Pacific Journal of Mathematics

THICK SUBCATEGORIES OVER ISOLATED SINGULARITIES

RYO TAKAHASHI

Volume 291 No. 1 November 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 291, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.291.183

THICK SUBCATEGORIES OVER ISOLATED SINGULARITIES

RYO TAKAHASHI

We study classifying thick subcategories of the category of finitely generated modules and its bounded derived category for a local ring with an isolated singularity.

1. Introduction

Let R be a commutative noetherian local ring. We denote by $\mathsf{mod}\, R$ the category of finitely generated R-modules, and by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of $\mathsf{mod}\, R$.

First, we consider classifying thick subcategories of the abelian category $\mathsf{mod}\, R$. In general, thick subcategories are much more than Serre subcategories; even when R is a hypersurface, the cardinality of thick subcategories of $\mathsf{mod}\, R$ containing R is equal to that of specialization-closed subsets of the singular locus [Takahashi 2010; 2013b], while the only Serre subcategory of $\mathsf{mod}\, R$ containing R is the whole category $\mathsf{mod}\, R$.

We prove the following structure theorem of thick closures:

Theorem 1.1. *Let R be a local ring with residue field k, and suppose that R has an isolated singularity. For each nonzero finitely generated R-module M one has*

$$\mathsf{thick}_{\mathsf{mod}\, R}\{k, M\} = \mathsf{thick}_{\mathsf{mod}\, R}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$$

of thick closures, provided that one of the following three conditions is satisfied.

(i) *M is locally free on the punctured spectrum of R.*

(ii) *R has (Krull) dimension at most two.*

(iii) *R has prime characteristic and M is (not necessarily maximal) Cohen–Macaulay.*

As a byproduct of the above theorem and its proof, we obtain the following result. Denote by $\operatorname{Nesc}(R)$ the set of nonempty specialization-closed subsets of $\operatorname{Spec} R$.

The author was partly supported by JSPS Grant-in-Aid for Scientific Research 25400038, 16K05098.
MSC2010: 13C60, 13D09, 18E30.
Keywords: Cohen–Macaulay ring/module, derived category, isolated singularity, module category, specialization-closed subset, thick subcategory.

Theorem 1.2. (1) *Let R be a local ring with residue field k. Suppose that R has an isolated singularity and dimension at most two. Then taking the supports gives a one-to-one correspondence between the set of thick subcategories of* $\operatorname{mod} R$ *containing k and* $\operatorname{Nesc}(R)$. *In particular, all the thick subcategories of* $\operatorname{mod} R$ *containing k are Serre.*

(2) *Let R be a regular local ring of positive characteristic. Then the thick closure in* $\operatorname{mod} R$ *of each nonzero R-module of finite length consists of all R-modules of finite length.*

Next, we consider classifying thick subcategories of the triangulated category $\mathsf{D}^{\mathsf{b}}(R)$. Stevenson [2014] completely classified the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ in the case where R is a complete intersection. Thus, our next goal is to classify the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ for a non-complete-intersection local ring R. However, this problem itself turns out to be quite hard, and it would be a reasonable approach to consider classifying the thick subcategories satisfying a certain condition which all the thick subcategories satisfy over complete intersections. The standard and costandard conditions are such ones; a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ is called *standard* (resp. *costandard*) if it contains a nonzero object of finite projective (resp. injective) dimension. Dwyer, Greenlees and Iyengar [Dwyer et al. 2006] showed that if R is a complete intersection, then every nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ is standard and costandard. We show the following classification theorem of standard and costandard thick subcategories:

Theorem 1.3. *Let R be a singular Cohen–Macaulay local ring with an isolated singularity. Assume that R is complete and has infinite residue field.*

(1) *If R is a hypersurface, then there is a one-to-one correspondence between the set of nonzero thick subcategories of* $\mathsf{D}^{\mathsf{b}}(R)$ *and the disjoint union of two copies of* $\operatorname{Nesc}(R)$.

(2) *If R has minimal multiplicity, then there is a one-to-one correspondence between the set of standard thick subcategories of* $\mathsf{D}^{\mathsf{b}}(R)$ *and the disjoint union of two copies of* $\operatorname{Nesc}(R)$.

(3) *If either R is non-Gorenstein and almost Gorenstein or R is of finite CM-representation type, then taking the supports gives a one-to-one correspondence between the set of standard and costandard thick subcategories of* $\mathsf{D}^{\mathsf{b}}(R)$ *and* $\operatorname{Nesc}(R)$.

In fact, the bijections in the first and second assertions are also explicitly described. The first assertion can also be deduced from [Stevenson 2014].

This paper is organized as follows. Section 2 is for preliminaries. The proof of Theorem 1.1 is divided into Sections 3, 4 and 5. In Section 6 we classify the thick subcategories of $\mathsf{D^b}(R)$ containing k. Applications of this, including Theorem 1.3, are given in Sections 7, 8 and 9.

2. Fundamental definitions

Throughout this paper, let R be a commutative noetherian ring. We assume that all modules are finitely generated, and that all subcategories are nonempty, full and closed under isomorphism. Denote by $\operatorname{mod} R$ the category of (finitely generated) R-modules, by $\mathsf{C^b}(R)$ the category of bounded complexes of (finitely generated) R-modules and by $\mathsf{D^b}(R)$ the bounded derived category of $\operatorname{mod} R$. Note that $\operatorname{mod} R$ and $\mathsf{C^b}(R)$ are abelian, while $\mathsf{D^b}(R)$ is triangulated.

Definition 2.1. (1) A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *Serre* if it is closed under submodules, quotient modules and extensions.

(2) A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ (resp. $\mathsf{C^b}(R)$, $\mathsf{D^b}(R)$) is called *thick* if it is closed under direct summands and satisfies the 2-out-of-3 property for short exact sequences of modules (resp. short exact sequences of complexes and closed under shifts, exact triangles).

(3) A subset S of $\operatorname{Spec} R$ is called *specialization-closed* if S contains $\mathrm{V}(\mathfrak{p})$ for all $\mathfrak{p} \in S$. Note that this is equivalent to saying that S is a union of closed subsets of $\operatorname{Spec} R$.

(4) (a) For each $M \in \operatorname{mod} R$ we denote by $\operatorname{Supp}_R M$ the set of prime ideals $\mathfrak{p}$ of R with $M_{\mathfrak{p}} \not\cong 0$ in $\operatorname{mod} R_{\mathfrak{p}}$, and call this the *support* of M in $\operatorname{mod} R$. This is a closed subset of $\operatorname{Spec} R$.

(b) The *support* of a subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is defined by $\operatorname{Supp}_R \mathcal{X} = \bigcup_{X\in\mathcal{X}} \operatorname{Supp}_R X$. This is a specialization-closed subset of $\operatorname{Spec} R$.

(c) For a subset S of $\operatorname{Spec} R$ we denote by $\operatorname{Supp}^{-1}_{\operatorname{mod} R} S$ the subcategory of $\operatorname{mod} R$ consisting of all modules whose supports are contained in S. This is a Serre subcategory of $\operatorname{mod} R$.

(d) The *support* of an object $X \in \mathsf{D^b}(R)$, denoted by $\operatorname{Supp}_R X$, is defined as the support of its homology $\mathrm{H}(X)$. Hence this is a closed subset.

(e) The *support* of a subcategory $\mathcal{X}$ of $\mathsf{D^b}(R)$ is defined by $\operatorname{Supp}_R \mathcal{X} = \bigcup_{X\in\mathcal{X}} \operatorname{Supp}_R X$. This is a specialization-closed subset of $\operatorname{Spec} R$.

(f) For a subset S of $\operatorname{Spec} R$ we denote by $\operatorname{Supp}^{-1}_{\mathsf{D^b}(R)} S$ the subcategory of $\mathsf{D^b}(R)$ consisting of objects whose supports are contained in S. This is a thick subcategory of $\mathsf{D^b}(R)$.

(5) A *perfect* complex is by definition (a complex quasi-isomorphic to) a bounded complex of finitely generated projective modules. We denote by $\mathsf{D_{perf}}(R)$ the subcategory of $\mathsf{D^b}(R)$ consisting of perfect complexes. This is a thick

subcategory of $\mathsf{D^b}(R)$, and hence a triangulated category. For each subset S of $\operatorname{Spec} R$ we set $\operatorname{Supp}^{-1}_{\mathsf{D_{perf}}(R)} S = (\operatorname{Supp}^{-1}_{\mathsf{D^b}(R)} S) \cap \mathsf{D_{perf}}(R)$.

(6) Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-module. Choose a minimal free resolution $F = (\cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to 0)$ of M. We define the *n-th syzygy* $\Omega^n_R M$ and the *transpose* $\operatorname{Tr}_R M$ of M by $\Omega^n_R M = \operatorname{Im}(\partial_n)$ and $\operatorname{Tr}_R M = \operatorname{Cok}(\partial_1^*)$, where we set $(-)^* = \operatorname{Hom}_R(-, R)$. One has $\Omega^n_R M \subseteq \mathfrak{m}F_{n-1}$ and $M^* \cong \Omega^2_R \operatorname{Tr}_R M \oplus R^{\oplus t}$ for some $t \geq 0$.

Since $\mathsf{C^b}(R)$ is abelian, we can define a complex over $\mathsf{C^b}(R)$. More precisely, a *complex* of objects of $\mathsf{C^b}(R)$ is a sequence $X = (\cdots \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \cdots)$ of morphisms $d_i : X_i \to X_{i-1}$ in $\mathsf{C^b}(R)$ with $d_i d_{i+1} = 0$. We introduce a Koszul complex on a complex of R-modules.

Definition 2.2. Let X be an object of $\mathsf{C^b}(R)$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R, and let $K = \mathrm{K}(\boldsymbol{x}, R) = (0 \to K_n \xrightarrow{\partial_n} K_{n-1} \to \cdots \to K_1 \xrightarrow{\partial_1} K_0 \to 0)$ be the Koszul complex of $\boldsymbol{x}$ on R. We define the *Koszul complex* $\mathrm{K}(\boldsymbol{x}, X)$ of $\boldsymbol{x}$ on X by

$$\mathrm{K}(\boldsymbol{x}, X) = (0 \to K_n \otimes_R X \xrightarrow{\partial_n \otimes_R X} K_{n-1} \otimes_R X \to \cdots \to K_1 \otimes_R X \xrightarrow{\partial_1 \otimes_R X} K_0 \otimes_R X \to 0),$$

where each $\partial_i \otimes_R X$ is a usual chain map, that is, a morphism in $\mathsf{C^b}(R)$. The Koszul complex $\mathrm{K}(\boldsymbol{x}, X)$ is a complex of objects of $\mathsf{C^b}(R)$.

Let $\mathcal{C}$ be one of the categories $\operatorname{mod} R$, $\mathsf{C^b}(R)$ and $\mathsf{D^b}(R)$. For a subcategory $\mathcal{M}$ of $\mathcal{C}$, the *thick closure* of $\mathcal{M}$ in $\mathcal{C}$, denoted by $\operatorname{thick}_{\mathcal{C}} \mathcal{M}$, is by definition the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{M}$. The proof of the following lemma is standard and omitted.

Lemma 2.3. (1) *Let $\mathcal{C}$ be one of the categories* $\operatorname{mod} R$, $\mathsf{C^b}(R)$ *and* $\mathsf{D^b}(R)$. *Let $\mathcal{X}$ be a thick subcategory of $\mathcal{C}$. Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a filtration in* $\operatorname{mod} R$. *If M_i/M_{i-1} is in $\mathcal{X}$ for each $1 \leq i \leq n$, then so is M. In particular, M is in* $\operatorname{thick}_{\mathcal{C}}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}$.

(2) *Let $\mathcal{X}$ be a thick subcategory of* $\operatorname{mod} R$, *and $X = (0 \to X_n \to \cdots \to X_1 \to X_0 \to 0)$ be a complex of R-modules in $\mathcal{X}$. If $\mathrm{H}_i(X) \in \mathcal{X}$ for all $1 \leq i \leq n$, then $\mathrm{H}_0(X) \in \mathcal{X}$.*

(3) *Let $\mathcal{X}$ be a thick subcategory of* $\mathsf{D^b}(R)$. *Let $C \in \mathsf{D^b}(R)$. If $\mathrm{H}(C)$ is in $\mathcal{X}$, then so is C.*

(4) *Let $X = (0 \to X^s \to X^{s+1} \to \cdots \to X^t \to 0)$ be a complex of R-modules. Then X belongs to* $\operatorname{thick}_{\mathsf{C^b}(R)}\{X^s, X^{s+1}, \dots, X^t\}$.

(5) *Let $\mathcal{X}$ be a thick subcategory of* $\mathsf{C^b}(R)$. *Let $X = (0 \to X_n \to \cdots \to X_0 \to 0)$ be a complex of objects of* $\mathsf{C^b}(R)$ *with $X_0, \dots, X_n \in \mathcal{X}$. If $\mathrm{H}_i(X) \in \mathcal{X}$ for all $1 \leq i \leq n$, then $\mathrm{H}_0(X) \in \mathcal{X}$.*

3. Modules locally free on the punctured spectrum

Let R be a local ring with residue field k. In this section, we study the structure of the thick closure of k and M in $\operatorname{mod} R$ when M is locally free on the punctured spectrum of R.

