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We consider the local Ginzburg–Rallis model over the complex field. We show that the multiplicity is always 1 for a majority of generic representations. We also have partial results on the real case for general generic representations. This is a continuation of our previous work in which we considered the *p*-adic case and the real case for tempered representations.

1. Introduction and main result

This paper is a continuation of [Wan 2016a; 2016b]. For an overview of the Ginzburg–Rallis model, see Section 1 of [Wan 2016a]. We recall from there the definition of the Ginzburg–Rallis models and conjectures.

Let F be a local field (p-adic or archimedean), D be the unique quaternion algebra over F if $F \neq \mathbb{C}$. Take $P = P_{2,2,2} = MU$ to be the standard parabolic subgroup of $G = \mathrm{GL}_6$ whose Levi part M is isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, and whose unipotent radical U consists of elements of the form

$$u = u(X, Y, Z) := \begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix}. \tag{1-1}$$

We define a character ξ on $U(F)$ by

$$\xi(u(X, Y, Z)) := \psi(\mathrm{tr}(X) + \mathrm{tr}(Y)), \tag{1-2}$$

where ψ is a nontrivial additive character on F. It's clear that the stabilizer of ξ is the diagonal embedding of $\mathrm{GL}_2(F)$ into $M(F)$, which is denoted by $H_0(F)$. For a given character χ of $F^{\times}$, one induces a one dimensional representation ω of $H_0(F)$ given by $\omega(h) := \chi(\det(h))$. We can extend the character ξ to the semidirect product

$$H(F) := H_0(F) \ltimes U(F) \tag{1-3}$$

by making it trivial on $H_0(F)$. Similarly we can extend the character ω to $H(F)$. It follows that the one dimensional representation $\omega \otimes \xi$ of $H(F)$ is well defined. The

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pair (G, H) is the Ginzburg–Rallis model, introduced in [Ginzburg and Rallis 2000]. Let π be an irreducible admissible representation of $G(F)$ with central character χ^2, we are interested in the Hom space $\mathrm{Hom}_{H(F)}(\pi, \omega \otimes \xi)$, the dimension of which is denoted by $m(\pi)$ and is called the multiplicity.

On the other hand, if $F \neq \mathbb{C}$, define $G_D = \mathrm{GL}_3(D)$. Similarly we can define U_D, $H_{0,D}$ and H_D. We also define the character $\omega_D \otimes \xi_D$ on $H_D(F)$ in the same way except that the trace in the definition of ξ is replaced by the reduced trace of the quaternion algebra D and the determinant in the definition of ω is replaced by the reduced norm of the quaternion algebra D. Then for an irreducible admissible representation π_D of $G_D(F)$ with central character χ^2, we can also talk about the Hom space $\mathrm{Hom}_{H_D(F)}(\pi_D, \omega_D \otimes \xi_D)$, whose dimension is denoted by $m(\pi_D)$.

The purpose of this paper is to study the multiplicity $m(\pi)$ and $m(\pi_D)$. First, it was proved by C.-F. Nien [2006] over a p-adic local field, and by D. Jiang, B. Sun and C. Zhu in [Jiang et al. 2011] for an archimedean local field that both multiplicities are less than or equal to 1: $m(\pi), m(\pi_D) \leq 1$. In other word, the pairs (G, H) and (G_D, H_D) are Gelfand pairs. In this paper, we are interested in the relation between $m(\pi)$ and $m(\pi_D)$ under the local Jacquet–Langlands correspondence established in [Deligne et al. 1984]. The local conjecture has been expected since the work of [Ginzburg and Rallis 2000], and was first discussed in detail by Jiang [2008].

Conjecture 1.1 [Jiang 2008]. *For any irreducible admissible representation π of $\mathrm{GL}_6(F)$, let π_D be the local Jacquet–Langlands correspondence of π to $\mathrm{GL}_3(D)$ if it exists, and zero otherwise. In particular, π_D is always 0 if $F = \mathbb{C}$. We still assume that the central character of π is χ^2. Then the following identity*:

$$m(\pi) + m(\pi_D) = 1 \tag{1-4}$$

holds for all irreducible generic representations π of $\mathrm{GL}_6(F)$.

Note that the assertion in Conjecture 1.1 can be formulated in terms of Vogan packets and pure inner forms of PGL_6. We refer to [Wan 2016a] for discussion.

Another aspect of the local conjecture is the epsilon dichotomy conjecture, which relates the multiplicity to the central value of the exterior cube epsilon factors. It can be stated as follows:

Conjecture 1.2. *With the same assumptions as in Conjecture 1.1, assume that the central character of π is trivial. Then we have*

$$m(\pi) = 1 \Longleftrightarrow \epsilon\left(\tfrac{1}{2}, \pi, \textstyle\bigwedge^3\right) = 1, \quad m(\pi) = 0 \Longleftrightarrow \epsilon\left(\tfrac{1}{2}, \pi, \textstyle\bigwedge^3\right) = -1.$$

In this paper, we always fix a Haar measure dx on F and an additive character ψ such that the Haar measure is self-dual for Fourier transform with respect to ψ. We use such dx and ψ in the definition of the ϵ factor. For simplicity, we will write the epsilon factor as $\epsilon(s, \pi, \rho)$ instead of $\epsilon(s, \pi, \rho, dx, \psi)$.

Remark 1.3. Conjecture 1.2 can also be formulated for general representations with nontrivial central character. To be specific, as in Conjecture 1.1, assume the central character of π is χ^2. Then the epsilon dichotomy conjecture for π becomes

$$m(\pi)=1 \Longleftrightarrow \epsilon\big(\tfrac{1}{2},\pi,\textstyle\bigwedge^3\otimes\chi^{-1}\big)=1,\quad m(\pi)=0 \Longleftrightarrow \epsilon\big(\tfrac{1}{2},\pi,\textstyle\bigwedge^3\otimes\chi^{-1}\big)=-1.$$

Here, $\epsilon\big(s,\pi,\bigwedge^3\otimes\chi^{-1}\big)$ is the epsilon factor of $\big(\bigwedge^3\phi_\pi\big)\otimes\chi^{-1}$ (*not* $\bigwedge^3(\phi_\pi\otimes\chi^{-1})$), where ϕ_π is the Langlands parameter of π. The proof of the epsilon dichotomy conjecture for representations with nontrivial central characters is the same as the trivial central character case. Hence for simplicity, in this paper, we will only consider the trivial central character case for the epsilon dichotomy conjecture. All our results can be easily extended to the nontrivial central character case.

