\alpha \in \Omega^k(M)$. Then $V^\flat \wedge \alpha = 0$ for every $V \in \mathcal{E}$.*

Proof. Following the discussion in Section 2C, every nc-Killing k-form α uniquely determines a parallel tractor $(k+1)$-form $\hat{\alpha}$. With respect to a metric g in the conformal class, decompose $\hat{\alpha}$ as in (8). Pointwise, $\hat{\alpha}$ is annihilated by the action # of $\mathfrak{hol}(M,c)$ on forms. In particular, one has for every $V \in \mathcal{E}_x$ that

$$(s_-^\flat \wedge V^\flat) \# \hat{\alpha}_x = 0.$$

Inserting (8), one immediately obtains that $V^\flat \wedge \alpha = 0$. □

Proposition 4.7. *Suppose M is orientable and the action of $\mathfrak{hol}(M,c)$ leaves invariant a nontrivial nondegenerate subspace of $\mathbb{R}^{p+1,q+1}$. Then $\mathcal{E} = 0$ on an open, dense subset of M.*

Proof. As the holonomy invariant space (of dimension $k+1$) is nondegenerate and M is orientable, there is actually a decomposable parallel tractor form in $\Omega^k \mathcal{T}^*$. The associated nc-Killing form α is of the form $\alpha = t_1 \wedge \cdots \wedge t_k$, defining a k-dimensional nondegenerate subspace $H \subset TM$ on an open, dense subset of M as follows from the discussion in [Leitner 2005], Thus, Proposition 4.6 implies that $\mathcal{E} \subset H$ on an open dense subset M' of M. On the other hand, by Proposition 4.2, $\mathcal{E}$ is over M' contained in a totally degenerate subspace. We conclude $\mathcal{E}_{|M'} = 0$. □

Proposition 4.8. *Suppose that* $\mathrm{Hol}(M,c)$ *fixes a totally lightlike* (*with respect to h*) *subbundle $\mathcal{H} \subset \mathcal{T}$. Then there is an open and dense subset of M and at least locally a metric $g \in c$ such that with respect to g*

$$\mathcal{H} = \mathbb{R}s_+ \oplus \mathcal{L}, \tag{45}$$

with $\mathcal{L} \subset TM$ a ∇^g-parallel distribution containing $\mathcal{E}$ and the image of Ric^g.

Proof. The existence of a parallel distribution $\mathcal{L} \subset TM$ containing the image of Ric^g was proven in [Lischewski 2015]. To see that at each $x \in M$, the fiber $\mathcal{L}_x$ contains $\mathcal{E}_x$, consider $V \in \mathcal{E}_x$ such that $s_-^\flat \wedge V^\flat \in \mathfrak{hol}_x(M,c)$. Then $(s_-^\flat \wedge V^\flat)(s_+) = V$ lies in $\mathcal{H}$, which shows that $\mathcal{E} \subset \mathcal{L}$. □

These results enable us to prove the main result of this section:

Theorem 4.9. *Let $U \subset M$ be a $\mathcal{E}$-adapted open set. Then exactly one of the following cases occurs on U:*

(1) *$\mathcal{E}$ is integrable.*

(2) *The dimension of M is 5 and $\mathcal{E}$ is a generic rank 2 distribution. Moreover, $\mathfrak{hol}(M,c) = \mathfrak{g}_2$ and hence the conformal structure $c = c_{\mathcal{E}}$ is defined by the generic distribution $\mathcal{E}$.*

(3) *The dimension of M is 6 and $\mathcal{E}$ is a generic rank 3 distribution. Moreover, $\mathfrak{hol}(M,c) = \mathfrak{spin}(3,4)$ and the conformal structure $c = c_{\mathcal{E}}$ is defined by the generic distribution $\mathcal{E}$.*

Proof. If $\mathfrak{hol}(M, c)$ is generic the statement is trivial as $\mathcal{E} = TM$ in this case. Thus, we may assume that the holonomy algebra is reduced and by the previous Proposition, $\mathcal{E}_x$ is a totally lightlike subspace of T_xM for $x \in U$.

Let V_1, V_2 be vector fields on U such that $s_-^\flat \wedge V_{i=1,2}^\flat \in \mathfrak{hol}_x(M, c)$ for $x \in U$. It follows that

$$(46)\quad \nabla^{\mathrm{nc}}_{V_1}(s_-^\flat \wedge V_2^\flat) - \nabla^{\mathrm{nc}}_{V_2}(s_-^\flat \wedge V_1^\flat) = -V_1^\flat \wedge V_2^\flat + s_-^\flat \wedge ([V_1, V_2])^\flat \in \mathfrak{hol}(M, c).$$

Moreover, let X be a vector field on U which is orthogonal to V_i for $i = 1, 2$. It follows from evaluating $[\nabla^{\mathrm{nc}}_X(s_-^\flat \wedge V_1^\flat), \nabla^{\mathrm{nc}}_X(s_-^\flat \wedge V_2^\flat)]$ that

$$(47)\qquad 2g(\nabla_X V_1, V_2)X^\flat \wedge s_-^\flat + g(X, X)V_1^\flat \wedge V_2^\flat \in \mathfrak{hol}(M, c).$$

Combining (46) and (47) it follows for X orthogonal to (V_1, V_2) that

$$(48)\qquad X \cdot g(\nabla_X V_1, V_2) \in \mathcal{E} \quad \text{for } g(X, X) = 0,$$

$$(49)\qquad [V_1, V_2] - \frac{2g(\nabla_X V_1, V_2)}{g(X, X)} \cdot X \in \mathcal{E} \quad \text{for } g(X, X) \neq 0.$$

Now we distinguish several cases: Obviously the statement is trivial in case $r^{\mathcal{E}} \leq 1$. Thus, we may assume that V_1, V_2 are linearly independent. Fix a local g-pseudoorthonormal basis $(s_1, \ldots, s_n)$ over U such that

$$(50)\qquad \mathcal{E} = \operatorname{span}(V_i := s_{2i-1} + s_{2i} \mid i = 1, \ldots, r^{\mathcal{E}}).$$

Moreover, let $V_i' := s_{2i-1} - s_{2i}$ for $i = 1, \ldots, e$. That is, $g(V_i, V_j') = 2\delta_{ij}$.

Case 1: $r^{\mathcal{E}} \geq 3$ and $n > 6$. In (48) let $X = V_3'$. It follows that $g(\nabla_{s_5} V_1, V_2) = g(\nabla_{s_6} V_1, V_2)$. But then letting $X = s_5, s_6$, (49) can only be true if $[V_1, V_2] - f \cdot V_3' \in \mathcal{E}$ for some function f. On the other hand, applying (49) to $X = s_n$ reveals that $[V_1, V_2] - h \cdot s_n \in \mathcal{E}$ for some function h. But this can only be true if $f = h = 0$, i.e., $[V_1, V_2] \in \mathcal{E}$.

Case 2: $r^{\mathcal{E}} = 2$ and $n > 5$. In complete analogy to the previous case, we obtain that $[V_1, V_2] - f s_5 \in \mathcal{E}$ for some function f as well as $[V_1, V_2] - h s_6 \in \mathcal{E}$ for some function h from which one has to conclude that $f = h = 0$.

Case 3: $r^{\mathcal{E}} = 2$ and $n = 4$. Necessarily, M is of signature $(2, 2)$. It follows from (48) that for $i, j, k \in \{1, 2\}$ we have $g(\nabla_{V_i}, V_j, V_k) = 0$. But this implies that $g(\nabla_{V_1} V_2 - \nabla_{V_2} V_1, V_k) = 0$, i.e., $[V_1, V_2] \in \mathcal{E}^\perp = \mathcal{E}$.

It remains to show that in signatures $(3, 2)$ with $\mathcal{E}$ of dimension 2 and in signature $(3, 3)$ with $\mathcal{E}$ being of dimension 3 and not integrable, $\mathcal{E}$ is generic.

First, let us consider signature $(3, 2)$ and assume that $\mathcal{E}$ is not integrable. In particular, $\mathcal{E}$ is of rank 2 on an open and dense set. One could proceed with the proof for this case analogously as with the $(3, 3)$ case below. However, as we are considering a conformal structure in *odd* dimension, one of the main results in [Čap et al. 2016] yields that $\mathfrak{hol}(M, c)$ is the holonomy algebra of a Ricci flat pseudo-Riemannian manifold of signature (4, 3). If the standard action of $\mathfrak{hol}(M, c)$ was reducible, then by Propositions 4.7 and 4.8, $\mathcal{E}$ would be either zero or contained in an integrable totally lightlike distribution, both contradicting the assumptions in the current case. Thus, the action of the holonomy algebra is irreducible and from $\mathcal{E} \neq TM$ and the pseudo-Riemannian version of the Berger list it follows that $\mathfrak{hol}(M, c) = \mathfrak{g}_2$, where $\mathfrak{g}_2$ denotes the noncompact simple Lie algebra of dimension 14. For this case, however, $\mathcal{E}$ is generic. This follows from the discussion of $\mathfrak{g}_2$-conformal structures in Section 2C in complete analogy to the proof of Corollary 5.11 in Section 5B.

Let us now treat the 6-dimensional case. Fix a local basis $(V_1, V_2, V_3, V_1', V_2', V_3')$ for TM over U as specified in (50) such that $g(V_i, V_j') = 2\delta_{ij}$. Moreover, without loss of generality, we may assume that

$$[V_1, V_2] \notin \mathcal{E}. \tag{51}$$

From (48) we obtain $g(\nabla_{V_3'} V_1, V_2) = 0$ and (49) applied to $X = V_3 + V_3'$ then yields

$$[V_1, V_2] - g(\nabla_{V_3} V_1, V_2) V_3' \in \mathcal{E}. \tag{52}$$

We conclude from (51) that $g(\nabla_{V_3} V_1, V_2) \neq 0$. Moreover, it follows from subtracting $\nabla_{V_2}(s_-^\flat \wedge V_1^\flat) \in \mathfrak{hol}(M, c)$ from (46) that

$$\nabla_{V_2} V_1 + [V_1, V_2] \in \mathcal{E}. \tag{53}$$

In complete analogy to the derivation of (52) we obtain $[V_1, V_3] - g(\nabla_{V_2} V_1, V_3) V_2' \in \mathcal{E}$. Inserting (53) and then using (51) and (52) reveals that the coefficient $g(\nabla_{V_2} V_1, V_3)$ is nonzero. The same argument applies to $[V_2, V_3]$ and we conclude that there are nowhere vanishing functions f_k for $k = 1, 2, 3$ such that

$$[V_i, V_j] = \epsilon_{ijk} f_k V_k' \mod \mathcal{E}.$$

In particular, $[\mathcal{E}, \mathcal{E}] = TM$.

It remains to show that in this case we have $\mathfrak{hol}(M, c) = \mathfrak{spin}(3, 4)$. Using (7), it is straightforward to compute that the 15 elements, $i, j = 1, 2, 3$,

$$s_-^\flat \wedge V_i^\flat, \quad \nabla^{\mathrm{nc}}_{V_i'}(s_-^\flat \wedge V_j^\flat) \quad \text{and} \quad \nabla^{\mathrm{nc}}_{V_i}(s_-^\flat \wedge V_j^\flat), \; i < j \tag{54}$$

are pointwise linearly independent in $\mathfrak{hol}(M, c) \cap \mathfrak{p}$. Then Proposition 4.5 comes into play, which ensures that $\mathfrak{so}(p+1, q+1) = \mathfrak{hol}(M, c) + \mathfrak{p}$ and hence that dim $\mathfrak{hol}(M, c) \geq 15 + 6 = 21$, which is the dimension of $\mathfrak{spin}(4, 3)$. Then the equality $\mathfrak{hol}(M, c) = \mathfrak{spin}(4, 3)$, and with it the last point in the theorem, follows from Lemma 4.10 below. □

Lemma 4.10. *Let* $\mathfrak{h} \subsetneq \mathfrak{so}(4,4)$ *be irreducible of dimension at least* 21. *Then* $\mathfrak{h} = \mathfrak{spin}(3,4)$.

Proof. Since $\mathfrak{h}$ acts irreducibly, it is reductive. Then either $\mathfrak{h}$ is semisimple and the complexified representation $\mathbb{C} \otimes \mathrm{R}^{4,4}$ is irreducible, or $\mathfrak{h} \subset \mathfrak{u}(2,2)$ and $\mathbb{C} \otimes \mathrm{R}^{4,4}$ is not irreducible (see for example [Di Scala and Leistner 2011, Section 2]). The second case however is excluded by the assumption $\dim(\mathfrak{h}) \geq 21$. Hence, we may consider $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{so}(8,\mathbb{C})$ semisimple acting irreducibly on $\mathbb{C}^8$. Inspecting the dimensions of simple complex Lie algebras below 28, it turns out that the only possibilities for $\mathfrak{h}$, apart from $\mathfrak{so}(7,\mathbb{C})$, are $\mathfrak{sl}_5\mathbb{C}$ and $\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_3\mathbb{C}$. Then $\mathfrak{sl}_5\mathbb{C}$ is excluded as it does not have an irreducible representation of dimension 8. On the other hand, any irreducible representation of $\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_3\mathbb{C}$ is a tensor product of irreducible representations, which is excluded as $\mathfrak{sl}_3\mathbb{C}$ does not have an irreducible representations of dimension 2 or 4. $\square$

Finally, we want to derive universal integrability conditions for the Weyl and Cotton tensors for conformal manifolds with reduced holonomy.

Proposition 4.11. *Let* (M, c) *be a conformal manifold with nongeneric holonomy. Locally, and off a singular set there is a totally degenerate subspace* $\mathcal{L} \subset TM$, *which is integrable if* $(p,q) \notin \{(3,2),(3,3)\}$, *such that*

$$W(\mathcal{L}, \mathcal{L}^{\perp}) = 0, \tag{55}$$

$$(n-4)C(\mathcal{L}, \mathcal{L}^{\perp}) = 0. \tag{56}$$

In even dimensions, one has $\mathrm{Im}(\mathcal{O}) \subset \mathcal{L}$. *In particular, if a conformal manifold in even dimension* ≥ 4 *admits a parallel tractor (of any type) other than the tractor metric, then the conformally invariant system (55) – (56) either becomes a nontrivial integrability condition on the curvature (and it couples* $\mathcal{O}$ *to the curvature) or* $\mathcal{O} = 0$.

Proof. We restrict the local analysis to $\mathcal{E}$-adapted open sets and let $\mathcal{L} = \mathcal{E}$. The conditions (55) and (56) are easily seen to be an equivalent reformulation of

$$[\mathrm{R}^{\mathrm{nc}}(X,Y), s_-^{\flat} \wedge V^{\flat}] \in \mathfrak{hol}(M,c), \tag{57}$$

$$[\mathrm{tr}_g \nabla.\mathrm{R}^{\mathrm{nc}}(\,\cdot\,, X), s_-^{\flat} \wedge V^{\flat}] \in \mathfrak{hol}(M,c), \tag{58}$$

where $X, Y \in TM$ and $V \in \mathcal{E}$. The statement follows from the definition of $\mathcal{E}$ and Theorems 3.1 and 4.9. $\square$

5. Applications to the obstruction tensor

Recall that according to Theorem 3.1 the image of the obstruction tensor $\mathcal{O}$ is contained in the holonomy distribution $\mathcal{E}$. In this section we apply the results about $\mathcal{E}$ to obtain the results in Corollaries 1.2 and 1.3. In the following we will always assume we have given a smooth conformal manifold (M, c) of even dimension n and with obstruction tensor $\mathcal{O}$. We view $\mathcal{O}$ as a $(1, 1)$-tensor by means of some $g \in c$ and define the rank of $\mathcal{O}$ at a point to be the rank of this $(1, 1)$-tensor. The holonomy reductions we will consider now were described in Section 2C.

5A. *The obstruction tensor and holonomy reductions.* We begin with a well-known case of a conformal holonomy reduction, the case of a parallel standard tractor. The existence of a parallel standard tractor is equivalent to the existence of an open dense subset in M, on which the conformal class contains local Einstein metrics. It is well known since [Fefferman and Graham 1985, Proposition 3.5], see also [Gover and Peterson 2006, Theorem 4.3] and [Fefferman and Graham 2012] that the existence of local Einstein metrics in the conformal class forces $\mathcal{O} = 0$. Our Theorem 3.1 provides us with an independent and alternative proof:

Corollary 5.1. *If locally on an open and dense subset of M there is an Einstein metric $g \in c$, then $\mathcal{O} = 0$.*

Proof. Given an Einstein metric on $U \subset M$ and splitting the tractor bundle over U with respect to g, there is on U a parallel standard tractor

$$T = -\frac{\mathrm{scal}^g}{2n(n-1)} s_- + s_+.$$

In particular, $\mathfrak{hol}_x(U, [g]) T_x = 0$. Theorem 3.1 yields $(s_-^\flat \wedge (X \lrcorner \mathcal{O}))(T) = \mathcal{O}(X) = 0$ on U for each $X \in TU$ which is equivalent to $\mathcal{O} = 0$ on U. $\square$

A weaker condition than admitting a parallel tractor is the existence of a subspace that is invariant under the conformal holonomy. In this situation Propositions 4.7 and 4.8 imply:

Corollary 5.2. *Suppose M is orientable and the action of* $\mathrm{Hol}(M, c)$ *leaves invariant a nontrivial subspace $\mathcal{H}$ of $\mathbb{R}^{p+1,q+1}$. Then we have the following alternatives (possibly replacing $\mathcal{H}$ with $\mathcal{H} \cap \mathcal{H}^\perp$ if it is degenerate):*

1. *If $\mathcal{H}$ is nondegenerate, then $\mathcal{O} = 0$.*
2. *If $\mathcal{H}$ is totally lightlike, then, locally on an open dense subset of M there is a metric $g \in c$ and a ∇^g-parallel distribution $\mathcal{L} \subset TM$ containing the image of* Ric^g *and of $\mathcal{O}$.*

Specializing the total lightlike case in this corollary further, in Section 5B we will consider Bryant's conformal structures as examples. Another example is the following:

Example. Suppose that M is of split signature (n, n) and that $\mathrm{Hol}(M, c)$ leaves invariant two complementary totally lightlike distributions $\mathcal{H} \oplus \mathcal{H}' = \mathcal{T}$, i.e., $\mathrm{Hol}(M, c) \subset \mathrm{GL}(n + 1, \mathbb{R}) \subset \mathrm{SO}(n + 1, n + 1)$. Such conformal structures arise from Fefferman type constructions starting with n-dimensional projective structures, see [Hammerl and Sagerschnig 2011a; Hammerl et al. 2015]. For $\mathcal{H}$ and $\mathcal{H}'$ define L and L' as above and fix a local metric g such that $\mathcal{H}$ is of the form (45) on some set $U \subset M$. Elementary linear algebra shows that on U the space $L \cap L'$ is at each point at most 1-dimensional. Moreover, we have from the conformal covariance of $\mathcal{O}$ and Corollary 5.2 that $\operatorname{Im}(\mathcal{O}) \subset L \cap L'$. It follows that the rank of $\mathcal{O}$ is less than or equal to one on an open, dense subset of M.

Proposition 5.3. *Let (M, c) be an even-dimensional conformal manifold admitting a twistor spinor φ. Then, at each point*

$$\operatorname{Im}(\mathcal{O}) \subset \mathcal{L}_\varphi. \tag{59}$$

In particular, $\mathcal{O}$ vanishes if there are twistor spinors whose associated subspaces L are transversal on an open and dense subset of M.

Proof. Combining Theorem 1.1 with relation (10) yields that

$$s_- \cdot \mathcal{O}(X) \cdot \psi = 0. \tag{60}$$

Filling in the technical details how ψ is related to φ by means of a metric in the conformal class as done in [Leitner 2007] reveals that (60) is equivalent to

$$\mathcal{O}(X) \cdot \varphi(x) = 0 \quad \text{for } \varphi(x) \neq 0, \tag{61}$$

which is clearly equivalent to (59). □

We continue by combining Theorem 3.1 with the results in Section 4C. In the nongeneric case, i.e., when $\mathfrak{hol}(M, c) \neq \mathfrak{so}(p+1, q+1)$, Proposition 4.4 shows that the image of $\mathcal{O}$ is lightlike over an open dense set in M, and hence everywhere:

Corollary 5.4. *If $\mathfrak{hol}(M, c)$ is not generic, then $\operatorname{Im}(\mathcal{O})$ is totally lightlike. In particular, if (M, c) is Riemannian and $\mathfrak{hol}(M, c)$ is not generic, then $\mathcal{O} = 0$.*

The statement in Corollary 5.4 about Riemannian conformal structure can be pieced together from several results in the literature: The decomposition theorem in [Armstrong 2007] states that a conformal structure with holonomy reduced from $\mathfrak{so}(1, n+1)$, locally over an open dense subset of M, contains an Einstein metric or a certain product of Einstein metrics. Corollary 5.1 and the results in [Gover and Leitner 2009] about products of Einstein metrics then ensure that (M, c) admits an ambient metric whose Ricci tensor vanishes to infinite order, and hence that the obstruction tensor vanishes. Our proof of $\mathcal{O} = 0$ for Riemannian nongeneric conformal classes in Corollary 5.4 is self-contained and does not make use of the results in the literature.

We consider now several options for the rank of $\mathcal{O}$. From Proposition 4.5 we get:

Corollary 5.5. *If* $\mathrm{Hol}^0(M,c)$ *has no open orbit on the lightcone* $\mathcal{N}\subset\mathbb{R}^{p+1,q+1}$, *then* $\mathrm{rk}(\mathcal{O})\leq 1$.

Indeed, if $\mathrm{Hol}^0(M,c)$ has no open orbit on the lightcone $\mathcal{N}\subset\mathbb{R}^{p+1,q+1}$, then by Proposition 4.5 the rank of $\mathcal{O}$ is ≤ 1 on an open dense set. Hence, the rank is ≤ 1 everywhere.

Again we refer to [Alt 2012], where conformal structures with a transitive and irreducible action of the conformal holonomy are classified. Moreover, Proposition 4.2 implies:

Corollary 5.6. *If the rank of* $\mathcal{O}$ *is maximal at some point* $x\in M$, *then* $\mathfrak{hol}(M,c)=\mathfrak{so}(p+1,q+1)$ *is generic. In particular, all parallel tractors are obtained from the tractor metric* h *only.*

Corollary 5.6 demonstrates that the ambient obstruction tensor $\mathcal{O}$ can also be interpreted as an obstruction to the existence of parallel tractors on (M,c) of any type. Namely for such a tractor to exist, $\mathcal{O}$ needs to have a nontrivial kernel everywhere. We analyze this phenomenon in more detail by focusing on parallel tractor forms and the associated normal conformal Killing forms (see Section 2C). Proposition 4.6 implies:

Corollary 5.7. *If* (M,c) *admits a nc-Killing form* $\alpha\in\Omega^k(M)$, *then* $\mathrm{Im}(\mathcal{O})\wedge\alpha=0$.

Corollary 5.8. *If* V *is a normal conformal vector field for* (M,c), *then* $\mathrm{Im}(\mathcal{O})\subset\mathbb{R}V$ *whenever* $V\neq 0$. *In particular,* $\mathcal{O}$ *vanishes if there is a normal conformal vector field that is not lightlike, or if the space of normal conformal vector fields has dimension greater than* 1.

In particular, Corollary 5.8 applies to Fefferman conformal structures (M,c) of signature $(2k+1,2r+1)$, i.e., $\mathrm{Hol}(M,c)\subset\mathrm{SU}(k+1,r+1)$. They admit a distinguished normal conformal Killing vector field V_F. Thus,

$$\mathrm{Im}\,\mathcal{O}\subset\mathbb{R}V_F, \tag{62}$$

for which an independent proof can be found in [Graham and Hirachi 2008]. For the Lorentzian case, i.e., $k=0$, any additional holonomy reduction will force $\mathcal{O}$ to vanish.

Proposition 5.9. *Let* (M,c) *be a Lorentzian conformal manifold of even dimension* n *with* $\mathfrak{hol}(M,c)\subsetneq\mathfrak{su}(1,\frac{n}{2})$. *Then* $\mathcal{O}=0$.

Proof. From the classification of irreducibly acting subalgebras of $\mathfrak{so}(2,n)$ in [Di Scala and Leistner 2011] and the results in [Alt et al. 2014] it follows that $\mathfrak{hol}(M,c)$ has to act with an invariant subspace. If the holonomy representation fixes a nondegenerate subspace or a lightlike line in $\mathbb{R}^{2,n}$ the result follows with Corollaries 5.1 and 5.2. Otherwise, $\mathfrak{hol}(M,c)$ fixes a totally lightlike 2-plane in

$\mathbb{R}^{2,n}$ and again Corollary 5.2 applies. That is, there is (at least locally) a metric $g \in c$ admitting a recurrent and nowhere vanishing null vector field U, i.e., $\nabla^g U = \theta \otimes U$ for some 1-form θ and $\mathrm{Im}(\mathcal{O}) \subset \mathbb{R}U$. Assume now that $\mathcal{O}$ is nonzero at some point. It follows from (62) that there is an open subset of M on which $\mathbb{R}V_F = \mathbb{R}U$. However, this contradicts the fact that the twist[1] of U is given by $\omega_U = \theta \wedge U^\flat \wedge U^\flat = 0$ but $\omega_{V_F} \neq 0$; see [Baum and Leitner 2004]. Thus, $\mathcal{O} \equiv 0$. $\square$

Remark. In similar fashion, Fefferman spaces over quaternionic contact structures, see [Alt 2008], admit 3 linearly independent $\mathrm{Hol}(M, c)$-invariant almost complex structures which descend to pointwise linear independent nc-vector fields (or 1-forms) on M. Thus $\mathcal{O} \equiv 0$ for this case by Corollary 5.8.

5B. ***The obstruction tensor for Bryant conformal structures.*** We now specialize to Bryant conformal structures in signature (3, 3) induced by a generic 3-distribution $\mathcal{D} \subset TM$ as in Section 2C, and deduce several new results about the relation of the generic distribution $\mathcal{D}$ and the image of $\mathcal{O}$.

Every Bryant conformal structure admits (and is equivalently characterized by) a parallel tractor 4-form $\hat{\alpha} \in \Gamma(M, \Lambda^4\mathcal{T})$ whose stabilizer under the SO(4, 4)-action at each point is isomorphic to $\mathrm{Spin}(4,3) \subset \mathrm{SO}(4,4)$. In particular, $\mathrm{Hol}(M, c) \subset \mathrm{Spin}(4, 3)$. For a fixed metric $g \in c$ and the corresponding splitting (8), i.e.,

$$\hat{\alpha} = s_+^\flat \wedge \alpha + \alpha_0 + \cdots, \tag{63}$$

one finds that $\alpha = l_1^\flat \wedge l_2^\flat \wedge l_3^\flat$ for $l_{i=1,2,3}$ some basis of $\mathcal{D}$ and α transforms conformally covariantly under a change of g. Using this, we can derive constraints on the obstruction tensor for Bryant conformal structures.

As an immediate consequence of Proposition 4.6 and Corollary 5.7 we obtain:

Corollary 5.10. *Let $(M, c_{\mathcal{D}})$ be a Bryant conformal structure induced by a generic 3-distribution $\mathcal{D} \subset TM$. Then $\mathcal{E} \subset \mathcal{D}$, and in particular,* $\mathrm{Im}(\mathcal{O}) \subset \mathcal{D}$.

Moreover:

Corollary 5.11. *If $\mathfrak{hol}(M, c) = \mathfrak{spin}(4, 3)$, then $\mathcal{D} = \mathcal{E}$ everywhere on M.*

Proof. The Lie algebra $\mathfrak{spin}(4, 3)$ equals the stabilizer algebra of a spinor ψ of nonzero length in signature (4, 4) which corresponds via some $g \in c$ to a twistor spinor φ with $L_\varphi = \mathcal{D}$ at every point (see Section 2C). Thus, $(s_-^\flat \wedge l^\flat) \cdot \psi = 0$ for every $l \in \mathcal{D}$, i.e., $\mathcal{D} \subset \mathcal{E}$. $\square$

Remark. This agrees with the curved orbit decomposition from [Čap et al. 2014], cf., the discussion in Section 4B for this particular case. Indeed, as discussed in that work for the general case, the curved orbits correspond to the Spin(4, 3)-orbits

[1] Recall that for a vector field $X \in \mathfrak{X}(M)$, its twist is the 3-form $\omega_X := dX^\flat \wedge X^\flat$. Clearly, the condition $d\omega_X = 0$ depends on $\mathbb{R}X$ only.

on $\mathrm{SO}(4,4)/\operatorname{Stab}_{\mathrm{SO}(4,4)}(\mathcal{L})$, where $\mathcal{L} \subset \mathbb{R}^{4,4}$ is a null line. However, there is only one such orbit as $\mathrm{Spin}(4,3)$ acts transitively on the projectivized lightcone in $\mathbb{R}^{4,4}$.

Proposition 5.12. *Assume that* $\mathfrak{hol}(M,c) \subsetneq \mathfrak{spin}(4,3) \subset \mathfrak{so}(4,4)$. *Then* $\operatorname{rk}(\mathcal{O}) \leq 1$.

Proof. Suppose first that there is an open set $U \subset M$ on which $\mathcal{E}$ has dimension 3, i.e., by Corollary 5.10 we have $\mathcal{E} = \mathcal{D}$ over U. By passing to a subset of U if necessary, we may assume that U is a $\mathcal{E}$-adapted open set. Let $V_{i=1,2,3}$ be a pointwise basis of $\mathcal{E}$ over U depending smoothly on x. Let V_i' be lightlike vector fields on U such that $g(V_i, V_j') = \delta_{ij}$. We have seen that in this case the 15 elements in (54) are pointwise linearly independent in $\mathfrak{hol}(M,c) \cap \mathfrak{p}$. But then it follows immediately from Proposition 4.5 that $\dim \mathfrak{hol}(M,c) \geq 15 + 6 = 21$, which is the dimension of $\mathfrak{spin}(4,3)$. Thus $\mathfrak{hol}(M,c)$ is no proper subalgebra of $\mathfrak{so}(4,4)$.

We have to conclude that the set on which $r^{\mathcal{E}} \leq 2$ is open and dense in M. In particular, $\operatorname{rk}(\mathcal{O}) < 3$ on an open and dense subset of M. However, the set on which $\operatorname{rk}(\mathcal{O}) < 3$ is also closed and since M is connected it follows that $\operatorname{rk}(\mathcal{O}) < 3$ on M. Assume next that there is $x \in M$ such that $\operatorname{rk}(\mathcal{O}) = 2$ at x. Since the subset on which $\operatorname{rk}(\mathcal{O}) \geq 2$ is open in M it follows that $\operatorname{rk}(\mathcal{O}) = 2$ on some open set U of M. After restricting U we may assume that U is a $\mathcal{E}$-adapted open set and $r^{\mathcal{E}} = 2$ on U. Thus, $\mathcal{E}$ is over U a 2-dimensional subbundle of $\mathcal{D}$. By Theorem 4.9, $\mathcal{E}$ is integrable over U which contradicts $\mathcal{D}$ being generic. Consequently, $\operatorname{rk}(\mathcal{O}) \leq 1$ everywhere. $\square$

Example. Proposition 5.12 applies to the situation when $\mathrm{Hol}(M,c)$ lies in the intersection of $\mathrm{Spin}(4,3)$ with the stabilizer of a totally degenerate subspace $\mathcal{H} \subset \mathbb{R}^{4,4}$. For $\dim \mathcal{H} = 4$, this intersection is isomorphic to

$$\mathfrak{spin}(3,4)_{\mathcal{H}} = \left\{ \begin{pmatrix} Z & X \\ 0 & -Z^{\top} \end{pmatrix} \;\middle|\; Z \in \mathfrak{csp}_2\mathbb{R},\ X \in \mathfrak{so}(4), \operatorname{tr}(X\boldsymbol{J}) = 0 \right\},$$

where

$$\boldsymbol{J} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix},$$

and

$$\mathfrak{csp}_2\mathbb{R} = \left\{ Z \in \mathfrak{gl}_4\mathrm{R} \mid Z^{\top}\boldsymbol{J} + \boldsymbol{J}Z - \tfrac{1}{2}\operatorname{tr}(Z)\boldsymbol{J} = 0 \right\} = \mathbb{R}\mathbf{1}_4 \oplus \mathfrak{sp}_2\mathbb{R}.$$

Moreover, since the Lie group $\mathrm{Spin}(3,4) \subset \mathrm{SO}(4,4)$ corresponding to $\mathfrak{spin}(3,4) \subset \mathfrak{so}(4,4)$ acts transitively on triples

$$\{(\mathrm{s}_+, \mathcal{H}, \mathrm{s}_-) \mid \mathcal{H} \text{ a totally null 4-plane, } \mathrm{s}_+ \in \mathcal{H},\ \mathrm{s}_- \in \mathbb{R}^8 \text{ null},\ g(\mathrm{s}_+, \mathrm{s}_-) = 1\},$$

we can express the stabilizer of $\mathcal{H}$ in conjunction with the $|1|$-grading $\mathfrak{spin}(3,4) = \boldsymbol{g}_{-1} \oplus \boldsymbol{g}_0 \oplus \boldsymbol{g}_1$ in a basis $(\mathrm{s}_+, \mathrm{e}_a, \mathrm{s}_-, \mathrm{e}_{\bar{a}})$ for $a = 1, 2, 3$ and $\bar{a} = a + 3$, as

$$\mathfrak{spin}(3,4)_{\mathcal{H}} = \left\{ \left(\begin{array}{cc|cc} r & w^\top & 0 & \overline{w}^\top \\ v^\top & Z & -\overline{w} & X \\ \hline 0 & 0 & -r & -v \\ 0 & 0 & -w & -Z^\top \end{array} \right) \;\middle|\; \begin{array}{c} w = (w^a) \in \mathrm{R}^3,\ \overline{w} = (\overline{w}^{\bar{a}}) \in \mathbb{R}^3,\ v = (v_{\bar{b}}) \in (\mathbb{R}^3)^*, \\ X = (X^{\bar{b}}{}_a) \in \mathfrak{so}(3),\ Z = (Z^b{}_a) \in \mathfrak{gl}_3\mathbb{R}, \\ w^3 = Z^2{}_1,\ w^1 = -Z^2{}_3,\ v_1 = -Z^3{}_2,\ v_3 = Z^1{}_2, \\ r = Z^1{}_1 - Z^2{}_2 + Z^3{}_3,\ \overline{w}^2 = -X^1{}_3. \end{array} \right\}.$$

Here (r, Z, X) corresponds to the $\boldsymbol{g}_0$ part whereas $(w, \overline{w})$ correspond tot he $\boldsymbol{g}_{-1}$ and v to the $\boldsymbol{g}_1$-part. In particular, the intersection $\mathfrak{p}_{\mathcal{H}}$ of $\mathfrak{spin}(3,4)_{\mathcal{H}}$ with the parabolic $\mathfrak{p}$ is given by setting w and $\overline{w}$ to zero, and the intersection $\mathcal{E}$ of $\mathfrak{spin}(3,4)_{\mathcal{H}}$ with $\boldsymbol{g}_1$ by requiring in addition that $X = Z = r = 0$. Note that $\mathcal{E}$ is one dimensional.

In regards to examples of this situation, we recall that in [Anderson et al. 2015] a certain class of Bryant's conformal structures was studied. They are defined by a rank 3 distribution $\mathcal{D}_f$ on $\mathbb{R}^6$ with coordinates $(x^1, x^2, x^3, y^1, y^2, y^3)$ given by the annihilator of three 1-forms

$$\theta_1 = dy^1 + x^2 dx^3, \quad \theta_2 = dy^2 + f dx^1, \quad \theta_3 = dy^3 + x^1 dx^2,$$

where $f = f(x^1, x^2, x^3)$ is a differentiable function of the variables (x^1, x^2, x^3) only. It was shown that, whenever f depends only on x^3 and x^1, the corresponding conformal class contains a metric for which the image of the Schouten tensor lies in a parallel rank 3 distribution, which implies [Lischewski 2015] that the conformal holonomy is contained in $\mathfrak{spin}(3,4)_{\mathcal{H}}$. In addition, these conformal structures turned out to have vanishing obstruction tensor, and therefore they admit ambient metrics. For the conformal class defined by $\mathcal{D}_f$ with $f = x^1(x^3)^2$, an ambient metric with holonomy equal to $\mathfrak{spin}(3,4)_{\mathcal{H}}$ was found, and for this example also the conformal holonomy is equal to $\mathfrak{spin}(3,4)_{\mathcal{H}}$.

