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We provide an explicit construction of star products on $\mathcal{U}(\mathfrak{g})$-module algebras by using the Fedosov approach. This allows us to give a constructive proof to Drinfeld's theorem and to obtain a concrete formula for Drinfeld twists. We prove that the equivalence classes of twists are in one-to-one correspondence with the second Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$. Finally, we show that for Lie algebras with Kähler structure we obtain a strongly positive universal deformation of *-algebras by using a Wick-type deformation. This results in a positive Drinfeld twist.

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1. Introduction

The concept of deformation quantization was defined by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [Bayen et al. 1978a; 1978b] based on Gerstenhaber's theory [1964] of associative deformations of algebra. A formal star product on a symplectic (or Poisson) manifold M is defined as a formal associative deformation $\star$ of the algebra of smooth functions $\mathscr{C}^\infty(M)$ on M. The existence as well as the classification of star products has been studied in many different settings, e.g., in [De Wilde and Lecomte 1983; Fedosov 1986; 1994; 1996; Kontsevich 2003; Nest and Tsygan 1995; Bertelson et al. 1997]; see also the textbooks [Esposito 2015;

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Waldmann 2007] for more details in deformation quantization. Quite parallel to this, Drinfeld introduced the notion of quantum groups and started the deformation of Hopf algebra; see, e.g., the textbooks [Kassel 1995; Chari and Pressley 1994; Etingof and Schiffmann 1998] for a detailed discussion.

It turned out that under certain circumstances one can give simple and fairly explicit formulas for associative deformations of algebras: whenever a Lie algebra $\mathfrak{g}$ acts on an associative algebra $\mathscr{A}$ by derivations, the choice of a *formal Drinfeld twist* $\mathcal{F} \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$ allows one to deform $\mathscr{A}$ by means of a *universal deformation formula*

$$a \star_{\mathcal{F}} b = \mu_{\mathscr{A}}(\mathcal{F} \triangleright (a \otimes b)) \tag{1-1}$$

for $a, b \in \mathscr{A}[[t]]$. Here

$$\mu_{\mathscr{A}} : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$$

is the algebra multiplication and $\triangleright$ is the action of $\mathfrak{g}$ extended to the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ and then to $\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$ acting on $\mathscr{A} \otimes \mathscr{A}$. Finally, all operations are extended $\mathsf{R}[[t]]$-multilinearly to formal power series. Recall that a formal Drinfeld twist [Drinfeld 1983; 1986] is an invertible element

$$\mathcal{F} \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$$

satisfying

$$(\Delta \otimes \mathsf{id})(\mathcal{F})(\mathcal{F} \otimes 1) = (\mathsf{id} \otimes \Delta)(\mathcal{F})(1 \otimes \mathcal{F}), \tag{1-2}$$

$$(\epsilon \otimes 1)\mathcal{F} = 1 = (1 \otimes \epsilon)\mathcal{F}, \tag{1-3}$$

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(t). \tag{1-4}$$

The properties of a twist are now easily seen to guarantee that (1-1) is indeed an associative deformation.

Yielding the explicit formula for the deformation universally in the algebra $\mathscr{A}$, Drinfeld twists are considered to be of great importance in deformation theory in general, and in fact, are used at many different places. We just mention a few recent developments, certainly not exhaustive: Giaquinto and Zhang studied the relevance of universal deformation formulas like (1-1) in great detail in the seminal paper [Giaquinto and Zhang 1998]. Bieliavsky and Gayral [2015] used universal deformation formulas also in a nonformal setting by replacing the notion of a Drinfeld twist with a certain integral kernel. This sophisticated construction leads to a wealth of new strict deformations having the above formal deformations as asymptotic expansions. But also beyond pure mathematics the universal deformation formulas found applications, e.g., in the construction of quantum field theories on noncommutative spacetimes; see, e.g., [Aschieri and Schenkel 2014].

In characteristic zero, there is one fundamental example of a Drinfeld twist in the case of an abelian Lie algebra $\mathfrak{g}$. Here one chooses any bivector $\pi \in \mathfrak{g} \otimes \mathfrak{g}$ and considers the formal exponential

$$\mathcal{F}_{\text{Weyl–Moyal}} = \exp(t\pi), \tag{1-5}$$

viewed as element in $(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$. An easy verification shows that this is indeed a twist. The corresponding universal deformation formula goes back at least till [Gerstenhaber 1968, Theorem 8] under the name of *deformation by commuting derivations*. In deformation quantization the corresponding star product is the famous Weyl–Moyal star product if one takes π to be antisymmetric.