In the previous papers [Wan 2016a; 2016b], we prove Conjecture 1.1 for the case that F is a p-adic local field or $\mathbb{R}$ and π is an irreducible tempered representation of $\mathrm{GL}_6(F)$. In [Wan 2016b], we also prove Conjecture 1.2 for the case $F=\mathbb{R}$ and π is tempered, together with the case when F is p-adic and π is tempered but not a discrete series or a parabolic induction of a discrete series of $\mathrm{GL}_4(F)\times\mathrm{GL}_2(F)$.

In this paper, we consider the case when $F=\mathbb{C}$. In this case, by the Langlands classification, any generic representation π is a principal series. In other words, let $B=M_0U_0$ be the Borel subgroup consisting of all the lower triangular matrices; here $M_0=(\mathrm{GL}_1)^6$ is just the diagonal matrix. Then π is of the form $I_B^G(\chi)$, where $\chi=\bigotimes_{i=1}^6\chi_i$ is a character on $M_0(F)$ and I_B^G is the normalized parabolic induction. For $1\le i\le 6$, we can find a unitary character σ_i and some real number $s_i\in\mathbb{R}$ such that $\chi_i=\sigma_i|\cdot|^{s_i}$. Without loss of generality, we assume that $s_i\le s_j$ for any $i\ge j$. Then if we combine those representations with the same exponents s_i, we can find a parabolic subgroup $Q=LU_Q$ containing B with $L=\times_{i=1}^k\mathrm{GL}_{n_i}$, a representation $\tau=\bigotimes_{i=1}^k\tau_i|\cdot|^{t_i}$ of $L(F)$, where τ_i are all tempered and the exponents t_i are strictly increasing (i.e., $t_1<t_2<\cdots<t_k$) such that $\pi=I_Q^G(\tau)$. On the other hand, we can also write π as $I_{\bar P}^G(\pi_0)$ with $\pi_0=\pi_1\otimes\pi_2\otimes\pi_3$, where π_i is the parabolic induction of $\chi_{2i-1}\otimes\chi_{2i}$. Here we want the representation to be induced from $\bar P$ instead of P because later in Sections 5 and 6, we would like to integrate the elements of the induced representation over the unipotent subgroup $U(F)$.

Theorem 1.4. *Assume that $F=\mathbb{C}$, with the same assumptions as in Conjecture 1.1 and with the notation above. Then we have the following*:

(1) *If $\bar P\subset Q$, Conjectures 1.1 and 1.2 hold. In particular, both conjectures hold for the tempered representations.*

(2) *If $Q\subsetneq\bar P$ and if π_0 satisfies the condition* (40) *in [Loke 2001], Conjectures 1.1 and 1.2 hold.*

There are two main ingredients in our proof. First we deal with the tempered representations. The idea is to construct an explicit element inside the Hom space given by integrating the matrix coefficient. Then we show that the nonvanishing property of this element is invariant under parabolic induction, which allows us to reduce to the torus case which is trivial. This idea already appears in [Wan 2016b] for the case when $F = \mathbb{R}$.

Then for general generic representations, we use the open orbit method to reduce our problems to the tempered case or the trilinear GL_2 model case. To be specific, if $\bar{P} \subset Q$, by applying the open orbit method, we can reduce to the model related to the Levi subgroup L. Then after twisting τ by some characters, we only need to deal with the tempered case which has already been proved. If $Q \subsetneq \bar{P}$, by applying the open orbit method, we reduce ourselves to the trilinear GL_2 model case. Then by applying the work of Loke [2001], we can prove our result. The extra condition in part (2) of Theorem 1.4 also comes from the same work.

It is worth mentioning that in Theorem 1.4(2), the requirements we made for the parabolic subgroup Q force some types of generalized Jacquet integrals to be absolutely convergent; this allows us to apply the open orbit method. If one can prove such integrals have holomorphic continuation, one can actually remove this restraint. This will be discussed in Section 7.

Finally, the open orbit method we use here can also be applied to the case when $F = \mathbb{R}$; this will gives us partial results about Conjecture 1.1 and Conjecture 1.2 for general generic representations. To be specific, let π be an irreducible generic representation of $G(F)$ with central character χ^2. By the Langlands classification, there is a parabolic subgroup $Q = LU_Q$ containing the lower Borel subgroup and an essential tempered representation $\tau = \bigotimes_{i=1}^k \tau_i |\cdot|^{s_i}$ of $L(F)$ with τ_i tempered, $s_i \in \mathbb{R}$ and $s_1 < s_2 < \cdots < s_k$ such that $\pi = I_Q^G(\tau)$. We say Q is nice if $Q \subset \bar{P}$ or $\bar{P} \subset Q$.

Theorem 1.5. *Let the notation be as above.*

(1) *If* $\pi_D = 0$ *and* Q *is nice, then Conjectures 1.1 and 1.2 hold.*

(2) *If* $\pi_D \neq 0$, *we have*

$$m(\pi) + m(\pi_D) \geq 1,$$

and if moreover if the central character of π *is trivial* (*as in* Conjecture 1.2), *we have*

$$\epsilon\left(\tfrac{1}{2}, \pi, \textstyle\bigwedge^3\right) = 1 \Rightarrow m(\pi) = 1; \quad m(\pi) = 0 \Rightarrow \epsilon\left(\tfrac{1}{2}, \pi, \textstyle\bigwedge^3\right) = -1.$$

As in the complex case, the assumption on Q can be removed if we can prove the holomorphic continuation of certain generalized Jacquet integrals. This will also be discussed in Section 7.