Lemma 3.1. *Let $\mathfrak{p}$ be a prime ideal of R.*

(1) *Suppose that $R_\mathfrak{p}$ is a regular local ring of dimension n. Then for each $0 \le i \le n$ there is an ideal $J = (x_1, \dots, x_i) \subseteq \mathfrak{p}$ with $\operatorname{ht} J = i$ such that $R_\mathfrak{p}/JR_\mathfrak{p}$ is a regular local ring of dimension $n - i$. In particular, there is an ideal $I = (x_1, \dots, x_n)$ of height n with $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$.*

(2) *Let I be an ideal of R with $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Then there exists an exact sequence $0 \to R/I \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/J \to 0$ of R-modules such that J strictly contains $\mathfrak{p}$.*

Proof. (1) We use induction on n. First of all, note that the assertion evidently holds for $i = 0$. When $n = 0$, we have $i = 0$, and we are done. Let $n \ge 1$. We may assume $1 \le i \le n$, so $0 \le i - 1 \le n - 1$. The induction hypothesis implies that there is an ideal $K = (x_1, \dots, x_{i-1}) \subseteq \mathfrak{p}$ with $\operatorname{ht} K = i - 1$ such that $R_\mathfrak{p}/KR_\mathfrak{p}$ is a regular local ring of dimension $n - i + 1$. Set $\bar{R} = R/K$ and $\bar{\mathfrak{p}} = \mathfrak{p}/K$. The local ring $\bar{R}_{\bar{\mathfrak{p}}}$ is regular and $\operatorname{ht} \bar{\mathfrak{p}} = \dim \bar{R}_{\bar{\mathfrak{p}}} = n - i + 1 > 0$. Nakayama's lemma shows that the symbolic power $\bar{\mathfrak{p}}^{(2)} = \bar{\mathfrak{p}}^2 \bar{R}_{\bar{\mathfrak{p}}} \cap \bar{R}_{\bar{\mathfrak{p}}}$ is strictly contained in $\bar{\mathfrak{p}}$. By prime avoidance we find an element $\overline{x_i} \in \bar{\mathfrak{p}}$ that is not contained in the union of ideals in $\operatorname{Min} \bar{R} \cup \{\bar{\mathfrak{p}}^{(2)}\}$. It is easy to see that the ideal $J := K + (x_i)$ has height i and $R_\mathfrak{p}/JR_\mathfrak{p} = \bar{R}_{\bar{\mathfrak{p}}}/\overline{x_i}\bar{R}_{\bar{\mathfrak{p}}}$ is a regular local ring of dimension $n - i$.

(2) Since $\mathfrak{p}$ is a minimal prime of I and $\mathfrak{p} = IR_\mathfrak{p} \cap R$, we see that $\mathfrak{p}$ is a $\mathfrak{p}$-primary component of I. Hence we can write $I = \mathfrak{p} \cap \mathfrak{q}$ for some ideal $\mathfrak{q}$ of R that is not contained in $\mathfrak{p}$ (when I is itself $\mathfrak{p}$-primary, we can take $\mathfrak{q} = R$). There is an exact sequence $0 \to R/I \to R/\mathfrak{p} \oplus R/\mathfrak{q} \to R/J \to 0$, where $J := \mathfrak{p} + \mathfrak{q}$ strictly contains $\mathfrak{p}$. □

The following result plays a key role in the proof of the main result of this section:

Lemma 3.2. *Suppose that R is locally Cohen–Macaulay on the punctured spectrum. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R generating an ideal of height n. Let M be an R-module locally free on the punctured spectrum of R. Then for each $i > 0$ the i-th Koszul homology $\mathrm{H}_i(\boldsymbol{x}, M)$ has finite length as an R-module.*

Proof. Pick any nonmaximal prime ideal $\mathfrak{p}$ of R. We want to show that $\mathrm{H}_i(\boldsymbol{x}, M)_\mathfrak{p}$ vanishes for all $i > 0$. This $R_\mathfrak{p}$-module is isomorphic to $\mathrm{H}_i(\boldsymbol{x}, M_\mathfrak{p})$, and $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module. Hence it suffices to show that $\mathrm{H}_i(\boldsymbol{x}, R_\mathfrak{p}) = 0$ for all $i > 0$. This holds true if $\mathfrak{p}$ does not contain $\boldsymbol{x}$, since $\boldsymbol{x}\,\mathrm{H}_i(\boldsymbol{x}, R_\mathfrak{p}) = 0$ for all $i \in \mathbb{Z}$. Let us consider the case where $\mathfrak{p}$ contains $\boldsymbol{x}$. We then have

$$n \geq \mathrm{ht}(\boldsymbol{x}R_{\mathfrak{p}}) = \inf\{\mathrm{ht}\, Q \mid Q \in \mathrm{V}(\boldsymbol{x}R_{\mathfrak{p}})\} = \inf\{\mathrm{ht}\, \mathfrak{q} \mid \mathfrak{q} \in \mathrm{V}(\boldsymbol{x}R),\ \mathfrak{q} \subseteq \mathfrak{p}\}$$
$$\geq \inf\{\mathrm{ht}\, \mathfrak{q} \mid \mathfrak{q} \in \mathrm{V}(\boldsymbol{x}R)\} = \mathrm{ht}(\boldsymbol{x}R) = n,$$

where the first inequality follows from Krull's height theorem. Hence the ideal $\boldsymbol{x}R_{\mathfrak{p}}$ generated by n elements has height n. Since $R_{\mathfrak{p}}$ is a Cohen–Macaulay local ring by assumption, $\boldsymbol{x}$ is an $R_{\mathfrak{p}}$-sequence. Therefore $\mathrm{H}_i(\boldsymbol{x}, R_{\mathfrak{p}}) = 0$ for all $i > 0$. □

Recall that a local ring R is said to have an *isolated singularity* if for every nonmaximal prime ideal $\mathfrak{p}$ of R the local ring $R_{\mathfrak{p}}$ is regular. The following is the main result of this section.

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring with an isolated singularity. Let M be a nonzero R-module which is locally free on the punctured spectrum of R. Then*

$$\mathsf{thick}_{\mathsf{mod}\, R}\{k, M\} = \mathsf{thick}_{\mathsf{mod}\, R}\{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R\, M\}.$$

Proof. As M is nonzero, the maximal ideal $\mathfrak{m}$ is in the support of M. Lemma 2.3(1) implies that the inclusion ($\subseteq$) holds. We show the opposite inclusion ($\supseteq$). Set $\mathcal{X} = \mathsf{thick}\{k, M\}$. The proof will be completed once we prove that $R/I \in \mathcal{X}$ for all ideals I of R with $\mathrm{V}(I) \subseteq \mathrm{Supp}\, M$. Suppose that this does not hold; we will be done if we derive a contradiction. The set of ideals

$$\{I \subseteq R \mid R/I \notin \mathcal{X},\ \mathrm{V}(I) \subseteq \mathrm{Supp}\, M\}$$

is nonempty, and this has a maximal element P with respect to the inclusion relation, as R is noetherian. We establish a claim.

Claim. One has $\mathfrak{m} \neq P \in \mathrm{Supp}\, M$. Every R-module L with $\mathrm{Supp}\, L \subseteq \mathrm{V}(P) - \{P\}$ is in $\mathcal{X}$.

Proof of Claim. Since P is in the above set of ideals, the module R/P is not in $\mathcal{X}$ and $\mathrm{V}(P)$ is contained in $\mathrm{Supp}\, M$. As k is in $\mathcal{X}$, we have $P \neq \mathfrak{m}$. It remains to show that P is a prime ideal. Take a filtration $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = R/P$ such that each N_i/N_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ in $\mathrm{Supp}_R(R/P) = \mathrm{V}(P)$. Assume that P is not a prime ideal. Then each $\mathfrak{p}_i$ strictly contains P, and the maximality of P implies $R/\mathfrak{p}_i \in \mathcal{X}$ for all $1 \leq i \leq n$. By Lemma 2.3(1) we have $R/P \in \mathcal{X}$. This contradiction shows that P is a prime ideal of R.

Take a filtration $0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_\ell = L$ such that for each i one has $L_i/L_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \mathrm{Supp}\, L \subseteq \mathrm{V}(P) - \{P\}$. The $\mathfrak{p}_i$ strictly contain P, and the maximality of P implies $R/\mathfrak{p}_i \in \mathcal{X}$, which forces L to be in $\mathcal{X}$ by Lemma 2.3(1). □

Since P is a nonmaximal prime ideal by the Claim and R is an isolated singularity, the localization R_P is a regular local ring. By Lemma 3.1 there is an exact sequence

$$0 \to R/(\boldsymbol{x}) \to R/P \oplus R/Q \to R/J \to 0,$$

where $\boldsymbol{x} = x_1, \ldots, x_n$ is a sequence of elements of R with $\mathrm{ht}(\boldsymbol{x}) = n$, and J strictly contains P. Applying the functor $- \otimes_R M$ to this gives rise to an exact sequence

$$\mathrm{Tor}_1^R(R/J, M) \xrightarrow{f} M/\boldsymbol{x}M \to M/PM \oplus M/QM \to M/JM \to 0.$$

The supports of M/JM and $\mathrm{Tor}_1^R(R/J, M)$ are contained in $\mathrm{V}(J)$, and so is the image C of the map f. As $\mathrm{V}(J)$ is contained in $\mathrm{V}(P) - \{P\}$, the Claim implies that M/JM and C are in $\mathcal{X}$.

Now, assume that M is locally free on the punctured spectrum. Then by Lemma 3.2 for each $i > 0$ the i-th Koszul homology $\mathrm{H}_i(\boldsymbol{x}, M)$ has finite length, and it is in $\mathcal{X}$. Each component of the Koszul complex $\mathrm{K}(\boldsymbol{x}, M)$ is a direct sum of copies of M, which is in $\mathcal{X}$. Lemma 2.3(2) implies $M/\boldsymbol{x}M = \mathrm{H}_0(\boldsymbol{x}, M) \in \mathcal{X}$. The induced exact sequence

$$0 \to C \to M/\boldsymbol{x}M \to M/PM \oplus M/QM \to M/JM \to 0$$

shows that M/PM is also in $\mathcal{X}$. As M/PM is a module over the domain R/P, it has a rank, say r. There is an exact sequence

$$0 \to (R/P)^{\oplus r} \to M/PM \to E \to 0$$

of R/P-modules with $\dim E < \dim R/P$. Since P is in the support of M by the Claim, Nakayama's lemma implies that it is also in the support of M/PM, and hence $r > 0$. It is easy to see that $\mathrm{Supp}_R E$ is contained in $\mathrm{V}(P) - \{P\}$, and the Claim implies $E \in \mathcal{X}$. As $M/PM \in \mathcal{X}$ and $r > 0$, the module R/P is in $\mathcal{X}$. This contradiction completes the proof of the theorem. □

Remark 3.4. We should remark that the equality in Theorem 3.3 is no longer true if we remove k from the left-hand side. The equality

$$\mathsf{thick}_{\mathsf{mod}\,R}\, M = \mathsf{thick}_{\mathsf{mod}\,R}\{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R M\}$$

holds for $M = R$ if and only if R is regular. This is one of the reasons why we consider thick subcategories containing k. See also Remark 6.4 stated later.

Applying Theorem 3.3 to $M = R$ and using Lemma 2.3(1), we obtain the following. This is a special case of [Schoutens 2003, Theorem VI.8] and [Krause and Stevenson 2013, Proposition 9], and includes [Takahashi 2010, Corollary 2.7].

Corollary 3.5. *If $(R, \mathfrak{m}, k)$ is an isolated singularity, then*

$$\mathsf{thick}_{\mathsf{mod}\,R}\{k, R\} = \mathsf{mod}\, R.$$

4. Rings of dimension at most two

In this section, we deal with the same problem as in the previous section for local rings with dimension at most 2.

Lemma 4.1. *Let* $(R, \mathfrak{m})$ *be local. Let* M, N *be* R*-modules, and* $\mathfrak{p}$ *a prime ideal. Assume* $M_\mathfrak{p} \cong N_\mathfrak{p}$ *and* $M_\mathfrak{q} = 0 = N_\mathfrak{q}$ *for all* $\mathfrak{q} \in \operatorname{Spec} R - \{\mathfrak{p}, \mathfrak{m}\}$. *Then*

$$\mathsf{thick}_{\mathsf{mod}\,R}\{k, M\} = \mathsf{thick}_{\mathsf{mod}\,R}\{k, N\}.$$

Proof. Since $M_\mathfrak{p}$ is isomorphic to $N_\mathfrak{p}$, there is an exact sequence

$$0 \to K \to M \to N \to C \to 0$$

such that $K_\mathfrak{p} = 0 = C_\mathfrak{p}$. We have $K_\mathfrak{q} = 0 = C_\mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Spec} R - \{\mathfrak{p}, \mathfrak{m}\}$, so K, C have finite length. Hence they are in both $\mathsf{thick}_{\mathsf{mod}\,R}\{k, M\}$ and $\mathsf{thick}_{\mathsf{mod}\,R}\{k, N\}$, and it is seen that $N \in \mathsf{thick}_{\mathsf{mod}\,R}\{k, M\}$ and $M \in \mathsf{thick}_{\mathsf{mod}\,R}\{k, N\}$. Thus the assertion follows. □

The next lemma is well known and also easy to prove, so we omit the proof.

