Pacific Journal of Mathematics

CHAIN TRANSITIVE HOMEOMORPHISMS ON A SPACE: ALL OR NONE

ETHAN AKIN AND JUHO RAUTIO

Volume 291 No. 1 November 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 291, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.291.1

CHAIN TRANSITIVE HOMEOMORPHISMS ON A SPACE: ALL OR NONE

ETHAN AKIN AND JUHO RAUTIO

Extending earlier work, we consider when a compact metric space can be realized as the omega limit set of a discrete time dynamical system. This is equivalent to asking when the space admits a chain transitive homeomorphism. We approach this problem in terms of various conditions on the connected components of the space. We also construct spaces where all homeomorphisms are chain transitive.

1. Introduction

Since long-term behavior is a central concern in dynamical systems theory, it is natural to consider the set of limit points for the orbit of a point in the state space, obtained as time tends to infinity. This omega limit set was explicitly defined for a real flow by Birkhoff [1927] and appears as well in the classic book [Andronov and Khaikin 1937]. The discrete time version, i.e., for systems obtained by iterating a homeomorphism, was studied in detail by Dowker [1953] and with extensions to maps by Sharkovsky [1965].

We will use the term *space* to mean a nonempty, compact, metrizable space unless otherwise mentioned. When a metric is required, we assume that one is chosen and fixed. The results are independent of the choice of metric. Broadly, our metric space ideas and constructions are really uniform space concepts, and a compact space has a unique uniformity.

We will use $[a, b]$, (a, b), $[a, b)$, etc. to denote intervals in $\mathbb{R}$, and so we will use $\langle a, b\rangle$ to represent points of $\mathbb{R}^2$.

MSC2010: 37B20, 37B25.
Keywords: omega set, chain transitive homeomorphism, rigid space, Slovak space.

The dynamical systems we will consider are pairs (X, f), where $f : X \to X$ is a continuous map on a space X. If $x \in X$, then the sequence $x, f(x), f^2(x), \dots$ is the *trajectory* of x and $\omega f(x)$ is the set of limit points of the trajectory sequence, i.e., $y \in \omega f(x)$ if and only if $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$.

The question when a system (X, f) can be embedded as a subsystem of some (Y, g) so that $X = \omega g(y)$ for some $y \in Y$ was answered by Dowker and Friedlander [1954] for homeomorphisms and by Sharkovsky [1965] in general. A system (X, f) is *f-connected* if, for any proper, nonempty, closed subset $U \subset X$, the intersection $f(U) \cap \overline{X \setminus U}$ is nonempty. They show that (X, f) can be embedded as the omega limit set in some larger system if and only if it is f-connected.

At the space level, the question now arises when a space X admits a map f so that (X, f) is an omega limit set subsystem. From the above results, this asks when X admits a map f with respect to which (X, f) is f-connected. Such a space X is called an *orbit enclosing omega limit set* when there exists a map f on X such that $X = \omega f(x)$ for some $x \in X$.

Considerable work, initiated by Sharkovsky, has been done on the related question of characterizing which subsets X of the unit interval I are omega limit sets for some map on the interval; see [Agronsky et al. 1989/90]. Notice that if X is a finite union of intervals or a Cantor set, then X is an orbit-enclosing omega limit set for some map f on X, so any extension via the Tietze extension theorem to a map on I will suffice. The result is more delicate for a general closed nowhere dense subset of the interval; see the elegant exposition in [Bruckner and Smítal 1992]. Later, Kolyada and Snoha [1992/93] extended these results by showing that a subset X of the unit interval is an omega limit set (or, equivalently, admits a chain transitive map) if and only if X is not a disjoint union of a finite number of nondegenerate intervals and a nonempty, countable set with the distance from this set to at least one of the intervals positive. Thus, this work is the first to consider the main question we will be addressing. For an early summary, see [Sharkovsky et al. 1989].

Recall that a *Peano space* is a compact, connected, locally connected space or, equivalently, a continuous image of the unit interval. In a pair of papers, Agronsky and Ceder [1991/92a; 1991/92b] proved that if X has finitely many components and each is a nontrivial, finite-dimensional Peano space, then X is an orbit enclosing omega limit set.

Our purpose here is to consider the related problem of when a space X admits a homeomorphism f so that (X, f) is the omega limit set in a larger system. As we will see, the results are somewhat different from the map case. First, we reinterpret the problem.

Given $\epsilon \geq 0$, a finite or infinite sequence $\{x_n \in X\}$ with at least two terms is an *ϵ-chain* for (X, f) if $d(f(x_k), x_{k+1}) \leq \epsilon$ for all terms x_k of the sequence (except the last one). The system (X, f) is called *chain transitive* when every pair of points of X can be connected by some finite ϵ-chain for every positive ϵ. A subset $A \subset X$ is called a *chain transitive subset* when it is closed and f-invariant (i.e., $f(A) = A$) and the subsystem (A, f) is chain transitive.

It is well known that any omega limit set is a chain transitive subset; see, e.g., [Akin 1993, Proposition 4.14]. On the other hand, as observed by Takens, it is easy to show that if (X, f) is chain transitive, then it can be embedded in a larger system in which it is an omega limit set [Akin 1993, Exercise 4.29]. We will review the proofs in Section 3. The construction uses a subset Y of $X \times [0, 1]$. In particular, if $X \subset [0, 1]^n$ then $Y \subset [0, 1]^{n+1}$. Applying the Tietze extension theorem in each coordinate, we can extend g to a continuous map on all of $[0, 1]^{n+1}$ and so obtain X as the omega limit set for a system on $[0, 1]^{n+1}$. Note, however, that even if g is a homeomorphism, we might not be able to extend it to a homeomorphism on $[0, 1]^{n+1}$.

These results are really just a restatement of the theorem of Dowker and Friedlander [1954].

To clarify the relationship between chain transitivity and f-connectedness, we recall the concept of an *attractor*, as described by Conley [1978] and with detailed exposition in [Akin 1993]. Call a closed set $U \subset X$ an *inward set* for f if $f(U)$ is contained in the interior U° or, equivalently, $f(U) \cap \overline{X \setminus U} = \varnothing$. Thus, (X, f) admits a proper, nonempty, inward subset if and only if it is not f-connected. If U is an inward set, then $A = \bigcap_{n=0}^{\infty} f^n(U)$ is called the associated attractor. For a number of equivalent descriptions of an attractor, see [Akin 1993, Theorem 3.3]. Theorem 4.12 of that paper says that (X, f) is chain transitive if and only if X is the only nonempty attractor and if and only if X is the only nonempty inward set. It follows that chain transitivity and f-connectedness are equivalent concepts.

The label “attractor” has been used for other ideas. Some authors refer to all omega limit sets as attractors. An attractor as defined above need not be chain transitive, and so there are attractors which are not omega limit sets. A more reasonable definition is that a set A is an attractor for f when it is f-invariant and, for every x in some neighborhood of A, we have $\omega f(x) \subset A$. This condition is necessary in order that A be an attractor à la Conley, but it is not sufficient. If X is the one-point compactification of $\mathbb{Z}$ and f is the extension to X of translation by 1 on $\mathbb{Z}$, then (X, f) is chain transitive. On the other hand, the point at infinity is the omega limit set of every point. By Theorem 3.6(a) of [Akin 1993], an f-invariant subset A is an attractor if and only if $\{x : \omega f(x) \subset A\}$ is a neighborhood of A and, in addition, A is stable, i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that if x is δ-close to A, then the forward orbit of x remains ϵ-close to A.

So the question we will address is when a space admits a chain transitive homeomorphism.

For a dynamical system (X, f), there exists $x \in X$ such that $\omega f(x) = X$ exactly when the system is topologically transitive with a recurrent transitive point. Thus, asking when X is an orbit-enclosing omega limit set is asking exactly when X admits f so that (X, f) is topologically transitive in this sense.

The identity map 1_X on X is chain transitive if and only if X is connected. Thus, a connected space admits a chain transitive homeomorphism and so is an omega limit set.

To illustrate the difference between the original problem and the homeomorphism version, consider the *tent map* on $[0, 1]$ defined by

$$T(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 1 - 2t & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases} \tag{1-1}$$

which is well known to be topologically transitive. Now let $X_0 = [0, 1] \times \{0, 1\}$, and define f_0 on X_0 by

$$f_0(t, 0) = \langle t, 1\rangle \quad \text{and} \quad f_0(t, 1) = \langle T(t), 0\rangle, \tag{1-2}$$

so that $f_0^2 = T \times 1_{\{0,1\}}$.

Let (X, f) be the quotient system obtained by identifying the points $\langle 0, 1\rangle = \langle 1, 1\rangle$ in X. Thus, X is the disjoint union of a circle and an interval. It easily follows that X does not admit a chain transitive homeomorphism. On the other hand, f is a topologically transitive map.

Let (X_1, f_1) be the quotient system obtained from (X, f) by identifying the points $\langle 0, 0\rangle = \langle 1, 1\rangle$ in X_1. Thus, X_1 consists of a circle and an interval joined at a point. As X_1 is connected, the identity is a chain transitive homeomorphism. Since the points of the interval other than $\langle 1, 0\rangle$ separate the space and the points of the circle other than the intersection point with the interval do not, it easily follows that X_1 does not admit a topologically transitive homeomorphism. On the other hand, f_1 is a topologically transitive map.

If X contains a proper, clopen, nonempty, f-invariant set A, then we say that X is *f-decomposable*. In this case (X, f) is not chain transitive, for if ϵ is smaller than the distance from A to its complement, then any ϵ-chain which begins in A remains in A. With $H(X)$ the group of homeomorphisms on X, we say that X is *$H(X)$-decomposable* if there is a proper, clopen, nonempty subset A of X such that A is invariant for every homeomorphism on X. If X is not f-decomposable (or not $H(X)$-decomposable), we will call it *f-indecomposable* (resp. *$H(X)$-indecomposable*).

Clearly, if X is $H(X)$-decomposable, then it admits no chain transitive homeomorphism. For example, let $\mathrm{Iso}(X)$ denote the (possibly empty) set of isolated points of X. If the closure $\overline{\mathrm{Iso}(X)}$ is a proper, clopen, nonempty subset of X, then X is $H(X)$-decomposable and so admits no chain transitive homeomorphism. In the zero-dimensional case, this is the only obstruction. We prove a slightly more general result in Section 3.

Theorem 1.1. *If X is a space such that $\overline{\mathrm{Iso}(X)}$ is not a proper, clopen subset of X and such that the open set $X \setminus \overline{\mathrm{Iso}(X)}$ is empty or zero-dimensional, then X admits a chain transitive homeomorphism.*

Corollary 1.2. *If the isolated points are dense in X, then X admits a chain transitive homeomorphism.*

Thus, the problems which remain come from the nontrivial components. A clopen component is called an *isolated component*. Clearly, if the closure of the union of isolated components is a proper, clopen, nonempty subset of X, then X is $H(X)$-decomposable.

If $\mathcal{Y}$ is a set of connected spaces, let $C_{\mathcal{Y}}$ denote the closure of the union of those components of X which are homeomorphic to some element of $\mathcal{Y}$. If $C_{\mathcal{Y}}$ is a proper, clopen, nonempty subset, then X is $H(X)$-decomposable. For example, if X has finitely many components, then either they are all homeomorphic, in which case X admits a periodic chain transitive map, or X is $H(X)$-decomposable. Contrast this with the map result described above.

We say that X satisfies the *diameter condition on isolated components* if for every $\epsilon > 0$ there are only finitely many isolated components with diameter greater than ϵ.

Theorem 1.3. *If X satisfies the diameter condition and the union of the isolated components is dense in X, then either X is $H(X)$-decomposable or else X admits a chain transitive homeomorphism.*

The rest of Section 3 consists of counterexamples to reasonable conjectures. We construct:

- A space X that is $H(X)$-decomposable, with all components homeomorphic, and no isolated components.
- A space X that is $H(X)$-indecomposable but f-decomposable for every $f \in H(X)$. The space can be chosen with the isolated components all homeomorphic and with a dense union.
- A space X that is f-indecomposable for some $f \in H(X)$ but admits no chain transitive homeomorphism. The space can be chosen with the isolated components all homeomorphic and with a dense union.

These examples rule out the obvious extension of the above corollary. The isolated components can be dense and all homeomorphic to one another, but nonetheless the space admits no chain transitive homeomorphism.

Having considered when there are no chain transitive homeomorphisms, we consider in Section 4 this question: When is every homeomorphism on a space X chain transitive? For such a space X, the identity map 1_X is chain transitive, and so X must be connected.

In [de Groot and Wille 1958], rigid spaces were defined and Peano space examples were constructed. A space X is *rigid* if 1_X is the only homeomorphism on X, i.e., the homeomorphism group $H(X)$ is trivial. For a connected rigid space, it is trivially true that all homeomorphisms are chain transitive.

Using rigid spaces one can construct more interesting examples. Following [de Groot 1959], we can begin with a finitely generated group G and use rigid spaces instead of intervals as edges in the Cayley graph. If X is the one-point compactification of this fattened Cayley graph, then $H(X)$ is isomorphic to G and every homeomorphism is chain transitive. We obtain examples with nondiscrete homeomorphism group and even with the path components nontrivial.

In these cases, the homeomorphism group does not act in a topologically transitive manner on the space. Distinct points in each rigid piece are homeomorphically distinct. It is possible to obtain examples with all homeomorphisms chain transitive and with the homeomorphism group acting in a topologically transitive manner. These are built using the recent, beautiful construction in [Downarowicz et al. 2017] of Slovak spaces. A *Slovak space* X has $H(X)$ isomorphic to $\mathbb{Z}$ and every element in it other than the identity 1_X acts minimally on X. We call a space *Slovakian* if every homeomorphism other than 1_X is topologically transitive. Using such Slovakian spaces we construct a space X such that $H(X)$ is topologically transitive on X, every element of $H(X)$ is chain transitive, and the homeomorphism group of the Cantor set occurs as a closed, topological subgroup of $H(X)$.

2. Relation dynamics

It will be convenient to use the dynamics of closed relations, and so we briefly review the ideas from [Akin 1993]. Recall that our spaces X, Y, etc. are assumed to be nonempty, compact, metrizable spaces with a fixed metric chosen when necessary.

For spaces X, Y, a relation $R : X \to Y$ is a subset of $X \times Y$. The set R is a *relation on* X when $Y = X$. A map is a relation such that $R(x) = \{y : (x, y) \in R\}$ is a singleton set for every $x \in X$. Notice that we are following the set theory convention for which a map is the set sometimes referred to as the graph of the map. Thus, for example, the identity map 1_X is the diagonal set $\{(x, x) : x \in X\}$.

For $A \subset X$, the *image* $R(A)$ is defined to be $\bigcup_{x\in A} R(x)$. Equivalently, $R(A)$ is the projection to Y of $R \cap (A \times Y) \subset X \times Y$. The inverse $R^{-1} : Y \to X$ is defined to be $\{(y, x) : (x, y) \in R\}$. For $B \subset Y$, we let $R^*(B) = \{x \in X : R(x) \subset B\} = X \setminus R^{-1}(Y \setminus B)$. So $R^*(B) \subset R^{-1}(B) \cup R^*(\varnothing)$. If R is a map, then $R^*(B) = R^{-1}(B)$.

For example,

$$\overline{V}_\epsilon = \{(x, y) \in X \times X : d(x, y) \le \epsilon\}$$

is a relation on X with $\overline{V}_\epsilon(x)$ the closed ball of radius ϵ and center x. When $\epsilon = 0$, $\overline{V}_\epsilon = 1_X$, the identity map on X.

If $R : X \to Y$ and $S : Y \to C$ are relations, then the composition $S \circ R : X \to C$ is the image under the projection to $X \times C$ of the set $(R \times C) \cap (X \times S) \subset X \times Y \times C$. Thus, $(x, c) \in S \circ R$ if and only if there exists $y \in Y$ such that $(x, y) \in R$ and $(y, c) \in S$. Composition is associative, and $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

A relation R on X is *reflexive* when $1_X \subset R$, *symmetric* when $R^{-1} = R$, and *transitive* when $R \circ R \subset R$.

For a relation R on X, we let $R^{n+1} = R^n \circ R$ and $R^{-n} = (R^{-1})^n$ for $n = 1, 2, \dots$, and let R^0 be the identity 1_X. We define the *cyclic set* $|R| = \{x : (x, x) \in R\}$.

For a relation R on X a subset A of X is called *forward R-invariant* (or *R-invariant*) if $R(A) \subset A$ (resp. $R(A) = A$).

For a relation R on X, the *orbit relation* is $\mathcal{O}R = \bigcup_{n=1}^{\infty} R^n$, and the *orbit closure relation* $\mathcal{R}R$ is defined by $\mathcal{R}R(x) = \overline{\mathcal{O}R(x)}$ for all $x \in X$, so that $\mathcal{R}R = \{(x, y) : x \in X,\ y \in \overline{\mathcal{O}R(x)}\}$. The *wandering relation* is $\mathcal{N}R = \overline{\mathcal{O}R}$. Even when R is a continuous map, $\mathcal{R}R$ is usually not closed and so is a proper subset of $\mathcal{N}R$.

The *chain relation* is

$$\mathcal{C}R = \bigcap_{\epsilon > 0} \mathcal{O}(\overline{V}_\epsilon \circ R \circ \overline{V}_\epsilon). \tag{2-1}$$

Both $\mathcal{O}R$ and $\mathcal{C}R$ are transitive relations. Since $(R^n)^{-1} = (R^{-1})^n$, it follows that $\mathcal{O}(R^{-1}) = (\mathcal{O}R)^{-1}$, $\mathcal{N}(R^{-1}) = (\mathcal{N}R)^{-1}$ and $\mathcal{C}(R^{-1}) = (\mathcal{C}R)^{-1}$. These operators on relations are monotone, i.e., they preserve inclusions, and $\mathcal{O}$ and $\mathcal{C}$ are idempotent. That is,

$$\mathcal{O}(\mathcal{O}R) = \mathcal{O}R \quad \text{and} \quad \mathcal{C}(\mathcal{C}R) = \mathcal{C}R. \tag{2-2}$$

It then follows that

$$\begin{aligned} \mathcal{R}(\mathcal{O}R)(x) &= \overline{\mathcal{O}(\mathcal{O}R)(x)} = \overline{\mathcal{O}R(x)} = \mathcal{R}R(x), \\ \mathcal{N}(\mathcal{O}R) &= \overline{\mathcal{O}(\mathcal{O}R)} = \overline{\mathcal{O}R} = \mathcal{N}R, \\ \mathcal{C}R &\subset \mathcal{C}(\mathcal{O}R) \subset \mathcal{C}(\mathcal{C}R) = \mathcal{C}R. \end{aligned} \tag{2-3}$$

If R is a transitive relation on X, then $R \cap R^{-1}$ is a symmetric, transitive relation which restricts to an equivalence relation on $|R|$. We call the equivalence classes the *basic sets* of R.

A *closed relation* R is a closed subset of $X \times Y$. If R is a map, then it is continuous if and only if it is a closed relation, i.e., its graph is a closed set.

The composition of closed relations is closed, and the image of a closed set by a closed relation is closed. So if B is open in Y, then $R^*(B)$ is open in X. For any relation R, the extensions $\mathcal{N}R$ and $\mathcal{C}R$ are closed relations.

It is easy to see that $\mathcal{C}R = \mathcal{C}\overline{R}$, where $\overline{R}$ is the closure of R. If R is a closed relation, then $\mathcal{C}R = \bigcap_{\epsilon > 0} \mathcal{O}(\overline{V}_\epsilon \circ R)$; see [Akin 1993, Proposition 1.18]. If R is a closed relation on X, then the cyclic set $|R|$ is a closed subset of X. If R is a closed, transitive relation, the basic sets $\{R(x) \cap R^{-1}(x) : x \in |R|\}$ are closed.

For a sequence $\{A_n\}$ of closed sets, $\limsup\{A_n\} = \bigcap_n \overline{\bigcup_{k \geq n} A_k}$. We have the identity $\overline{\bigcup_n A_n} = \left(\bigcup_n A_n\right) \cup \limsup\{A_n\}$. If R is a closed relation on X and A is a closed subset of X, then we define $\omega R[A] = \limsup\{R^n(A)\}$.

If A is forward R-invariant and R is closed, then the sequence of closed sets $\{R^n(A)\}$ is decreasing and $\omega R[A]$ is the intersection. Furthermore, if $y \in \omega R[A]$, then $R^{-1}(y)$ meets every $\{R^n(A)\}$, and so by compactness it meets $\omega R[A]$ itself. It follows that $\omega R[A]$ is the maximum R-invariant subset of the closed, forward R-invariant set A.

If R is closed, then for $x \in X$ we let

$$\omega R(x) = \omega R[x] = \limsup\{R^n(x)\},$$

defining the *omega limit set relation* ωR on X. We also define $\Omega R = \limsup\{R^n\}$ for a general relation R, so ΩR is a closed relation. When R is closed, we have the identities

$$\mathcal{R}R = \mathcal{O}R \cup \omega R \quad \text{and} \quad \mathcal{N}R = \mathcal{O}R \cup \Omega R. \tag{2-4}$$

Since $\mathcal{C}R$ is closed and transitive, the sequence $\{(\mathcal{C}R)^n\}$ is decreasing with intersection $\Omega\mathcal{C}R = \omega\mathcal{C}R$.