The paper is organized as follows: In Section 2, we review a well know result of the intertwining operator which is due to Harish-Chandra. We will also give a brief overview of the open orbit method which will be used in later sections. In Section 3, we show that for $F=\mathbb{C}$, Conjecture 1.1 implies Conjecture 1.2. In Section 4, we prove Theorem 1.4 for tempered representations. Then in Section 5, we prove it for general cases. In Section 6, we discuss the case for $F=\mathbb{R}$. In Section 7, we talk about how to remove the assumptions on Q based on the results on the holomorphic continuation of the generalized Jacquet integral due to Raul Gomez [$\geq$ 2017].

2. Preliminaries

2A. ***The intertwining operator.*** For every connected reductive algebraic group G defined over F, let A_G be the maximal split center of G and Z_G be the center of G. We denote by $X(G)$ the group of F-rational characters of G. Define $\mathfrak{a}_G=\mathrm{Hom}(X(G),\mathbb{R})$, and let $\mathfrak{a}_G^*=X(G)\otimes_{\mathbb{Z}}\mathbb{R}$ be the dual of $\mathfrak{a}_G$. We define a homomorphism $H_G:G(F)\to\mathfrak{a}_G$ by $H_G(g)(\chi)=\log(|\chi(g)|_F)$ for every $g\in G(F)$ and $\chi\in X(G)$.

Given a parabolic subgroup $P=MU$ of G and an admissible representation (τ,V_τ) of $M(F)$, let K be a maximal compact subgroup of $G(F)$ in good position with respect to M. Let $(I_P^G(\tau),I_P^G(V_\tau))$ be the normalized parabolic induced representation: $I_P^G(V_\tau)$ consists of smooth functions $e:G(F)\to V_\tau$ such that

$$e(mug)=\delta_P(m)^{1/2}\tau(m)e(g),\quad m\in M(F),\quad u\in U(F),\quad g\in G(F),$$

and the $G(F)$-action is just the right translation.

For $\lambda\in\mathfrak{a}_M^*\otimes_{\mathbb{R}}\mathbb{C}$, let τ_λ be the unramified twist of τ, i.e.,

$$\tau_\lambda(m)=\exp(\lambda(H_M(m)))\tau(m),$$

and let $I_P^G(\tau_\lambda)$ be the induced representation. By the Iwasawa decomposition, every function $e\in I_P^G(\tau_\lambda)$ is determined by its restriction on K, and that space is invariant under the unramified twist, i.e., for any λ, we can realize the representation $I_P^G(\tau_\lambda)$ on the space $I_{K\cap P}^K(\tau_K)$ which consists of functions $e_K:K\to V_\tau$ such that

$$e(mug)=\delta_P(m)^{1/2}\tau(m)e(g),\quad m\in M(F)\cap K,\quad u\in U(F)\cap K,\quad g\in K.$$

Here τ_K is the restriction of the representation τ to the group K.

Now we define the intertwining operator. For a Levi subgroup M of G, $P=MU$, $P'=MU'\in\mathcal{P}(M)$, and $\lambda\in\mathfrak{a}_M^*\otimes_{\mathbb{R}}\mathbb{C}$, define the intertwining operator

$$J_{P'|P}(\tau_\lambda):I_P^G(V_\tau)\to I_{P'}^G(V_\tau),\quad J_{P'|P}(\tau_\lambda)(e)(g)=\int_{(U(F)\cap U'(F))\backslash U'(F)}e(ug)\,du.$$

In general, the integral above is not absolutely convergent. But it is absolutely convergent when $\mathrm{Re}(\lambda)$ lies inside a positive cone, and it is $G(F)$-equivariant. By restricting to K, we can view $J_{P'|P}(\tau_\lambda)$ as a homomorphism from $I_{K\cap P}^K(V_{\tau_K})$ to $I_{K\cap P'}^K(V_{\tau_K})$. In general, $J_{P'|P}(\tau_\lambda)$ can be meromorphically continued to a function on $\mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, if we assume that τ is tempered, we have the following proposition which is due to Harish-Chandra. The proof of the proposition can be found in Proposition IV.2.1 of [Waldspurger 2003].

Proposition 2.1. *With the notation above, if τ is tempered, then the intertwining operator $J_{P'|P}$ is absolutely convergent for all $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ with $\langle \mathrm{Re}(\lambda), \check{\alpha} \rangle > 0$ for every $\alpha \in \Sigma(P) \cap \Sigma(\bar{P}')$. Here $\Sigma(P)$ is the subset of the roots of A_M that are positive with respect to P.*

We will use this proposition in later sections to show some generalized Jacquet integrals are absolutely convergent.

Finally, assume π is a unitary representation of $G(F)$. Let $\mathrm{End}(\pi)$ be the space of continuous endomorphisms of π. We define the norm on $\mathrm{End}(\pi)$ to be $\|T\| = \sup_{e\in\pi,\, |e|=1} |Te|$. Then it becomes a Banach space. It is also a continuous representation of $G(F) \times G(F)$ under the left and right translations. Let $\mathrm{End}(\pi)^\infty$ be the subspace of smooth vectors. We can define a locally convex topology on $\mathrm{End}(\pi)^\infty$ via the seminorms

$$\|T\|_{u,v} = \|\pi(u)T\pi(v)\|, \quad u, v \in \mathcal{U}(\mathfrak{g}), \quad T \in \mathrm{End}(\pi)^\infty.$$

Here $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra. This makes it a Fréchet space.