While this is an important example, it is not at all easy to find explicit formulas for twists in the general nonabelian case. A starting point is the observation that the antisymmetric part of the first order of a twist, $\mathcal{F}_1 - \mathsf{T}(\mathcal{F}_1)$, where T is the usual flip isomorphism, is first an element in $\Lambda^2\mathfrak{g}$ instead of $\Lambda^2\mathscr{U}(\mathfrak{g})$, and second a *classical r-matrix*. This raises the question whether one can go the opposite direction of a quantization: does every classical r-matrix $r \in \Lambda^2\mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ arise as the first order term of a formal Drinfeld twist? It is now a celebrated theorem of Drinfeld [1983, Theorem 6] that this is true.

But even more can be said: given a twist $\mathcal{F}$ one can construct a new twist by conjugating with an invertible element $S \in \mathscr{U}(\mathfrak{g})[[t]]$ starting with $S = 1 + \mathcal{O}(t)$ and satisfying $\epsilon(S) = 1$. More precisely,

$$\mathcal{F}' = \Delta(S)^{-1}\mathcal{F}(S \otimes S) \tag{1-6}$$

turns out to be again a twist. In fact, this defines an equivalence relation on the set of twists, preserving the semiclassical limit, i.e., the induced r-matrix. In the spirit of Kontsevich's formality theorem, and in fact building on its techniques, Halbout [2006] showed that the equivalence classes of twists quantizing a given classical r-matrix are in bijection with the equivalence classes of formal deformations of the r-matrix in the sense of r-matrices. In fact, this follows from Halbout's more profound result on formality for general Lie bialgebras; the quantization of r-matrices into twists is just a special case thereof. His theorem holds in a purely algebraic setting (in characteristic zero) but relies heavily on the fairly inexplicit formality theorems of Kontsevich [2003] and Tamarkin [1998] which in turn require a rational Drinfeld associator.

On the other hand, there is a simpler approach to the existence of twists in the case of real Lie algebras: in seminal work of Drinfeld [1983] he showed that a twist is essentially the same as a left G-invariant star product on a Lie group G with Lie algebra $\mathfrak{g}$, by identifying the G-invariant bidifferential operators on G with elements in $\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$. The associativity of the star product gives then immediately the properties necessary for a twist and vice versa. Moreover, an r-matrix is nothing else as a left G-invariant Poisson structure; see his Theorem 1. In that paper, Drinfeld also gives an existence proof of such G-invariant star products and therefore of twists; see Theorem 6. His argument uses the canonical star product on the dual of a central extension of the Lie algebra by the cocycle defined by the (inverse of the) r-matrix, suitably pulled back to the Lie group; see also Remark 5.8 for further details.

The equivalence of twists translates into the usual G-invariant equivalence of star products as discussed in [Bertelson et al. 1998]. Hence one can use the existence (and classification) theorems for invariant star products to yield the corresponding theorems for twists, a fact we learned from personal communication with Beliavsky. This is also the point of view taken by Dolgushev et al. in [Dolgushev et al. 2002], where the star product is constructed in a way inspired by Fedosov's construction of star products on symplectic manifolds.

A significant simplification concerning the existence comes from the observation that for every r-matrix $r \in \Lambda^2\mathfrak{g}$ there is a Lie subalgebra of $\mathfrak{g}$, namely

$$\mathfrak{g}_r = \{(\alpha \otimes \mathsf{id})(r) \mid \alpha \in \mathfrak{g}^*\}, \tag{1-7}$$

such that $r \in \Lambda^2\mathfrak{g}_r$ and r becomes *nondegenerate* as an r-matrix on this Lie subalgebra [Etingof and Schiffmann 1998, Propositions 3.2–3.3]. Thus it will always be sufficient to consider nondegenerate classical r-matrices when interested in the existence of twists. For the classification this is of course not true since a possibly degenerate r-matrix might be deformed into a nondegenerate one only in higher orders: here one needs Halbout's results for possibly degenerate r-matrices. However, starting with a nondegenerate r-matrix, one will have a much simpler classification scheme as well.

The aim of this paper is now twofold: On the one hand, we want to give a direct construction to obtain the universal deformation formulas for algebras acted upon by a Lie algebra with nondegenerate r-matrix. This will be obtained in a purely algebraic fashion for sufficiently nice Lie algebras and algebras over a commutative ring R containing the rationals. Our approach is based on a certain adaptation of the Fedosov construction of symplectic star products, which is in some sense closer to the original Fedosov construction compared to the approach of [Dolgushev et al. 2002] but yet completely algebraic. More precisely, the construction will not involve a twist at all but just the classical r-matrix. Moreover, it will be important to note that we can allow for a nontrivial symmetric part of the r-matrix, provided a certain technical condition on it is satisfied. This will produce deformations with more specific features: as in usual deformation quantization one is not only interested in the Weyl–Moyal like star products, but certain geometric circumstances require more particular star products like Wick-type star products on Kähler manifolds [Karabegov 1996; 2013; Bordemann and Waldmann 1997] or standard-ordered star products on cotangent bundles [Bordemann et al. 1998; 2003].