Lemma 4.2. (1) *Let* $x \in R$ *be a nonzerodivisor, and let* $n > 0$ *be an integer. Then there exists a short exact sequence*

$$0 \to R/(x^n) \to R/(x^{n+1}) \oplus R/(x^{n-1}) \to R/(x^n) \to 0,$$

where $x^0 := 1$

(2) *Let* S *be a multiplicatively closed subset of* R. *Let* $\sigma : 0 \to M_S \to X \to N_S \to 0$ *be an exact sequence of* R_S*-modules. Then there exists an exact sequence* $\tau : 0 \to M \to Y \to N \to 0$ *of* R*-modules such that* $X \cong Y_S$.

For a module M over a local ring R we denote by $\operatorname{Assh} M$ the set of prime ideals $\mathfrak{p}$ in the support of M with $\dim R/\mathfrak{p} = \dim M$. The following is a similar type of result to Theorem 3.3.

Theorem 4.3. *Let* $(R, \mathfrak{m}, k)$ *be a local ring with an isolated singularity. Suppose that* R *has Krull dimension at most* 2. *Then for any nonzero* R*-module* M *one has the equality*

$$\mathsf{thick}_{\mathsf{mod}\,R}\{k, M\} = \mathsf{thick}_{\mathsf{mod}\,R}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}.$$

Proof. The inclusion ($\subseteq$) follows from Lemma 2.3(1) and the fact that $\mathfrak{m}$ supports M, so we prove the opposite inclusion ($\supseteq$). We may assume that R, M have positive (Krull) dimension.

(1) If $\dim R = 1$, then the assumption that R has an isolated singularity forces M to be locally free on the punctured spectrum, and Theorem 3.3 shows the assertion.

(2) If $\dim R = 2$, then M has dimension either 1 or 2. Taking the $\mathfrak{m}$-torsion submodule of M, we see that there is an exact sequence $0 \to L \to M \to N \to 0$ such that L has finite length and N is a nonzero module of positive depth. We have $\operatorname{Supp} M = \operatorname{Supp} N$ and $\mathsf{thick}\{k, M\} = \mathsf{thick}\{k, N\}$. Replacing M with N, we may assume that M has positive depth.

(a) Suppose $\dim M = 1$. Then M is a 1-dimensional Cohen–Macaulay module, and it follows from [Bruns and Herzog 1998, Theorem 2.1.2(a)] that one has $\operatorname{Ass} M = \operatorname{Min} M = \operatorname{Assh} M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, where each prime ideal $\mathfrak{p}_i$ is such that $\dim R/\mathfrak{p}_i = 1$.

Let $0 = M_1 \cap \cdots \cap M_n$ be an irredundant primary decomposition of the zero submodule 0 of M such that M_i is $\mathfrak{p}_i$-primary for $1 \le i \le n$. There are exact sequences

$$0 \to M/N_{i-1} \to M/M_i \oplus M/N_i \to M/M_i + N_i \to 0 \quad (1 \le i \le n-1),$$

where $N_i := M_{i+1} \cap \cdots \cap M_n$. Each $M/M_i + N_i$ has finite length, and we get $\mathsf{thick}\{k, M\} = \mathsf{thick}\{k, M/M_1, \dots, M/M_n\}$. For each $1 \le i \le n$ we have $\operatorname{Ass} M/M_i = \{\mathfrak{p}_i\}$, which especially says that M/M_i is a 1-dimensional Cohen–Macaulay R-module whose support contains $\mathfrak{p}_i$.

Fix a prime ideal $\mathfrak{p}$ in the support of M. We want to show that $R/\mathfrak{p}$ is in $\mathsf{thick}\{k, M\}$. For this, we may assume $\mathfrak{p} \ne \mathfrak{m}$, and then we have $\mathfrak{p} = \mathfrak{p}_\ell$ for some $1 \le \ell \le n$. Replacing M with M/M_ℓ, we may assume $\operatorname{Ass} M = \{\mathfrak{p}\}$ and $\dim R/\mathfrak{p} = 1$. Then $M_\mathfrak{p}$ is a nonzero $R_\mathfrak{p}$-module of finite length and $\operatorname{Supp} M = \{\mathfrak{p}, \mathfrak{m}\}$. As $R_\mathfrak{p}$ is either a field or a discrete valuation ring, the structure theorem of finitely generated modules over principal ideal domains implies that

$$M_\mathfrak{p} \cong (R_\mathfrak{p}/\mathfrak{p}^{a_1} R_\mathfrak{p})^{\oplus b_1} \oplus \cdots \oplus (R_\mathfrak{p}/\mathfrak{p}^{a_t} R_\mathfrak{p})^{\oplus b_t}$$

for some $t > 0$, $a_1 > \cdots > a_t > 0$ and $b_1, \dots, b_t > 0$. Setting

$$E = (R/\mathfrak{p}^{a_1})^{\oplus b_1} \oplus \cdots \oplus (R/\mathfrak{p}^{a_t})^{\oplus b_t},$$

we have $M_\mathfrak{p} \cong E_\mathfrak{p}$ and $\operatorname{Supp} E = \{\mathfrak{p}, \mathfrak{m}\}$. Lemma 4.1 implies $\mathsf{thick}\{k, M\} = \mathsf{thick}\{k, E\}$.

We claim $R/\mathfrak{p}^n \in \mathsf{thick}_{\mathsf{mod}\, R}\{k, R/\mathfrak{p}^{n+1}\}$ for all $n > 0$. In fact, there is an exact sequence of $R_\mathfrak{p}$-modules

$$0 \to R_\mathfrak{p}/\mathfrak{p}^{n+1} R_\mathfrak{p} \to (R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}) \oplus (R_\mathfrak{p}/\mathfrak{p}^{n+2} R_\mathfrak{p}) \to R_\mathfrak{p}/\mathfrak{p}^{n+1} R_\mathfrak{p} \to 0;$$

this is trivial when $R_\mathfrak{p}$ is a field, and follows from Lemma 4.2(1) when $R_\mathfrak{p}$ is a discrete valuation ring. Put $V = R/\mathfrak{p}^n \oplus R/\mathfrak{p}^{n+2}$. Lemma 4.2(2) yields an exact sequence $0 \to R/\mathfrak{p}^{n+1} \to W \to R/\mathfrak{p}^{n+1} \to 0$ such that $V_\mathfrak{p} \cong W_\mathfrak{p}$. As $\operatorname{Supp} V = \operatorname{Supp} W = \{\mathfrak{p}, \mathfrak{m}\}$, Lemma 4.1 implies $\mathsf{thick}\{k, V\} = \mathsf{thick}\{k, W\}$. The claim follows.

Using the claim repeatedly, we observe that $R/\mathfrak{p}$ belongs to $\mathsf{thick}\{k, R/\mathfrak{p}^n\}$ for all $n > 0$. Hence $R/\mathfrak{p}$ is in $\mathsf{thick}\{k, E\}$, and therefore it is in $\mathsf{thick}\{k, M\}$, as desired.

(b) Suppose $\dim M = 2$. Set $(-)^* = \operatorname{Hom}_R(-, R)$, and let $\lambda : M \to M^{**}$ be the natural homomorphism. Extend this to the exact sequence

$$0 \to K \to M \xrightarrow{\lambda} M^{**} \to C \to 0.$$

The module M^{**} is a second syzygy, and we have $K \cong \operatorname{Ext}_R^1(\operatorname{Tr} M, R)$ and $C \cong \operatorname{Ext}_R^2(\operatorname{Tr} M, R)$ by [Auslander and Bridger 1969, Proposition (2.6)]. As R is a 2-dimensional isolated singularity, M^{**} is locally free on the punctured spectrum, K has dimension at most 1 and C has finite length. The image E of λ is nonzero and locally free on the punctured spectrum. Applying Theorem 3.3 to E yields

$$\mathsf{thick}\{k, E\} = \mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} E\}. \tag{4.3.1}$$

The above exact sequence induces a short exact sequence $\sigma : 0 \to K \to M \to E \to 0$. Hence

$$\operatorname{Supp} M = \operatorname{Supp} K \cup \operatorname{Supp} E. \tag{4.3.2}$$

Since M has positive depth, K is a Cohen–Macaulay R-module of dimension 1. By part **(a)** on the previous page we get

$$\mathsf{thick}\{k, K\} = \mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} K\}. \tag{4.3.3}$$

As E is locally free on the punctured spectrum, the R-module $\operatorname{Ext}_R^1(E, K)$ has finite length, and hence the annihilator $\mathfrak{a} = \operatorname{Ann}_R \operatorname{Ext}_R^1(E, K)$ is $\mathfrak{m}$-primary. Thus one can choose a K-regular element x in $\mathfrak{a}$. The choice of x implies that the exact sequence $x\sigma$ splits, and we observe that there is an exact sequence

$$0 \to M \to K \oplus E \to K/xK \to 0.$$

As K/xK has finite length, K and E belong to $\mathsf{thick}\{k, M\}$. The exact sequence σ implies that M is in $\mathsf{thick}\{K, E\}$, and hence

$$\mathsf{thick}\{k, M\} = \mathsf{thick}\{k, K, E\}. \tag{4.3.4}$$

Combining (4.3.1)–(4.3.4) implies $\mathsf{thick}\{k, M\} = \mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$. $\square$

Corollary 4.4. *Let* $(R, \mathfrak{m}, k)$ *be a local ring with* $\dim R \le 2$ *and having an isolated singularity.*

(1) *If* $\mathcal{X}$ *is a thick subcategory of* $\mathsf{mod}\, R$ *containing* k, *then*

$$\operatorname{Supp}_R \mathcal{X} = \{\mathfrak{p} \in \operatorname{Spec} R \mid R/\mathfrak{p} \in \mathcal{X}\}.$$

(2) *If* $\varnothing \ne S \subseteq \operatorname{Spec} R$ *is specialization-closed*, *then*

$$\operatorname{Supp}^{-1}_{\mathsf{mod}\, R} S = \mathsf{thick}_{\mathsf{mod}\, R}\{R/\mathfrak{p} \mid \mathfrak{p} \in S\}.$$

Proof. (1) Let $\mathfrak{p}$ be a prime ideal. If X is a module in $\mathcal{X}$ whose support contains $\mathfrak{p}$, then $R/\mathfrak{p}$ is in the thick closure of k and X by Theorem 4.3, and hence $R/\mathfrak{p}$ is in $\mathcal{X}$. Conversely, if $R/\mathfrak{p}$ is in $\mathcal{X}$, then the support of $\mathcal{X}$ contains that of $R/\mathfrak{p}$, which contains $\mathfrak{p}$. Now the assertion follows.

(2) Let X be a module whose support is contained in S. Then the thick closure of $\{R/\mathfrak{p} \mid \mathfrak{p} \in S\}$ contains that of $\{k, R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} X\}$, which contains X by Theorem 4.3. The set S contains $\mathrm{V}(\mathfrak{p}) = \operatorname{Supp}_R(R/\mathfrak{p})$ for each $\mathfrak{p} \in S$. Thus the assertion is shown. □

The following is the main result of this section, whose essential part is included in Theorem 4.3. Compare it with the similar results [Takahashi 2013a, Theorems 5.6 and 6.11] and [Takahashi 2013b, Theorem 5.1(2)].

Theorem 4.5. *Let* $(R, \mathfrak{m}, k)$ *be a local ring with* $\dim R \leq 2$ *and having an isolated singularity.*

(1) *Every thick subcategory of* $\operatorname{mod} R$ *containing* k *is Serre.*

(2) *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories of } \operatorname{mod} R \\ \textit{containing } k\end{matrix}\right\} \underset{g}{\overset{f}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Specialization-closed subsets of } \operatorname{Spec} R \\ \textit{containing } \mathfrak{m}\end{matrix}\right\},$$

where f *and* g *are defined by* $f(\mathcal{X}) = \operatorname{Supp}_R \mathcal{X}$ *and* $g(S) = \operatorname{Supp}^{-1}_{\operatorname{mod} R} S$.

Proof. (1) Let $\mathcal{X}$ be a thick subcategory of $\operatorname{mod} R$ containing k. It suffices to show that $\mathcal{X} = \operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{X})$, because this equality especially says that $\mathcal{X}$ is a Serre subcategory. It is obvious that $\mathcal{X}$ is contained in $\operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{X})$. Let M be an R-module whose support is contained in that of $\mathcal{X}$. Take any prime ideal $\mathfrak{p}$ in the support of M. Then there exists an R-module $X \in \mathcal{X}$ whose support contains $\mathfrak{p}$. Theorem 4.3 implies that $R/\mathfrak{p}$ is in the thick closure of k and X, which is contained in $\mathcal{X}$. By Lemma 2.3(1) we see that M is in $\mathcal{X}$. Thus $\mathcal{X}$ contains $\operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{X})$, and the above equality follows.