Remark. We point out that there is a large class of examples of Bryant conformal structures with f depending on three variables x^1, x^2, x^3 for which the obstruction tensor has rank 3, e.g., the one with $f = x^3 + x^1 x^2 + (x^2)^2 + (x^3)^2$ in [Anderson et al. 2015]. From our Proposition 5.12 it follows that these examples have $\mathfrak{hol}(M, c_{\mathcal{D}_f}) = \mathfrak{spin}(4,3)$.

More difficult is the question of finding examples with $\mathrm{rk}(\mathcal{O}) = 1$. Of course, a general conformal structure with holonomy $\mathfrak{su}(2,2) \subset \mathfrak{spin}(4,3)$ has $\mathrm{rk}(\mathcal{O}) = 1$, but we are not aware of an explicit example with $\mathrm{rk}(\mathcal{O}) = 1$ and $\mathfrak{hol}(M, c_{\mathcal{D}}) \subset \mathfrak{spin}(4,3)_{\mathcal{H}}$. Other examples with $\mathrm{rk}(\mathcal{O}) = 1$, not necessarily with $\mathfrak{hol}(M, c_{\mathcal{D}}) \subset \mathfrak{spin}(4,3)$, are given by pp-waves and their generalization to arbitrary signature [Leistner and Nurowski 2010; Anderson et al. 2017].

Finally, Theorem 4.9 implies:

Corollary 5.13. *Suppose* (M, c) *is of signature* $(3, 3)$ *and* $\mathrm{rk}(\mathcal{O}) \leq 3$ *on some open set and* $\mathrm{Im}(\mathcal{O})$ *is not integrable. Then* $\mathfrak{hol}(M, c)$ *is either equal to* $\mathfrak{so}(4, 4)$ *or to* $\mathfrak{spin}(4, 3)$.

Proof. From the assumptions, $\mathrm{rk}(\mathcal{O}) \geq 2$ on an open set. If $\mathrm{rk}(\mathcal{O}) = 2$ on an open set, it follows from Theorem 4.9 that $\mathcal{E}$ must have dimension at least 3 on this set. Otherwise the image of $\mathcal{O}$ would be integrable. But then the statement follows from Theorem 4.9. Otherwise the set on which $\mathrm{rk}(\mathcal{O}) \geq 3$ is open and dense and the statement is an immediate consequence of Theorem 4.9. □

References

[Alt 2008] J. Alt, *Fefferman constructions in conformal holonomy*, Ph.D. thesis, Humboldt University of Berlin, 2008, Available at http://tinyurl.com/jessealt.

[Alt 2012] J. Alt, "Transitive conformal holonomy groups", *Cent. Eur. J. Math.* **10**:5 (2012), 1710–1720. MR Zbl

[Alt et al. 2014] J. Alt, A. J. Di Scala, and T. Leistner, "Conformal holonomy, symmetric spaces, and skew symmetric torsion", *Differential Geom. Appl.* **33**:suppl. (2014), 4–43. MR Zbl

[Anderson et al. 2015] I. M. Anderson, T. Leistner, and P. Nurowski, "Explicit ambient metrics and holonomy", preprint, 2015. arXiv

[Anderson et al. 2017] I. Anderson, T. Leistner, A. Lischewski, and P. Nurowski, "Conformal classes with linear Fefferman–Graham equations", preprint, 2017. arXiv

[Armstrong 2007] S. Armstrong, "Definite signature conformal holonomy: a complete classification", *J. Geom. Phys.* **57**:10 (2007), 2024–2048. MR Zbl

[Armstrong and Leitner 2012] S. Armstrong and F. Leitner, "Decomposable conformal holonomy in Riemannian signature", *Math. Nachr.* **285**:2-3 (2012), 150–163. MR Zbl

[Bailey et al. 1994] T. N. Bailey, M. G. Eastwood, and A. R. Gover, "Thomas's structure bundle for conformal, projective and related structures", *Rocky Mountain J. Math.* **24**:4 (1994), 1191–1217. MR Zbl

[Baum 1981] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, Teubner-Texte zur Mathematik **41**, B. G. Teubner Verlag, Leipzig, 1981. MR Zbl

[Baum and Leitner 2004] H. Baum and F. Leitner, "The twistor equation in Lorentzian Spin geometry", *Math. Z.* **247**:4 (2004), 795–812. MR Zbl

[Baum et al. 1991] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik **124**, B. G. Teubner Verlag, Stuttgart, 1991. MR Zbl

[Biquard 2000] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265**, Société Mathématique de France, Paris, 2000. MR Zbl

[Bryant 2006] R. L. Bryant, "Conformal geometry and 3-plane fields on 6-manifolds", pp. 1–15 in *Developments of Cartan geometry and related mathematical problems*, RIMS Symposium Proceedings **1502**, Kyoto Univ., 2006.

[Čap and Gover 2003] A. Čap and A. R. Gover, "Standard tractors and the conformal ambient metric construction", *Ann. Global Anal. Geom.* **24**:3 (2003), 231–259. MR Zbl

[Čap and Gover 2010] A. Čap and A. R. Gover, "A holonomy characterisation of Fefferman spaces", *Ann. Global Anal. Geom.* **38**:4 (2010), 399–412. MR Zbl

[Čap and Sagerschnig 2009] A. Čap and K. Sagerschnig, "On Nurowski's conformal structure associated to a generic rank two distribution in dimension five", *J. Geom. Phys.* **59**:7 (2009), 901–912. MR Zbl

[Čap et al. 2014] A. Čap, A. R. Gover, and M. Hammerl, "Holonomy reductions of Cartan geometries and curved orbit decompositions", *Duke Math. J.* **163**:5 (2014), 1035–1070. MR Zbl

[Čap et al. 2016] A. Čap, A. R. Gover, C. R. Graham, and M. Hammerl, "Conformal holonomy equals ambient holonomy", *Pacific J. Math.* **285**:2 (2016), 303–318. MR Zbl

[Cartan 1893] E. Cartan, "Sur la structure des groupes simples finis et continus", *C. R. Acad. Sci. Paris* **116** (1893), 784–786. JFM

[Cartan 1910] E. Cartan, "Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre", *Ann. Sci. École Norm. Sup.* (3) **27** (1910), 109–192. MR

[Cartan 1924] E. Cartan, "Les espaces à connexion conforme", *Ann. Soc. Polon. Math.* **2** (1924), 171–221. JFM

[Di Scala and Leistner 2011] A. J. Di Scala and T. Leistner, "Connected subgroups of $SO(2, n)$ acting irreducibly on $\mathbb{R}^{2,n}$", *Israel J. Math.* **182** (2011), 103–121. MR Zbl

[Fefferman 1976] C. L. Fefferman, "Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains", *Ann. of Math.* (2) **103**:2 (1976), 395–416. Correction in **104**:3 (1976), 393–394. MR Zbl

[Fefferman and Graham 1985] C. Fefferman and C. R. Graham, "Conformal invariants", pp. 95–116 in *Élie Cartan et les mathématiques d'aujourd'hui* (Lyon, 1984), Astérisque **1985**, 1985. MR Zbl

[Fefferman and Graham 2012] C. Fefferman and C. R. Graham, *The ambient metric*, Annals of Mathematics Studies **178**, Princeton Univ. Press, 2012. MR Zbl

[Gauduchon 1990] P. Gauduchon, "Connexion canonique des structures de weyl en geometrie conforme", technical report CNRS UA766, Centre Nationnal de la Recherche Scientifique, 1990.

[Gover 2005] A. R. Gover, "Almost conformally Einstein manifolds and obstructions", pp. 247–260 in *Differential geometry and its applications*, edited by J. Bureš et al., Matfyzpress, Prague, 2005. MR Zbl

[Gover and Leitner 2009] A. R. Gover and F. Leitner, "A sub-product construction of Poincaré–Einstein metrics", *Internat. J. Math.* **20**:10 (2009), 1263–1287. MR Zbl

[Gover and Nurowski 2006] A. R. Gover and P. Nurowski, "Obstructions to conformally Einstein metrics in n dimensions", *J. Geom. Phys.* **56**:3 (2006), 450–484. MR Zbl

[Gover and Peterson 2006] A. R. Gover and L. J. Peterson, "The ambient obstruction tensor and the conformal deformation complex", *Pacific J. Math.* **226**:2 (2006), 309–351. MR Zbl

[Graham and Hirachi 2008] C. R. Graham and K. Hirachi, "Inhomogeneous ambient metrics", pp. 403–420 in *Symmetries and overdetermined systems of partial differential equations*, edited by M. Eastwood and W. Miller, Jr., IMA Vol. Math. Appl. **144**, Springer, New York, 2008. MR Zbl

[Graham and Willse 2012] C. R. Graham and T. Willse, "Parallel tractor extension and ambient metrics of holonomy split G_2", *J. Differential Geom.* **92**:3 (2012), 463–505. MR Zbl

[Hammerl and Sagerschnig 2011a] M. Hammerl and K. Sagerschnig, "A non-normal Fefferman-type construction of split-signature conformal structures admitting twistor spinors", preprint, 2011. arXiv

[Hammerl and Sagerschnig 2011b] M. Hammerl and K. Sagerschnig, "The twistor spinors of generic 2- and 3-distributions", *Ann. Global Anal. Geom.* **39**:4 (2011), 403–425. MR Zbl

[Hammerl et al. 2015] M. Hammerl, K. Sagerschnig, J. Šilhan, A. Taghavi-Chabert, and V. Žádník, "A projective-to-conformal Fefferman-type construction", preprint, 2015. arXiv

[Kobayashi and Nomizu 1963] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, I*, Wiley, New York, 1963. MR Zbl

[Leistner 2006] T. Leistner, "Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds", *Differential Geom. Appl.* **24**:5 (2006), 458–478. MR Zbl

[Leistner and Nurowski 2010] T. Leistner and P. Nurowski, "Ambient metrics for n-dimensional pp-waves", *Comm. Math. Phys.* **296**:3 (2010), 881–898. MR Zbl

[Leistner and Nurowski 2012] T. Leistner and P. Nurowski, "Conformal pure radiation with parallel rays", *Classical Quantum Gravity* **29**:5 (2012), art. id. 055007. MR Zbl

[Leitner 2004] F. Leitner, "Normal conformal Killing forms", preprint, 2004. arXiv

[Leitner 2005] F. Leitner, "Conformal Killing forms with normalisation condition", *Rend. Circ. Mat. Palermo* (2) *Suppl.* 75 (2005), 279–292. MR Zbl

[Leitner 2007] F. Leitner, *Applications of Cartan and tractor calculus to conformal and CR-geometry*, habilitation thesis, 2007, Available at http://tinyurl.com/crgeometry.

[Leitner 2008] F. Leitner, "A remark on unitary conformal holonomy", pp. 445–460 in *Symmetries and overdetermined systems of partial differential equations*, edited by M. Eastwood and W. Miller, Jr., IMA Vol. Math. Appl. **144**, Springer, New York, 2008. MR Zbl

[Lischewski 2015] A. Lischewski, "Reducible conformal holonomy in any metric signature and application to twistor spinors in low dimension", *Differential Geom. Appl.* **40** (2015), 252–268. MR Zbl

[Nijenhuis 1953a] A. Nijenhuis, "On the holonomy groups of linear connections, IA: General properties of affine connections", *Indagationes Math.* **56** (1953), 233–240. MR Zbl

[Nijenhuis 1953b] A. Nijenhuis, "On the holonomy groups of linear connections, IB: General properties of affine connections", *Indagationes Math.* **56** (1953), 241–249. MR Zbl

[Nijenhuis 1954] A. Nijenhuis, "On the holonomy groups of linear connections, II: Properties of general linear connections", *Indagationes Math.* **57** (1954), 17–25. MR Zbl

[Nurowski 2005] P. Nurowski, "Differential equations and conformal structures", *J. Geom. Phys.* **55**:1 (2005), 19–49. MR Zbl

Received January 20, 2016. Revised March 27, 2017.

THOMAS LEISTNER
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF ADELAIDE
ADELAIDE SA
AUSTRALIA

thomas.leistner@adelaide.edu.au

ANDREE LISCHEWSKI
INSTITUTE OF MATHEMATICS
HUMBOLDT UNIVERSITY BERLIN
BERLIN
GERMANY

lischews@math.hu-berlin.de

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.437

ON THE CLASSIFICATION OF POINTED FUSION CATEGORIES UP TO WEAK MORITA EQUIVALENCE

BERNARDO URIBE

A pointed fusion category is a rigid tensor category with finitely many isomorphism classes of simple objects which moreover are invertible. Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent if there exists an indecomposable right module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and $\mathcal{D}$ are tensor equivalent. We use the Lyndon–Hochschild–Serre spectral sequence associated to abelian group extensions to give necessary and sufficient conditions in terms of cohomology classes for two pointed fusion categories to be weakly Morita equivalent. This result allows one to classify the equivalence classes of pointed fusion categories of any given global dimension.

Introduction

Pointed fusion categories are rigid tensor categories with finitely many isomorphism classes of simple objects with the property that all simple objects are invertible. Any pointed fusion category $\mathcal{C}$ is equivalent to the fusion category $\mathrm{Vect}(G, \omega)$ of complex vector spaces graded by the finite group G together with the associativity constraint defined by the 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$. Whenever we have a right module category $\mathcal{M}$ over $\mathcal{C}$ we can define the dual category $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ which becomes a tensor category via composition of functors. Whenever $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is an indecomposable fusion category, the dual category $\mathcal{C}^*_{\mathcal{M}}$ is also a fusion category [Ostrik 2003a, §2.2]. An indecomposable module category $\mathcal{M}$ of $\mathrm{Vect}(G, \omega)$ may be defined by $\mathcal{M} = \mathcal{M}(K, \mu)$, where K is the space of cosets $K := A \setminus G$ for A a subgroup of G and $\mu \in C^2(G, \mathrm{Map}(K, \mathbb{C}^*))$ is a cochain that satisfies the equation $\delta_G \mu^{-1} = \omega$. Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent if there exists an indecomposable right module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{C}^*_{\mathcal{M}}$ and $\mathcal{D}$ are tensor equivalent [Müger 2003, Definition 4.2].

The author acknowledges the financial support of the Max Planck Institute of Mathematics in Bonn, Germany, and COLCIENCIAS through grant number FP44842-617-2014 of the Fondo Nacional de Financiamiento para la Ciencia, la Tecnología y la Inovación.
MSC2010: primary 18D10; secondary 20J06.
Keywords: tensor category, pointed tensor category, weak Morita equivalence, fusion category.

Now, if we have two pointed fusion categories $\mathrm{Vect}(G, \omega)$ and $\mathrm{Vect}(\hat{G}, \hat{\omega})$, what are the necessary and sufficient conditions for them to be weakly Morita equivalent? This question was raised in [Davydov 2000; Movshev 1993], it was answered by Davydov [2000, Corollary 6.2] for the case in which both ω and $\hat{\omega}$ were trivial, and the general case was answered by Naidu [2007, Theorem 5.8] in terms of the properties that A, ω and μ need to satisfy. Nevertheless these conditions were given in equations that a priori had no interpretation in terms of known cohomology classes.

We continue the work started by Naidu [2007] and frame all the calculations done there in the language of the double complex associated to an abelian group extension which induces the Lyndon–Hochschild–Serre (LHS) spectral sequence. By doing so we are able to obtain in Corollary 3.2 cohomological conditions on ω in order for the tensor category $\mathrm{Vect}(G, \omega)^*_{\mathcal{M}(A\backslash G,\mu)}$ to be pointed, namely that ω must be cohomologous to a cocycle appearing in $C^{2,1} \oplus C^{3,0}$ of the double complex which induces the Lyndon–Hochschild–Serre spectral sequence associated to the extension $1 \to A \to G \to K \to 1$.

With the previous result at hand, we construct explicit representatives of ω and μ in terms of coordinates and we determine explicitly the groups $\hat{G}$ and the cocycles $\hat{\omega}$. The main result of this paper is Theorem 3.9, in which we give the necessary and sufficient conditions for the categories $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ to be weakly Morita equivalent. We may summarize the conditions as follows: $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ are weakly Morita equivalent if and only if there exist isomorphisms of groups $\phi : A \rtimes_F K \xrightarrow{\cong} H$ and $\hat{\phi} : K \ltimes_{\hat{F}} \mathbb{A} \xrightarrow{\cong} \hat{H}$ for some finite group K acting on the abelian group A, with $F \in Z^2(K, A)$ and $\hat{F} \in Z^2(K, \mathbb{A})$ where $\mathbb{A} := \mathrm{Hom}(A, \mathbb{C}^*)$, such that both $[\hat{F}]$ and $[F]$ survive respectively the LHS spectral sequence for the groups $A \rtimes_F K$ and $K \ltimes_{\hat{F}} \mathbb{A}$, and such that $\phi^*\eta$ is cohomologous to

$$\omega((a_1, k_1), (a_2, k_2), (a_3, k_3)) := \hat{F}(k_1, k_2)(a_3)\epsilon(k_1, k_2, k_3)$$

and $\hat{\phi}^*\hat{\eta}$ is cohomologous to

$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)),$$

where $\epsilon : K^3 \to \mathbb{C}^*$ satisfies $\delta_K \epsilon = \hat{F} \wedge F$.

Theorem 3.9 may be used to determine the weak Morita equivalence classes of pointed fusion categories of a given global dimension but the cohomological calculations can become very elaborate and are beyond the scope of this article. Nevertheless in Section 4 we include a calculation in which we show how Theorem 3.9 can be used to prove that there are only seven weak Morita equivalence classes of pointed fusion categories of global dimension four and calculate the pointed fusion categories which are weakly Morita equivalent to $\mathrm{Vect}(Q_8, \eta)$ for the quaternion group Q_8.

1. Preliminaries

1A. ***Abelian group extensions.*** Consider the short exact sequence of finite groups

$$1 \to A \to G \to K \to 1 \tag{1-1}$$

with A abelian. Consider $u : K \to G$ any section of the projection map $p : G \to K$, $p(g) = (Ag)$ such that $u(1_K) = 1_G$ and denote the right G-action on K by

$$k \triangleleft g := p((u(k)g)$$

for $k \in K$ and $g \in G$. The elements $u(k)g$ and $u(k \triangleleft g)$ differ by an element $\kappa_{k,g} \in A$ satisfying the equation

$$u(k)g = \kappa_{k,g} u(k \triangleleft g), \tag{1-2}$$

which furthermore satisfies the relation

$$\kappa_{k,g_1 g_2} = \kappa_{k,g_1} \kappa_{k \triangleleft g_1, g_2}$$

for $k \in K$ and $g_1, g_2 \in G$. Since A is an abelian normal subgroup G, there is an induced K-left action on A by conjugation:

$${}^{k}a := u(k)au(k)^{-1} \quad \text{for } k \in K \quad \text{and} \quad a \in A.$$

Since the isomorphism class of the extension (1-1) can be classified by the cohomology class of the cocycle $F \in Z^2(K, A)$, i.e., a map $F : K \times K \to A$ such that

$$\delta_K F(k_1, k_2, k_3) = {}^{k_1}F(k_2, k_3) F(k_1 k_2, k_3)^{-1} F(k_1, k_2 k_3) F(k_1, k_2)^{-1} = 1,$$

without loss of generality we will further assume that

$$G := A \rtimes_F K,$$

where the product structure of G is given by the formula

$$(a_1, k_1)(a_2, k_2) := (a_1({}^{k_1}a_2)F(k_1, k_2), k_1 k_2).$$

With this explicit choice of the group G, we choose the function $u : K \to G$ to be $u(k) := (1_A, k)$ and therefore we have that

$$\kappa_{k_1,(a,k_2)} = {}^{k_1}a F(k_1, k_2),$$

thus obtaining $F(k_1, k_2) = \kappa_{k_1,(1,k_2)}$. We furthermore have that for $x \in K$ and $g = (a, k) \in G$,

$$x \triangleleft g = x \triangleleft (a, k) = xk.$$

Denote the dual group $\mathbb{A} := \mathrm{Hom}(A, \mathbb{C}^*)$ and note that there is an induced K-right action on $\mathbb{A}$ defined as $\rho^k(a) := \rho({}^k a)$ for $\rho \in \mathbb{A}$ and $k \in K$.

1B. ***Cohomology of groups and the LHS spectral sequence.*** In what follows we will construct an explicit double complex whose cohomology calculates the cohomology of the group G, and whose associated spectral sequence recovers the Lyndon–Hochschild–Serre (LHS) spectral sequence of the extension (1-1).

Endow the set $\mathrm{Map}(K, \mathbb{C}^*)$ with the left G-action $(g \triangleright f)(k) := f(k \triangleleft g)$, where $g \in G$, $k \in K$ and $f : K \to \mathbb{C}^*$, and consider the complex $C^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ with elements normalized chains

$$C^q(G, \mathrm{Map}(K, \mathbb{C}^*)) := \{ f : K \times G^q \to \mathbb{C}^* \mid f(k; g_1, \dots, g_q) = 1 \text{ whenever some } g_i = 1\}$$

and boundary map

$$(\delta_G f)(k; g_1, \dots, g_q) = f(k \triangleleft g_1; g_2, \dots, g_q) \prod_{i=1}^{q-1} f(k; g_1, \dots, g_i g_{i+1}, \dots, g_q)^{(-1)^i} f(k; g_1, \dots, g_{q-1})^{(-1)^q}. \tag{1-3}$$

Since the natural morphism of groupoids, defined by the inclusion of the group A into the action groupoid defined by the right action of G on K, is an equivalence of categories, we have that the restriction map

$$\psi : C^*(G, \mathrm{Map}(K, \mathbb{C}^*)) \to C^*(A, \mathbb{C}^*), \quad \psi(f)(a_1, \dots, a_q) := f(1_K; a_1, \dots, a_q),$$

is a morphism of complexes which induces an isomorphism in cohomology

$$\tilde{\psi} : H^*(G, \mathrm{Map}(K, \mathbb{C}^*)) \xrightarrow{\cong} H^*(A, \mathbb{C}^*).$$

The inverse map can be constructed at the level of cocycles as follows:

Lemma 1.1. *The map $\varphi : C^q(A, \mathbb{C}^*) \to C^q(G, \mathrm{Map}(K, \mathbb{C}^*))$,*

$$\varphi(\alpha)(k; g_1, \dots, g_q) := \alpha(\kappa_{k, g_1}, \kappa_{k \triangleleft g_1, g_2}, \dots, \kappa_{k \triangleleft g_1 g_2 \dots g_{q-1}, g_q}),$$

defines a map of complexes which induces an isomorphism in cohomology $\tilde{\varphi}$: $H^*(A, \mathbb{C}^*) \xrightarrow{\cong} H^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ which is the inverse of the map $\tilde{\psi}$.

Proof. On the one hand we have

$$\begin{aligned}
&\delta_G \varphi(\alpha)(k; g_1, \dots, g_p) \\
&= \varphi(\alpha)(k \triangleleft g_1; g_2, \dots, g_q) \prod_{i=1}^{q-1} \varphi(\alpha)(k; g_1, \dots, g_i g_{i+1}, \dots, g_q)^{(-1)^i} \varphi(\alpha)(k; g_1, \dots, g_{q-1})^{(-1)^q} \\
&= \alpha(\kappa_{k \triangleleft g_1, g_2}, \kappa_{k \triangleleft g_1 g_2, g_3}, \dots, \kappa_{k \triangleleft g_1 \dots g_{q-1}, g_q}) \\
&\qquad \prod_{i=1}^{q-1} \alpha(\kappa_{k, g_1}, \kappa_{k \triangleleft g_1, g_2}, \dots, \kappa_{k \triangleleft g_1 \dots g_{i-1}, g_i g_{i+1}}, \dots, \kappa_{k \triangleleft g_1 \dots g_{q-1}, g_q})^{(-1)^i} \\
&\qquad \alpha((\kappa_{k, g_1}, \kappa_{k \triangleleft g_1, g_2}, \dots, \kappa_{k \triangleleft g_1 \dots g_{q-2}, g_{q-1}})^{(-1)^q}
\end{aligned}$$

and on the other

$$
\begin{aligned}
&\varphi(\delta_G\alpha)(k;g_1,\dots,g_p)\\
&=\delta_G\alpha(\kappa_{k,g_1},\kappa_{k\triangleleft g_1,g_2},\dots,\kappa_{k\triangleleft g_1 g_2,\dots,g_{q-1},g_q})\\
&=\alpha(\kappa_{k\triangleleft g_1,g_2},\kappa_{k\triangleleft g_1 g_2,g_3},\dots,\kappa_{k\triangleleft g_1\dots g_{q-1},g_q})\\
&\qquad\prod_{i=1}^{q-1}\alpha(\kappa_{k,g_1},\kappa_{k\triangleleft g_1,g_2},\dots,\kappa_{k\triangleleft g_1\dots g_{i-1},g_i}\kappa_{k\triangleleft g_1\dots g_{i-1}g_i,g_{i+1}},\dots,\kappa_{k\triangleleft g_1\dots g_{q-1},g_q})^{(-1)^i}\\
&\qquad\qquad\alpha((\kappa_{k,g_1},\kappa_{k\triangleleft g_1,g_2},\dots,\kappa_{k\triangleleft g_1\dots g_{q-2},g_{q-1}})^{(-1)^q}.
\end{aligned}
$$

The equality $\delta_G\varphi(\alpha)=\varphi(\delta_G\alpha)$ follows from the identity

$$\kappa_{k\triangleleft g_1\dots g_{i-1},g_i g_{i+1}}=\kappa_{k\triangleleft g_1\dots g_{i-1},g_i}\kappa_{k\triangleleft g_1\dots g_{i-1}g_i,g_{i+1}}.$$

Finally, the composition $\psi(\varphi(\alpha))=\alpha$ follows from $\kappa_{1,a}=a$ for $a\in A$. □

The complex $C^*(A,\mathbb{C}^*)$ can be endowed with the structure of a right K-module by setting for $\alpha\in C^q(A,\mathbb{C}^*)$ and $k\in K$

$$\alpha^k(a_1,\dots,a_q):=\alpha(u(k)a_1u(k)^{-1},\dots,u(k)a_qu(k)^{-1}),$$

and the complex $C^*(G,\mathrm{Map}(K,\mathbb{C}^*))$ can also be endowed with the structure of a right K-module by setting for $f\in C^q(G,\mathrm{Map}(K,\mathbb{C}^*))$ and $k\in K$

$$(f\triangleleft k)(x;g_1,\dots,g_q):=f(kx;g_1,\dots,g_q).$$

The map φ fails to be a K-module map; nevertheless it induces a K-module map at the level of cohomology:

Lemma 1.2. *The isomorphism $\tilde{\varphi}:H^*(A,\mathbb{C}^*)\xrightarrow{\cong}H^*(G,\mathrm{Map}(K,\mathbb{C}^*))$ is an isomorphism of K-modules.*

Proof. Take $\alpha\in Z^q(A,\mathbb{C}^*)$ and $k\in K$. We claim that $\psi(\varphi(\alpha)\triangleleft k)=\alpha^k$, and since $\psi(\varphi(\alpha^k))=\alpha^k$, we conclude that $\varphi(\alpha)\triangleleft k$ and $\varphi(\alpha^k)$ are cohomologous. Now, let us calculate

$$
\begin{aligned}
\psi(\varphi(\alpha)\triangleleft k)(a_1,\dots,a_q)&=(\varphi(\alpha)\triangleleft k)(1;a_1,\dots,a_q)\\
&=\varphi(\alpha)(k;a_1,\dots,a_q)\\
&=\alpha(\kappa_{k,a_1},\kappa_{k\triangleleft a_1,a_2},\dots,\kappa_{k\triangleleft a_1 a_2,\dots,a_{q-1},a_q})\\
&=\alpha(\kappa_{k,a_1},\kappa_{k,a_2},\dots,\kappa_{k,a_q})\\
&=\alpha(u(k)a_1u(k)^{-1},u(k)a_2u(k)^{-1},\dots,u(k)a_qu(k)^{-1})\\
&=\alpha^k(a_1,a_2,\dots,a_q);
\end{aligned}
$$

the lemma follows. □

1B1. *Double complex.* Since $C^*(G, \mathrm{Map}(K, \mathbb{C}^*))$ is a complex of right K-modules, we can consider the complexes

$$C^*(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))),$$

with $C^p(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))$ consisting of normalized cochains

$$\{f : K^p \to C^q(G, \mathrm{Map}(K, \mathbb{C}^*)) \mid f(k_1, \dots, k_p) = 1 \text{ whenever some } k_i = 1\}$$

and whose differentials are

$$\begin{aligned}(\delta_K f)&(k_1, \dots, k_p)\\ &= f(k_2, \dots, k_p) \prod_{i=1}^{p-1} f(k_1, \dots, k_i k_{i+1}, \dots, k_p)^{(-1)^i} (f(k_1, \dots, k_{p-1}) \triangleleft k_p)^{(-1)^p}.\end{aligned}$$

These complexes assemble into a double complex

$$C^{p,q} := C^p(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))).$$

Let us denote by $\mathrm{Tot}(C^{*,*})$ the total complex associated to the double complex and let $\delta_{\mathrm{Tot}} := \delta_K \oplus (\delta_G)^{(-1)^p}$ be its differential.

We may filter the total complex by the degree of the G cochains, thus obtaining a spectral sequence whose first page becomes

$$E_1^{p,q} = H^p(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))).$$

Since the K-modules $C^q(G, \mathrm{Map}(K, \mathbb{C}^*))$ are free K-modules, we conclude that the first page localizes on the y-axis,

$$E_1^{0,q} = H^0(K, C^q(G, \mathrm{Map}(K, \mathbb{C}^*))) = C^q(G, \mathrm{Map}(K, \mathbb{C}^*))^K \cong C^q(G, \mathbb{C}^*)$$

and $E_1^{p,q} = 0$ for $p > 0$. The spectral sequence collapses at the second page, with the only surviving elements on the y-axis

$$E_2^{0,q} = H^q(G, \mathbb{C}^*).$$

Hence we have:

Proposition 1.3. *The inclusion of K-invariant cochains*

$$C^*(G, \mathrm{Map}(K, \mathbb{C}^*))^K \hookrightarrow \mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*))))$$

is a quasi-isomorphism. Therefore the cohomology groups

$$H^*(G, \mathbb{C}^*) \xrightarrow{\cong} H^*(\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*)))))$$

are canonically isomorphic.

Filtering the double complex by the degree of the K cochains we obtain the Lyndon–Hochschild–Serre spectral sequence associated to the group extension $1 \to A \to G \to K \to 1$; see [Evens 1991, §7.2]. The first page becomes

$$E_1^{p,q} = C^p(K, H^q(G, \mathrm{Map}(K, \mathbb{C}^*))),$$

and the second page becomes

$$E_2^{p,q} = H^p(K, H^q(G, \mathrm{Map}(K, \mathbb{C}^*))).$$

Since the projection map $\tilde{\psi} : H^q(G, \mathrm{Map}(K, \mathbb{C}^*)) \xrightarrow{\cong} H^q(A, \mathbb{C}^*)$ is an isomorphism of K-modules, we conclude:

Proposition 1.4 (LHS spectral sequence). *Filtering the total complex by the degree of the K-chains, we obtain a spectral sequence whose second page is*

$$E_2^{p,q} \cong H^p(K, H^q(A, \mathbb{C}^*))$$

and that converges to $H^*(G, \mathbb{C}^*)$.

We will denote by $d_i : E_i^{p,q} \to E_i^{p+i,q-i+1}$ the differentials of this spectral sequence.

1C. *Tensor categories.* Following [Bakalov and Kirillov 2001, §1], a tensor category consists of $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, α is the associativity constraint, i.e., a functorial isomorphism $\alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $1_{\mathcal{C}} \in \mathrm{Obj}(\mathcal{C})$ is a unit element and λ, ρ are functorial isomorphisms $\lambda_V : 1_{\mathcal{C}} \otimes V \xrightarrow{\sim} V$, $\rho_V : V \otimes 1_{\mathcal{C}} \xrightarrow{\sim} V$ satisfying the pentagon axiom

$$\begin{tikzcd}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[dl, "\alpha_{1,2,3} \otimes \mathrm{id}_4"'] \arrow[dr, "\alpha_{12,3,4}"] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[d, "\alpha_{1,23,4}"] & & (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[d, "\alpha_{1,2,34}"] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[rr, "\mathrm{id}_1 \otimes \alpha_{2,3,4}"'] & & (V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)))
\end{tikzcd}$$

and the triangle axiom

$$\begin{tikzcd}
(V_1 \otimes 1_{\mathcal{C}}) \otimes V_2 \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}"'] & & V_1 \otimes (1_{\mathcal{C}} \otimes V_2) \arrow[dl, "\mathrm{id} \otimes \lambda"] \\
 & V_1 \otimes V_2 &
\end{tikzcd}$$

1D. ***The fusion category* Vect(G, ω).** A fusion category over $\mathbb{C}$ is a rigid semisimple $\mathbb{C}$-linear tensor category, with only finitely many isomorphism classes of simple objects, such that the endomorphisms of the unit object is $\mathbb{C}$; see [Etingof et al. 2005].

For G a finite group and a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$, define the category $\mathrm{Vect}(G, \omega)$ whose objects are G-graded complex vector spaces $V = \bigoplus_{g \in G} V_g$, whose tensor product is $(V \otimes W)_g := \bigoplus_{hk=g} V_h \otimes W_k$, whose associativity constraint is

$$\alpha_{V_g, V_h, V_k} = \omega(g, h, k)\gamma \quad \text{with } \gamma((x \otimes y) \otimes z) = x \otimes (y \otimes z),$$

and whose left and right unit isomorphisms are

$$\lambda_{V_g} = \omega(1, 1, g)^{-1} \, \mathrm{id}_{V_g} \quad \text{and} \quad \rho_{V_g} = \omega(g, 1, 1) \, \mathrm{id}_{V_g}.$$

The category $\mathrm{Vect}(G, \omega)$ is a fusion category where the simple objects are the 1-dimensional vector spaces.

We will assume that all group cochains are normalized, and hence the left and right unit isomorphisms become identities.

For convenience we will work with a category $\mathcal{V}(G, \omega)$ which is *skeletal*, i.e., one on which isomorphic objects are equal, and which is equivalent to $\mathrm{Vect}(G, \omega)$. The category $\mathcal{V}(G, \omega)$ has for simple objects the elements g of the group G, the tensor product is $g \otimes h = gh$ and the associativity isomorphisms are $\omega(g, h, k) \, \mathrm{id}_{ghk}$.

A finite tensor category is called *pointed* if all its simple objects are invertible. It is thus easy to see that any finite tensor category which is pointed is equivalent to $\mathrm{Vect}(G, \omega)$ for some finite group G and some 3-cocycle ω.