On the other hand, we give an alternative construction of Drinfeld twists, again in the purely algebraic setting, based on the above correspondence to star products but avoiding the techniques from differential geometry completely in order to be able to work over a general field of characteristic zero. We also obtain a classification of the above restricted situation where the r-matrix is nondegenerate.

In fact, both questions turn out to be intimately linked since applying our universal deformation formula to the tensor algebra of $\mathscr{U}(\mathfrak{g})$ will yield a deformation of the tensor product which easily allows one to construct the twist. This is remarkable insofar as the tensor algebra is of course rigid, the deformation is equivalent to the undeformed tensor product, but the deformation is not the identity, allowing one therefore to consider nontrivial products of elements in $\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$.

We show that the universal deformation formula we construct in fact coincides with (1-1) for the twist we construct. However, it is important to note the detour via the twist is not needed to obtain the universal deformation of an associative algebra.

Finally, we add the notion of positivity: this seems to be new in the whole discussion of Drinfeld twists and universal deformation formulas so far. To this end we consider now an ordered ring R containing $\mathbb{Q}$ and its complex version $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathrm{i}^2 = -1$, and *-algebras over C with a *-action of the Lie algebra $\mathfrak{g}$, which is assumed to be a Lie algebra over R admitting a Kähler structure. Together with the nondegenerate r-matrix we can define a Wick-type universal deformation which we show to be *strongly positive*: every undeformed positive linear functional stays positive also for the deformation. Applied to the twist we conclude that the Wick-type twist is a convex series of positive elements.

The paper is organized as follows. In Section 2 we explain the elements of the (much more general) Fedosov construction which we will need. Section 3 contains the construction of the universal deformation formula. Here not only the deformation formula will be universal for all algebras $\mathscr{A}$ but also the construction itself will be universal for all Lie algebras $\mathfrak{g}$. In Section 4 we construct the Drinfeld twist while Section 5 contains the classification in the nondegenerate case. Finally, Section 6 discusses the positivity of the Wick-type universal deformation formula. In two Appendices we collect some more technical arguments and proofs. The results of this paper are partially based on the master thesis [Schnitzer 2016].

For symplectic manifolds with suitable polarizations one can define various types of star products with separation of variables [Karabegov 1996; 2013; Bordemann and Waldmann 1997; Donin 2003; Bordemann et al. 1998; 1999; 2003] which have specific properties adapted to the polarization. The general way to construct (and classify) them is to modify the Fedosov construction by adding suitable symmetric terms to the fiberwise symplectic Poisson tensor. We have outlined that this can be done for twists as well in the Kähler case, but there remain many interesting situations. In particular a more cotangent bundle-like polarization might be useful. We plan to come back to these questions in a future project.

2. The Fedosov Setup

In the following we present the Fedosov approach in the particular case of a Lie algebra $\mathfrak{g}$ with a nondegenerate r-matrix r. We follow the presentation of the Fedosov approach given in [Waldmann 2007] but replace differential geometric concepts by algebraic versions in order to be able to treat not only the real case. The setting for this work will be to assume that $\mathfrak{g}$ is a Lie algebra over a commutative ring R containing the rationals $\mathbb{Q} \subseteq \mathsf{R}$ such that $\mathfrak{g}$ is a finite-dimensional free module.

We denote by $\{e_1, \dots, e_n\}$ a basis of $\mathfrak{g}$ and by $\{e^1, \dots, e^n\}$ its dual basis of $\mathfrak{g}^*$. We also assume the r-matrix $r \in \Lambda^2\mathfrak{g}$ to be nondegenerate in the strong sense from the beginning, since, at least in the case of R being a field, we can replace $\mathfrak{g}$ by $\mathfrak{g}_r$ from (1-7) if necessary. Hence r induces the *musical isomorphism*

$$\sharp : \mathfrak{g}^* \to \mathfrak{g} \tag{2-1}$$

by pairing with r, the inverse of which we denote by $\flat$ as usual. Then the defining property of an r-matrix is $[\![r, r]\!] = 0$, where $[\![\,\cdot\,,\cdot\,]\!]$ is the unique extension of the Lie bracket to $\Lambda^\bullet\mathfrak{g}$ turning the Grassmann algebra into a Gerstenhaber algebra. Since we assume r to be (strongly) nondegenerate we have the inverse $\omega \in \Lambda^2\mathfrak{g}^*$ of r and $[\![r, r]\!] = 0$ becomes equivalent to the linear condition $\delta_{\mathrm{CE}}\omega = 0$, where δ_{CE} is the usual Chevalley–Eilenberg differential. Moreover, the musical isomorphisms intertwine δ_{CE} on $\Lambda^\bullet\mathfrak{g}^*$ with the differential $[\![r, \cdot\,]\!]$ on $\Lambda^\bullet\mathfrak{g}$. We refer to ω as the induced symplectic form.

