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## REGULARITY OF THE ANALYTIC TORSION FORM ON FAMILIES OF NORMAL COVERINGS


BING KWAN SO AND GUANGXIANG SU



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# REGULARITY OF THE ANALYTIC TORSION FORM ON FAMILIES OF NORMAL COVERINGS

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**We prove the smoothness of the $L^2$-analytic torsion form for fiber bundles with positive Novikov–Shubin invariant. We do so by generalizing the arguments of Azzali, Goette and Schick to an appropriate Sobolev space, and proving that the Novikov–Shubin invariant is also positive in the Sobolev setting, using an argument of Alvarez Lopez and Kordyukov.**

## 1. Introduction

Let $M$ be a closed Riemannian manifold and $F$ be a flat vector bundle on $M$. Ray and Singer [1971] introduced the analytic torsion, which is the analytic analogue of the combinatorial torsion (see [Milnor 1966]). Let $Z \to M \xrightarrow{\pi} B$ be a fiber bundle with connected closed fibers $Z_x = \pi^{-1}(x)$ and $F$ be a flat complex vector bundle on $M$ with a flat connection $\nabla^F$ and a Hermitian metric $h^F$. Let $T^HM$ be a horizontal distribution for the fiber bundle and $g^{TZ}$ be a vertical Riemannian metric. Bismut and Lott [1995, (3.118)] introduced the torsion form $\mathcal{T}(T^HM, g^{TZ}, h^F) \in \Omega(B)$ defined by

$$\mathcal{T}(T^HM, g^{TZ}, h^F) = -\int_0^{+\infty} \Big[ f^\wedge\big(C'_t, h^W\big) - \tfrac{1}{2}\chi'(Z; F) f'(0) \\ -\big(\tfrac{1}{4}\dim(Z)\,\mathrm{rk}(F)\,\chi(Z) - \tfrac{1}{2}\chi'(Z; F)\big) f'\big(\tfrac{1}{2} i\sqrt{t}\big)\Big] \frac{dt}{t}. \tag{1}$$

See [Bismut and Lott 1995] for the meaning of the terms in the integrand. To show the integral in the above formula is well defined, one must calculate the asymptotic of $f^\wedge(C'_t, h^W)$ as $t \to 0$ and the asymptotic as $t \to \infty$. For the asymptotic as $t \to 0$, they used the local index technique. For the asymptotic as $t \to \infty$, the key fact is that the fiber $Z$ is closed, so the fiberwise operators involved have uniform positive lower bound for positive eigenvalues. They also proved a $C^\infty$-analogue of the Riemann–Roch–Grothendieck theorem and proved that the torsion form is the transgression of the Riemann–Roch–Grothendieck theorem (see [Bismut and Lott 1995, Theorem 3.23]) and showed the zero degree part of $\mathcal{T}(T^HM, g^{TZ}, h^F) \in \Omega(B)$ is the Ray–Singer analytic torsion (see their Theorem 3.29).

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On the other hand, the $L^2$-analytic torsion was defined and studied by several authors; see [Carey and Mathai 1992], [Lott 1992], [Mathai 1992], etc. So it is natural to extend the $L^2$-analytic torsion to the family case, that is, to define and study the Bismut–Lott torsion form when the fiber $Z$ is noncompact. From the above we see that one must study the asymptotic of the $L^2$ analogue of $f^\wedge(C_t, h^W)$ as $t \to 0$ and $t \to \infty$. Since in the $L^2$ case $f^\wedge(C'_t, h^W)$ has the same asymptotic as $t \to 0$, this part is easy. But in general the integral at $\infty$ does not converge, since in the $L^2$ case the positive eigenvalues of the fiberwise operator involved in $f^\wedge(C'_t, h^W)$ may not have a positive lower bound. To overcome this difficulty, one considers the special case where the Novikov–Shubin invariant is (sufficiently) positive. Gong and Rothenberg [1996] defined the $L^2$-analytic torsion form and proved that the torsion form is smooth, under the condition that the Novikov–Shubin invariant is at least half of the dimension of the base manifold. Heitsch and Lazarov [2002] generalized essentially the same arguments to foliations. In [Azzali et al. 2015], Azzali, Goette and Schick proved that the integrand defining the $L^2$-analytic torsion form, as well as several other invariants related to the signature operator, converges, provided the Novikov–Shubin invariant is positive (or of determinant class and $L^2$-acyclic). However, they did not prove the smoothness of the $L^2$-analytic torsion form. To consider the transgression formula, they had to use weak derivatives.

The aim of this paper is to establish the regularity of the $L^2$-analytic torsion form in the case when the Novikov–Shubin invariant is positive. Our motivation comes from the study of analytic torsion on some “noncommutative” spaces (along the lines of [Gorokhovsky and Lott 2006], etc., for local index). In this case one considers universal differential forms (as in the same paper), and Duhamel’s formula for the heat operator having infinitely many terms. Instead, one makes essential use of the results of [Azzali et al. 2015] to ensure that (1) is well defined in the noncommutative case. We achieve this result by generalizing Azzali, Goette and Schick’s arguments to some Sobolev spaces.

The rest of the paper is organized as follows. In Section 2, we define Sobolev norms on the spaces of kernels on the fibered product groupoid. Unlike [Azzali et al. 2015], we consider Hilbert–Schmidt type norms on the space of smoothing operators. Given a kernel, the Hilbert–Schmidt norm can be explicitly written down. As a result, we are able to take into account derivatives in both the fiberwise and transverse directions, with the help of a splitting similar to [Heitsch 1995]. In Section 3, we turn to proving that having positive Novikov–Shubin invariant implies positivity of the Novikov–Shubin invariant in the Sobolev settings. We adapt an argument of Alvarez Lopez and Kordyukov [2001]. In Section 4, we apply the arguments in [Azzali et al. 2015] and conclude that the integral in equation (1) converges in all Sobolev norms, and hence obtain the regularity of the $L^2$-analytic torsion form.

## 2. Preliminaries

In this section, we will define Sobolev norms on the space of kernels on the fibered product groupoid.

**2A.** ***The geometric setting.*** Let $Z \to M \xrightarrow{\pi} B$ be a fiber bundle with connected fibers $Z_x = \pi^{-1}(x)$, $x \in B$. Let $E \xrightarrow{\wp} M$ be a vector bundle. We assume $B$ is compact. Let $V := \mathrm{Ker}(d\pi) \subset TM$.

We suppose that there is a finitely generated discrete group $G$ acting on $M$ from the right freely and properly discontinuously. We also assume that the group $G$ acts on $B$ such that the actions commute with $\pi$ and $M_0 := M/G$ is a compact manifold. Since the submersion $\pi$ is $G$-invariant, $M_0$ is also foliated and we denote its foliation $V_0$. Fix a distribution $H_0 \subset TM_0$ complementary to $V_0$. Fix a metric on $V_0$ and take a $G$-invariant metric on $B$, then these induce a Riemannian metric on $M_0$ by $g^{V_0} \oplus \pi^* g^{TB}$ on $TM_0 = V_0 \oplus H_0$.

Since the projection from $M$ to $M_0$ is a local diffeomorphism, one gets a $G$-invariant splitting $TM = V \oplus H$. Denote by $P^V$ and $P^H$ respectively the projections to $V$ and $H$. Moreover, one gets a $G$-invariant metric on $V$ and a Riemannian metric on $M$ by $g^{TM} = g^V \oplus \pi^* g^{TB}$ on $TM = V \oplus H$.

Given any vector field $X \in \Gamma^\infty(TB)$, denote the horizontal lift of $X$ by $X^H \in \Gamma^\infty(H) \subset \Gamma^\infty(TM)$. By our construction, $|X^H|_{g_M}(p) = |X|_{g_B}(\pi(p))$.

Denote by $\mu_x$ and $\mu_B$ respectively the Riemannian measures on $Z_x$ and $B$.

**Definition 2.1.** Let $E \xrightarrow{\wp} M$ be a complex vector bundle. We say that $E$ is a contravariant $G$-bundle if $G$ also acts on $E$ from the right, such that for any $v \in E$, $g \in G$, $\wp(vg) = \wp(v)g \in M$, and moreover $G$ acts as a linear map between the fibers.

The group $G$ then acts on sections of $E$ from the left by

$$s \mapsto g^* s, \quad (g^* s)(p) := s(pg)g^{-1} \in \wp^{-1}(p), \qquad \text{for all } p \in M.$$

We assume that $E$ is endowed with a $G$-invariant metric $g_E$, and a $G$-invariant connection $\nabla^E$ (which is obviously possible if $E$ is the pullback of some bundle on $M_0$). In particular, for any $G$-invariant section $s$ of $E$, $|s|$ is a $G$-invariant function on $M$.

In the following, for any vector bundle $F$ we denote its dual bundle by $F'$.

Recall that the "infinite dimensional bundle" over $B$ in the sense of Bismut is a vector bundle with typical fiber $\Gamma_c^\infty(E|_{Z_x})$ (or other function spaces) over each $x \in B$. We denote by $E_\flat$ such a Bismut bundle. The space of smooth sections on $E_\flat$ is, as a vector space, $\Gamma_c^\infty(E)$. Each element $s \in \Gamma_c^\infty(E)$ is regarded as a map

$$x \mapsto s|_{Z_x} \in \Gamma_c^\infty\big(E|_{Z_x}\big) \quad \text{for all } x \in B.$$

In other words, one defines a section on $E_\flat$ to be smooth if the images of all $x \in B$ fit together to form an element in $\Gamma_c^\infty(E)$. In particular, $\Gamma_c^\infty((M \times \mathbb{C})_\flat) = C_c^\infty(M)$, and one identifies $\Gamma_c^\infty(TB \otimes (M \times \mathbb{C})_\flat)$ with $\Gamma_c^\infty(H)$ by $X \otimes f \mapsto f X^H$.

**2B. *Covariant derivatives and Sobolev spaces.*** Let $\nabla^E$ be a $G$-invariant connection on $E$. Denote by $\nabla^{TM}$, $\nabla^{TB}$ the Levi-Civita connections (with respect to the metrics defined in the last section). Note that $[X^H, Y] \in \Gamma^\infty(V)$ for any vertical vector field $Y \in \Gamma^\infty(V)$. One naturally defines the connections

$$\begin{aligned} \nabla_X^{V_\flat} Y &:= [X^H, Y] \quad \text{for all } Y \in \Gamma^\infty(V_\flat) \cong \Gamma^\infty(V), \\ \nabla_X^{E_\flat} s &:= \nabla_{X^H}^E s \qquad \text{for all } s \in \Gamma^\infty(E_\flat) \cong \Gamma_c^\infty(E). \end{aligned}$$

**Definition 2.2.** The covariant derivative on $E_\flat$ is the map

$$\dot\nabla^{E_\flat} : \Gamma^\infty\Big(\otimes^\bullet T^*B \bigotimes \otimes^\bullet V_\flat' \bigotimes E_\flat\Big) \to \Gamma^\infty\Big(\otimes^{\bullet+1} T^*B \bigotimes \otimes^\bullet V_\flat' \bigotimes E_\flat\Big),$$

defined by

$$\begin{aligned} (2)\quad & \big(\dot\nabla^{E_\flat} s\big)(X_0, X_1, \dots, X_k; Y_1, \dots, Y_l) \\ & := \nabla_{X_0}^{E_\flat} s(X_1, \dots, X_k; Y_1, \dots, Y_l) - \sum_{j=1}^{l} s\big(X_1, \dots, X_k; Y_1, \dots, \nabla_{X_0}^{V_\flat} Y_j, \dots, Y_l\big) \\ & \qquad - \sum_{i=1}^{k} s\big(X_1, \dots, \nabla_{X_0}^{TB} X_i, \dots, X_k; Y_1, \dots Y_l\big), \end{aligned}$$

for any $k, l \in \mathbb{N}$, $X_0, \dots, X_k \in \Gamma^\infty(TB)$, $Y_1, \dots, Y_l \in \Gamma^\infty(V)$.

Clearly, taking the covariant derivative can be iterated, which we denote by $(\dot\nabla^{E_\flat})^m$, $m = 1, 2, \dots$. Note that $(\dot\nabla^{E_\flat})^m$ is a differential operator of order $m$.

Also, we define

$$\dot\partial^V : \Gamma^\infty\Big(\otimes^\bullet T^*B \bigotimes \otimes^\bullet V_\flat' \bigotimes E_\flat\Big) \to \Gamma^\infty\Big(\otimes^\bullet T^*B \bigotimes \otimes^{\bullet+1} V_\flat' \bigotimes E_\flat\Big)$$

by

$$\begin{aligned} (3)\quad & \big(\dot\partial^V s\big)(X_1, \dots, X_k; Y_0, Y_1, \dots, Y_l) \\ & := \nabla_{Y_0}^E s(X_1, \dots, X_k; Y_1, \dots, Y_l) - \sum_{j=1}^{l} s(X_1, \dots, X_k; Y_1, \dots, P^V(\nabla_{Y_0}^{TM} Y_j), \dots, Y_l). \end{aligned}$$

In the following definition, we regard $(\dot\nabla^{E_\flat})^i (\dot\partial^V)^j s \in \Gamma^\infty(\otimes^i H' \bigotimes \otimes^j V' \bigotimes E_\flat)$.

**Definition 2.3.** For $s \in \Gamma_c^\infty(E)$, we define its $m$-th Sobolev norm by

$$(4)\qquad \|s\|_m^2 := \sum_{i+j \le m} \int_{x \in B} \int_{y \in Z_x} \big|(\dot\nabla^{E_\flat})^i (\dot\partial^V)^j s\big|^2(x, y) \mu_x(y) \mu_B(x).$$

Denote by $\mathcal{W}^m(E)$ the Sobolev completion of $\Gamma_c^\infty(E)$ with respect to $\|\cdot\|_m$.

Recall that an operator $A$ is called $C^\infty$-bounded if in normal coordinates the coefficients and their derivatives are $C^\infty$-bounded.

Since $M$ is locally isometric to a compact space $M_0$, it is a manifold with bounded geometry (see [Shubin 1992, Appendix 1] for an introduction). Moreover, $\nabla^E$ is a $C^\infty$-bounded differential operator, because by $G$-invariance the Christoffel symbols of $\nabla^E$ and all their derivatives are uniformly bounded. Using normal coordinate charts and parallel transport with respect to $\nabla^E$ as the trivialization, one sees that $E$ is a bundle with bounded geometry.

Since the operators $\dot{\nabla}^{E_\flat}$ and $\dot{\partial}^V$ are just respectively the (0, 1) and (1, 0) parts of the usual covariant derivative operator, our Definition 2.3 is equivalent to the standard Sobolev norm [Shubin 1992, Appendix 1 (1.3)] (with $p = 2$ and $s$ nonnegative integers).