1E. ***Module categories.*** Following [Ostrik 2003b, §2.3], a right *module category* over the tensor category $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$ consists of $(\mathcal{M}, \otimes, \mu, \tau)$, where $\mathcal{M}$ is a category, $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is an exact bifunctor,

$$\mu_{M,X,Y} : M \otimes (X \otimes Y) \xrightarrow{\sim} (M \otimes X) \otimes Y$$

is a functorial associativity and $\tau_M : M \otimes 1_{\mathcal{C}} \xrightarrow{\sim} M$ is a unit isomorphism for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$, satisfying the pentagon axiom

(1-4)
$$\begin{tikzcd}
 & M \otimes ((X \otimes Y) \otimes Z) \arrow[dl, "\mathrm{id}_M \otimes \alpha_{X,Y,Z}"'] \arrow[dr, "\mu_{M, X \otimes Y, Z}"] & \\
M \otimes (X \otimes (Y \otimes Z)) \arrow[d, "\mu_{M,X,Y \otimes Z}"] & & (M \otimes (X \otimes Y)) \otimes Z \arrow[d, "\mu_{M,X,Y} \otimes \mathrm{id}_Z"] \\
(M \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\mu_{M \otimes X, Y, Z}"'] & & ((M \otimes X) \otimes Y) \otimes Z
\end{tikzcd}$$

and the triangle axiom

$$\begin{tikzcd} M\otimes(1_{\mathcal{C}}\otimes Y) \arrow[rr, "\mu_{M,1_{\mathcal{C}},Y}"] \arrow[dr, "\mathrm{id}_M\otimes\lambda_Y"'] & & (M\otimes 1_{\mathcal{C}})\otimes Y \arrow[dl, "\tau_M\otimes\mathrm{id}_Y"] \\ & M\otimes Y & \end{tikzcd} \tag{1-5}$$

A *module functor* $(F,\gamma):(\mathcal{M}_1,\mu^1,\tau^1)\to(\mathcal{M}_2,\mu^2,\tau^2)$ between two module categories consists of a functor $F:\mathcal{M}_1\to\mathcal{M}_2$ and a functorial isomorphism $\gamma_{M,X}:F(M\otimes X)\to F(M)\otimes X$ for any $X\in\mathcal{C},\ M\in\mathcal{M}$, satisfying the pentagon axiom

$$\begin{tikzcd} & F(M\otimes(X\otimes Y)) \arrow[dl, "F(\mu^1_{M,X,Y})"'] \arrow[dr, "\gamma_{M,X\otimes Y}"] & \\ F((M\otimes X)\otimes Y) \arrow[d, "\gamma_{M\otimes X,Y}"] & & F(M)\otimes(X\otimes Y) \arrow[d, "\mu^2_{F(M),X,Y}"] \\ F(M\otimes X)\otimes Y \arrow[rr, "\gamma_{M,X}\otimes\mathrm{id}_Y"'] & & (F(M)\otimes X)\otimes Y \end{tikzcd}$$

and the triangle axiom

$$\begin{tikzcd} F(M\otimes 1_{\mathcal{C}}) \arrow[rr, "F(\tau^1_M)"] \arrow[dr, "\gamma_{M,1_{\mathcal{C}}}"'] & & F(M) \\ & F(M)\otimes 1_{\mathcal{C}} \arrow[ur, "\tau^1_{F(M)}"'] & \end{tikzcd}$$

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exists a module functor between the two which is moreover an equivalence of categories. The *direct sum* $\mathcal{M}_1\oplus\mathcal{M}_2$ is the module category with the obvious structure. A module category is *indecomposable* if it is not equivalent to the direct sum of two nontrivial module categories.

A *natural module transformation* $\eta:(F^1,\gamma^1)\to(F^2,\gamma^2)$ consists of a natural transformation $\eta:F^1\to F^2$ such that the square

$$\begin{tikzcd} F^1(M\otimes X) \arrow[r, "\eta_{M\otimes X}"] \arrow[d, "\gamma^1_{M,X}"'] & F^2(M\otimes X) \arrow[d, "\gamma^2_{M,X}"] \\ F^1(M)\otimes X \arrow[r, "\eta_M\otimes\mathrm{id}_X"'] & F^2(M)\otimes X \end{tikzcd}$$

commutes for all $M\in\mathcal{M}$ and $X\in\mathcal{C}$.

1F. ***Indecomposable module categories over $\mathcal{V}(G,\omega)$.*** Let $\mathcal{M}$ be a skeletal right module category over $\mathcal{V}(G,\omega)$. The set of simple objects of $\mathcal{M}$ is a transitive right G-set and therefore it can be identified with the coset $K := A\setminus G$ for A a subgroup of G. The isomorphisms μ_{k,g_1,g_2} for $k \in K$ and $g_1, g_2 \in G$ are scalars, and we can assemble these scalars as an element

$$\mu \in C^2(G, \mathrm{Map}(K, \mathbb{C}^*)), \quad \mu(k; g_1, g_2) := \mu_{k,g_1,g_2}.$$

The pentagon axiom (1-4) translates into the equation

$$\omega(g_1, g_2, g_3)\mu(k; g_1, g_2g_3)\mu(k \triangleleft g_1; g_2, g_3) = \mu(k; g_1g_2, g_3)\mu(k; g_1, g_2),$$

which in view of the definition of the differential δ_G in (1-3) becomes

$$\delta_G\mu^{-1} = \pi^*\omega, \tag{1-6}$$

where $\pi^*\omega \in C^3(G, \mathrm{Map}(K, \mathbb{C}^*))^K$ is the K-invariant cocycle defined by ω, i.e.,

$$\pi^*\omega(k; g_1, g_2, g_3) := \omega(g_1, g_2, g_3).$$

Since μ is normalized and the unit constraint in $\mathcal{V}(G,\omega)$ is trivial, we have that the triangle axiom (1-5) implies that the unit constraint in $\mathcal{M}$ is trivial.

Denote this skeletal module category $\mathcal{M} = \mathcal{M}(A\setminus G, \mu)$. Note that two $\mathcal{V}(G,\omega)$-module categories $\mathcal{M}_1 = \mathcal{M}(A_1\setminus G, \mu_1)$ and $\mathcal{M}_2 = \mathcal{M}(A_2\setminus G, \mu_2)$ are equivalent if and only if there exist a right G-equivariant isomorphism $F : A_1\setminus G \xrightarrow{\cong} A_2\setminus G$ and an element $\gamma \in C^1(G, \mathrm{Map}(A_1\setminus G, \mathbb{C}^*))$ such that

$$\gamma(A_1g; g_1g_2)\mu_2(F(A_1g); g_1, g_2) = \mu_1(A_1g; g_1, g_2)\gamma(A_1gg_1; g_2)\gamma(A_1g; g_1).$$

This information implies that A_1 and A_2 are conjugate subgroups of G and that

$$\delta_G\gamma = \frac{F^*\mu_2}{\mu_1}.$$

In the case that $A = A_1 = A_2$, the G-equivariant isomorphisms are parametrized by the elements of the group $A\setminus N_G(A)$, and the equation $\delta_G\gamma = F^*\mu_2/\mu_1$ implies that $F^*\mu_2/\mu_1$ is trivial in $H^2(G, \mathrm{Map}(A\setminus G, \mathbb{C}^*))$. Since we know that $\tilde{\psi} : H^2(G, \mathrm{Map}(A\setminus G, \mathbb{C}^*)) \xrightarrow{\cong} H^2(A, \mathbb{C}^*)$ is an isomorphism, we can conclude that the isomorphism classes of module categories over $\mathcal{V}(G,\omega)$ may be parametrized (in a noncanonical manner) by pairs $([A], [\psi(\mu)])$, where $[A]$ is a conjugacy class of subgroups of G, and $[\psi(\mu)]$ is a representative of a cohomology class in the group of invariants $H^2(A, \mathbb{C}^*)/N_G(A)$.

1G. ***Dual category.*** Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ an indecomposable right module category. The dual category $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is the category whose objects are module functors from $\mathcal{M}$ to itself and whose morphisms are natural module transformations.

The category $\mathcal{C}^*_{\mathcal{M}}$ becomes a tensor category by composition of functors; namely for $(\gamma^1, F_1), (\gamma^2, F_2) \in \mathrm{Obj}(\mathcal{C}^*_{\mathcal{M}})$, where γ^1, γ^2 represent the module structures on the functors F_1 and F_2 respectively, we define the tensor structure by

$$(\gamma^1, F_1) \otimes (\gamma^2, F_2) := (\gamma, F_1 \circ F_2),$$

where the module structure γ is defined by $\gamma_{M,X} := \gamma^1_{F_2(M),X} \circ F_1(\gamma^2_{M,X})$ for $M \in \mathcal{M}$ and $X \in \mathcal{C}$. For two morphisms $\eta : (\gamma^1, F_1) \to (\gamma^2, F_2)$ and $\eta' : (\gamma'^1, F'_1) \to (\gamma'^2, F'_2)$ in $\mathcal{C}^*_{\mathcal{M}}$ their tensor product is $(\eta \otimes \eta')(M) := \eta_{F'_2(M)} \circ F_1(\eta'_M)$.

Whenever $\mathcal{C}$ and $\mathcal{M}$ are semisimple, the dual category $\mathcal{C}^*_{\mathcal{M}}$ is semisimple [Ostrik 2003a, §2.2]. Moreover, since $\mathcal{M}$ is itself a left module category over $\mathcal{C}^*_{\mathcal{M}}$ it has been shown in [Ostrik 2003b, Corollary 4.1] that the double dual is tensor equivalent to the original category, i.e., $(\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \simeq \mathcal{C}$. Furthermore, the module categories of $\mathcal{C}$ and of $\mathcal{C}^*_{\mathcal{M}}$ are in canonical bijection (Proposition 2.1 of the same work) by the following maps. For $\mathcal{M}_1$ a module category over $\mathcal{C}$, the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ of module functors from $\mathcal{M}_1$ to $\mathcal{M}$ is a left module category of $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ via the composition of functors. Conversely, if $\mathcal{M}_2$ is a left module category over $\mathcal{C}^*_{\mathcal{M}}$, then $\mathrm{Fun}_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}_2)$ is a right module category over $\mathrm{Fun}_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}) = (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \simeq \mathcal{C}$ via composition of functors. These maps are inverse from each other.

1H. *Center of a tensor category.* The *center* $\mathcal{Z}(\mathcal{C})$ of the tensor category $\mathcal{C}$ is the category whose objects are pairs (X, η), where X is an object in $\mathcal{C}$ and η is a functorial set of isomorphisms $\eta_Y : X \otimes Y \to Y \otimes X$ such that the hexagon diagram

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha"] \arrow[d, "\eta_Y \otimes 1"] & X \otimes (Y \otimes Z) \arrow[r, "\eta_{Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "\alpha"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha"] & Y \otimes (X \otimes Z) \arrow[r, "1 \otimes \eta_Z"] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
X \otimes 1_{\mathcal{C}} \arrow[rr, "\eta_{1_{\mathcal{C}}}"] \arrow[dr, "\rho"'] & & 1_{\mathcal{C}} \otimes X \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}$$

are commutative. A morphism $f : (X, \eta) \to (Y, \nu)$ consists of a morphism $f : X \to Y$ for which the diagram

$$\begin{tikzcd}
X \otimes Z \arrow[r, "\eta_Z"] \arrow[d, "f \otimes 1"'] & Z \otimes X \arrow[d, "1 \otimes f"] \\
Y \otimes Z \arrow[r, "\nu_Z"'] & Z \otimes Y
\end{tikzcd}$$

commutes for any object Z in $\mathcal{C}$. The tensor structure is defined as $(X, \eta)\otimes(Y, \nu) := (X \otimes Y, \gamma)$, where γ_Z is defined as the composition

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha"] & X \otimes (Y \otimes Z) \arrow[r, "1\otimes\nu_Z"] & X \otimes (Z \otimes Y) \arrow[dl, "\alpha^{-1}"'] & & \\
& (X \otimes Z) \otimes Y \arrow[r, "\eta_Z\otimes 1"] & (Z \otimes X) \otimes Y \arrow[r, "\alpha"] & Z \otimes (X \otimes Y)
\end{tikzcd}$$

The center $\mathcal{Z}(\mathcal{C})$ is moreover *braided* and the braiding for the pair (X, η), (Y, ν) is precisely the map η_Y.

The center $\mathcal{Z}(\mathrm{Vect}(G, \omega))$ of the tensor category $\mathrm{Vect}(G, \omega)$ contains the information necessary for constructing the quasi-Hopf algebra that is known as the twisted Drinfeld double $D^{\omega}(G)$ of the group G twisted by ω (see [Dijkgraaf et al. 1991, §3.2]).

1I. ***Weak Morita equivalence of tensor categories.*** Two tensor categories $\mathcal{C}$ and $\mathcal{D}$ are *weakly Morita equivalent* if there exists an indecomposable right module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{C}^*_{\mathcal{M}}$ and $\mathcal{D}$ are tensor equivalent [Müger 2003, Definition 4.2]. In Proposition 4.6 of the same work it is shown that weak Morita equivalence is an equivalence relation, and in [Etingof et al. 2011, Theorem 3.1] it is shown that two tensor categories are weak Morita equivalent if and only if their centers are braided equivalent. In particular we have that for $\mathcal{M}$ an indecomposable module category over $\mathcal{C}$ there is a canonical equivalence of braided tensor categories $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C}^*_{\mathcal{M}})$ [Ostrik 2003a, Proposition 2.2].

2. The dual of $\mathcal{V}(G, \omega)$ with respect to $\mathcal{M}(A \setminus G, \mu)$

Let us consider the tensor category $\mathcal{C} = \mathcal{V}(G, \omega)$ and the right module category $\mathcal{M} = \mathcal{M}(A \setminus G, \mu)$ described in Section 1F. In this chapter we will review the main results of [Naidu 2007], where explicit conditions are stated under which the dual category $\mathcal{C}^*_{\mathcal{M}}$ is pointed. For the sake of completeness and clarity we will review the constructions done in §3 and §4 of that work and we will reinterpret the equations given there in the terminology that we have set up in Section 1A and Section 1B.

2A. ***Conditions for $\mathcal{C}^*_{\mathcal{M}}$ to be pointed.*** Let us set up some notation for this section: let $K := A \setminus G$, $u : K \to G$ satisfy $p \circ u = 1_G$ and $u(p(1_G)) = 1_G$ for $p : G \to K$ the projection, $\kappa : K \times G \to A$ satisfy $u(k)g = \kappa_{k,g} u(k \triangleleft g)$ and K^A be the elements of K fixed under the conjugation by elements of A. The module category $\mathcal{M}(A \setminus G, \mu)$ is the skeletal category whose simple objects are the elements of $K = A \setminus G$, whose tensor structure is $k \otimes g := k \triangleleft g$ for $k \in K$ and $g \in G$ and whose associativity constraint μ satisfies $\delta_G \mu^{-1} = \pi^* \omega$; see (1-6). In what follows we will focus on parametrizing the invertible objects of $\mathcal{C}^*_{\mathcal{M}}$.

Following [Naidu 2007, Lemma 3.2] any invertible module functor in $\mathcal{C}^*_{\mathcal{M}}$ is of the form (F_y, γ), where the functor $F_y : \mathcal{M} \to \mathcal{M}$ is the one that extends the G-equivariant map $f_y : K \to K$, $f_y(k) = p(u(y)u(k))$, for $y \in K^A$, and γ is a functorial isomorphism $\gamma_{k,g} : F_y(k \otimes g) \xrightarrow{\cong} F_y(k) \otimes g$ that satisfies the pentagon axiom. Writing $\gamma_{k,g} := \gamma(k; g)\, \mathrm{id}_{p(u(y)u(k \triangleleft g))}$ for $\gamma \in C^1(G, \mathrm{Map}(K, \mathbb{C}^*))$ we have that the pentagon axiom of a module functor translates into the equation

$$\mu(k; g_1, g_2)\gamma(k \triangleleft g_1; g_2)\gamma(k; g_1) = \gamma(k; g_1 g_2)\mu(f_y(k); g_1, g_2),$$

which can also be written as

$$\delta_G \gamma(k; \gamma_1, \gamma_2) = \frac{\mu(f_y(k); g_1, g_2)}{\mu(k; g_1, g_2)}.$$

The inverse of (F_y, γ) is the module functor $(F_{p(u(y)^{-1})}, \bar{\gamma})$ with

$$\bar{\gamma}(k; g) := \gamma(p(u(y)^{-1}u(x))^{-1}; g)^{-1}.$$

Defining for each $y \in K^A$ the set

$$\mathrm{Fun}_y := \left\{ \gamma \in C^1(G, \mathrm{Map}(K, \mathbb{C}^*)) \,\middle|\, \delta_G \gamma(k; g_1, g_2) = \frac{\mu(f_y(k); g_1, g_2)}{\mu(k; g_1, g_2)} \right\}$$

for all $k \in K$ and $g_1, g_2 \in G$, we have that of invertible objects of $\mathcal{C}^*_{\mathcal{M}}$ are precisely the module functors (F_y, γ) where $y \in K^A$ and $\gamma \in \mathrm{Fun}_y$. To simplify the notation we will denote such a module functor by the pair (y, γ).

Two invertible module functors (y_1, γ^1) and (y_2, γ^2) in $\mathcal{C}^*_{\mathcal{M}}$ are isomorphic if and only if $y_1 = y_2$ and if there exists natural transformation parametrized by a map $\eta \in C^0(G, \mathrm{Map}(K, \mathbb{C}^*))$ satisfying the equation

$$\gamma^1(k; g)\eta(k) = \eta(k \triangleleft g)\gamma^2(k; g) \tag{2-1}$$

for all $k \in K$ and $g \in G$. These equations can be rewritten as the equation

$$\delta_G \eta = \frac{\gamma^2}{\gamma^1}$$

in $C^1(G, \mathrm{Map}(K, \mathbb{C}^*))$. Therefore for each $y \in K^A$ we may define an equivalence relation on the elements $\gamma^1, \gamma^2 \in \mathrm{Fun}_y$ by setting $\gamma^2 \simeq \gamma^1$ whenever there exists η such that $\delta_G \eta = \gamma^2/\gamma^1$; denote by $\overline{\mathrm{Fun}}_y$ the associated set of equivalence classes.

For each $y \in K^A$ let us choose an element $\gamma_y \in \mathrm{Fun}_y$, and note that the maps

$$\begin{aligned} \mathrm{Fun}_y \to Z^1(G, \mathrm{Map}(K, \mathbb{C}^*)), &\quad \beta \mapsto \frac{\beta}{\gamma_y}, \\ Z^1(G, \mathrm{Map}(k, \mathbb{C}^*)) \to \mathrm{Fun}_y, &\quad \epsilon \mapsto \epsilon\gamma_y \end{aligned}$$

are inverse to each other. Therefore we obtain bijections

$$\overline{\mathrm{Fun}}_y \cong H^1(G, \mathrm{Map}(K, \mathbb{C}^*)) \cong H^1(A, \mathbb{C}^*) = \mathbb{A},$$

which are realized by the maps

$$\zeta_y : \mathbb{A} \to \mathrm{Fun}_y, \quad \zeta_y(\rho) := \gamma_y \varphi(\rho), \qquad \theta_y : \mathrm{Fun}_y \to \mathbb{A}, \quad \theta_y(\beta) := \psi(\beta/\gamma_y). \tag{2-2}$$

Recall from [Etingof et al. 2005, Definition 2.2] that the *global dimension* $\dim(\mathcal{C})$ of a fusion category $\mathcal{C}$ is the sum of the squared norms of its simple objects, and note that by Theorem 2.15 of the same paper we have $\dim(\mathcal{C}^*_{\mathcal{M}}) = \dim(\mathcal{C})$ whenever $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is an indecomposable module category over $\mathcal{C}$.

Let us suppose now that the dual category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G, \omega)^*_{\mathcal{M}(A\backslash G,\mu)}$ is pointed. Therefore its global dimension

$$\dim(\mathcal{C}^*_{\mathcal{M}}) = |\mathbb{A}||K^A|$$

must be equal to the number of isomorphic classes of invertible objects, since on pointed categories all simple objects are invertible. On the other hand, by [Etingof et al. 2005, Theorem 2.15] we have $\dim(\mathcal{C}^*_{\mathcal{M}}) = \dim(\mathcal{C})$ and $\dim(\mathcal{C}) = |G|$. Therefore in order for the category $\mathcal{C}^*_{\mathcal{M}}$ to be pointed it is necessary that $|\mathbb{A}||K^A| = |G|$. Since $|G| = |A||K|$, $|\mathbb{A}| \le |A|$ and $|K^A| \le |K|$, the equality holds if and only if A is abelian, thus giving that $|\mathbb{A}| = |A|$, and if A is normal in G and $K^A = K$.

On the other hand, if A is abelian and normal on G, then the number of isomorphism classes of invertible objects in $\mathcal{C}^*_{\mathcal{M}}$ is $|\mathbb{A}||K| = |G|$. Since $\dim(\mathcal{C}^*_{\mathcal{M}}) = \dim(\mathcal{C}) = |G|$, we have that the set of isomorphism classes of invertible objects exhausts the set of simple elements, and therefore $\mathcal{C}^*_{\mathcal{M}}$ must be pointed.

Summarizing we have:

Theorem 2.1 [Naidu 2007, Theorem 3.4]. *The tensor category*

$$\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G, \omega)^*_{\mathcal{M}(A\backslash G,\mu)}$$

is pointed if and only if A is abelian and normal in G and the cohomology class $[(\mu \triangleleft y)/\mu]$ is trivial in $H^2(G, \mathrm{Map}(K, \mathbb{C}^*))$ for all $y \in K$.

Note that since A is normal in G, we may use the notation introduced in Section 1B so that $\mu(f_y(k); g_1, g_2) = \mu(yk; g_1, g_2) = (\mu \triangleleft y)(k; g_1, g_2)$. Since we have that $\delta_G \mu^{-1} = \pi^*\omega = \delta_G(\mu^{-1} \triangleleft y)$, the quotient $(\mu \triangleleft y)/\mu$ defines a cocycle in $Z^2(G, \mathrm{Map}(K, \mathbb{C}^*))$. The equation $\delta_G \gamma_y = (\mu \triangleleft y)/\mu$ implies that the quotient is trivial in cohomology.

2B. ***The Grothendieck ring of the pointed category $\mathcal{C}^*_{\mathcal{M}}$.*** From now on we will assume that the dual category $\mathcal{C}^*_{\mathcal{M}}$ is pointed. Therefore we have that A is abelian and normal in G and that we can choose elements $\gamma_y \in C^1(G, \mathrm{Map}(K, \mathbb{C}^*))$ for each $y \in K$ such that $\delta_G \gamma_y = (\mu \triangleleft y)/\mu$.

The Grothendieck ring $K_0(\mathcal{C}^*_{\mathcal{M}})$ of the category $\mathcal{C}^*_{\mathcal{M}}$ is the ring defined by the semiring whose elements are the isomorphism classes of objects and whose product is the one induced by the tensor product. Since $\mathcal{C}^*_{\mathcal{M}}$ is pointed, $K_0(\mathcal{C}^*_{\mathcal{M}})$ is isomorphic to the group ring $\mathbb{Z}[\Lambda]$ for some finite group Λ. In this section we will recall the construction of this isomorphism carried out in [Naidu 2007, Theorem 4.5].

The tensor product of two invertible elements (y_1, γ^1), (y_2, γ^2) in $\mathcal{C}^*_{\mathcal{M}}$ as defined in Section 1G is

$$(y_1, \gamma^1) \otimes (y_2, \gamma^2) = (y_1 y_2, (\gamma^1 \triangleleft y_2)\gamma^2).$$

This tensor product defines a group structure on the set of isomorphism classes of invertible objects

$$\Lambda := \bigcup_{y \in K} \{y\} \times \overline{\mathrm{Fun}}_y$$

by the equation $(y_1, [\gamma^1]) \star (y_2, [\gamma^2]) = (y_1 y_2, [(\gamma^1 \triangleleft y_2)\gamma^2])$, where $[\gamma]$ denotes the equivalence class of γ in Fun_y.

Define the element $\gamma \in C^1(K, C^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$ by the equation

$$\gamma(y) := \gamma_y$$

and note that the equations $\delta_G \gamma_y = (\mu \triangleleft y)/\mu$ are equivalent to the equation

$$\delta_G \gamma = \delta_K \mu.$$

Define the element $\tilde{\nu} := \delta_K \gamma$, i.e., $\tilde{\nu}(y_1, y_2) = (\gamma(y_2)\gamma(y_1) \triangleleft y_2)/\gamma(y_1 y_2)$, and note that

$$\delta_K \tilde{\nu} = \delta_K^2 \gamma = 1 \quad \text{and} \quad \delta_G \tilde{\nu} = \delta_G \delta_K \gamma = \delta_K \delta_G \gamma = \delta_K^2 \mu = 1.$$

Hence $\tilde{\nu} \in Z^2(K, Z^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$ and we may define

$$\nu := \psi \circ \tilde{\nu} \in Z^2(K, Z^1(A, \mathbb{C}^*)) = Z^2(K, \mathbb{A}), \tag{2-3}$$

thus having $\nu(y_1, y_2)(a) := \tilde{\nu}(y_1, y_2)(1; a)$.

With this 2-cocycle ν we may define the crossed product $K \ltimes_\nu \mathbb{A}$ by setting on pairs of elements of the set $K \times \mathbb{A}$

$$(y_1, \rho_1) \cdot (y_2, \rho_2) := (y_1 y_2, \rho_1^{y_2} \rho_2 \nu(y_1, y_2)).$$

Using the notation of (2-2) we have:

Theorem 2.2 [Naidu 2007, Theorem 4.5]. *The map*

$$T : K \ltimes_\nu \mathbb{A} \to \Lambda, \quad T((y, \rho)) = (y, [\zeta_y(\rho)]),$$

is an isomorphism of groups. Hence $K_0(\mathcal{C}^*_{\mathcal{M}}) \cong \mathbb{Z}[K \ltimes_\nu \mathbb{A}]$.

Proof. On the one hand we have

$$T((y_1,\rho_1)\cdot(y_2,\rho_2)) = T((y_1y_2, \rho_1^{y_2}\rho_2\nu(y_1,y_2))) = (y_1y_2, [\zeta_{y_1y_2}(\rho_1^{y_2}\rho_2\nu(y_1,y_2))])$$

and on the other

$$\begin{aligned} T((y_1,\rho_1))\star T((y_2,\rho_2)) &= (y_1,[\zeta_{y_1}(\rho_1)])\star(y_2,[\zeta_{y_2}(\rho_2)]) \\ &= (y_1y_2, [(\zeta_{y_1}(\rho_1)\triangleleft y_2)\zeta_{y_2}(\rho_2)]). \end{aligned}$$

The result follows if we check the equality

$$\theta_{y_1y_2}((\zeta_{y_1}(\rho_1)\triangleleft y_2)\zeta_{y_2}(\rho_2)) = \rho_1^{y_2}\rho_2\nu(y_1,y_2)$$

since this implies that $\zeta_{y_1y_2}((\rho_1\triangleleft y_2)\rho_2\nu(y_1,y_2))$ and $(\zeta_{y_1}(\rho_1)\triangleleft y_2)\zeta_{y_2}(\rho_2)$ are cohomologous; hence we have

$$\begin{aligned} \theta_{y_1y_2}((\zeta_{y_1}(\rho_1)\triangleleft y_2)\zeta_{y_2}(\rho_2))(a) &= \frac{((\zeta_{y_1}(\rho_1)\triangleleft y_2)(1;a))\zeta_{y_2}(\rho_2)(1;a)}{\gamma(y_1y_2)(1;a)} \\ &= \frac{(\gamma(y_1)\triangleleft y_2\varphi(\rho_1)\triangleleft y_2)(1;a)(\gamma(y_2)\varphi(\rho_2))(1;a)}{\gamma(y_1y_2)(1;a)} \\ &= \delta_K\gamma(y_1,y_2)(1;a)\rho_1^{y_2}(a)\rho_2(a) \\ &= (\nu(y_1,y_2)\rho_1^{y_2}\rho_2)(a). \end{aligned}$$ □

2C. ***A skeleton of the pointed category $\mathcal{C}^*_{\mathcal{M}}$.*** A skeleton $\mathrm{sk}(\mathcal{C}^*_{\mathcal{M}})$ of $\mathcal{C}^*_{\mathcal{M}}$ is a full subcategory of $\mathcal{C}^*_{\mathcal{M}}$ on which each object of $\mathcal{C}^*_{\mathcal{M}}$ is isomorphic to only one object in $\mathrm{sk}(\mathcal{C}^*_{\mathcal{M}})$. Let us choose for objects

$$\mathrm{Obj}(\mathrm{sk}(\mathcal{C}^*_{\mathcal{M}})) := \{(y,\zeta_y(\rho)) \mid (y,\rho)\in K\ltimes_\nu \mathbb{A}\}$$

and define its tensor product $\bullet$ by the one induced by $\star$, i.e.,

$$((y_1,\zeta_{y_1}(\rho_1))\bullet(y_2,\zeta_{y_2}(\rho_2)) := (y_1y_2, \zeta_{y_1y_2}(\nu(y_1,y_2)\rho_1^{y_2}\rho_1)).$$

For each pair of objects, choose isomorphisms in $\mathcal{C}^*_{\mathcal{M}}$

$$\begin{aligned} &f((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2)) \\ &\qquad : (y_1,\zeta_{y_1}(\rho_1))\bullet(y_2,\zeta_{y_2}(\rho_2)) \xrightarrow{\sim} (y_1,\zeta_{y_1}(\rho_1))\otimes(y_2,\zeta_{y_2}(\rho_2)), \end{aligned}$$

which by equation (2-1) satisfy

$$\begin{aligned} &((\zeta_{y_1}(\rho_1)\triangleleft y_2)\zeta_{y_1}(\rho_1))(k;g) \\ &\qquad = \frac{f((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2))(k\triangleleft g)}{f((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2))(k)} \times \zeta_{y_1y_2}(\nu(y_1,y_2)\rho_1^{y_2}\rho_1)(k;g). \end{aligned}$$

The tensor product $\otimes$ in $\mathcal{C}^*_{\mathcal{M}}$ is associative since it is defined by the composition of functors, but the tensor product $\bullet$ in its skeleton $\mathrm{sk}(\mathcal{C}^*_{\mathcal{M}})$ may fail to be associative.

The associativity constraint for $\mathrm{sk}(\mathcal{C}^*_{\mathcal{M}})$ is then

$$\begin{aligned}\hat{\omega}'((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2)),(y_3,\zeta_{y_3}(\rho_3)))\\ = \frac{f((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2))\otimes \mathrm{id}_{(\zeta_{y_3}(\rho_3),y_3)}}{f((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2))\bullet(y_3,\zeta_{y_3}(\rho_3)))}\\ \times\frac{f((y_1,\zeta_{y_1}(\rho_1))\bullet(y_2,\zeta_{y_2}(\rho_2)),(y_3,\zeta_{y_3}(\rho_3)))}{\mathrm{id}_{(\zeta_{y_1}(\rho_1),y_1)}\otimes f((y_2,\zeta_{y_2}(\rho_2)),(y_3,\zeta_{y_3}(\rho_3)))}.\end{aligned}$$

In [Naidu 2007, Theorem 4.9] it is shown that $\hat{\omega}'$ is K-invariant and moreover that it can be given in explicit form by the equation

$$\hat{\omega}'((y_1,\zeta_{y_1}(\rho_1)),(y_2,\zeta_{y_2}(\rho_2)),(y_3,\zeta_{y_3}(\rho_3))) = \tilde{\nu}(y_1,y_2)(1;u(y_3))\rho_1(\kappa_{y_2,u(y_3)}).$$

Therefore we may define the 3-cocycle on $K \ltimes_\nu \mathbb{A}$ by the equation

$$\hat{\omega}((y_1,\rho_1),(y_2,\rho_2),(y_3,\rho_3)) = \tilde{\nu}(y_1,y_2)(1;u(y_3))\rho_1(\kappa_{y_2,u(y_3)}),$$

and choosing $G = A \rtimes_F K$ and $u(y) = (1, y)$ as was done at the end of Section 1A, the 3-cocycle on $K \ltimes_\nu \mathbb{A}$ becomes

$$\hat{\omega}((y_1,\rho_1),(y_2,\rho_2),(y_3,\rho_3)) = \tilde{\nu}(y_1,y_2)(1;(1,y_3))\rho_1(F(y_2,y_3)). \tag{2-4}$$

Therefore the skeleton $\mathrm{sk}(\mathcal{C}^*_{\mathcal{M}})$ of $\mathcal{C}^*_{\mathcal{M}}$ becomes isomorphic to $\mathcal{V}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$, which is equivalent to $\mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$. Therefore we can conclude with:

Theorem 2.3 [Naidu 2007, Theorem 4.9]. *The fusion categories*

$$\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G,\omega)^*_{\mathcal{M}(A\backslash G,\mu)} \quad \textit{and} \quad \mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$$

are equivalent.

Applying the results of Section 1I we have:

Corollary 2.4. *The categories* $\mathrm{Vect}(A \rtimes_F K, \omega)$ *and* $\mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$ *are weakly Morita equivalent. Hence their centers*

$$\mathcal{Z}(\mathrm{Vect}(A \rtimes_F K, \omega)) \simeq \mathcal{Z}(\mathrm{Vect}(K \ltimes_\nu \mathbb{A}, \hat{\omega}))$$

are canonically equivalent as braided tensor categories.

3. Weak Morita equivalence classes of group-theoretical tensor categories

We are interested in classifying group-theoretical tensor categories of a specific global dimension up to weak Morita equivalence. For this purpose we will fix the group $G = A \rtimes_F K$ with A abelian and normal in G and $F \in Z^2(K, A)$, and we will give an explicit description of the cocycles $\omega \in Z^3(A \rtimes_F K, \mathbb{C}^*)$ and $\hat{\omega} \in Z^3(K \ltimes_\nu \mathbb{A}, \mathbb{C}^*)$ such that the tensor categories $\mathcal{V}(A \rtimes_F K, \omega)$ and $\mathcal{V}(K \ltimes_\nu \mathbb{A}, \hat{\omega})$ are weakly Morita equivalent.

3A. ***Description of ω, μ and γ.*** In Theorem 2.1 and in Section 2B we have seen the conditions needed for the tensor category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G,\omega)^*_{\mathcal{M}(A\backslash G,\mu)}$ to be pointed. In particular we have seen that we need the existence of $\gamma \in C^1(K, C^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$ such that

$$\delta_G \gamma = \delta_K \mu.$$

Since we also have that $\delta_G \mu^{-1} = \pi^* \omega$ we can obtain the following lemma:

Lemma 3.1. *In* $\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*))))$, *the cocycles* $\pi^*\omega$ *and* $\tilde{\nu}$ *are cohomologous.*

Proof. Recall the definition of the double complex $C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*)))$ given in Section 1B1, and note that we have $\pi^*\omega \in C^{0,3}$, $\mu \in C^{0,2}$, $\gamma \in C^{1,1}$ and $\tilde{\nu} = \delta_K \gamma \in C^{2,1}$, satisfying $\pi^*\omega \cdot \delta_G \mu = 1$ and $\delta_K \mu \cdot \delta_G \gamma^{-1} = 1$.

Consider the element $\mu \oplus \gamma \in \mathrm{Tot}^2$ and note that

$$\delta_{\mathrm{Tot}}(\mu \oplus \gamma) = (\delta_K \oplus \delta_G^{(-1)^p})(\mu \oplus \gamma) = \delta_G \mu \oplus \delta_K \mu \cdot \delta_G \gamma^{-1} \oplus \delta_K \gamma.$$

Therefore $\pi^*\omega \cdot \delta_{\mathrm{Tot}}(\mu \oplus \gamma) = \tilde{\nu}$. □

Lemma 3.1 implies further conditions on the cohomology class of ω for the tensor category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G,\omega)^*_{\mathcal{M}(A\backslash G,\mu)}$ to be pointed.

Corollary 3.2. *If the tensor category* $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G,\omega)^*_{\mathcal{M}(A\backslash G,\mu)}$ *is pointed then* ω *is cohomologous to a cocycle that lives in* $C^{2,1} \oplus C^{3,0}$ *of the double complex that induces the Lyndon–Hochschild–Serre spectral sequence.*

Remark 3.3. Note that this implies that the cohomology class of ω belongs to the subgroup of $H^3(G, \mathbb{C}^*)$ defined as

$$\Omega(G; A) := \ker(\ker(H^3(G, \mathbb{C}^*) \to E_\infty^{0,3}) \to E_\infty^{1,2}),$$

which fits into the short exact sequence

$$1 \to E_\infty^{3,0} \to \Omega(G; A) \to E_\infty^{2,1} \to 1.$$

The cohomology classes in $\Omega(G; A)$ are the only cohomology classes such that $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(G,\omega)^*_{\mathcal{M}(A\backslash G,\mu)}$ is pointed.

In what follows we will construct explicit representatives for ω and μ, but to do so we will start by constructing explicit 3-cocycles in $\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*))))$ which appear in $\Omega(G; A)$. Let us start by determining the second differential $d_2 : E_2^{2,1} \to E_2^{4,0}$.

Lemma 3.4. *The second differential* $d_2 : E_2^{2,1} \to E_2^{4,0}$ *is isomorphic to the homomorphism*

$$H^2(K, \mathbb{A}) \to H^4(K, \mathbb{C}^*), \quad [\hat{F}] \mapsto [(\hat{F} \wedge F)^{-1}],$$

where $(\hat{F} \wedge F)(k_1, k_2, k_3, k_4) := \hat{F}(k_1, k_2)(F(k_3, k_4))$.