(2) The assertion follows from (1) and Gabriel's classification [1962] of Serre subcategories. □

The assertion of Theorem 4.5 is no longer true for thick subcategories that do not contain k:

Example 4.6. Let R be a nonregular local ring with residue field k. Let $\mathcal{X}$ be the subcategory of $\operatorname{mod} R$ consisting of modules of finite projective dimension. Then $\mathcal{X}$ is a thick subcategory which does not contain k. There is an exact sequence $R \to k \to 0$, and we have $R \in \mathcal{X}$ and $k \notin \mathcal{X}$. This means that $\mathcal{X}$ is not a Serre subcategory of $\operatorname{mod} R$.

5. Rings of prime characteristic and Cohen–Macaulay modules

In this section, as in the prior two sections, we study the structure of $\operatorname{thick}_{\operatorname{mod} R}\{k, M\}$; we restrict ourselves to the case where R has prime characteristic and M is Cohen–Macaulay.

Let R be a ring of prime characteristic p, and let $q = p^e$ be a power of p. For a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of R we set $\boldsymbol{x}^q = x_1^q, \dots, x_n^q$. For an ideal I of R we denote by $I^{[q]}$ the ideal of R generated by the elements of the form a^q with $a \in I$. Note that if I is generated by a sequence $\boldsymbol{x}$ of elements of R, then $I^{[q]}$ is generated by the sequence $\boldsymbol{x}^q$. Note also that for each multiplicatively closed subset S of R one has $(IR_S)^{[q]} = I^{[q]}R_S$.

Lemma 5.1. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of characteristic $p > 0$. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a regular system of parameters of R. Let $q = p^e$ be a power of p. Let M be a nonzero R-module such that $\boldsymbol{x}^q M = 0$. Then there exists a nonzero free $R/(\boldsymbol{x}^q)$-module N possessing a filtration $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_t = N$ in $\mathsf{mod}\, R$ with $N_i/N_{i-1} \cong M$ for $1 \le i \le t$.*

Proof. We regard M as an $R/(\boldsymbol{x}^q)$-module. Cohen's structure theorem implies that the completion $\widehat{R}$ of R is isomorphic to $k[[x_1, \dots, x_d]]$. As $R/(\boldsymbol{x}^q)$ is artinian, it is complete. There are isomorphisms of k-algebras

$$\begin{aligned} R/(\boldsymbol{x}^q) &\cong \widehat{R/(\boldsymbol{x}^q)} \cong \widehat{R}/\boldsymbol{x}^q\widehat{R} \\ &\cong k[[x_1, \dots, x_d]]/(x_1^q, \dots, x_d^q) = k[x_1, \dots, x_d]/(x_1^q, \dots, x_d^q) \cong kG, \end{aligned}$$

where kG denotes the group algebra of the finite abelian p-group $G = (\mathbb{Z}/q\mathbb{Z})^{\oplus d}$; see [Iyengar 2004, (1.4)]. Hence one can identify $R/(\boldsymbol{x}^q)$ with kG. The tensor product $N := M \otimes_k kG$ is a kG-module via the diagonal action, and is projective; see Theorem (3.2) of the same work. Since kG is a (commutative) local ring, N is a nonzero finitely generated free kG-module. Tensoring over k the composition series of kG with M, we have a filtration of N as in the assertion. □

Denote by $\mathsf{fl}\, R$ the subcategory of $\mathsf{mod}\, R$ consisting of R-modules of finite length. Using the above lemma, we get a result on the structure of the thick closure of a finite length module.

Theorem 5.2. *Let R be a regular local ring of positive characteristic. Let M be a nonzero R-module of finite length. One then has $\mathsf{thick}_{\mathsf{mod}\, R}\, M = \mathsf{fl}\, R$.*

Proof. It is evident that the thick closure of M is contained in $\mathsf{fl}\, R$. As for the opposite inclusion relation, it is enough to show that the residue field k of R belongs to $\mathsf{thick}_{\mathsf{mod}\, R}\, M$. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a regular system of parameters of R. Let p be the characteristic of R, and let $q = p^e$ be a power of p such that $\boldsymbol{x}^q M = 0$. Lemma 5.1 shows that there exists a nonzero free $R/(\boldsymbol{x}^q)$-module N having a filtration $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_t = N$ in $\mathsf{mod}\, R$ with $N_i/N_{i-1} \cong M$ for $1 \le i \le t$. Note then that N is in the thick closure of M. We have only to show that k is in $\mathsf{thick}_{\mathsf{mod}\, R}\, R/(\boldsymbol{x}^q)$.

Let us do this by induction on $d = \dim R$. When $d = 0$, we have $R = k = R/(\boldsymbol{x}^q)$, and the statement trivially holds. Let $d > 0$, and put $\bar{R} = R/(x_1^q, \dots, x_{d-1}^q)$. There are exact sequences

$$0 \to \bar{R}/x_d^q\bar{R} \to \bar{R}/x_d^{q-1}\bar{R} \oplus \bar{R}/x_d^{q+1}\bar{R} \to \bar{R}/x_d^q\bar{R} \to 0,$$
$$0 \to \bar{R}/x_d^{q-1}\bar{R} \to \bar{R}/x_d^{q-2}\bar{R} \oplus \bar{R}/x_d^{q}\bar{R} \to \bar{R}/x_d^{q-1}\bar{R} \to 0,$$
$$\vdots$$
$$0 \to \bar{R}/x_d^2\bar{R} \to \bar{R}/x_d\bar{R} \oplus \bar{R}/x_d^{3}\bar{R} \to \bar{R}/x_d^2\bar{R} \to 0.$$

by Lemma 4.2(1). It can be seen from these exact sequences that $\bar{R}/x_d\bar{R}$ is in $\mathsf{thick}_{\mathsf{mod}\,R}\,\bar{R}/x_d^q\bar{R}$. Note $\bar{R}/x_d^q\bar{R} = R/(\boldsymbol{x}^q)$ and $\bar{R}/x_d\bar{R} = \widetilde{R}/(x_1^q,\dots,x_{d-1}^q)\widetilde{R}$, where $\widetilde{R} := R/(x_d)$. The induction hypothesis implies $k \in \mathsf{thick}_{\mathsf{mod}\,\widetilde{R}}\,\widetilde{R}/(x_1^q,\dots,x_{d-1}^q)\widetilde{R}$, and hence k is in $\mathsf{thick}_{\mathsf{mod}\,R}\,\widetilde{R}/(x_1^q,\dots,x_{d-1}^q)\widetilde{R}$. Consequently, we obtain $k \in \mathsf{thick}_{\mathsf{mod}\,R}\,R/(\boldsymbol{x}^q)$. □

Question 5.3. Does the assertion of Theorem 5.2 hold for any regular local ring R?

Remark 5.4. Using the Hopkins–Neeman theorem, one deduces that the derived category version of Theorem 5.2 holds: Let R be a regular ring. (We do not need to assume R is local or has prime characteristic.) Let $\mathsf{D}_{\mathsf{fl}}(R)$ stand for the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of complexes having finite length homology. Let M be a nonzero object of $\mathsf{D}_{\mathsf{fl}}(R)$. Then $\mathsf{D}^{\mathsf{b}}(R) = \mathsf{D}_{\mathsf{perf}}(R)$ and $\operatorname{Supp} M = \operatorname{Supp} \mathsf{D}_{\mathsf{fl}}(R) = \{\mathfrak{m}\}$, whence by [Neeman 1992, Theorem 1.5] we have $\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\,M = \mathsf{D}_{\mathsf{fl}}(R)$.

Let R be a local ring with residue field k. Recall that an R-module M is called *Cohen–Macaulay* if $\operatorname{Ext}_R^i(k, M) = 0$ for all $i < \dim M$ (i.e., $\operatorname{depth} M = \dim M$ or $M = 0$). Taking advantage of Lemma 5.1, we have the following similar theorem to Theorems 3.3 and 4.3:

Theorem 5.5. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p with an isolated singularity. Let $M \neq 0$ be a Cohen–Macaulay R-module. Then one has the equality*

$$\mathsf{thick}_{\mathsf{mod}\,R}\{k, M\} = \mathsf{thick}_{\mathsf{mod}\,R}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}.$$

Proof. Lemma 2.3(1) and the fact $\mathfrak{m} \in \operatorname{Supp} M$ guarantee that the right-hand side contains the left-hand side. Let us show the opposite inclusion relation by induction on $\dim M$. When $\dim M = 0$, the module M has finite length, and we are done. Let $\dim M \geq 1$. We will be done if we prove that R/I is in $\mathcal{X} := \mathsf{thick}_{\mathsf{mod}\,R}\{k, M\}$ for all ideals I with $\mathrm{V}(I) \subseteq \operatorname{Supp} M$. Suppose that this does not hold, and let P be a maximal element (with respect to the inclusion relation) among the ideals I with $\mathrm{V}(I) \subseteq \operatorname{Supp} M$ and $R/I \notin \mathcal{X}$. Similarly to the proof of Theorem 3.3, the ideal P is a nonmaximal prime ideal belonging to the support of M, the module R/P is not in $\mathcal{X}$, and every R-module whose support is contained in $\mathrm{V}(P) - \{P\}$ belongs to $\mathcal{X}$. Since M is Cohen–Macaulay, we have $\operatorname{Ass} M = \operatorname{Min} M = \operatorname{Assh} M$ by [Bruns and Herzog 1998, Theorem 2.1.2(a)]. Suppose that P is not an associated prime of M. Then we find an M-regular element x in P. The exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ shows that $\mathcal{X}$ contains $\mathsf{thick}_{\mathsf{mod}\, R}\{k, M/xM\}$. Note that M/xM is also a Cohen–Macaulay R-module whose support contains P. The induction hypothesis implies that R/P is in $\mathsf{thick}_{\mathsf{mod}\, R}\{k, M/xM\}$, and hence $\mathcal{X}$ contains R/P, which is a contradiction. Therefore P is an associated prime of M.

Let N be a P-primary component of the zero submodule 0 of M. Then $0 = N \cap L$ for some submodule L of M, which induces an exact sequence of R-modules $0 \to M \to M/N \oplus M/L \to M/N + L \to 0$. Observe that each prime ideal in the support of $M/N + L$ strictly contains P. Hence $\mathcal{X}$ contains $M/N + L$, and therefore $\mathcal{X}$ also contains M/N.

Since R_P is a regular local ring, by Lemma 3.1(1) one can choose a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements in P with $\operatorname{ht} P = n = \operatorname{ht}(\boldsymbol{x})$ and $PR_P = \boldsymbol{x}R_P$. Note the equality $\operatorname{Ass} M/N = \{P\}$ implies the R_P-module $(M/N)_P$ has finite length. Applying Lemma 5.1, we see that for large enough $q = p^e$ there is a free $R_P/P^{[q]}R_P$-module Z of rank $r > 0$ possessing a filtration $0 = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_t = Z$ in $\mathsf{mod}\, R_P$ with $Z_i/Z_{i-1} \cong (M/N)_P$ for $1 \le i \le t$. Using Lemma 4.2(2), one can inductively choose R-modules $W_0, \dots, W_t$ such that there exists a filtration $0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_t = W$ in $\mathsf{mod}\, R$ with $W_i/W_{i-1} \cong M/N$ for $1 \le i \le t$ and $W_P \cong Z \cong (R_P/P^{[q]}R_P)^{\oplus r}$. Then W is in the thick closure of M/N, and hence in $\mathcal{X}$. There is an exact sequence $0 \to K \to W \to (R/P^{[q]})^{\oplus r} \to C \to 0$ with $K_P = 0 = C_P$. Note that $\operatorname{Supp} W = \operatorname{Supp} M/N = \mathrm{V}(P) = \operatorname{Supp}(R/P^{[q]})^{\oplus r}$. Hence the supports of K, C are contained in $\mathrm{V}(P) - \{P\}$, which implies $\mathcal{X}$ contains K and C. Therefore the module $R/P^{[q]}$ is in $\mathcal{X}$.

There is an exact sequence

$$0 \to R_P/\boldsymbol{x}^q R_P \to (R_P/\mathfrak{a}_1 R_P) \oplus (R_P/\mathfrak{a}_2 R_P) \to R_P/\boldsymbol{x}^q R_P \to 0$$

by Lemma 4.2(1), where $\mathfrak{a}_1 = (x_1^q, \dots, x_{n-1}^q, x_n^{q-1})R$ and $\mathfrak{a}_2 = (x_1^q, \dots, x_{n-1}^q, x_n^{q+1})R$. Put $\mathfrak{b}_i = \mathfrak{a}_i R_P \cap R$ for $i = 1, 2$. Since P is a minimal prime of $\mathfrak{a}_i$, the ideal $\mathfrak{b}_i$ is the P-primary component of $\mathfrak{a}_i$. Note that $\mathfrak{a}_i R_P = \mathfrak{b}_i R_P$, and $\mathrm{V}(\mathfrak{b}_i) = \mathrm{V}(\sqrt{\mathfrak{b}_i}) = \mathrm{V}(P)$ for $i = 1, 2$. Setting $E = R/\mathfrak{b}_1 \oplus R/\mathfrak{b}_2$, we see from Lemma 4.2(2) that there is an exact sequence $0 \to R/P^{[q]} \to U \to R/P^{[q]} \to 0$ such that $U_P \cong E_P$. We have $\operatorname{Supp} E = \mathrm{V}(\mathfrak{b}_1) \cup \mathrm{V}(\mathfrak{b}_2) = \mathrm{V}(P) = \operatorname{Supp} U$. Choosing an exact sequence $0 \to K' \to U \to E \to C' \to 0$ with $K'_P = 0 = C'_P$, we see that the supports of K' and C' are contained in $\mathrm{V}(P) - \{P\}$, whence they are in $\mathcal{X}$. As U is in $\mathcal{X}$, so is E, and so is $R/\mathfrak{b}_1$.