Remark 2.1. For the Lie algebra $\mathfrak{g}$ there seems to be little gain in allowing a ring R instead of a field $\mathbb{K}$ of characteristic zero, as we have to require $\mathfrak{g}$ to be a free module and (2-1) to be an isomorphism. However, for the algebras which we would like to deform there will be no such restrictions later on. Hence allowing for algebras over rings in the beginning seems to be the cleaner way to do it, since after the deformation we will arrive at an algebra over a ring, namely $\mathsf{R}[\![t]\!]$ anyway.

Definition 2.2 (Formal Weyl algebra). The algebra $\big(\prod_{k=0}^{\infty} \mathrm{S}^k\mathfrak{g}^* \otimes \Lambda^\bullet\mathfrak{g}^*\big)[\![t]\!]$ is called the formal Weyl algebra where the product μ is defined by

$$(f \otimes \alpha) \cdot (g \otimes \beta) = \mu(f \otimes \alpha, g \otimes \beta) = f \vee g \otimes \alpha \wedge \beta. \tag{2-2}$$

for any factorizing tensors $f \otimes \alpha, g \otimes \beta \in \mathcal{W} \otimes \Lambda^\bullet$ and extended $\mathsf{R}[\![t]\!]$-bilinearly. We write $\mathcal{W} = \prod_{k=0}^{\infty} \mathrm{S}^k\mathfrak{g}^*[\![t]\!]$ and $\Lambda^\bullet = \Lambda^\bullet\mathfrak{g}^*[\![t]\!]$.

Since $\mathfrak{g}$ is assumed to be finite-dimensional we have

$$\mathcal{W} \otimes \Lambda^\bullet = \left(\prod_{k=0}^{\infty} \mathrm{S}^k\mathfrak{g}^* \otimes \Lambda^\bullet\mathfrak{g}^*\right)[\![t]\!]. \tag{2-3}$$

Since we will deform this product μ we shall refer to μ also as the *undeformed product* of $\mathcal{W} \otimes \Lambda^\bullet$. It is clear that μ is associative and graded commutative with respect to the antisymmetric degree. In order to handle this and various other degrees, it is useful to introduce the degree maps

$$\deg_{\mathrm{s}}, \deg_{\mathrm{a}}, \deg_t : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^\bullet, \tag{2-4}$$

defined by the conditions

$$\deg_{\mathrm{s}}(f \otimes \alpha) = kf \otimes \alpha \quad \text{and} \quad \deg_{\mathrm{a}}(f \otimes \alpha) = \ell f \otimes \alpha \tag{2-5}$$

for $f \in \mathrm{S}^k\mathfrak{g}^*$ and $\alpha \in \Lambda^\ell\mathfrak{g}^*$. We extend these maps to formal power series by $\mathsf{R}[[t]]$-linearity. Then we can define the degree map $\deg_t$ by

$$\deg_t = t\frac{\partial}{\partial t}, \tag{2-6}$$

which is, however, not $\mathsf{R}[[t]]$-linear. Finally, the *total degree* is defined by

$$\mathrm{Deg} = \deg_{\mathrm{s}} + 2\deg_t. \tag{2-7}$$

It will be important that all these maps are derivations of the undeformed product μ of $\mathcal{W} \otimes \Lambda^\bullet$. We denote by

$$\mathcal{W}_k \otimes \Lambda^\bullet = \bigcup_{r \geq k} \{a \in \mathcal{W} \otimes \Lambda^\bullet \mid \mathrm{Deg}\, a = ra\} \tag{2-8}$$

the subspace of elements which have total degree bigger or equal to $+k$. This endows $\mathcal{W} \otimes \Lambda^\bullet$ with a complete filtration, a fact which we shall frequently use in the sequel. Moreover, the filtration is compatible with the undeformed product (2-2) in the sense that

$$ab \in \mathcal{W}_{k+\ell} \otimes \Lambda^\bullet \quad \text{for } a \in \mathcal{W}_k \otimes \Lambda^\bullet \text{ and } b \in \mathcal{W}_\ell \otimes \Lambda^\bullet. \tag{2-9}$$

Following the construction of Fedosov, we define the operators δ and δ^* by

$$\delta = e^i \wedge \mathrm{i}_{\mathrm{s}}(e_i) \quad \text{and} \quad \delta^* = e^i \vee \mathrm{i}_{\mathrm{a}}(e_i), \tag{2-10}$$

where i_{s} and i_{a} are the symmetric and antisymmetric insertion derivations. Both maps are graded derivations of μ with respect to the antisymmetric degree: δ lowers the symmetric degree by one and raises the antisymmetric degree by one; for δ^* it is the other way round. For homogeneous elements $a \in \mathrm{S}^k\mathfrak{g}^* \otimes \Lambda^\ell\mathfrak{g}^*$ we define

$$\delta^{-1}(a) = \begin{cases} 0 & \text{if } k + \ell = 0, \\ 1/(k+\ell)\delta^*(a) & \text{else,} \end{cases} \tag{2-11}$$

and extend this $\mathsf{R}[[t]]$-linearly. Notice that this map is not the inverse of δ; instead we have the following properties:

Lemma 2.3. *For δ, δ^* and δ^{-1} defined above, $\delta^2 = (\delta^*)^2 = (\delta^{-1})^2 = 0$ and*

$$\delta\delta^{-1} + \delta^{-1}\delta + \sigma = \mathsf{id}, \tag{2-12}$$

where σ is the projection on the symmetric and antisymmetric degree zero.