2B. ***The open orbit method.*** In this section we will give a brief overview of the open orbit method. The purpose of this method is to study the distinction of induced representations; it is an application of the geometric lemma due to Bernstein and Zelevinsky [1977]. Let G be a connected reductive group defined over F, and $H \subset G$ be a closed subgroup such that $X = H \backslash G$ is a spherical variety of G (i.e., the Borel subgroup has an open orbit). Let $P = MU$ be a parabolic subgroup of G and (τ, V_τ) be an irreducible admissible representation of $M(F)$. We want to study the Hom space $\mathrm{Hom}_{H(F)}(I_P^G(\tau), \chi)$, where χ is some character of $H(F)$. We say $(\pi, V_\pi) = (I_P^G(\tau), I_P^G(V_\tau))$ is (H, χ)-distinguished (or just H-distinguished if χ is trivial) if the Hom space is nonzero. For simplicity, we assume that χ is trivial.

The geometric lemma [Bernstein and Zelevinsky 1977]. *There is an ordering $\{P(F)y_i H(F)\}_{i=1}^N$ on the double coset $H(F) \backslash G(F)/P(F)$ such that*

$$Y_i = \bigcup_{j=1}^{i} P(F)y_i H(F)$$

is open in $G(F)$ for any $1 \le i \le N$.

With the filtration above, for $1 \le i \le N$, define

$$V_i = \{f \in I_P^G(V_\tau) \mid \mathrm{supp}(f) \subset Y_i\}.$$

Then we have $V_1 \subset V_2 \subset \cdots \subset V_N = V_\pi$ and V_i is $H(F)$-invariant for all i. In particular, this implies that if $I_P^G(\tau)$ is H-distinguished, there exists i such that $\mathrm{Hom}_{H(F)}(V_i/V_{i-1}, \chi) \neq 0$ (here $V_0 = \{0\}$). Moreover, for any $1 \leq i \leq N$, it is easy to see that the map

$$f \in V_i \mapsto \phi_f(h) := f(y_i h)$$

is an isomorphism between V_i/V_{i-1} and $\mathrm{ind}_{H_i}^H(\delta_P^{1/2}\tau^{y_i}|_{H_i})$ ($\mathrm{ind}_{H_i}^H$ is the compact induction). Here $H_i = H(F) \cap y_i^{-1} P(F) y_i = y_i^{-1} P_i y_i$, with $P_i = P(F) \cap y_i H(F) y_i^{-1}$. By applying the reciprocity law, we have a necessary condition for $I_P^G(\tau)$ to be H-distinguished.

Proposition 2.2. *If $I_P^G(\tau)$ is H-distinguished, then there exists i such that τ is $(P_i, \delta_{P_i}\delta_P^{1/2})$-distinguished. Here we view τ as a representation of $P(F)$ by making it trivial on $U(F)$.*

What we are interested is the opposite direction of the proposition above. In other words, we want to have some sufficient conditions for $I_P^G(\tau)$ to be H-distinguished in terms of V_i/V_{i-1}. These are known as the open orbit method and the closed orbit method. For our purposes, we only consider the open orbit method.

Assume that τ is $(P_1, \delta_{P_1}\delta_P^{1/2})$-distinguished, we want to show that π is H-distinguished. For simplicity, assume that $H(F)P(F)$ is open in $G(F)$ and $y_1 = 1$. Choose a nonzero element l_0 in the Hom space for τ; it gives a nonzero element l in $\mathrm{Hom}_{H(F)}(V_1, 1)$ by integrating l_0 over $H_1(F)\backslash H(F)$. Then we would like to extend this integral to V_π, which will gives us a nonzero element in $\mathrm{Hom}_H(F)(V_\pi, 1)$. However, the integral will not be absolutely convergent in general; one needs to show that it has holomorphic continuation. In our case, the integral over $H(F)/H_1(F)$ will be some generalized Jacquet integral. In Sections 5 and 6, we will use Proposition 2.1 to show that the integral is absolutely convergent for some π with positive exponents. This will prove Theorem 1.4 and Theorem 1.5. Then in Section 7, we will talk about how to remove the restraints on the exponents by applying R. Gomez's result on the holomorphic continuation of generalized Jacquet integrals.

3. The relation between Conjectures 1.1 and 1.2

The goal of this section is to prove the following proposition:

Proposition 3.1. *If $F = \mathbb{C}$, then Conjecture 1.1 implies Conjecture 1.2.*

Proof. Since $F = \mathbb{C}$, π_D is always 0. Hence Conjecture 1.1 tells us that the multiplicity $m(\pi)$ is always 1. Therefore in order to prove Conjecture 1.2, it is enough to show that the epsilon factor $\epsilon\left(\frac{1}{2}, \pi, \bigwedge^3\right)$ equals 1 for any irreducible generic representations π of $\mathrm{GL}_6(F)$ with trivial central character.

By the Langlands classification, we can find a generic representation $\sigma = \sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_5(F) \times \mathrm{GL}_1(F)$ such that π is the parabolic induction of σ. Let ϕ be the

Langlands parameter of π and ϕ_i be the Langlands parameter of σ_i for $i = 1, 2$. We have $\phi = \phi_1 \oplus \phi_2$. This implies

$$\bigwedge^3(\phi) = \bigwedge^3(\phi_1 \oplus \phi_2) = \bigwedge^3(\phi_1) \oplus (\bigwedge^2(\phi_1) \otimes \phi_2).$$

Since the central character of π is trivial, $\det(\phi) = \det(\phi_1) \otimes \det(\phi_2) = 1$. Therefore $(\bigwedge^3(\phi_1))^\vee = \bigwedge^2(\phi_1) \otimes \det(\phi_1)^{-1} = \bigwedge^2(\phi_1) \otimes \det(\phi_2) = \bigwedge^2(\phi_1) \otimes \phi_2$, hence

$$\epsilon\left(\tfrac{1}{2}, \pi, \bigwedge^3\right) = \det(\bigwedge^3(\phi_1))(-1) = (\det(\phi_1))^6(-1) = 1.$$