Since $(R/\mathfrak{b}_1)_P = R_P/(x_1^q, \dots, x_{n-1}^q, x_n^{q-1})R_P$, the same argument as above shows that $R/\mathfrak{c}$ belongs to $\mathcal{X}$ with $\mathfrak{c} = (x_1^q, \dots, x_{n-1}^q, x_n^{q-2})R_P \cap R$ if $q > 2$. Iterating this procedure yields that $R/(x_1, \dots, x_n)R_P \cap R$ belongs to $\mathcal{X}$. (Here we use the fact that any permutation of a regular sequence on a local ring is again regular.) Since $(x_1, \dots, x_n)R_P \cap R = PR_P \cap R = P$, this means that R/P is in $\mathcal{X}$, which is a contradiction. This completes the proof of the theorem. □

6. Thick subcategories of derived categories containing the residue field

From this section to the end of this paper, we deal with thick subcategories of derived categories. In this section, we prove a classification theorem of thick subcategories containing the residue field over an isolated singularity.

We begin with a well-known statement. In view of this, it is reasonable to think of classifying, for a general local ring R, the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing the residue field.

Remark 6.1. Let R be a local ring with residue field k. The following are equivalent.

(1) The ring R is regular.

(2) Every nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ contains k.

(3) For each nonzero object X of $\mathsf{D}^{\mathsf{b}}(R)$, the thick closure of X contains k.

The following lemma helps us make the derived category version of Theorem 3.3:

Lemma 6.2. *Let R be an isolated singularity. Let X be a bounded complex of R-modules. Then X is quasi-isomorphic to a complex*

$$Y = (0 \to Y^s \to Y^{s+1} \to \cdots \to Y^t \to 0)$$

with $s \le t$ such that Y^i is free for all $s+1 \le i \le t$ and Y^s is locally free on the punctured spectrum of R.

Proof. Take a free resolution $F = (\cdots \xrightarrow{\delta^{t-2}} F^{t-1} \xrightarrow{\delta^{t-1}} F^t \to 0)$ of X. Choose an integer u such that $\mathrm{H}^i(F) = 0$ for all $i < u$, and put $d = \dim R$. Then $C := \mathrm{Cok}\, \delta^{u-d-1}$ is a d-th syzygy of $\mathrm{Cok}\, \delta^{u-1}$, which is locally free on the punctured spectrum. The complex X is quasi-isomorphic to the complex $(0 \to C \to F^{u-d+1} \to \cdots \to F^t \to 0)$. □

Now we can prove the following theorem analogous to Theorems 3.3, 4.3 and 5.5.

Theorem 6.3. *Let $(R, \mathfrak{m}, k)$ be a local ring with an isolated singularity. Let X be a nonacyclic bounded complex of R-modules. Then one has*

$$\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, X\} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R X\}.$$

Proof. The inclusion (⊆) follows from Lemma 2.3(1)(3) and the fact $\mathfrak{m} \in \mathrm{Supp}\, X$. Let us show the opposite inclusion (⊇). By Lemma 6.2 we may assume that X has the form $X = (0 \to X^s \to X^{s+1} \to \cdots \to X^t \to 0)$ such that the R-module X^i is free for all $s+1 \le i \le t$ and X^s is locally free on the punctured spectrum of R. Set $\mathcal{X} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, X\}$. It suffices to prove $R/I \in \mathcal{X}$ for all ideals I of R with $\mathrm{V}(I) \subseteq \operatorname{Supp} X$. Similarly to the proof of Theorem 3.3, we show this by contradiction. Assume that this does not hold, and choose a maximal element P of the set of ideals I with $R/I \notin \mathcal{X}$ and $\mathrm{V}(I) \subseteq \operatorname{Supp} X$. We then have:

Claim. (1) One has $\mathfrak{m} \neq P \in \operatorname{Supp} X$ and $R/P \notin \mathcal{X}$.

(2) Let C be an object of $\mathsf{D}^{\mathsf{b}}(R)$. If $\operatorname{Supp}_R C$ is contained in $\mathrm{V}(P) - \{P\}$, then C is in $\mathcal{X}$.

Proof of Claim. (1) This is similarly shown to the Claim in the proof of Theorem 3.3.

(2) Take any $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}(C) = \operatorname{Supp}_R C$. Then $\mathfrak{p}$ strictly contains P, and the maximality of P implies $R/\mathfrak{p} \in \mathcal{X}$. By Lemma 2.3(1), $\mathrm{H}(C)$ is in $\mathcal{X}$. Lemma 2.3(3) shows $C \in \mathcal{X}$. □

Since R is an isolated singularity, R_P is regular. By Lemma 3.1, there exists an exact sequence

$$0 \to R/(\boldsymbol{x}) \to R/P \oplus R/Q \to R/I \to 0, \tag{6.3.1}$$

where $\boldsymbol{x} = x_1, \ldots, x_n$ is a sequence in R generating an ideal of height n, and I is an ideal strictly containing P. Let

$$K = \mathrm{K}(\boldsymbol{x}, X) = (0 \to X^{\oplus\binom{n}{n}} \to X^{\oplus\binom{n}{n-1}} \to \cdots \to X^{\oplus\binom{n}{1}} \to X^{\oplus\binom{n}{0}} \to 0)$$

be the Koszul complex of $\boldsymbol{x}$ on X, which is a complex of objects of the abelian category $\mathsf{C}^{\mathsf{b}}(R)$. Put $\mathcal{Y} = \mathsf{thick}_{\mathsf{C}^{\mathsf{b}}(R)}\{k, X\}$. For each integer i the i-th homology $\mathrm{H}_i(K)$ of K is the complex $(0 \to \mathrm{H}_i(\boldsymbol{x}, X^s) \to \mathrm{H}_i(\boldsymbol{x}, X^{s+1}) \to \cdots \to \mathrm{H}_i(\boldsymbol{x}, X^t) \to 0)$ of R-modules, where $\mathrm{H}_i(\boldsymbol{x}, X^j)$ stands for the (usual) i-th Koszul homology of $\boldsymbol{x}$ on the R-module X^j. Lemma 3.2 implies that $\mathrm{H}_i(\boldsymbol{x}, X^j)$ has finite length for each $i > 0$ and $s \le j \le t$. By Lemma 2.3(4) we observe that $\mathrm{H}_i(K)$ belongs to $\mathcal{Y}$ for every $i > 0$, and by Lemma 2.3(5) the complex $\mathrm{H}_0(K) = R/(\boldsymbol{x}) \otimes_R X$ also belongs to $\mathcal{Y}$. Consequently, the complex $R/(\boldsymbol{x}) \otimes_R X$, as an object of $\mathsf{D}^{\mathsf{b}}(R)$, is in $\mathcal{X}$.

The short exact sequence (6.3.1) induces an exact sequence

$$\operatorname{Tor}_1^R(R/I, X) \xrightarrow{f} R/(\boldsymbol{x}) \otimes_R X \to (R/P \otimes_R X) \oplus (R/Q \otimes_R X) \to R/I \otimes_R X \to 0$$

in the abelian category $\mathsf{C}^{\mathsf{b}}(R)$, where $\operatorname{Tor}_1^R(R/I, X)$ stands for the induced complex $(0 \to \operatorname{Tor}_1^R(R/I, X^s) \to \operatorname{Tor}_1^R(R/I, X^{s+1}) \to \cdots \to \operatorname{Tor}_1^R(R/I, X^t) \to 0)$ of R-modules. Let Z be the image of the morphism f. Note that each component Z^i of the complex Z is a homomorphic image of the R-module $\operatorname{Tor}_1^R(R/I, X^i)$. Hence one has $\operatorname{Supp} Z^i \subseteq \mathrm{V}(I) \subseteq \mathrm{V}(P) - \{P\}$ for each i, and therefore $\operatorname{Supp} Z = \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} \mathrm{H}^i(Z) \subseteq \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} Z^i \subseteq \mathrm{V}(P) - \{P\}$. It follows from the Claim that Z, as an object of $\mathsf{D}^{\mathsf{b}}(R)$, belongs to $\mathcal{X}$. Similarly, it is seen that $R/I \otimes_R X \in \mathcal{X}$. The induced exact sequence

$$0 \to Z \to R/(\boldsymbol{x}) \otimes_R X \to (R/P \otimes_R X) \oplus (R/Q \otimes_R X) \to R/I \otimes_R X \to 0$$

shows that the subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ contains the complex

$$R/P \otimes_R X = (0 \to X^s/PX^s \to X^{s+1}/PX^{s+1} \to \cdots \to X^t/PX^t \to 0).$$

Since X^s/PX^s is a finitely generated module over the integral domain R/P, it has a rank, say r. There is an exact sequence $0 \to (R/P)^{\oplus r} \to X^s/PX^s \to C \to 0$ of R/P-modules such that $\dim C < \dim R/P$. We obtain a short exact sequence $0 \to W \to R/P \otimes_R X \to C[-s] \to 0$ in $\mathsf{C}^{\mathsf{b}}(R)$, where

$$W = (0 \to (R/P)^{\oplus r} \to X^{s+1}/PX^{s+1} \to \cdots \to X^t/PX^t \to 0).$$

As $PC = 0 = C_P$, the set $\operatorname{Supp}_R(C[-s]) = \operatorname{Supp}_R C$ is contained in $\mathrm{V}(P) - \{P\}$. The Claim yields that $C[-s]$ is in $\mathcal{X}$, and the above short exact sequence shows that W is in $\mathcal{X}$.

Note that W is a perfect complex of R/P-modules, and hence as an object of $\mathsf{D}^{\mathsf{b}}(R/P)$ it belongs to $\mathsf{D}_{\mathsf{perf}}(R/P)$. Since $C[-s]_P = 0$, we have isomorphisms $W_P = (R/P \otimes_R X)_P \cong \kappa(P) \otimes_{R_P} X_P \cong \kappa(P) \otimes^{\boldsymbol{L}}_{R_P} X_P$ in $\mathsf{D}^{\mathsf{b}}(R_P)$, where the last isomorphism follows from the fact that X_P is a perfect complex of R_P-modules. As P is in $\operatorname{Supp}_R X$, the complex W_P is not acyclic. This means that $\operatorname{Supp}_{R/P} W$ contains the zero ideal of R/P, and we obtain $\operatorname{Supp}_{R/P} W = \operatorname{Spec} R/P = \operatorname{Supp}_{R/P}(R/P)$. By [Neeman 1992, Theorem 1.5], R/P is in $\mathsf{thick}_{\mathsf{D}_{\mathsf{perf}}(R/P)} W = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R/P)} W$, and therefore it belongs to $\mathcal{X}$. This contradiction completes the proof. □

Remark 6.4. Similarly to Remark 3.4, the equality in Theorem 6.3 is no longer true if we remove k from the left-hand side; the equality

$$\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)} X = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}$$

holds for $X = R$ if and only if $\mathsf{D}_{\mathsf{perf}}(R) = \mathsf{D}^{\mathsf{b}}(R)$, if and only if R is regular. This is one of the reasons why we consider thick subcategories containing k.

Using Theorem 6.3, we get a derived category version of Corollary 4.4:

Corollary 6.5. *Let $(R, \mathfrak{m}, k)$ be a local ring with an isolated singularity.*

(1) *If $\mathcal{X}$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, then*

$$\operatorname{Supp}_R \mathcal{X} = \{\mathfrak{p} \in \operatorname{Spec} R \mid R/\mathfrak{p} \in \mathcal{X}\}.$$

(2) *If $S \neq \varnothing$ is a specialization-closed subset of* $\operatorname{Spec} R$, *then*

$$\operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S = \mathsf{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in S\}.$$

The following is the main theorem of this section:

Theorem 6.6. *Let $(R, \mathfrak{m}, k)$ be a local ring with an isolated singularity. The assignments $f : \mathcal{X} \mapsto \operatorname{Supp}_R \mathcal{X}$ and $g : S \mapsto \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ make mutually inverse bijections*

$$\left\{ \begin{matrix} \textit{Thick subcategories of } \mathsf{D}^{\mathsf{b}}(R) \\ \textit{containing } k \end{matrix} \right\} \underset{g}{\overset{f}{\underset{\leftarrow}{\overset{\rightarrow}{\scriptstyle 1-1}}}} \left\{ \begin{matrix} \textit{Specialization-closed subsets of } \operatorname{Spec} R \\ \textit{containing } \mathfrak{m} \end{matrix} \right\}.$$

Proof. Let $\mathcal{X}$ be a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, and let S be a specialization-closed subset of $\operatorname{Spec} R$ containing $\mathfrak{m}$.