In fact, this can be seen as the polynomial version of the Poincaré lemma: δ corresponds to the exterior derivative and δ^{-1} is the standard homotopy.

The next step consists of deforming the product μ into a noncommutative one: we define the *star product* $\circ_\pi$ for $a, b \in \mathcal{W} \otimes \Lambda^\bullet$ by

$$a \circ_\pi b = \mu \circ \mathrm{e}^{(t/2)\mathcal{P}}(a \otimes b), \quad \text{where} \quad \mathcal{P} = \pi^{ij}\, \mathrm{i}_{\mathrm{s}}(e_i) \otimes \mathrm{i}_{\mathrm{s}}(e_j), \tag{2-13}$$

for $\pi^{ij} = r^{ij} + s^{ij}$, where r^{ij} are the coefficients of the r-matrix and $s^{ij} = s(e^i, e^j) \in \mathsf{R}$ are the coefficients of a *symmetric* bivector $s \in \mathrm{S}^2\mathfrak{g}$. When taking $s = 0$ we denote $\circ_\pi$ simply by $\circ_{\mathrm{Weyl}}$.

Proposition 2.4. *The star product $\circ_\pi$ is an associative $\mathsf{R}[[t]]$-bilinear product on $\mathcal{W} \otimes \Lambda^\bullet$ deforming μ in zeroth order of t. Moreover, the maps δ, $\deg_{\mathrm{a}}$, and Deg are graded derivations of $\circ_\pi$ of antisymmetric degree $+1$ for δ and 0 for $\deg_{\mathrm{a}}$ and Deg, respectively.*

Proof. The associativity follows from the fact that the insertion derivations are commuting; see [Gerstenhaber 1968, Theorem 8]. The statement about δ, $\deg_{\mathrm{a}}$ and Deg are immediate verifications. □

Next, we will need the graded commutator with respect to the antisymmetric degree, denoted by

$$\mathrm{ad}(a)(b) = [a, b] = a \circ_\pi b - (-1)^{k\ell} b \circ_\pi a \tag{2-14}$$

for any $a \in \mathcal{W} \otimes \Lambda^k$ and $b \in \mathcal{W} \otimes \Lambda^\ell$ and extended $\mathbb{K}[[t]]$-bilinearly as usual. Since $\circ_\pi$ deforms the graded commutative product μ, all graded commutators $[a, b]$ will vanish in the zeroth order of t. This allows one to define graded derivations $(1/t)\,\mathrm{ad}(a)$ of $\circ_\pi$.

Lemma 2.5. *An element $a \in \mathcal{W} \otimes \Lambda^\bullet$ is central, that is $\mathrm{ad}(a) = 0$, if and only if $\deg_{\mathrm{s}}(a) = 0$.*

By definition, a *covariant derivative* is an arbitrary bilinear map

$$\nabla : \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{g}. \tag{2-15}$$

The idea is that in the geometric interpretation the covariant derivative is uniquely determined by its values on the left invariant vector fields: we want an invariant covariant derivative and hence it should take values again in $\mathfrak{g}$. An arbitrary covariant derivative is called *torsion-free* if

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{2-16}$$

for all $X, Y \in \mathfrak{g}$. Having a covariant derivative, we can extend it to the tensor algebra over $\mathfrak{g}$ by requiring the maps

$$\nabla_X : \mathrm{T}^\bullet\mathfrak{g} \to \mathrm{T}^\bullet\mathfrak{g} \tag{2-17}$$

to be derivations for all $X \in \mathfrak{g}$. We also extend ∇_X to elements in the dual by

$$(\nabla_X\alpha)(Y) = -\alpha(\nabla_X Y) \tag{2-18}$$

for all $X, Y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. Finally, we can extend ∇_X to $\mathrm{T}^\bullet\mathfrak{g}^*$ as a derivation, too. Acting on symmetric or antisymmetric tensors, ∇_X will preserve the symmetry type and yields a derivation of the $\vee$- and $\wedge$-products, respectively. The fact that we extended ∇ as a derivation in a way which is compatible with natural pairings will lead to relations like

$$[\nabla_X, \mathrm{i_s}(Y)] = \mathrm{i_s}(\nabla_X Y) \tag{2-19}$$

for all $X, Y \in \mathfrak{g}$, as one can easily check on generators.