If R is closed, then the following useful identities hold for the chain relation:

$$\begin{gathered} \mathcal{C}R = \mathcal{O}R \cup \Omega\mathcal{C}R, \\ R \cup ((\mathcal{C}R) \circ R) = \mathcal{C}R = R \cup (R \circ (\mathcal{C}R)); \end{gathered} \tag{2-5}$$

see [Akin 1993, Proposition 2.4(c), Proposition 1.11(d)].

It is not usually true that $(\omega R)^{-1} = \omega(R^{-1})$. We write αR for $\omega(R^{-1})$, defining the *alpha limit set relation*.

The points of $|\mathcal{O}R|$ are called *periodic points* for R, $|\mathcal{R}R|$ are the *recurrent points*, $|\mathcal{N}R|$ are the *nonwandering points*, and $|\mathcal{C}R|$ are the *chain recurrent points*. When R is a closed relation, we can apply the inclusion $|\mathcal{O}R| \subset |\omega R|$ to (2-4) and (2-5) to obtain $|\mathcal{R}R| = |\omega R|$, $|\mathcal{N}R| = |\Omega R|$ and $|\mathcal{C}R| = |\Omega\mathcal{C}R|$. We call x a *transitive point* for R when $\mathcal{R}R(x) = X$. We denote by $\mathrm{Trans}(R)$ the (possibly empty) set of transitive points.

On $|\mathcal{C}R|$, $\mathcal{C}R \cap \mathcal{C}R^{-1}$ is a closed equivalence relation. We call the equivalence classes, i.e., the basic sets for $\mathcal{C}R$, the *chain components* of R.

For a relation R on X, we will say that R is *minimal* when $\mathcal{R}R = X \times X$, *topologically transitive* when $\mathcal{N}R = X \times X$, and *chain transitive* when $\mathcal{C}R = X \times X$. We call R *central* when $|\mathcal{N}R| = X$ and *chain recurrent* when $|\mathcal{C}R| = X$. From (2-3) it follows that any one of these properties holds for R if and only if it holds for $\mathcal{O}R$.

Notice that if R is minimal, then X contains no proper closed, forward R-invariant subset. If R is a continuous map, then the converse is true as well since $\omega R(x)$ is then R-invariant and nonempty for every x. Thus, if R is a continuous map, it is minimal if and only if $X = \text{Trans}(R)$.

A set U is *inward* for a closed relation R on X if U is closed and $R(U) \subset U^\circ$, the interior of U. Since $R(U)$ has a positive distance from the complement of U, it easily follows that U is inward for $\mathcal{C}R$ when it is inward for R. An inward set is therefore forward $\mathcal{C}R$-invariant. In general, a closed set A is forward $\mathcal{C}R$-invariant if and only if it has a neighborhood base consisting of inward sets; see [Akin 1993, Theorem 3.3(c)]. If U is an inward set, then $A_+ = \omega R[U] = \bigcap_{n\in\mathbb{N}} R^n(U)$ is the associated *attractor*. The set $X \setminus U^\circ$ is inward for R^{-1}, and $A_- = \omega(R^{-1})[X \setminus U^\circ]$ is the *dual repeller* for R. The pair A_+, A_- is called an *attractor-repeller pair*. If $x \in X \setminus (A_+ \cup A_-)$, then $\Omega\mathcal{C}R(x) \subset A_+$ and $\Omega\mathcal{C}R^{-1}(x) \subset A_-$; see [Akin 1993, Proposition 3.9].

Notice that a clopen set is inward if and only if it is forward invariant. If an inward set is invariant, then it is clopen.

Proposition 2.1. *Let R be a closed relation on X.*

(a) *If A_+, A_- is an attractor-repeller pair, then $|\mathcal{C}R| \subset A_+ \cup A_-$ and the intersections $A_+ \cap |\mathcal{C}R|$ and $A_- \cap |\mathcal{C}R|$ are each clopen in $|\mathcal{C}R|$.*

(b) *If $x \in X \setminus |\mathcal{C}R|$, then there exists an attractor-repeller pair A_+, A_- such that $x \notin A_+ \cup A_-$.*

(c) *If $x, y \in |\mathcal{C}R|$, then $y \in \mathcal{C}R(x)$ if and only if, for all attractors A, $x \in A$ implies $y \in A$.*

(d) *The space of chain components, i.e., the quotient space of $|\mathcal{C}R|$ by the equivalence relation $\mathcal{C}R \cap \mathcal{C}R^{-1}$, is a compact zero-dimensional metric space.*

Proof. (a) If $x \in X \setminus (A_+ \cup A_-)$, then $\Omega\mathcal{C}R(x) \subset A_+$, and so $x \notin \Omega\mathcal{C}R(x)$. If U is an inward set with $\omega R[U] = A_+$, then $U^\circ \cap |\mathcal{C}R| = A_+ \cap |\mathcal{C}R|$.

(b), (c) These are part of Proposition 3.11 of [Akin 1993].

(d) From (b) and (c) applied to R and R^{-1} it follows that every chain component C is the intersection of $\{A \cap |\mathcal{C}R| : A$ is an attractor or repeller and $C \subset A\}$. Hence, the attractors and repellers induce a clopen subbase on the space of chain components. □

For a relation R on a space X we say that X is *R-decomposable* if there is a proper, forward R-invariant decomposition $\{A_1, A_2\}$ of X, i.e., each is a nonempty, closed, forward R-invariant subset and $A_1 \cap A_2 = \varnothing$, $A_1 \cup A_2 = X$. Such a pair of proper, clopen sets is called an *R-decomposition*. Since the sets are clopen and forward R-invariant, they are inward for R. Hence, an R-decomposition is a $\mathcal{C}R$-decomposition. So the notion of decomposability is the same for R, $\mathcal{O}R$, $\mathcal{R}R$, $\mathcal{N}R$ and $\mathcal{C}R$. If no such decomposition exists, X is said to be *R-indecomposable*.

For any relation R on X let $R_\pm = R \cup 1_X \cup R^{-1}$. This is a reflexive and symmetric relation on X so that $\mathcal{O}(R_\pm)$ and $\mathcal{C}(R_\pm)$ are equivalence relations on X.

Proposition 2.2. *For a relation R on X, the following conditions are equivalent*:

(i) *The space X is R-indecomposable.*

(ii) *The space X is $R_\pm$-indecomposable.*

(iii) *The relation $R_\pm$ is chain transitive.*

Proof. (i) $\Leftrightarrow$ (ii): If a set is forward R-invariant, then its complement is forward R^{-1}-invariant. Hence, an R-decomposition is an $R_\pm$-decomposition. Clearly, the reverse is true since $R \subset R_\pm$.

(iii) $\Rightarrow$ (ii): If $\{A_1, A_2\}$ is a decomposition, then each is inward for $R_\pm$ and therefore forward $\mathcal{C}R_\pm$-invariant. Hence, $R_\pm$ is not chain transitive.

(ii) $\Rightarrow$ (iii): If $R_\pm$ is not chain transitive, the equivalence relation $\mathcal{C}R_\pm$ on X has more than one equivalence class. By Proposition 2.1(d) the space of equivalence classes is zero-dimensional. Hence, there is a pair of disjoint, nonempty, clopen sets $\{A_1, A_2\}$ which cover X such that each is a union of equivalence classes. Since each equivalence class is $\mathcal{C}R_\pm$-invariant, it follows that $\{A_1, A_2\}$ is a $R_\pm$-decomposition. $\square$

Definition 2.3. Let $\pi : X_1 \to X_2$ be a continuous map between spaces, and let R_j be a relation on X_j for $j = 1, 2$. We say that π *maps R_1 to R_2* if $\pi \circ R_1 \subset R_2 \circ \pi$ and that π *is a semiconjugacy from R_1 to R_2* if $\pi \circ R_1 = R_2 \circ \pi$.

Proposition 2.4. *Let $\pi : X_1 \to X_2$ be a continuous map between spaces*, *and let R_j be a relation on X_j for $j = 1, 2$.*

(a) *The function π maps R_1 to R_2 if and only if $(\pi \times \pi)(R_1) \subset R_2$. If π is a semiconjugacy from R_1 to R_2 and $\pi(X_1) = X_2$, i.e., π is surjective, then $(\pi \times \pi)(R_1) = R_2$.*

(b) *If R_j is a mapping for $j = 1, 2$, then π is a semiconjugacy from R_1 to R_2 if it maps R_1 to R_2.*

(c) *If π maps R_1 to R_2, then π maps R_1^n to R_2^n for all $n \in \mathbb{Z}$ and maps $\mathcal{A}R_1$ to $\mathcal{A}R_2$ for $\mathcal{A} = \mathcal{O}, \mathcal{R}, \mathcal{N}$ and $\mathcal{C}$.*

(d) *Assume π is surjective and maps R_1 to R_2. If R_1 is minimal*, *topologically transitive*, *central*, *chain transitive or chain recurrent*, *then R_2 satisfies the corresponding property.*

(e) *If π is a semiconjugacy from R_1 to R_2 and π is an open map*, *then π is a semiconjugacy from $\mathcal{A}R_1$ to $\mathcal{A}R_2$ for $\mathcal{A} = \mathcal{O}, \mathcal{R}, \mathcal{N}$ and $\mathcal{C}$.*

Proof. (a) Firstly, π maps the relation R_1 to R_2 if and only if $(x, y) \in R_1$ implies $(\pi(x), \pi(y)) \in R_2$, and so the first equivalence is clear. The second result is easy to check.

(b) An inclusion between functions is an equation.

(c) Given $\epsilon > 0$, there exists $\delta > 0$ such that $(\pi \times \pi)(\overline{V}_\delta) \subset V_\epsilon$ since π is uniformly continuous. If $\{(x_n, y_n) \in R_1\}$ is a finite or infinite sequence with $(y_n, x_{n+1}) \in \overline{V}_\delta$, then $\{(\pi(x_n), \pi(y_n)) \in R_2\}$ with $(\pi(y_n), \pi(x_{n+1})) \in \overline{V}_\epsilon$. That is, δ-chains for R_1 are mapped to ϵ-chains for R_2. It follows that π maps $\mathcal{C}R_1$ to $\mathcal{C}R_2$. The others are easy to check.

(d) If π is surjective and maps $\mathcal{A}R_1$ to $\mathcal{A}R_2$, then $X_1 \times X_1 = \mathcal{A}R_1$ implies $X_2 \times X_2 = \mathcal{A}R_2$, and $1_{X_1} \subset \mathcal{A}R_1$ implies $1_{X_2} \subset \mathcal{A}R_2$.

(e) The function $1_{X_1} \times \pi$ maps 1_{X_1} to $\pi \subset X_1 \times X_2$. If π is open and $\epsilon > 0$, then $1_{X_1} \times \pi$ maps $\overline{V}_\epsilon$ to a neighborhood of π and so contains $\overline{V}_\delta \circ \pi$ for some $\delta > 0$. That is, $\overline{V}_\delta \circ \pi \subset \pi \circ \overline{V}_\epsilon$. If $\{(u_n, v_n) \in R_2\}$ is a finite or infinite sequence with $(v_n, u_{n+1}) \in \overline{V}_\delta$ and $\pi(x_n) = u_n$, then because π is a semiconjugacy on R_1, there exists $y_n \in X_1$ such that $(x_n, y_n) \in R_1$ and $\pi(y_n) = v_n$, and there exists x_{n+1} such that $(y_n, x_{n+1}) \in \overline{V}_\epsilon$ and $\pi(x_{n+1}) = u_{n+1}$. Thus, every δ-chain for R_2 can be lifted to an ϵ-chain for R_1 with a given initial lift. That is, π is a semiconjugacy from $\mathcal{C}R_1$ to $\mathcal{C}R_2$. The cases of $\mathcal{A} = \mathcal{O}, \mathcal{R}$ and $\mathcal{N}$ are similar. In fact, for $\mathcal{O}$ and $\mathcal{R}$ it is not necessary that the map be open. □

Remark. Suppose $R_j = 1_{X_j}$ with $X_2 = [0, 1]$ and $X_1 = \{-1\} \cup [0, 1]$, and let $\pi : X_1 \to X_2$ be an extension of the identity on $[0, 1]$, mapping -1 to some point of $[0, 1]$. The map π is a semiconjugacy from R_1 to R_2 and maps $\mathcal{C}R_1$ onto $\mathcal{C}R_2$, but it is not a semiconjugacy from $\mathcal{C}R_1$ to $\mathcal{C}R_2$.

As indicated in the Introduction, our concern is with homeomorphisms. However, we will apply this machinery to three different relations.

Let $H(X)$ be the homeomorphism group of X. For $f \in H(X)$, we define $f_\pm$ as the closed relation $f \cup 1_X \cup f^{-1}$. Clearly, we have

$$\begin{aligned}
\mathcal{O}f &= \{f^n : n \in \mathbb{N}\}, \\
\mathcal{O}f_\pm &= \{f^n : n \in \mathbb{Z}\} = \mathcal{O}f \cup 1_X \cup \mathcal{O}f^{-1}, \\
\mathcal{R}f(x) &= \overline{\{f^n(x) : n \in \mathbb{N}\}}, \\
\mathcal{R}f_\pm(x) &= \overline{\{f^n(x) : n \in \mathbb{Z}\}} \quad \text{for } x \in X, \\
\mathcal{N}f_\pm &= \mathcal{N}f \cup 1_X \cup \mathcal{N}f^{-1}, \\
\mathcal{R}f_\pm &= \mathcal{R}f \cup 1_X \cup \mathcal{R}(f^{-1}).
\end{aligned} \tag{2-6}$$

On the other hand, $\mathcal{C}f_\pm$ is in general larger than $\mathcal{C}f \cup 1_X \cup \mathcal{C}f^{-1}$. The latter relation need not be transitive. See Example 2.8 below.

Since $1_X \subset \mathcal{O}f_\pm$, every point is recurrent for $f_\pm$.

An action of a group G on X is a homomorphism $\rho : G \to H(X)$. If G is a subgroup of $H(X)$, then G acts on X by evaluation. That is, ρ is the inclusion.

With the action understood we let $h_G = \bigcup\{f \in \rho(G)\}$. This is just the orbit relation of the action of G on X. It is an equivalence relation but usually not closed. Since it is an equivalence relation, $\mathcal{O}h_G = h_G$ and $\mathcal{N}h_G = \overline{h_G}$. Since h_G is reflexive, every point is recurrent for h_G.

If G is the cyclic group generated by a homeomorphism f, then $h_G = \mathcal{O}f_{\pm}$.

When $G = H(X)$, we write h_X for h_G. Clearly, $H(X)$-decomposability as described in the Introduction is just h_X-decomposability.

For a space X, let $\mathrm{Iso}(X)$ denote the set of isolated points of X.

Proposition 2.5. *Let $f \in H(X)$ and $\rho : G \to H(X)$ be an action of a group G on X. Let $\mathcal{B}$ be a countable base of nonempty open sets for X, and let $\mathcal{B}^*$ be the collection of finite covers of X by elements of $\mathcal{B}$.*

(a) *The homeomorphism f is central if and only if the G_δ set of recurrent points, $|\omega f|$, is dense:*

$$|\omega f| = \bigcap_{\mathcal{A}\in\mathcal{B}^*,\, n\in\mathbb{N}} \ \bigcup_{U\in\mathcal{A},\, i\geq n} \{U \cap f^{-i}(U)\}. \tag{2-7}$$

If f is central, then any isolated point x of X is a periodic point for f.

(b) *The homeomorphism f is topologically transitive if and only if the set of transitive points,* $\mathrm{Trans}(f)$, *is nonempty, in which case*

$$\mathrm{Trans}(f) = \{x : \omega f(x) = X\} \tag{2-8}$$

$$= \bigcap_{U\in\mathcal{B},\, n\in\mathbb{N}} \ \bigcup_{i\geq n} \{f^{-i}(U)\} \tag{2-9}$$

is a dense G_δ subset of X. If f is topologically transitive, then f^{-1} is topologically transitive. If f is topologically transitive, then X is perfect or consists of a single periodic orbit for f.

(c) *The relation $f_{\pm}$ is topologically transitive if and only if the set of transitive points,* $\mathrm{Trans}(f_{\pm})$, *is nonempty, in which case*

$$\mathrm{Trans}(f_{\pm}) = \bigcap_{U\in\mathcal{B}} \ \bigcup_{n\in\mathbb{Z}} \{f^{-n}(U)\} \tag{2-10}$$

is a dense G_δ subset of X. If $f_{\pm}$ is topologically transitive and x is an isolated point of X, then

$$\mathrm{Trans}(f_{\pm}) = \mathcal{O}f_{\pm}(x) = \mathrm{Iso}(X). \tag{2-11}$$

Thus, if $f_{\pm}$ is topologically transitive, then either $\mathrm{Iso}(X) = \emptyset$, *so X is perfect, or else* $\mathrm{Iso}(X)$ *is dense. If* $\mathrm{Iso}(X)$ *is finite and nonempty, then X consists of a single periodic orbit.*

(d) *The following are equivalent*:

(i) f *is topologically transitive*;

(ii) $f_\pm$ *is topologically transitive and* f *is central*;

(iii) $f_\pm$ *is topologically transitive and* X *is either perfect or consists of a single periodic orbit for* f.

In that case, $\mathrm{Trans}(f_\pm) = \mathrm{Trans}(f) \cup \mathrm{Trans}(f^{-1})$.

(e) *The relation* h_G *is topologically transitive if and only if the set of transitive points*, $\mathrm{Trans}(h_G)$, *is nonempty, in which case*

$$\mathrm{Trans}(h_G) = \bigcap_{U \in \mathcal{B}} \bigcup_{f \in G} \{f^{-1}(U)\} \tag{2-12}$$

is a dense G_δ *subset of* X.

Proof. For a relation R on X and subsets $U, V \subset X$, let

$$\begin{aligned} N_R(U, V) &= \{n \in \mathbb{N} : R^n(U) \cap V \neq \varnothing\} \\ &= \{n \in \mathbb{N} : U \cap R^{-n}(V) \neq \varnothing\}. \end{aligned} \tag{2-13}$$

Clearly, R is central if and only if $N_R(U, U) \neq \varnothing$ for all nonempty open subsets U, and R is topologically transitive if and only if $N_R(U, V) \neq \varnothing$ for all nonempty open subsets U, V. If $R = f \in H(X)$ and $N_f(U, V)$ is finite for some open U, V, then with $n = \max N_f(U, V)$ we let $W = U \cap f^{-n}(V)$. Then W is a nonempty open set and $N_f(W, V) = \varnothing$. Hence, if f is central, then $N_f(U, U)$ is infinite, and if f is topologically transitive, then $N_f(U, V)$ is infinite for all nonempty open U, V.

The equations (2-7)–(2-13) are easy to check, and density follows from the Baire category theorem.

Now assume that $f_\pm$ is topologically transitive and that x is an isolated point of X. If V is a nonempty open set, then $N_{f_\pm}(\{x\}, V) = \varnothing$ unless V meets the orbit $\mathcal{O} f_\pm(x)$. Hence, $x \in \mathrm{Trans}(f_\pm)$. If y is another isolated point, then $N_{f_\pm}(\{x\}, \{y\}) \neq \varnothing$ implies that y is in the $f_\pm$ orbit of x. Thus, (2-11) holds. In particular, $\mathrm{Iso}(X)$ is dense if it is nonempty. If it is finite and nonempty, then $X = \mathrm{Iso}(X)$ because the latter is closed and dense. Since the elements of $\mathrm{Iso}(X)$ lie on a single orbit, X consists of a single periodic orbit.

If f is central and $x \in \mathrm{Iso}(X)$, then $N_f(\{x\}, \{x\}) \neq \varnothing$ if and only if x is a periodic point. Thus, if $f_\pm$ is transitive and f is central, then either X is perfect or it consists of a single periodic orbit, proving the implication (ii) $\Rightarrow$ (iii) in (d). Since (i) obviously implies (ii) in (d), it follows that if f is topologically transitive, then either X is perfect or it consists of a single periodic orbit. So if $\mathcal{R} f(x) = X$, then $\overline{\{f^k(x) : k \geq n\}} = X$ for every $n \in \mathbb{N}$. Intersecting, we see that $\omega f(x) = X$. Thus, $\mathrm{Trans}(f) = \{x : \omega f(x) = X\}$.