One has elliptic regularity for these Sobolev spaces:

**Lemma 2.4** [Shubin 1992, Lemma 1.4]. *Let $A$ be any $C^\infty$-bounded, uniformly elliptic differential operator of order $m$. For any $i, j \geq 0$, there exists a constant $C$ such that for any $s \in \Gamma_c^\infty(E)$*

$$\|s\|_{i+m} \leq C(\|As\|_i + \|s\|_j).$$

**Remark 2.5.** Throughout this paper, by an "elliptic operator" on a manifold, we mean elliptic in all directions, without taking any foliation structure into consideration. We use the term "fiberwise elliptic operators" to refer to differential operators that are fiberwise and elliptic restricted to fibers.

## 2C. *The fibered product.*

**Definition 2.6.** The fibered product of the manifold $M$ is

$$M \times_B M := \{(p, q) \in M \times M : \pi(p) = \pi(q)\},$$

and with the maps from $M \times_B M$ to $M$ defined by

$$\boldsymbol{s}(p, q) := q \quad \text{and} \quad \boldsymbol{t}(p, q) := p.$$

The manifold $M \times_B M$ is a fiber bundle over $B$, with typical fiber $Z \times Z$. One naturally has the splitting [Heitsch 1995, Section 2]

$$T(M \times_B M) = \hat{H} \oplus V_{\boldsymbol{t}} \oplus V_{\boldsymbol{s}},$$

where $V_{\boldsymbol{s}} := \operatorname{Ker}(d\boldsymbol{t})$ and $V_{\boldsymbol{t}} := \operatorname{Ker}(d\boldsymbol{s})$.

Note that $V_{\boldsymbol{s}} \cong \boldsymbol{s}^* V$ and $V_{\boldsymbol{t}} \cong \boldsymbol{t}^* V$. As in Section 2A, we endow $M \times_B M$ with a metric by lifting the metrics on $H_0$ and $V_0$. Then $M \times_B M$ is a manifold with bounded geometry.

**Notation 2.7.** With some abuse in notation, we shall often write elements in $M\times_B M$ as a triple $(x, y, z)$ and $\boldsymbol{s}(x, y, z) = (x, z)$, $\boldsymbol{t}(x, y, z) = (x, y) \in M$, where $x \in B$ and $y, z \in Z_x$.

Let $G$ act on $M \times_B M$ by the diagonal action $(p, q)g := (pg, qg)$. Let $E \to M$ be a contravariant $G$-vector bundle and $E'$ be its dual. We shall consider

$$\hat{E} \to M \times_B M := \boldsymbol{t}^*E \otimes \boldsymbol{s}^*E'.$$

Given a $G$-invariant connection $\nabla^E$ on $E$, let

$$\nabla^{\hat{E}} := \boldsymbol{t}^*\nabla^E \otimes \mathrm{id}_{\boldsymbol{s}^*E'} + \mathrm{id}_{\boldsymbol{t}^*E} \otimes \boldsymbol{s}^*\nabla^{E'}$$

be the tensor product of the pullback connections. Fix any local base $\{e_1, \dots e_r\}$ of $E'$ on some $U \subset M$. Any section can be written as

$$s = \sum_{i=1}^{r} u_i \otimes \boldsymbol{s}^*e_i$$

on $\boldsymbol{s}^{-1}(U)$, where $u_i \in \Gamma^\infty(\boldsymbol{t}^*E)$. Then by definition we have for any vector $X$ on $M$,

$$\nabla^{\hat{E}}_X\left(\sum_{i=1}^{r} u_i \otimes \boldsymbol{s}^*e_i\right) = \sum_{i=1}^{r}(\nabla^{\boldsymbol{t}^*E}_X u_i) \otimes \boldsymbol{s}^*e_i + u_i \otimes \boldsymbol{s}^*(\nabla^{E'}_{ds(X)}e_i). \tag{5}$$

Similarly to Definition 2.2, we define the covariant derivative operators on $\Gamma^\infty(\otimes^\bullet T^*B \bigotimes \otimes^\bullet(V'_{\boldsymbol{t}})_\flat \bigotimes \otimes^\bullet(V'_{\boldsymbol{s}})_\flat \bigotimes \hat{E}_\flat)$.

**Definition 2.8.** Define

$$\begin{aligned}
\big(\dot\nabla^{\hat{E}_\flat}\psi\big)&(X_0, X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&:= \nabla^{\hat{E}_\flat}_{X_0}\psi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&\quad - \sum_{1\le j\le l} \psi(X_1, \dots, X_k; Y_1, \dots, \nabla^{V_\flat}_{X_0}Y_j, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&\quad - \sum_{1\le j\le l'} \psi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, \nabla^{V_\flat}_{X_0}Z_j, \dots, Z_{l'}) \\
&\quad - \sum_{1\le i\le k} \psi(X_1, \dots, \nabla^{TB}_{X_0}X_i, \dots, X_k; Y_1, \dots Y_l, Z_1, \dots, Z_{l'}),
\end{aligned}$$

and

$$\begin{aligned}
\big(\dot{\partial}^s\psi\big)&(X_1,\dots,X_k;Y_0,Y_1,\dots,Y_l,Z_1,\dots,Z_{l'}) \\
&:= \nabla^{\hat{E}}_{Y_0}\psi(X_1,\dots,X_k;Y_1,\dots,Y_l,Z_1,\dots,Z_{l'}) \\
&\quad - \sum_{1\le j\le l}\psi(X_1,\dots,X_k;Y_1,\dots,P^{V^s}(\nabla^{TM}_{Y_0}Y_j),\dots,Y_l,Z_1,\dots,Z_{l'}) \\
&\quad - \sum_{1\le j\le l'}\psi(X_1,\dots,X_k;Y_1,\dots,Y_l,Z_1,\dots,P^{V^t}[Y_0,Z_j],\dots,Z_{l'}),
\end{aligned}$$

and

$$\begin{aligned}
\big(\dot{\partial}^t\psi\big)&(X_1,\dots,X_k;Y_1,\dots,Y_l,Z_0,Z_1,\dots,Z_{l'}) \\
&:= \nabla^{\hat{E}}_{Y_0}\psi(X_1,\dots,X_k;Y_1,\dots,Y_l,Z_0,Z_1,\dots,Z_{l'}) \\
&\quad - \sum_{1\le j\le l}\psi(X_1,\dots,X_k;Y_1,\dots,P^{V^s}[Z_0,Y_j],\dots,Y_l,Z_1,\dots,Z_{l'}) \\
&\quad - \sum_{1\le j\le l'}\psi(X_1,\dots,X_k;Y_1,\dots,Y_l,Z_1,\dots,P^{V^t}(\nabla^{TM}_{Z_0}Z_j),\dots,Z_{l'}).
\end{aligned}$$

Given any vector fields $Y, Z \in V$, let $Y^s$, $Z^t$ be respectively the lifts of $Y$ and $Z$ to $V_s$ and $V_t$. Then $[Y^s, Z^t] = 0$. It follows that as differential operators, $[\dot{\partial}^s, \dot{\partial}^t] = 0$. Also, it is straightforward to verify that $[\dot{\nabla}^{\hat{E}_\flat}, \dot{\partial}^s]$ and $[\dot{\nabla}^{\hat{E}_\flat}, \dot{\partial}^t]$ are both zeroth order differential operators (i.e., smooth bundle maps).

Fix a local trivialization

$$\boldsymbol{x}_\alpha : \pi^{-1}(B_\alpha) \to B_\alpha \times Z, \quad p \mapsto (\pi(p), \varphi^\alpha(p)),$$

where $B = \bigcup_\alpha B_\alpha$ is a finite open cover (since $B$ is compact), and $\varphi^\alpha|_{\pi^{-1}(x)} : Z_x \to Z$ is a diffeomorphism. Such a trivialization induces a local trivialization of the fiber bundle $M \times_B M \xrightarrow{t} M$ by $M = \bigcup M_\alpha$, $M_\alpha := \pi^{-1}(B_\alpha)$,

$$\hat{\boldsymbol{x}}_\alpha : \boldsymbol{t}^{-1}(M_\alpha) \to M_\alpha \times Z, \quad (p,q) \mapsto (p, \varphi^\alpha(q)).$$

On $M_\alpha \times Z$ the source and target maps are explicitly given by

$$\boldsymbol{s}\circ(\hat{\boldsymbol{x}}_\alpha)^{-1}(p,z) = (\boldsymbol{x}_\alpha)^{-1}(\pi(p),z) \text{ and } \boldsymbol{t}\circ(\hat{\boldsymbol{x}}_\alpha)^{-1}(p,z) = p. \tag{6}$$

For such a trivialization, one has the natural splitting

$$T(M_\alpha \times Z) = H^\alpha \oplus V^\alpha \oplus TZ,$$

where $H^\alpha$ and $V^\alpha$ are respectively $H$ and $V$ restricted to $M_\alpha \times \{z\}$, $z \in Z$. It follows from (6) that

$$V^\alpha = d\hat{\boldsymbol{x}}_\alpha(V_s) \quad \text{and} \quad TZ = d\hat{\boldsymbol{x}}_\alpha(V_t).$$

Given any vector field $X$ on $B$, let $X^H$, $X^{\hat{H}}$ be respectively the lifts of $X$ to $H$ and $\hat{H}$. Since $d\boldsymbol{t}(X^{\hat{H}}) = d\boldsymbol{s}(X^{\hat{H}}) = X^H$, it follows that

$$d\hat{\boldsymbol{x}}_\alpha(X^{\hat{H}}) = X^{H^\alpha} + d\varphi^\alpha(X^H).$$

Note that $d\varphi^\alpha(X^H) \in TZ \subseteq T(M_\alpha \times Z)$.

Corresponding to the splitting $T(M_\alpha \times Z) = H^\alpha \oplus V^\alpha \oplus TZ$, one can define the covariant derivative operators. Let $\nabla^{TM_\alpha}$ be the Levi-Civita connection on $M_\alpha$ and $\nabla^{TZ}$ be the Levi-Civita connection on $Z$. Define for any smooth section $\phi \in \Gamma^\infty(\otimes^\bullet T^*B \bigotimes \otimes^\bullet (V^\alpha)'_\flat \bigotimes \otimes^\bullet T^*Z_\flat \bigotimes (\hat{\boldsymbol{x}}_\alpha^{-1})^* \hat{E}_\flat)$,

$$\begin{aligned}
\big(\dot{\nabla}^\alpha \phi\big)&(X_0, X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
:= {}& (\boldsymbol{x}_\alpha^* \nabla^{\hat{E}_\flat})_{X_0^{H^\alpha}} \phi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&- \sum_{1 \le j \le l} \phi(X_1, \dots, X_k; Y_1, \dots, [X_0^{H^\alpha}, Y_j], \dots, Y_l, , Z_1, \dots, Z_{l'}) \\
&- \sum_{1 \le j \le l'} \phi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, [X_0^{H^\alpha} Z_j], \dots, Z_{l'}) \\
&- \sum_{1 \le i \le k} \phi(X_1, \dots, \nabla^{TB}_{X_0} X_i, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}),
\end{aligned}$$

and

$$\begin{aligned}
\big(\dot{\partial}^\alpha \phi\big)&(X_1, \dots, X_k; Y_0, Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
:= {}& (\boldsymbol{x}_\alpha^* \nabla^{\hat{E}_\flat})_{Y_0} \phi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&- \sum_{1 \le j \le l} \phi(X_1, \dots, X_k; Y_1, \dots, P^{V^\alpha}(\nabla^{TM_\alpha}_{Y_0} Y_j), \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&- \sum_{1 \le j \le l'} \phi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, P^{TZ}[Y_0, Z_j], \dots, Z_{l'}),
\end{aligned}$$

and

$$\begin{aligned}
\big(\dot{\partial}^Z \phi\big)&(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_0, Z_1, \dots, Z_{l'}) \\
:= {}& (\boldsymbol{x}_\alpha^* \nabla^{\hat{E}_\flat})_{Z_0} \phi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_0, Z_1, \dots, Z_{l'}) \\
&- \sum_{1 \le j \le l} \phi(X_1, \dots, X_k; Y_1, \dots, P^{V^\alpha}[Z_0, Y_j], \dots, Y_l, Z_1, \dots, Z_{l'}) \\
&- \sum_{1 \le j \le l'} \phi(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, \nabla^{TZ}_{Z_0} Z_j, \dots, Z_{l'}).
\end{aligned}$$

Consider the special case when $\phi = u \otimes \boldsymbol{s}^* e$, where $u \in \Gamma^\infty(\otimes^\bullet T^*B \bigotimes \otimes^\bullet (V^\alpha)'_\flat \otimes \boldsymbol{t}^* E)$ and $e \in \Gamma^\infty(\otimes^\bullet T^*Z_\flat \otimes E')$.

**Lemma 2.9.** *For* $(x, y, z) \in M_\alpha \times Z$, *one has*

$$\dot\nabla^\alpha(u \otimes \boldsymbol{s}^* e)(x, y, z) = \big(\dot\nabla^{E_\flat} u|_{M_\alpha\times\{z\}}(x, y)\big) \otimes \boldsymbol{s}^*(e(x, z)) + u \otimes \boldsymbol{s}^*\big(\nabla^{E'_\flat} e(x, z)\big)$$

*and* $\dot\partial^\alpha(u \otimes \boldsymbol{s}^* e)(x, y, z) = (\dot\partial^V u|_{M_\alpha\times\{z\}}(x, y)) \otimes \boldsymbol{s}^*(e(x, z))$.