Sometimes it will be advantageous to use the basis of $\mathfrak{g}$ for computations. With respect to the basis we define the *Christoffel symbols*

$$\Gamma_{ij}^k = e^k(\nabla_{e_i} e_j) \tag{2-20}$$

of a covariant derivative, where $i, j, k = 1, \dots, n$. Clearly, ∇ is uniquely determined by its Christoffel symbols. Moreover, ∇ is torsion-free if and only if

$$\Gamma_{ij}^k - \Gamma_{ji}^k = C_{ij}^k, \tag{2-21}$$

with the usual structure constants $C_{ij}^k = e^k([e_i, e_j]) \in \mathsf{R}$ of the Lie algebra $\mathfrak{g}$.

As in symplectic geometry, the Hess trick [1981] shows the existence of a *symplectic* torsion-free covariant derivative:

Proposition 2.6 (Hess trick). *Let $(\mathfrak{g}, r)$ be a Lie algebra with nondegenerate r-matrix r and inverse ω. Then there exists a torsion-free covariant derivative ∇ such that for all $X \in \mathfrak{g}$ we have*

$$\nabla_X\omega = 0 \quad \textit{and} \quad \nabla_X r = 0. \tag{2-22}$$

Proof. The idea is to start with the half-commutator connection as in the geometric case and make it symplectic by means of the Hess trick. The covariant derivative

$$\tilde{\nabla} : \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto \tfrac{1}{2}[X, Y] \in \mathfrak{g}$$

is clearly torsion-free. Since ω is nondegenerate, we can determine a map ∇_X uniquely by

$$\omega(\nabla_X Y, Z) = \omega(\tilde{\nabla}_X Y, Z) + \tfrac{1}{3}(\tilde{\nabla}_X\omega)(Y, Z) + \tfrac{1}{3}(\tilde{\nabla}_Y\omega)(X, Z). \tag{2-23}$$

It is then an immediate computation using the closedness $\delta_{\mathrm{CE}}\omega = 0$ of ω, that this map satisfies all requirements. $\square$

The curvature $\tilde{R}$ corresponding to ∇ is defined by

$$\tilde{R} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \ni (X, Y, Z) \mapsto \tilde{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \in \mathfrak{g}. \tag{2-24}$$

For a symplectic covariant derivative, we contract $\tilde{R}$ with the symplectic form ω and get

$$R : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \ni (Z, U, X, Y) \mapsto \omega(Z, \tilde{R}(X, Y)U) \in \mathsf{R}, \tag{2-25}$$

which is symmetric in the first two components and antisymmetric in the last ones: this follows at once from ∇ being torsion-free and symplectic. In other words, $R \in \mathrm{S}^2(\mathfrak{g}^*) \otimes \Lambda^2 \mathfrak{g}^*$ becomes an element of the formal Weyl algebra satisfying

$$\deg_{\mathrm{s}} R = 2R = \operatorname{Deg} R, \quad \deg_{\mathrm{a}} R = 2R, \quad \text{and} \quad \deg_t R = 0. \tag{2-26}$$

In the following, we will fix a symplectic torsion-free covariant derivative, the existence of which is granted by Proposition 2.6. Since ∇_X acts on all types of tensors already, we can use ∇ to define the following derivation D on the formal Weyl algebra

$$D : \mathcal{W} \otimes \Lambda^\bullet \ni (f \otimes \alpha) \mapsto \nabla_{e_i} f \otimes e^i \wedge \alpha + f \otimes e^i \wedge \nabla_{e_i} \alpha \in \mathcal{W} \otimes \Lambda^{\bullet+1}. \tag{2-27}$$

Notice that we do not use the explicit expression of ∇ given in (2-23). In fact, any other symplectic torsion-free covariant derivative will do the job as well.

For every torsion-free covariant derivative ∇ it is easy to check that

$$e^i \wedge \nabla_{e_i} \alpha = \delta_{\mathrm{CE}} \alpha \tag{2-28}$$

holds for all $\alpha \in \Lambda^\bullet \mathfrak{g}^*$: indeed, both sides define graded derivations of antisymmetric degree $+1$ and coincide on generators in $\mathfrak{g}^* \subseteq \Lambda^\bullet \mathfrak{g}^*$. Therefore, we can rewrite D as

$$D(f \otimes \alpha) = \nabla_{e_i} f \otimes e^i \wedge \alpha + f \otimes \delta_{\mathrm{CE}} \alpha. \tag{2-29}$$

From now on, unless clearly stated, we refer to $[\,\cdot\,,\,\cdot\,]$ as the supercommutator with respect to the antisymmetric degree.