Finally, if X is a single periodic orbit, then f is topologically transitive and every point is a transitive point for f and f^{-1}. Now assume that X is perfect and that $f_{\pm}$ is transitive. If $x \in \text{Trans}(f_{\pm})$, then $\overline{\{f^k(x) : k \in \mathbb{Z},\ |k| \geq n\}} = X$ for all $n \in \mathbb{N}$. Intersecting, we obtain that $\omega f(x) \cup \alpha f(x) = X$. In particular, x is in one of these. If x is contained in a closed, invariant set A like $\omega f(x)$ or $\alpha f(x)$, then $X = \mathcal{R}(f_{\pm})(x) \subset A$. Thus, $x \in \text{Trans}(f) \cup \text{Trans}(f^{-1})$, so either f or f^{-1} is topologically transitive. But $\mathcal{N}(f^{-1}) = (\mathcal{N}f)^{-1}$ implies that f^{-1} is topologically transitive if f is. □

Remark. There are various, slightly conflicting, definitions of topological transitivity. These are sorted out in [Akin and Carlson 2012]. We are following [Akin 1993].

Lemma 2.6. (a) *Let $f \in H(X)$ and $x \in X$. The sets $\omega f(x)$ and $\alpha f(x)$ are f-invariant. If $x \in |\mathcal{C}f|$, then $\mathcal{C}f(x)$ and $\mathcal{C}f^{-1}(x)$ are f-invariant.*

(b) *The chain components of 1_X are the components of X.*

Proof. (a) For a bijection f on X, a subset A is f-invariant if and only if it is f^{-1}-invariant if and only if it is forward invariant for both f and f^{-1}.

When f is a continuous map on X, then $y \in \omega f(x)$ when there is a subsequence $\{f^{n_i}(x)\}$ of the orbit sequence which converges to y. Then $\{f^{n_i+1}(x)\}$ converges to $f(y)$, and if a subsequence of $\{f^{n_i-1}(x)\}$ converges to z, then $f(z) = y$. That is, $\omega f(x)$ is a nonempty, closed, f-invariant subset of X when f is a continuous map on X. So when f is a homeomorphism, the same is true for $\alpha f(x)$.

For $f \in H(X)$ we obtain from the identity (2-5) that $1_X \cup \mathcal{C}f = f^{-1} \circ \mathcal{C}f$. If $x \in \mathcal{C}f(x)$, then $\mathcal{C}f(x) = \{x\} \cup \mathcal{C}f(x) = f^{-1}(\mathcal{C}f(x))$. That is, $\mathcal{C}f(x)$ is f^{-1}-invariant and so is f-invariant. Since $\mathcal{C}(f^{-1}) = (\mathcal{C}f)^{-1}$ the same is true for $\mathcal{C}f^{-1}(x)$.

(b) The space of chain components being zero-dimensional by Proposition 2.1(d), every connected set contained in $|\mathcal{C}R|$ is entirely contained in a single chain component. In particular, when R is 1_X, every component is a subset of a chain component. On the other hand, when $R = 1_X$, every clopen subset is inward for R and so contains any chain component that it meets. It follows that the components are the chain components. □

Proposition 2.7. *Let $f \in H(X)$ and $\rho : G \to H(X)$ be an action of a group G on X.*

(a) *The following are equivalent*:
  (i) *X is f-indecomposable*;
  (ii) *X is $f_{\pm}$-indecomposable*;
  (iii) *$f_{\pm}$ is chain transitive.*

(b) *The relation h_G is chain transitive if and only if h_G is indecomposable.*

(c) *If f is chain recurrent*, *then $\mathcal{C}f = \mathcal{C}(f_{\pm})$.*

(d) *The following are equivalent*:

(i) f *is chain transitive*;

(ii) f *is chain recurrent and* $f_\pm$ *is chain transitive*;

(iii) f *is chain recurrent and indecomposable.*

(e) *If* X *is connected*, *then* f *is chain transitive if and only if it is chain recurrent. In particular*, 1_X *is chain transitive.*

Proof. (a), (b) Since $h_G = (h_G)_\pm$, these results are a special case of Proposition 2.2 applied with $R = f$ and $R = h_G$.

(c) If f is chain recurrent, then $|\mathcal{C}f| = X$ and $1_X \subset \mathcal{C}f$. For any $x \in X$, we have $x \in \mathcal{C}f(x)$, and this set is f-invariant by Lemma 2.6. Hence, $f^{-1}(x) \in \mathcal{C}f(x)$. Thus, $f^{-1} \subset \mathcal{C}f$. Since $f \subset f_\pm \subset \mathcal{C}f$, it follows from (2-2) that $\mathcal{C}f \subset \mathcal{C}(f_\pm) \subset \mathcal{C}\mathcal{C}f = \mathcal{C}f$.

(d) If f is chain transitive, then it is clearly chain recurrent, and $f \subset f_\pm$ implies that $f_\pm$ is chain transitive. The converse follows from (c). This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from (a).

(e) If X is connected, then by Lemma 2.6(b) X consists of a single chain component for 1_X, and so 1_X is chain transitive. So if f is chain recurrent, then $X \times X = \mathcal{C}1_X \subset \mathcal{C}\mathcal{C}f = \mathcal{C}f$ by (2-2) again. □

Example 2.8. On $\mathbb{Z}$ let t be the translation bijection given by $n \mapsto n+1$. Let $\mathbb{Z}^*$ be the one-point compactification adjoining the point $\pm\infty$ to $\mathbb{Z}$, and let $\mathbb{Z}^{**}$ be the two-point compactification adjoining the points $-\infty, +\infty$ to $\mathbb{Z}$. Let t^* and t^{**} be the homeomorphisms extending t to $\mathbb{Z}^*$ and $\mathbb{Z}^{**}$, respectively. Both $t^*_\pm$ and $t^{**}_\pm$ are topologically transitive with $\mathbb{Z}$ the orbit of isolated points, and so $\mathbb{Z} = \text{Trans}(t^*_\pm) = \text{Trans}(t^{**}_\pm)$. Of course, both $t^*_\pm$ and $t^{**}_\pm$ are chain transitive, but t^* is also chain transitive while t^{**} is not.

Let X be the quotient space of $Z^{**} \times \{0, 1\}$ with the fixed points $\langle +\infty, 0\rangle$ and $\langle +\infty, 1\rangle$ identified. Let f be the homeomorphism on X induced by $t^{**} \times 1_{\{0,1\}}$. Clearly, $f_\pm$ is chain transitive. But $\mathcal{C}f \cup 1_X \cup \mathcal{C}f^{-1}$ is contained in the image of $(Z^{**} \times \{0\})^2 \cup (Z^{**} \times \{1\})^2$ in X^2 and so is a proper subset of $\mathcal{C}(f_\pm) = X^2$. Furthermore, it is easy to check that in this case $\mathcal{C}(f \cup 1_X) = (\mathcal{C}f) \cup 1_X$. Hence, $R = f \cup 1_X$ is a closed relation such that R is chain recurrent and $R_\pm$ is chain transitive, but R is not chain transitive. Thus, (c) and (d) in Proposition 2.7 do not extend to general relations. □

Finally, recall the *uniqueness of Cantor*: any compact, perfect, zero-dimensional, metrizable space is homeomorphic to the Cantor set in [0, 1]. We will call any such space a Cantor set. We will need the following well-known result, and we provide a brief sketch of the proof.

Proposition 2.9. *For any space* X *there exists a surjective continuous map* $\pi : C \to X$ *with* C *a Cantor set.*

Proof. With $\mathcal{B}$ a countable basis for X, let Z be the closure in $X \times \{0, 1\}^{\mathcal{B}}$ of the set of pairs $\{(x, z) : z_U = 1 \Leftrightarrow x \in U\}$. The projection to X is clearly onto, and because $\mathcal{B}$ is a basis, the projection to $\{0, 1\}^{\mathcal{B}}$ is easily seen to be injective. It follows that Z is compact and zero-dimensional. If C_0 is a Cantor set, then $C = Z \times C_0$ is perfect as well as zero-dimensional and so is a Cantor set. Let π be the composition of projections $C \to Z \to X$. □

3. Spaces which admit chain transitive maps

We begin with the relationship between omega limit sets and chain transitive subsets which was described in the Introduction.

Proposition 3.1. *If f is a homeomorphism on a space X and $x \in X$, then $\omega f(x)$ and $\alpha f(x)$ are chain transitive subsets, i.e., the restriction of f to each of these nonempty, closed, invariant sets is chain transitive.*

Proof. Let $y, y' \in \omega f(x)$ and let $\epsilon > 0$. Choose $\delta > 0$ an $\epsilon/2$-modulus of uniform continuity for f with $\delta < \epsilon/2$. There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $f^n(x) \in \overline{V}_\delta(\omega f(x))$. There exists $n \geq N$ such that $d(f^n(x), y) < \delta$ and $k \in \mathbb{N}$ such that $d(f^{n+k}(x), y') < \delta$. For $j = 0, \dots, k$ choose $y_j \in \omega f(x)$ such that $d(f^{n+j}(x), y_j) \leq \delta$ with $y_0 = y, y_k = y'$. Hence, $d(f^{n+j+1}(x), f(y_j)) \leq \epsilon/2$ and so $d(y_{j+1}, f(y_j)) \leq \epsilon$. That is, $\{y_j\}$ is an ϵ-chain from y to y'. It follows that $\omega f(x)$ is a chain transitive subset. Applying the result to f^{-1}, we see that $\alpha f(x) = \omega(f^{-1})(x)$ is a chain transitive subset as well, since f is chain transitive if and only if f^{-1} is. □

We prove, conversely, in Theorem 3.13(a) below that a chain transitive homeomorphism is the restriction to an omega limit set of a homeomorphism in a larger system.

In the constructions which follow, we will repeatedly use the process of *attachment*. Assume that A is a nonempty, closed, nowhere dense subset of a space X and that $h : A \to B$ is a continuous surjection. We may assume that X and B are disjoint. Otherwise, replace X by $X \times \{0\}$ and B by $B \times \{1\}$. Define X/h, the space with *X attached to B via h* as follows: Let $\tilde{h}$ denote the continuous retraction $h \cup 1_B : A \cup B \to B$. Let $E_h = 1_X \cup (\tilde{h}^{-1} \circ \tilde{h}) = 1_X \cup (\tilde{h} \times \tilde{h})^{-1}(1_B)$, a closed equivalence relation on $X \cup B$. Let X/h be the quotient space with projection $q_h : X \cup B \to X/h$. Since q_h is injective on B, we may regard it as an identification and so regard B as a subset of X/h. Furthermore, q_h restricts to a surjection $X \to X/h$ which maps A onto B via h and which is a homeomorphism between the dense open sets $X \setminus A \subset X$ and $(X/h) \setminus B \subset X/h$.

When B is a singleton, we write $q_A : X \to X/A$ for the quotient map and describe the result as *smashing A to a point*.

We will use some results of E. R. Lorch [1981; 1982] (see also [Tsankov 2006]), which we will briefly review.

For a locally compact space W, a *compactification* of W is a compact space Y together with a dense embedding of W into Y, i.e., a homeomorphism of W onto a dense subset of Y. Because W is locally compact, its image is an open, dense subset of Y, so $X = Y \setminus W$ is a nowhere dense, closed subset of Y. Reversing the point of view, we call Y an *extension of* X if Y is a compact space and X is a closed, nowhere dense subset of Y.

By a *pair of spaces* (Y, X) we will mean a space and a closed subset, respectively. Recall our default assumption that a space is a nonempty, compact, metrizable space. We will call (Y, X) an *extension pair* when Y is an extension of X, i.e., when X is nowhere dense in Y.

A continuous map $f : Y_1 \to Y_2$ is a map of pairs $f : (Y_1, X_1) \to (Y_2, X_2)$ when $f(X_1) \subset X_2$. If $X_1 = X_2 = X$, we will say that $f : (Y_1, X) \to (Y_2, X)$ is a map *rel* X if it restricts to the identity on X and if, in addition, $f(Y_1 \setminus X) \subset Y_2 \setminus X$. These conditions imply that $1_X = (f \times f)^{-1}(1_X)$. So by intersecting over $\delta > 0$ and using compactness, we see that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{gathered}\text{if } (u, v) \in Y_1 \times Y_1 \text{ and } d_2(f(u), f(v)), d_2(f(u), X), d_2(f(v), X) \leq \delta, \\ \text{then } (u, v) \in \overline{V}_\epsilon^{d_1}, \\ \text{and so } f^{-1}(\overline{V}_\delta^{d_2}(x)) \subset \overline{V}_\epsilon^{d_1}(x) \text{ for all } x \in X,\end{gathered} \tag{3-1}$$

where d_1, d_2 are metrics on Y_1 and Y_2. Note that in considering different extensions Y_1, Y_2 of the same space X we do not assume that the metrics d_1 and d_2 agree on X, although they are, of course, uniformly equivalent on X.

Definition 3.2. We call $X^{(p)}$ an *isolated point extension* of X, or just a *point extension* of X, if $X^{(p)}$ is an extension of X with each point of $X^{(p)} \setminus X$ isolated. We then call $(X^{(p)}, X)$ a *point extension pair.*

A pair (Y, A) is a point extension pair if and only if Y is infinite, $\mathrm{Iso}(Y)$ is dense and $A = Y \setminus \mathrm{Iso}(Y)$. Since Y is separable, $\mathrm{Iso}(Y)$ is denumerable, and so Lorch uses the term *denumerable extension* instead of point extension. Thus, Y is a compactification of the denumerable discrete set $\mathrm{Iso}(Y)$.

Lemma 3.3. *Let $\pi : (Y, X) \to (X^{(p)}, X)$ be a surjective map of pairs rel X with $X^{(p)}$ a point extension of X. For every $\epsilon > 0$, the set $\{x \in X^{(p)} : \operatorname{diam} \pi^{-1}(x) > \epsilon\}$ is finite.*

Proof. By (3-1) there exists $\delta > 0$ such that for $x \in X^{(p)}$ with $x \in \overline{V}_\delta(X)$, $\operatorname{diam} \pi^{-1}(x) \leq \epsilon$. So the set $\{x \in X^{(p)} : \operatorname{diam} \pi^{-1}(x) > \epsilon\}$ is contained in the complement of the open δ-neighborhood of X in $X^{(p)}$. This is a compact set of isolated points and so is finite. □

For a point extension $(X^{(p)}, X)$, we define a *canonical retraction* $r : X^{(p)} \to X$ so that for all $x \in X^{(p)}$

$$d(x, r(x)) = d(x, X). \tag{3-2}$$

The choice depends on the metric. Even for a fixed metric there may be more than one closest to X point $r(x)$. For each x we fix a choice to define r. Clearly,

$$d(r(x_1), r(x_2)) \leq d(x_1, X) + d(x_1, x_2) + d(x_2, X). \tag{3-3}$$

For every $\epsilon > 0$ there are only finitely many points $x \in X^{(p)}$ with $d(x, X) \geq \epsilon$, so continuity of r at the points of X follows. Continuity at the isolated points is trivial.

Notice that if N_0 is a cofinite subset of $X^{(p)} \setminus X$, then the closure of N_0 contains X, so $r(N_0)$ is dense in X.

We recall the elegant proof of *Lorch's uniqueness theorem* [1981, Proposition 10].

Theorem 3.4. *Every space X has an essentially unique isolated point extension. That is, if (Y, X) and (Y', X) are point extension pairs, then there is a homeomorphism $f : (Y, X) \to (Y', X)$ rel X.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of not necessarily distinct points in X such that $\{x_n : n \geq N\}$ is dense in X for all $N \in \mathbb{N}$. Let

$$X^{(p)} = X \times \{0\} \cup \{(x_n, n^{-1}) : n \in \mathbb{N}\} \subset X \times [0, 1],$$

and identify X with $X \times \{0\}$. Clearly, $(X^{(p)}, X)$ is a point extension pair, so X has at least one point extension.

Let (Y, X) and (Y', X) be point extension pairs. Fix metrics d and d' on Y and Y', respectively. Define a metric d'' on X as the pointwise maximum of d and d'. The three metrics d, d' and d'' are uniformly equivalent on X. Let $r : Y \to X$ and $r' : Y' \to X$ be canonical retractions. Let $N = \mathrm{Iso}(Y) = Y \setminus X$ and $N' = \mathrm{Iso}(Y') = Y' \setminus X$. Use a counting of N and N' to impose orderings which are order isomorphic to $\mathbb{N}$ and so are well-orderings.

Let a_1 be the first element of N, and choose b_1 to be the first element of N' which satisfies

$$d''(r(a_1), r'(b_1)) < d(a_1, X). \tag{3-4}$$

Let b_2 be the first element of $N' \setminus \{b_1\}$, and choose a_2 to be the first element of $N \setminus \{a_1\}$ such that

$$d''(r(a_2), r'(b_2)) < d'(b_2, X). \tag{3-5}$$

Proceed inductively. If n is even, let a_{n+1} be the first element of $N \setminus \{a_1, \dots, a_n\}$, and if n is odd, let b_{n+1} be the first element of $N' \setminus \{b_1, \dots, b_n\}$. We can then choose b_{n+1} or a_{n+1} so that

$$d''(r(a_{n+1}), r'(b_{n+1})) < \max\{d(a_{n+1}, X), d'(b_{n+1}, X)\}. \tag{3-6}$$

By construction, $\{a_n\}$ and $\{b_n\}$ are renumberings of the sets N and N'. Define the mapping $f : Y \to Y'$ as an extension of the identity on X by putting $f(a_n) = b_n$ for all n. Let $m_n = \max\{d(a_n, X), d'(b_n, X)\}$. Observe that $m_n \to 0$ as $n \to \infty$.

Continuity of f is clear at the isolated points. Suppose $x \in X$ and $a_{n_i} \to x$ so that $n_i \to \infty$. Then

$$\begin{aligned} d(r(a_{n_i}), x) &\le d(a_{n_i}, x) + d(r(a_{n_i}), a_{n_i}) \\ &\le d(a_{n_i}, x) + m_{n_i} \to 0, \end{aligned} \tag{3-7}$$

and so $d'(r(a_{n_i}), x) \to 0$. Hence,

$$\begin{aligned} d'(b_{n_i}, x) &\le d'(r(b_{n_i}), b_{n_i}) + d'(r(b_{n_i}), r(a_{n_i})) + d'(r(a_{n_i}), x) \\ &\le 2m_{n_i} + d'(r(a_{n_i}), x) \to 0. \end{aligned} \tag{3-8}$$

Continuity of f follows. The result for f^{-1} is similar and also follows from compactness. □

Corollary 3.5. *Let $(X^{(p)}, X)$ and $(Y^{(p)}, Y)$ be point extension pairs and $h : X \to Y$ a surjective continuous map. There exists a continuous map $H : (X^{(p)}, X) \to (Y^{(p)}, Y)$ which restricts to h on X and to a homeomorphism of* $\mathrm{Iso}(X^{(p)}) = X^{(p)} \setminus X$ *onto* $\mathrm{Iso}(Y^{(p)}) = Y^{(p)} \setminus Y$. *In particular, if h is a homeomorphism, so is H.*

Proof. We attach Y to $X^{(p)}$ using h, letting $q_h : X^{(p)} \to X^{(p)}/h$ be the quotient map. We regard Y as a subset of $X^{(p)}/h$ so that $(X^{(p)}/h, Y)$ is a point extension pair and $q_h : X^{(p)} \to X^{(p)}/h$ is an extension of h which is a homeomorphism from $X^{(p)} \setminus X$ onto $X^{(p)}/h \setminus Y$. By Theorem 3.4 there is a homeomorphism $f : (X^{(p)}/h, Y) \to (Y^{(p)}, Y)$ rel Y. Let $H = f \circ q_h$. □

Lemma 3.6. *Suppose that $(X_1^{(p)}, X_1)$ and $(X_2^{(p)}, X_2)$ are point extension pairs and that $H : (X_1^{(p)}, X_1) \to (X_2^{(p)}, X_2)$ is a continuous map of pairs. Let (Y_1, X_1) and (Y_2, X_2) be extension pairs with $\pi_1 : (Y_1, X_1) \to (X_1^{(p)}, X_1)$ and $\pi_2 : (Y_2, X_2) \to (X_2^{(p)}, X_2)$ pair maps rel X_1 and rel X_2, respectively. Assume that $\widetilde{H} : Y_1 \to Y_2$ is a function such that $\pi_2 \circ \widetilde{H} = H \circ \pi_1$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} Y_1 & \xrightarrow{\widetilde{H}} & Y_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1^{(p)} & \xrightarrow[H]{} & X_2^{(p)} \end{array}$$

If for each $x \in \mathrm{Iso}(X_1^{(p)})$ the restriction $\widetilde{H} : \pi_1^{-1}(x) \to \pi_2^{-1}(H(x))$ is continuous, then $\widetilde{H}$ is continuous on Y_1.

Proof. If $x \in \mathrm{Iso}(X_1^{(p)})$, then $\pi_1^{-1}(x)$ is a clopen set, and thus $\widetilde{H}$ is continuous at the points of $\pi_1^{-1}(\mathrm{Iso}(X_1^{(p)}))$ by hypothesis.