This finishes the proof of the proposition. □

4. The tempered case

In this section, we prove our main theorem for the tempered case; the method is very similar to the case $F = \mathbb{R}$ we proved in [Wan 2016b]. Let π be a tempered representation of $G = \mathrm{GL}_6(F)$ with central character χ^2. Our goal is to show that $m(\pi) = 1$. Since we already know that $m(\pi) \leq 1$, it is enough to show that

$$m(\pi) \neq 0. \tag{4-1}$$

For all $T \in \mathrm{End}(\pi)^\infty$, define

$$\mathcal{L}_\pi(T) = \int^*_{Z_H(F)\backslash H(F)} \mathrm{tr}(\pi(h^{-1})T)\omega \otimes \xi(h)\, dh.$$

Here $\int^*_{Z_H(F)\backslash H(F)}$ is the normalized integral defined in Proposition 5.1 of [Wan 2016b]. Note the arguments in the loc. cit. is for the case when $F = \mathbb{R}$, but they also work for $F = \mathbb{C}$. For details, see the proof of Proposition 6.1.1 of [Wan 2017]. By Lemma 5.2 of [Wan 2016b] or Lemma 6.1.2 of [Wan 2017], for any $h, h' \in H(F)$,

$$\mathcal{L}_\pi(\pi(h)T\pi(h')) = \omega \otimes \xi(hh')\mathcal{L}_\pi(T). \tag{4-2}$$

For $e, e' \in \pi$, define $T_{e,e'} \in \mathrm{End}(\pi)^\infty$ by $e_0 \in \pi \mapsto (e_0, e')e$. Set $\mathcal{L}_\pi(e, e') = \mathcal{L}_\pi(T_{e,e'})$. Then

$$\mathcal{L}_\pi(e, e') = \int^*_{Z_H(F)\backslash H(F)} (e, \pi(h)e')\omega \otimes \xi(h)\, dh.$$

If we fix e', by (4-2), the map $e \in \pi \to \mathcal{L}_\pi(e, e')$ belongs to $\mathrm{Hom}_H(\pi, \omega \otimes \xi)$. Since $\mathrm{Span}\{T_{e,e'} \mid e, e' \in \pi\}$ is dense in $\mathrm{End}(\pi)^\infty$, we have $\mathcal{L}_\pi \neq 0 \Rightarrow m(\pi) \neq 0$. Hence in order to show the multiplicity $m(\pi)$ is nonzero, it is enough to show that the operator $\mathcal{L}_\pi$ is nonzero.

Since we are in the complex case, only $\mathrm{GL}_1(F)$ has discrete series; hence π is a principal series. Let $R = M_R U_R$ be a good minimal parabolic subgroup of G in the sense that RH is Zariski open in G. The existence of such R is proved in Proposition 4.2 of [Wan 2016b]. It is also proved in the same proposition that for all such R, we have $H_R := H \cap R = Z_G$. Hence the reduced model associated to R is just (M_R, Z_G). Since π is a principal series, there is a unitary character τ of $M_R(F)$ such that $\pi = I_R^G(\tau)$. For $T_0 \in \mathrm{End}(\tau)^\infty$, define

$$\mathcal{L}_\tau(T_0) = \mathrm{tr}(T_0).$$

By Proposition 5.9 of [Wan 2016b], the nonvanishing property of $\mathcal{L}_\pi$ is invariant under the parabolic induction, hence we have $\mathcal{L}_\pi \neq 0 \iff \mathcal{L}_\tau \neq 0$. Here the arguments in the loc. cit. is for the case when $F = \mathbb{R}$, but they also work for $F = \mathbb{C}$. Since $\mathcal{L}_\tau$ is obviously nonzero, we have $\mathcal{L}_\pi \neq 0$. This proves $m(\pi) \neq 0$ and hence finishes the proof of Theorem 1.4 for tempered representations.

5. The proof of Theorem 1.4

5A. *The case when $\bar{P} \subset Q$.* In this section, we prove the first part of Theorem 1.4. In other words, we assume that $\bar{P} \subset Q$. Then there are four possibilities for Q: type (6), type (4, 2), type (2, 4) or type (2, 2, 2). The idea is to first reduce our problem to the reduced model $(L, H \cap Q)$ by the open orbit method, then reduce it to the tempered case which was considered in the previous section.

If $Q = G$ is of type (6), by twisting π by some characters, we can assume that π is tempered. Note that twisting by characters will not change the multiplicities. Then by applying the result in the last section, we know that $m(\pi) \neq 0$ and this proves Theorem 1.4.

If Q is of type (4, 2), then $L(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ and $H_Q(F) = H(F) \cap Q(F)$ is of the form

$$H_Q(F) = H_0(F) \ltimes U_{0,Q}(F),$$

where

$$U_{0,Q}(F) = \left\{ u = u(X) := \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; X \in M_2(F) \right\}.$$

The restriction of the character ξ on $U_{0,Q}(F)$ is just $\xi(u(X)) = \psi(\mathrm{tr}(X))$ and the character ω on $H_0(F)$ is defined as usual. The model (L, H_Q) is the middle model introduced in [Wan 2016a]; it can be understood as the model between the Ginzburg–Rallis model and the trilinear GL_2 model. By the definition of Q, π is of the form $I_Q^G(\tau_1|\cdot|^{t_1} \otimes \tau_2|\cdot|^{t_2})$, where τ_1, τ_2 are tempered and $t_1 < t_2$. Hence any element $f \in \pi$ is a smooth function $f : G(F) \to \tau = \tau_1|\cdot|^{t_1} \otimes \tau_2|\cdot|^{t_2}$ such that

$$f(lug) = \delta_Q(l)^{1/2}\tau(l)f(g) \tag{5-1}$$

for all $l \in L(F)$, $u \in U_Q(F)$ and $g \in G(F)$. Here we use the letters π, σ, τ to denote both the representations and the underlying vector spaces. Let $\bar{Q} = LU_{\bar{Q}}$ be the opposite parabolic subgroup of Q. It is easy to see that $U_{\bar{Q}} \subset U$ and $U = U_{\bar{Q}} U_{0,Q}$. For any $f \in \pi$, define

$$J_Q(f) = \int_{U_{\bar{Q}}(F)} f(u)\xi^{-1}(u)\,du. \tag{5-2}$$

By Proposition 2.1 together with the assumption that $t_1 < t_2$, the integral above is absolutely convergent.