Proof. First recall that

$$E_2^{2,1} = H^2(K, H^1(G, \mathrm{Map}(K, \mathbb{C}^*))) \cong H^2(K, \mathrm{Hom}(A, \mathbb{C}^*)) = H^2(K, \mathbb{A}),$$
$$E_2^{4,0} = H^4(K, H^0(G, \mathrm{Map}(K, \mathbb{C}^*))) = H^4(K, \mathrm{Map}(K, \mathbb{C}^*)^G) \cong H^4(K, \mathbb{C}^*).$$

Take $\hat{F} \in Z^2(K, \mathbb{A})$ and use the map φ of Lemma 1.1 to lift this cocycle to $\varphi(\hat{F}) \in C^2(K, Z^1(G, \mathrm{Map}(K, \mathbb{C}^*)))$; in coordinates:

$$\begin{aligned}\varphi(\hat{F})(k_1, k_2)(x_1, (a_2, x_2)) &= \hat{F}(k_1, k_2)(\kappa_{x_1,(a_2,x_2)})\\ &= \hat{F}(k_1, k_2)({}^{x_1}a_2 F(x_1, x_2))\\ &= \hat{F}(k_1, k_2)({}^{x_1}a_2)\hat{F}(k_1, k_2)(F(x_1, x_2)).\end{aligned}$$

Its boundary is

$$\begin{aligned}&\delta_k\varphi(\hat{F})(k_1,k_2,k_3)(x_1,(a_2,x_2))\\ &\quad= \hat{F}(k_2,k_3)({}^{x_1}a_2F(x_1,x_2))\hat{F}(k_1k_2,k_3)({}^{x_1}a_2F(x_1,x_2))^{-1}\\ &\qquad\quad \hat{F}(k_1,k_2k_3)({}^{x_1}a_2F(x_1,x_2))\hat{F}(k_1,k_2)({}^{k_3x_1}a_2F(k_3x_1,x_2))^{-1}\\ &\quad= \hat{F}(k_1,k_2)^{k_3}(F(x_1,x_2))\hat{F}(k_1,k_2)(F(k_3x_1,x_2))^{-1}\\ &\quad= \hat{F}(k_1,k_2)\left(\frac{F(k_3,x_1)}{F(k_3,x_1x_2)}\right),\end{aligned}$$

and we can define $u \in C^3(K, C^0(G, \mathrm{Map}(K, \mathbb{C}^*)))$ as

$$u(k_1, k_2, k_3)(x) := \hat{F}(k_1, k_2)(F(k_3, x)).$$

On the one hand we have

$$\begin{aligned}\delta_G u(k_1, k_2, k_3)(x_1, (a_2, x_2)) &= u(k_1, k_2, k_3)(x_1x_2)u(k_1, k_2, k_3)(x_1)^{-1}\\ &= \hat{F}(k_1, k_2)\left(\frac{F(k_3, x_1x_2)}{F(k_3, x_1)}\right)\end{aligned}$$

and on the other

$$\begin{aligned}&\delta_K u(k_1, k_2, k_3, k_4)(x)\\ &\quad= \hat{F}(k_2, k_3)(F(k_4, x))\hat{F}(k_1k_2, k_3)(F(k_4, x))^{-1}\hat{F}(k_1, k_2k_3)(F(k_4, x))\\ &\qquad\qquad \hat{F}(k_1, k_2)(F(k_3k_4, x))^{-1}\hat{F}(k_1, k_2)(F(k_3, k_4x))\\ &\quad= \hat{F}(k_1, k_2)^{k_3}(F(k_4, x))\hat{F}(k_1, k_2)(F(k_3k_4, x))^{-1}\hat{F}(k_1, k_2)(F(k_3, k_4x))\\ &\quad= \hat{F}(k_1, k_2)(F(k_3, k_4)).\end{aligned}$$

Since $\delta_G u = \delta_K\varphi(\hat{F})$ we have that

$$\delta_{\mathrm{Tot}}(\varphi(\hat{F}) \oplus u^{-1}) = \delta_K\varphi(\hat{F})\delta_G u \oplus \delta_k u^{-1} = (\hat{F} \wedge F)^{-1};$$

therefore $d_2[\varphi(\hat{F})] = [(\hat{F} \wedge F)^{-1}]$. □

Suppose that $d_2[\varphi(\hat{F})]=0$; hence there is $\epsilon \in C^3(K, \mathbb{C}^*)$ such that $\delta_K \epsilon = \hat{F} \wedge F$. Define $\bar{\epsilon} \in C^3(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ by the equation

$$\bar{\epsilon}(k_1, k_2, k_3)(x) := \epsilon(k_1, k_2, k_3)$$

and note $\delta_K \bar{\epsilon} = \hat{F} \wedge F$ and $\delta_G \bar{\epsilon} = 1$. Hence the class $\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1} \in C^{2,1} \oplus C^{3,0}$ defines a 3-cocycle in the total complex:

$$\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1} \in Z^3 \,\mathrm{Tot}(C^*(K, C^*(G, \mathrm{Map}(K, \mathbb{C}^*)))).$$

Define $\beta \in C^2(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ by the equation

$$\beta(k_1, k_2)(x) := \epsilon(k_1, k_2, x)$$

and note that

$$\begin{aligned} \delta_K \beta(k_1, k_2, k_3)(x) &= \epsilon(k_2, k_3, x)\epsilon(k_1k_2, k_3, x)^{-1}\epsilon(k_1, k_2k_3, x)\epsilon(k_1, k_2, k_3x)^{-1} \\ &= \delta_K \epsilon(k_1, k_2, k_3, x)\epsilon(k_1, k_2, k_3)^{-1} \\ &= \hat{F}(k_1, k_2)((F(k_3, x))\bar{\epsilon}(k_1, k_2, k_3)(x)^{-1}. \end{aligned}$$

Therefore $\delta_K \beta \bar{\epsilon} u^{-1} = 1$; hence we have that the class $\varphi(\hat{F})\delta_G \beta \in C^{2,1}$ is a 3-cocycle in the total complex and moreover that it is cohomologous to the class $\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1}$, in coordinates:

$$\begin{aligned} &(\varphi(\hat{F})\delta_G \beta)(k_1, k_2)(x_1, (a_2, x_2)) \\ &\quad = \hat{F}(k_1, k_2)({}^{x_1}a_2)\hat{F}(k_1, k_2)(F(x_1, x_2))\epsilon(k_1, k_2, x_1x_2)\epsilon(k_1, k_2, x_1)^{-1}. \end{aligned} \tag{3-1}$$

Summarizing the previous results:

Proposition 3.5. *Every cohomology class which appears in $\Omega(G; A)$ can be represented by a 3-cocycle $\varphi(\hat{F})\delta_G \beta \in C^{2,1}$ with $\hat{F} \in Z^2(K, \mathbb{A})$, $\beta(k_1, k_2)(x) = \epsilon'(k_1, k_2, x)$ and $\delta_K \epsilon' = \hat{F} \wedge F$.*

Proof. Take $[\omega] \in \Omega(G; A)$ and let $[\hat{F}] \in E_2^{2,1}$ be a representative of the cohomology class of the image of $[\omega]$ in $E_\infty^{2,1}$. Since $d_2[\varphi(\hat{F})]=0$ we know that the cohomology class $[\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1}]$ constructed above belongs to $\Omega(G; A)$. Therefore we have

$$[\omega^{-1}] \cdot [\varphi(\hat{F}) \oplus \bar{\epsilon} u^{-1}] \in E_\infty^{3,0}.$$

Hence we can choose a representative cocycle $[\tau] \in H^3(K, \mathbb{C}^*) \cong E_2^{3,0}$ such that

$$[\omega] = [\varphi(\hat{F}) \oplus \bar{\epsilon}\bar{\tau} u^{-1}],$$

with $\bar{\tau} \in C^3(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ defined as

$$\bar{\tau}(k_1, k_2, k_3)(x) := \tau(k_1, k_2, k_3).$$

Let $\epsilon' := \epsilon\tau$ and define $\beta \in C^2(K, C^0(G, \mathrm{Maps}(K, \mathbb{C}^*)))$ by the equation

$$\beta(k_1, k_2)(x) := \epsilon'(k_1, k_2, x).$$

Equation (3-1) implies that $\delta_K\beta = (\bar{\epsilon}\bar{\tau})^{-1}u$ and therefore the proposition follows from the equation

$$(\varphi(\hat{F}) \oplus \bar{\epsilon}\bar{\tau}u^{-1})\delta_{\mathrm{Tot}}\beta = \varphi(\hat{F})\delta_G\beta \oplus \delta_K\beta\bar{\epsilon}\bar{\tau}u^{-1} = \varphi(\hat{F})\delta_G\beta. \qquad \square$$

Now we need to find an explicit description of $\omega \in Z^3(G, \mathbb{C}^*)$ such that $\pi^*\omega$ and $\varphi(\hat{F})\delta_G\beta$ are cohomologous.

Theorem 3.6. *Let $G = A \rtimes_F K$ and consider $\omega \in C^3(G, \mathbb{C}^*)$, $\mu \in C^{0,2}$ and $\gamma \in C^{1,1}$ defined by the following equations*:

$$\begin{aligned} \omega((a_1, x_1), (a_2, x_2), (a_3, x_3)) &:= \hat{F}(x_1, x_2)(a_3)\epsilon(x_1, x_2, x_3), \\ \mu(x_1, (a_2, x_2), (a_3, x_3)) &= (\hat{F}(x_1, x_2)(a_3)\epsilon(x_1, x_2, x_3))^{-1}, \\ \gamma(y)(x_1, (a_2, x_2)) &= \hat{F}(y, x_1)(a_2)\epsilon(y, x_1, x_2,). \end{aligned}$$

Then $\pi^*\omega \cdot (\delta_{\mathrm{Tot}}\mu \oplus \gamma) = \varphi(\hat{F})\delta_G\beta$.

Proof. Let us calculate:

$$\begin{aligned} &\delta_G\mu(x_1, (a_2, x_2), (a_3, x_3), (a_4, x_4)) \\ &\quad= \mu(x_1x_2, (a_3, x_3), (a_4, x_4))\mu(x_1, ({a_2}^{x_2}a_3F(x_2, x_3), x_2x_3), (a_3, x_3))^{-1} \\ &\qquad \mu(x_1, (a_2, x_2)({a_3}^{x_3}a_4F(x_3, x_4), x_3x_4))\ \mu(x_1, (a_2, x_2), (a_3, x_3))^{-1} \\ &\quad= \hat{F}(x_1x_2, x_3)(a_4)^{-1}\hat{F}(x_1, x_2x_3)(a_4)\hat{F}(x_1, x_2)({a_3}^{x_3}a_4F(x_3, x_4))^{-1} \\ &\qquad \hat{F}(x_1, x_2)(a_3)\epsilon(x_2, x_3, x_4)^{-1}\delta_K\epsilon(x_1, x_2, x_3, x_4) \\ &\quad= \hat{F}(x_2, x_3)(a_4)^{-1}\epsilon(x_2, x_3, x_4)^{-1}, \end{aligned}$$

and

$$\begin{aligned} \pi^*\omega(x_1, (a_2, x_2), (a_3, x_3), (a_4, x_4)) &= \omega((a_2, x_2), (a_3, x_3), (a_4, x_4)) \\ &= \hat{F}(x_2, x_3)(a_4)\epsilon(x_2, x_3, x_4); \end{aligned}$$

hence we have that $\delta_G\mu \cdot \pi^*\omega = 1$.

Now

$$\begin{aligned} \delta_K\mu(y)(x_1, (a_2, x_2), (a_3, x_3)) &= \mu(x_1, (a_2, x_2), (a_3, x_3))\mu(yx_1, (a_2, x_2), (a_3, x_3))^{-1} \\ &= \frac{\hat{F}(yx_1, x_2)(a_3)\epsilon(yx_1, x_2, x_3)}{\hat{F}(x_1, x_2)(a_3)\epsilon(x_1, x_2, x_3)}, \end{aligned}$$

and

$$\begin{aligned}
&\delta_G\gamma(y)(x_1,(a_2,x_2),(a_3,x_3))\\
&\quad=\gamma(y)(x_1x_2,(a_3,x_3))\gamma(y)(x_1,({a_2}^{x_2}a_3F(x_2,x_3),x_2x_3))^{-1}\gamma(y)(x_1,(a_2,x_2))\\
&\quad=\hat{F}(y,x_1x_2)(a_3)\hat{F}(y,x_1)({a_2}^{x_2}a_3F(x_2,x_3))^{-1}\hat{F}(y,x_1)(a_2)\\
&\qquad\qquad\epsilon(y,x_1x_2,x_3)\epsilon(y,x_1,x_2x_3)^{-1}\epsilon(y,x_1,x_2)\\
&\quad=\hat{F}(yx_1,x_2)(a_3)\hat{F}(x_1,x_2)(a_3)^{-1}\epsilon(yx_1,x_2,x_3)\epsilon(x_1,x_2,x_3)^{-1};
\end{aligned}$$

hence we have that

$$\delta_K\mu\cdot\delta_G\gamma^{-1}=1.$$

Finally we calculate

$$\begin{aligned}
&\delta_K\gamma(k_1,k_2)(x_1,(a_2,x_2))\\
&\quad=\gamma(k_2)(x_1,(a_2,x_2))\gamma(k_1k_2)(x_1,(a_2,x_2))^{-1}\gamma(k_1)(k_2x_1,(a_2,x_2))\\
&\quad=\hat{F}(k_2,x_1)(a_2)\hat{F}(k_1k_2,x_1)(a_2)^{-1}\hat{F}(k_1,k_2x_2)(a_2)\\
&\qquad\qquad\epsilon(k_2,x_1,x_2)\epsilon(k_1k_2,x_1,x_2)^{-1}\epsilon(k_1,k_2x_1,x_2)\\
&\quad=\hat{F}(k_1,k_2)({}^{x_1}a_2)\delta_K\epsilon(k_1,k_2,x_1,x_2)\epsilon(k_1,k_2,x_1x_2)\epsilon(k_1,k_2,x_1)^{-1}\\
&\quad=\hat{F}(k_1,k_2)({}^{x_1}a_2)\hat{F}(k_1,k_2)(F(x_1,x_2))\epsilon(k_1,k_2,x_1x_2)\epsilon(k_1,k_2,x_1)^{-1},
\end{aligned}$$

and since by equation (3-1) we have that

$$\begin{aligned}
&(\varphi(\hat{F})\delta_G\beta)(k_1,k_2)(x_1,(a_2,x_2))\\
&\quad=\hat{F}(k_1,k_2)({}^{x_1}a_2)\hat{F}(k_1,k_2)(F(x_1,x_2))\epsilon(k_1,k_2,x_1x_2)\epsilon(k_1,k_2,x_1)^{-1},
\end{aligned}$$

we have that

$$\delta_K\gamma=\varphi(\hat{F})\delta_G\beta.$$

Hence $\pi^*\omega\cdot(\delta_{\mathrm{Tot}}\mu\oplus\gamma)=\varphi(\hat{F})\delta_G\beta$. □

3B. *Description of $\hat{\omega}$ and ν.* Assuming the explicit descriptions of ω, μ and γ given in Theorem 3.6, we see that $\tilde{\nu}=\varphi(\hat{F})\delta_G\beta$. Applying this explicit description of $\tilde{\nu}$ to the definition of ν given in (2-3) and of $\hat{\omega}$ given in (2-4) we obtain

$$\nu(k_1,k_2)(a):=\tilde{\nu}(k_1,k_2)(1,(a,1))=\hat{F}(k_1,k_2)(a),$$

which implies that $\nu=\hat{F}$ and

$$\begin{aligned}
\hat{\omega}((k_1,\rho_1),(k_2,\rho_2),(k_3,\rho_3))&:=\tilde{\nu}(k_1,k_2)(1;(1,k_3))\rho_1(F(k_2,k_3))\\
&=\epsilon(k_1,k_2,k_3)\rho_1(F(k_2,k_3)).
\end{aligned}$$

After applying Corollary 2.4 to the previous explicit construction of $\hat{\omega}$ we obtain the following theorem:

Theorem 3.7. *Let K be a finite group acting on the finite abelian group A. Consider cocycles $F \in Z^2(K, A)$ and $\hat{F} \in Z^2(K, \mathbb{A})$ such that $\hat{F} \wedge F$ is trivial in cohomology, i.e., there exists $\epsilon \in C^3(K, \mathbb{C}^*)$ such that $\delta_K \epsilon = \hat{F} \wedge F$. Define the 3-cocycles $\omega \in Z^3(A \rtimes_F K, \mathbb{C}^*)$ and $\hat{\omega} \in Z^3(K \ltimes_{\hat{F}} \mathbb{A}, \mathbb{C}^*)$ by the equations*:

$$\omega((a_1, k_1), (a_2, k_2), (a_3, k_3)) := \hat{F}(k_1, k_2)(a_3)\epsilon(k_1, k_2, k_3)$$
$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)).$$

Then the tensor categories $\mathrm{Vect}(A \rtimes_F K, \omega)$ *and* $\mathrm{Vect}(K \ltimes_{\hat{F}} \mathbb{A}, \hat{\omega})$ *are weakly Morita equivalent, and therefore their centers are braided equivalent*:

$$\mathcal{Z}(\mathrm{Vect}(A \rtimes_F K, \omega)) \simeq \mathcal{Z}(\mathrm{Vect}(K \ltimes_{\hat{F}} \mathbb{A}, \hat{\omega})).$$

Note that we may have taken a different choice of μ and γ in Section 3A thus producing different $\tilde{\nu}$ and $\hat{\omega}$. The description of $\hat{\omega}$ depends on the choice of cohomology class $[\hat{F}] \in H^2(K, \mathbb{A}) \cong E_2^{2,1}$ in the second page representing the image of $[\omega]$ in $E_3^{2,1} = E_\infty^{2,1}$. This choice may be changed by elements in the image of the second differential $d_2 : E_2^{0,2} \to E_2^{2,1}$.

Changing ω by a coboundary $\omega' = \omega\delta_G\alpha$, and writing ω' explicitly as

$$\omega'((a_1, x_1), (a_2, x_2), (a_3, x_3)) := \hat{F}'(x_1, x_2)(a_3)\epsilon'(x_1, x_2, x_3) \tag{3-2}$$

produces a $\hat{\omega}'$ which becomes

$$\hat{\omega}'((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon'(k_1, k_2, k_3)\rho_1(F(k_2, k_3)). \tag{3-3}$$

Applying Theorem 3.7 and using the equivalence of categories

$$\mathrm{Vect}(A \rtimes_F K, \omega) \simeq \mathrm{Vect}(A \rtimes_F K, \omega')$$

we obtain that the tensor categories $\mathrm{Vect}(A \rtimes_F K, \omega)$ and $\mathrm{Vect}(K \ltimes_{\hat{F}'} \mathbb{A}, \hat{\omega}')$ are also weakly Morita equivalent. The previous argument permits us to conclude the following corollary:

Corollary 3.8. *Suppose that the fusion category $\mathcal{C}^*_{\mathcal{M}} = \mathcal{V}(A \rtimes_F K, \omega)^*_{\mathcal{M}(K,\mu)}$ is pointed. Then it is equivalent to the category* $\mathrm{Vect}(K \ltimes_{\hat{F}'} \mathbb{A}, \hat{\omega}')$, *where $\hat{\omega}'$ and ω' are the cocycles defined in (3-2) and (3-3) respectively and ω' is cohomologous to ω.*

3C. ***Classification theorem.*** Now we are ready to state the key result in order to establish the weak Morita equivalence classes of group theoretical tensor categories.

Theorem 3.9. *Let H and $\hat{H}$ be finite groups, $\eta \in Z^3(H, \mathbb{C}^*)$ and $\hat{\eta} \in Z^3(\hat{H}, \mathbb{C}^*)$. Then the tensor categories* $\mathrm{Vect}(H, \eta)$ *and* $\mathrm{Vect}(\hat{H}, \hat{\eta})$ *are weakly Morita equivalent if and only if the following conditions are satisfied*:

- *There exist isomorphisms of groups*

$$\phi : G = A \rtimes_F K \xrightarrow{\cong} H, \quad \hat{\phi} : \hat{G} = K \ltimes_{\hat{F}} \mathbb{A} \xrightarrow{\cong} \hat{H}$$

  for some finite group K acting on the abelian group A, with $F \in Z^2(K, A)$ and $\hat{F} \in Z^2(K, \mathbb{A})$ where $\mathbb{A} := \mathrm{Hom}(A, \mathbb{C}^*)$.

- *There exists $\epsilon : K^3 \to \mathbb{C}^*$ such that $\hat{F} \wedge F = \delta_K \epsilon$.*

- *The cohomology classes satisfy the equations $[\phi^* \eta] = [\omega]$ and $[\hat{\phi}^* \hat{\eta}] = [\hat{\omega}]$ with*

$$\omega((a_1, k_1), (a_2, k_2), (a_3, k_3)) := \hat{F}(k_1, k_2)(a_3)\epsilon(k_1, k_2, k_3),$$
$$\hat{\omega}((k_1, \rho_1), (k_2, \rho_2), (k_3, \rho_3)) := \epsilon(k_1, k_2, k_3)\rho_1(F(k_2, k_3)).$$

Proof. Suppose that $\mathrm{Vect}(H, \eta)$ and $\mathrm{Vect}(\hat{H}, \hat{\eta})$ are weakly Morita equivalent. Then $\mathrm{Vect}(\hat{H}, \hat{\eta})$ is equivalent to the dual category $\mathcal{V}(H, \eta)^*_{\mathcal{M}(A \backslash H, \mu)}$ with $K := A \backslash H$, $\phi : G = A \rtimes_F K \xrightarrow{\cong} H$ and $\mathcal{M}(A \backslash H, \mu)$ some module category of $\mathcal{V}(H, \eta)$. By Corollary 3.8 the tensor category $\mathrm{Vect}(\hat{H}, \hat{\eta})$ is furthermore equivalent to $\mathrm{Vect}(K \ltimes_{\hat{F}'} \mathbb{A}, \hat{\omega}')$, where ω' and $\hat{\omega}'$ are the cocycles defined in equations (3-2) and (3-3) respectively, and such that ω' is cohomologous to $\phi^* \eta$. In particular we have that $\hat{\phi} : \hat{G} = K \ltimes_{\hat{F}} \mathbb{A} \xrightarrow{\cong} \hat{H}$ and that $\hat{\phi}^* \hat{\eta}$ is cohomologous to $\hat{\omega}'$.

The converse is the statement of Theorem 3.7. □

In the case that both ω and $\hat{\omega}$ are cohomologically trivial, we conclude that $\mathrm{Vect}(A \rtimes_F K, 1)$ and $\mathrm{Vect}(K \ltimes_{\hat{F}} \mathbb{A}, 1)$ are weakly Morita equivalent if and only if the cohomology class $[\hat{F}] \in H^2(K, \mathbb{A})$ lies in the image of the second differential of the spectral sequence $d_2 : H^2(A, \mathbb{C}^*)^K \to H^2(K, \mathbb{A})$. This result was originally proved in [Davydov 2000, Corollary 6.2].

4. Examples

4A. ***Pointed fusion categories of global dimension 4.*** We can now calculate the weakly Morita equivalence classes of pointed fusion categories of global dimension 4.

For $G = \mathbb{Z}/4$ we have that $H^*(\mathbb{Z}/4, \mathbb{Z}) \cong \mathbb{Z}[u]/4u$ with $|u| = 2$ and that the nontrivial automorphism of $\mathbb{Z}/4$ maps u to $-u$; therefore

$$H^4(\mathbb{Z}/4, \mathbb{Z}) / \mathrm{Aut}(\mathbb{Z}/4) = \langle u^2 \rangle = \mathbb{Z}/4.$$

For $G = (\mathbb{Z}/2)^2$ we have that

$$H^4((\mathbb{Z}/2)^2, \mathbb{Z}) \cong \ker(\mathrm{Sq}^1 : H^4((\mathbb{Z}/2)^2, \mathbb{F}_2) \to H^5((\mathbb{Z}/2)^2, \mathbb{F}_2)) = \langle x^4, x^2 y^2, y^4 \rangle,$$

where $H^*((\mathbb{Z}/2)^2, \mathbb{F}_2) = \mathbb{F}_2[x, y]$ and Sq^1 is the Steenrod operation, and up to automorphisms of $(\mathbb{Z}/2)^2$ we get

$$H^4((\mathbb{Z}/2)^2, \mathbb{Z})/\operatorname{Aut}((\mathbb{Z}/2)^2) = \begin{cases} 0, \\ (x^4) = \{x^4, y^4, x^4 + y^4\}, \\ (x^2y^2) = \{x^2y^2, x^2y^2 + x^4, x^2y^2 + y^4\}, \\ (x^4 + x^2y^2 + y^4) = \{x^4 + x^2y^2 + y^4\}. \end{cases}$$

Since we have a clear description for a base of $H^4((\mathbb{Z}/2)^2, \mathbb{Z})$, we will abuse notation and denote with the symbols of $H^4((\mathbb{Z}/2)^2, \mathbb{Z})$ the elements of $H^3((\mathbb{Z}/2)^2, \mathbb{C}^*)$. With this clarification, the relevant terms of the second page of the LHS spectral sequence of the extension $1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$ become

$$\begin{tikzcd}
3 & \mathbb{Z}/2 = \langle y^4 \rangle \\
2 & 0 & 0 \\
1 & \mathbb{Z}/2 & \mathbb{Z}/2 = \langle yx \rangle \arrow[drr, "\cong"] & \mathbb{Z}/2 = \langle yx^2 \rangle \\
0 & \mathbb{C}^* & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 = \langle x^4 \rangle & 0 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

where the second differential is defined by the assignment $d_2(yx^k) = \mathrm{Sq}^1(x^{k+2})$ with the class x^2 classifying the extension. We conclude that the only weak Morita equivalence that appears, which does not come from an automorphism of a group, is

$$\mathrm{Vect}(\mathbb{Z}/4, 0) \simeq \mathrm{Vect}((\mathbb{Z}/2)^2, x^2y^2).$$

Therefore we see that there are exactly seven weak Morita equivalence classes of pointed fusion categories of global dimension 4, namely the three for $\mathbb{Z}/4$:

$$\mathrm{Vect}(\mathbb{Z}/4, u^2), \quad \mathrm{Vect}(\mathbb{Z}/4, 2u^2), \quad \mathrm{Vect}(\mathbb{Z}/4, 3u^2);$$

the three for $(\mathbb{Z}/2)^2$:

$$\mathrm{Vect}((\mathbb{Z}/2)^2, 0), \quad \mathrm{Vect}((\mathbb{Z}/2)^2, x^4), \quad \mathrm{Vect}((\mathbb{Z}/2)^2, x^4 + y^4 + x^2y^2);$$

and the one that we have just constructed

$$\mathrm{Vect}(\mathbb{Z}/4, 0) \simeq_M \mathrm{Vect}((\mathbb{Z}/2)^2, x^2y^2).$$

4B. ***Nontrivial action of $\mathbb{Z}/2$ on $\mathbb{Z}/4$.*** Consider the nontrivial action of $\mathbb{Z}/2$ on $\mathbb{Z}/4$ and the abelian extension $1 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/2 \to 1$. The group G is either the dihedral group D_8 in the case that the extension is a split extension or the quaternion group Q_8 in the case that the extension is a nonsplit extension.

In the case of D_8 the relevant elements of the second page of the LHS spectral sequence associated to the extension are

$$
\begin{array}{c|ccccc}
3 & \mathbb{Z}/4 = \langle a \rangle & & & & \\
2 & 0 & 0 & & & \\
1 & \mathbb{Z}/2 & \mathbb{Z}/2 = \langle e \rangle & \mathbb{Z}/2 = \langle b \rangle & & \\
0 & \mathbb{C}^* & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 = \langle c \rangle & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

and they all survive to the page at infinity. Since $H^3(D_8, \mathbb{C}^*) = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ we may say that $H^3(D_8, \mathbb{C}^*) \cong \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$, and since $D_8 \cong \mathbb{Z}/4 \rtimes \mathbb{Z}/2$ we have that $F = 0$. The element $b \in H^2(\mathbb{Z}/2, \mathbb{Z}/4)$ defines the nontrivial extension $Q_8 \cong \mathbb{Z}/2 \ltimes_b \mathbb{Z}/4$.

The second page of the LHS spectral sequence of the extension $Q_8 \cong \mathbb{Z}/2 \ltimes_b \mathbb{Z}/4$ becomes

$$
\begin{tikzcd}
3 & \mathbb{Z}/4 = \langle \alpha \rangle & & & & \\
2 & 0 & 0 & & & \\
1 & \mathbb{Z}/2 & \mathbb{Z}/2 = \langle e \rangle \arrow[drr, "\cong"] & \mathbb{Z}/2 = \langle 4\alpha \rangle & & \\
0 & \mathbb{C}^* & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 = \langle c \rangle & 0 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

where $d_2 : E_2^{1,1} \xrightarrow{\cong} E_2^{3,0}$ is an isomorphism and $H^3(Q_8, \mathbb{C}^*) = \langle \alpha \rangle = \mathbb{Z}/8$.

Therefore for these extensions we only have the weak Morita equivalences

$$\mathrm{Vect}(D_8, b) \simeq_M \mathrm{Vect}(Q_8, 0) \simeq_M \mathrm{Vect}(D_8, b \oplus c),$$

where the equivalence of the right is obtained from the fact that c does not survive the spectral sequence for the group Q_8, and the self-Morita equivalence

$$\mathrm{Vect}(Q_8, 4\alpha) \simeq_M \mathrm{Vect}(Q_8, 4\alpha).$$

4C. ***Extension of $\mathbb{Z}/2 \times \mathbb{Z}/2$ by $\mathbb{Z}/2$.*** Consider the nonabelian extensions of the form

$$1 \to \mathbb{Z}/2 \to G \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1,$$

namely D_8 and Q_8.

The second page of the LHS spectral sequence for these extensions becomes

$$
\begin{array}{c|ccccc}
3 & \mathbb{Z}/2 & & & & \\
2 & 0 & 0 & & & \\
1 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^3 & & \\
0 & \mathbb{C}^* & (\mathbb{Z}/2)^2 & \mathbb{Z}/2 & (\mathbb{Z}/2)^3 & (\mathbb{Z}/2)^2 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

and we need only to concentrate on the differentials $d_2 : E_2^{p,1} \to E_2^{p+2,0}$ between the first two rows since we know that $E_2^{0,3} = \mathbb{Z}/2$ survives the spectral sequence in all the groups.

First we will determine the differential $\bar{d}_2^G$ in the LHS spectral sequence for coefficients in the field of two elements $\mathbb{F}_2$. In this case

$$E_2 \cong H^*(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(\mathbb{Z}/2, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, e],$$

and $\bar{d}_2^G e \in H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{F}_2)$ represents the class that defines the extension G. It is known that the class $x^2 + xy + y^2$ defines Q_8 [Adem and Milgram 1994, Lemma 2.10], the classes $x^2 + xy$, $xy + y^2$, xy define D_8 (p. 130 of the same book) and the classes x^2, y^2, $x^2 + y^2$ define $\mathbb{Z}/2 \times \mathbb{Z}/4$.

Second we use the fact that for the group $(\mathbb{Z}/2)^2$ we have the isomorphism

$$H^j((\mathbb{Z}/2)^2, \mathbb{Z}) \cong \ker(\mathrm{Sq}^1 : H^j((\mathbb{Z}/2)^2, \mathbb{Z}/2) \to H^{j+1}((\mathbb{Z}/2)^2, \mathbb{Z}/2)),$$

where Sq^1 is the first Steenrod square. This implies that the canonical map

$$H^j((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \to H^j((\mathbb{Z}/2)^2, \mathbb{C}^*)$$

can be seen as the map

$$\begin{aligned} H^j((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \xrightarrow{\mathrm{Sq}^1} \ker(\mathrm{Sq}^1 : H^{j+1}&((\mathbb{Z}/2)^2, \mathbb{Z}/2) \longrightarrow \\ &H^{j+2}((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \cong H^{j+1}((\mathbb{Z}/2)^2, \mathbb{Z}) \cong H^j((\mathbb{Z}/2)^2, \mathbb{C}^*). \end{aligned}$$

Therefore the second differential

$$d_2^G : H^{p-2}((\mathbb{Z}/2)^2, \mathbb{Z}/2) \to H^p((\mathbb{Z}/2)^2, \mathbb{C}^*)$$

is isomorphic to the composite map

$$\begin{aligned} d_2^G : H^{p-2}((\mathbb{Z}/2)^2, \mathbb{Z}/2) \longrightarrow \ker(\mathrm{Sq}^1 : H^{p+1}&((\mathbb{Z}/2)^2, \mathbb{Z}/2) \longrightarrow \\ &H^{p+2}((\mathbb{Z}/2)^2, \mathbb{Z}/2)) \cong H^{p+1}((\mathbb{Z}/2)^2, \mathbb{Z}) \cong H^p((\mathbb{Z}/2)^2, \mathbb{C}^*) \end{aligned}$$

taking z to $\mathrm{Sq}^1(z \cup \bar{d}_2^G e)$.

Without loss of generality we may choose $\bar{d}_2^G e = xy + x^2$ for calculating the LHS spectral sequence for D_8. Applying the differential d_2^G to the elements 1, x, y, x^2, xy, y^2 we obtain that the surviving terms in the infinite page of the LHS spectral sequence for D_8 become

$$
\begin{array}{c|ccccc}
3 & \mathbb{Z}/2 & & & & \\
2 & 0 & 0 & & & \\
1 & 0 & \mathbb{Z}/2=\langle e(y)\rangle & \mathbb{Z}/2=\langle e(xy+x^2)\rangle & & \\
0 & \mathbb{C}^* & (\mathbb{Z}/2)^2=\langle x^2, y^2\rangle & 0 & (\mathbb{Z}/2)^2=\dfrac{\langle x^4, x^2y^2, y^4\rangle}{\langle x^2y^2+x^4\rangle} & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Here we are abusing the notation and we are using the explicit base of $H^4((\mathbb{Z}/2)^2, \mathbb{Z})$ to denote the elements in $H^3((\mathbb{Z}/2)^2, \mathbb{C}^*)$. Since $E_3^{2,1} = \langle e(xy + x^2)\rangle$, the weak Morita equivalences that we obtain in the extension are

$$
\begin{aligned}
\mathrm{Vect}(D_8, 0) &\simeq_M \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(xy + x^2))),\\
\mathrm{Vect}(D_8, x^4) &\simeq_M \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(xy + x^2)) + x^4),\\
\mathrm{Vect}(D_8, y^4) &\simeq_M \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(xy + x^2)) + y^4),
\end{aligned}
$$

and the self-equivalence

$$\mathrm{Vect}(D_8, e(xy + x^2)) \simeq \mathrm{Vect}(D_8, e(xy + x^2)).$$

The surviving terms for Q_8 with $\bar{d}_2^G e = x^2 + xy + y^2$ are

$$
\begin{array}{c|ccccc}
3 & \mathbb{Z}/2 & & & & \\
2 & 0 & 0 & & & \\
1 & 0 & 0 & \mathbb{Z}/2 = \langle e(x^2 + xy + y^2)\rangle & & \\
0 & \mathbb{C}^* & (\mathbb{Z}/2)^2 = \langle x^2, y^2\rangle & 0 & \mathbb{Z}/2 = \langle x^2y^2\rangle & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

with $E_\infty^{0,3} = \mathbb{Z}/2 = \langle\alpha\rangle$, $\langle x^2 + xy + y^2\rangle = \langle 2\alpha\rangle$ and $\langle x^2y^2\rangle = \langle 4\alpha\rangle$, where α is a generator $\langle\alpha\rangle = H^3(Q_8, \mathbb{C}^*)$ that was defined in section Section 4B.