Suppose $x \in X_1$, so $y = H(x) \in X_2$. Given $\epsilon > 0$, there exists by (3-1) $\delta > 0$ so that $\pi_2^{-1}(V_\delta^{d_2}(y)) \subset V_\epsilon^{d_2'}(y)$, where d_2 and d_2' are the metrics on $X_2^{(p)}$ and Y_2, respectively. By the continuity of H there exists $\gamma > 0$ so that $H(V_\gamma^{d_1}(x)) \subset V_\delta^{d_2}(y)$. If $x_1 \in \pi_1^{-1}(V_\gamma^{d_1}(x))$, then $\widetilde{H}(x_1) \in \pi_2^{-1}(V_\delta^{d_2}(y)) \subset V_\epsilon^{d_2'}(y)$. Hence, $\pi_1^{-1}(V_\gamma^{d_1}(x))$ is a neighborhood of x in Y_1 which is mapped by $\widetilde{H}$ into the neighborhood $V_\epsilon^{d_2'}(y)$ in Y_2, and continuity at x follows. □

Definition 3.7. For a space K and a pair of spaces (Y, X), we call Y a *K-extension* of X if there exist a point extension pair $(X^{(p)}, X)$ and a pair map $\pi : (Y, X) \to (X^{(p)}, X)$ rel X such that $\pi^{-1}(x)$ is homeomorphic to K for every $x \in \mathrm{Iso}(X^{(p)})$. We then call (Y, X) a *K-extension pair*, and the space Y is denoted by $X^{(K)}$.

We extend Lorch's theorem.

Theorem 3.8. *For any given space K, every space X has an essentially unique K-extension pair $(X^{(K)}, X)$. Furthermore, if $(X^{(K)}, X)$ and $(Y^{(K)}, Y)$ are K-extension pairs and $h : X \to Y$ is a surjective continuous map, then there exists a continuous map $H : (X^{(K)}, X) \to (Y^{(K)}, Y)$ which restricts to h on X and to a homeomorphism of $X^{(K)} \setminus X$ onto $Y^{(K)} \setminus Y$. In particular, if h is a homeomorphism, then so is H.*

Proof. Let $(X^{(p)}, X)$ be a point extension of X. Let $\pi : (X^{(p)} \times K, X \times K) \to (X^{(p)}, X)$ be the map of pairs given by the first coordinate projection. Attach $X^{(p)} \times K$ to X by using $h = \pi|_{X \times K}$. The map π factors through the quotient map q_h to define a map $((X^{(p)} \times K)/h, X) \to (X^{(p)}, X)$ rel X. Thus, $((X^{(p)} \times K)/h, X)$ is a K-extension pair.

Now let $\pi_X : (X^{(K)}, X) \to (X^{(p)}, X)$ and $\pi_Y : (Y^{(K)}, Y) \to (Y^{(p)}, Y)$ be maps rel X and Y, respectively, with each fiber over $X^{(p)} \setminus X$ and $Y^{(p)} \setminus Y$ homeomorphic to K. Use Corollary 3.5 to get an extension $H^{(p)} : (X^{(p)}, X) \to (Y^{(p)}, Y)$ which maps $X^{(p)} \setminus X$ to $Y^{(p)} \setminus Y$ homeomorphically. Define H by choosing for each $x \in X^{(p)} \setminus X$ an arbitrary homeomorphism from $\pi_X^{-1}(x)$ to $\pi_Y^{-1}(H^i(x))$. These exist because each fiber is homeomorphic to K. By Lemma 3.6, the resulting H is continuous.

In particular, if $Y = X$ and $h = 1_X$, then it follows that the K-extension is essentially unique. □

Two cases are of special interest to us. Recall that a component of a space X is an *isolated component* if it is a clopen subset of X.

Theorem 3.9. *Let (Y, X) be an extension pair.*

(a) *If K is a Cantor set, then (Y, X) is a K-extension pair if and only if Y is perfect and the dense, open set $Y \setminus X$ is zero-dimensional.*

(b) *If K is a connected space, then (Y, X) is a K-extension pair if and only if*

- *the union of the isolated components is dense in Y;*
- *each isolated component is homeomorphic to K;*
- (diameter condition) *for every $\epsilon > 0$ there are only finitely many isolated components with diameter greater than ϵ.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a pairwise disjoint sequence of nonempty clopen subsets of Y with union $Y \setminus X$. If for every $\epsilon > 0$ only finitely many of the sets A_n have diameter greater than ϵ, then

$$E = 1_X \cup \bigcup_n \{A_n \times A_n\} \tag{3-9}$$

is a closed equivalence relation. If $q : Y \to Y/E$ is the quotient space projection, then $(Y/E, X)$ is a point extension pair, q is a map of pairs rel X, and the fibers of q over the isolated points are the sets A_n. Conversely, if $\pi : (Y, X) \to (X^{(p)}, X)$ is a map of pairs rel X, then by (3-1) there are only finitely many fibers $\pi^{-1}(x)$ with diameter at least ϵ.

(a) If $\pi : (Y, X) \to (X^{(p)}, X)$ is a map rel X with each fiber over a point of $X^{(p)} \setminus X$ a Cantor set, then as a countable disjoint union of Cantor sets, $Y \setminus X$ is zero-dimensional and $\text{Iso}(Y) = \varnothing$.

Conversely, if the locally compact space $Y \setminus X$ is zero-dimensional with no isolated points, then we can express it as the union of a pairwise disjoint sequence $\{C_n : n \in \mathbb{N}\}$ of nonempty, clopen subsets of Y each of which is thus a Cantor set. Let $\{C_{n,i} : i = 1, \ldots, N_n\}$ be a partition of C_n by nonempty clopen subsets of diameter less than n^{-1}. If $\{A_n\}$ is a counting of the collection $\{C_{n,i} : n \in \mathbb{N}, i = 1, \ldots, N_n\}$, then with E as in (3-9) the projection $q : (Y, X) \to (Y/E, X)$ is a map rel X with each fiber over a point of $(Y/E) \setminus X$ a Cantor set. Thus, (Y, X) is a Cantor set extension pair.

(b) If $\pi : (Y, X) \to (X^{(p)}, X)$ is a map rel X with each fiber connected, then $\{\pi^{-1}(x) : x \in X^{(p)} \setminus X\}$ is the set of isolated components, so the conditions of (b) are necessary; see Lemma 3.3.

Conversely, if they hold, then we let $\{A_n\}$ be the sequence of isolated components. There are infinitely many isolated components because X is nonempty and $Y \setminus X$ is dense. With E as in (3-9) again, $q : (Y, X) \to (Y/E, X)$ is the required map rel X. □

Corollary 3.10. *If Y and X are Cantor sets with closed, nowhere dense subsets $Y_1 \subset Y$ and $X_1 \subset X$, then for any surjective continuous map $h : Y_1 \to X_1$ there is a continuous map $H : Y \to X$ which extends h and which restricts to a homeomorphism of $Y \setminus Y_1$ to $X \setminus X_1$. In particular, if h is a homeomorphism, then so is H.*

Proof. By Theorem 3.9(a), (Y, Y_1) and (X, X_1) are Cantor set extension pairs. The existence of H then follows from Theorem 3.8. □

Remark. This is a classical theorem of Knaster and Reichbach [1953], extended by Gutek [1979].

Proposition 3.11. *Let $(X^{(p)}, X)$ be a point extension pair, $(X^{(C)}, X)$ a Cantor set extension pair and $(X^{(K)}, X)$ a K-extension pair for spaces X and K. If $\pi : (X^{(C)}, X) \to (X^{(p)}, X)$ is a surjective map of pairs rel X, then there exist $\pi_1 : (X^{(C)}, X) \to (X^{(K)}, X)$ and $\pi_2 : (X^{(K)}, X) \to (X^{(p)}, X)$ surjective maps of pairs rel X such that $\pi = \pi_2 \circ \pi_1$. That is, π factors through the pair $(X^{(K)}, X)$.*

Proof. For each $x \in \mathrm{Iso}(X^{(p)})$, $\pi^{-1}(x)$ is a clopen subset of $X^{(C)} \setminus C$ and so is compact, perfect and zero-dimensional, i.e., it is a Cantor set. By Proposition 2.9 there exists a continuous surjection h_x from $\pi^{-1}(x)$ onto K, so $E_x = (h_x \times h_x)^{-1}(1_K)$ is a closed equivalence relation on $\pi^{-1}(x)$ such that the surjection h_x factors to give a homeomorphism from the quotient space $\pi^{-1}(x)/E_x$ onto K. Let

$$E = 1_{X^{(C)}} \cup \bigcup_{x \in \mathrm{Iso}(X^{(p)})} E_x.$$

From Lemma 3.3 it follows that E is a closed equivalence relation on $X^{(C)}$. Let $q_E : X^{(C)} \to X^{(C)}/E$ be the quotient map. From the construction there exists $q : X^{(C)}/E \to X^{(p)}$ such that $\pi = q \circ q_E$. The E equivalence class of each $x \in X$ is a singleton and so, identifying X with $q_E(X)$, we can regard $q_E : (X^{(C)}, X) \to (X^{(C)}/E, X)$ and $q : (X^{(C)}/E, X) \to (X^{(p)}, X)$ as maps rel X. Since each $\pi^{-1}(x)/E_x$ is homeomorphic to K, $(X^{(C)}/E, X)$ is a K extension pair. By uniqueness, there exists $h : (X^{(C)}/E, X) \to (X^{(K)}, X)$ a homeomorphism rel X. Let $\pi_1 = h \circ q_E$ and $\pi_2 = q \circ h^{-1}$. □

Now we apply these results.

Lemma 3.12. *Let $(X^{(p)}, X)$ be a point extension pair, (Y, X) an extension pair and $\pi : (Y, X) \to (X^{(p)}, X)$ a surjective map of pairs rel X.*

(a) *The map $\pi : Y \to X^{(p)}$ is open.*
(b) *Assume that $H : (Y, X) \to (Y, X)$ and $H^{(p)} : (X^{(p)}, X) \to (X^{(p)}, X)$ are homeomorphisms with π mapping H to $H^{(p)}$, i.e., $\pi \circ H = H^i \circ \pi$. If $H^{(p)}$ is chain transitive, then H is chain transitive.*

Proof. (a) If $U \subset Y$ is open and $x \in \pi(U) \cap (X^{(p)} \setminus X)$, then $\pi(U)$ is a neighborhood of x because x is an isolated point.

If $x \in \pi(U) \cap X$, then there exist $\epsilon > 0$ and $\delta > 0$ so that $\overline{V}^{d_Y}_{\epsilon}(x) \subset U$ and $\pi^{-1}(\overline{V}^{d_{X^{(p)}}}_{\delta}(x)) \subset \overline{V}^{d_Y}_{\epsilon}(x)$. Because π is surjective, $\pi(\overline{V}^{d_Y}_{\epsilon}(x)) \supset \overline{V}^{d_{X^{(p)}}}_{\delta}(x)$, so $\pi(U)$ is a neighborhood of x.

(b) Since π is open, Proposition 2.4(e) implies that π is a semiconjugacy from $\mathcal{C}H$ to $\mathcal{C}H^{(p)}$ and from $\mathcal{C}H^{-1}$ to $\mathcal{C}(H^{(p)})^{-1}$. Fix $x \in X$. For any $y \in Y$ we have $x \in \mathcal{C}H^{(p)}(\pi(y))$ and $x \in \mathcal{C}(H^{(p)})^{-1}(\pi(y))$ because $H^{(p)}$ is chain transitive. Since $\{x\} = \pi^{-1}(x)$, it follows from the semiconjugacy that $x \in \mathcal{C}H(y)$ and $x \in \mathcal{C}H^{-1}(y)$. That is, every point of Y is chain equivalent to x, and so transitivity of $\mathcal{C}H$ implies that H is chain transitive. □

Theorem 3.13. *Let $(X^{(p)}, X)$ be a point extension pair, $(X^{(C)}, X)$ a Cantor set extension pair and $(X^{(K)}, X)$ a K-extension pair for spaces X and K.*

(a) *If f is a chain transitive homeomorphism on X, then there exists a homeomorphism F on $X^{(p)}$ which extends f on X such that*
   - *F is chain transitive,*
   - *$F_{\pm}$ is topologically transitive, and*
   - *if $x \in X^{(p)} \setminus X$, then $\omega F(x) = X = \alpha F(x)$.*

(b) *There exists $G^{(C)}$ a topologically transitive homeomorphism on $X^{(C)}$ which extends 1_X.*

(c) *There exists $G^{(K)}$ a chain transitive homeomorphism on $X^{(K)}$ which extends 1_X.*

Proof. (a) By concatenating ϵ-chains which are ϵ-dense in X, we can obtain an infinite sequence $\{x_k : k \in \mathbb{Z}\}$ with $d(f(x_k), x_{k+1}) \to 0$ as $|k| \to \infty$ and so that for any $N \in \mathbb{N}$ the tails $\{x_k : k \geq N\}$ and $\{x_{-k} : k \geq N\}$ are dense in X. Define the sequence $\{y_k \in X \times [0,1] : k \in \mathbb{Z}\}$ by

$$y_k = \begin{cases} (x_k, (2k+1)^{-1}) & \text{for } k \geq 0, \\ (x_k, (2|k|)^{-1}) & \text{for } k < 0. \end{cases} \tag{3-10}$$

Let $Y = X \times \{0\} \cup \{y_k : k \in \mathbb{Z}\}$, and define $\bar{F}(x,0) = (f(x), 0)$ and $\bar{F}(y_k) = y_{k+1}$ for $k \in \mathbb{Z}$. It is easy to see that Y is an isolated point extension of $X = X \times \{0\}$, $\bar{F}$ is a homeomorphism on Y, and $X \times \{0\} = \omega\bar{F}(y_k) = \alpha\bar{F}(y_k)$ for any $k \in \mathbb{Z}$. Since the orbit $\mathcal{O}(\bar{F}_{\pm})(y_0) = \{y_k : k \in \mathbb{Z}\}$ is dense, it follows that $\bar{F}_{\pm}$ is topologically transitive. The homeomorphism $\bar{F}$ is chain transitive on Y because f is chain transitive on X and because $X = \omega\bar{F}(y_k) \subset \mathcal{C}\bar{F}(y_k)$ and also $X = \alpha\bar{F}(y_k) \subset \mathcal{C}\bar{F}^{-1}(y_k)$.

By Lorch's uniqueness theorem (Theorem 3.4) there exists a homeomorphism $H : (X^{(p)}, X) \to (Y, X)$ rel X. Let $F = H^{-1} \circ \bar{F} \circ H$.

(b) We begin with G a topologically transitive homeomorphism on a Cantor set with a Cantor set C of fixed points. By topological transitivity, C is necessarily nowhere dense. To be specific, let G be the shift homeomorphism on the product space $C^{\mathbb{Z}}$. Let $c : C \to C^{\mathbb{Z}}$ be the embedding with $c(x)_i = x$ for all $i \in \mathbb{Z}$. Thus, c is a homeomorphism onto the set of fixed points. Since C is nowhere dense, $(C^{\mathbb{Z}}, C)$ is a Cantor set extension pair by Theorem 3.9(a).

For an arbitrary space X, there exists by Proposition 2.9 a continuous surjection $h : C \to X$. We attach $C^{\mathbb{Z}}$ to X using h. Let $Y = (C^{\mathbb{Z}})/h$. Now, (Y, X) is a Cantor set extension pair, and G factors to define a topologically transitive homeomorphism $\overline{G}$ which restricts to the identity on X.

By Theorem 3.8 there exists a homeomorphism $H : (X^{(C)}, X) \to (Y, X)$ rel X. Let $G^{(C)} = H^{-1} \circ \overline{G} \circ H$.

(c) First we consider the case where X is the usual Cantor set C in $[0, 1]$ with $0, 1 \in C$. Then the complement consists of a pairwise disjoint, countable collection $\{(a_i, b_i) : i \in \mathbb{N}\}$ of open intervals in $(0, 1)$. Let $\ell_i = b_i - a_i$, so $\ell_i > 0$ for all $i \in \mathbb{N}$, but for any $\epsilon > 0$ there are only finitely many with $\ell_i \geq \epsilon$. Let

$$C^{(p)} = C \times \{0\} \cup \bigcup_{i,m \in \mathbb{N}} \{\langle a_i, \ell_i/m\rangle, \langle b_i, \ell_i/m\rangle\} \subset C \times [0, 1]. \tag{3-11}$$

Since the set of endpoints is dense in C, it follows that $C^{(p)}$ is a point extension of $C = C \times \{0\}$. Now we relabel the isolated points. For $i \in \mathbb{N}$, $k \in \mathbb{Z}$, define

$$\begin{aligned} u_{i,k} &= \begin{cases} \langle a_i, \ell_i \cdot (2k+1)^{-1}\rangle & \text{for } k \geq 0, \\ \langle b_i, \ell_i \cdot (2|k|)^{-1}\rangle & \text{for } k < 0, \end{cases} \\ v_{i,k} &= \begin{cases} \langle b_i, \ell_i \cdot (2k+1)^{-1}\rangle & \text{for } k \geq 0, \\ \langle a_i, \ell_i \cdot (2|k|)^{-1}\rangle & \text{for } k < 0. \end{cases} \end{aligned} \tag{3-12}$$

Define G as an extension of 1_C so that $u_{i,k} \overset{G}{\mapsto} u_{i,k+1}$ and $v_{i,k} \overset{G}{\mapsto} v_{i,k+1}$. Thus, above each endpoint a_i the $u_{i,k}$'s run up the $\langle a_i, \ell_i/m\rangle$'s with m even, jump from $\langle a_i, \ell_i/2\rangle$ to $\langle b_i, \ell_i\rangle$, and then move down the $\langle b_i, \ell_i/m\rangle$'s with m odd. The $v_{i,k}$'s provide a similar path from b_i to a_i. Since for any $\epsilon > 0$ at most finitely many points move a distance more than ϵ, it follows that G and its inverse are continuous.

It is clear that for any i the points of $\{\langle a_i, \ell_i/m\rangle, \langle b_i, \ell_i/m\rangle : m \in \mathbb{N}\}$ all lie in a single chain component. Given $\epsilon > 0$, it is clear that we can get from a point $x \in C$ to a point $y \in C$ by an ϵ-chain jumping across the gaps of length less than ϵ which occur between x and y. For the finite number of remaining gaps we use the isolated point orbits to get across. Hence, G is chain transitive on $C^{(p)}$.

For an arbitrary space X, we again use a continuous surjection $h : C \to X$ from Proposition 2.9. Let $X^{(p)}$ be the quotient space $C^{(p)}/h$ obtained by attaching X via h. Then, $G \cup 1_X$ on $C^{(p)} \cup X$ factors through the quotient map q_h to define a homeomorphism $G^{(p)}$. Because q_h maps G on $C^{(p)}$ onto $G^{(p)}$ on $X^{(p)}$ it follows that $G^{(p)}$ is chain transitive by Proposition 2.4(d). Because $q_h : C^{(p)} \setminus C \to X^{(p)} \setminus X$ is a homeomorphism, it follows that $X^{(p)}$ is an isolated point extension of X.

Because $(X^{(K)}, X)$ is a K-extension pair and the point extension is essentially unique, there exists $\pi : (X^{(K)}, X) \to (X^{(p)}, X)$, a map of pairs rel X, such that the fiber $\pi^{-1}(x)$ is homeomorphic to K for every $x \in X^{(p)} \setminus X$. For each such x let $G^{(K)}$ restrict to a homeomorphism from $\pi^{-1}(x)$ to $\pi^{-1}(G^{(p)}(x))$. By Lemma 3.6 $G^{(K)}$

and its inverse are continuous. By Lemma 3.12 $G^{(K)}$ is chain transitive because $G^{(p)}$ is. □

Remark. By Proposition 2.5(b) there is no topologically transitive homeomorphism on a space with infinitely many isolated points. Hence, the result in (a) above is the best we can hope for. In particular, we see that any chain transitive homeomorphism on X can be extended to a system in which X is an omega limit set.

Recall from Proposition 2.7(e) that if X is connected, then 1_X is chain transitive. Now we can prove a slight extension of Theorem 1.1.

Corollary 3.14. *For a space X let X_1 be the closure of the union of all components which meet $\overline{\text{Iso}(X)}$. If X_1 is a proper, clopen, nonempty subset of X, then X is $H(X)$-decomposable and so X admits no chain transitive homeomorphism. If X_1 is not a proper, clopen subset of X and the open set $X \setminus X_1$ is empty or zero-dimensional, then X admits a chain transitive homeomorphism.*

Proof. The sets $\overline{\text{Iso}(X)}$ and X_1 are h_X-invariant, so if X_1 is proper, clopen and nonempty, then X is h_X-decomposable.

Assume that X_1 is not a proper, clopen subset of X. If $\text{Iso}(X)$ is finite and nonempty, then $X = X_1 = \text{Iso}(X)$ and we can define f so that X consists of a single periodic orbit. If $\text{Iso}(X) = \varnothing$, then $X_1 = \varnothing$ and $X = X \setminus X_1$ is zero-dimensional and perfect, and so X is a Cantor set. Hence, X admits a topologically transitive homeomorphism.