Proposition 5.1. (1) *For all $f \in \pi$, $u \in U_{\bar{Q}}(F)$ and $l \in H_Q(F)$, we have*

$$J_Q(\pi(u)f) = \xi(u)J(f) \tag{5-3}$$

and

$$J_Q(\pi(l)f) = \tau(l)J(f). \tag{5-4}$$

(2) *The function*

$$J_Q : \pi \to \tau, \quad f \to J_Q(f)$$

is surjective.

Proof. Part (1) follows from (5-1) and changing variables in the integral (5-2). For part (2), fix a function $\varphi \in C_c^\infty(U_{\bar{Q}}(F))$ such that $\int_{U_{\bar{Q}}(F)} \varphi(u)\psi^{-1}(u)\,du = 1$. For any $v \in \tau$, since $Q(F)U_{\bar{Q}}(F)$ is open in $G(F)$, the function

$$f(g) = \begin{cases} \delta_Q(l)^{1/2}\tau(l)\varphi(u)v & \text{if } g = u'lu \text{ with } l \in L(F),\, u \in U_{\bar{Q}}(F),\, u' \in U_Q(F), \\ 0 & \text{else} \end{cases}$$

lies inside π. Then we have

$$J_Q(f) = \int_{U_{\bar{Q}}(F)} f(u)\psi^{-1}(u)\,du = \int_{U_{\bar{Q}}(F)} \varphi(u)\psi^{-1}(u)v\,du = v.$$

This proves (2). $\square$

We consider the Hom space $\mathrm{Hom}_{H_Q(F)}(\tau, (\omega \otimes \xi)|_{H_Q(F)})$ and let $m(\tau)$ be the dimension of that space. The following proposition tells us the relation between $m(\pi)$ and $m(\tau)$:

Proposition 5.2. $$m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0.$$

Proof. If $m(\tau) \neq 0$, choose $0 \neq l_0 \in \mathrm{Hom}_{H_Q(F)}(\tau, (\omega \otimes \xi)|_{H_Q(F)})$. Define an operator l on π to be

$$l(f) = l_0(J_Q(f)).$$

Since $l_0 \neq 0$ and J_Q is surjective, we have $l \neq 0$. Hence we only need to show that $l \in \mathrm{Hom}_{H(F)}(\pi, \omega \otimes \xi)$.

For $h \in H(F)$, we can write $h = h_1 u_1$ with $h_1 \in H_Q(F)$ and $u_1 \in U_{\bar{Q}}(F)$. By (5-3) and (5-4), we have

$$\begin{aligned} l(\pi(h)f) = l_0(J_Q(\pi(h_1u_1)f)) &= l_0(\tau(h_1)J_Q(\pi(u_1)f)) \\ = \omega\otimes\xi(h_1)l_0(J_Q(\pi(u_1)f)) &= \omega\otimes\xi(h_1)l_0(\xi(u_1)J_Q(f)) \\ = \omega\otimes\xi(h)l_0(J_Q(f)) &= \omega\otimes\xi(h)l(f). \end{aligned}$$

This implies $l \in \mathrm{Hom}_{H(F)}(\pi, \omega\otimes\xi)$ and finishes the proof of the proposition. □

By the proposition above, we only need to show that $m(\tau) \neq 0$. It is easy to see that the multiplicity $m(\tau)$ is invariant under the unramified twist, hence we may assume that τ is tempered (note that originally τ is of the form $\tau_1|\cdot|^{t_1} \otimes \tau_2|\cdot|^{t_2}$ with τ_1 and τ_2 being tempered). Then by applying the argument in the previous section to the middle model case, we can show that the multiplicity $m(\tau)$ is always nonzero for all tempered representations τ. This proves Theorem 1.4.

If Q is of type $(2, 4)$, the argument is the same as the $(4, 2)$ case; we skip it here.

If Q is of type $(2, 2, 2)$, the argument is still similar to the $(4, 2)$ case: we first reduce to the trilinear GL_2 model case by the open orbit method. Then after twisting by some characters we only need to consider the tempered case. Finally, by applying the argument in the previous section to the trilinear GL_2 model case, we can show that the multiplicity is nonzero and this proves Theorem 1.4. We skip the details here.

Now the proof of Theorem 1.4(1) is complete.

5B. ***The case when $Q \subsetneq \bar{P}$.*** In this section, we prove part (2) of Theorem 1.4. Recall that in Section 1 we assume that $\pi = I_B^G\big(\bigotimes_{i=1}^6 \chi_i\big)$, where B is the lower Borel subgroup, $\chi_i = \sigma_i|\cdot|^{s_i}$, σ_i are unitary characters, and s_i are real numbers with $s_1 \le s_2 \le \cdots \le s_6$. By the assumption $Q \subsetneq \bar{P}$, we have $s_2 < s_3$ and $s_4 < s_5$. Also as in Section 1, we write $\pi = I_{\bar{P}}^G(\pi_0)$, with $\pi_0 = \pi_1 \otimes \pi_2 \otimes \pi_3$ and π_i be the parabolic induction of $\chi_{2i-1} \otimes \chi_{2i}$. Then π consists of smooth functions $f \to \pi_0$ such that

$$f(mug) = \delta_{\bar{P}}(m)^{1/2}\pi_0(m)f(g) \tag{5-5}$$

for all $m \in M(F)$, $u \in \bar{U}(F)$ and $g \in G(F)$. We still want to apply the open orbit method. For $f \in \pi$, define

$$J(f) = \int_{U(F)} f(ug)\xi^{-1}(u)\,du. \tag{5-6}$$

By Proposition 2.1 together with the assumption on the exponents s_i, the integral above is absolutely convergent. Similarly as in the previous section, we can show

$$m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0. \tag{5-7}$$

Here $m(\pi_0)$ is the multiplicity for the trilinear GL_2 model. In fact, for $0 \neq l_0 \in \mathrm{Hom}_{H_0(F)}(\pi_0, \omega)$. By a similar argument as in Proposition 5.2, we know that

$$l(f) := l_0(J(f))$$

is a nonzero element in $\mathrm{Hom}_{H(F)}(\pi, \omega \otimes \xi)$. This proves (5-7). Now by our assumption on π_0 together with the work by Loke [2001] for the trilinear GL_2 model, we know that $m(\pi_0) \neq 0$. This implies $m(\pi) \neq 0$ and finishes the proof of Theorem 1.4.