*Proof.* It suffices to consider the case when $Y_j$, $Z_{j'}$ are respectively vector fields on $M_\alpha$ and $Z$ lifted to $M_\alpha \times Z$. From this assumption it follows that $[Y_j, Z_{j'}] = [X_0^{H^\alpha}, Z_{j'}] = 0$. The lemma follows by a simple computation. □

We express the (pullback of) covariant derivatives $\dot\nabla^{\hat E_\flat}\psi$, $\dot\partial^{\boldsymbol{s}}\psi$ and $\dot\partial^{\boldsymbol{t}}\psi$ in terms of $\dot\nabla^\alpha\psi^\alpha$, $\dot\partial^\alpha\psi^\alpha$ and $\dot\partial^Z\psi^\alpha$, where $\psi^\alpha := (\boldsymbol{x}_\alpha^{-1})^*\psi$. One directly verifies

$$\begin{aligned}
(7)\quad &\big(\dot\nabla^{E_\flat}\psi\big)(X_0, X_1, \dots, X_k; Y_1, \dots, Y_l, Z_1, \dots, Z_{l'})\\
&= (\boldsymbol{x}_\alpha^{-1})^*\Bigg(\nabla^\alpha_{(X_0^{H^\alpha}+d\varphi^\alpha(X_0^H))}\psi^\alpha\big(X_1,\dots,X_k; d\boldsymbol{x}_\alpha(Y_1,\dots,Y_l,Z_1,\dots,Z_{l'})\big)\\
&\qquad - \sum_{1\le j\le l}\psi^\alpha\big(X_1,\dots,X_k; d\boldsymbol{x}_\alpha Y_1,\dots,[X_0^{H^\alpha}, d\boldsymbol{x}_\alpha Y_j],\dots,d\boldsymbol{x}_\alpha Y_l, d\boldsymbol{x}_\alpha(Z_1,\dots,Z_{l'})\big)\\
&\quad - \sum_{1\le j\le l'}\psi^\alpha\big(X_1,\dots,X_k; d\boldsymbol{x}_\alpha(Y_1,\dots,Y_l), d\boldsymbol{x}_\alpha Z_1,\dots,[X_0^{H^\alpha}+d\varphi^\alpha(X_0^H), d\boldsymbol{x}_\alpha Z_j],\dots,d\boldsymbol{x}_\alpha Z_{l'}\big)\\
&\qquad\qquad - \sum_{1\le i\le k}\psi^\alpha\big(X_1,\dots,\nabla^{TB}_{X_0}X_i,\dots,X_k; Y_1,\dots,Y_l, Z_1,\dots,Z_{l'}\big)\Bigg)\\
&= (\boldsymbol{x}_\alpha^{-1})^*\big(\dot\nabla^\alpha\psi^\alpha(X_0, X_1,\dots,X_k; Y_1,\dots,Y_l, Z_1,\dots,Z_{l'})\\
&\qquad\qquad + \dot\partial^Z\psi^\alpha(X_1,\dots,X_k; Y_1,\dots,Y_l, d\varphi^\alpha(X_0^H), Z_1,\dots,Z_{l'})\\
&\quad + \sum_{1\le j\le l'}\psi^\alpha\big(X_1,\dots,X_k; d\boldsymbol{x}_\alpha(Y_1,\dots,Y_l), d\boldsymbol{x}_\alpha Z_1,\dots,(\nabla^{TZ}d\varphi^\alpha(X_0^H))(d\boldsymbol{x}_\alpha Z_j),\dots,d\boldsymbol{x}_\alpha Z_{l'}\big)\big).
\end{aligned}$$

By similar computations for $\dot\partial^{\boldsymbol{s}}$ and $\dot\partial^{\boldsymbol{t}}$, one gets:

$$\begin{aligned}
(8)\quad &\big(\dot\partial^{\boldsymbol{s}}\psi\big)(X_1, \dots, X_k; Y_0, Y_1, \dots Y_l, Z_1, \dots Z_{l'})\\
&\qquad = (\boldsymbol{x}_\alpha^{-1})^*\big(\dot\partial^\alpha\psi^\alpha(X_1,\dots,X_k; d\boldsymbol{x}_\alpha(Y_0, Y_1,\dots,Y_l, Z_1,\dots,Z_{l'})\big),
\end{aligned}$$

and

$$\begin{aligned}
(9)\quad &\big(\dot\partial^{\boldsymbol{t}}\psi\big)(X_1, \dots, X_k; Y_1, \dots, Y_l, Z_0, Z_1, \dots, Z_{l'})\\
&\qquad = (\boldsymbol{x}_\alpha^{-1})^*\big(\dot\partial^Z\psi^\alpha(X_1,\dots,X_k; d\boldsymbol{x}_\alpha(Y_1,\dots,Y_l, Z_0,\dots,Z_{l'}))\big)\\
&\qquad\quad + \sum_{1\le j\le l'}\psi^\alpha\big(X_1,\dots,X_k; d\boldsymbol{x}_\alpha(Y_1,\dots,Y_l), d\boldsymbol{x}_\alpha Z_1,\\
&\qquad\qquad \dots,\big(\nabla^{TZ}_{d\boldsymbol{x}_\alpha Z_0} d\boldsymbol{x}_\alpha Z_j - d\boldsymbol{x}_\alpha(P^{V_t}\nabla^{TM}_{Z_0}Z_j),\dots,d\boldsymbol{x}_\alpha Z_{l'}\big)\big).
\end{aligned}$$

**2D.** ***Smoothing operators.*** For any $(x, y, z) \in M \times_B M$, let $\boldsymbol{d}(x, y, z)$ be the Riemannian distance between $y, z \in Z_x$. We regard $\boldsymbol{d}$ as a continuous, nonnegative function on $M \times_B M$.

**Definition 2.10.** (See [Nistor et al. 1999]). As a vector space,

$$\Psi^{-\infty}_{\infty}(M \times_B M, E) := \left\{ \psi \in \Gamma^{\infty}(\hat{E}) : \begin{array}{l} \text{For any } m \in \mathbb{N}, \varepsilon > 0, \text{ there exists } C_m > 0 \\ \text{such that for all } i + j + k \leq m, \\ |(\dot{\nabla}^{\hat{E}_\flat})^i (\dot{\partial}^{V_s})^j (\dot{\partial}^{V_t})^k \psi| \leq C_m e^{-\varepsilon \boldsymbol{d}}. \end{array} \right\}.$$

The convolution product structure on $\Psi^{-\infty}_{\infty}(M \times_B M, E)$ is defined by

$$\psi_1 \star \psi_2(x, y, z) := \int_{Z_x} \psi_1(x, y, w)\psi_2(x, w, z)\mu_x(w).$$

We introduce a Sobolev type generalization of the Hilbert–Schmidt norm on $\Psi^{-\infty}_{\infty}(M \times_B M, E)^G$, the space of $G$-invariant kernels. Since $G$ is a finitely generated discrete group and acts on $M$ freely and properly discontinuously, then there exists a smooth compactly supported function $\chi \in C^{\infty}_c(M)$, such that

$$\sum_{g \in G} g^* \chi = 1.$$

In particular, one may construct $\chi$ as follows. Denote by $\pi_G$ the projection $M \to M_0 = M/G$. There exists some $r > 0$ and a finite collection of geodesic balls $B(p_\alpha, r)$ of radius $r$ such that $B(p_\alpha, r)$ is diffeomorphic to its image in $M_0$ under $\pi_G$, and moreover $\left\{B\left(p_\alpha, \frac{1}{3}r\right)\right\}$ covers $M_0$ (since $M_0$ is compact). Since $G$ acts on $M$ by isometry, $\pi_G(B(p_\alpha g, r)) = \pi_G(B(p_\alpha, r))$ for all $g \in G$. Thus one may without loss of generality assume that $B(p_\alpha, r)$ are mutually disjoint.

Define the functions $f \in C^{\infty}(\mathbb{R})$, $F_\alpha, F \in C^{\infty}_c(M)$ by

$$\begin{aligned} f(t) &:= e^{-1/t^2} \text{ if } t > 0, \quad 0 \text{ if } t \leq 0, \\ F_\alpha(p) &:= f\Big(1 - \frac{2}{r}\boldsymbol{d}(p, p_\alpha)\Big)\Big(f\Big(\frac{3}{r}\boldsymbol{d}(p, p_\alpha) - 1\Big) + f\Big(1 - \frac{2}{r}\boldsymbol{d}(p, p_\alpha)\Big)\Big)^{-1}, \ p \in M, \\ F &:= \sum_\alpha F_\alpha. \end{aligned}$$

Note that $F$ is well defined because $F_\alpha$ is supported on $B(p_\alpha, r)$, which is locally finite. Since by construction

$$\left\{ \bigcup_\alpha B\Big(p_\alpha g, \frac{r}{3}\Big) \right\}_{g \in G}$$

is a locally finite cover of $M$, $\sum_g g^* F$ is also well defined. Define

$$\chi := F\Big(\sum_g g^* F\Big)^{-1}.$$

Then clearly $\chi$ is the required partition of unity. Moreover, observe that $\chi^{1/2}$ is a smooth function because $f^{1/2}$ is smooth and all denominators are uniformly bounded away from 0.

For any $G$-invariant $\psi \in \Psi^{-\infty}_{\infty}(M \times_B M, E)^G$, recall that the standard trace of $\psi$ is

$$\operatorname{tr}_\Psi(\psi)(x) := \int_{z \in Z_x} \chi(x,z)\operatorname{tr}(\psi(x,z,z))\mu_x(z) \in C^\infty(B).$$

The definition does not depend on the choice of $\chi$. The corresponding Hilbert–Schmidt norm is

$$\begin{aligned}(10)\quad &\int_B \big(\operatorname{tr}_\Psi(\psi\psi^*)(x)\big)^2 \mu_B(x)\\ &\qquad = \int_B \int_{Z_x} \chi(x,z) \int_{Z_x} \operatorname{tr}(\psi(x,z,y)\psi^*(x,y,z))\mu_x(y)\mu_x(z)\mu_B(x).\end{aligned}$$

Note that equation (10) coincides with the $L^2$-norm of $\psi$. Generalizing (10) to taking into account derivatives, we define:

**Definition 2.11.** The $m$-th Hilbert–Schmidt norm on $\Psi^{-\infty}_{\infty}(M \times_B M, E)^G$ is defined to be

$$\|\psi\|^2_{\mathrm{HS}m} := \sum_{i+j+k \le m} \int_B \int_{Z_x} \chi(x,z) \int_{Z_x} \big|(\dot{\nabla}^{\hat{E}_\flat})^i (\dot{\partial}^{\boldsymbol{s}})^j (\dot{\partial}^{\boldsymbol{t}})^k \psi\big|^2 (x,y,z)\mu_x(y)\mu_x(z)\mu_B(x),$$

for any $G$-invariant element $\psi$. Let $\bar{\Psi}^{-\infty}_m(M \times_B M, E)^G$ be the completion of $\Psi^{-\infty}_{\infty}(M \times_B M, E)^G$ with respect to $\|\cdot\|_{\mathrm{HS}m}$.

Similar to Lemma 2.4, one has elliptic regularity for the Hilbert–Schmidt norm:

**Lemma 2.12.** *Let $A$ be a $G$-invariant, first order elliptic differential operator, then for any $m = 0, 1, \dots,$ there exists a constant $C > 0$ such that*

$$\|\psi\|_{\mathrm{HS}m+1} \le C(\|A\psi\|_{\mathrm{HS}m} + \|\psi\|_m),$$

*for all $\psi \in \Psi^{-\infty}_{\infty}(M \times_B M, E)^G$.*

*Proof.* Define

$$S := \{g \in G : \chi(g^*\chi) \ne 0\}.$$

Then $S$ is finite because $\{g^*\chi\}$ is a locally finite partition of unity.

Consider $(\chi(x,z))^{1/2}\psi$. By the Leibniz rule, one has

$$(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\chi^{1/2}\psi=\chi^{1/2}(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\psi$$

modulo terms involving lower derivatives in $\psi$. Since $(\chi(x,z))^{1/2}$ is smooth with bounded derivatives, there exists some $C_1>0$ such that for any $(x,y,z)\in M\times_B M$,

$$\Big|\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\chi^{1/2}\psi\big|^2-\chi\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\psi\big|^2\Big|(x,y,z)\\ \le\sum_{g\in S}g^*\chi\Big(C_1\sum_{i+j+k\le m-1}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\psi\big|^2\Big)(x,y,z). \tag{11}$$

Similarly, since $A\chi^{1/2}-\chi^{1/2}A$ is a $C^\infty$-bounded tensor, one has

$$\Big|\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k(A\chi^{1/2}\psi)\big|^2-\chi\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^kA\psi\big|^2\Big|\\ \le\sum_{g\in S}g^*\chi\Big(C_2\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\psi\big|^2\Big). \tag{12}$$

Since the integrand is $G$-invariant, for any $g\in G$

$$\int_{M\times_B M}g^*\chi\sum_{i+j+k\le m-1}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k\psi\big|^2\mu_x(y)\mu_x(z)\mu_B(x)=\|\psi\|^2_{\mathrm{HS}m-1}.$$

Observe that $A$ being $G$-invariant implies $A$ is uniformly elliptic and $C^\infty$-bounded. Therefore applying Lemma 2.4 for $(\chi(x,z))^{1/2}\psi$, there exists a constant $C_3>0$ such that

$$\int_{M\times_B M}\sum_{i+j+k\le m+1}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k(\chi^{1/2}\psi)\big|^2\mu_x(y)\mu_x(z)\mu_B(x)\\ \le C_3\Big(\int_{M\times_B M}\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k(A\chi^{1/2}\psi)\big|^2\mu_x(y)\mu_x(z)\mu_B(x)\\ +\int_{M\times_B M}\sum_{i+j+k\le m}\big|(\dot\nabla^{\hat E_\flat})^i(\dot\partial^s)^j(\dot\partial^t)^k(\chi^{1/2}\psi)\big|^2\mu_x(y)\mu_x(z)\mu_B(x)\Big).$$

Then by equations (11) and (12), we get the lemma. □

**2E.** ***Fiberwise operators.*** We turn to considering another class of operators and a different norm.

**Definition 2.13.** A fiberwise operator is a linear operator $A : \Gamma_c^\infty(E_\flat) \to \mathcal{W}^0(E)$ such that for all $x \in B$, and any sections $s_1, s_2 \in \Gamma_c^\infty(E_\flat)$,

$$(As_1)(x) = (As_2)(x),$$

whenever $s_1(x) = s_2(x)$.

We say that $A$ is smooth if $A(\Gamma_c^\infty(E)) \subseteq \Gamma^\infty(E)$. A smooth fiberwise operator $A$ is said to be bounded of order $m$ if $A$ extends to a bounded map from $\mathcal{W}^m(E)$ to itself.

Denote by $\|A\|_{\mathrm{op}m}$ the operator norm of $A : \mathcal{W}^m(E) \to \mathcal{W}^m(E)$.

**Example 2.14.** Examples of smooth fiberwise operators are $\Psi_\infty^{-\infty}(M \times_B M, E)$, acting on $\mathcal{W}^m(E)$ by vector representation, i.e.,

$$(\Psi s)(x, y) := \int_{Z_x} \psi(x, y, z) s(x, z) \mu_x(z).$$

**Notation 2.15.** For the fiberwise operator $A : \Gamma_c^\infty(E_\flat) \to \mathcal{W}^0(E)$ which is of the form given by Example 2.14, we denote its kernel by $A(x, y, z)$. We shall write

$$\|A\|_{\mathrm{HS}m} := \|A(x, y, z)\|_{\mathrm{HS}m},$$

provided $A(x, y, z) \in \bar{\Psi}_m^{-\infty}(M \times_B M, E)$.

The following lemma enables one to construct more fiberwise operators:

**Lemma 2.16.** *Let $A$ be any first order, $C^\infty$-bounded differential operator on $M$ and $\Psi \in \Psi_\infty^{-\infty}(M \times_B M, E)$ be as in Example 2.14. Then $[A, \Psi]$ is a fiberwise operator in $\Psi_\infty^{-\infty}(M \times_B M, E)$.*

*Proof.* Since multiplication by a tensor or differentiation along $V$ is fiberwise, all that remains is to consider operators of the form $\nabla_{X^H}^E$, for some vector field $X$ on $B$. Let $L_{X^H}^{\nabla^E} = d^{\nabla^E} i_{X^H} + i_{X^H} d^{\nabla^E}$, where $d^{\nabla^E}$ is the twisted de Rham operator. In the remainder of this paper, the Lie derivatives are all defined in this way.