Now assume that $\text{Iso}(X)$ is infinite, so $A = \overline{\text{Iso}(X)} \setminus \text{Iso}(X)$ is nonempty. Clearly, $(\overline{\text{Iso}(X)}, A)$ is a point extension pair, and by Theorem 3.13(c) there exists a chain transitive homeomorphism f_1 on $\overline{\text{Iso}(X)}$ which is the identity on A. Extend f_1 to be the identity on $X_1 \setminus \text{Iso}(X)$. Thus, f_1 is the identity on every nontrivial component of X which meets $\overline{\text{Iso}(X)}$. It follows from Proposition 2.7(e) that all of these are contained in the chain component of f_1 which contains all of $\overline{\text{Iso}(X)}$. As this chain component is closed, it must contain all of X_1. That is, f_1 on X_1 is chain transitive. If $X = X_1$, then we are done.

Otherwise, the nonempty, open, zero-dimensional set $X \setminus X_1$ is not closed and contains no isolated points. So $X_2 = \overline{X \setminus X_1}$ is perfect and $B = X_2 \cap X_1$ is nonempty subset of X_1 disjoint from $\text{Iso}(X)$. We see that (X_2, B) is a Cantor set extension pair, so by Theorem 3.13(b) there exists a topologically transitive homeomorphism f_2 on X_2 which restricts to the identity on B.

The concatenation $f = f_1 \cup f_2$ is a homeomorphism on X. Since f_1 and f_2 are each chain transitive and $X_1 \cap X_2 \neq \varnothing$ it follows that all of X is contained in a single chain component, i.e., f is chain transitive. □

To extend these results we need some simple lifting facts.

Lemma 3.15. *Let $f_i \in H(X_i)$ for $i = 1, 2$ and let $\pi : X_1 \to X_2$ be a continuous surjection mapping f_1 to f_2. Assume that $(\pi \times \pi)^{-1}(1_{X_2}) \subset \mathcal{C}f_1$. That is, each fiber of π is entirely contained in a single chain component of f_1.*

(a) *Both f_1 and f_2 are chain recurrent, i.e.,*

$$1_{X_1} \subset \mathcal{C}f_1 \quad \text{and} \quad 1_{X_2} \subset \mathcal{C}f_2.$$

(b) *The space X_1 is f_1-decomposable if and only if X_2 is f_2-decomposable.*

(c) *The chain relations satisfy*

$$\mathcal{C}f_2 = (\pi \times \pi)(\mathcal{C}f_1) \quad \text{and} \quad \mathcal{C}f_1 = (\pi \times \pi)^{-1}(\mathcal{C}f_2).$$

(d) *The homeomorphism f_1 is chain transitive if and only if f_2 is chain transitive.*

Proof. (a) Let $E_\pi = (\pi \times \pi)^{-1}(1_{X_2})$. It is a closed equivalence relation and so contains 1_{X_1}. Hence, $1_{X_1} \subset E_\pi \subset \mathcal{C}f_1$.

Because π is surjective, $1_{X_2} = (\pi \times \pi)(1_{X_1})$. Since π maps f_1 to f_2, it maps $\mathcal{C}f_1$ to $\mathcal{C}f_2$ by Proposition 2.4(d). Hence,

$$1_{X_2} = (\pi \times \pi)(1_{X_1}) \subset (\pi \times \pi)(\mathcal{C}f_1) \subset \mathcal{C}f_2.$$

(b) If B and its complement are proper, clopen, forward f_2-invariant subsets of X_2 then because π is surjective, $\pi^{-1}(B)$ and its complement are proper, clopen, forward f_1-invariant subsets of X_1.

Now assume that A and its complement are proper, clopen, forward f_1-invariant subsets of X_1. Since each is forward $\mathcal{C}f_1$-invariant, it follows that each is saturated by the equivalence relation $E_\pi \subset \mathcal{C}f_1$. Hence, $\pi(A)$ and $\pi(X_1 \setminus A)$ are disjoint closed sets with union $\pi(X_1) = X_2$. That is, they are complementary clopen sets. Furthermore, they are forward f_2-invariant.

(c) As mentioned above, $(\pi \times \pi)(\mathcal{C}f_1) \subset \mathcal{C}f_2$. Now assume $(x_1, x_2) \notin \mathcal{C}f_1$. Since $X_1 = |\mathcal{C}f_1|$ by (a), Proposition 2.1(c) implies there is an attractor A for f_1 which contains x_1 but not x_2, and by Proposition 2.1(d) A is a clopen $\mathcal{C}f_1$-invariant set. Hence, it is saturated by E_π. So $\pi(A)$ is clopen and f_2-invariant, and therefore also $\mathcal{C}f_2$-invariant. Now, $\pi(x_1) \in \pi(A)$, and $x_2 \notin A = \pi^{-1}(\pi(A))$ implies $\pi(x_2) \notin \pi(A)$. It follows that $(\pi(x_1), \pi(x_2)) \notin \mathcal{C}f_2$. Thus, the complement of $\mathcal{C}f_1$ in $X_1 \times X_1$ maps into the complement of $\mathcal{C}f_2$. Since $\pi \times \pi$ is surjective, the equations of (c) follow.

(d) Immediate from (c) and the surjectivity of π. □

For any space X, the chain relation $\mathcal{C}1_X$ is a closed equivalence relation and by Lemma 2.6(b) the equivalence classes are the components of X. Let $[X]$ be the zero-dimensional space of components, the quotient space for this equivalence relation with quotient map $\pi_X : X \to [X]$; see Lemma 2.6(b) and Proposition 2.1(d). There is a natural homomorphism $[\cdot] : H(X) \to H([X])$ with π_X mapping f to

$[f]$ for $f \in H(X)$. Thus, $H(X)$ acts on $[X]$ and we let $[h_X] = \bigcup\{[h] : h \in H(X)\}$ be the associated relation on $[X]$.

Proposition 3.16.

(a) *A space X is h_X-decomposable if and only if $[X]$ is $[h_X]$-decomposable.*

(b) *If f is a chain recurrent homeomorphism on a space X, then $(\pi_X \times \pi_X)^{-1}(1_{[X]}) \subset \mathcal{C}f$. In that case, the following are equivalent:*

(i) *The map f is chain transitive.*

(ii) *The space X is f-indecomposable.*

(iii) *The map $[f]$ is chain transitive.*

(iv) *The space $[X]$ is $[f]$-indecomposable.*

Proof. (a) This is clear because any clopen set is saturated by the equivalence relation $\mathcal{C}1_X$.

(b) Since the space of chain components is zero-dimensional, every connected set of chain recurrent points is contained in a single chain component. If f is chain recurrent, then every point is chain recurrent, so each component is contained in a single chain component. It follows that $(\pi_X \times \pi_X)^{-1}(1_{[X]}) \subset \mathcal{C}f$. Since f and $[f]$ are chain recurrent, the equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from Proposition 2.7(d). The implication (i) $\Rightarrow$ (iii) holds because π maps f to $[f]$. The converse, (iii) $\Rightarrow$ (i), follows from Lemma 3.15. □

Thus, a chain transitive homeomorphism on $[X]$ lifts to a chain transitive homeomorphism if and only if it lifts to a chain recurrent homeomorphism.

Recall that a component K of X is an *isolated component* if it is a clopen subset of X. For a space X let $\mathcal{I}_X$ denote the set of isolated components. Two isolated components K_1 and K_2 are *$H(X)$-equivalent* if they are homeomorphic or, equivalently, if there exists $g \in H(X)$ such that $g(K_1) = K_2$. Let I_X be the set of $H(X)$-equivalence classes in $\mathcal{I}_X$. For $i \in I_X$ let Q_i be the union of the isolated components in i, and let Q be the union of all of the isolated components, so that Q is the disjoint union of the Q_i. Thus, Q and all of the Q_i are open subsets of X.

Lemma 3.17. *If A is a clopen $H(X)$-invariant set which meets some $\overline{Q_i}$, then it contains $\overline{Q_i}$. In particular, if all the isolated components are homeomorphic to one another and the union of the isolated components is dense, then X is $H(X)$-indecomposable.*

Proof. Since A is open, it meets some isolated component $K \in i$. Since it is clopen, it contains K. If $K_1 \in Q_i$, then there exists $g \in H(X)$ such that $g(K) = K_1$, and so $H(X)$-invariance implies $K_1 \subset A$. Since $Q_i \subset A$ and A is closed, we get $\overline{Q_i} \subset A$. □

We say that X satisfies the *diameter condition on isolated components* if for every $\epsilon > 0$ there are only finitely many isolated components with diameter greater than ϵ.

The following is the furthest we can extend Corollary 3.14.

Theorem 3.18. *For a space X, let X_1 be the closure of the union of all components which meet the closure of the union of all isolated components. If X satisfies the diameter condition on isolated components and the open set $X \setminus X_1$ is empty or zero-dimensional, then*

(a) *either X is $H(X)$-decomposable or X admits a chain transitive homeomorphism;*

(b) *if X_1 is a proper, clopen subset of X, then X is $H(X)$-decomposable, and so X admits no chain transitive homeomorphism;*

(c) *if X_1 is not a proper, clopen subset of X and all the isolated components are homeomorphic to one another, then X admits a chain transitive homeomorphism.*

Proof. (b) Obvious since X_1 is $H(X)$-invariant.

In (a) and (c) the extension from the closure of the isolated components to the rest of X proceeds just as in Corollary 3.14. So from now on we will assume that the union Q of the isolated components is dense.

(c) If there are only n isolated components, then their union Q is clopen and so is all of X. By assumption they are all homeomorphic to some common space K. We can choose a homeomorphism with $f^n = 1_X$, and so that each periodic orbit meets each component. Clearly, f is chain transitive.

Now we may assume that $\mathcal{I}_X$ is infinite, and let $A = X \setminus Q$. Thus, A is a nonempty, closed, nowhere dense set. By Theorem 3.9(b) (X, A) is a K-extension pair. By Theorem 3.13(c) X admits a chain transitive homeomorphism rel A.

(a) We may assume that there is more than one equivalence class in I_X, for otherwise we are in case (c). If any $i \in I_X$ is finite, then Q_i is a clopen $H(X)$-invariant set, so X is $H(X)$-decomposable since $X \neq Q_i$ by the assumption that I_X contains more than one class.

Now assume that every equivalence class in I_X is infinite. Then the closure of each open $H(X)$-invariant set Q_i meets A, and we let $A_i = \overline{Q_i} \cap A$. If $i \neq j$, then $\overline{Q_i} \cap \overline{Q_j} \subset A$, and so this intersection equals $A_i \cap A_j$.

Applying the argument for (c) to $\overline{Q_i}$, there exists a homeomorphism f_i on $\overline{Q_i}$ which is chain transitive and which restricts to the identity on A_i.

Let f on X equal f_i on $\overline{Q_i}$ and the identity on A. Because the diameter condition holds, we can apply Lemma 3.12 to see that f is a homeomorphism on X. Since each f_i is chain transitive, $\overline{\bigcup_i \{Q_i \times Q_i\}} \subset \mathcal{C}f$.

Let $x \in X$ and $g \in H(X)$. Since $Q = \bigcup_i Q_i$ is dense in X, there is a sequence $\{x_k \in Q_{i_k}\}$ which converges to x. Then $g(x_k) \in Q_{i_k}$, and so $(x, g(x)) \in \overline{\bigcup_i Q_i \times Q_i}$. Hence, $h_X \subset \mathcal{C}f$. In particular, $1_X \subset h_X$ implies that f is chain recurrent. So by Proposition 2.7(d) f is chain transitive if and only if X is f-indecomposable. Since $h_X \subset \mathcal{C}f$, an f-decomposition, which is a $\mathcal{C}f$-decomposition, is also an h_X-decomposition. Hence, if X is f-indecomposable, then it is h_X-indecomposable, i.e., X is $H(X)$-indecomposable. On the other hand, if X is $H(X)$-decomposable, then there does not exist any chain transitive homeomorphism on X. □

As we will see below, the diameter condition on isolated components is essential for this result.

Example 3.19. We construct X so that

- the connected components of X are all homeomorphic to $[0, 1]$;
- the space X is $H(X)$-decomposable;
- there are no isolated components.

Let $C \subset [0, 1]$ be a Cantor set and $S = \{a_n : n \in \mathbb{N}\}$ a sequence of distinct points in C with closure A in C. Define

$$I_n = \{\langle a_n, t\rangle : 0 \le t \le n^{-1}\} \subset C \times [0, 1] \quad \text{for } n \in \mathbb{N},$$

$$C_0 = C \times \{0\}, \quad C_+ = C_0 \cup \bigcup_n I_n, \quad C_\pm = C_+ \cup C \times [-1, 0], \tag{3-13}$$

$$A_0 = A \times \{0\}, \quad A_+ = \overline{\bigcup_n I_n}, \quad A_\pm = A_+ \cup A \times [-1, 0].$$

Each $I_n^\circ = I_n \setminus C \times \{0\}$ is open in $C_\pm$, and so the points of each I_n° have connected neighborhoods. Hence, A, A_+ and $A_\pm$ are $H(C_\pm)$-invariant. Thus, if A is a proper, clopen subset of C, then $C_\pm$ is $H(C_\pm)$-decomposable. Observe that every component is homeomorphic to the unit interval and the first coordinate projection maps $C_\pm$ onto the Cantor set C.

Also, if A is a proper, clopen set, then $C \times [-1, 0]$ admits chain transitive homeomorphisms, but the factor $X = C \cup A \times [-1, 0]$ is $H(X)$-decomposable.

A homeomorphism f on C can be extended to a homeomorphism F_+ of C_+ if and only if A is f-invariant. In that case, we can then define F_+ by using any orientation-preserving homeomorphism from I_x to $I_{f(x)}$, i.e., one which maps $\langle x, 0\rangle$ to $\langle f(x), 0\rangle$. Here $I_x = I_n$ for $x = a_n$ and $= \{(x, 0)\}$ if $x \notin S$. Continuity at points of A_+° is clear, and if $x \in C$, then for every ϵ there is a neighborhood U of x in C so that $y \in U \setminus \{x\}$ implies that the length of the interval $I_{f(y)}$ is less than ϵ. This implies continuity at $(x, 0)$. Notice that if $\mathcal{O}f(x)$ is infinite, then

$$\lim_{|n|\to\infty} |I_{f^n(x)}| = 0, \tag{3-14}$$

where $|J|$ denotes the length of an interval J. This says that any pair $(x, t_1), (x, t_2) \in I_x$ is *asymptotic* for F_+ and $(F_+)^{-1}$ with

$$\begin{aligned} \omega F_+(x, t_1) = \omega F_+(x, t_2) = \omega f(x) \times \{0\} \subset C_0, \\ \alpha F_+(x, t_1) = \alpha F_+(x, t_2) = \alpha f(x) \times \{0\} \subset C_0. \end{aligned} \tag{3-15}$$

Now assume that A is not clopen. If f is chain transitive and every point of A has an infinite orbit, then any extension F_+ to C_+ is chain transitive. Observe that F_+ is chain transitive if, whenever a_n is a periodic point for f, we define F_+ by using the unique linear, orientation-preserving homeomorphism from I_{a_n} to $I_{f(a_n)}$. On the other hand, if $f(a_1) = a_1$ and on I_{a_1} we define F_+ by $(a_1, t) \mapsto (a_1, t^2)$ then $(a_1, 1)$ is a repelling fixed point for F_+ on I_{a_1} and hence for F_+ on C_+. If F_+ is chain transitive, then we can obtain a chain transitive extension $F_\pm$ on $C_\pm$ by using $f \times 1_{[-1,0]}$ on $C \times [-1, 0]$. □

Example 3.20. We construct a space X so that

- the connected components of X are all homeomorphic to $[0, 1]$;
- the space of connected components, $[X]$, consists of a convergent sequence and its limit, and the union of isolated components in X is dense;
- it is f-decomposable for any $f \in H(X)$ but not $H(X)$-decomposable.

The space X we construct is $H(X)$-indecomposable by the first two properties and by Lemma 3.17.

For every $n \in \mathbb{N}$ we define $I_n = [0, n^{-1}]$ and a continuous function $t_n : I_n \to I = [0, 1]$ so that, for integers $i = 0, \dots, (2n)!$,

$$t_n\left(\frac{i}{n(2n)!}\right) = \begin{cases} 0 & \text{when } i \text{ is even,} \\ 1 & \text{when } i \text{ is odd,} \end{cases}$$

and the rest of the values are defined by linear interpolations.

Each interval I_n contains $(2n)!$ intervals $\{I_n^i : i = 1, \dots, (2n)!\}$ of equal length, each of which is mapped by t_n onto I. We can further subdivide each I_n^i into intervals $\{I_n^{i,j} : j = 1, \dots, n\}$ of equal length so that each is mapped to a subinterval of I of length n^{-1} by t_n. The corresponding restrictions of t_n are denoted by $t_{n,i} = t_n|_{I_n^i}$ and $t_{n,i,j} = t_n|_{I_n^{i,j}}$.

Recall that we identify a function with its graph, so the functions t_n, $t_{n,i}$ and $t_{n,i,j}$ are all closed subsets of $I_n \times I$. We define

$$\begin{aligned} X_n &= \{n^{-1}\} \times t_n, \quad n \in \mathbb{N}, \\ X_\infty &= \{\langle 0, 0\rangle\} \times [0, 1], \\ X &= \bigcup_n X_n \cup X_\infty. \end{aligned} \tag{3-16}$$

The space X is clearly a closed, bounded subset of $\mathbb{R}^3$, and the space $[X]$ can be identified with $\pi(X)$, where $\pi : X \to \{n^{-1} : n \in \mathbb{N}\} \cup \{0\}$ is the projection to the first coordinate. The union of the isolated components is clearly dense in X. In addition, for all appropriate n, i and j, we define

$$X_n^i = \{n^{-1}\} \times t_{n,i} \quad \text{and} \quad X_n^{i,j} = \{n^{-1}\} \times t_{n,i,j}. \tag{3-17}$$

Note that each X_n is a union of the line segments X_n^i. Similarly, each X_n^i is a union of the line segments $X_n^{i,j}$. The diameter of each X_n^i is greater than 1, and the diameter of each $X_n^{i,j}$ is less than $2n^{-1}$. The arc length of X_n is greater than $(2n)!$.

Suppose that $h : X_n \to X_m$ is a homeomorphism for some $n < m < \infty$. Then for some $i = 1, \dots, (2n)!$, $j = 1, \dots, n$ and $k = 1, \dots, (2m)!$, it must happen that $h(X_n^{i,j}) \supset X_m^k$. If not, then each $h(X_n^{i,j})$ meets at most two of the segments X_m^i, so the arc length of each mapped segment $h(X_n^{i,j})$ is less than 4. The arc length of $h(X_n)$, which is the sum of the arc lengths of the $(2n)!n$ mapped segments $h(X_n^{i,j})$, is less than $4(2n)!n$. Since $4(2n)!n < (2m)!$, the map h cannot be surjective. It follows that there exists a pair of points $u, v \in X_n$ with $d(u, v) \le 2n^{-1}$ but with $d(h(u), h(v)) \ge 1$.

Now if $f \in H(X)$, then it follows that the induced homeomorphism $[f]$ on the space $[X]$ is the identity on all but finitely many points. For if not, then by replacing f by f^{-1} if necessary, we can assume that there are sequences (m_i) and (n_i) in $\mathbb{N}$ tending to infinity such that $f(X_{n_i}) = X_{m_i}$ and $m_i > n_i$ for all i. Hence, there exist $u_i, v_i \in X_{n_i}$ with $d(u_i, v_i) \le 2n_i^{-1}$ but with $d(f(u_i), f(v_i)) \ge 1$. Thus, there are convergent subsequences of $\{u_i\}$ and $\{v_i\}$ with a common limit in X_∞. Hence, f cannot extend to a continuous function on all of X.

Thus, there exists $N \in \mathbb{N}$ such that $f(X_n) = X_n$ for all $N \le n \le \infty$. Since the isolated components X_n are invariant for n large enough, X is f-decomposable. □

Example 3.21. We construct spaces X and X^+ so that

- the isolated components of X and X^+ are all homeomorphic to one another and their union is dense;
- X^+ is $H(X^+)$-indecomposable but it is f-decomposable for all $f \in H(X^+)$;
- there exists $f \in H(X)$ such that X is f-indecomposable, but no $f \in H(X)$ is chain transitive.

Let $Z = \{x_n : n \in \mathbb{Z}\} \subset I = [0, 1]$ with

$$x_n = \begin{cases} 1 - (n+2)^{-1} & \text{for } n = 0, 1, \dots, \\ (|n|+2)^{-1} & \text{for } n = -1, -2, \dots. \end{cases}$$

Define for $n \in \mathbb{Z}$

$$a_n = \langle x_n, 0 \rangle, \quad b_n = \langle x_n, (|n|+1)^{-1} \rangle, \qquad I_n = \{\langle x_n, t \rangle : 0 \le t \le (|n|+1)^{-1}\}. \tag{3-18}$$

Define

$$\begin{gathered} J = I \times \{0\}, \quad H = [-1, 0] \times \{0\}, \\ C = H \cup J \cup \bigcup \{I_n : n \in \mathbb{Z}\}, \\ Z_0 = \{a_n : n \in \mathbb{Z}\}, \quad Z_1 = \{b_n : n \in \mathbb{Z}\}, \\ e_0 = \langle 0, 0 \rangle, \quad e_1 = \langle 1, 0 \rangle, \quad e_{-1} = \langle -1, 0 \rangle. \end{gathered} \tag{3-19}$$

We will call C a *comb* with handle H.