Remark 5.3. The assumption $Q \subsetneq \bar{P}$ is only used to make the generalized Jacquet integral $J(f)$ be absolutely convergent. Hence in general, if one can prove the holomorphic continuation of the generalized Jacquet integral $J(f)$, then the assumption $Q \subsetneq \bar{P}$ in Theorem 1.4(2) can be removed. This will be discussed in Section 7.

6. The proof of Theorem 1.5

In this section, by applying the open orbit method to the case when $F = \mathbb{R}$, we prove Theorem 1.5. Let π be an irreducible generic representation of $G(F)$ with central character χ^2. With the notation as in Section 1, there is a parabolic subgroup $Q = LU_Q$ containing the lower Borel subgroup and an essential tempered representation $\tau = \bigotimes_{i=1}^k \tau_i|\cdot|^{s_i}$ of $L(F)$ with τ_i tempered, $s_i \in \mathbb{R}$ and $s_1 < s_2 < \cdots < s_k$ such that $\pi = I_Q^G(\tau)$.

6A. ***The case when $\pi_D = 0$.*** In this section we assume that $\pi_D = 0$. Then by our assumptions in Theorem 1.5, Q is nice. If $Q \subset \bar{P}$, let $\pi_0 = I_{Q\cap M}^M(\tau)$. It is a generic representation of $M(F)$ and we have $\pi = I_{\bar{P}}^G(\pi_0)$. By the same argument as in Section 5B, we can show that

$$m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0, \tag{6-1}$$

where $m(\pi_0)$ is the multiplicity of the trilinear GL_2 model. Since $\pi_D = 0$, the Jacquet–Langlands correspondence of π_0 from $M(F) = (\mathrm{GL}_2(F))^3$ to $(\mathrm{GL}_1(D))^3$ is zero. By applying the result for the trilinear GL_2 model in [Prasad 1990] and [Loke 2001], we have $m(\pi_0) = 1$. Combining with (6-1), we know $m(\pi) \neq 0$. Hence $m(\pi) = 1$, since we already know $m(\pi) \leq 1$. Therefore

$$m(\pi) + m(\pi_D) = m(\pi) = 1.$$

This proves Conjecture 1.1. For Conjecture 1.2, we only need to show that when $\pi_D = 0$, the epsilon factor $\epsilon\left(\frac{1}{2}, \pi, \bigwedge^3\right)$ is always 1. Since $\pi_D = 0$, by the local Jacquet–Langlands correspondence in [Deligne et al. 1984], π_0 is not an essential discrete series (i.e., discrete series twisted by characters), hence at least one of the π_i $(i = 1, 2, 3)$ is a principal series. Therefore we can find a generic representation

$\sigma=\sigma_1\otimes\sigma_2$ of $\mathrm{GL}_5(F)\times\mathrm{GL}_1(F)$ such that π is the parabolic induction of σ. Then by the same argument as in Section 3, we can show that

$$\epsilon\big(\tfrac{1}{2},\pi,\wedge^3\big)=1.$$

This finishes the proof of Conjecture 1.2.

If $\bar{P}\subset Q$, there are only four possibilities for Q: type (6), (4, 2), (2, 4) and (2, 2, 2). If Q is type (6), by twisting π by some characters we can assume that π is tempered. Then both Conjectures 1.1 and 1.2 are proved in [Wan 2016b]. If Q is type (4, 2) or (2, 4), by the same argument as in Section 5A, we can reduce to the middle model case by the open orbit method. Then by twisting some characters, we only need to consider the tempered case which has already been proved in [Wan 2016b]. If Q is type (2, 2, 2), the argument is similar except replacing the middle model by the trilinear GL_2 model.

Now the proof of Theorem 1.5(1) is complete.

6B. ***The case when $\pi_D\neq 0$.*** In this section we assume that $\pi_D\neq 0$. As a result, $\pi=I_{\bar{P}}^{G}(\pi_0)$ is the parabolic induction of some essential discrete series

$$\pi_0=\pi_1|\cdot|^{s_1}\otimes\pi_2|\cdot|^{s_2}\otimes\pi_3|\cdot|^{s_3}$$

of $M(F)$, where the π_i are discrete series of $\mathrm{GL}_2(F)$ and s_i are real numbers. As usual, we assume that $s_1\leq s_2\leq s_3$. We can write π_D in the form $I_{\bar{P}_D}^{G_D}(\pi_{0,D})$, where $\pi_{0,D}=\pi_{1,D}|\cdot|^{s_1}\otimes\pi_{2,D}|\cdot|^{s_2}\otimes\pi_{3,D}|\cdot|^{s_3}$ is the Jacquet–Langlands correspondence of π_0 from $M(F)$ to $M_D(F)$. Let $m(\pi_0)$ (resp. $m(\pi_{0,D})$) be the multiplicity of the trilinear $\mathrm{GL}_2(F)$ (resp. $\mathrm{GL}_1(D)$) model.