Let $s \in \Gamma_c^\infty(E)$ be arbitrary. We first suppose that $Z$ is orientable and $\mu_x$ is a volume form. By the decay condition in Definition 2.10, one can differentiate under the integral sign to get

$$\begin{aligned} A\Psi s(x, z) &= \int_{Z_x} L_{X^{\hat{H}}}^{\nabla^{\hat{E}}} (\psi(x, y, z) s(x, y) \mu_x(y)) \\ &= \int_{Z_x} \big(L_{X^{\hat{H}}}^{\nabla^{\hat{E}}} \psi(x, y, z)\big) s(x, y) \mu_x(y) + \int_{Z_x} \psi(x, y, z) \big(L_{X^H}^{\nabla^E} s(x, y)\big) \mu_x(y) \\ &\qquad + \int_{Z_x} \psi(x, y, z) s(x, y) \big(L_{X^H}^{\nabla^E} \mu_x(y)\big). \end{aligned}$$

The second term in the last line is just $\Psi A s$. Hence the result.

For the general case, one can take a suitable partition of unity and integrate over local volume forms. Then one obtains a similar equation. □

Let $A$ be a smooth fiberwise operator on $\Gamma_c^\infty(E_\flat)$. Then $A$ induces a fiberwise operator $\hat{A}$ on $\Gamma_c^\infty(\hat{E}_\flat)$ by

$$\hat{A}(u \otimes \boldsymbol{s}^* e) := A(u|_{M_\alpha \times \{z\}}) \otimes (\boldsymbol{s}^* e) \tag{13}$$

on $\boldsymbol{t}^{-1}(M_\alpha) \cong M_\alpha \times Z$, for any sections $e \in \Gamma^\infty(E')$ and $u \in \Gamma^\infty(\boldsymbol{t}^* E)$, and $\psi = u \otimes \boldsymbol{s}^* e \in \Gamma_c^\infty(\hat{E})$.

Note that $\hat{A}$ is independent of trivialization since $A$ is fiberwise, and for any $\alpha$, $\beta$ and $z \in Z$, the transition function $\boldsymbol{x}_\beta \circ (\boldsymbol{x}_\alpha)^{-1}$ maps the submanifold $Z_x \times \{z\}$ to $Z_x \times \{\varphi_x^\beta \circ (\varphi_x^\alpha)^{-1}(z)\}$ as the identity diffeomorphism. If $\Psi$ is a kernel and $A$ is a fiberwise smooth operator, $A\Psi$ is also a kernel and is given by $\hat{A}\Psi$.

**2F. *The main theorem.*** Suppose that $A$ is smooth and bounded of order $m$ for all $m \in \mathbb{N}$. Consider the covariant derivatives of $\hat{A}\psi$.

**Theorem 2.17.** *For any smooth bounded $G$-invariant operator $A$, there exist constants $C'_{1,1}$, $C'_{0,0} > 0$ such that for any $\psi \in \Psi_\infty^{-\infty}(M \times_B M)^G$ one has $\hat{A}\psi \in \Psi_\infty^{-\infty}(M \times_B M)^G$ and*

$$\|\hat{A}\psi\|_{\mathrm{HS}\,1} \le (C'_{1,1}\|A\|_{\mathrm{op}\,1} + C'_{1,0}\|A\|_{\mathrm{op}\,0})\|\psi\|_{\mathrm{HS}\,1}.$$

*Proof.* Fix a partition of unity $\{\theta_\alpha\} \in C_c^\infty(B)$ subordinate to $\{B_\alpha\}$. We still denote by $\{\theta_\alpha\}$ its pullback to $M$ and $M \times_B M$. Fix any Riemannian metric on $Z$ and denote the corresponding Riemannian measure by $\mu_Z$. Then one writes

$$(\hat{\boldsymbol{x}}_\alpha)_\star(\mu_x \mu_B) = J_\alpha \mu_B \mu_Z,$$

for some smooth positive function $J_\alpha$. Moreover, over any compact subset of $B_\alpha \times Z$, $1/J_\alpha$ is bounded.

Given any $\psi \in \Psi_\infty^{-\infty}(M \times_B M)^G$, let $\psi^\alpha := \hat{\boldsymbol{x}}_\alpha^*(\psi)$. The theorem clearly follows from the inequalities

$$\int_{B_\alpha}\int_{Z_x} \chi(x,z) \int_{Z_x} |\dot{\nabla}^\alpha \hat{A}(\theta_\alpha \psi^\alpha)|^2 \mu_x(y)\mu_x(z)\mu_B(x) \le (C_1\|A\|^2_{\mathrm{op}\,1} + C_2\|A\|^2_{\mathrm{op}\,0})\|\psi\|^2_{\mathrm{HS}\,1}, \tag{14}$$

$$\int_{B_\alpha}\int_{Z_x} \chi(x,z) \int_{Z_x} |\dot{\partial}^\alpha \hat{A}(\theta_\alpha \psi^\alpha)|^2 \mu_x(y)\mu_x(z)\mu_B(x) \le (C_1\|A\|^2_{\mathrm{op}\,1} + C_2\|A\|^2_{\mathrm{op}\,0})\|\psi\|^2_{\mathrm{HS}\,1}, \tag{15}$$

$$\int_{B}\int_{Z_x} \chi(x,z) \int_{y \in Z_x} |\dot{\partial}^Z \hat{A}(\theta_\alpha \psi^\alpha)|^2 \mu_x(y)\mu_x(z)\mu_B(x) \le \|A\|^2_{\mathrm{op}\,0}\|\psi\|^2_{\mathrm{HS}\,1}. \tag{16}$$

Let $Z = \bigcup_\lambda Z_\lambda$ be a locally finite cover. Then the support of $\chi\theta_\alpha$ lies in some finite subcover. Let $\chi_\alpha$ be the characteristic function

$$\chi_\alpha(x, z) = \begin{cases} 1 & \text{if } (\chi\theta_\alpha)(x, z) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume $E'|_{Z_\lambda}$ are all trivial. For each $\lambda$ fix an orthonormal basis $\{e_r^\lambda\}$ of $E'|_{B_\alpha \times Z_\lambda}$, and write $\psi^\alpha := \sum_r u_r^\lambda \otimes \boldsymbol{s}^* e_r^\lambda$. Using Lemma 2.9, one estimates the integrand of the left-hand side of equation (14). Then there exits a constant $C_3 > 0$ such that

$$\begin{aligned} &\big|\dot\nabla^\alpha(\hat A\theta_\alpha\psi^\alpha)\big|^2(x, y, z) \\ &\qquad = \Big|\sum_r (\dot\nabla^{E_\flat} A\theta_\alpha(u_r^\lambda|_{M_\alpha\times\{z\}})(x, y)) \otimes \boldsymbol{s}^* e_r^\lambda + (A\theta_\alpha u_r^\lambda) \otimes \boldsymbol{s}^*(\nabla^E e_r^\lambda)\Big|^2 \\ &\qquad \le C_3 \sum_r \Big(\big|\dot\nabla^{E_\flat} A\theta_\alpha(u_r^\lambda|_{M_\alpha\times\{z\}})(x, y)\big|^2 + \big|(A\theta_\alpha u_r^\lambda) \otimes \boldsymbol{s}^*(\nabla^E e_r^\lambda)\big|^2\Big). \end{aligned}$$

By integrating, one gets for some constants $C_q$, $q = 4, \dots, 10$, that

$$\begin{aligned} &\int_{B_\alpha}\int_{Z_x} \chi(x, z) \int_{Z_x} |\dot\nabla^\alpha \hat A(\theta_\alpha\psi^\alpha)|^2 \mu_x(y)\mu_x(z)\mu_B(x) \\ &\quad \le C_4 \sum_\lambda \int_{Z_\lambda}\int_{B_\alpha}\int_{Z_x} \sum_r \big(\big|\dot\nabla^{E_\flat} A\theta_\alpha(u_r^\lambda|_{M_\alpha\times\{z\}})(x, y)\big|^2 \\ &\qquad\quad + \big|(A\theta_\alpha u_r^\lambda) \otimes \boldsymbol{s}^*(\nabla^E e_r^\lambda)\big|^2\big)\mu_x(y)\mu_B(x)\mu_Z(z) \\ &\quad \le \sum_\lambda \int_{Z_\lambda}\int_{B_\alpha}\int_{Z_x} \sum_r \big(C_5\|A\|^2_{\mathrm{op}\,1}\big(\big|\dot\nabla^{E_\flat}\theta_\alpha(u_r^\lambda|_{M_\alpha\times\{z\}})(x, y)\big|^2 \\ &\qquad\qquad\qquad \big|\dot\partial^V\theta_\alpha(u_r^\lambda|_{M_\alpha\times\{z\}})(x, y)\big|^2 + \big|\theta_\alpha(u_r^\lambda|_{M_\alpha\times\{z\}})(x, y)\big|^2\big) \\ &\qquad\quad + C_6\|A\|_{\mathrm{op}\,0}\big|\theta_\alpha u_r^\lambda\big|^2\big)\mu_x(y)\mu_B(x)\mu_Z(z) \\ &\quad \le \sum_\lambda \int_{Z_\lambda}\int_{B_\alpha}\int_{Z_x} J_\alpha(C_7\|A\|^2_{\mathrm{op}\,1} + C_8\|A\|_{\mathrm{op}\,0})\big(\big|\dot\nabla^\alpha\theta_\alpha\psi_\alpha\big|^2 \\ &\qquad\quad + \big|\dot\partial^\alpha\theta_\alpha\psi_\alpha\big|^2 + \big|\dot\partial^Z\theta_\alpha\psi_\alpha\big|^2 + \big|\theta_\alpha\psi_\alpha\big|^2\big)\mu_x(y)\mu_B(x)\mu_Z(z) \\ &\quad \le \int_B\int_{Z_x} \chi_\alpha \int_{Z_x} \big(C_9\|A\|^2_{\mathrm{op}\,1} + C_{10}\|A\|_{\mathrm{op}\,0}\big)\big(\big|\dot\nabla^{\hat E_\flat}\boldsymbol{x}^*_\alpha(\theta_\alpha\psi)\big|^2 \\ &\qquad\quad + \big|\dot\partial^{\boldsymbol{s}}\boldsymbol{x}^*_\alpha(\theta_\alpha\psi)\big|^2 + \big|\dot\partial^{\boldsymbol{t}}\boldsymbol{x}^*_\alpha(\theta_\alpha\psi)\big|^2 + \big|\boldsymbol{x}^*_\alpha(\theta_\alpha\psi)\big|^2\big)\mu_x(y)\mu_x(z)\mu_B(x). \end{aligned}$$

Now we use an argument similar to the proof of Lemma 2.12. Namely, write the integrand as a sum

$$\begin{aligned}&\chi_\alpha\big(\big|\dot{\nabla}^{\hat{E}_\flat}\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\dot{\partial}^s\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\dot{\partial}^t\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2\big)\\&=\sum_{g\in S}\chi_\alpha g^*\chi\big(\big|\dot{\nabla}^{\hat{E}_\flat}\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\dot{\partial}^s\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\dot{\partial}^t\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2\big).\end{aligned}$$

Then since for all $g$

$$\int g^*\chi\big(\big|\dot{\nabla}^{\hat{E}_\flat}\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\dot{\partial}^s\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\dot{\partial}^t\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2+\big|\boldsymbol{x}_\alpha^*(\theta_\alpha\psi)\big|^2\big)=\|\psi\|_{\mathrm{HS}\,1},$$

equation (14) follows.

Using the same arguments with $\dot{\partial}^\alpha$ in place of $\dot{\nabla}^\alpha$, one gets (15).

As for the last inequality, since $\boldsymbol{t}^*E|_{M_\alpha\times\{z\}}$ and the connection $(\boldsymbol{x}_\alpha^{-1})^*\nabla^{\boldsymbol{s}^*E'}$ are trivial along $\exp tZ_0$, one can write

$$\begin{aligned}\nabla^\alpha_{Z_0}(\hat{A}u\otimes\boldsymbol{s}^*e)&=\frac{d}{dt}\Big|_{t=0}Au\big|_{M_\alpha\times\{\exp tZ\}}\otimes\boldsymbol{s}^*e+u\otimes\nabla^{\boldsymbol{s}^*E'}_{Z_0}\boldsymbol{s}^*e\\&=A\bigg(\frac{d}{dt}\Big|_{t=0}u\big|_{M_\alpha\times\{\exp tZ\}}\bigg)\otimes\boldsymbol{s}^*e+u\otimes\nabla^{\boldsymbol{s}^*E'}_{Z_0}\boldsymbol{s}^*e=\hat{A}\big(\nabla^\alpha_{Z_0}(u\otimes\boldsymbol{s}^*e)\big).\end{aligned}$$

It follows that

$$\dot{\partial}^Z\hat{A}\psi^\alpha=\hat{A}(\dot{\partial}^Z\psi^\alpha),$$

from which (16) follows. □

Clearly, the arguments leading to Theorem 2.17 can be repeated and we obtain:

**Corollary 2.18.** *For any smooth bounded operator $A$ and $m=0,1,\ldots$, there exists $C'_{m,l}>0$ such that for any $\psi\in\Psi^{-\infty}_\infty(M\times_B M)^G$ one has*

$$\|\hat{A}\psi\|_{\mathrm{HS}\,m}\le\bigg(\sum_{0\le l\le m}C_{m,l}\|A\|_{\mathrm{op}\,l}\bigg)\|\psi\|_{\mathrm{HS}\,m}.$$

**Notation 2.19.** In view of Corollary 2.18, we shall denote

$$\|A\|_{\mathrm{op}'m}:=\bigg(\sum_{0\le l\le m}C_{m,l}\|A\|_{\mathrm{op}\,l}\bigg).$$

We may assume without loss of generality that $C_{m,l}\ge 2$. Then one still has

$$\|A_1A_2\|_{\mathrm{op}'m}\le\|A_1\|_{\mathrm{op}'m}\|A_2\|_{\mathrm{op}'m}.\tag{17}$$

## 3. Large time behavior of the heat operator

In this section we will prove that under the condition of the positivity of the Novikov–Shubin invariant, the heat operator also convergences to the projection operator under the norm $\|\cdot\|_{\mathrm{HS}\,m}$.

**3A.** ***The Novikov–Shubin invariant.*** Let $M \to B$ be a fiber bundle with a $G$ action, and $TM = H \oplus V$ be the $G$-invariant splitting, as defined in Section 2A. Recall that we assumed the metric on $H \cong \pi^* TB$ is given by pulling back some Riemannian metric on $B$. In other words, $V$ is a Riemannian foliation.

Let $E \to M$ be a flat, contravariant $G$-vector bundle, and $\nabla$ be an invariant flat connection on $E$. Denote $E^\bullet := \wedge^\bullet V' \otimes E$.

Since the vertical distribution $V$ is integrable, the de Rham differential $d_V^{\nabla^E}$ along $V$ is well defined. Write $\eth_0 := d_V^{\nabla^E} + \big(d_V^{\nabla^E}\big)^*$, $\Delta := \eth_0^2$, and denote by $e^{-t\Delta}$ the heat operator and $\Pi_0$ the orthogonal projection onto $\mathrm{Ker}(\Delta)$.