(1) The set $\operatorname{Supp} \mathcal{X}$ is specialization-closed. Since the residue field k is in $\mathcal{X}$, the maximal ideal $\mathfrak{m}$ is in the support of $\mathcal{X}$. Hence f is a well-defined map.

(2) The subcategory $\operatorname{Supp}^{-1} S$ is thick. The support of k is contained in $\{\mathfrak{m}\}$, which is contained in S. Hence k is in $\operatorname{Supp}^{-1} S$, and g is a well-defined map.

(3) It is obvious that $\operatorname{Supp} \operatorname{Supp}^{-1} S$ is contained in S. Let $\mathfrak{p}$ be a prime ideal in S. We have $\operatorname{Supp}_R R/\mathfrak{p} = \mathrm{V}(\mathfrak{p})$, which is contained in S as S is specialization-closed. Hence $\mathfrak{p}$ is in $\operatorname{Supp}_R R/\mathfrak{p}$ and $R/\mathfrak{p}$ is in $\operatorname{Supp}^{-1} S$. Thus we obtain $S = \operatorname{Supp} \operatorname{Supp}^{-1} S$.

(4) Clearly, the subcategory $\operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X}$ contains $\mathcal{X}$. Let C be an object of $\mathsf{D}^{\mathsf{b}}(R)$ whose support is contained in that of $\mathcal{X}$. Take a prime ideal $\mathfrak{p} \in \operatorname{Supp} C$. Then $\mathfrak{p}$ is in the support of X for some $X \in \mathcal{X}$. Theorem 6.3 implies that $R/\mathfrak{p}$ belongs to the thick closure of k and X, which is contained in $\mathcal{X}$. Thus $R/\mathfrak{p}$ is in $\mathcal{X}$ for all prime ideals $\mathfrak{p}$ in the support of C. Using Theorem 6.3 again, we observe that C belongs to $\mathcal{X}$. Consequently, we obtain $\mathcal{X} = \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X}$.

Getting the above **(1)**–**(4)** together completes the proof of the theorem. □

Remark 6.7. An anonymous referee has pointed out that Theorem 6.6 can also be shown as follows: Let $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ be the punctured spectrum of R. The assumption that R has an isolated singularity implies that U is a regular scheme. On one hand, by [Thomason 1997, Theorem 3.15] the thick subcategories of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} U)$ correspond to the specialization-closed subsets of U, which are the same as the specialization-closed subsets of $\operatorname{Spec} R$ containing $\mathfrak{m}$. On the other hand, since $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} U)$ is equivalent to $\mathsf{D}^{\mathsf{b}}(R)/\operatorname{thick} k$ by [Orlov 2011, Lemma 2.2], the thick subcategories of $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} U)$ correspond to the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing k.

This is a simpler proof, using techniques in algebraic geometry. Our methods are purely ring-theoretic, and also essentially the same as those in the proof of Theorem 3.3, for which the approach the referee mentions does not seem to work. It is thus worth giving our methods.

Unless R has an isolated singularity, Theorem 6.6 does not necessarily hold:

Remark 6.8. Let $(R, \mathfrak{m}, k)$ be a local ring, and suppose that R does not have an isolated singularity. Set $\mathcal{X} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, R\}$. Then $\mathcal{X}$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, but $\mathcal{X} \neq \mathrm{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ for all subsets S of $\operatorname{Spec} R$.

One has a classification theorem of thick subcategories without using prime ideals:

Corollary 6.9. *If R is a local ring with an isolated singularity, one has a* 1-1 *correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories of } \mathsf{D}^{\mathsf{b}}(R) \\ \textit{containing } k\end{matrix}\right\} \underset{\psi}{\overset{\phi}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Nonzero thick subcategories} \\ \textit{of } \mathsf{D}_{\mathsf{perf}}(R)\end{matrix}\right\},$$

where ϕ, ψ are defined by $\phi(\mathcal{X}) = \mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R)$ and $\psi(\mathcal{Y}) = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\})$ for subcategories $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ and $\mathcal{Y}$ of $\mathsf{D}_{\mathsf{perf}}(R)$.

Proof. Let S be a specialization-closed subset of $\operatorname{Spec} R$ containing $\mathfrak{m}$. Take a system of generators $\boldsymbol{x}$ of $\mathfrak{m}$. Then $\mathrm{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} S$ contains the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$, and hence it is a nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. Conversely, for any nonzero thick subcategory $\mathcal{Y}$ of $\mathsf{D}_{\mathsf{perf}}(R)$, the support $\mathrm{Supp}_R \mathcal{Y}$ contains $\mathfrak{m}$. Thus, [Neeman 1992, Theorem 1.5] implies that Supp_R and $\mathrm{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)}$ make mutually inverse bijections between the nonzero thick subcategories of $\mathsf{D}_{\mathsf{perf}}(R)$ and the specialization-closed subsets of $\operatorname{Spec} R$ containing $\mathfrak{m}$.

Let $\mathcal{X}$ be a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, and let $\mathcal{Y}$ be a nonzero thick subcategory of $\mathsf{D}_{\mathsf{perf}}(R)$. Combining our Theorem 6.6 with the above one-to-one correspondence, one has only to verify the equalities

(1) $\mathrm{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} \mathrm{Supp}\, \mathcal{X} = \mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R)$,

(2) $\mathrm{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} \mathrm{Supp}\, \mathcal{Y} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\})$.

We have $\mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R) \subseteq \mathrm{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)}(\mathrm{Supp}\, \mathcal{X}) \subseteq \mathrm{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathrm{Supp}\, \mathcal{X}) = \mathcal{X}$, where the last equality follows from Theorem 6.6. This shows (1). On the other hand, it holds that $\mathrm{Supp}\, \mathcal{Y} = \mathrm{Supp}(\mathcal{Y} \cup \{k\}) = \mathrm{Supp}(\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\}))$, where the second equality follows from the fact that $\mathcal{Y}$ is nonzero. Applying $\mathrm{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)}$ and using Theorem 6.6, we obtain (2). □

7. Hypersurfaces and Cohen–Macaulay rings with minimal multiplicity

In this section, using the classification obtained in the previous section, we explore thick subcategories over hypersurfaces and Cohen–Macaulay rings with minimal multiplicity.

Definition 7.1. (1) A local ring R is called a *hypersurface* if the completion of R is isomorphic to a quotient of a regular local ring by a nonzero element.

(2) Let R be a Cohen–Macaulay local ring. Then R satisfies the inequality

$$\mathrm{e}(R) \geq \operatorname{edim} R - \dim R + 1, \tag{7.1.1}$$

where $\mathrm{e}(R)$ and $\operatorname{edim} R$ denote the multiplicity of R and the embedding dimension of R, respectively. We say that R has *minimal multiplicity* (or *maximal embedding dimension*) if the equality of (7.1.1) holds.

(3) Let A_1, A_2 be sets whose intersection is possibly nonempty. The *disjoint union* of A_1 and A_2 is defined as

$$A_1 \sqcup A_2 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}) = \{(x, 1), (y, 2) \mid x \in A_1, y \in A_2\}.$$

In the case where $A_1 \cap A_2$ is empty, the set $A_1 \sqcup A_2$ is identified with the union $A_1 \cup A_2$, namely, it is the usual disjoint union.

Below is the main result of this section. See Section 1 for the definition of standardness.

Theorem 7.2. *Let R be a nonregular local ring with an isolated singularity, which is either*

(1) *a hypersurface, or*

(2) *a Cohen–Macaulay ring with minimal multiplicity and infinite residue field.*

Then there is a one-to-one correspondence

$$\left\{\begin{matrix}\textit{Standard thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{matrix}\right\} \underset{\Gamma}{\overset{\Lambda}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Nonempty}\\ \textit{specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\} \sqcup \left\{\begin{matrix}\textit{Nonempty}\\ \textit{specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\}.$$

Here, the maps Λ and Γ are defined by:

$$\Lambda(\mathcal{X}) = \begin{cases} (\operatorname{Supp} \mathcal{X}, 1) & \text{if } \mathcal{X} \subseteq \mathsf{D}_{\mathsf{perf}}(R), \\ (\operatorname{Supp} \mathcal{X}, 2) & \text{if } \mathcal{X} \nsubseteq \mathsf{D}_{\mathsf{perf}}(R), \end{cases}$$

$$\Gamma((S, i)) = \begin{cases} (\operatorname{Supp}^{-1} S) \cap \mathsf{D}_{\mathsf{perf}}(R) & \text{if } i = 1, \\ \operatorname{Supp}^{-1} S & \text{if } i = 2. \end{cases}$$

We shall give a proof of this theorem at the end of this section, after preparing several necessary tools. Here are some examples of a ring satisfying the assumption of Theorem 7.2(2).

Example 7.3. Let k be an infinite field, and let x, y be indeterminates over k. Then it is easy to observe that $k[[x, y]]/(x^2, xy, y^2)$, $k[[x, y, z]]/(x^2-yz, y^2-zx, z^2-xy)$ and $k[[x^3, x^2y, xy^2, y^3]]$ are non-Gorenstein rings satisfying the condition (2) in Theorem 7.2. In general, normal local rings of dimension two with rational singularities satisfy Theorem 7.2(2); see [Huneke and Watanabe 2015, Theorem 3.1].

Remark 7.4. (1) Theorem 7.2(1) can also be deduced from [Stevenson 2014, Theorem 4.9].

(2) Theorem 7.2(2) especially says the following.

Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring with an isolated singularity, and assume k is infinite and R has minimal multiplicity. Let $\mathcal{X}$ be a standard thick subcategory of $\mathsf{D^b}(R)$ which is not contained in $\mathsf{D_{perf}}(R)$. Then $\mathcal{X}$ contains k.

This statement is no longer true without the assumption that R has minimal multiplicity. Indeed, let $R = k[x, y]/(x^2, y^2)$ with k a field, and let $\mathcal{X}$ be the thick closure of R and $R/(x)$ in $\mathsf{D^b}(R)$. Then R is an artinian complete intersection local ring, and $\mathcal{X}$ is a thick subcategory of $\mathsf{D^b}(R)$. As $R \in \mathcal{X}$, it is standard. Since $R/(x)$ has infinite projective dimension as an R-module, $\mathcal{X}$ is not contained in $\mathsf{D_{perf}}(R)$. Both R and $R/(x)$ have complexity at most 1, and the subcategory of $\mathsf{D^b}(R)$ consisting of objects having complexity at most 1 is thick. Hence any object in $\mathcal{X}$ has complexity at most 1. The fact that k has complexity 2 shows $k \notin \mathcal{X}$.

Thus, the assumption in Theorem 7.2(2) that R has minimal multiplicity is indispensable.

We state a general lemma on triangulated categories, whose proof is standard and omitted.

Lemma 7.5. *Let $\mathcal{T}$ be an essentially small triangulated category.*

(1) *Let $\mathcal{U}$ be a thick subcategory of $\mathcal{T}$. Let $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{U}$ be the canonical functor. Let T be an object of $\mathcal{T}$ and $\mathcal{X}$ a subcategory of $\mathcal{T}$. Then T is in $\mathsf{thick}_{\mathcal{T}}(\mathcal{U} \cup \mathcal{X})$ if and only if πT is in $\mathsf{thick}_{\mathcal{T}/\mathcal{U}}(\pi\mathcal{X})$.*

(2) *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$. For each object $T \in \mathsf{thick}_{\mathcal{T}}\,\mathcal{C}$ there exist a finite number of objects $C_1, \dots, C_n \in \mathcal{C}$ such that $T \in \mathsf{thick}_{\mathcal{T}}\{C_1, \dots, C_n\}$.*

The *stable derived category* $\mathsf{D_{sg}}(R)$ of R, which is also called the *singularity category* of R, is defined as the Verdier quotient of $\mathsf{D^b}(R)$ by $\mathsf{D_{perf}}(R)$. The following proposition says that a standard thick subcategory generating the singularity category contains the residue field.

Proposition 7.6. *Let R be a local ring with residue field k. Let $\mathcal{X}$ be a standard thick subcategory of $\mathsf{D^b}(R)$. Suppose that the equality $\mathsf{thick}_{\mathsf{D_{sg}}(R)}(\pi\mathcal{X}) = \mathsf{D_{sg}}(R)$ holds, where $\pi : \mathsf{D^b}(R) \to \mathsf{D_{sg}}(R)$ stands for the canonical functor. Then $\mathcal{X}$ contains k.*

Proof. Lemma 7.5(1) implies $\mathsf{thick}_{\mathsf{D^b}(R)}(\{R\} \cup \mathcal{X}) = \mathsf{D^b}(R)$. By Lemma 7.5(2) there is an object $X \in \mathcal{X}$ such that k belongs to the thick closure of R and X. Since $\mathcal{X}$ is standard, it contains a nonacyclic perfect complex P. Tensoring P shows that $P \otimes_R^{\boldsymbol{L}} k$ belongs to the thick closure of P and $P \otimes_R^{\boldsymbol{L}} X$, which is contained in $\mathcal{X}$. As P is not acyclic, the maximal ideal is in the support of P in $\mathsf{D^b}(R)$, which means $P \otimes_R^{\boldsymbol{L}} k \neq 0$ in $\mathsf{D^b}(R)$. Thus $P \otimes_R^{\boldsymbol{L}} k$ contains $k[n]$ as a direct summand for some $n \in \mathbb{Z}$, and it follows that k is in $\mathcal{X}$. □

For every triangulated category $\mathcal{T}$, the zero subcategory $\mathbf{0}$ and the whole category $\mathcal{T}$ are thick subcategories of $\mathcal{T}$. We call these two thick subcategories *trivial*, and the other thick subcategories *nontrivial*. The assumption of Theorem 7.2 comes from the fact that the following proposition holds under it.