Proposition 2.7. *Let ∇ be a symplectic torsion-free covariant derivative. If in addition s is covariantly constant, i.e., if $\nabla_X s = 0$ for all $X \in \mathfrak{g}$, the map $D : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ is a graded derivation of antisymmetric degree $+1$ of the star product $\circ_\pi$, i.e.,*

$$D(a \circ_\pi b) = D(a) \circ_\pi b + (-1)^k a \circ_\pi D(b) \tag{2-30}$$

for $a \in \mathcal{W} \otimes \Lambda^k$ and $b \in \mathcal{W} \otimes \Lambda^\bullet$. In addition, we have

$$\delta R = 0, \quad DR = 0, \quad [\delta, D] = \delta D + D\delta = 0, \quad D^2 = \frac{1}{2}[D, D] = \frac{1}{t}\operatorname{ad}(R). \tag{2-31}$$

Proof. For the operator $\mathcal{P}$ from (2-13) we have

$$
\begin{aligned}
(\mathsf{id} \otimes \nabla_{e_k} + \nabla_{e_k} \otimes \mathsf{id})\mathcal{P}(a \otimes b)
&= \pi^{ij}\, \mathrm{i_s}(e_i)a \otimes \nabla_{e_k} \mathrm{i_s}(e_j)b + \pi^{ij} \nabla_{e_k} \mathrm{i_s}(e_i)a \otimes \mathrm{i_s}(e_j)b \\
&\overset{(a)}{=} (\pi^{\ell j}\Gamma^i_{k\ell} + \pi^{i\ell}\Gamma^j_{k\ell})\, \mathrm{i_s}(e_i)a \otimes \mathrm{i_s}(e_j)b + \mathcal{P}(\mathsf{id} \otimes \nabla_{e_k} + \nabla_{e_k} \otimes \mathsf{id})(a \otimes b) \\
&= \mathcal{P}(\mathsf{id} \otimes \nabla_{e_k} + \nabla_{e_k} \otimes \mathsf{id})(a \otimes b)
\end{aligned}
$$

for $a, b \in \mathcal{W} \otimes \Lambda^\bullet$. Here we used the relation $[\nabla_X, \mathrm{i_s}(Y)] = \mathrm{i_s}(\nabla_X Y)$ as well as the definition of the Christoffel symbols in (a). In the last step we used $\pi^{\ell j}\Gamma^i_{k\ell} + \pi^{i\ell}\Gamma^j_{k\ell} = 0$ which follows from $\nabla(r+s) = 0$. Therefore we have

$$
\nabla_{e_i} \circ \mu \circ \mathrm{e}^{\frac{1}{2}t\mathcal{P}} = \mu \circ (\mathsf{id} \otimes \nabla_{e_i} + \nabla_{e_i} \otimes \mathsf{id}) \circ \mathrm{e}^{\frac{1}{2}t\mathcal{P}} = \mu \circ \mathrm{e}^{\frac{1}{2}t\mathcal{P}} \circ (\mathsf{id} \otimes \nabla_{e_i} + \nabla_{e_i} \otimes \mathsf{id}).
$$

By $\wedge$-multiplying by the corresponding e^i it follows that D is a graded derivation of antisymmetric degree $+1$. Let $f \otimes \alpha \in \mathcal{W} \otimes \Lambda^\bullet$. Just using the definition of δ, (2-29) and the fact that ∇ is torsion-free we get

$$
\begin{aligned}
\delta D(f \otimes \alpha) &= \delta(\nabla_{e_k} f \otimes e^k \wedge \alpha + f \otimes \delta_{\mathrm{CE}}\alpha) \\
&= -D\delta(f \otimes \alpha) + \tfrac{1}{2}(\Gamma^\ell_{ik} - \Gamma^\ell_{ki} - C^\ell_{ik})\, \mathrm{i_s}(e_\ell) f \otimes e^i \wedge e^k \wedge \alpha \\
&= -D\delta(f \otimes \alpha).
\end{aligned}
$$

Using a similar computation in coordinates, we get $D^2 = \frac{1}{2}[D, D] = (1/t)\,\mathrm{ad}(R)$. Finally, from the Jacobi identity of the graded commutator we get $(1/2t)\,\mathrm{ad}(\delta R) = [\delta, [D, D]] = 0$. Hence δR is central. Since δR has symmetric degree $+1$, this can only happen if $\delta R = 0$. With the same argument, $0 = [D, [D, D]]$ yields that DR is central, which again gives $DR = 0$ by counting degrees. □

Remark 2.8. In principle, we will mainly be interested in the case $s = 0$ in the following. However, if the Lie algebra allows for a covariantly constant s it might be interesting to incorporate this into the universal construction: already in the abelian case this leads to the freedom of choosing a different ordering than the Weyl ordering (total symmetrization). Here in particular the Wick ordering is of significance due to the better positivity properties; see [Bursztyn and Waldmann 2000] for a universal deformation formula in this context.