Hence the only Morita equivalences that we obtain are

$$\mathrm{Vect}(Q_8, 0) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(x^2+xy+y^2)))$$
$$\mathrm{Vect}(Q_8, 4\alpha) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \mathrm{Sq}^1(e(x^2+xy+y^2))+x^2y^2)$$

and the self-Morita equivalences

$$\mathrm{Vect}(Q_8, 2\alpha) \simeq_M \mathrm{Vect}(Q_8, 2\alpha) \quad \text{and} \quad \mathrm{Vect}(Q_8, 6\alpha) \simeq_M \mathrm{Vect}(Q_8, 6\alpha).$$

Bundling up the previous results for the group Q_8 we obtain the following result:

Proposition 4.1. *Let us suppose that* $\mathrm{Vect}(Q_8, k\alpha)$ *is weakly Morita equivalent to* $\mathrm{Vect}(G, \eta)$. *Then*:

- *For k odd or $k = 2, 6$, the group G must be isomorphic to Q_8 and η must correspond to $j\alpha$ with j odd or $j = 2, 6$.*
- *For $k = 4$, G must be isomorphic to Q_8 or $(\mathbb{Z}/2)^3$.*
- *For $k = 0$, G must be isomorphic to Q_8, D_8 or $(\mathbb{Z}/2)^3$.*

Proof. First note the action of $\mathrm{Aut}(Q_8)$ on $H^3(Q_8, \mathbb{C}^*)$ is trivial. Second note the only normal subgroups of Q_8 are its center and the cyclic ones generated by roots of unity and that they all fit into the central extension $1 \to \mathbb{Z}/2 \to Q_8 \to (\mathbb{Z}/2)^2 \to 1$ or the nonsplit extension $1 \to \mathbb{Z}/4 \to Q_8 \to \mathbb{Z}/2 \to 1$ that we have studied before. Since any weak Morita equivalence between pointed fusion categories comes from a normal and abelian subgroup of Q_8, the classification that we have done before exhausts all possibilities. For k odd we know that $k\alpha$ survives to the restriction to the center and to the cyclic subgroups isomorphic to $\mathbb{Z}/4$ and therefore G can only be Q_8. The classes 2α and 6α trivialize on the center of Q_8 but these classes define extensions of $(\mathbb{Z}/2)^2$ by $\mathbb{Z}/2$ which are isomorphic to Q_8 and define cohomology classes which are precisely 2α and 6α. The class 4α trivializes in all normal and abelian subgroups; in the case of the subgroup $\mathbb{Z}/4$ the only group that may appear is Q_8, and in the case of the center we may obtain the weak Morita equivalence

$$\mathrm{Vect}(Q_8, 4\alpha) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \quad \mathrm{Sq}^1(e(x^2+xy+y^2))+x^2y^2).$$

Finally, the trivial class produces only the group D_8 in the case of the subgroup $\mathbb{Z}/4$ and $(\mathbb{Z}/2)^3$ in the case of the center; some weak Morita equivalences are

$$\mathrm{Vect}(Q_8, 0) \simeq \mathrm{Vect}((\mathbb{Z}/2)^3, \quad \mathrm{Sq}^1(e(x^2+xy+y^2))) \simeq_M \mathrm{Vect}(D_8, b). \qquad \square$$

References

[Adem and Milgram 1994] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Grundlehren der Math. Wissenschaften **309**, Springer, Berlin, 1994. MR Zbl

[Bakalov and Kirillov 2001] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series **21**, American Mathematical Society, Providence, RI, 2001. MR Zbl

[Davydov 2000] A. A. Davydov, "Finite groups with the same character tables, Drinfeld algebras and Galois algebras", pp. 99–111 in *Algebra* (Moscow, 1998), edited by Y. Bahturin, de Gruyter, Berlin, 2000. MR Zbl arXiv

[Dijkgraaf et al. 1991] R. Dijkgraaf, V. Pasquier, and P. Roche, "Quasi Hopf algebras, group cohomology and orbifold models", *Nuclear Phys. B Proc. Suppl.* **18B** (1991), 60–72. MR Zbl

[Etingof et al. 2005] P. Etingof, D. Nikshych, and V. Ostrik, "On fusion categories", *Ann. of Math.* (2) **162**:2 (2005), 581–642. MR Zbl

[Etingof et al. 2011] P. Etingof, D. Nikshych, and V. Ostrik, "Weakly group-theoretical and solvable fusion categories", *Adv. Math.* **226**:1 (2011), 176–205. MR Zbl

[Evens 1991] L. Evens, *The cohomology of groups*, Oxford Univ. Press, 1991. MR Zbl

[Movshev 1993] M. V. Movshev, "Twisting in group algebras of finite groups", *Funktsional. Anal. i Prilozhen.* **27**:4 (1993), 17–23. In Russian; translated in *Funct. Anal. Appl.* **27**:4 (1993), 240–244. MR Zbl

[Müger 2003] M. Müger, "From subfactors to categories and topology, I: Frobenius algebras in and Morita equivalence of tensor categories", *J. Pure Appl. Algebra* **180**:1-2 (2003), 81–157. MR Zbl

[Naidu 2007] D. Naidu, "Categorical Morita equivalence for group-theoretical categories", *Comm. Algebra* **35**:11 (2007), 3544–3565. MR Zbl

[Ostrik 2003a] V. Ostrik, "Module categories over the Drinfel'd double of a finite group", *Int. Math. Res. Not.* **2003**:27 (2003), 1507–1520. MR Zbl

[Ostrik 2003b] V. Ostrik, "Module categories, weak Hopf algebras and modular invariants", *Transform. Groups* **8**:2 (2003), 177–206. MR Zbl

Received July 20, 2016. Revised February 7, 2017.

BERNARDO URIBE
DEPARTAMENTO DE MATEMÁTICAS Y ESTADÍSTICA
UNIVERSIDAD DEL NORTE
BARRANQUILLA
COLOMBIA

bjongbloed@uninorte.edu.co

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.467

LENGTH-PRESERVING EVOLUTION OF IMMERSED CLOSED CURVES AND THE ISOPERIMETRIC INEQUALITY

XIAO-LIU WANG, HUI-LING LI AND XIAO-LI CHAO

It is shown that all immersed closed, locally convex curves with total curvature of $2m\pi$ and n-fold rotational symmetry ($m/n \le 1$) finally evolve into m-fold circles under the length-preserving curvature flow. Sufficient conditions for the occurrence of the finite-time singularities in the flow are also established. As a byproduct, an isoperimetric inequality for rotationally symmetric, locally convex curves is proved via the flow method.

1. Introduction

In this paper we investigate the evolution of immersed closed curves $X(p,t)$ parametrized by p and driven by the inner normal speed

$$V(p,t) = \left(-\int_{X(\cdot,t)} k^2\,ds \Big/ \int_{X(\cdot,t)} k\,ds + k(p,t)\right)\boldsymbol{n}(p,t), \tag{1-1}$$

where $k(p,t)$ denotes the curvature of $X(p,t)$ with respect to inner normal $\boldsymbol{n}(p,t)$. Denote by X_0 the given smooth closed initial curve. When X_0 is a simple convex closed curve ($m=1$), this flow has been studied by Ma and Zhu [2012]. It is shown that the flow preserves convexity and length while it increases the enclosed area, finally converging to a round circle in the C^∞ metric.

When X_0 is an immersed, locally convex closed curve, it is not difficult to show that the convexity and length of evolving curves are still preserved under the flow, and the enclosed algebraic area is increasing. Moreover, in [Wang and Wo 2014], two special classes of rotationally symmetric, locally convex closed initial curves, which both enclose a positive algebraic area, are found to guarantee the convergence of the flow (1-1) to m-fold circles. One class consists of highly symmetric convex curves. Specifically, they are locally convex closed curves with total curvature $2m\pi$ and n-fold rotational symmetry where $n > 2m$. The other is Abresch–Langer type convex curves, which still have total curvature of $2m\pi$ and n-fold rotational symmetry but with $n < 2m$ and some additional conditions on the curvature (see

MSC2010: 35B40, 35K59, 53C44.
Keywords: curvature flow, nonlocal, blow-up, convergence, isoperimetric inequality.

its definition in [Wang and Wo 2014]). Note that Abresch–Langer curves [1986] belong to the later class.

One may naturally ask about the behavior for a general rotationally symmetric curve under the flow (1-1). Furthermore, is there any possibility of the occurrence of singularity in the flow (1-1)? We devote this short paper to answering these questions. For the convenience of the reader, we use the following notation:

ds	the differential element of arclength,
θ	the normal angle of $X(\cdot\,,t)$,
$L(t)$	the length of $X(\cdot\,,t)$,
$A(t)$	the algebraic area of $X(\cdot\,,t)$ defined by $-\frac{1}{2}\int_X \langle X, \boldsymbol{n}\rangle\, ds$,
$k(\cdot\,,t)$	the curvature of $X(\cdot\,,t)$ with respect to $\boldsymbol{n}$.

Here, we always take the orientation of $X(\cdot\,,t)$ to be counterclockwise.

Define

$$\bar{k} = \frac{\int_X k^2\, ds}{\int_X k\, ds} = \frac{\int_X k^2\, ds}{2m\pi}.$$

We write down the evolution of various geometric quantities along the flow (1-1). They can be deduced from the general formulas in [Chou and Zhu 2001].

$$\frac{\partial k}{\partial t} = k_{ss} + k^2(k-\bar{k}), \quad \frac{dL}{dt} = -\int_X k(k-\bar{k})\, ds = 0, \quad \frac{dA}{dt} = -\int_X (k-\bar{k})\, ds \geq 0.$$

Here, it can be seen that the length of the evolving curves is preserved while the enclosed algebraic area is increasing.

Each point on the locally convex solution $X(\cdot\,,t)$ has a unique tangent and one can use the tangent angle $\theta \in S^1_m := \mathbb{R}/2m\pi\mathbb{Z}$ to parametrize it. Generally speaking, θ is a function depending on t. One can make θ independent of time t by adding a tangential component to the velocity vector $\partial X/\partial t$, which does not affect the geometric shape of the evolving curve (see, for instance, [Gage 1986]). Then the evolution equations can be expressed in the coordinates of θ and t. If we denote by $k(\theta,t)$ the curvature function of $X(\theta,t)$, the evolution problem of (1-1) can be reformulated equivalently into equations of the curvature k:

$$\begin{cases} k_t = k^2(k_{\theta\theta} + k - \bar{k}), & (\theta,t) \in I \times (0,T), \\ k(\theta,0) = k_0(\theta), & \theta \in I, \end{cases} \tag{1-2}$$

where k_0 is the curvature of X_0 and T is the maximal existence time of the flow. Here and after, I always denotes the circle S^1_m. In terms of the new coordinates, we have

$$\bar{k} = \frac{\int_I k(\theta,t)\, d\theta}{2m\pi}.$$

The first main theorem is:

Theorem 1. *If the initial curve is locally convex, closed and its curvature $k_0(\theta)$ satisfies*

$$\int_I (k_0-\bar{k}_0)^2\,d\theta \ge \int_I (k_{0\theta})^2\,d\theta, \tag{1-3}$$

and $k_0(\theta)$ is nonconstant in I, then the solution $k(\theta,t)$ to problem (1-2) blows up in some finite time and a singularity appears during the evolution of the flow (1-1).

We note that the condition (1-3) is not void since by the Poincaré inequality,

$$\int_I (k_0-\bar{k}_0)^2\,d\theta \le m^2\int_I (k_{0\theta})^2\,d\theta.$$

If the curvature of initial curve does not satisfy (1-3), how about the behavior of the flow? In fact, we find a large class of initial curves which do not satisfy the condition (1-3) and can evolve into m-fold circles under the flow. This is our second main theorem.

Theorem 2. *If the initial curve is locally convex, closed and has total curvature of $2m\pi$ and n-fold rotational symmetry with $m/n \le 1$, then the flow (1-1) exists globally and converges to an m-fold circle in the C^∞-metric as time goes to infinity.*

When the initial curve is simple closed and convex, it can be regarded as the case of $m=n=1$ in Theorem 2. In addition, its curvature cannot satisfy the condition (1-3) except by being a constant, in view of the Poincaré inequality.

The third theorem gives an isoperimetric condition such that the singularity appears.

Theorem 3. *Assume the initial curve X_0 is locally convex, closed and has total curvature of $2m\pi$. If X_0 satisfies*

$$L_0^2 < 4m\pi A_0, \tag{1-4}$$

where L_0 and A_0 denote its length and enclosed algebraic area respectively, then the solution $k(\theta,t)$ to problem (1-2) blows up in some finite time and a singularity appears during the evolution of the flow (1-1)

As a result, we can present a new proof of the following isoperimetric inequality for the rotationally symmetric and locally convex curves, which was proven in [Chou 2003] and [Süssmann 2011]:

Proposition 4. *For the rotationally symmetric and locally convex curves, with total curvature of $2m\pi$ and n-fold symmetry $(m/n<1)$, the length L and the enclosed algebraic area A satisfy*

$$L^2 \ge 4m\pi A. \tag{1-5}$$

We give some remarks on the above theorems and the nonlocal flow. As an interesting variant of the popular curve shortening flow [Gage and Hamilton 1986; Angenent 1991; Andrews 1998; Chou and Zhu 2001], the nonlocal curvature flow, arising in many application fields [Sapiro and Tannenbaum 1995; Capuzzo Dolcetta et al. 2002; Xu and Yang 2014], such as phase transitions, image processing, etc., has received much attention in recent years. Before the work of Ma and Zhu [2012], there was an original study by Gage [1986], where an area-preserving flow was investigated with its inner normal velocity given by

$$V=\Big(-\int_{X(\,\cdot\,,t)} k\,ds\Big/\int_{X(\,\cdot\,,t)} ds+k\Big)\boldsymbol{n}. \tag{1-6}$$

After that, there are a lot of papers on the nonlocal flow for simple convex curves, including [Jiang and Pan 2008; Lin and Tsai 2012]. In the higher dimensional case, people also consider nonlocal flows. For example, there are volume-preserving mean curvature flows; see [Huisken 1987; McCoy 2005; Cabezas-Rivas and Sinestrari 2010]. And also there are surface area-preserving mean curvature flows, see [McCoy 2003]. Recently, the study of nonlocal flow extends to the case of Riemannian manifolds; see [Xu et al. 2014].

In all of the papers mentioned above, the main concern is the global existence and convergence of the flow. For a study of the singularity, one can refer to [Escher and Ito 2005], or to [Wang and Kong 2014], where the area-preserving flow of immersed curves is studied and some geometric initial conditions are given to guarantee the occurrence of singularity. This urges us to carry the present work on the length-preserving flow of immersed curves.

One interesting aspect of this paper is that we have obtained the sufficient conditions for the flow (1-1) to yield the singularity. Moreover, the geometric condition (1-4) given in Theorem 3 can be interpreted as

$$\int_I (h_0-\bar h_0)^2\,d\theta>\int_I (h_{0\theta})^2\,d\theta, \tag{1-7}$$

where $h_0(\theta)$ is the support function of the initial curve X_0, defined by $h_0(\theta)=-\langle X_0(\theta),\boldsymbol{n}_0(\theta)\rangle$ with $\boldsymbol{n}_0$ being the inner normal of X_0, and $\bar h_0=\int_I h_0\,d\theta/(2m\pi)$. Indeed, we can deduce (1-7) from the following observations:

$$k_0=(h_0+h_{0\theta\theta})^{-1},\quad L_0=\int_I\frac{d\theta}{k_0}=\int_I h_0\,d\theta,$$

and

$$A_0=\frac12\int_{X_0} h_0\,ds=\frac12\int_I h_0(h_0+h_{0\theta\theta})\,d\theta,$$

where k_0 is the curvature function of X_0.

Another interesting aspect is that we have refined the results of [Wang and Wo 2014] in Theorem 2 and showed that the convergence result holds for all rotationally symmetric, locally convex immersed curves whether the enclosed algebraic area A_0 of the initial curve X_0 is negative or not. This differs with the flow (1-6), since a singularity must happen in the flow (1-6) if $A_0 < 0$, see [Escher and Ito 2005] for reference. One may also compare it with the different evolution of rotationally symmetric curves in the curve shortening flow, see [Au 2010].

We organize this paper in the following way. Some basic and useful lemmas are prepared in Section 2. Then we prove Theorems 1 and 2 in Section 3, and prove Theorem 3 in Section 4.

2. Lemmas

In this section, we present some lemmas for later use. The first lemma shows the flow exists as long as its curvature is bounded.

Lemma 2.1. *When the initial curve is immersed closed, locally convex and smooth, problem (1-1) has a unique smooth, locally convex solution in a time interval* $[0, T)$ *for some* $T > 0$, *which can be continued as long as the curvature of evolving curves is finite.*

Proof. The unique existence of the flow can be proven by applying the classical Leray–Schauder fixed point theorem to problem (1-2). See details in [Mao et al. 2013], where a general area-preserving flow is studied. One can also find the relative references in [McCoy 2003; 2005; Cabezas-Rivas and Sinestrari 2010], where the nonlocal flows in higher dimensions are discussed. The preserved convexity will be proved in the next lemma. □

By the maximum principle, we can show that the local convexity of the initial curve is preserved by the flow (1-1).

Lemma 2.2. *If the initial curve* X_0 *is locally convex, then* $X(\cdot\,, t)$ *is locally convex as long as the flow exists.*

Proof. By the continuity, $\min_{\theta\in I} k(\theta, t)$ remains positive on a small time interval. Assume that the time span of the flow is T. Suppose to the contrary that the conclusion is not true. Then there must be a first time, say $t_1 < T$, such that

$$\min_{\theta\in I} k(\theta, t_1) = 0. \tag{2-1}$$

We will deduce a contradiction. Consider the quantity

$$\Phi(\theta, t) = \frac{1}{k(\theta, t)} - \frac{L(t)}{2m\pi} - \frac{1}{2m\pi}\int_0^t \int_0^{2m\pi} k(\theta, \tau)\, d\theta\, d\tau,$$

with $(\theta, t) \in I \times [0, t_1)$. By (1-2), we have

$$\Phi_t(\theta, t) = -k_{\theta\theta} - k \leq k^2(\theta, t)\Phi_{\theta\theta}(\theta, t).$$

Hence by the maximum principle,

$$\frac{1}{k(\theta, t)} \leq \max_{\theta \in I}\left(\frac{1}{k_0(\theta)}\right) + \frac{L(t)-L(0)}{2m\pi} + \frac{1}{2m\pi}\int_0^t \int_0^{2m\pi} k(\theta, \tau)\, d\theta\, d\tau$$

for all $(\theta, t) \in I \times [0, t_1)$, where we note that $L(t) = L(0)$ for all time t and

$$\sup_{(\theta,t)\in I\times[0,t_1)} k(\theta, t) \leq C_1(t_1) < \infty$$

for some constant $C_1(t_1)$. Therefore,

$$\inf_{\theta \in I} k(\theta, t) \geq C_2(t_1) > 0 \quad \text{for all } t \in [0, t_1)$$

for some constant $C_2(t_1)$. This is a contraction with (2-1)! The proof is done. □

The following lemma is the gradient estimate.

Lemma 2.3. *Along the flow* (1-1), *we have*

$$\int_I (k_\theta)^2\, d\theta \leq \int_I k^2\, d\theta + C$$

for some constant C independent of time.

Proof. From (1-2), we have

$$\frac{1}{2}\frac{d}{dt}\int_I [(k_\theta)^2 - k^2 + 2\bar{k}k]\, d\theta = -\int_I k^2(k_{\theta\theta} + k - \bar{k})^2 + \frac{d\bar{k}}{dt}\int_I k\, d\theta \leq \frac{d\bar{k}}{dt}\int_I k\, d\theta.$$

Hence,

$$\frac{d}{dt}\int_I (k_\theta)^2\, d\theta \leq \frac{d}{dt}\int_I (k^2 - 2\bar{k}k)\, d\theta + 2\frac{d\bar{k}}{dt}\int_I k\, d\theta,$$

and the integration yields

$$\begin{aligned}
\int_I (k_\theta)^2 &\leq \int_I (k^2 - 2\bar{k}k)\, d\theta + \frac{1}{2m\pi}\int_0^t \frac{d}{d\tau}\left(\int_I k\, d\theta\right)^2 d\tau + C_1 \\
&= \int_I (k^2 - 2\bar{k}k)\, d\theta + \frac{1}{2m\pi}\left(\int_I k\, d\theta\right)^2 + C_2 \\
&= \int_I k^2\, d\theta - \bar{k}\int_I k\, d\theta + C_2 \\
&\leq \int_I k^2\, d\theta + C_2,
\end{aligned}$$

where C_1, C_2 only depend on the initial data. The proof is done. □

By the obtained gradient estimate, if the curvature k blows up, we can show that the blow-up set for k must contain at least some open interval.

Denote
$$k_{\max}(t) = \max_{\theta\in I} k(\theta, t), \quad t \in [0, T).$$

Lemma 2.4. *Assume that $k_{\max}(t) = k(\theta_t, t)$ for some $\theta_t \in [0, 2m\pi]$. Then for any small $\varepsilon > 0$, there exists a number $\delta > 0$, depending only on ε, such that*
$$(1-\varepsilon)k_{\max}(t) \le k(\theta, t) + \sqrt{2m\pi|C|}$$
for all $\theta \in (\theta_t - \delta^2, \theta_t + \delta^2)$ and all $t \in (0, T)$, where C is the constant in Lemma 2.3.

Proof. An easy integration combined with the Hölder inequality shows that
$$k_{\max}(t) = k(\theta, t) + \int_\theta^{\theta_t} k_\theta(\theta, t)\, d\theta \le k(\theta, t) + |\theta_t - \theta|^{1/2}\left(\int_\theta^{\theta_t} k_\theta^2\, d\theta\right)^{1/2}.$$
Then from Lemma 2.3 we have
$$\begin{aligned}
k_{\max}(t) &\le k(\theta, t) + |\theta_t - \theta|^{1/2}\left(\int_I k^2\, d\theta + |C|\right)^{1/2} \\
&\le k(\theta, t) + |\theta_t - \theta|^{1/2}(2m\pi k_{\max}^2(t) + |C|)^{1/2} \\
&\le k(\theta, t) + |\theta_t - \theta|^{1/2}\sqrt{2m\pi}k_{\max}(t) + |\theta_t - \theta|^{1/2}|C|^{1/2} \\
&\le k(\theta, t) + |\theta_t - \theta|^{1/2}\sqrt{2m\pi}k_{\max}(t) + \sqrt{2m\pi|C|}.
\end{aligned}$$
Take δ such that $|\theta_t - \theta|^{1/2} \le \delta := \varepsilon/\sqrt{2m\pi}$ and the lemma is proved. □

We need the following lemma, proven in [Wang and Wo 2014], to conclude the convergence of the flow after we obtain the a priori estimate for the curvature.

Lemma 2.5. *If there is a constant C independent of time such that*
$$\max_{\theta\in I} k(\theta, t) \le C, \quad t \in [0, T),$$
with T being the maximal existence time, then the flow (1-1) must exist for all time and converge smoothly to an m-fold circle as time goes to infinity.

3. Proofs of Theorems 1 and 2

First, we deduce a sufficient condition for the occurrence of the singularity at some finite time. The following two lemmas are useful in the proof.

Lemma 3.1. *If the flow (1-1) exists for all time, then there exists a sequence $\{t_j\}_{j=1}^\infty \to \infty$ such that*
$$\int_I k(\theta, t_j)\, d\theta \le C$$
for some constant C independent of time.

Proof. We have

$$\frac{dA}{dt} = -\int_I (k-\bar{k})\,ds = \frac{L(t)}{2m\pi}\int_I k\,d\theta - 2m\pi \geq 0.$$

Since an isoperimetric inequality of Rado (see [Osserman 1978]) says that

$$L(t)^2 \geq 4\pi A(t)$$

and $L(t) = L_0$, we know that $A(t)$ is uniformly bounded from above. Notice that $A(t)$ is increasing in time. We have $\int_0^\infty (dA/d\tau)\,d\tau < \infty$. Thus for any small $\varepsilon > 0$, there exists a sequence $\{t_j\}_{j=1}^\infty \to \infty$, such that

$$\frac{dA}{dt}(t_j) < \varepsilon,$$

that is,

$$\int_I k(\theta, t_j)\,d\theta < \frac{2m\pi}{L_0}(\varepsilon + 2m\pi).$$

Then we can draw the conclusion by fixing an $\varepsilon > 0$. $\square$

Denote

$$E(t) = \int_I (k_\theta)^2\,d\theta - \int_I k^2\,d\theta + \frac{1}{2m\pi}\left(\int_I k\,d\theta\right)^2.$$

That is,

$$E(t) = \int_I (k_\theta)^2\,d\theta - \int_I (k-\bar{k})^2\,d\theta.$$

Lemma 3.2. *For the energy $E(t)$ defined as above, we have*

$$\frac{dE(t)}{dt} \leq 0.$$

Proof. From the equation (1-2), we have

$$\int_I \frac{(k_t)^2}{k^2}\,d\theta = \int_I (k_{\theta\theta} + k - \bar{k})k_t\,d\theta = -\frac{1}{2}\frac{d}{dt}\int_I [(k_\theta)^2 - k^2]\,d\theta - \bar{k}\int_I k_t\,d\theta,$$

where

$$\bar{k}\int_I k_t\,d\theta = \frac{d}{dt}\int_0^t \bar{k}(\tau)\int_I k_\tau\,d\theta d\tau = \frac{1}{4m\pi}\frac{d}{dt}\int_0^t \frac{d}{d\tau}\left(\int_I k\,d\theta\right)^2 d\tau.$$

Thus,

$$-\frac{1}{2}\frac{dE(t)}{dt} = \int_I \frac{(k_t)^2}{k^2}\,d\theta \geq 0,$$

and the proof is done. $\square$

Proof of Theorem 1. Using the equation (1-2) and integrating by parts yield

$$\frac{d}{dt}\int_I \ln k\,d\theta = \int_I k(k_{\theta\theta}+k-\bar{k})\,d\theta = -E(t).$$

From Lemma 3.2, we have

$$\frac{d}{dt}\int_I \ln k\,d\theta \geq -E(0) = -\int_I (k_{0\theta})^2\,d\theta + \int_I (k_0-\bar{k}_0)^2\,d\theta.$$

First, we consider the case of $E(0)<0$. If we suppose to the contrary the flow exists for all time, then $\lim_{t\to\infty}\int_I \ln k\,d\theta=\infty$. This implies that for any $t>0$, we can find a $\theta_t\in I$, such that $\lim_{t\to\infty} k(\theta_t,t)=\infty$. Then by Lemma 2.4, we have $\lim_{t\to\infty}\int_I k(\theta,t)\,d\theta=\infty$, which is a contradiction to Lemma 3.1. Thus the flow must exist for some finite time.

If $E(0)=0$, we claim that $k_{0\theta\theta}+k_0-\bar{k}_0\neq 0$ must hold at some point of I and hence in some interval of I by the continuity. Indeed, if $k_{0\theta\theta}+k_0-\bar{k}_0=0$ holds everywhere in I, we set $w=k_0-\bar{k}_0$ and w satisfies

$$w_{\theta\theta}+w=0 \quad \text{in } I,$$

which implies that w is a 2π-periodic function and so is k_0. Hence, $E(0)$=0 tells us that k_0 is a constant function in view of the Poincaré inequality, a contradiction with the assumption! Thus we have shown that $k_{0\theta\theta}+k_0-\bar{k}_0\neq 0$ must hold in some interval of I. Then by recalling the proof of Lemma 3.2, we have

$$\frac{dE(t)}{dt} = -2\int_I \frac{(k_t)^2}{k^2}\,d\theta < 0,$$

which implies that $E(t)<0$ for $t>0$. At last, we can still show the conclusion holds via a similar method to the one above. The proof is finished. □

One may naturally ask what happens if the condition (1-3) does not hold for the initial curve. A large class of rotationally symmetric curves belong to this case. In fact, the Poincaré inequality tells us the following lemma:

Lemma 3.3. *If a curve is locally convex, closed and has total curvature of* $2m\pi$ *and* n*-fold rotational symmetry with* $m/n\leq 1$, *then its curvature* $k(\theta)$ *satisfies*

$$\int_I (k-\bar{k})^2\,d\theta \leq \left(\frac{m}{n}\right)^2\int_I (k_\theta)^2\,d\theta.$$

Proof. By the Poincaré inequality, we have

$$\int_0^{2m\pi/n} (k-\bar{k})^2\,d\theta \leq \left(\frac{m}{n}\right)^2\int_0^{2m\pi/n} (k_\theta)^2\,d\theta,$$

and then the conclusion follows. □

Proof of Theorem 2. By equation (1-2) and integration by parts, we have

$$\frac{d}{dt}\int_I \ln k\,d\theta = \int_I k(k_{\theta\theta}+k-\bar{k})\,d\theta = -\int_I (k_\theta)^2\,d\theta + \int_I (k-\bar{k})^2\,d\theta.$$

From Lemma 3.3, we have $d\big(\int_I \ln k\,d\theta\big)/dt \le 0$. Thus there is a constant C_1 independent of time, such that $\int_I \ln k(\theta,t)\,d\theta \le C_1$ for all $t \in [0,T)$. This implies that there is a constant C_2 independent of time, such that

$$\max_{\theta\in I} k(\theta,t) \le C_2 \tag{3-1}$$

for all $t \in [0,T)$. Indeed, for $m/n < 1$, using Lemma 3.3 and the fact that $E(t) \le E(0)$, we can deduce an estimate of k_θ, which implies (3-1) holds. As a result of the a priori estimate (3-1), we can show the flow's global existence and its smooth convergence to an m-fold circle as time goes to infinity by using Lemma 2.5. □

4. Proof of Theorem 3

To prove Theorem 3, we need to show the following lemma holds, which states a subconvergence of the global flow without any a priori estimate on the curvature like that in Lemma 2.5.

Lemma 4.1. *If the flow (1-1) starts from a locally convex closed curve and exists for all time, then it subconverges to an m-fold circle in C^2 sense, that is, there exists a time sequence $\{t_j\}_{j=1}^\infty \to \infty$ such that $k(\theta,t_j)$ converges to a positive constant function in the L^∞ norm.*

Proof. Notice that a careful choice of $\{t_j\}_{j=1}^\infty$ in Lemma 3.1 can guarantee that $(dA/dt)(t_j) \to 0$ as $j \to \infty$, that is,

$$\frac{L_0}{2m\pi}\int_I k(\theta,t_j)\,d\theta \to 2m\pi, \qquad j\to\infty. \tag{4-1}$$

We claim that along the sequence $\{t_j\}_{j=1}^\infty$ we have

$$\max_{\theta\in I} k(\theta,t_j) \le C_1 \tag{4-2}$$

for some constant C_1 independent of time. Suppose $\limsup_{j\to\infty}\max_{\theta\in I} k(\theta,t_j) = \infty$. Then we can find a subsequence, still denoted by $\{t_j\}_{j=1}^\infty$, and a sequence $\{\theta_j\}_{j=1}^\infty \subset I$, such that $t_j \to \infty$ and $k(\theta_j,t_j) \to \infty$. By Lemma 2.4, $\int_I k(\theta,t_j)\,d\theta \to \infty$, contradicting Lemma 3.1! Thus we have (4-2). Furthermore, by Lemma 2.3,

$$\int_I (k_\theta)^2(\theta,t_j)\,d\theta \le C_2, \tag{4-3}$$

for some constant C_2 independent of time. Combining (4-2) with (4-3) we obtain

$$\|k(\cdot,t_j)\|_{W^{1,2}(I)} \le C_3$$

for some constant C_3 independent of time. The compactness yields a subsequence of $\{k(\theta, t_j)\}_{j=1}^{\infty}$, still denoted by $\{k(\theta, t_j)\}_{j=1}^{\infty}$, which converges to a continuous function $k_\infty(\theta)$ in the L^∞ norm as $j \to \infty$. Taking the limit in (4-1) along the time sequence $\{t_j\}_{j=1}^{\infty}$, we have

$$\frac{L_0}{2m\pi}\int_I k_\infty(\theta)\, d\theta = 2m\pi. \tag{4-4}$$

By Fatou's lemma,

$$\int_I \frac{d\theta}{k_\infty(\theta)} \le \int_I \frac{d\theta}{k(\theta, t_j)} = L_0. \tag{4-5}$$

Thus, substituting (4-5) into (4-4) yields

$$\int_I k_\infty\, d\theta \int_I \frac{d\theta}{k_\infty(\theta)} \le (2m\pi)^2.$$

We notice that

$$(2m\pi)^2 = \left(\int_I 1\, d\theta\right)^2 \le \int_I k_\infty\, d\theta \int_I \frac{d\theta}{k_\infty}.$$

Thus k_∞ must be a constant function, i.e., the sequence $\{k(\theta, t_j)\}_{j=1}^{\infty}$ converges to a constant function in L^∞ norm as $j \to \infty$. □

Proof of Theorem 3. Assume the initial curve satisfies

$$L_0^2 < 4m\pi A_0.$$

Since $dL(t)/dt \equiv 0$ and $dA(t)/dt \ge 0$, we have $L_0 = L(\infty) := \lim_{t\to\infty} L(t)$ and $A_0 \le A(\infty) := \lim_{t\to\infty} A(t)$. Thus,

$$L^2(\infty) < 4m\pi A(\infty). \tag{4-6}$$

Suppose to the contrary that the flow exists for all time. Then by Lemma 4.1 the flow converges to an m-fold circle along some time sequence $\{t_j\}_{j=1}^{\infty} \to \infty$, implying

$$L^2(\infty) = 4m\pi A(\infty).$$

This contradicts (4-6)! Thus, the singularity must happen at some finite time during the evolution of the flow. □

As a result of Theorem 3, we can give a proof for Proposition 4.

Proof of Proposition 4. On one hand, by Theorem 2 the flow (1-1) starting from such rotationally symmetric curves must converge to m-fold circles at $t \to \infty$. However, on the other hand, if (1-5) does not hold, then by Theorem 3 there is a finite-time singularity during the evolution. This contradiction shows (1-5) holds. □

Acknowledgments

We thank the referees for their careful reading and useful comments which have helped improve the presentation of this paper. Our work is partially supported by the Fundamental Research Funds for the Central Universities 2242015R30012, the NSF of China 11101078, the NSF of China 11171064 and the Natural Science Foundation of Jiangsu Province BK20161412.

References

[Abresch and Langer 1986] U. Abresch and J. Langer, "The normalized curve shortening flow and homothetic solutions", *J. Differential Geom.* **23**:2 (1986), 175–196. MR Zbl

[Andrews 1998] B. Andrews, "Evolving convex curves", *Calc. Var. Partial Differential Equations* **7**:4 (1998), 315–371. MR Zbl

[Angenent 1991] S. Angenent, "On the formation of singularities in the curve shortening flow", *J. Differential Geom.* **33**:3 (1991), 601–633. MR Zbl

[Au 2010] T. K.-K. Au, "On the saddle point property of Abresch–Langer curves under the curve shortening flow", *Comm. Anal. Geom.* **18**:1 (2010), 1–21. MR Zbl

[Cabezas-Rivas and Sinestrari 2010] E. Cabezas-Rivas and C. Sinestrari, "Volume-preserving flow by powers of the mth mean curvature", *Calc. Var. Partial Differential Equations* **38**:3-4 (2010), 441–469. MR Zbl

[Capuzzo Dolcetta et al. 2002] I. Capuzzo Dolcetta, S. Finzi Vita, and R. March, "Area-preserving curve-shortening flows: from phase separation to image processing", *Interfaces Free Bound.* **4**:4 (2002), 325–343. MR Zbl

[Chou 2003] K.-S. Chou, "A blow-up criterion for the curve shortening flow by surface diffusion", *Hokkaido Math. J.* **32**:1 (2003), 1–19. MR Zbl

[Chou and Zhu 2001] K.-S. Chou and X.-P. Zhu, *The curve shortening problem*, Chapman & Hall, Boca Raton, FL, 2001. MR Zbl

[Escher and Ito 2005] J. Escher and K. Ito, "Some dynamic properties of volume preserving curvature driven flows", *Math. Ann.* **333**:1 (2005), 213–230. MR Zbl

[Gage 1986] M. Gage, "On an area-preserving evolution equation for plane curves", pp. 51–62 in *Nonlinear problems in geometry* (Mobile, AL, 1985), edited by D. M. DeTurck, Contemp. Math. **51**, American Mathematical Society, Providence, RI, 1986. MR Zbl

[Gage and Hamilton 1986] M. Gage and R. S. Hamilton, "The heat equation shrinking convex plane curves", *J. Differential Geom.* **23**:1 (1986), 69–96. MR Zbl

[Huisken 1987] G. Huisken, "The volume preserving mean curvature flow", *J. Reine Angew. Math.* **382** (1987), 35–48. MR Zbl

[Jiang and Pan 2008] L. Jiang and S. Pan, "On a non-local curve evolution problem in the plane", *Comm. Anal. Geom.* **16**:1 (2008), 1–26. MR Zbl

[Lin and Tsai 2012] Y.-C. Lin and D.-H. Tsai, "Application of Andrews and Green–Osher inequalities to nonlocal flow of convex plane curves", *J. Evol. Equ.* **12**:4 (2012), 833–854. MR Zbl

[Ma and Zhu 2012] L. Ma and A. Zhu, "On a length preserving curve flow", *Monatsh. Math.* **165**:1 (2012), 57–78. MR Zbl

[Mao et al. 2013] Y. Mao, S. Pan, and Y. Wang, "An area-preserving flow for closed convex plane curves", *Internat. J. Math.* **24**:4 (2013), art. id. 1350029, 31 pp. MR Zbl

[McCoy 2003] J. McCoy, "The surface area preserving mean curvature flow", *Asian J. Math.* **7**:1 (2003), 7–30. MR Zbl

[McCoy 2005] J. A. McCoy, "Mixed volume preserving curvature flows", *Calc. Var. Partial Differential Equations* **24**:2 (2005), 131–154. MR Zbl

[Osserman 1978] R. Osserman, "The isoperimetric inequality", *Bull. Amer. Math. Soc.* **84**:6 (1978), 1182–1238. MR Zbl

[Sapiro and Tannenbaum 1995] G. Sapiro and A. Tannenbaum, "Area and length preserving geometric invariant scale-spaces", *IEEE Trans. Pattern Anal. Machine Intel.* **17**:1 (1995), 67–72.