Proposition 7.7. *Let R be a local ring with an isolated singularity. Suppose that R is either*

(1) *a hypersurface*, *or*

(2) *a Cohen–Macaulay ring with minimal multiplicity and infinite residue field.*

Then $\mathsf{D_{sg}}(R)$ has no nontrivial thick subcategory.

Proof. (1) By virtue of [Takahashi 2010, Main Theorem], the thick subcategories of $\mathsf{D_{sg}}(R)$ bijectively correspond to the specialization-closed subsets of the singular locus $\operatorname{Sing} R$ of R, i.e., the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is not regular. Since R has an isolated singularity, $\operatorname{Sing} R$ is trivial. Thus there exist only trivial thick subcategories of $\mathsf{D_{sg}}(R)$.

(2) Let $\mathcal{X}$ be a nonzero thick subcategory of $\mathsf{D_{sg}}(R)$. Then there exists a bounded R-complex C of infinite projective dimension such that πC is in $\mathcal{X}$, where $\pi : \mathsf{D^b}(R) \to \mathsf{D_{sg}}(R)$ is the canonical functor. One finds an exact triangle $P \to C \to M[n] \rightsquigarrow$ in $\mathsf{D^b}(R)$ with $P \in \mathsf{D_{perf}}(R)$ and $n \in \mathbb{Z}$ such that M is the $(d+1)$-st syzygy of an R-module, where $d = \dim R$. As C has infinite projective dimension, M is a nonzero module. The object πC is isomorphic to $\pi M[n]$ in $\mathsf{D_{sg}}(R)$, whence πM belongs to $\mathcal{X}$.

There is a maximal Cohen–Macaulay R-module N such that $M \cong \Omega_R N$. Since R has minimal multiplicity and the residue field of R is infinite, we find a parameter ideal $Q = (x_1, \dots, x_d)$ of R such that $\mathfrak{m}^2 = Q\mathfrak{m}$; see [Bruns and Herzog 1998, Exercise 4.6.14]. Note that $\boldsymbol{x} := x_1, \dots, x_d$ is a regular sequence on R, and hence on N. We see that M/QM is isomorphic to $\Omega_{R/Q}(N/QN)$, which is contained in $\mathfrak{m}L$ for some free R/Q-module L. Since $\mathfrak{m}^2$ is contained in Q, the module $\Omega_{R/Q}(N/QN)$ is annihilated by $\mathfrak{m}$, which implies that M/QM is a nonzero k-vector space.

In the derived category $\mathsf{D^b}(R)$ the module M/QM is isomorphic to the Koszul complex $K := \mathrm{K}(\boldsymbol{x}, M)$. Since K is a bounded complex of direct sums of copies of M, the object πK belongs to the thick closure of πM (see Lemma 2.3(4)), and hence πK belongs to $\mathcal{X}$. Consequently, the object πk is in $\mathcal{X}$. As R has an isolated singularity, $\mathsf{D}_{\mathsf{sg}}(R)$ coincides with the thick closure of πk by Corollary 3.5. This implies $\mathcal{X} = \mathsf{D}_{\mathsf{sg}}(R)$, which is what we want. □

We give a lemma on elementary set theory, whose proof is also elementary and omitted.

Lemma 7.8. *Let A_1, A_2, B_1, B_2 be sets. Let $f_i : A_i \to B_i$ be a bijection for each $i = 1, 2$. Define the map $g : A_1 \sqcup A_2 \to B_1 \sqcup B_2$ by $g((a, i)) = (f_i(a), i)$ for $a \in A_i$ and $i = 1, 2$. Then g is a bijection.*

Now we can prove Theorem 7.2:

Proof of Theorem 7.2. Let A_1 be the set of nonzero thick subcategories of $\mathsf{D}_{\mathsf{perf}}(R)$. Let A_2 be the set of standard thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. Then $A_1 \cap A_2$ is empty, and $A_1 \sqcup A_2$ coincides with the set of standard thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$. Let B be the set of nonempty specialization-closed subsets of Spec R. By [Neeman 1992, Theorem 1.5] there is a one-to-one correspondence $f : A_1 \rightleftarrows B : g$ defined by $f(\mathcal{X}) = \operatorname{Supp} \mathcal{X}$ and $g(S) = (\operatorname{Supp}^{-1} S) \cap \mathsf{D}_{\mathsf{perf}}(R)$. In view of Lemma 7.8, it suffices to show that there is a one-to-one correspondence $p : A_2 \rightleftarrows B : q$ defined by $p(\mathcal{X}) = \operatorname{Supp} \mathcal{X}$ and $q(S) = \operatorname{Supp}^{-1} S$. By Theorem 6.6, we have only to show that a thick subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ contains the residue field k if and only if $\mathcal{X}$ is a standard thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ not contained in $\mathsf{D}_{\mathsf{perf}}(R)$.

To show the "only if" part, suppose that $\mathcal{X}$ contains k. As R is not regular, k does not belong to $\mathsf{D}_{\mathsf{perf}}(R)$, whence $\mathcal{X}$ is not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. The thick closure $\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\, k$ contains the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$, where $\boldsymbol{x}$ is a system of generators of the maximal ideal of R. Hence $\mathcal{X}$ contains the nonacyclic perfect complex $\mathrm{K}(\boldsymbol{x}, R)$, which implies that $\mathcal{X}$ is standard.

To show the "if" part, assume that $\mathcal{X}$ is standard and not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. Then the image of $\mathcal{X}$ in $\mathsf{D}_{\mathsf{sg}}(R)$ is nonzero, and hence its thick closure coincides with $\mathsf{D}_{\mathsf{sg}}(R)$ by Proposition 7.7. Therefore $\mathcal{X}$ contains k by Proposition 7.6.

Thus, the proof of the theorem is completed. □

Remark 7.9. Theorem 7.2(2), Proposition 7.7(2) and the statement (written in italic) in Remark 7.4 are valid if one replaces the assumption that *R has minimal multiplicity and k is infinite* with the assumption that *there exists a parameter ideal Q of R with $\mathfrak{m}^2 = Q\mathfrak{m}$*. In fact, the same proofs work under this assumption.

8. Almost Gorenstein rings and Cohen–Macaulay rings of finite CM-representation type

In this section, as another application of Theorem 6.6, we study classifying standard and costandard thick subcategories over an almost Gorenstein ring and a Cohen–Macaulay ring of finite CM-representation type. We start by recalling the definitions:

Definition 8.1. Let R be a Cohen–Macaulay local ring.

(1) We say that R is *almost Gorenstein* if there exists an exact sequence

$$0 \to R \to \omega \to C \to 0$$

of R-modules such that ω is a canonical module of R and C is an *Ulrich* module, that is, C is a Cohen–Macaulay R-module whose multiplicity is equal to the minimal number of generators. For the details of almost Gorenstein local rings, we refer the reader to [Goto et al. 2015].

(2) We say R is of *finite CM-representation type* if there exist only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules.

The main result of this section is the following theorem. (The definitions of standard and costandard thick subcategories are given in Section 1.)

Theorem 8.2. *Let R be a non-Gorenstein local ring. Suppose that R is either*

(1) *an almost Gorenstein ring with an isolated singularity*, *or*

(2) *an excellent Cohen–Macaulay ring with canonical module and finite CM-representation type.*

Then there is a one-to-one correspondence

$$\left\{ \begin{matrix} \textit{Standard and costandard} \\ \textit{thick subcategories of } \mathsf{D}^{\mathsf{b}}(R) \end{matrix} \right\} \underset{\text{Supp}^{-1}}{\overset{\text{Supp}}{\underset{1-1}{\rightleftarrows}}} \left\{ \begin{matrix} \textit{Nonempty specialization-closed} \\ \textit{subsets of } \operatorname{Spec} R \end{matrix} \right\}.$$

We state examples, remarks and propositions related to this theorem, and prove the theorem.

Example 8.3. Let k be an algebraically closed field of characteristic zero. Let t, x, y be indeterminates over k.

(1) Both the numerical semigroup ring $k[[t^3, t^4, t^5]]$ and the Veronese subring $k[[x^3, x^2y, xy^2, y^3]]$ satisfy all the conditions (2) in Theorem 7.2 and (1), (2) in Theorem 8.2.

(2) Consider the numerical semigroup rings $k[[t^4, t^5, t^7]]$, $k[[t^4, t^7, t^9]]$ and the residue ring $k[[x, y, z, s]]/I$, where I is the ideal generated by the 2-minors of the matrix $\left(\begin{smallmatrix} x^2 & y^2-s^{10} & z \\ y & z & x \end{smallmatrix}\right)$. (All of these rings are the completions of positively graded k-algebras.) These rings satisfy Theorem 8.2(1), but do not satisfy Theorem 8.2(2) or Theorem 7.2(2).

For the proofs, see [Goto et al. 2015, Examples 3.16, 7.5, Corollary 11.4 and Theorem 12.1] and [Yoshino 1990, Theorems (9.2) and (10.14)].

In view of this example, it seems that there exist a lot of examples of rings satisfying Theorem 8.2(1). Here is a remark on Theorem 8.2(2).

Remark 8.4. According to [Schreyer 1987, (7.1)], all known examples of a nonhypersurface Cohen–Macaulay complete local $\mathbb{C}$-algebra of finite CM-representation type have minimal multiplicity. Hence, at least for these examples, the one-to-one correspondence in Theorem 8.2 is obtained by restricting that in Theorem 7.2.

The following two propositions play a crucial role in the proof of Theorem 8.2.

Proposition 8.5. *Let R be a local ring with residue field k and dualizing complex D. Assume that k belongs to* $\mathsf{thick}_{\mathsf{D^b}(R)}\{R, D\}$. *Let P (resp. I) be a nonacyclic R-complex of finite projective (resp. injective) dimension. Then k belongs to* $\mathsf{thick}_{\mathsf{D^b}(R)}\{P, I\}$.

Proof. The Foxby equivalence theorem [Avramov and Foxby 1997, Theorem (3.2)] implies that the complex $Q := \mathbf{R}\mathrm{Hom}_R(D, I)$ has finite projective dimension and I is isomorphic to $D \otimes^{\boldsymbol{L}}_R Q$ in $\mathsf{D^b}(R)$. As k is in the thick closure of R and D, applying the functor $- \otimes^{\boldsymbol{L}}_R Q \otimes^{\boldsymbol{L}}_R P$ shows that $k \otimes^{\boldsymbol{L}}_R Q \otimes^{\boldsymbol{L}}_R P$ is in the thick closure of $Q \otimes^{\boldsymbol{L}}_R P$ and $I \otimes^{\boldsymbol{L}}_R P$. Note that $Q \otimes^{\boldsymbol{L}}_R P$ and $I \otimes^{\boldsymbol{L}}_R P$ belong to the thick closures of P and I, respectively. Hence $k \otimes^{\boldsymbol{L}}_R Q \otimes^{\boldsymbol{L}}_R P$ belongs to the thick closure of P and I. Since P and I are not acyclic, $k \otimes^{\boldsymbol{L}}_R Q \otimes^{\boldsymbol{L}}_R P$ is nonzero in $\mathsf{D^b}(R)$, whence it contains $k[n]$ as a direct summand for some integer n. Thus the assertion follows. □

A local ring R is called *G-regular* if the totally reflexive modules over R are the free modules. For the details of G-regular local rings, we refer the reader to [Takahashi 2008].

Proposition 8.6. *Let R be a non-Gorenstein local ring with canonical module ω, being either*

(1) *an almost Gorenstein ring with an isolated singularity*, *or*

(2) *an excellent Cohen–Macaulay ring with canonical module and finite CM-representation type.*

Then $\mathsf{thick}_{\mathsf{mod}\,R}\{R, \omega\} = \mathsf{mod}\,R$. *In particular*, *$R$ is a G-regular local ring.*

Proof. Let k be the residue field of R. We first prove $\mathsf{thick}_{\mathsf{mod}\,R}\{R, \omega\} = \mathsf{mod}\,R$.

(1) Since R is an isolated singularity, we have $\mathsf{thick}_{\mathsf{mod}\,R}\{R, k\} = \mathsf{mod}\,R$ by Corollary 3.5. According to [Goto et al. 2015, Corollary 4.5], there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to k^{r-1} \to 0$, where r is the Cohen–Macaulay type of R and each X_i is a finite direct sum of copies of R and ω. We have $r \geq 2$ since R is not Gorenstein, and it is seen that k belongs to the thick closure of R and ω. Thus the equality follows.

(2) It follows from [Huneke and Leuschke 2002, Corollary 2] that R has an isolated singularity, and so does the completion of R since R is excellent (see [Takahashi 2010, Proposition 3.4]). Using [Takahashi 2013a, Corollary 6.9], one sees that the thick closure of R and ω must be the whole category $\mathsf{mod}\,R$.