The following result is classical: See, for example, [Bismut 1986, Proposition 2.8] and [Heitsch 1995, Proposition 3.5].

**Lemma 3.1.** *The heat operator $e^{-t\Delta}$ is given by a smooth kernel. Moreover, for any first order differential operator $A$, one has the Duhamel type formula*

$$[A, e^{-t\Delta}] = -\int_0^t e^{-(t-t')\Delta}[A, \Delta] e^{-t'\Delta}\, dt'. \tag{18}$$

From Lemma 3.1, it follows that:

**Corollary 3.2** [Heitsch 1995, Corollary 3.11]. *For any $i, j, k$, there exist $C, M > 0$ such that*

$$\big|(\dot\nabla^{E_\flat})^i (\dot\partial^s)^j (\dot\partial^t)^k e^{-t\eth_0^2}\big|(x, y, z) \le C e^{-M d(y,z)^2}.$$

Hence $e^{-t\eth_0^2} \in \Psi_\infty^{-\infty}(M \times_B M, E^\bullet)^G$.

As for $\Pi_0$, one has

**Lemma 3.3.** *The kernel of $\Pi_0$ lies in $\bar\Psi_0^{-\infty}(M \times_B M, E^\bullet)^G$.*

*Proof.* By [Gong and Rothenberg 1996, Theorem 2.2] $\Pi_0$ is also represented by a smooth kernel $\Pi_0(x, y, z)$. Moreover by the same theorem and the fact that $\Pi_0 = \Pi_0^2$, one has

$$\sup_{x \in B}\left\{\int_{Z_x} \chi(x, z) \int_{Z_x} |\Pi_0(x, y, z)|^2 \mu_x(y)\mu_x(z)\right\} = \|\Pi_0\|_\tau < \infty,$$

where $\|\cdot\|_\tau$ is the $\tau$-trace norm defined in [Gong and Rothenberg 1996] (see also [Azzali et al. 2015]).

Hence we are left to consider $\chi_n(x, y, z)\Pi_0(x, y, z)$, where $\chi_n \in C^\infty(M \times_B M)^G$ is a sequence of smooth functions such that

(1) $0 \le \chi_n \le 1$;

(2) $\chi_n$ is increasing and converges pointwise to 1;

(3) $\chi_n(x, y, z) = 0$ whenever $\boldsymbol{d}(y, z) > nr$ for some $r > 0$.

To construct $\chi_n$, let $r > 0$ to be the infimum of the injective radius of the fibers $Z_x$, and $\phi_1$ be a nonnegative smooth function such that $\phi_1(t) = 1$ if $t < \frac{1}{2}r$, $\phi_1(t) = 0$ if $t > r$. Then $\chi_1 := \phi_1 \circ \boldsymbol{d}(y, z)$ is $G$-invariant. Define

$$\tilde{\chi}_n := \chi_1 \star \cdots \star \chi_1 \text{ (convolution by } n \text{ times).}$$

Note that $\tilde{\chi}_n(x, y, z) > 0$ whenever $\boldsymbol{d}(y, z) < \frac{1}{2}nr$. Moreover, $\tilde{\chi}_n$ is $G$-invariant and $\tilde{\chi}_n(x, y, z) = 0$ whenever $\boldsymbol{d}(y, z) > nr$. Since $\tilde{\chi}_{n+1}$ is bounded away from 0 on the support of $\tilde{\chi}_n$, clearly one can find smooth functions $\phi_n$ such that $\chi_n := \phi_n \circ \tilde{\chi}_n$ satisfies conditions (1)–(3). □

Because of Corollary 3.2 and Lemma 3.3, it makes sense to define:

**Definition 3.4.** We say that $\Delta$ has positive Novikov–Shubin invariant if there exist $\gamma > 0$ and $C_0 > 0$ such that for sufficiently large $t$,

$$\sup_{x \in B} \left\{ \int_{Z_x} \chi(x, z) \int_{Z_x} \left| (e^{-t\Delta} - \Pi_0)(x, y, z) \right|^2 \mu_x(y) \mu_x(z) \right\} \le C_0 t^{-\gamma}.$$

**Remark 3.5.** The positivity of the Novikov–Shubin invariant is independent of the metrics defining the operator $\Delta$.

**Remark 3.6.** Since $e^{-(t/2)\Delta} - \Pi_0$ is nonnegative, self adjoint and $(e^{-(t/2)\Delta} - \Pi_0)^2 = e^{-t\Delta} - \Pi_0$, one has

$$\sup_{x \in B} \left\{ \int_{Z_x} \chi(x, z) \int_{Z_x} \left| (e^{-\frac{t}{2}\Delta} - \Pi_0)(x, y, z) \right|^2 \mu_x(y) \mu_x(z) \right\} = \| e^{-t\Delta} - \Pi_0 \|_\tau.$$

Hence our definition of having positive Novikov–Shubin invariant is equivalent to that of [Azzali et al. 2015]. Our argument here is similar to the proof of [Bismut et al. 2017, Theorem 7.7].

In this paper, we shall always assume $\Delta$ has positive Novikov–Shubin invariant. From this assumption, it follows by integration over $B$ that there exist constants $\gamma > 0$ and $C > 0$ such that for $t$ large enough

$$\| e^{-t\Delta} - \Pi_0 \|_{\mathrm{HS}\,0} < C t^{-\gamma}. \tag{19}$$

**3B. *Example: The Bismut superconnection.***

**Definition 3.7.** A standard flat Bismut superconnection is an operator of the form

$$d^{\nabla^E} := d_V^{\nabla^E} + \nabla^{E_\flat^\bullet} + \iota_\Theta,$$

where $\Theta$ is the $V$-valued horizontal 2-form defined by

$$\Theta(X_1^H, X_2^H) := -P^V[X_1^H, X_2^H] \quad \text{for all } X_1, X_2 \in \Gamma^\infty(TB),$$

and $\iota_\Theta$ is the contraction with $\Theta$. Note that $P^V$ is not canonical and it depends on the splitting $TM = V \oplus H$.

Observe that the adjoint of the Bismut superconnection,

$$\big(d^{\nabla^E}\big)' = \big(d_V^{\nabla^E}\big)^* + \big(\nabla^{E_\flat^\bullet}\big)' - \Lambda_{\Theta^*},$$

is also flat. It follows that

$$\big(\nabla^{E_\flat^\bullet}\big)'\big(d_V^{\nabla^E}\big)^* + \big(d_V^{\nabla^E}\big)^*\big(\nabla^{E_\flat^\bullet}\big)' = 0.$$

Define

$$\Omega := \tfrac{1}{2}\big(\big(\nabla^{E_\flat^\bullet}\big)' - \nabla^{E_\flat^\bullet}\big).$$

Observe that $\Omega$ is a tensor (see [Álvarez López and Kordyukov 2001] for an explicit formula for $\Omega$). Moreover one has

$$\nabla^{E_\flat^\bullet}\big(d_V^{\nabla^E}\big)^* + \big(d_V^{\nabla^E}\big)^*\nabla^{E_\flat^\bullet} = 2\Omega\big(d_V^{\nabla^E}\big)^* + 2\big(d_V^{\nabla^E}\big)^*\Omega.$$

Also, observe that $\big(d_V^{\nabla^E}\big) + \big(d_V^{\nabla^E}\big)^* + \nabla^{E_\flat^\bullet} + \big(\big(\nabla^{E_\flat^\bullet}\big)'\big)^*$ is an elliptic operator.

**3C.** ***The regularity result of Alvarez Lopez and Kordyukov.*** We first recall that an operator $A$ is called $C^\infty$-bounded if in normal coordinates the coefficients and their derivatives are uniformly bounded. As in [Álvarez López and Kordyukov 2001], we make the more general assumption that there exists $C^\infty$-bounded first order differential operator $Q$, and zero degree operators $R_1, R_2, R_3, R_4$, all $G$-invariant, such that $d_V^{\nabla^E} + \big(d_V^{\nabla^E}\big)^* + Q$ is elliptic, and

$$Q\, d_V^{\nabla^E} + d_V^{\nabla^E} Q = R_1\, d_V^{\nabla^E} + d_V^{\nabla^E} R_2, \tag{20}$$
$$Q\big(d_V^{\nabla^E}\big)^* + \big(d_V^{\nabla^E}\big)^* Q = R_3\big(d_V^{\nabla^E}\big)^* + \big(d_V^{\nabla^E}\big)^* R_4.$$

Clearly, in our example, $\nabla^{E_\flat^\bullet} + \big(\big(\nabla^{E_\flat^\bullet}\big)'\big)^*$ satisfies Equation (20).

Write $\eth_0 := d_V^{\nabla^E} + (d_V^{\nabla^E})^*$, $\varDelta := \eth_0^2$, and denote by $\varPi_{d_V}$ and $\varPi_{d_V^*}$ respectively the orthogonal projections onto the range of $d_V^{\nabla^E}$ and $\big(d_V^{\nabla^E}\big)^*$, which we shall denote by $\mathrm{Rg}(d_V)$ and $\mathrm{Rg}(d_V^*)$.

In this section, we shall consider the operators

$$B_1 := R_1\varPi_{d_V} + R_3\varPi_{d_V^*}, \quad B_2 := \varPi_{d_V^*} R_2 + \varPi_{d_V} R_4, \quad B := B_2\varPi_0 + B_1(\mathrm{id} - \varPi_0).$$

We recall some elementary formulas regarding these operators from [Álvarez López and Kordyukov 2001]:

**Lemma 3.8** [Álvarez López and Kordyukov 2001, Lemma 2.2]. *One has*

$$\begin{aligned} Q\,d_V^{\nabla^E}+d_V^{\nabla^E}Q &= B_1 d_V^{\nabla^E}+d_V^{\nabla^E}B_2,\\ Q\big(d_V^{\nabla^E}\big)^*+\big(d_V^{\nabla^E}\big)^*Q &= B_1\big(d_V^{\nabla^E}\big)^*+\big(d_V^{\nabla^E}\big)^*B_2,\\ [Q,\Delta] &= B_1\Delta-\Delta B_2-\eth_0(B_1-B_2)\eth_0. \end{aligned}$$

One can furthermore estimate the derivatives of $\Pi_0$. First, recall that

**Lemma 3.9.** *One has* (*see* [Álvarez López and Kordyukov 2001, Corollary 2.8])

$$[Q+B,\Pi_0]=0.$$

*Proof.* Here we give a different proof. From definition we have

$$B=(\Pi_{d_V^*}R_2+\Pi_d R_4)\Pi_0+R_1\Pi_{d_V}+R_3\Pi_{d_V^*},$$

where we used $\Pi_{d_V}\Pi_0=\Pi_{d_V^*}\Pi_0=0$. Hence

$$B\Pi_0-\Pi_0 B=(\Pi_{d_V^*}R_2+\Pi_{d_V}R_4)\Pi_0-\Pi_0R_1\Pi_{d_V}-\Pi_0R_3\Pi_{d_V^*}.$$

For any $s$ one has

$$\Pi_{d_V}s=\lim_{n\to\infty}d\tilde{s}_n,$$

for some sequence $\tilde{s}_n$ (in some suitable function spaces). It follows that

$$\Pi_0R_1\Pi_{d_V}s=\lim_{n\to\infty}\Pi_0R_1d\tilde{s}_1=\lim_{n\to\infty}\Pi_0\big(Qd_V^{\nabla^E}+d_V^{\nabla^E}Q-d_V^{\nabla^E}R_2\big)\tilde{s}_1=\Pi_0Q\Pi_{d_V}s.$$

Similarly, one has $\Pi_0R_3\Pi_{d_V^*}=\Pi_0Q\Pi_{d_V^*}$ and by considering the adjoint,

$$\Pi_{d_V^*}R_2\Pi_0=\Pi_{d_V^*}Q\Pi_0\quad\text{and}\quad\Pi_{d_V^*}R_4\Pi_0=\Pi_{d_V^*}Q\Pi_0.$$

It follows that

$$[Q+B,\Pi_0]=(\mathrm{id}-\Pi_{d_V}-\Pi_{d_V^*})Q\Pi_0-\Pi_0Q(\mathrm{id}-\Pi_{d_V}-\Pi_{d_V^*})=0.\qquad\square$$

In other words, regarding $[Q,\Pi_0]$ and $[B,\Pi_0]$ as kernels, one has

$$\|[Q,\Pi_0]\|_{\mathrm{HS}m}=\|[B,\Pi_0]\|_{\mathrm{HS}m},$$

provided the right-hand side is finite. Hence, using elliptic regularity and the same arguments as Lemma 3.3, one can prove inductively that

$$\Pi_0(x,y,z)\in\bar{\Psi}_m^{-\infty}(M\times_B M,E^\bullet)\quad\text{for all } m.$$

Next, we recall the main result of [Álvarez López and Kordyukov 2001]

**Lemma 3.10.** *For any* $m=0,1,\dots,$

(1) *The heat operator* $e^{-t\Delta}$, *and the operators* $\eth_0 e^{-t\Delta}$, $\Delta e^{-t\Delta}$ *map* $\mathcal{W}^m(E)$ *to itself as bounded operators. Moreover, there exist constants* $C_m^0, C_m^1, C_m^2 > 0$ *such that*

$$\|e^{-t\Delta}\|_{\mathrm{op}m} \le C_m^0, \quad \|\eth_0 e^{-t\Delta}\|_{\mathrm{op}m} \le t^{-\frac{1}{2}} C_m^1, \quad \|\Delta e^{-t\Delta}\|_{\mathrm{op}m} \le t^{-1} C_m^2,$$

*for all* $t > 0$.

(2) *As* $t \to \infty$, $e^{-t\Delta}$ *strongly converges as an operator on* $\mathcal{W}^m(E)$. *Moreover,* $(t, s) \mapsto e^{-t\Delta}s$ *is a continuous map from* $[0, \infty] \times \mathcal{W}^m(E)$ *to* $\mathcal{W}^m(E)$.

(3) *One has the Hodge decomposition*

$$\mathcal{W}^m(E) = \mathrm{Ker}(\Delta) + \overline{\mathrm{Rg}(\Delta)} = \mathrm{Ker}(\eth_0) + \overline{\mathrm{Rg}(\eth_0)},$$

*where the kernel, image and closure are in* $\mathcal{W}^m(E)$.

Note that our case is slightly different from that of [Álvarez López and Kordyukov 2001], where $M$ is assumed to be compact (but with possibly noncompact fibers). However, the same arguments clearly apply because our $M$ is of bounded geometry.

We recall more results in [Álvarez López and Kordyukov 2001, Section 2].