[Süssmann 2011] B. Süssmann, "Isoperimetric inequalities for special classes of curves", *Differential Geom. Appl.* **29**:1 (2011), 1–6. MR Zbl

[Wang and Kong 2014] X.-L. Wang and L.-H. Kong, "Area-preserving evolution of nonsimple symmetric plane curves", *J. Evol. Equ.* **14**:2 (2014), 387–401. MR Zbl

[Wang and Wo 2014] X. Wang and W. Wo, "Length-preserving evolution of non-simple symmetric plane curves", *Math. Methods Appl. Sci.* **37**:6 (2014), 808–816. MR Zbl

[Xu and Yang 2014] G. Xu and X. Yang, "Construction of several second- and fourth-order geometric partial differential equations for space curves", *Comput. Aided Geom. Design* **31**:2 (2014), 63–80. MR Zbl

[Xu et al. 2014] H. Xu, Y. Leng, and E. Zhao, "Volume-preserving mean curvature flow of hypersurfaces in space forms", *Internat. J. Math.* **25**:3 (2014), art. id. 1450021, 19 pp. MR Zbl

Received January 21, 2016. Revised February 8, 2017.

XIAO-LIU WANG
SCHOOL OF MATHEMATICS
SOUTHEAST UNIVERSITY
JIULONGHU CAMPUS, JIANGNING
211189 NANJING
CHINA
xlwang@seu.edu.cn

HUI-LING LI
SCHOOL OF MATHEMATICS
SOUTHEAST UNIVERSITY
JIULONGHU CAMPUS, JIANGNING
211189 NANJING
CHINA
huilingli@seu.edu.cn

XIAO-LI CHAO
SCHOOL OF MATHEMATICS
SOUTHEAST UNIVERSITY
JIULONGHU CAMPUS, JIANGNING
211189 NANJING
CHINA
xlchao@seu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 290, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.290.481

CALABI–YAU PROPERTY UNDER MONOIDAL MORITA–TAKEUCHI EQUIVALENCE

XINGTING WANG, XIAOLAN YU AND YINHUO ZHANG

Let H and L be two Hopf algebras such that their comodule categories are monoidally equivalent. We prove that if H is a twisted Calabi–Yau (CY) Hopf algebra, then L is a twisted CY algebra when it is homologically smooth. In particular, if H is a Noetherian twisted CY Hopf algebra and L has finite global dimension, then L is a twisted CY algebra.

Introduction

In noncommutative projective algebraic geometry, what is now called an Artin–Schelter (AS) regular algebra $A = \bigoplus_{i\geq 0} A_i$ of dimension n was introduced in [Artin and Schelter 1987] as a homological analogue of a polynomial algebra with n variables. The connected graded noncommutative algebra A is considered as the homogeneous coordinate ring of some noncommutative projective space $\mathbb{P}^n$.

In lecture notes, Manin [1988] constructed the quantum general linear group $\mathcal{O}_A(\mathrm{GL})$ that universally coacts on an AS regular algebra A. Similarly, we can define the quantum special linear group of A, denoted by $\mathcal{O}_A(\mathrm{SL})$, by requiring the homological codeterminant of the Hopf coaction to be trivial; see [Walton and Wang 2016, Section 2.1] for details. As pointed out in that work, it is conjectured that these universal quantum groups should possess the same homological properties of A, among which the Calabi–Yau (CY) property is the most interesting, since A is always twisted CY according to [Reyes et al. 2014, Lemma 2.1] (see Section 1.2 for the definition of a twisted CY algebra). Moreover, many classical quantized coordinate rings can be realized as universal quantum groups associated to AS regular algebras via the above construction [Chirvasitu et al. 2016; Walton and Wang 2016], whose CY property and rigid dualizing complexes have been discussed in [Brown and Zhang 2008; Goodearl and Zhang 2007].

Now let us look at a nontrivial example, which is the motivation for our paper. Let $\Bbbk$ be a field. AS regular algebras of global dimension 2 (not necessarily Noetherian)

MSC2010: 16E65, 16W30, 16W35.
Keywords: Morita–Takeuchi equivalence, Calabi–Yau algebra, cogroupoid.

were classified by Zhang [1998]. They are the algebras (assume they are generated in degree one)

$$A(E) = \Bbbk\langle x_1, x_2, \ldots, x_n\rangle / \Big(\sum_{1\leqslant i,j\leqslant n} e_{ij} x_i x_j \Big)$$

for $E = (e_{ij}) \in \mathrm{GL}_n(\Bbbk)$ with $n \geqslant 2$. It is shown in [Walton and Wang 2016, Corollary 2.17] that $\mathcal{O}_{A(E)}(\mathrm{SL}) \cong \mathcal{B}(E^{-1})$ as Hopf algebras, where $\mathcal{B}(E^{-1})$ was defined by Dubois-Violette and Launer [1990] as the quantum automorphism group of the nondegenerate bilinear form associated to E^{-1}. In particular, when

$$E = \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E^{-1} = E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \quad \text{for some } q \in \Bbbk^{\times},$$

we have $A(E) = A_q = k\langle x_1, x_2\rangle/(x_2x_1 + qx_1x_2)$ is the quantum plane and $\mathcal{O}_{A_q}(\mathrm{SL}) = \mathcal{B}(E_q) = \mathcal{O}_q(\mathrm{SL}_2)$ is the quantized coordinate ring of $\mathrm{SL}_2(\Bbbk)$.

Two Hopf algebras are called monoidally Morita–Takeuchi equivalent, if their comodule categories are monoidally equivalent. Bichon [2003, Theorem 1.1] obtained that $\mathcal{B}(E)$ (for any $E \in \mathrm{GL}_n(\Bbbk)$ with $n \geq 2$) and $\mathcal{O}_q(\mathrm{SL}_2)$ are monoidally Morita–Takeuchi equivalent when $q^2 + \mathrm{tr}(E^t E^{-1})q + 1 = 0$. By applying this monoidal equivalence, Bichon obtained a free Yetter–Drinfeld module resolution (Definition 2.2.4) of the trivial Yetter–Drinfeld module $\Bbbk$ over $\mathcal{B}(E)$. This turns out to be the key ingredient to prove the CY property of $\mathcal{B}(E)$; see that work or [Walton and Wang 2016]. Note that the quantized coordinate ring $\mathcal{O}_q(\mathrm{SL}_2)$ is well known to be twisted CY [Brown and Zhang 2008, Section 6.5 and 6.6]. Thus it is natural to ask the following question:

Question 1. *Let H and L be two Hopf algebras that are monoidally Morita–Takeuchi equivalent. Suppose H is twisted CY. Is L always twisted CY?*

The monoidal equivalence between the comodule categories of various universal quantum groups have been widely observed [Bichon 2003; 2014; Mrozinski 2014; Chirvasitu et al. 2016] by using the language of cogroupoids. In recent papers, Raedschelders and Van den Bergh [2015; 2017] proved that, for a Koszul AS regular algebra A, the monoidal structure of the comodule category of $\mathcal{O}_A(\mathrm{GL})$ only depends on the global dimension of A and not on A itself [Raedschelders and Van den Bergh 2017, Theorem 1.2.6]. We expect a positive answer to Question 1, which should play an important role in investigating the CY property of these universal quantum groups associated to AS regular algebras.

The following is our main result, showing that in order to answer Question 1, it suffices to prove that the homologically smooth condition is a monoidally Morita–Takeuchi invariant.

Theorem 2 (Theorem 2.4.5). *Let H and L be two monoidally Morita–Takeuchi equivalent Hopf algebras. If H is twisted CY of dimension d and L is homologically smooth, then L is twisted CY of dimension d as well.*

Note that for Hopf algebras, there are several equivalent descriptions of the homological smoothness stated in Proposition A.2. Now Question 1 is reduced to the following question:

Question 3. *Let H and L be two monoidally Morita–Takeuchi equivalent Hopf algebras. Suppose H is homologically smooth. Is L always homologically smooth?*

Though we can not fully answer Question 3, it is true in certain circumstances. We obtain the following result:

Theorem 4 (Theorem 2.4.7). *Let H be a twisted CY Hopf algebra of dimension d, and L a Hopf algebra monoidally Morita–Takeuchi equivalent to H. If one of the following conditions holds, then L is also twisted CY of dimension d.*

(i) *H admits a finitely generated relative projective Yetter–Drinfeld module resolution for the trivial Yetter–Drinfeld module $\Bbbk$ and L has finite global dimension.*

(ii) *H admits a bounded finitely generated relative projective Yetter–Drinfeld module resolution for the trivial Yetter–Drinfeld module $\Bbbk$.*

(iii) *H is Noetherian and L has finite global dimension.*

(iv) *L is Noetherian and has finite global dimension.*

Relative projective Yetter–Drinfeld modules and resolutions will be explained in Section 2.2. The trivial module $\Bbbk$ over $\mathcal{O}_q(\mathrm{SL}_2)$ admits a finitely generated free Yetter–Drinfeld resolution of length 3 [Bichon 2013, Theorem 5.1]. Every free Yetter–Drinfeld module resolution is a relative projective Yetter–Drinfeld module resolution. According to our result above, this immediately implies that $\mathcal{B}(E)$ is twisted CY since $\mathcal{B}(E)$ and $\mathcal{O}_q(\mathrm{SL}_2)$ are monoidally Morita–Takeuchi equivalent as mentioned above.

Twisted CY algebras, of course, have finite global dimensions. Theorem 4 leads to the last question about whether the global dimension is a monoidally Morita–Takeuchi invariant. A similar question was asked by Bichon [2016] concerning the Hochschild dimension, and the two questions are essentially the same by Proposition A.1.

Question 5. *Let H and L be two monoidally Morita–Takeuchi equivalent Hopf algebras. Does* $\mathrm{gldim}(H) = \mathrm{gldim}(L)$, *or at least*, $\mathrm{gldim}(H) < \infty$ *if and only if* $\mathrm{gldim}(L) < \infty$?

If the answer is positive, then the finite global dimension assumptions in conditions (i), (iii), and (iv) of Theorem 4 can be dropped. This will partially answer Question 1 under the assumption that one of the Hopf algebras is Noetherian. As a consequence of Theorem 4, we provide a partial answer to Question 5 answer under the assumption that both Hopf algebras are twisted CY.

Theorem 6 (Corollary 2.4.8). *Let H and L be two monoidally Morita–Takeuchi equivalent Hopf algebras. If both H and L are twisted CY, then* $\mathrm{gldim}(H) = \mathrm{gldim}(L)$.

Monoidal Morita–Takeuchi equivalence can be described by the language of cogroupoids. If H and L are two Hopf algebras that are monoidally Morita–Takeuchi equivalent, then there exists a connected cogroupoid with 2 objects X, Y such that $H = \mathcal{C}(X, X)$ and $L = \mathcal{C}(Y, Y)$. In this case, $\mathcal{C}(X, Y)$ is just the H-L-bigalois object (see Section 1.1 for details). Throughout, we will use the language of cogroupoids to discuss Hopf algebras whose comodule categories are monoidally equivalent. We generalize many definitions and results in [Brown and Zhang 2008] to the level of cogroupoids (see Section 2.4). Especially for Hopf–Galois objects, we define the left (resp. right) winding automorphisms of $\mathcal{C}(X, Y)$ using the homological integrals of $\mathcal{C}(X, X)$ (resp. $\mathcal{C}(Y, Y)$). We also generalize the famous Radford S^4 formula for finite dimensional Hopf algebras to Hopf–Galois object $\mathcal{C}(X, Y)$ by assuming both $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ are AS-Gorenstein Hopf algebras.

Theorem 7 (Theorem 2.4.9 and Remark 2.4.10). *Let $\mathcal{C}$ be a connected cogroupoid. If X and Y are two objects such that $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ are both AS-Gorenstein Hopf algebras. Then for the Hopf–Galois object $\mathcal{C}(X, Y)$ we have*

$$(S_{Y,X} \circ S_{X,Y})^2 = \gamma \circ \phi \circ \xi^{-1}, \tag{1}$$

where ξ and ϕ are respectively the left and right winding automorphisms given by the left integrals of $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$, and γ is an inner automorphism.

At last, we provide two examples in Section 3. One is the connected cogroupoid associated to $\mathcal{B}(E)$ and the other is the connected cogroupoid associated to a generic datum of finite Cartan type $(\mathcal{D}, \lambda)$.

1. Preliminaries

We work over a fixed field $\Bbbk$. Unless stated otherwise all algebras and vector spaces are over $\Bbbk$. The unadorned tensor $\otimes$ means $\otimes_{\Bbbk}$ and Hom means $\mathrm{Hom}_{\Bbbk}$.

Given an algebra A, we write A^{op} for the *opposite algebra* of A and A^e for the *enveloping algebra* $A \otimes A^{\mathrm{op}}$. The category of left (resp. right) A-modules is denoted by $\mathrm{Mod}\, A$ (resp. $\mathrm{Mod}\, A^{\mathrm{op}}$). An A-bimodule can be identified with an A^e-module, that is, an object in $\mathrm{Mod}\, A^e$.

For an A-bimodule M and two algebra automorphisms μ and ν, we let ${}^{\mu}M^{\nu}$ denote the A-bimodule such that ${}^{\mu}M^{\nu} \cong M$ as vector spaces, and the bimodule structure is given by

$$a \cdot m \cdot b = \mu(a) m \nu(b),$$

for all $a, b \in A$ and $m \in M$. If one of the automorphisms is the identity, we will omit it. It is well known that $A^{\mu} \cong A$ as A-bimodules if and only if μ is an inner automorphism of A.

For a Hopf algebra H, as usual, we use the symbols Δ, ε and S respectively for its comultiplication, counit, and antipode. We use Sweedler's (sumless) notation for the comultiplication and coaction of H. The category of right H-comodules is denoted by $\mathcal{M}^H$. We write ${}_{\varepsilon}\Bbbk$ (resp. $\Bbbk_{\varepsilon}$) for the left (resp. right) trivial module defined by the counit ε of H.

1.1. *Cogroupoid.* We first recall the definition of a cogroupoid.

Definition 1.1.1. A *cocategory* $\mathcal{C}$ consists of:

- A set of objects $\mathrm{ob}(\mathcal{C})$,
- For any $X, Y \in \mathrm{ob}(\mathcal{C})$, an algebra $\mathcal{C}(X, Y)$,
- For any $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, algebra homomorphisms

$$\Delta^Z_{XY} : \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \text{ and } \varepsilon_X : \mathcal{C}(X, X) \to \Bbbk$$

such that for any $X, Y, Z, T \in \mathrm{ob}(\mathcal{C})$, the following diagrams commute:

$$\begin{tikzcd}
\mathcal{C}(X, Y) \arrow[r, "\Delta^Z_{X,Y}"] \arrow[d, "\Delta^T_{X,Y}"'] & \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \arrow[d, "\Delta^T_{X,Z} \otimes 1"] \\
\mathcal{C}(X, T) \otimes \mathcal{C}(T, Y) \arrow[r, "1 \otimes \Delta^Z_{T,Y}"] & \mathcal{C}(X, T) \otimes \mathcal{C}(T, Z) \otimes \mathcal{C}(Z, Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(X, Y) \arrow[d, "\Delta^Y_{X,Y}"] \arrow[dr, equal] & \\
\mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Y) \arrow[r, "1 \otimes \varepsilon_Y"] & \mathcal{C}(X, Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(X, Y) \arrow[d, "\Delta^X_{X,Y}"] \arrow[dr, equal] & \\
\mathcal{C}(X, X) \otimes \mathcal{C}(X, Y) \arrow[r, "\varepsilon_X \otimes 1"] & \mathcal{C}(X, Y).
\end{tikzcd}$$

Thus a cocategory with one object is just a bialgebra.

A cocategory $\mathcal{C}$ is said to be *connected* if $\mathcal{C}(X, Y)$ is a nonzero algebra for any $X, Y \in \mathrm{ob}(\mathcal{C})$.

Definition 1.1.2. A *cogroupoid* $\mathcal{C}$ consists of a cocategory $\mathcal{C}$ together with, for any $X, Y \in \mathrm{ob}(\mathcal{C})$, linear maps

$$S_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{C}(Y, X)$$

such that for any $X, Y \in \mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
\mathcal{C}(X,X) \arrow[r, "\varepsilon_X"] \arrow[d, "\Delta^Y_{X,X}"'] & \Bbbk \arrow[r, "u"] & \mathcal{C}(X,Y) \\
\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,X) \arrow[rr, "1\otimes S_{Y,X}"] & & \mathcal{C}(X,Y)\otimes\mathcal{C}(X,Y) \arrow[u, "\cdot"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(X,X) \arrow[r, "\varepsilon_X"] \arrow[d, "\Delta^Y_{X,X}"'] & \Bbbk \arrow[r, "u"] & \mathcal{C}(Y,X) \\
\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,X) \arrow[rr, "S_{X,Y}\otimes 1"] & & \mathcal{C}(Y,X)\otimes\mathcal{C}(Y,X). \arrow[u, "\cdot"]
\end{tikzcd}$$

From the definition, we can see $\mathcal{C}(X, X)$ is a Hopf algebra for each object $X \in \mathcal{C}$.

We use Sweedler's notation for cogroupoids. Let $\mathcal{C}$ be a cogroupoid. For any $a^{X,Y} \in \mathcal{C}(X, Y)$, we write

$$\Delta^Z_{X,Y}(a^{X,Y}) = a_1^{X,Z} \otimes a_2^{Z,Y}.$$

The following lemma describes properties of the "antipodes":

Lemma 1.1.3 [Bichon 2014, Proposition 2.13]. *Let $\mathcal{C}$ be a cogroupoid and let $X, Y \in \mathrm{ob}(\mathcal{C})$.*

(i) $S_{Y,X} : \mathcal{C}(Y, X) \to \mathcal{C}(X, Y)^{\mathrm{op}}$ *is an algebra homomorphism.*

(ii) *For any $Z \in \mathrm{ob}(\mathcal{C})$ and $a^{Y,X} \in \mathcal{C}(Y, X)$,*

$$\Delta^Z_{X,Y}(S_{Y,X}(a^{Y,X})) = S_{Z,X}(a_2^{Z,X}) \otimes S_{Y,Z}(a_1^{Y,Z}).$$

For other basic properties of cogroupoids, we refer to the same work.

Bichon [2014] reformulated Schauenburg's [1996] results by cogroupoids. This theorem shows that discussing two Hopf algebras with monoidally equivalent comodule categories is equivalent discussing connected cogroupoids. In what follows, unless otherwise stated, we assume that the cogroupoids mentioned are *connected*.

Theorem 1.1.4 [Bichon 2014, Theorem 2.10, 2.12]. *If $\mathcal{C}$ is a connected cogroupoid, then for any $X, Y \in \mathcal{C}$, we have equivalences of monoidal categories that are inverse to each other*

$$\mathcal{M}^{\mathcal{C}(X,X)} \cong^{\otimes} \mathcal{M}^{\mathcal{C}(Y,Y)} \qquad \mathcal{M}^{\mathcal{C}(Y,Y)} \cong^{\otimes} \mathcal{M}^{\mathcal{C}(X,X)}$$

$$V \mapsto V\square_{\mathcal{C}(X,X)}\mathcal{C}(X,Y) \qquad V \mapsto V\square_{\mathcal{C}(Y,Y)}\mathcal{C}(Y,X).$$

Conversely, if H and L are Hopf algebras such that $\mathcal{M}^H \cong^{\otimes} \mathcal{M}^L$, then there exists a connected cogroupoid with 2 objects X, Y such that $H = \mathcal{C}(X, X)$ and $L = \mathcal{C}(Y, Y)$.

This monoidal equivalence can be extended to categories of Yetter–Drinfeld modules.

Lemma 1.1.5 [Bichon 2014, Proposition 6.2]. *Let $\mathcal{C}$ be a cogroupoid, $X, Y \in \mathrm{ob}(\mathcal{C})$ and V a right $\mathcal{C}(X, X)$-module.*

(i) *$V \otimes \mathcal{C}(X, Y)$ has a right $\mathcal{C}(Y, Y)$-module structure defined by*

$$(v \otimes a^{X,Y}) \leftarrow b^{Y,Y} = v \cdot b_2^{X,X} \otimes S_{Y,X}(b_1^{Y,X}) a^{X,Y} b_3^{X,Y}.$$

Together with the right $\mathcal{C}(Y, Y)$-comodule structure defined by $1 \otimes \Delta_{X,Y}^{Y}$, $V \otimes \mathcal{C}(X, Y)$ is a Yetter–Drinfeld module over $\mathcal{C}(Y, Y)$.

(ii) *If moreover V is a Yetter–Drinfeld module, then $V \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$ is a Yetter–Drinfeld submodule of $V \otimes \mathcal{C}(X, Y)$.*

Theorem 1.1.6 [Bichon 2014, Theorem 6.3]. *Let $\mathcal{C}$ be a connected cogroupoid. Then for any $X, Y \in \mathrm{ob}(\mathcal{C})$, the functor*

$$\mathcal{YD}_{\mathcal{C}(X,X)}^{\mathcal{C}(X,X)} \to \mathcal{YD}_{\mathcal{C}(Y,Y)}^{\mathcal{C}(Y,Y)} \quad V \mapsto V \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$$

is a monoidal equivalence.

1.2. *Calabi–Yau algebras.* In this subsection, we recall the definition of (twisted) Calabi–Yau algebras.

Definition 1.2.1. An algebra A is a *twisted Calabi–Yau algebra of dimension d* if

(i) A is *homologically smooth*, that is, A has a bounded resolution by finitely generated projective A^e-modules;

(ii) There is an automorphism μ of A such that

$$\mathrm{Ext}_{A^e}^{i}(A, A^e) \cong \begin{cases} 0, & i \neq d, \\ A^{\mu}, & i = d, \end{cases} \tag{2}$$

as A^e-modules.

If such an automorphism μ exists, it is unique up to an inner automorphism and is called the *Nakayama automorphism* of A. In the definition, the dimension d is usually called the Calabi–Yau dimension of A. A *Calabi–Yau algebra* in the sense of Ginzburg [2007] is a twisted Calabi–Yau algebra whose Nakayama automorphism is an inner automorphism. In what follows, Calabi–Yau is abbreviated to CY.

Twisted CY algebras include CY algebras as a subclass. They are the natural algebraic analogues of Bieri and Eckmann's [1973] duality groups. The twisted CY property of noncommutative algebras has been studied under other names for many years, even before the definition of a CY algebra. Rigid dualizing complexes of noncommutative algebras were studied in [Van den Bergh 1997]. The twisted CY property was called "rigid Gorenstein" in [Brown and Zhang 2008] and was called "skew Calabi–Yau" in a recent paper [Reyes et al. 2014].

2. Calabi–Yau property

2.1. ***Artin–Schelter Gorenstein Hopf algebras.*** Let H be a Hopf algebra. We denote the Hochschild dimension of H by $\mathrm{Hdim}(H)$. In the Appendix, it is shown that the left global dimension and the right global dimension of H are always equal. We denote the global dimension of H by $\mathrm{gldim}(H)$. The left adjoint functor L : $\mathrm{Mod}\, H^e \to \mathrm{Mod}\, H$ is defined by the algebra homomorphism $(\mathrm{id} \otimes S) \circ \Delta : H \to H^e$. Similarly, the algebra homomorphism $\tau \circ (S \otimes \mathrm{id}) \circ \Delta : H \to (H^e)^{\mathrm{op}} = H^e$ defines the right adjoint functor $R : \mathrm{Mod}(H^e)^{\mathrm{op}} \to \mathrm{Mod}\, H^{\mathrm{op}}$, where $\tau : H^{\mathrm{op}} \otimes H \to H \otimes H^{\mathrm{op}}$ is the flip map. Let M be an H-bimodule. Then $L(M)$ is a left H-module defined by the action

$$x \rightarrow m = x_1 m S(x_2) \quad \text{for any } x \in H,$$

while $R(M)$ is a right H-module defined by the action

$$m \leftarrow x = S(x_1) m x_2 \quad \text{for any } x \in H.$$

The algebra H^e is a left and right H^e-module with left action

$$(a \otimes b) \rightarrow (x \otimes y) = ax \otimes yb, \tag{3}$$

and right action

$$(x \otimes y) \leftarrow (a \otimes b) = xa \otimes by. \tag{4}$$

for any $x \otimes y$ and $a \otimes b \in H^e$. So $L(H^e)$ and $R(H^e)$ are H-H^e and H^e-H-bimodules, where the corresponding H-module structures are given by

$$a \rightarrow (x \otimes y) = a_1 x \otimes y S(a_2) \quad \text{and} \quad (x \otimes y) \leftarrow a = x a_2 \otimes S(a_1) y$$

for any $a \in H$ and $x \otimes y \in H^e$, respectively.

Let ${}_*H \otimes H$ be the free left H-module, where the structure is given by the left multiplication of the first factor H. Similarly, let $H_* \otimes H$ be the free right H-module defined by the right multiplication of the first factor H. Moreover, we give ${}_*H \otimes H$ a right H^e-module structure such that

$$(x \otimes y) \leftarrow (a \otimes b) = x a_1 \otimes b y S^2(a_2) \tag{5}$$

and $H_* \otimes H$ a left H^e-module structure via

$$(a \otimes b) \rightarrow (x \otimes y) = a_2 x \otimes S^2(a_1) y b \tag{6}$$

for any $x \otimes y \in {}_*H \otimes H$ or $H_* \otimes H$ and $a \otimes b \in H^e$.

Lemma 2.1.1. *Retain the above notation. Then we have*:

(i) $L(H^e) \cong {}_*H \otimes H$ *as* H-H^e-*bimodules.*

(ii) $R(H^e) \cong H_* \otimes H$ *as* H^e-H-*bimodules.*

Proof. It is straightforward to check the corresponding isomorphisms of bimodules are given by the following four homomorphisms:

$$L(H^e) \to {}_*H \otimes H, \quad x \otimes y \mapsto x_1 \otimes yS^2(x_2)$$

with inverse

$$_*H \otimes H \to L(H^e), \quad x \otimes y \mapsto x_1 \otimes yS(x_2),$$

and

$$R(H^e) \to H_* \otimes H, \quad x \otimes y \mapsto x_2 \otimes S^2(x_1)y$$

with inverse

$$H_* \otimes H \to R(H^e), \quad x \otimes y \mapsto x_2 \otimes S(x_1)y. \qquad \square$$

Lemma 2.1.2. *Let H be a Hopf algebra and B an algebra.*

(i) *Let M be an H^e-B-bimodule. Then* $\mathrm{Ext}^i_{H^e}(H, M) \cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, L(M))$ *as right B-modules for all $i \geqslant 0$.*

(ii) *Let M be an B-H^e-bimodule. Then* $\mathrm{Ext}^i_{H^e}(H, M) \cong \mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, R(M))$ *as left B-modules for all $i \geqslant 0$.*

Proof. We only prove (i); the proof of (ii) is quite similar. With Lemma 2.4 in [Brown and Zhang 2008], we only need to prove that for an H^e-B-bimodule N, there is an H^e-B-bimodule monomorphism $0 \to N \to I$, such that I is injective as an H^e-module. The H^e-B-bimodule N can be viewed as an $H^e \otimes B^{\mathrm{op}}$-module. It can be embedded into an injective $H^e \otimes B^{\mathrm{op}}$-module I. We have

$$\begin{aligned} \mathrm{Hom}_{H^e}(-, I) &\cong \mathrm{Hom}_{H^e}(-, \mathrm{Hom}_{H^e \otimes B^{\mathrm{op}}}((H^e \otimes B^{\mathrm{op}})_{H^e}, I)) \\ &\cong \mathrm{Hom}_{H^e \otimes B^{\mathrm{op}}}((H^e \otimes B^{\mathrm{op}})_{H^e} \otimes -, I). \end{aligned}$$

$H^e \otimes B^{\mathrm{op}}$ is clearly free as an H^e-module. Therefore, the functor $\mathrm{Hom}_{H^e}(-, I)$ is exact. That is, I is injective as an H^e-module. This completes the proof. $\square$

It is well known that there is an equivalence of categories between the category of left H^e-modules and the category of right H^e-modules for $(H^e)^{\mathrm{op}} = H^e$. As a consequence, $\mathrm{Ext}^i_{H^e}(H, H^e)$ can be computed both by using the left and the right H^e-module structures on H^e defined in (3) and (4).

Proposition 2.1.3. *Let H be a Hopf algebra such that it is homologically smooth. We have*

$$\mathrm{Ext}^i_{H^e}(H, H^e) \cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) \otimes H \cong \mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \otimes H$$

as H^e-modules for all $i \geq 0$, where the H^e-module structures on $\mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) \otimes H$ *and on* $\mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \otimes H$ *are induced by (5) and (6), respectively.*

Proof. We prove the isomorphism $\mathrm{Ext}^i_{H^e}(H, H^e) \cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) \otimes H$. The proof of the isomorphism $\mathrm{Ext}^i_{H^e}(H, H^e) \cong \mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \otimes H$ is quite similar.

Since H is homologically smooth, the trivial module ${}_\varepsilon\Bbbk$ admits a bounded projective resolution $\boldsymbol{P}_\bullet \to {}_\varepsilon\Bbbk \to 0$, with each term finitely generated (Proposition A.2). Now we have the following H^e-module isomorphisms:

$$\begin{aligned}\mathrm{Ext}^i_{H^e}(H, H^e) &\cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, L(H^e)) \ \cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, {}_*H \otimes H)\\ &\cong \mathrm{H}^i(\boldsymbol{P}_\bullet, {}_*H \otimes H) \quad \cong \mathrm{H}^i(\boldsymbol{P}_\bullet, H) \otimes H\\ &\cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) \otimes H.\end{aligned}$$

The first and the second isomorphism follows from Lemma 2.1.2 and 2.1.1, respectively. The fourth isomorphism holds since $\boldsymbol{P}_\bullet \to {}_\varepsilon\Bbbk \to 0$ is a bounded projective resolution with each term finitely generated. □

Now we recall the definition of an Artin–Schelter (AS) Gorenstein algebra.

Definition 2.1.4 (cf. [Brown and Zhang 2008, Definition 1.2]). Let H be a Hopf algebra.

(i) The Hopf algebra H is said to be *left AS-Gorenstein* if
  - (a) $\operatorname{injdim}{}_H H = d < \infty$,
  - (b) $\mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) = 0$ for $i \neq d$ and $\mathrm{Ext}^d_H({}_\varepsilon\Bbbk, H) = \Bbbk$.

(ii) The Hopf algebra H is said to be *right AS-Gorenstein* if
  - (c) $\operatorname{injdim} H_H = d < \infty$,
  - (d) $\mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = 0$ for $i \neq d$ and $\mathrm{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = \Bbbk$.

(iii) If H is both left and right AS-Gorenstein (relative to the same augmentation map ε), then H is called *AS-Gorenstein*.

(iv) If, in addition, the global dimension of H is finite, then H is called *AS-regular*.

Remark 2.1.5. In above definitions, we do not require the Hopf algebra H to be Noetherian. For AS-regularity, the right global dimension always equals the left global dimension by Proposition A.1. Moreover, when H is AS-Gorenstein and homologically smooth, the right injective dimension always equals the left injective dimension, which are both given by the integer d such that $\mathrm{Ext}^d_{H^e}(H, H^e) \neq 0$ by Proposition 2.1.3.

Homological integrals for an AS-Gorenstein Hopf algebra introduced in [Lu et al. 2007] are a generalization of integrals for finite dimensional Hopf algebras [Sweedler 1969]. The concept was further extended to any AS-Gorenstein algebra in [Brown and Zhang 2008].

Let A be a left AS-Gorenstein algebra of injective dimension d with augmentation $\varepsilon : A \to \Bbbk$. One sees that $\mathrm{Ext}^d_A({}_\varepsilon\Bbbk, A)$ is a one-dimensional right A-module. Any nonzero element in $\mathrm{Ext}^d_A({}_\varepsilon\Bbbk, A)$ is called a *left homological integral* of A. Usually,

$\mathrm{Ext}^d_A({}_\varepsilon\Bbbk, A)$ is denoted by $\int^l_A$. Similarly, if A is a right AS-Gorenstein algebra of injective dimension d, any nonzero element in $\mathrm{Ext}^d_{A^{\mathrm{op}}}(\Bbbk_\varepsilon, A)$ is called a *right homological integral*. And $\mathrm{Ext}^d_{A^{\mathrm{op}}}(\Bbbk_\varepsilon, A)$ is denoted by $\int^r_A$. Abusing the language slightly, $\int^l_A$ (resp. $\int^r_A$) is also called the left (resp. right) homological integral.

A Noetherian Hopf algebra H with bijective antipode is AS-regular in the sense of [Brown and Zhang 2008, Definition 1.2] if and only if H is twisted CY [Reyes et al. 2014, Lemma 1.3]. If H is not necessarily Noetherian, we have the following result:

Proposition 2.1.6. *Let H be a Hopf algebra with bijective antipode such that it is homologically smooth. Then the following are equivalent*:

(i) *H is a twisted CY algebra of dimension d.*

(ii) *There is an integer d such that*

$$\mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) = 0 \quad \textit{for } i \neq d \quad \textit{and} \quad \dim \mathrm{Ext}^d_H({}_\varepsilon\Bbbk, H) = 1.$$

(iii) *There is an integer d such that*

$$\mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = 0 \quad \textit{for } i \neq d \quad \textit{and} \quad \dim \mathrm{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = 1.$$

(iv) $\mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H)$ *and* $\mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H)$ *are finite dimensional for $i \geqslant 0$ and there is an integer d such that* $\dim \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) = \dim \mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = 0$ *for $i > d$, and* $\dim \mathrm{Ext}^d_H({}_\varepsilon\Bbbk, H) \neq 0$ *or* $\dim \mathrm{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \neq 0$.

In these cases, we have $\mathrm{gldim}(H) = \mathrm{injdim}\, H_H = \mathrm{injdim}\, {}_H H = d$.

Proof. (i) ⇒(ii), (iii) This proof can be found for example in [Yu et al. 2016, Lemma 2.15].

(ii)⇒ (i) By Proposition 2.1.3, $\mathrm{Ext}^i_{H^e}(H, H^e) \cong \mathrm{Ext}^i_H({}_\varepsilon\Bbbk, H) \otimes H$ for all $i \geq 1$ as H^e-modules. Since $\mathrm{Ext}^d_H({}_\varepsilon\Bbbk, H)$ is a one-dimensional right H-module, we simply write it as $\Bbbk_\xi$, for some algebra homomorphism $\xi : H \to \Bbbk$. Therefore,

$$\mathrm{Ext}^i_{H^e}(H, H^e) = 0 \quad \text{for } i \neq d \quad \text{and} \quad \mathrm{Ext}^d_{H^e}(H, H^e) \cong \Bbbk_\xi \otimes H \overset{(a)}{\cong} H^\mu,$$

where μ is defined by $\mu(h) = \xi(h_1)S^2(h_2)$ for any $h \in H$. The isomorphism (a) holds because the H^e-module structure on $\Bbbk_\xi \otimes H$ is induced by the equation (5) according to Proposition 2.1.3. Moreover, it is easy to check that μ is an algebra automorphism of H with inverse given by $\mu^{-1}(h) = \xi(S(h_1))S^{-2}(h_2)$ for any $h \in H$.