Now let us show the last assertion. Let G be a totally reflexive R-module. Let $\mathcal{M}$ be the subcategory of $\mathsf{mod}\, R$ consisting of modules M such that $\mathrm{Ext}_R^{\gg 0}(G, M) = 0$. By definition we have $\mathrm{Ext}_R^{>0}(G, R) = 0$, and moreover $\mathrm{Ext}_R^{>0}(G, \omega) = 0$ since G is maximal Cohen–Macaulay. Therefore R and ω belong to $\mathcal{M}$. It is easy to see that $\mathcal{M}$ is a thick subcategory of $\mathsf{mod}\, R$, whence it contains $\mathsf{thick}_{\mathsf{mod}\, R}\{R, \omega\}$, which coincides with $\mathsf{mod}\, R$. Thus k is in $\mathcal{M}$, which implies that G has finite projective dimension, so that G is free. □

Remark 8.7. In Proposition 8.6(2), the excellence can be replaced with the condition that the completion of R is an isolated singularity.

Now we can give a proof of the main result of this section:

Proof of Theorem 8.2. Let $\mathcal{X}$ be a standard and costandard thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. Then by Propositions 8.6 and 8.5, $\mathcal{X}$ contains the residue field k of R.

Conversely, let $\mathcal{X}$ be a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k. Then $\mathcal{X}$ contains the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$ with $\boldsymbol{x}$ a system of generators of the maximal ideal of R, whence $\mathcal{X}$ is standard. By assumption, R admits a canonical module ω. Let $\boldsymbol{y}$ be a system of parameters of R. Then $\boldsymbol{y}$ is a regular sequence on R, and hence on ω. The module $\omega/\boldsymbol{y}\omega$ has finite injective dimension as an R-module, and belongs to $\mathcal{X}$ because it is in $\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\, k$. Therefore $\mathcal{X}$ is costandard.

The assertion follows from the above argument and Theorem 6.6. □

9. Gorenstein rings with almost minimal multiplicity

This section is devoted to exploring thick subcategories over a Gorenstein local ring having relatively small multiplicity. Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring. We say that R has *almost minimal multiplicity* if the following equality holds:

$$\mathrm{e}(R) = \mathrm{edim}\, R - \dim R + 2.$$

Assume that k is infinite. Then there is a minimal reduction Q of $\mathfrak{m}$. Note that Q is a parameter ideal of R satisfying $\mathfrak{m}^2/Q\mathfrak{m} \cong k$, and hence $\mathfrak{m}^3 \subseteq Q$. Only assuming this inclusion, we have the following classification of thick subcategories, which is the main result of this section:

Theorem 9.1. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein nonregular local ring with an isolated singularity. Let Q be a parameter ideal of R containing $\mathfrak{m}^3$. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R) \textit{ containing}\\ \textit{a nonacyclic perfect } \bar{R}\textit{-complex}\end{matrix}\right\} \underset{\mathrm{Supp}^{-1}}{\overset{\mathrm{Supp}}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Nonempty specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\},$$

where $\bar{R} = R/(Q : \mathfrak{m})$.

We first prepare lemmas to show this theorem.

Lemma 9.2. *Let $(R, \mathfrak{m}, k)$ be an artinian Gorenstein local ring which is not a field. Then* $\mathsf{thick}_{\mathsf{mod}\, R}(R/\operatorname{Soc} R) = \mathsf{mod}\, R$.

Proof. Denote by $(-)^*$ the R-dual functor $\operatorname{Hom}_R(-, R)$. Since R is artinian and Gorenstein, R is an injective R-module and $k^* \cong k$. Applying $(-)^*$ to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we have an exact sequence

$$0 \to k \to R \to R/\operatorname{Soc} R \to 0 \tag{9.2.1}$$

and see that $\mathfrak{m}^*$ is isomorphic to $R/\operatorname{Soc} R$.

Let $x_1, x_2, \dots, x_n$ be a minimal system of generators of $\mathfrak{m}$. As R is not a field, the integer n is positive. Let $I = (x_1^2, x_2, \dots, x_n)$ be an ideal. Then $\mathfrak{m}/I$ is isomorphic to k, and there exists an exact sequence $0 \to k \to R/I \to k \to 0$. Taking the first syzygies, we obtain an exact sequence $0 \to \mathfrak{m} \to R \oplus I \to \mathfrak{m} \to 0$. Applying $(-)^*$ gives rise to an exact sequence $0 \to \mathfrak{m}^* \to R \oplus I^* \to \mathfrak{m}^* \to 0$.

Thus, there is an exact sequence

$$0 \to R/\operatorname{Soc} R \to R \oplus I^* \to R/\operatorname{Soc} R \to 0. \tag{9.2.2}$$

It follows from (9.2.2) that $\mathsf{thick}_{\mathsf{mod}\, R}(R/\operatorname{Soc} R)$ contains R, and from (9.2.1) that it contains k. Since R is artinian, $\mathsf{thick}_{\mathsf{mod}\, R}(R/\operatorname{Soc} R)$ coincides with the whole module category $\mathsf{mod}\, R$. □

We need one more lemma, whose proof is straightforward.

Lemma 9.3. *Let $\mathcal{T}, \mathcal{U}$ be triangulated categories, and let $F : \mathcal{T} \to \mathcal{U}$ be a triangle functor. Let $\mathcal{X}$ be a thick subcategory of $\mathcal{U}$. Denote by $F^{-1}(\mathcal{X})$ the subcategory of $\mathcal{T}$ consisting of objects $T \in \mathcal{T}$ with $F(T) \in \mathcal{X}$. Then $F^{-1}(\mathcal{X})$ is a thick subcategory of $\mathcal{T}$.*

Now we can prove our Theorem 9.1:

Proof of Theorem 9.1. By Theorem 6.6, it suffices to show that a thick subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ contains a nonacyclic perfect $\bar{R}$-complex if and only if $\mathcal{X}$ contains the residue field $k = R/\mathfrak{m}$.

The "if" part: If k is in $\mathcal{X}$, then all R-modules of finite length are in $\mathcal{X}$, whence $\bar{R} \in \mathcal{X}$.

The "only if" part: Assume that $\mathcal{X}$ contains a nonacyclic perfect $\bar{R}$-complex L. Let $F : \mathsf{D}^{\mathsf{b}}(\bar{R}) \to \mathsf{D}^{\mathsf{b}}(R)$ be the natural triangle functor. Lemma 9.3 implies that $F^{-1}(\mathcal{X})$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(\bar{R})$, and it is standard since it contains L. As $\mathfrak{m}^3$ is contained in Q, the square of the maximal ideal of $\bar{R}$ is zero. Using Remarks 7.4 and 7.9, we observe that $F^{-1}(\mathcal{X})$ either contains k or is contained in $\mathsf{D}_{\mathsf{perf}}(\bar{R})$. As to the former case, k belongs to $\mathcal{X}$.

Let us consider the latter case. Note that $F^{-1}(\mathcal{X})$ is a thick subcategory of $\mathsf{D}_{\mathsf{perf}}(\bar{R})$, and that $\operatorname{Spec}\bar{R}$ consists of the maximal ideal. By [Neeman 1992, Theorem 1.5], $F^{-1}(\mathcal{X})$ coincides with either the zero category $\mathbf{0}$ or the whole category $\mathsf{D}_{\mathsf{perf}}(\bar{R})$. Because the nonacyclic complex L is in $F^{-1}(\mathcal{X})$, we have $F^{-1}(\mathcal{X}) = \mathsf{D}_{\mathsf{perf}}(\bar{R})$. In particular, $\mathcal{X}$ contains $\bar{R}$. Note that R/Q is an artinian Gorenstein ring that is not a field. Applying Lemma 9.2 to the ring R/Q, we have $\mathsf{thick}_{\mathsf{mod}\, R/Q}(\bar{R}) = \mathsf{mod}\, R/Q$. Hence k is in $\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R/Q)}(\bar{R})$. Sending this containment by the natural triangle functor $\mathsf{D}^{\mathsf{b}}(R/Q) \to \mathsf{D}^{\mathsf{b}}(R)$ shows $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\bar{R})$. Thus k belongs to $\mathcal{X}$. □

Acknowledgments

Part of this work was done during the visit of the author to the Centre de Recerca Matemàtica (CRM) to attend the research program "IRTATCA: Interactions between Representation Theory, Algebraic Topology and Commutative Algebra" held from January to June in 2015. The author is grateful for their great atmosphere and kind hospitality. Also, the author very much thanks Mitsuyasu Hashimoto for giving helpful advice on the proof of Lemma 3.2, Shunsuke Takagi for indicating the reference [Huneke and Watanabe 2015], an anonymous referee for valuable comments on the proof of Theorem 6.6 (see Remark 6.7), and another anonymous referee for a lot of useful comments and suggestions throughout the paper.

References

[Auslander and Bridger 1969] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society **94**, American Mathematical Society, Providence, RI, 1969. MR Zbl

[Avramov and Foxby 1997] L. L. Avramov and H.-B. Foxby, "Ring homomorphisms and finite Gorenstein dimension", *Proc. London Math. Soc.* (3) **75**:2 (1997), 241–270. MR Zbl

[Bruns and Herzog 1998] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, 2nd ed., Cambridge Studies in Advanced Mathematics **39**, Cambridge Univ. Press, 1998. Zbl

[Dwyer et al. 2006] W. Dwyer, J. P. C. Greenlees, and S. Iyengar, "Finiteness in derived categories of local rings", *Comment. Math. Helv.* **81**:2 (2006), 383–432. MR Zbl

[Gabriel 1962] P. Gabriel, "Des catégories abéliennes", *Bull. Soc. Math. France* **90** (1962), 323–448. MR Zbl

[Goto et al. 2015] S. Goto, R. Takahashi, and N. Taniguchi, "Almost Gorenstein rings: towards a theory of higher dimension", *J. Pure Appl. Algebra* **219**:7 (2015), 2666–2712. MR Zbl

[Huneke and Leuschke 2002] C. Huneke and G. J. Leuschke, "Two theorems about maximal Cohen–Macaulay modules", *Math. Ann.* **324**:2 (2002), 391–404. MR Zbl

[Huneke and Watanabe 2015] C. Huneke and K.-i. Watanabe, "Upper bound of multiplicity of F-pure rings", *Proc. Amer. Math. Soc.* **143**:12 (2015), 5021–5026. MR Zbl

[Iyengar 2004] S. Iyengar, "Modules and cohomology over group algebras: one commutative algebraist's perspective", pp. 51–86 in *Trends in commutative algebra*, edited by L. L. Avramov et al., Math. Sci. Res. Inst. Publ. **51**, Cambridge Univ. Press, 2004. MR Zbl

[Krause and Stevenson 2013] H. Krause and G. Stevenson, "A note on thick subcategories of stable derived categories", *Nagoya Math. J.* **212** (2013), 87–96. MR Zbl

[Neeman 1992] A. Neeman, "The chromatic tower for $D(R)$", *Topology* **31**:3 (1992), 519–532. MR Zbl

[Orlov 2011] D. Orlov, "Formal completions and idempotent completions of triangulated categories of singularities", *Adv. Math.* **226**:1 (2011), 206–217. MR Zbl

[Schoutens 2003] H. Schoutens, "Projective dimension and the singular locus", *Comm. Algebra* **31**:1 (2003), 217–239. MR Zbl

[Schreyer 1987] F.-O. Schreyer, "Finite and countable CM-representation type", pp. 9–34 in *Singularities, representation of algebras, and vector bundles* (Lambrecht, 1985), edited by G.-M. Greuel and G. Trautmann, Lecture Notes in Math. **1273**, Springer, Berlin, 1987. MR Zbl

[Stevenson 2014] G. Stevenson, "Duality for bounded derived categories of complete intersections", *Bull. Lond. Math. Soc.* **46**:2 (2014), 245–257. MR Zbl

[Takahashi 2008] R. Takahashi, "On G-regular local rings", *Comm. Algebra* **36**:12 (2008), 4472–4491. MR Zbl

[Takahashi 2010] R. Takahashi, "Classifying thick subcategories of the stable category of Cohen–Macaulay modules", *Adv. Math.* **225**:4 (2010), 2076–2116. MR Zbl

[Takahashi 2013a] R. Takahashi, "Classifying resolving subcategories over a Cohen–Macaulay local ring", *Math. Z.* **273**:1-2 (2013), 569–587. MR Zbl

[Takahashi 2013b] R. Takahashi, "Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra", *J. Math. Soc. Japan* **65**:2 (2013), 357–374. MR Zbl

[Thomason 1997] R. W. Thomason, "The classification of triangulated subcategories", *Compositio Math.* **105**:1 (1997), 1–27. MR Zbl

[Yoshino 1990] Y. Yoshino, *Cohen–Macaulay modules over Cohen–Macaulay rings*, London Mathematical Society Lecture Note Series **146**, Cambridge Univ. Press, 1990. MR Zbl

Received May 15, 2016. Revised March 28, 2017.

RYO TAKAHASHI
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA
JAPAN

takahashi@math.nagoya-u.ac.jp
http://www.math.nagoya-u.ac.jp/~takahashi/

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 291 No. 1 November 2017

0030-8730(201711)291:1;1-Y