**Lemma 3.11** [Álvarez López and Kordyukov 2001, Lemma 2.4]. *For any* $m \ge 0$, *there exists a constant* $C_m^3 > 0$ *such that*

$$\|[Q, e^{-t\Delta}]\|_{\mathrm{op}m} \le C_m^3.$$

*Proof.* Using the third equation of Lemma 3.8, equation (18) becomes

$$[Q, e^{-t\Delta}] = \int_0^t e^{-(t-t')\Delta}\eth_0(B_1 - B_2)\eth_0 e^{-t'\Delta}\, dt' - \int_0^t e^{-(t-t')\Delta}(B_1\Delta - \Delta B_2)e^{-t'\Delta}\, dt'.$$

Using Lemma 3.10, we estimate the first integral

$$\begin{aligned}\left\| \int_0^t e^{-(t-t')\Delta}\eth_0(B_1 - B_2)\eth_0 e^{-t'\Delta}\, dt' \right\|_{\mathrm{op}m} &\le \|B_1 - B_2\|_{\mathrm{op}m}(C_m^1)^2 \int_0^t \frac{dt'}{\sqrt{(t-t')t'}} \\ &= \|B_1 - B_2\|_{\mathrm{op}m}(C_m^1)^2 \pi.\end{aligned}$$

As for the second integral, we split the domain of integration into $\left[0, \frac{1}{2}t\right]$ and $\left[\frac{1}{2}t, t\right]$, and then integrate by part to get

$$\begin{aligned}&\int_0^t e^{-(t-t')\Delta}(B_1\Delta - \Delta B_2)e^{-t'\Delta}\, dt' \\ &\quad = \int_0^{t/2} e^{-(t-t')\Delta}\Delta(-B_1 - B_2)e^{-t'\Delta}\, dt' - \int_{t/2}^t e^{-(t-t')\Delta}(B_1 - B_2)\Delta e^{-t'\Delta}\, dt' \\ &\qquad + e^{-(t-t')\Delta}B_1 e^{-t'\Delta}\Big|_{t'=0}^{t/2} - e^{-(t-t')\Delta}B_2 e^{-t'\Delta}\Big|_{t'=t/2}^{t}.\end{aligned}$$

Again using Lemma 3.10, its $\|\cdot\|_{\mathrm{op}\,m}$-norm is bounded by

$$C_m^0 C_m^1(\|B_1\|_{\mathrm{op}\,m}+\|B_2\|_{\mathrm{op}\,m})\left(\int_0^{t/2}\frac{dt'}{t-t'}+\int_{t/2}^{t}\frac{dt'}{t'}\right)+C_m^0(C_m^0+1)(\|B_1\|_{\mathrm{op}\,m}+\|B_2\|_{\mathrm{op}\,m}),$$

which is uniformly bounded because $\int_0^{t/2} 1/(t-t')\,dt' = \int_{t/2}^{t} 1/t'\,dt' = \log 2$. $\square$

Lemma 3.9 suggests that $[Q+B, e^{-t\Delta}]$ converges to zero as $t\to\infty$. Indeed, we shall prove a stronger result, namely, $[Q+B, e^{-t\Delta}]$ decays polynomially in the $\|\cdot\|_{\mathrm{HS}\,m}$-norm for all $m$.

**Lemma 3.12.** *Suppose there exist $C_m,\gamma>0$ such that $\|e^{-t\Delta}-\Pi_0\|_{\mathrm{HS}\,m}\le C_m t^{-\gamma}$, then there exist $C'_m,\gamma_m>0$ such that*

$$\|[Q+B, e^{-t\Delta}]\|_{\mathrm{HS}\,m}=\|[Q+B, e^{-t\Delta}-\Pi_0]\|_{\mathrm{HS}\,m}\le C'_m t^{-\gamma_m}.$$

*Proof.* We follow the proof of [Álvarez López and Kordyukov 2001, Lemma 2.6]. By Lemma 3.8, we get

$$[Q+B,\Delta]=(\Delta(B_1+B_2)+\eth_0(B_1-B_2)\eth_0)(\mathrm{id}-\Pi_0).$$

It follows that $\Pi_0[Q+B, e^{-(t/2)\Delta}]=[Q+B, e^{-(t/2)\Delta}]\Pi_0=0$. Write

$$\begin{aligned}[Q+B, e^{-t\Delta}]&=[Q+B, e^{-\frac{1}{2}t\Delta}]e^{-\frac{1}{2}t\Delta}+e^{-\frac{1}{2}t\Delta}[Q+B, e^{-\frac{1}{2}t\Delta}]\\&=[Q+B, e^{-\frac{1}{2}t\Delta}](e^{-\frac{1}{2}t\Delta}-\Pi_0)+(e^{-\frac{1}{2}t\Delta}-\Pi_0)[Q+B, e^{-\frac{1}{2}t\Delta}].\end{aligned}$$

Taking $\|\cdot\|_{\mathrm{HS}\,m}$ and using Corollary 2.18 and Lemma 3.11, the claim follows. $\square$

**Theorem 3.13.** *Suppose $\|e^{-t\Delta}-\Pi_0\|_{\mathrm{HS}\,0}\le C_0 t^{-\gamma}$ for some $\gamma>0$, $C_0>0$. Then for any $m$, there exists $C''_m>0$ such that*

$$\|e^{-t\Delta}-\Pi_0\|_{\mathrm{HS}\,m}\le C''_m t^{-\gamma}\quad \textit{for all } t>1.$$

*Proof.* We prove the theorem by induction. The case $m=0$ is given. Suppose that for some $m$, $\|e^{-t\Delta}-\Pi_0\|_{\mathrm{HS}\,m}\le C_m t^{-\gamma}$. Consider $\|e^{-t\Delta}-\Pi_0\|_{\mathrm{HS}\,m+1}$.

Since $Q$ is a first order differential operator, for any kernel $\psi\in\Psi_\infty^{-\infty}(M\times_B M, E^\bullet)^G$, $[Q,\psi]$ is also a kernel lying in $\Psi_\infty^{-\infty}(M\times_B M, E^\bullet)^G$, that is in particular, given by a composition of the covariant derivatives $\dot\nabla^{\hat E_\flat}$, $\dot\partial^s$, $\dot\partial^t$ and some tensors acting on $\psi$. Since $\|\psi\|_{\mathrm{HS}\,m}$ is by definition the $\|\cdot\|_{\mathrm{HS}\,0}$ norm of the $m$-th derivatives of $\psi$, elliptic regularity (Lemma 2.12) implies

$$\|\psi\|_{\mathrm{HS}\,m+1}\le \tilde C_m(\|\psi\|_{\mathrm{HS}\,m}+\|\eth_0\psi\|_{\mathrm{HS}\,m}+\|\psi\eth_0\|_{\mathrm{HS}\,m}+\|[Q,\psi]\|_{\mathrm{HS}\,m}),$$

for some constant $\tilde{C}_m > 0$. Put $\psi = e^{-t\Delta} - \Pi_0$. The theorem then follows from the estimates

$$\begin{aligned}
\|\eth_0(e^{-t\Delta} - \Pi_0)\|_{\mathrm{HS}\,m} &= \|(e^{-t\Delta} - \Pi_0)\eth_0\|_{\mathrm{HS}\,m} \\
&\le \Big(\sum_{0\le l\le m} C'_{m,l}\|\eth_0(e^{-(t/2)\Delta} - \Pi_0)\|_{\mathrm{op}\,l}\Big)\|e^{-(t/2)\Delta} - \Pi_0\|_{\mathrm{HS}\,m} \\
&\le \Big(\sum_{0\le l\le m} C'_{m,l}C^1_l\big(\tfrac{1}{2}t\big)^{-1/2}\Big)C_m\big(\tfrac{1}{2}t\big)^{-\gamma}, \\
\|[Q, e^{-t\Delta} - \Pi_0]\|_{\mathrm{HS}\,m} &\le \|[Q + B, e^{-t\Delta} - \Pi_0]\|_{\mathrm{HS}\,m} + \|[B, e^{-t\Delta} - \Pi_0]\|_{\mathrm{HS}\,m} \\
&\le C'_m t^{-\gamma} + 2\Big(\sum_{0\le l\le m} C'_{m,l}\|B\|_{\mathrm{op}\,l}\Big)C_m t^{-\gamma}.
\end{aligned}$$

Note that we used Lemma 3.12 for the last inequality. □

## 4. Sobolev convergence

In this section we will use the method of [Azzali et al. 2015] to prove that under the condition of positivity of the Novikov–Shubin invariant the $L^2$-analytic torsion form is a smooth form.

Let $\nabla^E$ be a flat connection on $E$. Define the number operators on $\wedge^\bullet H'\otimes\wedge^\bullet V'\otimes E$ by

$$N_\Omega|_{\wedge^q H'\otimes\wedge^{q'} V'\otimes E} := q \quad\text{and}\quad N|_{\wedge^q H'\otimes\wedge^{q'} V'\otimes E} := q'.$$

In this section, we consider the rescaled Bismut superconnection [Berline et al. 1992, Chapter 9.1]

$$\begin{aligned}
\eth(t) &:= \tfrac{1}{2}t^{1/2}t^{-N_\Omega/2}(d + d^*)t^{N_\Omega/2} \\
&= \tfrac{1}{2}\big(t^{1/2}(d_V + d_V^*) + \big(\nabla^{E_\flat} + (\nabla^{E_\flat})'\big) + t^{-1/2}(-\Lambda_{\Theta^*} + \iota_\Theta)\big).
\end{aligned}$$

Denote

$$\begin{aligned}
D_0 &:= -\tfrac{1}{2}(d_V - d_V^*), \quad \Omega_t := -\tfrac{1}{2}\big(\nabla^{E_\flat} - (\nabla^{E_\flat})'\big) - \tfrac{1}{2}t^{-1/2}(-\Lambda_{\Theta^*} - \iota_\Theta), \\
D(t) &:= t^{1/2}D_0 + \Omega_t.
\end{aligned}$$

The curvature of $\eth(t)$ can be expanded in the form:

$$\eth(t)^2 = -D(t)^2 = t\Delta + t^{1/2}\Omega_t D_0 + t^{1/2}D_0\Omega_t + \Omega_t^2.$$

Hence as a consequence of Duhamel's expansion (see [Berline et al. 1992]), we have

$$e^{-\eth(t)^2} = e^{D(t)^2} = e^{-t\Delta} + \sum_{n=1}^{\dim B} \int_{(r_0,\dots,r_k)\in\Sigma^n} e^{-r_0 t\Delta}(t^{1/2}\Omega_t D_0 + t^{1/2} D_0 \Omega_t + \Omega_t^2) e^{-r_1 t\Delta} \cdots (t^{1/2}\Omega_t D_0 + t^{1/2} D_0\Omega_t + \Omega_t^2) e^{-r_n t\Delta}\, d\Sigma^n,$$

where $\Sigma^n := \{(r_0, r_1, \dots, r_n) \in [0, 1]^{n+1} : r_0 + \dots + r_n = 1\}$.

**4A.** ***The large time estimate of the rescaled heat operator.*** In this section, we follow [Azzali et al. 2015, Section 4] to estimate the Hilbert–Schmidt norms of $e^{-\eth(t)^2}$ (see Theorem 4.4 below).

Let $\gamma' := 1 - (1 + 2\gamma/(\dim B + 2 + 2\gamma))^{-1}$, $\bar{r}(t) := t^{-\gamma'}$. Fix $\bar{t}$ such that $\bar{r}(\bar{t}) < (\dim B + 1)^{-1}$. One has the following counterparts of [Azzali et al. 2015, Lemma 4.2]:

**Lemma 4.1.** *For $c = 0, 1, 2$, there exists a constant $C_m$ such that*

$$\big\| (\sqrt{t}\eth_0)^{c/2} e^{rt(D_0)^2} \big\|_{\mathrm{op}'m} \le C_m r^{-c/2} \text{ for any } t > \bar{t},\ 0 < r < 1 \text{ (by Lemma 3.10)};$$

*and for any $t > \bar{t}$, $\bar{r}(t) < r < 1$,*

$$\big\| e^{rt(D_0)^2} \big\|_{\mathrm{HS}m} \le C_m (rt)^{-\gamma} \qquad \text{(by Theorem 3.13)},$$

$$\big\| (\sqrt{t}\eth_0)^{c/2} e^{rt(D_0)^2} \big\|_{\mathrm{HS}m} \le C_m r^{-c/2} (rt)^{-\gamma} \text{ if } c = 1, 2 \text{ (by Corollary 2.18)}.$$

We furthermore observe that the arguments leading to the main result [Azzali et al. 2015, Theorem 4.1] still hold if one replaces the operator and $\|\cdot\|_\tau$ norm respectively by $\|\cdot\|_{\mathrm{op}'m}$ and $\|\cdot\|_{\mathrm{HS}m}$ for any $m$.

The arguments in [Azzali et al. 2015, Section 4] are elementary, so we shall only recall some key steps.

First, one splits the domain of integration $\Sigma^n = \bigcup_{I \neq \{0,\dots,n\}} \Sigma^n_{\bar{r}(t),I}$, where

$$\Sigma^n_{\bar{r}(t),I} := \{(r_0, \dots, r_n) : r_i \le \bar{r}(t), \text{ for all } i \in I,\ r_j \ge \bar{r}(t), \text{ for all } j \notin I\}.$$

Define

$$K(t, n, I, c_0, \dots, c_n; a_1, \dots a_n) := \int_{\Sigma^n_{\bar{r}(t),I}} (t^{1/2} D_0)^{c_0} e^{-r_0 t\Delta} \prod_{i=1}^{n} \big(\Theta_t^{a_i} (t^{1/2} D_0)^{c_i} e^{-r_i t\Delta}\big)\, d\Sigma^n, \tag{21}$$

for $c_i = 0, 1, 2$, $a_j = 1, 2$. Then one has

$$e^{-\eth(t)^2} = e^{D(t)^2} = \sum K(t, n, I, c_0, \dots, c_n; a_1, \dots, a_n),$$

by grouping terms involving $D_0$ together.

We shall consider the kernels $K(t, n, I, c_0, \dots, c_n; a_1, \cdots, a_n)(x, y, z)$ of the terms in the summation above. Consider the special case when $c_i = 0, 1$. One has the analogue of [Azzali et al. 2015, Proposition 4.6]:

**Lemma 4.2.** *There exists $\varepsilon > 0$ such that as $t \to \infty$,*

$$K(t, n, I, c_0, \dots, c_n, a_1, \dots, a_n)(x, y, z) = \begin{cases} \left(\frac{\Pi_0 \Omega^{a_1} \Pi_0 \cdots \Pi_0}{n!}\right)(x, y, z) + O(t^{-\varepsilon}) & \text{if } I = \varnothing, \text{ all } c_i = 0 \\ O(t^{-\varepsilon}) & \text{otherwise} \end{cases}$$

*in the $\|\cdot\|_{\mathrm{HS}m}$-norm.*

*Proof.* We first consider the case $I = \varnothing$. Suppose furthermore $c_q = 1$ for some $q$. By Corollary 2.18, The $\|\cdot\|_{\mathrm{HS}m}$-norm of the integrand on the right-hand side of (21) is bounded by

$$\begin{aligned} \left\|(t^{1/2} D_0)^{c_0} e^{-r_0 t \Delta}\right\|_{\mathrm{op}'m} \cdots \left\|\Omega_t^{a_q}\right\|_{\mathrm{op}'m} \left\|(t^{1/2} D_0) e^{-r_q t \Delta}\right\|_{\mathrm{HS}m} \cdots \left\|(t^{1/2} D_0)^{c_n} e^{-r_n t \Delta}\right\|_{\mathrm{op}'m} \\ \le C'_m r_0^{-c_0/2} \cdots r_q^{-c_q/2} (r_q t)^{-\gamma} \cdots r_n^{-c_n/2} \\ \le C'_m \bar{r}(t)^{-n/2-\gamma} t^{-\gamma}. \end{aligned}$$

Integrating, we have the estimate

$$\|K(t, n, c_0, \dots, c_n; a_1, \dots, a_n)(x, y, z)\|_{\mathrm{HS}m} \le C'_m t^{-\gamma + \gamma'(n/2+\gamma)} \int d\Sigma^n,$$

which is $O(t^{-\varepsilon})$ with $\varepsilon = \gamma(1 - (\dim B + 2\gamma)/(\dim B + 2 + 2\gamma))$.