The group $H(C)$ fixes e_{-1}, e_0 and e_1. Each of the sets Z_0 and Z_1 is a single $H(C)$-orbit. The closed sets J and H are $H(C)$-invariant.

If $h \in H(C)$, then for some k

$$h(b_n) = b_{n+k} \;\Leftrightarrow\; h(a_n) = a_{n+k} \;\Leftrightarrow\; h(I_n) = I_{n+k}.$$

In that case $h(a_{n\pm1}) = a_{n\pm1+k}$, and so h induces a translation by k on the sequences Z_0 and Z_1. If $k > 0$, then e_1 is an attractor with complementary repeller H, and the reverse is true if $k < 0$. If $k = 0$, then h fixes each point of Z_0 and of Z_1.

On C define T by

$$\begin{aligned} T(e_0) &= e_0, & T(e_{\pm1}) &= e_{\pm1}, \\ T(a_n) &= a_{n+1}, & T(b_n) &= b_{n+1}, \end{aligned} \tag{3-20}$$

and with $T : [a_{n-1}, a_n] \to [a_n, a_{n+1}]$, $T : I_n \to I_{n+1}$ and $T : H \to H$ linear for all $n \in \mathbb{Z}$. Thus, $T \in H(C)$ induces a translation by 1 on Z_0 and Z_1 and is the identity on H.

For $n \in \mathbb{Z}$ let

$$\begin{aligned} I_n^* &= I_n \cup \left\{(x, (|n|+1)^{-1}) : \tfrac{1}{2}(x_{n-1} + x_n) \le x \le \tfrac{1}{2}(x_{n+1} + x_n)\right\}, \\ C_n &= C \cup I_n^*, \end{aligned} \tag{3-21}$$

a comb with a queer tooth I_n^* at n replacing I_n. Observe that C and all the C_n are connected.

If $f : C_n \to C_m$ is a homeomorphism, then $f(I_n^*) = I_m^*$. If $h \in H(C)$, then h extends to a homeomorphism from C_n to C_m if and only if $h(I_n) = I_m$. Thus, C_n and C_m are homeomorphic for all $n, m \in \mathbb{Z}$.

In $\mathbb{R}^2 \times I$ we define

$$\begin{aligned} X &= C \times \{0, 1\} \cup \bigcup \{C_n \times \{x_n\} : n \in \mathbb{Z}\}, \\ X^+ &= C \times \{1\} \cup \bigcup \{C_n \times \{x_n\} : n \in \mathbb{N}\}, \end{aligned} \tag{3-22}$$

Thus, X and X^+ both have a dense union of isolated components, and each isolated component is homeomorphic to C_0. It follows from Lemma 3.17 that X is $H(X)$-indecomposable and X^+ is $H(X^+)$-indecomposable.

Any homeomorphism in $H(X^+)$ restricts to a homeomorphism of $C \times \{1\}$ and so restricts to translation by k on $Z_0 \times \{1\}$. This means that h must map $C_n \times \{x_n\}$ to $C_{n+k} \times \{x_{n+k}\}$ when $n \ge N$ for N sufficiently large, and we may suppose $N + k > 0$. This implies that h maps the set of N complementary components $\{C_n \times \{x_n\} : 0 \le n < N\}$ to the set of $N + k$ components $\{C_n \times \{x_n\} : 0 \le n < N + k\}$. This requires that $k = 0$. Hence, each isolated component $C_n \times \{x_n\}$ with $n \ge N$ is invariant, and so X^+ is h-decomposable.

Thus, X^+ is an example of a space which is $H(X^+)$-indecomposable but such that X^+ is h-decomposable for every $h \in H(X^+)$.

In the case of X, we start by assuming that h fixes each of the two nonisolated components, i.e., h maps $C \times \{\epsilon\}$ to itself for $\epsilon = 0, 1$. As before, since h translates $Z_0 \times \{\epsilon\}$ by some k_ϵ for $\epsilon = 0, 1$, there exists $N \in \mathbb{N}$ large enough that h maps $C_n \times \{x_n\}$ to $C_{n+k_1} \times \{x_{n+k_1}\}$ for $n \geq N$ and $C_n \times \{x_n\}$ to $C_{n+k_0} \times \{x_{n+k_0}\}$ for $-n \geq N$. Again we can choose N large enough that $N + k_\epsilon > 0$ for $\epsilon = 0, 1$. This implies that h maps the set of complementary components $C_n \times \{x_n\}$ for $-N < n < N$ to the set of components $C_n \times \{x_n\}$ for $-N + k_0 < n < N + k_1$. This requires that $k_0 = k_1$, and we let $k = k_0 = k_1$.

If $k > 0$, then $C \times \{1\}$ is an attractor and $C \times \{0\}$ is a repeller with the reverse if $k < 0$. Hence, h is not chain transitive. If $k = 0$, then each isolated component $C_n \times \{x_n\}$ is invariant for $|n| \geq N$, and so X is h-decomposable.

The remaining possibility is that h interchanges the two limit components $C \times \{\epsilon\}$ for $\epsilon = 0, 1$ and then each is invariant for h^2. Applying the previous argument to h^2, we see that for a large N the components are translated by h^2 with a common k. If $k > 0$, then $C \times \{1\}$ is an attractor for h^2 while $C \times \{0\}$ is a repeller. But since h commutes with h^2 this would imply that $C \times \{0\} = h(C \times \{1\})$ would be an attractor for h^2 as well, which it is not. Similarly, $k < 0$ leads to a contradiction. It follows that in this interchange case $k = 0$, and so $C_n \times \{x_n\}$ is h^2-invariant for $|n|$ sufficiently large. Hence, for each such n the set $C_n \times \{x_n\} \cup h(C_n \times \{x_n\})$ is clopen and h-invariant, so X is h-decomposable when h interchanges the ends.

Finally, extend $T : C \to C$ by

$$T_n : C_n \times \{x_n\} \to C_{n+1} \times \{x_{n+1}\}$$

for all $n \in \mathbb{Z}$ and by $T \times 1_{\{0,1\}}$ on $C \times \{0, 1\}$ to obtain a homeomorphism (with $k = 1$) with respect to which X is not decomposable.

Thus, X is an example of a space with an element $h \in H(X)$ such that X is not h-decomposable, but nonetheless there is no chain transitive homeomorphism in $H(X)$. □

4. Spaces with all homeomorphisms chain transitive

Having considered spaces which admit no chain transitive homeomorphisms, we turn to the opposite extreme to consider spaces such that every homeomorphism is chain transitive. By Proposition 2.7(e) the identity 1_X is chain transitive if and only if X is connected, and if $f \in H(X)$ with X connected, then f is chain transitive if and only if it is chain recurrent.

A space X is called *rigid* if 1_X is the only homeomorphism on X, i.e., the group $H(X)$ is trivial. Such spaces were introduced and constructed by de Groot and Wille

[1958]. For a connected rigid space the only element of $H(X)$ is chain transitive. We construct some more interesting examples by using rigid spaces as tools. We need a pairwise disjoint sequence $\{Z_n\}$ of connected, locally connected spaces such that

(RIG) for any nonempty open subset U of Z_n and any disk I^k, there does not exist a homeomorphism of $U \times I^k$ onto any subset of $(Z_n \setminus U) \times I^k$ or onto any subset of $Z_m \times I^k$ with $m \neq n$;

(CON) for every $n \in \mathbb{N}$ and for any finite $F \subset Z_n$, the set $Z_n \setminus F$ has only finitely many components, and for any positive integer N there is a subset $F \subset Z_n$ of cardinality N such that $Z_n \setminus F$ is connected.

Condition RIG is a slight strengthening of the condition on a space called *strongly chaotic* in [Charatonik and Charatonik 1996]. We construct such a sequence in the Appendix. For each Z_n we choose a pair of distinct points $e_n^-, e_n^+ \in Z_n$. Our examples are obtained by using such rigid spaces instead of the unit interval in some common constructions.

Example 4.1. Suppose that G is a finitely generated (and hence countable) group. Following de Groot [1959], we construct a space X so that

- the homeomorphism group $H(X)$ is isomorphic to G;
- every $f \in H(X)$ is chain transitive.

First, consider the case $G = \mathbb{Z}$. We can think of the real line as a graph with $\mathbb{Z}$ as the set of vertices and intervals $[n, n+1]$ as edges. Now let $Z = Z_1$ be one of the chaotic spaces described above with points $e^-, e^+ \in Z$. We replace each edge by a copy of Z. That is, let X_0 be the quotient space of $\mathbb{Z} \times Z$ with (n, e^+) identified with $(n+1, e^-)$. If t is the translation homeomorphism on $\mathbb{Z}$ with $t(n) = n+1$, then t has a unique extension t to X_0 which is the quotient of $t \times 1_Z$. The only homeomorphisms on X_0 are the iterates t^n. Let X be the one-point compactification of X_0 with the additional point ∞. Let $t \in H(X)$ be the unique homeomorphism extension of t on X_0 and so of t on $\mathbb{Z}$. Since X is connected, 1_X is chain transitive. For any $n \neq 0$, we have $\{\infty\} = \omega(t^n)(x) = \alpha(t^n)(x)$ for all $x \in X$, and so t^n is chain transitive. In this case, $H(X)$ is isomorphic to $\mathbb{Z}$.

In general, suppose that G is generated by $\{g_1, \dots, g_n\}$. Let X_0 be the Cayley graph with rigid spaces as linking edges. That is, let $\{g_1, \dots, g_n\}$ be a list of generators for G, and let $\{Z_1, \dots, Z_n\}$ be distinct strongly chaotic spaces as above, each with a chosen pair of points. Let X_0 be the quotient space of $G \cup \big[G \times \big(\bigcup_{i=1}^n Z_i\big)\big]$ with (g, e_i^+) identified with $(g_i g, e_i^-)$ for $g \in G$, $i = 1, \dots, n$ and (g, e_i^-) identified with $g \in G$ for $i = 1, \dots, n$. Because there are only finitely many generators, the space X_0 is locally compact and the set of vertices $\{g \in G\}$ is invariant with respect to any homeomorphism h. Furthermore, if $g \in G$, then $h(g_i g) = g_i h(g)$, and so h commutes with all left translations. It follows that, on G, the mapping h is the right translation r_v, where $v = h(u)$ and u is the identity element of G. Thus, on X_0, the mapping h is the quotient of the map $r_v \cup [r_v \times 1_{\bigcup_i Z_i}]$. Let X be the one-point compactification of X_0, and let r_v denote the extension of h to X. If v is of finite order k, then $(r_v)^k = 1_X$ and so r_v is chain transitive on X. If v is of infinite order, then $\{\infty\} = \omega(r_v)(x) = \alpha(r_v)(x)$, and so r_v is chain transitive on X in this case as well. The group $H(X)$ is isomorphic to the discrete group G by $v \mapsto r_v^{-1}$. □

In these cases, the homeomorphism group is discrete. It is possible to obtain rather large nondiscrete groups. We first review some standard topology constructions.

A *pointed space* is a pair (X, x) consisting of a space with a chosen *base point* $x \in X$. We let $H(X, x)$ denote the closed subgroup of $H(X)$ consisting of those homeomorphisms which fix x. A space Y can be regarded as a pointed space with base point an isolated point not in Y. If (X_1, x_1) and (X_2, x_2) are pointed spaces and $f : X_1 \to X_2$ is a function, we use the notation $f : (X_1, x_1) \to (X_2, x_2)$ to mean that $f(x_1) = x_2$.

If A is a nonempty closed subset of a space X, then the space X/A with A *smashed to a point* is the quotient space of X with respect to the closed equivalence relation $1_X \cup A \times A$. Thus, the quotient map $q : X \to X/A$ is a homeomorphism between the open sets $X \setminus A$ and $X/A \setminus \{x_A\}$ with x_A the point which is the image of A.

Given two pointed spaces (X_1, x_1), (X_2, x_2), their *smash product* is

$$(X_1, x_1) \# (X_2, x_2) = (X_{12}, x_{12}),$$

a pointed space consisting of the product $X_1 \times X_2$ with the *wedge* $X_1 \times \{x_2\} \cup \{x_1\} \times X_2$ smashed to the point x_{12}. We can also define the smash product of a pointed space (X_1, x_1) and any space X_2 as $(X_1, x_1) \# X_2 = (X_{12}, x_{12})$, where X_{12} is the product $X_1 \times X_2$ with $\{x_1\} \times X_2$ smashed to the point x_{12}. Notice that in this case we can regard the space X_{12} as the one-point compactification of $(X_1 \setminus x_1) \times X_2$. The projections $\pi_1 : (X_1, x_1) \# X_2 \to (X_1, x_1)$ and $\pi_2 : X_1 \setminus \{x_1\} \times X_2 \to X_2$ are open and surjective.

We can define the smash product of two continuous functions once we fix base points from the domains. For $i = 1, 2$, let X_i and Y_i be spaces, $f_i : X_i \to Y_i$ a continuous function and $x_i \in X_i$ a base point. We set $y_i = f_i(x_i)$ to obtain a pointed space (Y_i, y_i) for $i = 1, 2$. Let $(X_{12}, x_{12}) = (X_1, x_1) \# (X_2, x_2)$ and $(Y_{12}, y_{12}) = (Y_1, y_1) \# (Y_2, y_2)$, and let $q : X_1 \times X_2 \to X_{12}$ and $r : Y_1 \times Y_2 \to Y_{12}$ be the quotient maps. We define a continuous function $f = (f_1, x_1) \# (f_2, x_2)$ from (X_{12}, x_{12}) to (Y_{12}, y_{12}) by the formula $f \circ q = r \circ (f_1 \times f_2)$. We can also define the smash product of two functions when only one of the domains has a base point. If $(V, v) = (X_1, x_1) \# X_2$ and $(W, w) = (Y_1, y_1) \# Y_2$ and if $s : X_1 \times X_2 \to V$ and $t : Y_1 \times Y_2 \to W$ are the quotient maps, then we define the continuous function $g = (f_1, x_1) \# f_2$ from (V, v) to (W, w) by $g \circ s = t \circ (f_1 \times f_2)$.

Example 4.2. We construct a space X so that

- the homeomorphism group $H(X)$ contains a nontrivial path-connected subgroup;
- every $f \in H(X)$ is chain transitive.

Let (Z, e) be a chaotic space Z, as above, with base point $e \in Z$ such that $Z \setminus \{e\}$ is connected. Let W be a connected, compact manifold (perhaps with boundary) of positive dimension k. Let $(X, e_X) = (Z, e) \# W$. If $(z, w) \in (Z \setminus \{e\}) \times W = X \setminus \{e_X\}$ and h is a homeomorphism from an open set containing (z, w) into X, then $h(z, w) = (z_1, w_1)$ with $z_1 \in Z \setminus \{e\}$ implies $z = z_1$. If not, then we can choose disk neighborhoods of w and w_1 each homeomorphic to I^k, and we can choose disjoint open neighborhoods U of z and U_1 of z_1 so that h induces a homeomorphism from $U \times I^k$ onto a subset of $U_1 \times I^k$, and this contradicts condition RIG. If $h(z, w) = e_X$, then h would map points (z_2, w_2) close to (z, w) in $X \setminus \{e_X\}$ to points (z_1, w_1) with z_1 in $X \setminus \{e_X\}$ close to e. This does not happen by the previous argument. Thus, it follows that $\pi_1 \circ h = \pi_1$, where π_1 is the projection to (Z, e). This implies that $H(X) = H(X, e_X)$, and π_1 maps every $h \in H(X)$ to 1_Z.

Since the homeomorphisms of X leave the preimages of π_1 invariant, it follows that every $h \in H(X)$ is of the form $h(z, w) = (z, q(z)(w))$ with $q : Z \setminus \{e\} \to H(W)$ a continuous map. Thus, the space of continuous maps $C(Z \setminus \{e\}, H(W))$ with the obvious group structure is isomorphic as a group with $H(X)$. Notice that we need not worry about behavior as z approaches e in Z since all of $\{e\} \times W$ is smashed to a point. Thus, the isomorphism is topological if we choose an increasing sequence $\{K_n\}$ of compacta in Z with union $Z \setminus \{e\}$ and define the metric $d(q_1, q_2) = \sup_n 2^{-n} d_H(q_1|_{K_n}, q_2|_{K_n})$ with d_H the uniform metric on $H(W)$.

In particular, the constant maps q yield $H(W)$ as a subgroup of $H(X)$. Since W is a manifold of positive dimension, it follows that the path components of the identity in $H(W)$ and hence in $H(X)$ are nontrivial subgroups. The remaining path components are cosets and so are nontrivial as well.

Let $h \in H(X)$. Given $\epsilon > 0$ and $(z, w) \in X \setminus \{e_X\}$, there is an ϵ-chain $z_0, \dots, z_N$ for 1_Z with $z_0 = z$ and $z_N = e$ and $z_i \neq e$ for $i < N$. That is, $d(z_{n+1}, z_n) \leq \epsilon$. Define $\{w_0, \dots, w_N\}$ by $w_0 = w$ and $h(z_i, w_i) = (z_i, w_{i+1})$ for $i < N$. Clearly, $(z_0, w_0), \dots, (z_{N-1}, w_{N-1}), e_X$ is an ϵ-chain for h from (z, w) to e_X. Similarly, there is an ϵ-chain for h^{-1} from (z, w) to e_X. It follows that h is chain recurrent. Since X is connected, h is chain transitive. □

Now let (Y, e) be a pointed space such that Y and $Y \setminus \{e\}$ are connected and, for every $y \in Y$, the open set $Y \setminus \{y\}$ has only finitely many components. Let C be a zero-dimensional space, and let $(X, e_X) = (Y, e) \# C$. Since C is zero-dimensional, the components of $X \setminus \{e_X\}$ are the sets $\{(Y \setminus \{e\}) \times \{c\} : c \in C\}$. If $y \in Y \setminus \{e\}$ and $c \in C$, then the components of $X \setminus \{(y, c)\}$ which do not contain e are all of the form $D \times \{c\}$, where D is a component of $Y \setminus \{y\}$ which does not contain e. Thus, if C is infinite, $X \setminus \{e_X\}$ has infinitely many components, while $X \setminus \{(y, c)\}$ has only finitely many components for $y \in Y \setminus \{e\}$, $c \in C$. Hence, if C is infinite, then any homeomorphism of X fixes e_X. We will assume that, even with C finite, the space Y is such that the point e_X is fixed by every homeomorphism of X.

It follows that for any homeomorphism on h on X, the projection $\pi_2 : X \setminus \{e_X\} \to C$ maps the restriction of h on $X \setminus \{e_X\}$ to a homeomorphism on C. The map on C is continuous because with $y_0 \neq e$ fixed, the map is given by $c \mapsto \pi_2(h(y_0, c))$. Similarly, the inverse is continuous. Thus, we obtain $(\pi_2)_* : H(X) \to H(C)$, a continuous, surjective homomorphism of topological groups. This splits via the continuous injection $j : H(C) \to H(X)$ given by $j(k) = (1_Y, e) \# k$.

Now suppose that $h \in H(X)$ is in the kernel of $(\pi_2)_*$, that is, it projects to 1_C. This means that every $(Y \setminus \{e\}) \times \{c\}$ is h-invariant. It follows that $h(y, c) = (q(c)(y), c)$, where $q : C \to H(Y, e)$ is a continuous map. That is, the kernel is $C(C, H(Y, e))$ with the obvious topological group structure. Thus, $H(X)$ is the semidirect product of $H(C)$ with $C(C, H(Y, e))$. The adjoint action of $j(H(C))$ is just the action $H(C) \times C(C, H(Y, e)) \to C(C, H(Y, e))$ given by $(k, q) \mapsto q \circ (k^{-1})$.

Example 4.3. We construct a space X so that

- the homeomorphism group $H(X)$ is isomorphic to the homeomorphism group of the Cantor set;
- every $f \in H(X)$ is chain transitive.

Let (Z, e) be a pointed chaotic space as before, and let $(X, e_X) = (Z, e) \# C$ with C a zero-dimensional space. We first check that even if C is finite, any homeomorphism h on X fixes e_X. If not, then there exist points $x_1, x_2 \in Z$, distinct from each other and distinct from e, and points $a_1, a_2 \in C$ such that $h(x_1, a_1) = (x_2, a_2)$. This implies that h induces a homeomorphism between sufficiently small neighborhoods U_1 of x_1 and U_2 of x_2. Choosing these as disjoint neighborhoods, we obtain a contradiction of condition RIG.

In this case $H(Z, e) = H(Z) = \{1_Z\}$. That is, the group $H(Z, e)$ is trivial, and so the group $C(C, H(Z, e))$ is trivial. This means that $(p_2)_* : H(X) \to H(C)$ and $j : H(C) \to H(X)$ are inverse isomorphisms, so every homeomorphism on X is mapped by π_1 to 1_Z. Just as in Example 4.2, it follows that every homeomorphism is chain transitive.