Proposition 6.1. *With the notation above, in order to prove Theorem 1.5(2), it is enough to show that*

$$m(\pi_0)\neq 0\Rightarrow m(\pi)\neq 0;\quad m(\pi_{0,D})\neq 0\Rightarrow m(\pi_D)\neq 0. \tag{6-2}$$

Proof. By Prasad's result for the trilinear GL_2 model, we have

$$m(\pi_0)+m(\pi_{0,D})=1. \tag{6-3}$$

Moreover, if we assume the central character of π_0 is trivial on $H_0(F)$, we have

$$m(\pi_0)=1\Longleftrightarrow\epsilon\big(\tfrac{1}{2},\pi_0\big)=1;\quad m(\pi)=0\Longleftrightarrow\epsilon\big(\tfrac{1}{2},\pi_0\big)=-1. \tag{6-4}$$

Combining (6-2) and (6-3), we have $m(\pi)+m(\pi_D)\geq 1$; this proves the first part of Theorem 1.5(2). For the second part, assume that the central character of π is trivial. In Section 6.2 of [Wan 2016b], we proved that

$$\epsilon\big(\tfrac{1}{2},\pi,\wedge^3\big)=\epsilon\big(\tfrac{1}{2},\pi_0\big). \tag{6-5}$$

Now if $\epsilon\left(\frac{1}{2}, \pi, \bigwedge^3\right) = 1$, by (6-5), we have $\epsilon\left(\frac{1}{2}, \pi_0\right) = 1$. Combining with (6-4), we have $m(\pi_0) = 1$, therefore $m(\pi) = 1$ by (6-2). On the other hand, if $m(\pi) = 0$, by (6-2), we have $m(\pi_0) = 0$. Combining with (6-4), we have $\epsilon\left(\frac{1}{2}, \pi_0\right) = -1$, therefore $\epsilon\left(\frac{1}{2}, \pi, \bigwedge^3\right) = -1$ by (6-5). This finishes the proof of Theorem 1.5(2). $\square$

By the proposition above, it is enough to prove (6-2). If $s_1 = s_2 = s_3$, by twisting π by some characters, we may assume that π is tempered (note that the multiplicities for both the Ginzburg–Rallis model and the trilinear GL_2 model are invariant under twisting by characters). Then the relation (6-2) has already been proved in Corollary 5.13 of [Wan 2016b]. In fact, in this case, we even have $m(\pi) = m(\pi_0)$ and $m(\pi_D) = m(\pi_{0,D})$.

If $s_1 < s_2 = s_3$, let $\pi_{2,3}$ be the parabolic induction of $\pi_2 \otimes \pi_3$; it is a tempered representation of $\mathrm{GL}_4(F)$. We also know that π will be the parabolic induction of $\pi' = \pi_1|\cdot|^{s_1} \otimes \pi_{2,3}|\cdot|^{s_2}$. Let $m(\pi')$ be the multiplicity for the middle model. By applying the open orbit method as in Section 5A, we have

$$m(\pi') \neq 0 \Rightarrow m(\pi) \neq 0.$$

Hence in order to prove $m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0$, it is enough to show that $m(\pi_0) \neq 0 \Rightarrow m(\pi') \neq 0$. Again by twisting π' by some characters, we may assume that π' is tempered. Then by Corollary 5.13 of [Wan 2016b], we have $m(\pi_0) = m(\pi')$, which implies $m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0$. The proof of the quaternion version is similar. This proves (6-2).

If $s_1 = s_2 < s_3$, the argument is the same as the case above; we skip it here.

If $s_1 < s_2 < s_3$, (6-2) follows directly from the open orbit method as in Section 5A.

Now the proof of Theorem 1.5(2) is complete.

7. Holomorphic continuation of the generalized Jacquet integral

In the previous sections, we have already seen that the extra conditions on Q in Theorem 1.4(2) and Theorem 1.5(1) can be removed if the generalized Jacquet integral $J(f)$ defined in (5-6) has holomorphic continuation. In this section, we are going to remove the condition on Q based on the following hypothesis:

Hypothesis: The generalized Jacquet integrals have holomorphic continuation for all parabolic subgroups whose unipotent radical is abelian.

The hypothesis has been proved by Gomez and Wallach [2012] for the case when the stabilizer of the unipotent character is compact, and proved by Gomez [$\geq$ 2017] for the general case. The second paper is still in preparation; this is why we write it as a hypothesis.

Let F be $\mathbb{R}$ or $\mathbb{C}$ and π be a generic representation of $\mathrm{GL}_6(F)$ of the form $\pi = I_P^G(\pi_0)$ for some generic representation π_0 of $M(F) = (\mathrm{GL}_2(F))^3$. By the discussion in Section 5B and 6A, we know that in order to prove Theorem 1.4(2) and Theorem 1.5(1) for π, it is enough to show that

$$m(\pi_0) \neq 0 \Rightarrow m(\pi) \neq 0, \tag{7-1}$$

where $m(\pi_0)$ is the multiplicity for the trilinear GL_2 model.

Let $Q_{4,2} = L_{4,2}U_{4,2}$ be the parabolic subgroup of $\mathrm{GL}_6(F)$ containing $\bar{P}$ of type (4, 2), and let $\pi_1 = I_{\bar{P}\cap L_{4,2}}^{L_{4,2}}(\pi_0)$. Then in order to prove (7-1), it is enough to show

$$m(\pi_0) \neq 0 \Rightarrow m(\pi_1) \neq 0, \quad m(\pi_1) \neq 0 \Rightarrow m(\pi) \neq 0, \tag{7-2}$$

where $m(\pi_1)$ is the multiplicity for the middle model defined in Section 5A. Note that the unipotent radicals of $Q_{4,2}$ and $\bar{P} \cap L_{4,2}$ are all abelian. Therefore by the hypothesis, the generalized Jacquet integrals associated to $Q_{4,2}$ and $\bar{P} \cap L_{4,2}$ have holomorphic continuation. This allows us to apply the open orbit method as in Sections 5 and 6, which give the relations in (7-2). This proves (7-1), and finishes the proof of Theorem 1.4(2) and Theorem 1.5(1) without the assumptions on Q.

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