Next, suppose $I = \varnothing$ and $c_i = 0$ for all $i$. Write $e^{-r_0 t \Delta - \Pi_0} + \Pi_0$ and split the integrand

$$(e^{-r_0 t \Delta} \Omega_t^{a_1} e^{-r_1 t \Delta} \cdots e^{-r_n t \Delta})(x, y, z)$$

into $2^{n+1}$ terms. If any term contains a $e^{-r_i t \Delta} - \Pi_0$ factor, similar arguments as above shows that it is $O(t^{-\gamma})$. Hence the only term that does not converge to 0 is

$$(\Pi_0 \Omega^{a_1} \Pi_0 \cdots \Pi_0)(x, y, z).$$

Since the volume of $\Sigma^n_{\bar{r}(t), I}$ converges to $\frac{1}{n!}$ as $t \to \infty$, the claim follows.

We are left to consider the case when $I$ is nonempty. Write $I = \{i_1, \dots, i_s\}$, $\{0, \dots, n\} \setminus I =: \{k_1, \dots, k_{s'}\} \neq \varnothing$. For $t$ sufficiently large $I \neq \{0, \dots, n\}$. Suppose $c_q = 1$ for some $q \notin I$. Then we take $\|\cdot\|_{\mathrm{HS}m}$-norm for $(t^{1/2} D_0) e^{-r_q t \Delta}$ term, and estimate

$$\begin{aligned}\|K(t,n,c_0,\ldots,c_n;a_1,\ldots,a_n)(x,y,z)\|_{\mathrm{HS}m} \\ \le \int_0^{\bar r(t)}\cdots\int_0^{\bar r(t)}\Big(\int_{\{(r_{k_1},\ldots,r_{k'_s}):(r_0,\ldots,r_n)\in\Sigma^n_{\bar r(t),I}\}} C'_m r_0^{-c_0/2}\cdots r_q^{-c_q/2}(r_q t)^{-\gamma}\cdots r_n^{-c_n/2} \\ d(r_{k_1}\cdots r_{k_{s'}})\Big)\,dr_{i_1}\cdots dr_{i_s}.\end{aligned}$$

As in the $I=\varnothing$ case, the integral over $\{(r_{k_1}),\ldots,r_{k_{s'}}:(r_0,\ldots,r_n)\in\Sigma^n_{\bar r(t),I}\}$ is $O(t^{-\varepsilon})$, while $\int_0^{\bar r(t)} r_i^{c_i/2}\,dr_i = O(t^{-\gamma'(1-c_i/2)})$. Again the claim is verified.

Finally if $c_i=0$ for all $i\in I$, then

$$\begin{aligned}\|K(t,n,c_0,\ldots,c_n;a_1,\ldots,a_n)(x,y,z)\|_{\mathrm{HS}m} \\ \le \int_0^{\bar r(t)}\cdots\int_0^{\bar r(t)}\Big(\int_{\{(r_{k_1},\ldots,r_{k'_s}):(r_0,\ldots,r_n)\in\Sigma^n_{\bar r(t),I}\}} C''_m r_0^{-c_{i_1}/2}\cdots r_n^{-c_{i_s}/2} d(r_{k_1}\cdots r_{k_{s'}})\Big)dr_{i_1}\cdots dr_{i_s} \\ = O(t^{-\gamma'(1-c_i/2)}).\end{aligned}$$

□

One then turns to the case for some $i$, $c_i=2$. If $I$ and $J$ are disjoint subsets of $\{0,\ldots,n\}$ with $I=\{i_1,\ldots,i_r\}$, and $\{0,\ldots,n\}\setminus(I\cup J)=:\{k_0,\ldots,k_q\}\neq\varnothing$, denote

$$\Sigma^n_{\bar r(t),I,J} := \{(r_0,\ldots,r_n)\in\Sigma^n_{\bar r(t),I} : r_j=\bar r(t),\ \text{whenever } j\in J\},$$

and define

$$\begin{aligned}K(t,n,I,J,c_0,\ldots,c_n;a_1,\ldots,a_n) := \int_0^{\bar r(t)}\cdots\int_0^{\bar r(t)}\int_{\{(r_{k_0},\ldots r_{k_q}):(r_0,\ldots,r_n)\in\Sigma^n_{\bar r(t),I}\}}(t^{1/2}D_0)^{c_0}e^{-r_0t\Delta} \\ \prod_{i=1}^n\big(\Theta_t^{a_i}(t^{1/2}D_0)^{c_i}e^{-r_it\Delta}\big)\big|_{\Sigma^n_{\bar r(t),I,J}} d^q(r_{k_0},\ldots,r_{k_q})\,dr_1\cdots dr_r.\end{aligned}$$

Using integration by parts, one gets [Azzali et al. 2015, Equation (4.17)],

$$K(t,n,I\cup\{i_p\},J;\ldots,2,\ldots,c_{k_0},\ldots;\ldots,a_{i_p},a_{i_{p+1}},\ldots) \tag{22}$$
$$= \begin{cases} K(t,n,I,J\cup\{i_p\};\ldots,0,\ldots,c_{k_0},\ldots;\ldots,a_{i_p},a_{i_{p+1}},\ldots) & \\ -K(t,n-1,I,J;\ldots,\ldots,c_{k_0},\ldots;\ldots,a_{i_0}+a_{i_{p+1}},\ldots) & q>0,\\ +K(t,n,I\cup\{i_p\},J\cup\{k_0\};\ldots,0,\ldots,c_{k_0},\ldots;\ldots,a_{i_p},a_{i_{p+1}},\ldots) & \\ +K(t,n,I\cup\{i_p\},J;\ldots,0,\ldots,c_{k_0}+2,\ldots;\ldots,a_{i_p},a_{i_{p+1}},\ldots) & \\ K(t,n,I,J\cup\{i_p\};\ldots,0,\ldots,c_{k_0},\ldots;\ldots,a_{i_p},a_{i_{p+1}},\ldots) & \\ -K(t,n-1,I,J;\ldots,\ldots,c_{k_0},\ldots;\ldots,a_{i_0}+a_{i_{p+1}},\ldots) & q=0.\\ +K(t,n,I\cup\{i_p\},J;\ldots,0,\ldots,c_{k_0}+2,\ldots;\ldots,a_{i_p},a_{i_{p+1}},\ldots) & \end{cases}$$

We remark that the proof of [Azzali et al. 2015, Equation (4.17)] does not involve any norm, and therefore we omit the details here.

Using equation (22) repeatedly, one eliminates all terms with $c_i = 2$.

On the other hand one has the following straightforward generalization of Lemma 4.2 (compare with [Azzali et al. 2015, Proposition 4.7]):

**Lemma 4.3.** *Suppose $c_i = 0, 1$. As $t \to \infty$,*

$$K(t,n,I,J,c_0,\dots c_n;a_1,\dots,a_n)(x,y,z) = \begin{cases} \left(\frac{1}{(n-|J|)!}\Pi_0\Omega^{a_1}\Pi_0\cdots\Pi_0\right)(x,y,z) + O(t^{-\gamma'}) & \text{if } I=\varnothing, c_0,\dots,c_n=0 \\ O(t^{-\gamma'}) & \text{otherwise,} \end{cases}$$

*for some $\gamma' > 0$, in the $\|\cdot\|_{\mathrm{HS}m}$-norm.*

Thus the term $K(t,n,I,c_0,\dots c_n;a_1,\dots a_n)$ converges to 0 unless

$$c_i = 0 \text{ whenever } i \in I, \quad c_i = 2 \text{ whenever } i \notin I.$$

Then one follows exactly as [Azzali et al. 2015, Section 4.5] to compute the limit, and concludes with the following analogue of their Theorem 4.1:

**Theorem 4.4.** *For $k = 0, 1, 2$ and any $m \in \mathbb{N}$,*

$$\lim_{t\to\infty} D(t)^k e^{-\eth(t)^2}(x,y,z) = \Pi_0(\Omega\Pi_0)^k e^{(\Omega\Pi_0)^2}(x,y,z)$$

*in the $\|\cdot\|_{\mathrm{HS}m}$-norm, where $\Omega := -\frac{1}{2}(\nabla^{E_\flat} - (\nabla^{E_\flat})^*)$. Moreover, there exits $\varepsilon' > 0$ such that as $t \to \infty$,*

$$\left\|\left(D(t)^k e^{-\eth(t)^2} - \Pi_0(\Omega\Pi_0)^k e^{(\Omega\Pi_0)^2}\right)(x,y,z)\right\|_{\mathrm{HS}m} = O(t^{-\varepsilon'}).$$

**4B. *Application: the $L^2$-analytic torsion form.*** Our main application of this theorem is in establishing the smoothness and transgression formula of the $L^2$-analytic torsion form. Here, we briefly recall the definitions.

On $\wedge^\bullet T^*M \otimes E \cong \wedge^\bullet H' \otimes \wedge^\bullet V' \otimes E$, define $N_\Omega$, $N$ to be the number operators of $\wedge^\bullet H' \cong \pi^{-1}(\wedge^\bullet T^*B)$ and $\wedge^\bullet V'$ respectively.

Define

$$F^\wedge(t) := (2\pi\sqrt{-1})^{-N_\Omega/2}\,\mathrm{str}_\Psi(2^{-1}N(1+2D(t)^2)e^{-\eth(t)^2}).$$

Then under the positivity of the Novikov–Shubin invariant, we have the following well-defined $L^2$-analytic torsion form:

**Definition 4.5** [Azzali et al. 2015].

$$\tau := \int_0^\infty \Big\{-F^\wedge(t) + \tfrac{1}{2}\,\mathrm{str}_\Psi(N\Pi_0) + \Big(\tfrac{1}{4}\dim(Z)\,\mathrm{rk}(E)\,\mathrm{str}_\Psi(\Pi_0) - \tfrac{1}{2}\,\mathrm{str}_\Psi(N\Pi_0)\Big)(1-2t)e^{-t}\Big\}\,\frac{dt}{t}.$$

In [Azzali et al. 2015], it is only shown that the form $\tau$ is continuous. Next we will show that indeed the form $\tau$ is smooth.

**Theorem 4.6.** *The form $\tau$ is smooth, i.e., $\tau \in \Gamma^\infty(\wedge^\bullet T^*B)$.*

*Proof.* Using [Berline et al. 1992, Proposition 9.24], the derivatives of the $t$-integrand are bounded as $t \to 0$. It follows that its integral over $[0, 1]$ is smooth.

We turn to studying the large time behavior. Consider $\mathrm{str}(2^{-1}N(e^{-\eth(t)^2} - \Pi_0))$. Using the semigroup property, we can write $e^{-\eth(t)^2} = 2^{-N_\Omega/2} e^{-\eth(t/2)^2} e^{-\eth(t/2)^2} 2^{N_\Omega/2}$. Also, since $\mathrm{str}(N\Pi_0(\Omega\Pi_0)^{2j}) = \mathrm{str}([N\Pi_0(\Omega\Pi_0), \Pi_0(\Omega\Pi_0)^{2j-1}]) = 0$ for any $j \geq 1$ one has

$$\mathrm{str}(N\Pi_0) = \mathrm{str}(N\Pi_0 e^{(\Omega\Pi_0)^2}) = 2^{-N_\Omega/2}\, \mathrm{str}(N\Pi_0 e^{(\Omega\Pi_0)^2} \Pi_0 e^{(\Omega\Pi_0)^2}).$$

Therefore

$$\begin{aligned}\mathrm{str}\big(2^{-1}N\big(e^{-\eth(t)^2} - \Pi_0\big)\big) &= 2^{-N_\Omega/2}\mathrm{str}\big(2^{-1}N\big(e^{-\eth(t/2)^2}e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\Pi_0 e^{(\Omega\Pi_0)^2}\big)\big) \\ &= 2^{-N_\Omega/2}\mathrm{str}\big(2^{-1}Ne^{-\eth(t/2)^2}\big(e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big)\big) \\ &\quad + 2^{-N_\Omega/2}\mathrm{str}\big(2^{-1}N\big(e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big)\Pi_0 e^{(\Omega\Pi_0)^2}\big).\end{aligned}$$

Now consider the $L^2(B)$-norm of $\mathrm{str}_\Psi\big(2^{-1}Ne^{-\eth(t/2)^2}\big(e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big)\big)$. To shorten notations, denote $G := 2^{-1}Ne^{-\eth(t/2)^2}\big(e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big)$. Writing $G$ as a convolution product, then there exists a constant $C_0 > 0$ such that

$$\begin{aligned}&\int_B \left| \int_{Z_x} \chi(x, z)\, \mathrm{str}(G(x, z, z))\mu_x(z) \right|^2 \mu_B(x) \\ &= \int_B \left| \int_{Z_x} \chi\, \mathrm{str}\left( \frac{N}{2} \int_{y \in Z_x} e^{-\eth(t/2)^2}(x, z, y)\big(e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big)(x, y, z)\mu_x(y) \right)\mu_x(z) \right|^2 \mu_B(x) \\ &\leq C_0 \int_B \left( \int_{Z_x} \chi \int_{y \in Z_x} \big|e^{-\eth(t/2)^2}\big|(x, z, y)\big|e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big|(x, y, z)\mu_x(y)\mu_x(z) \right)^2 \mu_B(x) \\ &\leq C_0 \big\|e^{-\eth(t/2)^2}\big\|^2_{\mathrm{HS}\,0} \big\|e^{-\eth(t/2)^2} - \Pi_0 e^{(\Omega\Pi_0)^2}\big\|^2_{\mathrm{HS}\,0},\end{aligned}$$

where we used the Cauchy–Schwarz inequality three times. Since $\|e^{-\eth(t/2)^2}\|_{\mathrm{HS}\,0}$ is bounded for $t$ large (by the triangle inequality), the expression above is $O(t^{-\gamma'})$.