When C is a Cantor set, we obtain an example with homeomorphism group isomorphic to the homeomorphism group of the Cantor set. □

Because of the rigidity of the connecting links, it is not true in these examples that $H(X)$ acts transitively on X. We can obtain examples which satisfy this additional condition by using the beautiful construction of Slovak spaces due to Downarowicz, Snoha, and Tywoniuk in [Downarowicz et al. 2017].

Let g be a totally transitive homeomorphism on a Cantor set W. That is, g^n is topologically transitive for all $n \in \mathbb{Z} \setminus \{0\}$. The construction begins with the *suspension* of g. That is, let $Y = W \times [0, 1]$ with $(x, 1)$ identified with $(g(x), 0)$ for all $x \in W$. On Y we define the real flow $\phi : \mathbb{R} \times Y \to Y$, the associated time-$t$ map $\phi^t : Y \to Y$, and the path map $\phi_x : \mathbb{R} \to Y$ for $t \in \mathbb{R}$, $x \in W$ by

$$\begin{aligned} \phi(t, (x, s)) = \phi^t(x, s) &= (g^{[t+s]}(x), \{t + s\}), \\ \phi_x(t) &= \phi^t(x, 0), \end{aligned} \tag{4-1}$$

where $[a]$ and $\{a\}$ are the integer part and fractional part, respectively, of the real number a. Identifying W with $W \times \{0\} \subset Y$, we see that g on W is identified with the time-one map ϕ^1 restricted to W.

Observe that the time-s map of the flow ϕ^s restricts to a homeomorphism from $\left[-\frac{1}{3}, \frac{1}{3}\right] \times W$ onto a neighborhood of $W \times \{s\}$ in Y for $s \in [0, 1]$. It follows that the path components of Y are exactly the $\mathbb{R}$-orbits of the flow. For $x \in W$ we let $\mathbb{R}x = \phi_x(\mathbb{R})$ denote the $\mathbb{R}$-orbit through $(x, 0)$.

In Y there are three types of path components:

Type 1: If x is a periodic point for g, then $\mathbb{R}x$ is a circle embedded in Y_0. This is a *circle type* path component.

Type 2: If x is recurrent for neither g nor g^{-1}, i.e., $x \notin \omega g(x) \cup \alpha g(x)$, then $\mathbb{R}x$ is an embedded copy of $\mathbb{R}$. That is, ϕ_x is a homeomorphism from $\mathbb{R}$ onto its image in Y. This is an *embedded* $\mathbb{R}$ *type* path component.

Type 3: If x is not periodic but $x \in \omega g(x) \cup \alpha g(x)$, then ϕ_x is a continuous injection which is not a homeomorphism onto its image. In fact, $\mathbb{R}x \subset Y_0$ is not locally connected. This is an *injected* $\mathbb{R}$ *type* path component.

The Slovak space construction is based on the following result:

Theorem 4.4 [Downarowicz et al. 2017, Lemma 4.3]. *Let f be a homeomorphism on a space Y with y_0 a point of Y which is not a periodic point for f. Let $\{a_n : n \in \mathbb{Z}\}$ be a sequence of positive reals such that $\sum_n a_n = 1$ and the set $\{|\ln(a_n) - \ln(a_{n-1})| : n \in \mathbb{Z}\}$ is bounded, and let $u : Y \setminus \{y_0\} \to [0, 1]$ be a continuous function. Let $Y' = Y \setminus \mathcal{O} f_{\pm}(y_0)$, and on Y' define the continuous function $u' = \sum_n a_n u \circ f^n$ so that the graph of u' is a closed subset of $Y' \times [0, 1]$ with the first coordinate projection $p : u' \to Y'$ a homeomorphism. Define on the set u' the homeomorphism $f' = (p)^{-1} \circ f \circ (p)$. Let X be the closure of u' in $Y \times [0, 1]$.*

The homeomorphism f' and its inverse are uniformly continuous on u', and so f' extends to a homeomorphism h on X. The first coordinate projection $p : X \to Y$ maps h on X to f on Y. If $y \in Y'$, then $(y, u'(y))$ is the unique point of X which is mapped by p to y. Hence, h is an almost one-to-one extension of f.

Now, following [Downarowicz et al. 2017], we apply this construction to the situation above, i.e., with Y the suspension of the Cantor set W via a totally transitive homeomorphism g.

Let x_0 be an element of the dense G_δ set $\bigcap\{\mathrm{Trans}(g^n) : n \in \mathbb{Z} \setminus \{0\}\}$. By [Akin 1993, Proposition 6.3(a)] the set of $\tau \in (0, \infty)$ such that $\omega\phi^{n\tau}(x_0, 0) = Y = \alpha\phi^{n\tau}(x_0, 0)$ is residual in $(0, \infty)$. Since the irrationals are also residual, we may fix such a τ, irrational in $(0, 1)$, and let $f = \phi^\tau$. Thus, $y_0 = (x_0, 0)$ is a transitive point for f^n on Y for all $n \in \mathbb{Z} \setminus \{0\}$. Since τ is irrational, f has no periodic points.

We first define u on a piece of the orbit of the point $y_0 = (x_0, 0)$:

$$u(\phi_{x_0}(t)) = \begin{cases} 0 & \text{for } -\frac{1}{2} \le t < 0, \\ \frac{1}{2}\bigl(1 - \cos\bigl(\frac{\pi}{t}\bigr)\bigr) & \text{for } 0 < t \le \frac{1}{2}. \end{cases} \tag{4-2}$$

Apply the Tietze extension theorem to obtain the continuous function $u : Y \setminus \{y_0\} \to [0, 1]$.

Apply Theorem 4.4 to Y with the homeomorphism f. We obtain an almost one-to-one lift h on $X = \overline{u'} \subset Y \times [0, 1]$. All of the path components of X are mapped by p homeomorphically onto the path components of Y, except that the Type 3 path component $\mathbb{R}y_0$ is cut into a sequence of path components of a new type.

Type 4: The path component Comp_n of $h^n(y_0)$ is mapped by p onto the set $\phi_{x_0}\bigl(((n-1)\tau, n\tau]\bigr)$. As $t \searrow (n-1)\tau$, above the open interval end, there is a topologist's sine which projects homeomorphically. Above the $n\tau$ endpoint there is a vertical segment $J_n = \{h^n(y_0)\} \times [0, a_{-n}]$ to which the oscillating end of the path component Comp_{n+1} converges. That is, $J_n = \mathrm{Comp}_n \cap \overline{\mathrm{Comp}_{n+1}}$. Thus, each path component Comp_n is a homeomorphic image of $\mathbb{R}_+ = [0, \infty)$. Each is an *embedded* $\mathbb{R}_+$ path component.

Theorem 4.5. *The homeomorphism group of X is $H(X) = \{h^n : n \in \mathbb{Z}\}$. For all $n \neq 0$ the homeomorphism h^n is topologically transitive.*

Proof. If $n \neq 0$, then by choice of τ, the homeomorphism $f^n = \phi^{n\tau}$ is topologically transitive on Y. Because p mapping h to f is an almost one-to-one lift, it follows that h^n is topologically transitive on X.

If h_1 is any homeomorphism on X, then the Type 4 component Comp_n is mapped to some Type 4 component Comp_{n+k}. Furthermore, $J_n = \mathrm{Comp}_n \cap \overline{\mathrm{Comp}_{n+1}}$ is mapped to J_{n+k}. It follows that p projects h_1 to a continuous map on Y which agrees with f^k on the dense set $\mathcal{O}(f_\pm)(y_0)$. It follows that it projects to f^k. Since p is almost one-to-one, it follows that $h_1 = h^k$. □

Downarowicz, Snoha, and Tywoniuk begin with g on W minimal and observe that, for a residual set of positive reals τ, the homeomorphism ϕ^τ is minimal on Y.

Choosing one such τ, they have f minimal on Y. All of the components of Y are then of Type 3. Then h^n is minimal on X for all $n \neq 0$.

We recall and extend their definition of a Slovak space.

Definition 4.6. (a) A space X is a *Slovak space* if X contains at least three points, $H(X)$ is isomorphic to $\mathbb{Z}$ and every $h \in H(X) \setminus \{1_X\}$ is minimal.

(b) A space X is *Slovakian* if X contains at least three points, $H(X)$ is nontrivial and every $h \in H(X) \setminus \{1_X\}$ is topologically transitive.

We extend [Downarowicz et al. 2017, Theorem 4] with essentially the same proof.

Theorem 4.7. *A Slovakian space is connected, and its homeomorphism group has no elements of finite order other than the identity.*

Proof. Let h be a topologically transitive homeomorphism on a space X. Suppose that X contains a proper, clopen, nonempty subset A. If $h^{-1}(A) \subset A$, then for any $x \in \mathrm{Trans}(h)$, we have $h^n(x) \in A$ for some $n \in \mathbb{N}$, and so $x \in A$. Thus, the dense set $\mathrm{Trans}(h)$ is contained in A, contradicting the assumption that A is a proper, clopen set. It follows that $B = h^{-1}(A) \setminus A$ is a proper, clopen, nonempty set, and $B \cap h(B) = \varnothing$. Define

$$g(x) = \begin{cases} h(x) & \text{for } x \in B, \\ h^{-1}(x) & \text{for } x \in h(B), \\ x & \text{for } x \in X \setminus (B \cup h(B)). \end{cases} \tag{4-3}$$

The points of the nonempty set $B \cup h(B)$ are periodic with period 2, and so $g \neq 1_X$. Since $g^2 = 1_X$, it is clear that g is not topologically transitive.

Since X is nontrivial and connected, it is perfect and therefore uncountable. On such a space, no topologically transitive homeomorphism has finite order. □

Questions. Does there exists a Slovakian space X for which $H(X)$ is not discrete? More generally, does there exist a nontrivial space X such that the topologically transitive homeomorphisms are dense in $H(X)$? If X is such a space, then every $h \in H(X)$ is chain transitive. In particular, since 1_X is chain transitive, X is connected. On the other hand, 1_X is not topologically transitive, but it is a limit of topologically transitive homeomorphisms, and so $H(X)$ is not discrete.

The only Slovakian spaces we know of are variations on the original construction of [Downarowicz et al. 2017]. All of these have homeomorphism group isomorphic to $\mathbb{Z}$.

Now we extend the above construction, which was built on a totally transitive homeomorphism g on a Cantor space W. Suppose that we are given B, a proper, closed, g-invariant subset of W, and that $r : B \to A$ is a continuous surjection which maps the restriction $g|_B$ to 1_A, the identity on the space A. That is, $r^{-1}(a)$ is a closed, invariant set in B for every $a \in A$. Since B is proper, closed and invariant, it is disjoint from $\mathrm{Trans}(g)$. In particular, $x_0 \notin B$.

Let $\widehat{B}$ be the quotient of $B \times [0, 1]$ in Y. This is just the suspension of $g|_B$. Since $r \circ g = r$ on B, it follows that $r \circ \pi_1 : B \times [0, 1] \to A$ factors to define the surjection $\hat{r} : \widehat{B} \to A$, and each $\hat{r}^{-1}(a)$ is a ϕ-invariant closed subset of Y. Since $x_0 \notin B$ and B is g-invariant, $\widehat{B} \subset Y'$.

The preimage of $\widehat{B}$ via the homeomorphism $p : u' \to Y'$ is a compact f'-invariant subset of $X = \overline{u'}$. Thus, $p^{-1}(\widehat{B})$ is a closed h-invariant subset of X.

Now we attach Y and X to A, using the maps $\hat{r}$ and $\hat{r} \circ p$. That is, we define

$$\begin{aligned} E_{r,Y} &= 1_Y \cup [(\hat{r})^{-1} \circ \hat{r}], \\ E_{r,X} &= 1_X \cup [(\hat{r} \circ p)^{-1} \circ (\hat{r} \circ p)], \end{aligned} \tag{4-4}$$

closed equivalence relations on Y and X, respectively. Let $q_r^Y : Y \to Y_r$ and $q_r^X : X \to X_r$ be the projections to the quotient spaces. The homeomorphisms f and h induce homeomorphisms f_r and h_r on the quotient spaces. The projection $p : X \to Y$ induces $p_r : X_r \to Y_r$, a continuous, almost one-to-one surjection which maps h_r to f_r. As usual, we will regard the homeomorphisms induced by $\hat{r} : \widehat{B} \to A$ and $\hat{r} \circ p : p^{-1}(\widehat{B}) \to A$ as identifications, so A is thought of as a subset of Y_r and also as a subset of X_r. Recall that f has no periodic points and so h does not either. It follows that $A \subset Y$ and $A \subset X$ are the sets of fixed points for f_r and h_r, respectively.

For $a \in A$ and $x \in W$, we say that $q_r^Y(\mathbb{R}x)$ is an *a-orbit* if $\omega g(x) \subset r^{-1}(a)$ or $\alpha g(x) \subset r^{-1}(a)$. If $\omega g(x) \subset r^{-1}(a)$, then the map $q_r^Y \circ \phi_x : \mathbb{R} \to Y_r$ extends continuously to $\mathbb{R} \cup \{+\infty\}$ by mapping $+\infty$ to a. If $x \in B$, then x is an a-orbit if and only if $r(x) = a$, in which case $q_r^Y(\mathbb{R}x) = \{a\}$. If K_1 and K_2 are path components of A, they are *linked* if there exists an $x \in W$ which is both an a_1-orbit and an a_2-orbit for some $a_1 \in K_1$, $a_2 \in K_2$, i.e., if $\alpha g(x) \subset r^{-1}(a_1)$ and $\omega g(x) \subset r^{-1}(a_2)$ or vice versa. Two path components are *linkage equivalent* if there is a finite sequence $K_1, \ldots, K_N$ joining them with each K_i linked to its successor.

In Y_r we have a new path component type:

Type 5: Let $[K]$ be a linkage equivalence class of path components of A. The $[K]$-*component* in Y is the union of all of the a-orbits for $a \in K_1 \in [K]$ and of the path components $K_1 \in [K]$.

It is possible for a $[K]$-component to be of embedded $\mathbb{R}_+$ type. The only way this can happen is if the embedding of $[0, \infty)$ onto the $[K]$-component maps 0 to a point of A. The *endpoint* of this $\mathbb{R}_+$ type component lies in A.

Theorem 4.8. *The homeomorphism group of X_r is $H(X_r) = \{h_r^n : n \in \mathbb{Z}\}$. The homeomorphism h_r^n is topologically transitive for all $n \neq 0$.*

Proof. The surjection q_r^X maps the totally transitive homeomorphism h onto the totally transitive homeomorphism h_r.

If h_1 is any homeomorphism on X_r, we proceed just as in the proof of Theorem 4.5. There is, however, one tricky bit. It is possible that a Type 5 path component is of embedded $\mathbb{R}_+$ type. If Comp_n maps to Comp_{n+k}, then just as before, $J_n = \text{Comp}_n \cap \overline{\text{Comp}_{n+1}}$ is mapped to J_{n+k}, and $h_1 = h^k$ as before.

Suppose instead that Comp_n is mapped to some Type 5 path component, which we call Q_n. Then Comp_{n+1} will have to be mapped to a Type 5 path component Q_{n+1} with J_n mapping to $Q_n \cap \overline{Q_{n+1}}$, and this set contains the endpoint of Q_n which is in A. Thus, there is a sequence $\{x_n \in J_n : n \in \mathbb{Z}\}$ with $h_1(x_n) \in A$. But the sequence $\{x_n\}$ projects to the f_r-orbit of y_0, and this is dense in Y_r. Since $p_r : X_r \to Y_r$ is an almost one-to-one map, the sequence $\{x_n\}$ is dense in X_r. Since the proper closed subset A contains the sequence $\{h_1(x_n)\}$ and h_1 is a homeomorphism, we obtain a contradiction. □

If X is one of the examples of Slovakian spaces as constructed above, then $X \setminus D$ is connected for any countable subset D of X. This is because we can choose a countable invariant subset $D_0 \subset W$ with $x_0 \in D_0$ such that $D \subset D_0 \times [0, 1]$. Since $\text{Trans}(g) \cap \text{Trans}(g^{-1})$ is residual and thus uncountable, we can choose $x \in (\text{Trans}(g) \cap \text{Trans}(g^{-1})) \setminus D_0$. Then the orbit $\mathbb{R}x$ is connected, dense in X, and contained in $X \setminus D$. We don't know whether this property holds for all Slovakian spaces.

Now for our final construction.

Example 4.9. Let C be a space which is countable or the Cantor set. We construct a space K so that

- the homeomorphism group $H(K)$ is isomorphic as a topological group to the semidirect product of $H(C)$ with $C(C, \mathbb{Z})$;
- the action of $H(K)$ on K is topologically transitive;
- every element of $H(K)$ is chain transitive;
- if C is either finite or the Cantor set, then there exists a topologically transitive homeomorphism in $H(K)$.

Let g be a totally transitive homeomorphism on a Cantor set W with a fixed point $e \in W$. Let $r : \{e\} \to \{e\}$ be the identity. The space Y_r is the suspension of g with the circle $\{e\} \times [0, 1]$ smashed to the point e, and ϕ_r is the associated real flow on Y_r. Then X_r is the Slovakian space over Y_r with $p_r : X_r \to Y_r$ the almost one-to-one projection. The group $H(X_r)$ is cyclic with generator h_r mapped by p to ϕ_r^τ.

Let (K, e_K) be the smash product $(X_r, e) \# C$, and let $(L, e_L) = (Y_r, e) \# C$. We have the projections

$$\begin{aligned} \pi_L &: L \setminus \{e_L\} = (Y_r \setminus \{e\}) \times C \to C, \\ \pi = \pi_L \circ (p \times 1_C) &: K \setminus \{e_K\} = (X_r \setminus \{e\}) \times C \to C, \end{aligned} \tag{4-5}$$

and $P : (K, e_K) \to (L, e_L)$ is the smash product $(p, e) \# 1_C$.

If C is a singleton, then (K, e_K) is just (X_r, e), $H(C)$ is trivial, and $C(C, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and to $H(K)$. The identity is chain transitive, and all other elements of $H(K)$ are topologically transitive. So the result is clear in this case. Now assume that C has at least two points. This implies that e_K disconnects K, while no other point does since no point of X_r disconnects X_r. Hence, every homeomorphism of K preserves e_K.

From the discussion preceding Example 4.3 it follows that $H(K)$ is the semi-direct product of $H(C)$ and $C(C, H(X_r))$, which is essentially $C(C, \mathbb{Z})$ since X_r is Slovakian. The projection $\pi : K \setminus \{e_K\} \to C$ induces the group surjection $\pi_* : H(K) \to H(C)$, which is split by the injection $j : H(C) \to H(K)$. If $k \in H(C)$, then $j(k)$ is $(1_{X_r}, e) \# k$. In this case, the subgroup $C(C, \mathbb{Z})$ is commutative. If $q \in C(C, \mathbb{Z})$, then the associated element of $H(K)$ is the projection of $(x, c) \mapsto (h_r^{q(c)}(x), c)$. The constant elements, the homeomorphisms $(h^n, e) \# 1_C$, commute with all the elements of $H(K)$ since they commute with the members of the subgroup $j(H(C))$. For $(n, k) \in \mathbb{Z} \times H(C)$ let $J(n, k) = (h^n, e) \# k$. Thus, $J : \mathbb{Z} \times H(C) \to H(K)$ is a topological embedding and a group homomorphism.

Note that P maps $(h_r^{q(c)}(x), c)$ to $(\phi_r^{q(c)\tau}(x), c)$ and maps $(y, k(c))$ to $(p(y), k(c))$. It follows that every homeomorphism of K projects by P to a homeomorphism of L.

Case 1 (C is finite): Let k be a cyclic permutation on C so that C consists of a single periodic orbit under k. Since h is totally transitive, the product $h \times k$ on $X_r \times C$ is topologically transitive, and so it projects to a topologically transitive element of $\mathbb{Z} \times H(C) \subset H(K)$.

Let F be an arbitrary homeomorphism on K with $k = \pi_*(F)$. Since k is a permutation of a finite set, there exists $N \in \mathbb{N}$ such that k^N is the identity. This means that F^N projects to the identity on C and so preserves each fiber $X_r \times \{c\}$. So there exists $n_c \in \mathbb{Z}$ such that F^N restricts to h^{n_c} on the fiber. This is chain transitive on the fiber, topologically transitive if $n_c \neq 0$, and so every point is chain recurrent for F^n and hence for F. Since K is connected, F is chain transitive on K.

Case 2 (C is countably infinite): Since C is countable, the isolated points are dense in C. If $x_1, x_2 \in C$ are distinct isolated points, let k interchange these two points and fix the remaining points of C. By Case 1, $J(1, k) = (h, e) \# k$ restricts to a topologically transitive homeomorphism on the closed subset $(X_r, e) \# \{x_1, x_2\}$. This implies that $H(K)$ acts in a topologically transitive manner on the set $(X_r \setminus \{e\}) \times \mathrm{Iso}(C)$. This is a dense open subset of K, and so $H(K)$ is topologically transitive on K.