(iii)⇒(i) The proof is similar to that of (ii)⇒ (i).

(ii), (iii)⇒(iv) This is obvious.

(iv)⇒(ii), (iii) The proof of [Brown and Zhang 2008, Lemma 3.2] works generally for this case. Suppose $\dim \mathrm{Ext}^d_H({}_\varepsilon\Bbbk, H) \neq 0$; the case for $\dim \mathrm{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \neq 0$

is similar. Since H is homologically smooth, by Proposition A.2 and [Brown and Goodearl 1997, Lemma 1.11], we can apply Ischebeck's spectral sequence

$$\operatorname{Ext}^p_{H^{\mathrm{op}}}(\operatorname{Ext}^{-q}_H({}_\varepsilon\Bbbk, H), H) \Longrightarrow \operatorname{Tor}^H_{-p-q}(H, {}_\varepsilon\Bbbk)$$

to obtain $\dim \operatorname{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = 0$ for $i \neq d$. From the proof of [Brown and Goodearl 1997, Lemma 1.11], $\dim \operatorname{Ext}^d_H(M, H) = \dim M \cdot \dim \operatorname{Ext}^d_H({}_\varepsilon\Bbbk, H)$ for any finite dimensional left H-module M. Thus by the finite dimensional assumption,

$$\dim \operatorname{Ext}^d_H(\operatorname{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H), H) = \dim \operatorname{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \cdot \dim \operatorname{Ext}^d_H({}_\varepsilon\Bbbk, H).$$

Again by the Ischebeck's spectral sequence, $\operatorname{Ext}^d_H(\operatorname{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H), H) \cong \Bbbk$. Hence,

$$\dim \operatorname{Ext}^d_H({}_\varepsilon\Bbbk, H) = \dim \operatorname{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) = 1.$$

Now (ii) and (iii) are proved.

Finally, we can apply the same proof of [Berger and Taillefer 2007, Proposition 2.2] to show that for a twisted CY Hopf algebra H of dimension d, we have $\operatorname{Hdim}(H) = d$. Hence $\operatorname{gldim}(H) = d$ by Proposition A.1. The equality of the injective dimension of H is easy to see since it is always bounded by $\operatorname{gldim}(H) = d$ and we have $\dim \operatorname{Ext}^d_H({}_\varepsilon\Bbbk, H) \neq 0$ or $\dim \operatorname{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_\varepsilon, H) \neq 0$. □

Corollary 2.1.7. *Let H be a Hopf algebra with bijective antipode. Then the following are equivalent*:

- *H is twisted CY.*
- *H is left AS-Gorenstein and the left trivial module ${}_\varepsilon\Bbbk$ admits a bounded projective resolution with each term finitely generated.*
- *H is right AS-Gorenstein and the right trivial module $\Bbbk_\varepsilon$ admits a bounded projective resolution with each term finitely generated.*

Proof. It follows from Proposition A.2 and Proposition 2.1.6. □

2.2. ***Yetter–Drinfeld modules.*** In this subsection, we recall some definitions related to Yetter–Drinfeld modules.

Definition 2.2.1. Let H be a Hopf algebra. A *(right-right) Yetter–Drinfeld module* V over H is simultaneously a right H-module and a right H-comodule satisfying the compatibility condition

$$\delta(v \cdot h) = v_{(0)} \cdot h_2 \otimes S(h_1) v_{(1)} h_3 \quad \text{for any } v \in V,\ h \in H.$$

We denote by $\mathcal{YD}^H_H$ the category of Yetter–Drinfeld modules over H with morphisms given by H-linear and H-collinear maps. Endowed with the usual tensor product of modules and comodules, $\mathcal{YD}^H_H$ is a monoidal category, with unit the trivial Yetter–Drinfeld module $\Bbbk$.

We can always construct a Yetter–Drinfeld module from a right comodule.

Lemma-Definition 2.2.2 [Bichon 2013, Proposion 3.1, Definition 3.2]. Let H be a Hopf algebra and V a right H-comodule. Endow $V \otimes H$ with the right H-module structure defined by multiplication on the right. Then the linear map

$$V \otimes H \to V \otimes H \otimes H, \quad v \otimes h \mapsto v_{(0)} \otimes h_2 \otimes S(h_1)v_{(1)}h_3$$

endows $V \otimes H$ with a right H-comodule structure, and with a right-right Yetter–Drinfeld module structure. We denote by $V \boxtimes H$ the resulting Yetter–Drinfeld module.

A Yetter–Drinfeld module over H is said to be *free* if it is isomorphic to $V \boxtimes H$ for some right H-comodule V.

A free Yetter–Drinfeld module is obviously free as a right H-module. We call a free Yetter–Drinfeld module $V \boxtimes H$ *finitely generated* if V is finite dimensional.

Bichon [2016] introduced the notion of relative projective Yetter–Drinfeld module, corresponding to the notion of relative projective Hopf bimodule considered in [Shnider and Sternberg 1993] via the monoidal equivalence between Yetter–Drinfeld modules and Hopf bimodules.

Lemma-Definition 2.2.3 [Bichon 2016, Definition 4.1, Proposition 4.2]. Let P be a Yetter–Drinfeld module over a Hopf algebra H. The following are equivalent:

(1) The functor $\mathrm{Hom}_{\mathcal{YD}_H^H}(P, -)$ transforms exact sequences of Yetter–Drinfeld modules that splits as sequences of comodules to exact sequences of vector spaces.

(2) Any epimorphism of Yetter–Drinfeld modules $f : M \to P$ that admits a comodule section admits a Yetter–Drinfeld module section.

(3) P is a direct summand of a free Yetter–Drinfeld module.

A Yetter–Drinfeld module is said to be *relative projective* if it satisfies one of the above equivalent conditions.

It is clear that a relative projective Yetter–Drinfeld module is a projective module. We call a relative projective Yetter–Drinfeld module *finitely generated* if it is a direct summand of a finitely generated free Yetter–Drinfeld module.

Definition 2.2.4. Let H be a Hopf algebra and let $M \in \mathcal{YD}_H^H$. A *free (resp. relative projective) Yetter–Drinfeld module resolution* of M consists of a complex of free (resp. relative projective) Yetter–Drinfeld modules

$$\boldsymbol{P_\bullet} : \cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to 0$$

for which there exists a Yetter–Drinfeld module morphism $\epsilon : P_0 \to M$ such that

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

is an exact sequence in $\mathcal{YD}_H^H$.

If each P_i, $i \geqslant 0$, is a finitely generated free (resp. relative projective) Yetter–Drinfeld module, we call this complex $\boldsymbol{P}_{\bullet}$ a finitely generated free (resp. relative projective) Yetter–Drinfeld module resolution.

Of course each free Yetter–Drinfeld module resolution is a free resolution and each relative projective Yetter–Drinfeld module resolution is a projective resolution.

Lemma 2.2.5. *Let $\mathcal{C}$ be a cogroupoid and $X, Y \in \mathrm{ob}(\mathcal{C})$. The equivalence functor $-\square_{\mathcal{C}(X,X)}\mathcal{C}(X, Y)$ sends any relative projective Yetter–Drinfeld module resolution $\boldsymbol{P}_{\bullet}$ of the trivial Yetter–Drinfeld module $\Bbbk$ over $\mathcal{C}(X, X)$ to a relative projective Yetter–Drinfeld module resolution $\boldsymbol{P}_{\bullet}\square_{\mathcal{C}(X,X)}\mathcal{C}(X, Y)$ of the trivial Yetter–Drinfeld module $\Bbbk$ over $\mathcal{C}(Y, Y)$. In particular, if $\boldsymbol{P}_{\bullet}$ is finitely generated (resp. bounded), then $\boldsymbol{P}_{\bullet}\square_{\mathcal{C}(X,X)}\mathcal{C}(X, Y)$ is also finite generated (resp. bounded).*

Proof. Following from Lemma-Definition 2.2.3 and Section 4 in [Bichon 2013], we see that the functor $-\square_{\mathcal{C}(X,X)}\mathcal{C}(X, Y)$ is exact and sends a relative projective Yetter–Drinfeld module over $\mathcal{C}(X, X)$ to a relative projective Yetter–Drinfeld module over $\mathcal{C}(Y, Y)$. So $\boldsymbol{P}_{\bullet}\square_{\mathcal{C}(X,X)}\mathcal{C}(X, Y)$ is a relative projective Yetter–Drinfeld module resolution.

Lemma-Definition 2.2.3 and [Bichon 2014, Proposition 1.16] guarantee that if $\boldsymbol{P}_{\bullet}$ is finitely generated, then $\boldsymbol{P}_{\bullet}\square_{\mathcal{C}(X,X)}\mathcal{C}(X, Y)$ is also finite generated. The argument for boundedness is clear. $\square$

2.3. ***Homological properties of cogroupoids.*** From now on we assume that the Hopf algebras mentioned have *bijective* antipodes. We also assume that any cogroupoid $\mathcal{C}$ mentioned satisfies that $S_{X,Y}$ is *bijective* for any $X, Y \in \mathrm{ob}(\mathcal{C})$. This assumption is to make sure that $S_{Y,X} \circ S_{X,Y}$ is an algebra automorphism of $\mathcal{C}(X, Y)$. Actually, if $\mathcal{C}$ is a connected cogroupoid such that for some object X, $\mathcal{C}(X, X)$ is a Hopf algebra with bijective antipode, then $S_{X,Y}$ is bijective for any objects X, Y (see Remark 2.6 in [Yu 2016]).

Let $\mathcal{C}$ be a cogroupoid and $X, Y \in \mathrm{ob}(\mathcal{C})$. Both the morphisms $\Delta^{Y}_{X,X} : \mathcal{C}(X, X) \to \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, X)$ and $S_{Y,X} : \mathcal{C}(Y, X) \to \mathcal{C}(X, Y)^{\mathrm{op}}$ are algebra homomorphisms (Lemma 1.1.3), so the composition of the morphisms

$$\text{(7)}\quad \mathcal{C}(X, X) \xrightarrow{\Delta^{Y}_{X,X}} \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, X) \xrightarrow{\mathrm{id} \otimes S_{Y,X}} \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, X)^{\mathrm{op}} (= \mathcal{C}(X, Y)^{e})$$

is an algebra homomorphism. This induces a functor

$$\mathcal{L}_X : \mathrm{Mod}\,\mathcal{C}(X, Y)^{e} \to \mathrm{Mod}\,\mathcal{C}(X, X).$$

The functor $\mathcal{L}_X$ is just the functor $\mathcal{L}$ defined in [Yu 2016]. Let M be a $\mathcal{C}(X, Y)$-bimodule. The left $\mathcal{C}(X, X)$-module structure of $\mathcal{L}_X(M)$ is given by

$$x \to m = x_1^{X,Y} m S_{Y,X}(x_2^{Y,X}) \quad \text{for any } m \in M,\ x \in \mathcal{C}(X, X).$$

From the cogroupoid $\mathcal{C}$, we define a coopposite cogroupoid $\mathcal{C}'$ as follows:

- $\mathrm{ob}(\mathcal{C}') = \mathrm{ob}(\mathcal{C})$.
- For any objects Y, X, the algebra $\mathcal{C}'(Y, X)$ is the algebra $\mathcal{C}(X, Y)$.
- For any objects Y, X and Z, the algebra homomorphism $\Delta'^{Z}_{YX} : \mathcal{C}'(Y, X) \to \mathcal{C}'(Y, Z) \otimes \mathcal{C}'(Z, X)$ is the algebra homomorphism $\tau \circ \Delta^{Z}_{XY} : \mathcal{C}(X, Y) \to \mathcal{C}(Z, Y) \otimes \mathcal{C}(X, Z)$ in $\mathcal{C}$, where $\tau : \mathcal{C}(X, Z) \otimes \mathcal{C}(Z, Y) \to \mathcal{C}(Z, Y) \otimes \mathcal{C}(X, Z)$ is the flip map.
- For any object X, $\varepsilon'_X : \mathcal{C}'(X, X) \to \Bbbk$ is the same as $\varepsilon_X : \mathcal{C}(X, X) \to \Bbbk$ in $\mathcal{C}$.
- For any objects Y, X, the morphism $S'_{Y,X} : \mathcal{C}'(Y, X) \to \mathcal{C}'(X, Y)$ is the morphism $S^{-1}_{Y,X} : \mathcal{C}(X, Y) \to \mathcal{C}(Y, X)$.

It is easy to check that this indeed defines a cogroupoid.

For any objects $X, Y \in \mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}')$, the algebras $\mathcal{C}(X, Y)$ and $\mathcal{C}(Y, Y)$ in $\mathcal{C}$ are just the algebras $\mathcal{C}'(Y, X)$ and $\mathcal{C}'(Y, Y)$ in $\mathcal{C}'$. So we have a functor

$$\mathcal{L}'_Y : \mathrm{Mod}\, \mathcal{C}(X, Y)^e \to \mathrm{Mod}\, \mathcal{C}(Y, Y).$$

If M is a $\mathcal{C}(X, Y)$-bimodule, the left $\mathcal{C}(Y, Y)$-module structure of $\mathcal{L}'_Y(M)$ is given by

$$y \to m = y_2^{X,Y} m S^{-1}_{X,Y}(y_1^{Y,X}) \quad \text{for any } m \in M \quad \text{and} \quad y \in \mathcal{C}(Y, Y).$$

As usual, we view $\mathcal{C}(X, Y)^e$ as a left and a right $\mathcal{C}(X, Y)^e$-module respectively in the following ways:

$$(a \otimes b) \to (x \otimes y) = ax \otimes yb, \tag{8}$$

and

$$(x \otimes y) \leftarrow (a \otimes b) = xa \otimes by, \tag{9}$$

for any $x \otimes y$ and $a \otimes b \in \mathcal{C}(X, Y)^e$. Then we have the modules $\mathcal{L}_X(\mathcal{C}(X, Y)^e)$ and $\mathcal{L}'_Y(\mathcal{C}(X, Y)^e)$. They are all free modules.

Let ${}_*\mathcal{C}(X, X) \otimes \mathcal{C}(X, Y)$ be the left $\mathcal{C}(X, X)$-module defined by the left multiplication of the factor $\mathcal{C}(X, X)$, and ${}_*\mathcal{C}(Y, Y) \otimes \mathcal{C}(X, Y)$ be the left $\mathcal{C}(Y, Y)$-module defined by the left multiplication of the factor $\mathcal{C}(Y, Y)$. Then we have the following:

Lemma 2.3.1. (i) *$\mathcal{L}_X(\mathcal{C}(X, Y)^e) \cong {}_*\mathcal{C}(X, X) \otimes \mathcal{C}(X, Y)$ as left $\mathcal{C}(X, X)$-modules. The isomorphism is given by*

$$\mathcal{L}_X(\mathcal{C}(X, Y)^e) \to {}_*\mathcal{C}(X, X) \otimes \mathcal{C}(X, Y), \quad x \otimes y \mapsto x_1^{X,X} \otimes y S_{Y,X}(S_{X,Y}(x_2^{X,Y})).$$

(ii) *$\mathcal{L}'_Y(\mathcal{C}(X, Y)^e) \cong {}_*\mathcal{C}(Y, Y) \otimes \mathcal{C}(X, Y)$ as left $\mathcal{C}(Y, Y)$-modules. The isomorphism is given by*

$$\mathcal{L}'_Y(\mathcal{C}(X, Y)^e) \to {}_*\mathcal{C}(Y, Y) \otimes \mathcal{C}(X, Y), \quad x \otimes y \mapsto x_2^{Y,Y} \otimes y S^{-1}_{X,Y}(S^{-1}_{Y,X}(x_1^{X,Y})).$$

Proof. (i) is Lemma 2.1 in [Yu 2016]. (ii) can be obtained by applying (i) to the coopposite cogroupoid $\mathcal{C}'$. □

Lemma 2.3.2. *Let $\mathcal{C}$ be a cogroupoid, $X, Y \in \mathrm{ob}(\mathcal{C})$ and B another algebra. Let M be a $\mathcal{C}(X, Y)^e$-B-bimodule.*

(i) $\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X,Y), M) \cong \mathrm{Ext}^i_{\mathcal{C}(X,X)}({}_\varepsilon\Bbbk, \mathcal{L}_X(M))$ *as right B-bimodules for all $i \geqslant 0$.*

(ii) $\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X,Y), M) \cong \mathrm{Ext}^i_{\mathcal{C}(Y,Y)}({}_\varepsilon\Bbbk, \mathcal{L}'_Y(M))$ *as right B-bimodules for all $i \geqslant 0$.*

Proof. By applying Lemma 2.2 in [Yu 2016] to the cogroupoid $\mathcal{C}$ and its coopposite cogroupoid $\mathcal{C}'$, we obtain vector space isomorphisms

$$\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X, Y), M) \cong \mathrm{Ext}^i_{\mathcal{C}(X,X)}({}_\varepsilon\Bbbk, \mathcal{L}_X(M))$$

and

$$\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X, Y), M) \cong \mathrm{Ext}^i_{\mathcal{C}(Y,Y)}({}_\varepsilon\Bbbk, \mathcal{L}'_Y(M))$$

for all $i \geqslant 0$. By a quite similar discussion to that in the proof of Lemma 2.1.2, we can see that the isomorphisms above are B-linear. □

2.4. *Main results.* In order to state our main results we need to define winding automorphisms of cogroupoids.

Let $\mathcal{C}$ be a cogroupoid and $X, Y \in \mathrm{ob}(\mathcal{C})$. Let $\xi : \mathcal{C}(X, X) \to \Bbbk$ be an algebra homomorphism. The *left winding automorphism* $[\xi]^l_{X,Y}$ of $\mathcal{C}(X, Y)$ associated to ξ is defined to be

$$[\xi]^l_{X,Y}(a^{X,Y}) = \xi(a_1^{X,X})a_2^{X,Y} \quad \text{for any } a \in \mathcal{C}(X, Y).$$

Let $\eta : \mathcal{C}(Y, Y) \to \Bbbk$ be an algebra homomorphism. Similarly, the *right winding automorphism* of $\mathcal{C}(X, Y)$ associated to η is defined to be

$$[\eta]^r_{X,Y}(a^{X,Y}) = a_1^{X,Y}\eta(a_2^{Y,Y}) \quad \text{for any } a \in \mathcal{C}(X, Y).$$

Lemma 2.4.1. *Let $\mathcal{C}$ be a cogroupoid and $X, Y \in \mathrm{ob}(\mathcal{C})$, let $\xi : \mathcal{C}(X, X) \to \Bbbk$, and $\eta : \mathcal{C}(Y, Y) \to \Bbbk$ be algebra homomorphisms. Then*

(i) $([\xi]^l_{X,Y})^{-1} = [\xi S_{X,X}]^l_{X,Y}$.

(ii) $\xi S^2_{X,X} = \xi$, *so* $[\xi]^l_{X,Y} = [\xi S^2_{X,X}]^l_{X,Y}$.

(iii) $[\xi]^l_{X,Y} \circ S_{Y,X} \circ S_{X,Y} = S_{Y,X} \circ S_{X,Y} \circ [\xi]^l_{X,Y}$.

(i') $([\eta]^r_{X,Y})^{-1} = [\eta S_{Y,Y}]^r_{X,Y}$.

(ii') $\eta S^2_{Y,Y} = \eta$, *so* $[\eta]^r_{X,Y} = [\eta S^2_{Y,Y}]^r_{X,Y}$.

(iii') $[\eta]^r_{X,Y} \circ S_{Y,X} \circ S_{X,Y} = S_{Y,X} \circ S_{X,Y} \circ [\eta]^r_{X,Y}$.

Proof. (i) can be proved directly and (ii) is just Lemma 2.5 (c) in [Brown and Zhang 2008]. Now, we give a proof of (iii). For $x \in \mathcal{C}(X, Y)$,

$$S_{Y,X} \circ S_{X,Y} \circ [\xi]^l_{X,Y}(a^{X,Y}) = \xi(a_1^{X,X}) S_{Y,X}(S_{X,Y}(a_2^{X,Y})).$$

Since $\Delta^X_{X,Y}(S_{Y,X}(S_{X,Y}(a^{X,Y}))) = S^2_{X,X}(a_1^{X,X}) \otimes S_{Y,X}(S_{X,Y}(a_2^{X,Y}))$,

$$[\xi]^l_{X,Y} \circ S_{Y,X} \circ S_{X,Y}(a^{X,Y}) = \xi S^2_{X,X}(a_1^{X,X}) S_{Y,X}(S_{X,Y}(a_2^{X,Y})).$$

By (ii), $\xi S^2_{X,X} = \xi$, so

$$S_{Y,X} \circ S_{X,Y} \circ [\xi]^l(a^{X,Y}) = [\xi]^l \circ S_{Y,X} \circ S_{X,Y}(a^{X,Y}).$$

Therefore, $S_{Y,X} \circ S_{X,Y} \circ [\xi]^l_{X,Y} = [\xi]^l_{X,Y} \circ S_{Y,X} \circ S_{X,Y}$.

(i’), (ii’) and (iii’) hold symmetrically to (i), (ii) and (iii), respectively. □

The following is the main result of [Yu 2016]:

Theorem 2.4.2. *Let $\mathcal{C}$ be a connected cogroupoid and let $X \in \mathrm{ob}(\mathcal{C})$ such that $\mathcal{C}(X, X)$ is a twisted CY algebra of dimension d with left homological integral $\int^l_{\mathcal{C}(X,X)} = \Bbbk_\xi$, where $\xi : \mathcal{C}(X, X) \to \Bbbk$ is an algebra homomorphism. Then for any $Y \in \mathrm{ob}(\mathcal{C})$, $\mathcal{C}(X, Y)$ is a twisted CY algebra of dimension d with Nakayama automorphism μ defined as $\mu = S_{Y,X} \circ S_{X,Y} \circ [\xi]^l_{X,Y}$. That is,*

$$\mu(a) = \xi(a_1^{X,X}) S_{Y,X}(S_{X,Y}(a_2^{X,Y}))$$

for any $x \in \mathcal{C}(X, Y)$.

Though we do not say that the CY-dimension of $\mathcal{C}(X, X)$ and $\mathcal{C}(X, Y)$ are the same in the statement of [Yu 2016, Theorem 2.5], it is easy to see from its proof. Applying Theorem 2.4.2 to the coopposite cogroupoid $\mathcal{C}'$, we obtain the following corollary:

Corollary 2.4.3. *Let $\mathcal{C}$ be a connected cogroupoid and let $Y \in \mathrm{ob}(\mathcal{C})$ such that $\mathcal{C}(Y, Y)$ is a twisted CY algebra of dimension d with left homological integral $\int^l_{\mathcal{C}(Y,Y)} = \Bbbk_\eta$, where $\eta : \mathcal{C}(Y, Y) \to \Bbbk$ is an algebra homomorphism. Then for any $X \in \mathrm{ob}(\mathcal{C})$, $\mathcal{C}(X, Y)$ is a twisted CY algebra of dimension d with Nakayama automorphism μ' defined as $\mu' = S^{-1}_{X,Y} \circ S^{-1}_{Y,X} \circ [\eta]^r_{X,Y}$. That is,*

$$\mu'(a) = S^{-1}_{X,Y}(S^{-1}_{Y,X}(a_1^{X,Y}))\eta(a_2^{Y,Y})$$

for any $x \in \mathcal{C}(X, Y)$.

Theorem 2.4.4. *Let $\mathcal{C}$ be a connected cogroupoid and let X be an object in $\mathcal{C}$ such that $\mathcal{C}(X, X)$ is a twisted CY Hopf algebra of dimension d. Then for any $Y \in \mathrm{ob}(\mathcal{C})$ such that $\mathcal{C}(Y, Y)$ is homologically smooth, $\mathcal{C}(Y, Y)$ is a twisted CY algebra of dimension d as well.*

Proof. Let Y be an object in $\mathcal{C}$ such that $\mathcal{C}(Y,Y)$ is homologically smooth. We need to compute the Hochschild cohomology of $\mathcal{C}(Y,Y)$. By Lemma 2.3.2,

$$\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X,Y),\mathcal{C}(X,Y)^e)\cong \mathrm{Ext}^i_{\mathcal{C}(Y,Y)^{\mathrm{op}}}({}_{\varepsilon}\Bbbk,\mathcal{L}'_Y(\mathcal{C}(X,Y)^e))$$

for all $i\geqslant 0$. $\mathcal{L}'_Y(\mathcal{C}(X,Y)^e)$ is a $\mathcal{C}(Y,Y)$-$\mathcal{C}(X,Y)^e$-bimodule. The left $\mathcal{C}(Y,Y)$-module isomorphism

$$\mathcal{L}'_Y(\mathcal{C}(X,Y)^e)\to {}_*\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y),\quad x\otimes y\mapsto x_2^{Y,Y}\otimes yS^{-1}_{X,Y}(S^{-1}_{Y,X}(x_1^{X,Y}))$$

in Lemma 2.3.1 is also an isomorphism of left $\mathcal{C}(X,Y)^e$-modules if we endow a right $\mathcal{C}(X,Y)^e$-module structure on ${}_*\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y)$ as follows:

$$(x\otimes y)\leftarrow(a\otimes b)=xa_2^{Y,Y}\otimes byS^{-1}_{X,Y}(S^{-1}_{Y,X}(a_1^{X,Y}))$$

for any $x\otimes y\in {}_*\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y)$ and $a\otimes b\in\mathcal{C}(X,Y)^e$. Therefore, we obtain the following left $\mathcal{C}(X,Y)^e$-module isomorphisms:

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X,Y),\mathcal{C}(X,Y)^e)&\cong \mathrm{Ext}^i_{\mathcal{C}(Y,Y)}({}_{\varepsilon}\Bbbk,\mathcal{L}'_Y(\mathcal{C}(X,Y)^e))\\&\cong \mathrm{Ext}^i_{\mathcal{C}(Y,Y)}({}_{\varepsilon}\Bbbk,{}_*\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y))\\&\cong \mathrm{Ext}^i_{\mathcal{C}(Y,Y)}({}_{\varepsilon}\Bbbk,\mathcal{C}(Y,Y))\otimes\mathcal{C}(X,Y)\end{aligned}$$

for $i\geqslant 0$. The third isomorphism follows from the fact that $\mathcal{C}(Y,Y)$ is homologically smooth, the trivial module ${}_{\varepsilon}\Bbbk$ admits a bounded projective resolution with each term finitely generated (Proposition A.2). The right $\mathcal{C}(X,Y)^e$-module structure on $\mathrm{Ext}^i_{\mathcal{C}(Y,Y)^{\mathrm{op}}}(\Bbbk_{\varepsilon},\mathcal{C}(Y,Y))\otimes\mathcal{C}(X,Y)$ induced by the isomorphisms above is given by

$$(x\otimes y)\leftarrow(a\otimes b)=xa_2^{Y,Y}\otimes byS^{-1}_{X,Y}(S^{-1}_{Y,X}(a_1^{X,Y}))$$

for any $x\otimes y\in\mathrm{Ext}^i_{\mathcal{C}(Y,Y)^{\mathrm{op}}}({}_{\varepsilon}\Bbbk,\mathcal{C}(Y,Y))\otimes\mathcal{C}(X,Y)$ and $a\otimes b\in\mathcal{C}(X,Y)^e$. Note that the right $\mathcal{C}(Y,Y)$-module structure of $\mathcal{C}(Y,Y)$ induces a right $\mathcal{C}(Y,Y)$-module structure on $\mathrm{Ext}^i_{\mathcal{C}(Y,Y)^{\mathrm{op}}}(\Bbbk_{\varepsilon},\mathcal{C}(Y,Y))$.

It follows from Theorem 2.4.2 that $\mathcal{C}(X,Y)$ is a twisted CY algebra of dimension d with Nakayama automorphism $\mu=S_{Y,X}\circ S_{X,Y}\circ[\xi]^l_{X,Y}$. So

$$\mathrm{Ext}^i_{\mathcal{C}(X,Y)^e}(\mathcal{C}(X,Y),\mathcal{C}(X,Y)^e)=\begin{cases}0, & i\neq d,\\ \mathcal{C}(X,Y)^{\mu}, & i=d.\end{cases}$$

Now we arrive at the isomorphism of left $\mathcal{C}(X,Y)^e$-modules

$$\mathrm{Ext}^i_{\mathcal{C}(Y,Y)}({}_{\varepsilon}\Bbbk,\mathcal{C}(Y,Y))\otimes\mathcal{C}(X,Y)\cong\begin{cases}0, & i\neq d,\\ \mathcal{C}(X,Y)^{\mu}, & i=d.\end{cases}$$

A right $\mathcal{C}(X,Y)^e$-module can be viewed as a $\mathcal{C}(X,Y)$-bimodule. The left module structure of $\mathrm{Ext}^i_{\mathcal{C}(Y,Y)}(\Bbbk_{\varepsilon},\mathcal{C}(Y,Y))\otimes\mathcal{C}(X,Y)$ is just the left multiplication to the factor $\mathcal{C}(X, Y)$. So especially, as left $\mathcal{C}(X, Y)$-modules,

$$\operatorname{Ext}^i_{\mathcal{C}(Y,Y)}({}_\varepsilon\Bbbk, \mathcal{C}(Y, Y)) \otimes \mathcal{C}(X, Y) \cong \begin{cases} 0, & i \neq d, \\ \mathcal{C}(X, Y), & i = d. \end{cases}$$

This shows that $\operatorname{Ext}^i_{\mathcal{C}(Y,Y)}({}_\varepsilon\Bbbk, \mathcal{C}(Y, Y)) = 0$ for $i \neq d$. Moreover, for degree d, we denote $V = \operatorname{Ext}^d_{\mathcal{C}(Y,Y)}({}_\varepsilon\Bbbk, \mathcal{C}(Y, Y))$. Then $V \otimes \mathcal{C}(X, Y) \cong \mathcal{C}(X, Y)$ as free left $\mathcal{C}(X, Y)$-modules. Hence $0 < \dim V < \infty$ (note that we do not know whether $\mathcal{C}(X, Y)$ has the FBN property). Similarly, $\operatorname{Ext}^i_{\mathcal{C}(Y,Y)^{\mathrm{op}}}(\Bbbk_\varepsilon, \mathcal{C}(Y, Y)) = 0$ for $i \neq d$ and $\operatorname{Ext}^d_{\mathcal{C}(Y,Y)^{\mathrm{op}}}(\Bbbk_\varepsilon, \mathcal{C}(Y, Y))$ is finite dimensional as well. Hence $\mathcal{C}(Y, Y)$ is twisted CY of dimension d by Proposition 2.1.6. □

Theorem 2.4.5. *Let H and L be two monoidally Morita–Takeuchi equivalent Hopf algebras. If H is twisted CY of dimension d and L is homologically smooth, then L is twisted CY of dimension d as well.*

Proof. This directly follows from Theorem 1.1.4 and Theorem 2.4.4. □

Before we present our next theorem, we need the following lemma:

Lemma 2.4.6. *If H be a Noetherian Hopf algebra, then the trivial Yetter–Drinfeld module $\Bbbk$ admits a finitely generated free Yetter–Drinfeld module resolution.*

Proof. First we have an epimorphism $\epsilon : \Bbbk \boxtimes H \to \Bbbk$, $1 \otimes h \mapsto \varepsilon(h)$ of Yetter–Drinfeld modules. Set $P_0 = \Bbbk \boxtimes H$. Since H is Noetherian, $\operatorname{Ker}\epsilon$ is finitely generated as a module over H. Say it is generated by a finite dimensional subspace V_1 of P_0. That is, there exists an epimorphism $V_1 \otimes H \to \operatorname{Ker}\epsilon \to 0$ given by $v \otimes h \mapsto vh$ for any $v \in V_1$ and $h \in H$. Let C_1 be the subcomodule of $\operatorname{Ker}\epsilon$ generated by V_1. We know C_1 is finite dimensional since V_1 is finite dimensional by the fundamental theory of comodules. Construct the epimorphism $C_1 \boxtimes H \to \operatorname{Ker}\epsilon \to 0$ via $c \otimes h \mapsto ch$ for any $c \in C_1$ and $h \in H$. It is easy to check that it is a morphism of Yetter–Drinfeld modules. Set $P_1 = C_1 \boxtimes H$, we have the exact sequence $P_1 \to P_0 \to \Bbbk \to 0$. Note that P_1 is again a Noetherian H-module. Hence we can do the procedure recursively to obtain a finitely generated free Yetter–Drinfeld module resolution of $\Bbbk$. □

Theorem 2.4.7. *Let H be a twisted CY Hopf algebra of dimension d, and L a Hopf algebra monoidally Morita–Takeuchi equivalent to H. If one of the following conditions holds, then L is also twisted CY of dimension d.*

(i) *H admits a finitely generated relative projective Yetter–Drinfeld module resolution for the trivial Yetter–Drinfeld module $\Bbbk$ and L has finite global dimension.*

(ii) *H admits a bounded finitely generated relative projective Yetter–Drinfeld module resolution for the trivial Yetter–Drinfeld module $\Bbbk$.*

(iii) *H is Noetherian and L has finite global dimension.*

(iv) *L is Noetherian and has finite global dimension.*

Proof. By Theorem 2.4.4, we only need to prove that if one of the conditions listed in the theorem holds, then L is homologically smooth.

In case (i) We use the language of cogroupoids. Since H and L are monoidally Morita–Takeuchi equivalent, there exists a connected cogroupoid with 2 objects X, Y such that $H = \mathcal{C}(X, X)$ and $L = \mathcal{C}(Y, Y)$ (Theorem 1.1.4). By Proposition A.2, to show $L = \mathcal{C}(Y, Y)$ is homologically smooth, we only need to show that the trivial module $\Bbbk_\varepsilon$ admits a bounded projective resolution with each term finitely generated. By assumption, the trivial Yetter–Drinfeld module $\Bbbk$ over the Hopf algebra $H = \mathcal{C}(X, X)$ admits a finitely generated relative projective Yetter–Drinfeld module resolution

$$\cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0. \tag{10}$$

By Lemma 2.2.5,

$$\begin{aligned}\cdots \to P_i \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \xrightarrow{\delta_i \square \mathcal{C}(X,Y)} P_{i-1} \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \to \cdots \\ \cdots \to P_1 \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \to P_0 \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \to \Bbbk \to 0.\end{aligned} \tag{11}$$

is a finitely generated relative projective Yetter–Drinfeld module resolution of the trivial Yetter–Drinfeld module $\Bbbk$ over $\mathcal{C}(Y, Y)$. So each $P_i \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$ is a finite generated projective $\mathcal{C}(Y, Y)$-module. By assumption, the global dimension of $\mathcal{C}(Y, Y)$ is finite, say n. Set $K_n = \mathrm{Ker}(\delta_{n-1} \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y))$. Following from Lemma 4.1.6 in [Weibel 1994], K_n is projective, so it is a direct summand of $P_n \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$. Since $P_n \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y)$ is finitely generated, K_n is finitely generated as well. Therefore,

$$\begin{aligned}0 \to K_n \to P_{n-1} \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \to \cdots \\ \cdots \to P_1 \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \to P_0 \square_{\mathcal{C}(X,X)} \mathcal{C}(X, Y) \to \Bbbk \to 0\end{aligned}$$

is a bounded projective resolution with each term finitely generated. Hence, $L = \mathcal{C}(Y, Y)$ is homologically smooth.

The proof in case (ii) uses a similar argument as in case (i) since equations (10) and (11) now are bounded finitely generated projective resolutions for $\Bbbk$.

Case (iii) is a direct consequence of Lemma 2.4.6 and (i).

That the Hopf algebra L is homologically smooth in case (iv) follows from [Brown and Zhang 2008, Lemma 5.2]. ☐

Corollary 2.4.8. *Let H and L be two monoidally Morita–Takeuchi equivalent Hopf algebras. If both H and L are twisted CY, then* $\mathrm{gldim}(H) = \mathrm{gldim}(L)$.