We turn to estimating its derivatives. For any vector field $X$ on $B$,

$$\begin{aligned}\nabla^{TB}_X \mathrm{str}_\Psi(G) = \int (L_{X^H}\chi(x, z))\, \mathrm{str}(G(x, z, z))\mu_x(z) \\ + \int \chi(x, z)\big(L^{\nabla^{\pi^{-1}TB}}_{X^H} \mathrm{str}(G(x, z, z))\big)\mu_x(z) \\ + \int \chi(x, z)\, \mathrm{str}(G(x, z, z))(L_{X^H}\mu_x(z)).\end{aligned}$$

Differentiating under the integral sign is valid because we knew a priori that the integrands are all $L^1$. Since $L_{X^H}\mu_x(z)$ equals $\mu_x(z)$ multiplied by some bounded functions, it follows that the last term $\int \chi(x,z)\,\mathrm{str}(G(x,z,z))(L_{X^H}\mu_x(z))$ is $O(t^{-\gamma'})$.

For the first term, we write $L_{X^H}\chi(x,z)=\sum_{g\in G}(g^*\chi)(x,z)(L_{X^H}\chi)(x,z)$. The sum is finite because $L_{X^H}\chi$ is compactly supported. By $G$-invariance,

$$\int (g^*\chi)(x,z)\,\mathrm{str}(G(x,z,z))\mu_x(z)=\int \chi(x,z)\,\mathrm{str}(G(x,z,z))\mu_x(z).$$

Since $(L_{X^H}\chi)(x,z)$ is bounded, it follows that $\int (L_{X^H}\chi)(x,z)\,\mathrm{str}(G(x,z,z))\mu_x(z)$ is also $O(t^{-\gamma'})$.

As for the second term, we differentiate under the integral sign, then use the Leibniz rule to get that there exists a constant $C_1>0$ such that

$$\begin{aligned}
&\big|L_{X^H}^{\nabla^{\pi^{-1}TB}}\,\mathrm{str}(G(x,z,z))\big|\\
&\quad\le C_1\bigg(\int_{Z_x}\big|L_{X^H}^{\nabla^{\wedge^\bullet H'\otimes\wedge^\bullet V'\otimes\hat{E}}}e^{-\eth(t/2)^2}(x,z,y)\big|\big|e^{-\eth(t/2)^2}-\Pi_0e^{(\Omega\Pi_0)^2}(x,y,z)\big|\mu_x(y)\\
&\qquad+\int_{Z_x}\big|e^{-\eth(t/2)^2}(x,z,y)\big|\big|L_{X^H}^{\nabla^{\wedge^\bullet H'\otimes\wedge^\bullet V'\otimes\hat{E}}}(e^{-\eth(t/2)^2}-\Pi_0e^{(\Omega\Pi_0)^2})(x,y,z)\big|\mu_x(y)\\
&\qquad+\int_{Z_x}\big|e^{-\eth(t/2)^2}(x,z,y)\big|\big|e^{-\eth(t/2)^2}-\Pi_0e^{(\Omega\Pi_0)^2}(x,y,z)\big|\sup|L_{X^H}\mu|\mu_x(y)\bigg),
\end{aligned}$$

and

$$\begin{aligned}
&\int_B\bigg|\int_{Z_x}\chi(x,z)\big(L_{X^H}^{\nabla^{\pi^{-1}TB}}\,\mathrm{str}(G(x,z,z))\big)\mu_x(z)\bigg|^2\mu_B(x)\\
&\quad\le C_1\big(\|e^{-\eth(t/2)^2}\|^2_{\mathrm{HS}\,1}\|e^{-\eth(t/2)^2}-\Pi_0e^{(\Omega\Pi_0)^2}\|^2_{\mathrm{HS}\,0}+\|e^{-\eth(t/2)^2}\|^2_{\mathrm{HS}\,0}\|e^{-\eth(t/2)^2}\\
&\qquad-\Pi_0e^{(\Omega\Pi_0)^2}\|^2_{\mathrm{HS}\,1}+\sup|L_{X^H}\mu|\|e^{-\eth(t/2)^2}\|^2_{\mathrm{HS}\,1}\|e^{-\eth(t/2)^2}-\Pi_0e^{(\Omega\Pi_0)^2}\|^2_{\mathrm{HS}\,0}\big)\\
&\quad=O(t^{-\gamma'}).
\end{aligned}$$

Clearly the above arguments can be repeated and one concludes that all Sobolev norms of $\mathrm{str}_\Psi(G)$ are $O(t^{-\gamma'})$.

By exactly the same arguments, we have as $t\to\infty$,

$$\mathrm{str}_\Psi\big(2^{-1}N(e^{-\eth(t/2)^2}-\Pi_0e^{(\Omega\Pi_0)^2})\Pi_0e^{(\Omega\Pi_0)^2}\big)=O(t^{-\gamma'}),$$

in all Sobolev norms.

As for $\mathrm{str}_\Psi(2^{-1}N(D(t)^2e^{-\eth(t)^2}))$, one has $D(t)^2 = 2(2^{-N_\Omega/2}D\big(\frac{t}{2}\big)^2 2^{N_\Omega/2})$. Thus

$$\begin{aligned}
&\mathrm{str}_\Psi\big(\tfrac{1}{2}N\big(D(t)^2e^{-\eth(t)^2}\big)\big)\\
&\quad= 2^{-N_\Omega/2}\,\mathrm{str}_\Psi\big(N\big(D\big(\tfrac{1}{2}t\big)^2e^{-\eth(t/2)^2}e^{-\eth(t/2)^2} - \Pi_0(\Omega\Pi_0)^2e^{(\Omega\Pi_0)^2}e^{(\Omega\Pi_0)^2}\big)\big)\\
&\quad= 2^{-N_\Omega/2}\,\mathrm{str}_\Psi\big(N\big(D\big(\tfrac{1}{2}t\big)^2e^{-\eth(t/2)^2} - \Pi_0(\Omega\Pi_0)^2e^{(\Omega\Pi_0)^2}\big)e^{-\eth(t/2)^2}\big)\\
&\qquad\qquad - 2^{-N_\Omega/2}\,\mathrm{str}_\Psi\big(N\Pi_0(\Omega\Pi_0)^2e^{(\Omega\Pi_0)^2}(e^{-\eth(t/2)^2} - e^{(\Omega\Pi_0)^2})\big),
\end{aligned}$$

which is also $O(t^{-\gamma'})$ as $t \to \infty$ by similar arguments.

By the Sobolev embedding theorem (for the compact manifold $B$), it follows that

$$-F^\wedge(t) + \tfrac{1}{2}\,\mathrm{str}_\Psi(N\Pi_0) + \big(\tfrac{1}{4}\dim(Z)\,\mathrm{rk}(E)\,\mathrm{str}_\Psi(\Pi_0) - \tfrac{1}{2}\,\mathrm{str}_\Psi(N\Pi_0)\big)(1-2t)e^{-t}$$

and all its derivatives are $O(t^{-\gamma'})$ uniformly.

Finally, since all derivatives of the $t$-integrand in Definition 4.5 are $L^1$, derivatives of $\tau$ exist and equal differentiations under the $t$-integration sign. Hence we conclude that the torsion $\tau$ is smooth. □

**Remark 4.7.** If $Z$ is $L^2$-acyclic and of determinant class (see [Azzali et al. 2015, Def. 6.3]), the analogue of Remark 3.6 reads

$$\int_0^\infty \|e^{-t\Delta}\|^2_{\mathrm{HS}\,0}\,\frac{dt}{t} = \int_0^\infty \|e^{-t\Delta}\|_\tau\,\frac{dt}{t} < \infty$$

(note that $\Pi_0 = 0$ by hypothesis). Unlike having positive Novikov–Shubin invariant, the heat operator is not of determinant class in $\|\cdot\|_{\mathrm{HS}\,0}$.

Take a power series $f(x) = \sum a_j x^j$. For clarity, let $h$ be the metric on $\wedge^\bullet V \otimes E$ and denote

$$f\big(\nabla^{\wedge^\bullet V'\otimes E}, h\big) := \mathrm{str}\Big(\sum_j a_j\big(\tfrac{1}{2}\big(\nabla^{\wedge^\bullet V'\otimes E} - (\nabla^{\wedge^\bullet V'\otimes E})^*\big)\big)^j\Big) \in \Gamma^\infty(\wedge^\bullet T^*M),$$

$$f\big(\nabla^{\wedge^\bullet V'\otimes E}, h\big)_{H^\bullet(Z,E)} := \mathrm{str}_\Psi\Big(\sum_j a_j\big(\tfrac{1}{2}\Pi_0\big(\nabla^{\wedge^\bullet V'_\flat\otimes E_\flat} - (\nabla^{\wedge^\bullet V'_\flat\otimes E_\flat})^*\big)\Pi_0\big)^j\Big) \in \Gamma^\infty(\wedge^\bullet T^*B).$$

Note that the summations are only up to $\dim M$.

Let $TZ$ be the vertical tangent bundle of the fiber bundle $M \to B$ and recall that we have chosen a splitting of $TM$ and defined a Riemannian metric on $TM$. Let $P^{TZ}$ denote the projection from $TM$ to $TZ$. Let $\nabla^{TM}$ be the corresponding Levi-Civita connection on $TM$ and define $\nabla^{TZ} = P^{TZ}\nabla^{TM}P^{TZ}$, a connection on $TZ$. The restriction of $\nabla^{TZ}$ to a fiber coincides with the Levi-Civita connection of the fiber. Let $R^{TZ}$ be the curvature of $\nabla^{TZ}$.

For $N$ even, let $\mathrm{Pf} : \mathfrak{so}(N) \to \mathbb{R}$ denote the Pfaffian and put

$$e\left(TZ, \nabla^{TZ}\right) := \begin{cases} \mathrm{Pf}\left[\frac{R^{TZ}}{2\pi}\right] & \text{if } \dim(Z) \text{ is even,} \\ 0 & \text{if } \dim(Z) \text{ is odd.} \end{cases} \tag{23}$$

A classical argument [Bismut and Lott 1995; Ma and Zhang 2008; Azzali et al. 2015] then gives:

**Corollary 4.8.** *If* $\dim Z = 2n$ *is even one has the transgression formula*

$$d\tau(x) = \int_{Z_x} \chi(x, z) e(TZ, \nabla^{TZ}) f\left(\nabla^{\wedge^\bullet V' \otimes E}\right) - f\left(\nabla^{\wedge^\bullet V' \otimes E}\right)_{H^\bullet(Z,E)},$$

*with* $f(x) = xe^{x^2}$.

Now let $h_l$ be a family of $G$-invariant metrics on $\wedge^\bullet V \otimes E$, $l \in [0, 1]$. Define

$$\tilde{f}\left(\nabla^{\wedge^\bullet V' \otimes E}, h_l\right) := \int_0^1 (2\pi\sqrt{-1})^{N_\Omega/2} \operatorname{str}\left((h_l)^{-1} \frac{dh_l}{dl} f'\left(\nabla^{\wedge^\bullet V' \otimes E}, h_l\right)\right) dl,$$

and similarly for $\tilde{f}\left(\nabla^{\wedge^\bullet V' \otimes E}, h_l\right)_{H^\bullet(Z,E)}$. Note that $f'\left(\nabla^{\wedge^\bullet V' \otimes E}, h_l\right)$ uses the adjoint connection with respect to $h_l$.

Let $\hat{e}\left(TZ, \nabla^{TZ,0}, \nabla^{TZ,1}\right) \in Q^M / Q^{M,0}$ (see [Bismut and Lott 1995]) be the secondary class associated to the Euler class. Its representatives are forms of degree $\dim(Z) - 1$ such that

$$d\hat{e}\left(TZ, \nabla^{TZ,0}, \nabla^{TZ,1}\right) = e\left(TZ, \nabla^{TZ,1}\right) - e\left(TZ, \nabla^{TZ,0}\right). \tag{24}$$

If $\dim(Z)$ is odd, we take $\hat{e}\left(TZ, \nabla^{TZ,0}, \nabla^{TZ,1}\right)$ to be zero.

One has an anomaly formula [Bismut and Lott 1995, Theorem 3.24].

**Lemma 4.9.** *Modulo exact forms*

$$\begin{aligned} \tau_1 - \tau_0 = &\int_{Z_x} \chi(x, z) \hat{e}(TZ, \nabla^{TZ,0}, \nabla^{TZ,1}) f\left(\nabla^{\wedge^\bullet V' \otimes E}, h_0\right) \\ &+ \int_{Z_x} \chi(x, z) e(TZ, \nabla^{TZ,1}) \tilde{f}\left(\nabla^{\wedge^\bullet V' \otimes E}, h_l\right) - \tilde{f}\left(\nabla^{\wedge^\bullet V'_\flat \otimes E_\flat}, h_l\right)_{H^\bullet(Z,E)}. \end{aligned} \tag{25}$$

In particular, the degree 0 part of equation (25) is the anomaly formula for the $L^2$-Ray–Singer analytic torsion, which is a special case of [Zhang 2005, Theorem 3.4].

**Remark 4.10.** Let $Z_0 \to M_0 \to B$ be a fiber bundle with compact fiber $Z_0$, $Z \to M \to B$ be the normal covering of the fiber bundle $Z_0 \to M_0 \to B$. Then one can define the Bismut–Lott and $L^2$-analytic torsion form $\tau_{M_0 \to B}, \tau_{M_0 \to B} \in \Gamma^\infty(\wedge^\bullet T^* B)$, and one has the respective transgression formulas

$$d\tau_{M_0\to B} = \int_{\pi_0^{-1}(x)} e(TZ_0, \nabla^{TZ_0}) f\big(\nabla^{\wedge^\bullet V_0'\otimes E_0}\big) - f\big(\nabla^{\wedge^\bullet V_0'\otimes E_0}\big)_{H^\bullet(Z_0,E_0)},$$

$$d\tau_{M\to B} = \int_{Z_x} \chi(x,z) e(TZ, \nabla^{TZ}) f\big(\nabla^{\wedge^\bullet V'\otimes E}\big) - f\big(\nabla^{\wedge^\bullet V'\otimes E}\big)_{H^\bullet(Z,E)}.$$

Suppose further that the de Rham cohomologies are trivial:

$$H^\bullet(Z_0, E|_{Z_0}) = H^\bullet_{L^2}(Z, E|_Z) = \{0\}.$$

Then $d(\tau_{M\to B} - \tau_{M_0\to B}) = 0$. Hence $\tau_{M\to B} - \tau_{M_0\to B}$ defines some class in the de Rham cohomology of $B$. We also remark that this form was also mentioned in [Azzali et al. 2015, Remark 7.5], as a weakly closed form.

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BING KWAN SO
MATHEMATICS INSTITUTE
JILIN UNIVERSITY
2699 QIANJIN STREET
CHANGCHUN, 130012
CHINA
bkso@graduate.hku.hk

GUANGXIANG SU
CHERN INSTITUTE OF MATHEMATICS AND LPMC
NANKAI UNIVERSITY
TIANJIN, 300071
CHINA
guangxiangsu@gmail.com

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