Case 3 (C is a Cantor set): A homeomorphism k on C is *topologically mixing* if, for all nonempty open sets $U_1, U_2 \subset C$, there exists $N \in \mathbb{N}$ such that $N_k(U_1, U_2) = \{n : k^n(U_1) \cap U_2 \neq \varnothing\}$ contains every $m \geq N$. The shift homeomorphism on $\{0, 1\}^{\mathbb{Z}}$ is topologically mixing, and the product of a topologically mixing and a topologically transitive homeomorphism is topologically transitive. It follows that if $k \in H(C)$ is topologically mixing, then $J(1, k)$ is a topologically transitive element of $H(K)$.

It remains to show that every element of $H(K)$ is chain transitive. It suffices to show that, for an arbitrary $F \in H(K)$ and an arbitrary point y of K, we have $e_K \in \mathcal{C}F(y)$. Applying this to F and F^{-1}, we see that every point of K is chain recurrent for F.

With the homeomorphism g on W chosen arbitrarily subject to the above conditions, we do not know whether this is always true. We prove it by imposing further restrictions on the homeomorphism g with which we began.

Call a homeomorphism g *semiminimal* if it has a fixed point e and if for every $x \neq e$ the orbit $\mathcal{O}g_{\pm}(x) = \{g^n(x) : n \in \mathbb{Z}\}$ is dense in W. Such semiminimal homeomorphisms exist. Topologically mixing examples of semiminimal homeomorphisms on the Cantor set are constructed explicitly in [Akin 2016, Theorem 4.19], and Theorem 4.16 of that paper shows that the semiminimal homeomorphisms form a dense G_δ subset of the set of those chain transitive homeomorphisms on a Cantor set which admit a fixed point. Now assume that g is a semiminimal homeomorphism which is topologically mixing and so is totally transitive. This implies that if $x \neq e$, then the real orbit $\mathbb{R}x$ is dense in Y_r.

Let $F \in H(K)$ and $y \in K$. We must show that y chains to e_K. Let $k = \pi_*(F)$ be the induced homeomorphism on C and $G = P_*(F)$ the induced homeomorphism on L.

The invariant set $\omega F(y)$ contains a closed subset M such that the restriction of F to M is minimal. Of course, $M \subset \mathcal{C}F(y)$. So it suffices to prove that $e_K \in \mathcal{C}F(M)$. This is obvious if $M = \{e_K\}$. Now assume M is not equal to $\{e_K\}$ and so does not contain e_K since distinct minimal sets are disjoint.

Hence, M is a compact subset of $K \setminus \{e_K\}$. Since π maps F to k, the subset $Q = \pi(M)$ of C is compact and invariant on which k restricts to a minimal homeomorphism. Let $(\widehat{K}, e_K) = (X_r, e)\#Q$ and $(\widehat{L}, e_K) = (Y_r, e)\#Q$, so that $\widehat{K}$ is a closed F-invariant subset of K and $\widehat{L}$ is a closed G-invariant subset of L. The restriction of F to $M \subset \widehat{K}$ is minimal, and so if $\widetilde{M} = P(M) \subset \widehat{L}$, then G on $\widetilde{M}$ is minimal. On $\widetilde{L}$ (but not on $\widetilde{K}$) the homeomorphism $(\phi_r^t, e)\#1_Q$ is defined for every t, and each such homeomorphism commutes with G. It follows that $\widetilde{M}_t = ((\phi_r^t, e)\#1_Q)(\widetilde{M})$ is a G-invariant subset on which G is minimal. If $a \in A$, there exists $((x, s), a) \in \widetilde{M}$ for some $x \in W \setminus \{e\}$ and $s \in [0, 1)$ because π_L maps $\widetilde{M}$ onto A. Because $x \neq e$, the real orbit $\mathbb{R}x$ is dense in Y_r. It follows that $\bigcup_t \widetilde{M}_t$ is dense in every fiber $\pi_L^{-1}(a)$. This implies that the union of the minimal subsets of G is dense in $\widehat{L}$. Above each $\widetilde{M}_t$ there is a minimal subset $M_t \subset \widehat{K}$ with $P(M_t) = \widetilde{M}_t$. Because P is an almost one-to-one map, it follows that the union $\bigcup_t M_t$ is dense in $\widehat{K}$. This implies that the recurrent points for F are dense in $\widehat{K}$, and so every point of $\widehat{K}$ is chain recurrent. Since $\widehat{K}$ is connected, F on $\widehat{K}$ is chain transitive and, in particular, $e_K \in \mathcal{C}F(M)$, as required. □

Remark. Notice that an almost one-to-one lift of a chain transitive map need not be chain transitive. Let t be translation by 1 on $\mathbb{Z}$ and t^*, t^{**} the extensions to the one-point compactification $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$ and the two-point compactification $\mathbb{Z}^{**} = \mathbb{Z} \cup \{+\infty, -\infty\}$. The map $p : \mathbb{Z}^{**} \to \mathbb{Z}^*$ which maps both $+\infty$ and $-\infty$ to ∞ is an almost one-to-one map, but while t^* is chain transitive, t^{**} is not.

In general, if A is a nowhere dense, closed, invariant set which contains $|\mathcal{C}f|$ for a homeomorphism f on X, then smashing A to the point e in X/A, the induced homeomorphism f_A on X/A has e as the unique minimal point, so f_A is chain transitive, the projection $q_A : X \to X/A$ is almost one-to-one since A is nowhere dense, and f is not chain recurrent since $|\mathcal{C}f|$ is a proper subset of X. For example, with $X = I^2$ and $f(x, y) = (x^2, y^2)$, the corner points are the only chain recurrent points. Let A be the boundary $I \times \{0, 1\} \cup \{0, 1\} \times I$.

Appendix: Chaotic spaces

The rigid and strongly chaotic spaces constructed in [de Groot and Wille 1958] and [Charatonik and Charatonik 1996] are dendrites or subsets of $\mathbb{R}^2$. It will be convenient for our purposes to use infinite-dimensional examples for the sequence of spaces $\{Z_n\}$ which satisfy RIG and CON.

Let M be an infinite subset of $\mathbb{N} \setminus \{1\}$, and let $m : \mathbb{N} \to M$ be the unique order-preserving bijection. For $n = 0, 1, \ldots$, let $M^n = \{m(2^n(2k-1)) : k \in \mathbb{N}\}$ so that $\{M^n\}$ is a partition of M by a pairwise disjoint sequence of infinite sets. For all $i \in \mathbb{N}$ let S^i be the sphere in $\mathbb{R}^{i+1}$ of radius i^{-1} centered at $(i^{-1}, 0, \ldots, 0)$ so that the origin 0 is a point of S^i. Let $S(n, k) = S^{m(2^n(2k-1))}$ for $n = 0, 1, \ldots$ and $k \in \mathbb{N}$.

Let Z^0 be the two-torus $S^1 \times S^1$, and let $A^0 = \{a^0_k : k \in \mathbb{N}\}$ be a dense sequence of distinct points in Z^0. Let Z^1 be Z^0 with a copy of $S(0, k)$ attached to Z^0 with $0 \in S(0, k)$ identified with the *attachment point* a^0_k for each $k \in \mathbb{N}$ so that the attached spheres are disjoint in Z^1. Let $r^1 : Z^1 \to Z^0$ be the retraction with each $S(0, k)$ mapped to the point a_k. For each $k \in \mathbb{N}$ let $r(0, k) : Z^1 \to S(0, k)$ be the retraction mapping $Z^1 \setminus S(0, k)$ to the point a_k. Since the diameters of the spheres tend to zero, the space Z^1 is compact and metrizable and the retractions are continuous. The open set $Z^1 \setminus Z^0$ is dense in Z^1.

For $n \geq 1$ assume that Z^n has been defined with a retraction $r^n : Z^n \to Z^{n-1}$ and with retractions $r(n-1, k) : Z^n \to S(n-1, k)$ onto the attached spheres and so that $Z^n \setminus Z^{n-1}$ is dense in Z^n. Let $A^n = \{a^n_k : k \in \mathbb{N}\}$ be a sequence of distinct points dense in $Z^n \setminus Z^{n-1}$. Let Z^{n+1} be Z^n with a copy of $S(n, k)$ attached to Z^n with $0 \in S(n, k)$ identified with the attachment point a^n_k for each $k \in \mathbb{N}$ so that the attached spheres are disjoint in Z^{n+1}. Let $r^{n+1} : Z^{n+1} \to Z^n$ be the retraction which maps the new spheres to their attachment points, and let $r(n,k) : Z^{n+1} \to S(n,k)$ be the retraction mapping $Z^{n+1} \setminus S(n,k)$ to a^n_k. Notice that for each $x \in Z^n \setminus A^n$ the set $(r^{n+1})^{-1}(x)$ equals $\{x\}$.

Let Z be the inverse limit of the system $\{r^n : Z^n \to Z^{n-1}, n \in \mathbb{N}\}$. The inclusions $i_n : Z^n \to Z^m$ with $m > n$ commute with the retractions. We obtain a limiting inclusion and so can regard $\{Z^n\}$ as an increasing sequence of subsets of Z. The projection $r_n : Z \to Z^n$ is then a retraction with $r^n \circ r_n = r_{n-1}$. We write $r(n,k) : Z \to S(n,k)$ for the composition of retractions $r(n,k) \circ r_{n+1}$. Notice that if we pick z from the set $Z^* = Z \setminus \left(\bigcup_n Z^n\right)$, then there is a unique sequence of attachment points $\{a^n_{k_n}\}$ converging to z with $r^n(a^n_{k_n}) = a^{n-1}_{k_{n-1}}$ for all n. Thus, Z^*, while a dense G_δ, is totally disconnected.

Observe that $(r_{n+1})^{-1}(S(n,k))$ and $\{a^n_k\} \cup (r_n)^{-1}(Z^n \setminus \{a^n_k\})$ are connected sets with union Z and which meet only at a^n_k. Thus, each attachment point disconnects Z.

If $F \subset Z$ is a finite set containing no attachment points, then $Z \setminus F$ is connected. For any finite $F \subset Z$, the set $Z \setminus F$ contains only finitely many components. In fact, the number of components is exactly one more than the number of attachment points in F. Thus, we obtain the condition CON.

Lemma A.1. *If W is a closed, connected, nontrivial subset of $Z \times I^N$ such that $W \setminus A$ is connected for any $A \subset W$ with topological dimension at most N, then either $W \subset Z^0 \times I^N$ or there exists a unique attached sphere $S(n,k)$ such that $W \subset S(n,k) \times I^N$.*

Proof. Notice first that the dimension of W is at least $N+2$. For if U is a nonempty open set with $\overline{U}$ a proper subset of W, then the topological boundary of U disconnects W and so has dimension at least $N+1$. This implies that the dimension of W is at least $N+2$.

The set W is not contained in $Z^* \times I^N$ since Z^* is totally disconnected and the components of $Z^* \times I^N$ have dimension N. Assume W is not a subset of $Z^0 \times I^N$.

If W meets $(Z^{n+1} \setminus Z^n) \times I^N$ for some $n \geq 0$, then X meets $(S(n,k) \setminus \{a^n_k\}) \times I^N$ for some k. Since $a^n_k \times I^N$ disconnects $Z \times I^N$ and no such set disconnects W, it follows that $W \subset r_n^{-1}(S(n,k)) \times I^N$. For the attachment points a^{n+1}_j in $S(n,k) \setminus \{a^n_k\}$, the sets $a^{n+1}_j \times I^N$ also disconnect Z. We see that either $W \subset S(n,k) \times I^N$ or else $W \subset r_{n+1}^{-1}(b) \times I^N$, where b is one of the attachment points in $S(n,k) \setminus \{a^n_k\}$.

If this process does not halt with W contained in the product of I^N with some attached sphere, then there is a sequence of attachment points $\{b^{n_i}_{k_i}\}$ with $W \subset r^{-1}_{n_i+1}(b^{n_i}_{k_i}) \times I^N$ and $r_{n_i+1}(b^{n_{i+1}}_{k_{i+1}}) = b^{n_i}_{k_i}$ with $n_i \to \infty$. The limit of such a sequence $\{b^{n_i}_{k_i}\}$ is a point of Z^*, and this would imply that W is a subset of $Z^* \times I^N$. ☐

If A and X are spaces, a *nonnull embedding* of A in X is a continuous injective function $j : A \to X$ which is not homotopic to a constant map in X.

Corollary A.2. *If S is a sphere of dimension at least two and $j : S \times I^N \to Z \times I^N$ is a nonnull embedding, then for a unique pair (n, k), we have $j(S \times I^N) \subset S(n, k) \times I^N$. Furthermore, the dimension of S equals the dimension of $S(n, k)$. On the other hand, if $j_0 : S \to S(n, k)$ is a homeomorphism, then $j = j_0 \times 1_{I^N} : S \times I^N \to Z \times I^N$ is a nonnull embedding.*

Proof. Corollary 1 of Theorem IV.4 in [Hurewicz and Wallman 1941] says that a connected manifold of dimension at least $N+2$ cannot be disconnected by a subset of dimension N. Hence, Lemma A.1 applies to $W = j(S \times I^N)$, and so $j(S \times I^N) \subset S(n, k) \times I^N$ for some pair (n, k) which is clearly unique. The space $S \times I^N$ cannot be embedded in a manifold of smaller dimension, and so $\dim S \le \dim S(n, k)$. If the inequality were strict then the map j would be homotopically trivial since the homotopy groups of a sphere vanish below its dimension. Hence, the dimension of S must be $m(2^n(2k-1)) = \dim S(n, k)$.

If $j_0 : S \to S(n, k)$ is a homeomorphism, then $j = j_0 \times 1_{I^N} : S \times I^N \to S(n, k) \times I^N$ is not homotopically trivial. Since $S(n, k)$ is a retract of Z, the embedding $j : S \times I^N \to Z \times I^N$ is not homotopically trivial. □

Thus, we can associate to any open subset $U \subset Z$ the set $\delta(U) = \{\dim S(n, k) : S(n, k) \subset U\} \subset M$. Since the diameters of the attaching spheres tend to zero, it follows that if U is nonempty, then $\delta(U)$ is infinite.

Corollary A.2 says that for a sphere S of dimension at least two there exists a nonnull embedding of $S \times I^N$ into $U \times I^N$ if and only if $\dim S \in \delta(U)$. That is, $\delta(U)$ is a topological invariant for the sets $U \times I^N$. Observe that if U_1 and U_2 are disjoint, nonempty, open sets in Z, then $\delta(U_1) \cap \delta(U_2) = \varnothing$ since distinct attached spaces have distinct dimensions.

If we begin by partitioning $\mathbb{N} \setminus \{1\}$ into a pairwise disjoint sequence $\{M_n\}$ of infinite subsets, then we can do this construction associating Z_n with M_n. That is, $\delta(Z_n) = M_n$. It follows that if $U_1 \subset Z_{n_1}$ and $U_2 \subset Z_{n_2}$ are nonempty open subsets with $n_1 \ne n_2$, then $\delta(U_1) \cap \delta(U_2) = \varnothing$.

Condition RIG of Section 4 is thus proved for this sequence of spaces.

Acknowledgements

We would like to express our appreciation to the referee for the helpful discussion of the history of the omega limit set problems and for suggestions concerning the exposition.

References

[Agronsky and Ceder 1991/92a] S. Agronsky and J. Ceder, "What sets can be ω-limit sets in E^n?", *Real Anal. Exchange* **17**:1 (1991/92), 97–109. MR Zbl

[Agronsky and Ceder 1991/92b] S. Agronsky and J. G. Ceder, “Each Peano subspace of E^k is an ω-limit set”, *Real Anal. Exchange* **17**:1 (1991/92), 371–378. MR Zbl

[Agronsky et al. 1989/90] S. J. Agronsky, A. M. Bruckner, J. G. Ceder, and T. L. Pearson, “The structure of ω-limit sets for continuous functions”, *Real Anal. Exchange* **15**:2 (1989/90), 483–510. MR Zbl

[Akin 1993] E. Akin, *The general topology of dynamical systems*, Graduate Studies in Mathematics **1**, American Mathematical Society, Providence, RI, 1993. MR Zbl

[Akin 2016] E. Akin, “Conjugacy in the Cantor set automorphism group”, pp. 1–42 in *Ergodic theory, dynamical systems, and the continuing influence of John C. Oxtoby*, edited by J. Auslander et al., Contemp. Math. **678**, American Mathematical Society, Providence, RI, 2016. MR Zbl

[Akin and Carlson 2012] E. Akin and J. D. Carlson, “Conceptions of topological transitivity”, *Topology Appl.* **159**:12 (2012), 2815–2830. MR Zbl

[Andronov and Khaikin 1937] A. A. Andronov and S. E. Khaikin, Теория колебаний, Ob. Nauch.-Tekh. Izd. NKTP SSSR, Moscow and Leningrad, 1937. Translated as *Theory of oscillations*, Princeton Univ. Press, 1949. An enlarged second edition was published in 1959 by Fitmatgiz, Moscow, with A. A. Witt as a coauthor; translation: Pergamon Press, Oxford, 1966.

[Birkhoff 1927] G. D. Birkhoff, *Dynamical systems*, American Mathematical Society Colloquium Publications **9**, American Mathematical Society, Providence, RI, 1927. MR JFM

[Bruckner and Smítal 1992] A. M. Bruckner and J. Smítal, “The structure of ω-limit sets for continuous maps of the interval”, *Math. Bohem.* **117**:1 (1992), 42–47. MR Zbl

[Charatonik and Charatonik 1996] J. J. Charatonik and W. J. Charatonik, “Strongly chaotic dendrites”, *Colloq. Math.* **70**:2 (1996), 181–190. MR Zbl

[Conley 1978] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics **38**, American Mathematical Society, Providence, RI, 1978. MR Zbl

[Dowker 1953] Y. N. Dowker, “The mean and transitive points of homeomorphisms”, *Ann. of Math.* (2) **58** (1953), 123–133. MR Zbl

[Dowker and Friedlander 1954] Y. N. Dowker and F. G. Friedlander, “On limit sets in dynamical systems”, *Proc. London Math. Soc.* (3) **4** (1954), 168–176. MR Zbl

[Downarowicz et al. 2017] T. Downarowicz, L. Snoha, and D. Tywoniuk, “Minimal spaces with cyclic group of homeomorphisms”, *J. Dynam. Differential Equations* **29**:1 (2017), 243–257. MR

[de Groot 1959] J. de Groot, “Groups represented by homeomorphism groups, I”, *Math. Ann.* **138** (1959), 80–102. MR Zbl

[de Groot and Wille 1958] J. de Groot and R. J. Wille, “Rigid continua and topological group-pictures”, *Arch. Math.* **9**:5 (1958), 441–446. MR Zbl

[Gutek 1979] A. Gutek, “On extending homeomorphisms on the Cantor set”, pp. 105–116 in *Topological structures, II, Part I* (Amsterdam, 1978), edited by P. C. Baayen and J. van Mill, Math. Centre Tracts **115**, Math. Centrum, Amsterdam, 1979. MR Zbl

[Hurewicz and Wallman 1941] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Mathematical Series **4**, Princeton Univ. Press, 1941. MR Zbl

[Knaster and Reichbach 1953] B. Knaster and M. Reichbach, “Notion d’homogénéité et prolongements des homéomorphies”, *Fund. Math.* **40** (1953), 180–193. MR Zbl

[Kolyada and Snoha 1992/93] S. F. Kolyada and L. Snoha, “On ω-limit sets of triangular maps”, *Real Anal. Exchange* **18**:1 (1992/93), 115–130. MR Zbl

[Lorch 1981] E. R. Lorch, “On some properties of the metric subalgebras of l^∞”, *Integral Equations Operator Theory* **4**:3 (1981), 422–434. MR Zbl

[Lorch 1982] E. R. Lorch, "Certain compact spaces and their homeomorphism groups", *Rend. Sem. Mat. Fis. Milano* **52**:1 (1982), 75–86. MR Zbl

[Sharkovsky 1965] O. M. Sharkovskiĭ, "On attracting and attracted sets", *Dokl. Akad. Nauk SSSR* **160** (1965), 1036–1038. In Russian; translated in *Soviet Math. Dokl.* **6** (1965), 268–270. MR Zbl

[Sharkovsky et al. 1989] A. N. Sharkovskiĭ, S. F. Kolyada, A. G. Sivak, and V. V. Fedorenko, Динамика одномерных отображений, Naukova Dumka, Kiev, 1989. Translated as *Dynamics of one-dimensional maps*, Kluwer, Dordrecht, 1997. MR Zbl

[Tsankov 2006] T. Tsankov, "Compactifications of $\mathbb{N}$ and Polishable subgroups of S_∞", *Fund. Math.* **189**:3 (2006), 269–284. MR Zbl

Received December 2, 2015. Revised February 3, 2017.

ETHAN AKIN
MATHEMATICS DEPARTMENT
THE CITY COLLEGE
NEW YORK, NY
UNITED STATES
ethanakin@earthlink.net

JUHO RAUTIO
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF OULU
FINLAND
juhok.rautio@mail.suomi.net

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 291 No. 1 November 2017