Proof. It follows from Theorem 2.4.7 and the fact that for twisted CY Hopf algebras the CY dimension always equals the global dimension by Proposition 2.1.6. ☐

Now we discuss the relation between the homological integrals of $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ when both of them are twisted CY.

Theorem 2.4.9. *Let $\mathcal{C}$ be a connected cogroupoid. If X and Y are two objects such that $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ are both twisted CY algebras, then we have*

$$(S_{Y,X} \circ S_{X,Y})^2 = [\eta]^r_{X,Y} \circ ([\xi]^l_{X,Y})^{-1} \circ \gamma, \tag{12}$$

where $\xi : \mathcal{C}(X, X) \to \Bbbk$ and $\eta : \mathcal{C}(Y, Y) \to \Bbbk$ are algebra homomorphisms given by the left homological integrals of $\mathcal{C}(X, X) : \int^l_{\mathcal{C}(X,X)} = \Bbbk_\xi$ and $\mathcal{C}(Y, Y) : \int^l_{\mathcal{C}(Y,Y)} = \Bbbk_\eta$ respectively, and γ is an inner automorphism of $\mathcal{C}(X, Y)$.

Proof. From Theorem 2.4.2 and Corollary 2.4.3, it is easy to see that the CY-dimensions of $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ are equal. Moreover, $\mu = S_{Y,X} \circ S_{X,Y} \circ [\xi]^l$ and $\mu' = S^{-1}_{X,Y} \circ S^{-1}_{Y,X} \circ [\eta]^r$ are the Nakayama automorphisms of $\mathcal{C}(X, Y)$. Since Nakayama automorphisms are unique up to inner automorphisms,

$$S_{Y,X} \circ S_{X,Y} \circ [\xi]^l_{X,Y} = S^{-1}_{X,Y} \circ S^{-1}_{Y,X} \circ [\eta]^r_{X,Y} \circ \gamma,$$

for some inner automorphism γ of $\mathcal{C}(X, Y)$. The automorphism $[\xi]^l_{X,Y}$ commutes with $S_{Y,X} \circ S_{X,Y}$ (Lemma 2.4.1), we obtain that

$$(S_{Y,X} \circ S_{X,Y})^2 = ([\xi]^l_{X,Y})^{-1} \circ [\eta]^r_{X,Y} \circ \gamma. \qquad \square$$

Remark 2.4.10. The three maps composed to give $(S_{Y,X} \circ S_{X,Y})^2$ in (12) commute with each other. This can be proved as in [Brown and Zhang 2008, Proposition 4.6] with the help of Lemma 2.4.1. It is not hard to see that Theorem 2.4.9 holds when $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ are both AS-Gorenstein. The equation (12) is just (4.6.1) in the same work when $X = Y$. Since the inner automorphism $\gamma = (S_{Y,X} \circ S_{X,Y})^2 \circ ([\eta]^r_{X,Y})^{-1} \circ [\xi]^l_{X,Y}$ is intrinsic in $\mathcal{C}(X, Y)$, it prompts us to generalize their Question 4.6 to the Hopf-bigalois object $\mathcal{C}(X, Y)$ when both $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ are AS-Gorenstein.

Question 2.4.11. What is the inner automorphism in Theorem 2.4.9?

3. Examples

In this section, we provide some examples.

3.1. ***Example 1.*** We take the field $\Bbbk$ to be $\mathbb{C}$ in this subsection. Let $E \in \mathrm{GL}_m(\mathbb{C})$ with $m \geqslant 2$ and let $\mathcal{B}(E)$ be the algebra presented by generators $(u_{ij})_{1 \leqslant i,j \leqslant m}$ and relations

$$E^{-1} u^t E u = I_m = u E^{-1} u^t E,$$

where u is the matrix $(u_{ij})_{1 \leqslant i,j \leqslant m}$, u^t is the transpose of u and I_m is the identity matrix. The algebra $\mathcal{B}(E)$ is a Hopf algebra and was defined by Dubois-Violette and

Launer [1990] as the quantum automorphism group of the nondegenerate bilinear form associated to E. When

$$E = E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix},$$

$\mathcal{B}(E_q)$ is just the algebra $\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$, the quantized coordinate algebra of $\mathrm{SL}_2(\mathbb{C})$.

In order to describe Hopf algebras whose comodule categories are monoidally equivalent to the one of $\mathcal{B}(E)$, we recall the cogroupoid $\mathcal{B}$.

Let $E \in \mathrm{GL}_m(\mathbb{C})$ and let $F \in \mathrm{GL}_n(\mathbb{C})$. The algebra $\mathcal{B}(E, F)$ is defined to be the algebra with generators u_{ij}, $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$, subject to the relations:

$$F^{-1}u^t E u = I_n; \quad u F^{-1} u^t E = I_m. \tag{13}$$

The generators u_{ij} in $\mathcal{B}(E, F)$ is denoted by u_{ij}^{EF} to express the dependence on E and F when needed. It is clear that $\mathcal{B}(E) = \mathcal{B}(E, E)$.

For any $E \in \mathrm{GL}_m(\mathbb{C}),\ F \in \mathrm{GL}_n(\mathbb{C})$ and $G \in \mathrm{GL}_p(\mathbb{C})$, define the following maps:

$$\Delta^G_{E,F}: \quad \mathcal{B}(E, F) \to \mathcal{B}(E, G) \otimes \mathcal{B}(G, F), \qquad u_{ij} \mapsto \sum_{k=1}^{p} u_{ik} \otimes u_{kj}, \tag{14}$$

$$\varepsilon_E: \quad \mathcal{B}(E) \to \mathbb{C}, \qquad u_{ij} \mapsto \delta_{ij}, \tag{15}$$

$$S_{E,F}: \quad \mathcal{B}(E, F) \to \mathcal{B}(F, E)^{\mathrm{op}}, \qquad u \mapsto E^{-1}u^t F. \tag{16}$$

It is clear that $S_{E,F}$ is bijective.

Lemma 3.2 in [Bichon 2014] ensures that with these morphisms we have a cogroupoid. The cogroupoid $\mathcal{B}$ is defined as follows:

(i) $\mathrm{ob}(\mathcal{B}) = \{E \in \mathrm{GL}_m(\mathbb{C}), m \geqslant 1\}$.

(ii) For $E, F \in \mathrm{ob}(\mathcal{B})$, the algebra $\mathcal{B}(E, F)$ is the algebra defined as in (13).

(iii) The structural maps $\Delta^{\bullet}_{\bullet,\bullet}$, $\varepsilon_{\bullet}$ and $S_{\bullet,\bullet}$ are defined in (14), (15) and (16), respectively.

Lemma 3.1.1 [Bichon 2014, Lemma 3.4, Corollary 3.5]. *Let $E \in \mathrm{GL}_m(\mathbb{C}),\ F \in \mathrm{GL}_n(\mathbb{C})$ with $m, n \geqslant 2$. Then $\mathcal{B}(E, F) \neq (0)$ if and only if $\mathrm{tr}(E^{-1}E^t) = \mathrm{tr}(F^{-1}F^t)$. Consequently, let $\lambda \in \mathbb{C}$, and $\mathcal{B}^\lambda$ the full subcogroupoid of $\mathcal{B}$ with objects*

$$\mathrm{ob}(\mathcal{B}^\lambda) = \{E \in \mathrm{GL}_n(\mathbb{C}), m \geqslant 2, \mathrm{tr}(E^{-1}E^t) = \lambda\}.$$

Then $\mathcal{B}^\lambda$ is a connected cogroupoid.

Thus, if $E \in \mathrm{GL}_m(\mathbb{C}),\ F \in \mathrm{GL}_n(\mathbb{C})$ with $m, n \geqslant 2$ satisfy $\mathrm{tr}(E^{-1}E^t) = \mathrm{tr}(F^{-1}F^t)$, then the comodule categories of $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are monoidally equivalent.

The Calabi–Yau property of the algebras $\mathcal{B}(E)$ was discussed in [Bichon 2013, Section 6] (see also [Walton and Wang 2016] and [Yu 2016]). Theorem 2.4.7 provides a more simplified way to prove that the algebras $\mathcal{B}(E)$ are twisted CY algebras.

Actually, by Lemma 5.6 in [Bichon 2013], the trivial Yetter–Drinfeld module over the algebra $\mathcal{B}(E_q)$ admits a bounded finitely generated free Yetter–Drinfeld module resolution and $\mathcal{B}(E_q)$ twisted CY of dimension 3 with left homological integral $\int^l_{\mathcal{B}(E_q)} = \mathbb{C}_\eta$ given by

$$\eta(u) = \begin{pmatrix} q^{-2} & 0 \\ 0 & q^2 \end{pmatrix}.$$

For any $E \in \mathrm{GL}_m(\mathbb{C})$ ($m \geqslant 2$), there is a $q \in \mathbb{C}^\times$ such that

$$\mathrm{tr}(E^{-1}E^t) = -q - q^{-1} = \mathrm{tr}(E_q^{-1}E_q^t),$$

so $\mathcal{B}(E)$ and $\mathcal{B}(E_q)$ are monoidally Morita–Takeuchi equivalent. Therefore, the algebra $\mathcal{B}(E)$ is twisted CY by Theorem 2.4.7. Let $\int^l_{\mathcal{B}(E)} = \mathbb{C}_\xi$ be the left homological integral of $\mathcal{B}(E)$, where $\xi : \mathcal{B}(E) \to \mathbb{C}$ is an algebra homomorphism. Since there are no nontrivial units in $\mathcal{B}(E, E_q)$. Then ξ and η satisfy the equation

$$(S_{E_q,E} \circ S_{E,E_q})^2 = [\eta]^r_{E,E_q} \circ ([\xi]^l_{E,E_q})^{-1}$$

by Theorem 2.4.9. So ξ is defined by $\xi(u^E) = (E^t)^{-1}E(E^t)^{-1}E$. Hence, the Nakayama automorphism of $\mathcal{B}(E)$ is defined by $\mu(u) = (E^t)^{-1}Eu(E^t)^{-1}E$ [Reyes et al. 2014, Lemma 1.3].

3.2. ***Example 2.*** In this subsection, we want to present a class of Hopf algebras such that the inner automorphism in Theorem 2.4.9 can be calculated. We first recall the definition of the 2-cocycle cogroupoid.

Let H be a Hopf algebra with bijective antipode. A *(right) 2-cocycle* on H is a convolution invertible linear map $\sigma : H \otimes H \to \Bbbk$ satisfying

$$\sigma(h_1, k_1)\sigma(h_2k_2, l) = \sigma(k_1, l_1)\sigma(h, k_2l_2), \quad \sigma(h, 1) = \sigma(1, h) = \varepsilon(h)$$

for all $h, k, l \in H$. The set of 2-cocycles on H is denoted $Z^2(H)$. They define the 2-cocycle cogroupoid of H.

Let $\sigma, \tau \in Z^2(H)$. The algebra $H(\sigma, \tau)$ is defined to be the vector space H together with the multiplication given by

$$x \centerdot y = \sigma(x_1, y_1)x_2y_2\tau^{-1}(x_3, y_3) \quad \text{for any } x, y \in H. \tag{17}$$

The Hopf algebra $H(\sigma, \sigma)$ is just the *cocycle deformation* H^σ of H defined by Doi [1993]. The comultiplication of H^σ is the same as the comultiplication of H. However, the multiplication and the antipode are deformed:

$$h \centerdot k = \sigma(h_1, k_1)h_2k_2\sigma^{-1}(h_3, k_3), \quad S_{\sigma,\sigma}(h) = \sigma(h_1, S(h_2))S(h_3)\sigma^{-1}(S(h_4), h_5)$$

for any $h, k \in H^\sigma$.

Now we recall the necessary structural maps for the 2-cocycle cogroupoid of H. For any $\sigma, \tau, \omega \in Z^2(H)$, define the following maps:

(18) $\Delta^{\omega}_{\sigma,\tau} = \Delta : H(\sigma,\tau) \to H(\sigma,\omega) \otimes H(\omega,\tau), \quad x \mapsto x_1 \otimes x_2.$

(19) $\varepsilon_{\sigma} = \varepsilon : H(\sigma,\sigma) \to \Bbbk.$

(20) $S_{\sigma,\tau} : H(\sigma,\tau) \to H(\tau,\sigma), \quad x \mapsto \sigma(x_1, S(x_2)) S(x_3) \tau^{-1}(S(x_4), x_5).$

It is routine to check that the inverse of $S_{\sigma,\tau}$ is given as follows:

(21) $S^{-1}_{\sigma,\tau} : H(\tau,\sigma) \to H(\sigma,\tau), \quad x \mapsto \sigma^{-1}(x_5, S^{-1}(x_4)) S^{-1}(x_3) \tau(S^{-1}(x_2), x_1).$

The 2-*cocycle cogroupoid* of H, denoted $\underline{H}$, is the cogroupoid defined as follows:

(i) $\mathrm{ob}(\underline{H}) = Z^2(H)$.

(ii) For $\sigma, \tau \in Z^2(H)$, the algebra $\underline{H}(\sigma,\tau)$ is the algebra $H(\sigma,\tau)$ defined in (17).

(iii) The structural maps $\Delta^{\bullet}_{\bullet,\bullet}$, $\varepsilon_{\bullet}$ and $S_{\bullet,\bullet}$ are defined in (18), (19) and (20) respectively.

Following [Bichon 2014, Lemma 3.13], the morphisms $\Delta^{\bullet}_{\bullet,\bullet}$, $\varepsilon_{\bullet}$ and $S_{\bullet,\bullet}$ indeed satisfy the conditions required for a cogroupoid. It is clear that a 2-cocycle cogroupoid is connected.

Now we recall the definition of the pointed Hopf algebras $U(\mathcal{D}, \lambda)$. For a group Γ, we denote by ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ the category of Yetter–Drinfeld modules over the group algebra $\Bbbk\Gamma$. If Γ is an abelian group, then it is well known that a Yetter–Drinfeld module over the algebra $\Bbbk\Gamma$ is just a Γ-graded Γ-module.

We fix the following terminology.

- a free abelian group Γ of finite rank s;
- a Cartan matrix $\mathbb{A} = (a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$ of finite type, where $\theta \in \mathbb{N}$. Let $(d_1, \dots, d_\theta)$ be a diagonal matrix of positive integers such that $d_i a_{ij} = d_j a_{ji}$, which is minimal with this property;
- a set $\mathcal{X}$ of connected components of the Dynkin diagram corresponding to the Cartan matrix $\mathbb{A}$. If $1 \leqslant i, j \leqslant \theta$, then $i \sim j$ means that they belong to the same connected component;
- a family $(q_I)_{I\in\mathcal{X}}$ of elements in $\Bbbk$ which are *not* roots of unity;
- elements $g_1, \dots, g_\theta \in \Gamma$ and characters $\chi_1, \dots, \chi_\theta \in \hat{\Gamma}$ such that

(22) $$\chi_j(g_i)\chi_i(g_j) = q_I^{d_i a_{ij}}, \quad \chi_i(g_i) = q_I^{d_i} \quad \text{for all } 1 \leqslant i, j \leqslant \theta,\ I \in \mathcal{X}.$$

For simplicity, we write $q_{ji} = \chi_i(g_j)$. Then equation (22) reads as follows:

(23) $$q_{ii} = q_I^{d_i} \quad \text{and} \quad q_{ij}q_{ji} = q_I^{d_i a_{ij}} \quad \text{for all } 1 \leqslant i, j \leqslant \theta,\ I \in \mathcal{X}.$$

Let $\mathcal{D}$ be the collection $\mathcal{D}(\Gamma, (a_{ij})_{1\leqslant i,j\leqslant\theta}, (q_I)_{I\in\mathcal{X}}, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta})$. A *linking datum* $\lambda = (\lambda_{ij})$ for $\mathcal{D}$ is a collection of elements $(\lambda_{ij})_{1\leqslant i<j\leqslant\theta, i\nsim j} \in \Bbbk$ such that $\lambda_{ij} = 0$ if $g_i g_j = 1$ or $\chi_i\chi_j \neq \varepsilon$. We write the datum $\lambda = 0$, if $\lambda_{ij} = 0$ for all $1 \leqslant i < j \leqslant \theta$. The datum $(\mathcal{D}, \lambda) = (\Gamma, (a_{ij}), q_I, (g_i), (\chi_i), (\lambda_{ij}))$ is called a *generic datum of finite Cartan type* for group Γ.

A generic datum of finite Cartan type for a group Γ defines a Yetter–Drinfeld module over the group algebra $\Bbbk\Gamma$. Let V be a vector space with basis $\{x_1, x_2, \ldots, x_\theta\}$. We set

$$|x_i| = g_i, \quad g(x_i) = \chi_i(g)x_i, \quad 1 \leqslant i \leqslant \theta, g \in \Gamma,$$

where $|x_i|$ denotes the degree of x_i. This makes V a Yetter–Drinfeld module over the group algebra $\Bbbk\Gamma$. We write $V = \{x_i, g_i, \chi_i\}_{1\leqslant i\leqslant\theta} \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. The braiding is given by

$$c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \quad 1 \leqslant i, j \leqslant \theta.$$

The tensor algebra $T(V)$ on V is a natural graded braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. The smash product $T(V)\#\Bbbk\Gamma$ is a usual Hopf algebra. It is also called a bosonization of $T(V)$ by $\Bbbk\Gamma$.

Definition 3.2.1. Given a generic datum of finite Cartan type $(\mathcal{D}, \lambda)$ for a group Γ, define $U(\mathcal{D}, \lambda)$ as the quotient Hopf algebra of the smash product $T(V)\#\Bbbk\Gamma$ modulo the ideal generated by

$$\begin{aligned} (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) &= 0, & 1 \leqslant i \neq j \leqslant \theta, \quad & i \sim j, \\ x_i x_j - \chi_j(g_i)x_j x_i &= \lambda_{ij}(g_i g_j - 1), & 1 \leqslant i < j \leqslant \theta, \quad & i \nsim j, \end{aligned}$$

where ad_c is the braided adjoint representation defined in [Andruskiewitsch and Schneider 2004, Sec. 1].

To present the CY property of the algebras $U(\mathcal{D}, \lambda)$, we recall the concept of root vectors. Let Φ be the root system corresponding to the Cartan matrix $\mathbb{A}$ with $\{\alpha_1, \ldots, \alpha_\theta\}$ a set of fixed simple roots, and $\mathcal{W}$ the Weyl group. We fix a reduced decomposition of the longest element $w_0 = s_{i_1} \cdots s_{i_p}$ of $\mathcal{W}$ in terms of the simple reflections. Then the positive roots are precisely the following:

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}).$$

For $\beta_i = \sum_{i=1}^{\theta} m_i\alpha_i$, we write $g_{\beta_i} = g_1^{m_1} \cdots g_\theta^{m_\theta}$ and $\chi_{\beta_i} = \chi_1^{m_1} \cdots \chi_\theta^{m_\theta}$.

Lusztig [1993] defined the root vectors for a quantum group $U_q(\mathfrak{g})$. Up to a nonzero scalar, each root vector can be expressed as an iterated braided commutator. In [Andruskiewitsch and Schneider 2002, Sec. 4.1], the root vectors were generalized on a pointed Hopf algebras $U(\mathcal{D}, \lambda)$. For each positive root β_i, $1 \leqslant i \leqslant p$, the root vector x_{β_i} is defined by the same iterated braided commutator of the elements $x_1, \ldots, x_\theta$, but with respect to the general braiding.

Remark 3.2.2. If $\beta_j = \alpha_l$, then we have $x_{\beta_j} = x_l$. That is, $x_1, \ldots, x_\theta$ are the simple root vectors.

Lemma 3.2.3 [Yu et al. 2016, Lemma 3.3]. *Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan type for a group Γ, and H the Hopf algebra $U(\mathcal{D}, \lambda)$. Let s be the rank of Γ and p the number of the positive roots of the Cartan matrix.*

(i) *The algebra H is Noetherian AS-regular of global dimension $p+s$. The left homological integral module $\int_H^l$ of H is isomorphic to $\Bbbk_\xi$, where $\xi : H \to \Bbbk$ is an algebra homomorphism defined by $\xi(g) = \big(\prod_{i=1}^p \chi_{\beta_i}\big)(g)$ for all $g \in \Gamma$ and $\xi(x_k) = 0$ for all $1 \leqslant k \leqslant \theta$.*

(ii) *The algebra H is twisted CY with Nakayama automorphism μ defined by $\mu(x_k) = q_{kk} x_k$ for all $1 \leqslant k \leqslant \theta$, and $\mu(g) = \big(\prod_{i=1}^p \chi_{\beta_i}\big)(g)$ for all $g \in \Gamma$.*

Let $(\mathcal{D}, \lambda)$ be a generic datum of finite Cartan type for a group Γ. The algebra $U(\mathcal{D}, \lambda)$ is a cocycle deformation of $U(\mathcal{D}, 0)$. That is $U(\mathcal{D}, \lambda) = U(\mathcal{D}, 0)^\sigma$, where σ is the cocycle defined by

$$\begin{aligned} &\sigma(g, g') = 1, \\ &\sigma(g, x_i) = \sigma(x_i, g) = 0, \quad 1 \leqslant i \leqslant \theta, g, g' \in \Gamma, \\ &\sigma(x_i, x_j) = \begin{cases} \lambda_{ij}, & i < j, i \nsim j, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{24}$$

Lemma 3.2.3 shows that both $U(\mathcal{D}, 0)$ and its cocycle deformation $U(\mathcal{D}, \lambda)$ are twisted CY. The algebras $U(\mathcal{D}, \lambda)$ are Noetherian with finite global dimension by Lemma 2.1 in [Yu and Zhang 2013]. Therefore, Theorem 2.4.7 explains why for this class of Hopf algebras, cocycle deformation preserves the CY property.

With Lemma 3.2.3, we can write the inner automorphism in Theorem 2.4.9 explicitly.

Proposition 3.2.4. *Let H be $U(\mathcal{D}, 0)$, then $U(\mathcal{D}, \lambda) = H^\sigma$, where σ is the cocycle as defined in (24). Let $\int_H^l = \Bbbk_\xi$ and $\int_{H^\sigma}^r = \Bbbk_\eta$ be left homological integral of H and H^σ respectively, where $\xi : H \to \Bbbk$ and $\eta : H^\sigma \to \Bbbk$ are algebra homomorphisms. Then the following equation holds.*

$$(S_{\sigma,1} \circ S_{1,\sigma})^2 = [\eta]^r_{1,\sigma} \circ ([\xi]^l_{1,\sigma})^{-1} \circ \gamma,$$

where γ is the inner automorphism defined by $\gamma(x_k) = \big[\prod_{i=1}^p g_{\beta_i}\big]^{-1}(x_k)\big[\prod_{i=1}^p g_{\beta_i}\big]$ for $1 \leqslant k \leqslant \theta$ and $\gamma(g) = g$ for any $g \in \Gamma$.

Appendix

We list two basic homological properties of Hopf algebras. They are well known, but due to a lack of convenient references, we provide in most cases their proofs. We do not require bijectivity of antipode or Noetherianity of a Hopf algebra.

First we want to show that for a Hopf algebra, the left global dimension always equals the right global dimension.

Let H be a Hopf algebra. We denote the left global dimension, the right global dimension and the Hochschild dimension of H by $\operatorname{lgldim}(H)$, $\operatorname{rgldim}(H)$ and $\operatorname{Hdim}(H)$, respectively. We have the left adjoint functor $L : \operatorname{Mod} H^e \to \operatorname{Mod} H$ and the right adjoint functor $R : \operatorname{Mod}(H^e)^{\mathrm{op}} \to \operatorname{Mod} H^{\mathrm{op}}$. Let M be an H-bimodule. Then $L(M)$ is a left H-module defined by the action

$$x \to m = x_1 m S(x_2) \quad \text{for any } x \in H.$$

While $R(M)$ is a right H-module defined by the action

$$m \leftarrow x = S(x_1) m x_2 \quad \text{for any } x \in H.$$

Proposition A.1. *Let H be a Hopf algebra. Then*

$$\operatorname{projdim} \Bbbk_\varepsilon = \operatorname{projdim} {}_\varepsilon\Bbbk = \operatorname{rgldim}(H) = \operatorname{lgldim}(H) = \operatorname{Hdim}(H).$$

Proof. That $\operatorname{projdim} \Bbbk_\varepsilon = \operatorname{rgldim}(H)$ and $\operatorname{projdim} {}_\varepsilon\Bbbk = \operatorname{lgldim}(H)$ follows from [Lorenz and Lorenz 1995, Section 2.4]. We know from [Cartan and Eilenberg 1956, IX.7.6] that $\operatorname{rgldim}(H)$ and $\operatorname{lgldim}(H)$ are bounded by $\operatorname{Hdim}(H)$. Let M be any H-bimodule. By Lemma 2.4 in [Brown and Zhang 2008], there are isomorphisms $\operatorname{Ext}^i_{H^e}(H, M) \cong \operatorname{Ext}^i_H({}_\varepsilon\Bbbk, L(M))$ for $i \geqslant 0$. This shows that $\operatorname{Hdim}(H) \leqslant \operatorname{lgldim}(H)$. Similarly, for $i \geqslant 0$, the isomorphisms $\operatorname{Ext}^i_{H^e}(H, M) \cong \operatorname{Ext}^i_H(\Bbbk_\varepsilon, R(M))$ hold. So $\operatorname{Hdim}(H) \leqslant \operatorname{rgldim}(H)$. Therefore, we have $\operatorname{rgldim}(H) = \operatorname{lgldim}(H) = \operatorname{Hdim}(H)$. In conclusion, we obtain that

$$\operatorname{projdim} \Bbbk_\varepsilon = \operatorname{projdim} {}_\varepsilon\Bbbk = \operatorname{rgldim}(H) = \operatorname{lgldim}(H) = \operatorname{Hdim}(H). \qquad \square$$

Next we want to show that to see whether a Hopf algebra H is homologically smooth it is enough to investigate the projective resolution of the trivial module.

Proposition A.2. *Let H be a Hopf algebra. The following are equivalent*:

(i) *The algebra H is homologically smooth.*

(ii) *The left trivial module ${}_\varepsilon\Bbbk$ admits a bounded projective resolution with each term finitely generated.*

(iii) *The right trivial module $\Bbbk_\varepsilon$ admits a bounded projective resolution with each term finitely generated.*

Proof. We only need to show that (i) and (ii) are equivalent. (i)$\Longleftrightarrow$(iii) can be proved symmetrically.

(i)$\Rightarrow$(ii) Suppose that H is homologically smooth. That is, H has a resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to H \to 0$$

such that each term is a finitely generated projective H^e-module. Following from Lemma 2.4 in [Berger and Taillefer 2007],

$$0 \to P_n \otimes_H {}_\varepsilon\Bbbk \to P_{n-1} \otimes_H {}_\varepsilon\Bbbk \to \cdots \to P_1 \otimes_H {}_\varepsilon\Bbbk \to P_0 \otimes_H {}_\varepsilon\Bbbk \to {}_\varepsilon\Bbbk \to 0$$

is a projective resolution of ${}_\varepsilon\Bbbk$. Clearly, it is a bounded projective resolution with each term finitely generated as left H-module.

(ii)⇒(i) View H^e as an H^e-H-bimodule via

$$a \otimes b \to x \otimes y = ax \otimes yb, \quad (x \otimes y) \leftarrow a = xa_1 \otimes S(a_2)y$$

for any $a \otimes b, x \otimes y \in H^e$ and $a \in H$. Let $H \otimes H_*$ be the free right H-module defined by multiplication to the second factor H. The morphism

$$H^e \to H \otimes H_*, \quad x \otimes y \mapsto x_2 y \otimes x_1$$

is an isomorphism of right H-modules with inverse

$$H \otimes H_* \to H^e, \quad x \otimes y \mapsto y_1 \otimes S(y_2)x.$$

That is, $H^e \cong H \otimes H_*$ as right H-modules. So the functor $H^e \otimes_H - : \operatorname{Mod} H \to \operatorname{Mod} H^e$ is exact. This functor clearly sends projective H-modules to projective H^e-modules. Moreover, $H^e \otimes {}_\varepsilon\Bbbk \cong H$ as left H^e-modules. The isomorphism $H^e \otimes {}_\varepsilon\Bbbk \to H$ is defined by $x \otimes y \mapsto xy$. Therefore, if the left trivial module ${}_\varepsilon\Bbbk$ admits a bounded projective resolution $\boldsymbol{Q_\bullet}$ with each term finitely generated, then $H^e \otimes_H \boldsymbol{Q_\bullet}$ is a bounded projective resolution of H over H^e with each term finitely generated. That is, H is homologically smooth. $\square$

Acknowledgement

The authors sincerely thank the referee for his/her valuable comments and suggestions that helped them to improve the paper quite a lot. Wang and Yu are grateful to the Department of Mathematics of Zhejiang Normal University for the hospitality they received during a visit in summer 2016. Wang is supported by AMS-Simons travel grant. Yu is supported by grants from NSFC (No. 11301126, No. 11571316, No. 11671351).

References

[Andruskiewitsch and Schneider 2002] N. Andruskiewitsch and H.-J. Schneider, "Finite quantum groups over abelian groups of prime exponent", *Ann. Sci. École Norm. Sup.* (4) **35**:1 (2002), 1–26. MR Zbl

[Andruskiewitsch and Schneider 2004] N. Andruskiewitsch and H.-J. Schneider, "A characterization of quantum groups", *J. Reine Angew. Math.* **577** (2004), 81–104. MR Zbl

[Artin and Schelter 1987] M. Artin and W. F. Schelter, "Graded algebras of global dimension 3", *Adv. in Math.* **66**:2 (1987), 171–216. MR Zbl

[Berger and Taillefer 2007] R. Berger and R. Taillefer, "Poincaré–Birkhoff–Witt deformations of Calabi–Yau algebras", *J. Noncommut. Geom.* **1**:2 (2007), 241–270. MR Zbl

[Bichon 2003] J. Bichon, "The representation category of the quantum group of a non-degenerate bilinear form", *Comm. Algebra* **31**:10 (2003), 4831–4851. MR Zbl

[Bichon 2013] J. Bichon, "Hochschild homology of Hopf algebras and free Yetter–Drinfeld resolutions of the counit", *Compos. Math.* **149**:4 (2013), 658–678. MR Zbl

[Bichon 2014] J. Bichon, "Hopf–Galois objects and cogroupoids", *Rev. Un. Mat. Argentina* **55**:2 (2014), 11–69. MR Zbl

[Bichon 2016] J. Bichon, "Gerstenhaber–Schack and Hochschild cohomologies of Hopf algebras", *Doc. Math.* **21** (2016), 955–986. MR Zbl

[Bieri and Eckmann 1973] R. Bieri and B. Eckmann, "Groups with homological duality generalizing Poincaré duality", *Invent. Math.* **20** (1973), 103–124. MR Zbl

[Brown and Goodearl 1997] K. A. Brown and K. R. Goodearl, "Homological aspects of Noetherian PI Hopf algebras and irreducible modules and maximal dimension", *J. Algebra* **198**:1 (1997), 240–265. MR Zbl

[Brown and Zhang 2008] K. A. Brown and J. J. Zhang, "Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras", *J. Algebra* **320**:5 (2008), 1814–1850. MR Zbl

[Cartan and Eilenberg 1956] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956. MR Zbl

[Chirvasitu et al. 2016] A. Chirvasitu, C. Walton, and X. Wang, "On quantum groups associated to a pair of preregular forms", preprint, 2016. arXiv

[Doi 1993] Y. Doi, "Braided bialgebras and quadratic bialgebras", *Comm. Algebra* **21**:5 (1993), 1731–1749. MR Zbl

[Dubois-Violette and Launer 1990] M. Dubois-Violette and G. Launer, "The quantum group of a nondegenerate bilinear form", *Phys. Lett. B* **245**:2 (1990), 175–177. MR Zbl

[Ginzburg 2007] V. Ginzburg, "Calabi–Yau algebras", preprint, 2007. arXiv

[Goodearl and Zhang 2007] K. R. Goodearl and J. J. Zhang, "Homological properties of quantized coordinate rings of semisimple groups", *Proc. Lond. Math. Soc.* (3) **94**:3 (2007), 647–671. MR Zbl

[Lorenz and Lorenz 1995] M. E. Lorenz and M. Lorenz, "On crossed products of Hopf algebras", *Proc. Amer. Math. Soc.* **123**:1 (1995), 33–38. MR Zbl

[Lu et al. 2007] D.-M. Lu, Q.-S. Wu, and J. J. Zhang, "Homological integral of Hopf algebras", *Trans. Amer. Math. Soc.* **359**:10 (2007), 4945–4975. MR Zbl

[Lusztig 1993] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics **110**, Birkhäuser, Boston, 1993. MR Zbl

[Manin 1988] Y. I. Manin, "Quantum groups and noncommutative geometry", lecture notes, Centre de Recherches Mathématiques, Université de Montréal, 1988. MR

[Mrozinski 2014] C. Mrozinski, "Quantum groups of GL(2) representation type", *J. Noncommut. Geom.* **8**:1 (2014), 107–140. MR Zbl

[Raedschelders and Van den Bergh 2015] T. Raedschelders and M. Van den Bergh, "The representation theory of noncommutative $\mathcal{O}(\mathrm{GL}_2)$", preprint, 2015. arXiv

[Raedschelders and Van den Bergh 2017] T. Raedschelders and M. Van den Bergh, "The Manin Hopf algebra of a Koszul Artin–Schelter regular algebra is quasi-hereditary", *Adv. Math.* **305** (2017), 601–660. MR Zbl

[Reyes et al. 2014] M. Reyes, D. Rogalski, and J. J. Zhang, "Skew Calabi–Yau algebras and homological identities", *Adv. Math.* **264** (2014), 308–354. MR Zbl

[Schauenburg 1996] P. Schauenburg, "Hopf bi-Galois extensions", *Comm. Algebra* **24**:12 (1996), 3797–3825. MR Zbl

[Shnider and Sternberg 1993] S. Shnider and S. Sternberg, *Quantum groups: from coalgebras to Drinfel'd algebras*, International Press, Cambridge, MA, 1993. MR Zbl

[Sweedler 1969] M. E. Sweedler, "Integrals for Hopf algebras", *Ann. of Math.* (2) **89** (1969), 323–335. MR Zbl

[Van den Bergh 1997] M. Van den Bergh, "Existence theorems for dualizing complexes over non-commutative graded and filtered rings", *J. Algebra* **195**:2 (1997), 662–679. MR Zbl

[Walton and Wang 2016] C. Walton and X. Wang, "On quantum groups associated to non-Noetherian regular algebras of dimension 2", *Math. Z.* **284**:1-2 (2016), 543–574. MR Zbl

[Weibel 1994] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge Univ. Press, 1994. MR Zbl

[Yu 2016] X. Yu, "Hopf–Galois objects of Calabi–Yau Hopf algebras", *J. Algebra Appl.* **15**:10 (2016), art. id. 1650194. MR Zbl

[Yu and Zhang 2013] X. Yu and Y. Zhang, "Calabi–Yau pointed Hopf algebras of finite Cartan type", *J. Noncommut. Geom.* **7**:4 (2013), 1105–1144. MR Zbl

[Yu et al. 2016] X. Yu, F. Van Oystaeyen, and Y. Zhang, "Cleft extensions of Koszul twisted Calabi–Yau algebras", *Israel J. Math.* **214**:2 (2016), 785–829. MR Zbl

[Zhang 1998] J. J. Zhang, "Non-Noetherian regular rings of dimension 2", *Proc. Amer. Math. Soc.* **126**:6 (1998), 1645–1653. MR Zbl

Received October 12, 2016. Revised February 17, 2017.

XINGTING WANG
DEPARTMENT OF MATHEMATICS
TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122
UNITED STATES

xingting@temple.edu

XIAOLAN YU
DEPARTMENT OF MATHEMATICS
HANGZHOU NORMAL UNIVERSITY
310036 HANGZHOU
CHINA

xlyu@hznu.edu.cn

YINHUO ZHANG
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF HASSELT
UNIVERSITAIRE CAMPUS
3590 DIEPEENBEEK
BELGIUM

yinhuo.zhang@uhasselt.be

CONTENTS

Volume 290, no. 1 and no. 2

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 290 No. 2 October 2017