The core of Fedosov's construction is now to turn $-\delta + D$ into a differential: due to the curvature R the derivation $-\delta + D$ is not a differential directly. Nevertheless, from the above discussion we know that it is an inner derivation. Hence the idea is to compensate the defect of being a differential by inner derivations, leading to the following statement:

Proposition 2.9. *Let $\Omega \in t\Lambda^2\mathfrak{g}^*[[t]]$ be a series of δ_{CE}-closed two-forms. Then there is a unique $\varrho \in \mathcal{W}_2 \otimes \Lambda^1$, such that*

$$\delta\varrho = R + D\varrho + \tfrac{1}{t}\varrho \circ_\pi \varrho + \Omega \tag{2-32}$$

and

$$\delta^{-1}\varrho = 0. \tag{2-33}$$

Moreover, the derivation $\mathscr{D}_{\mathrm{F}} = -\delta + D + (1/t)\,\mathrm{ad}(\varrho)$ satisfies $\mathscr{D}_{\mathrm{F}}^2 = 0$.

Proof. Let us first assume that (2-32) is satisfied and apply δ^{-1} to (2-33). This yields

$$\delta^{-1}\delta\varrho = \delta^{-1}\Big(R + Dx + \frac{1}{t}\varrho \circ_\pi \varrho + \Omega\Big).$$

From the Poincaré Lemma as in Lemma 2.3 we have

$$\varrho = \delta^{-1}\Big(R + D\varrho + \frac{1}{t}\varrho \circ_\pi \varrho + \Omega\Big). \tag{2-34}$$

Let us define the operator $B : \mathcal{W} \otimes \Lambda^1 \to \mathcal{W} \otimes \Lambda^1$ by

$$B(a) = \delta^{-1}\Big(R + Da + \frac{1}{t}a \circ_\pi a + \Omega\Big).$$

Thus the solutions of (2-33) coincide with the fixed points of the operator B. Now we want to show that B has indeed a unique fixed point. By a careful but straightforward counting of degrees we see that B maps $\mathcal{W}_2 \otimes \Lambda^1$ into $\mathcal{W}_2 \otimes \Lambda^1$. Second, we note that B is a contraction with respect to the total degree. Indeed, for $a, a' \in \mathcal{W}_2 \otimes \Lambda^1$ with $a - a' \in \mathcal{W}_k \otimes \Lambda^1$ we have

$$\begin{aligned} B(a) - B(a') &= \delta^{-1}D(a-a') + \frac{1}{t}(a \circ_\pi a - a' \circ_\pi a') \\ &= \delta^{-1}D(a-a') + \frac{1}{t}\delta^{-1}((a-a') \circ_\pi a' + a \circ_\pi (a-a')). \end{aligned}$$

The first term $\delta^{-1}D(a-a')$ is an element of $\mathcal{W}_{k+1} \otimes \Lambda^1$, because D does not change the total degree and δ^{-1} increases it by $+1$. Since Deg is a $\circ_\pi$-derivation and since a, a' have total degree at least 2 and their difference has total degree at least k, the second term has total degree at least $k+1$, as $1/t$ has total degree -2 but δ^{-1} raises the total degree by $+1$. This allows one to apply the Banach fixed-point theorem for the complete filtration by the total degree: we have a unique fixed-point $B(\varrho) = \varrho$ with $\varrho \in \mathcal{W}_2 \otimes \Lambda^1$, i.e., ϱ satisfies (2-34). Finally, we show that this ϱ fulfills (2-33). Define

$$A = \delta\varrho - R - D\varrho - \frac{1}{t}\varrho \circ_\pi \varrho - \Omega.$$

Applying δ to A and using Proposition 2.7, we obtain

$$\begin{aligned}
\delta A &= -\delta D\varrho - \frac{1}{t}(\delta\varrho \circ_\pi \varrho - \varrho \circ_\pi \delta\varrho) \\
&= D\delta\varrho + \frac{1}{t}\operatorname{ad}(\varrho)\delta\varrho \\
&= D\Big(A + R + D\varrho + \frac{1}{t}\varrho \circ_\pi \varrho + \Omega\Big) + \frac{1}{t}\operatorname{ad}(\varrho)\Big(A + R + D\varrho + \frac{1}{t}\varrho \circ_\pi \varrho + \Omega\Big) \\
&\overset{(a)}{=} DA + \frac{1}{t}\operatorname{ad}(\varrho)(A).
\end{aligned}$$

In (a) we used that $(-\delta + D + (1/t)\operatorname{ad}(\varrho))(R + D\varrho + (1/t)\varrho \circ_\pi \varrho + \Omega) = 0$, which can be seen as a version of the second Bianchi identity for $-\delta + D + (1/t)\operatorname{ad}(\varrho)$. This follows by an explicit computation for arbitrary ϱ. On the other hand

$$\delta^{-1}A = \delta^{-1}\Big(\delta\varrho - R - D\varrho - \frac{1}{t}\varrho \circ_\pi \varrho - \Omega\Big) = \delta^{-1}\delta\varrho - \varrho = \delta\delta^{-1}\varrho = 0$$

for ϱ being the fixed-point of the operator B. In other words,

$$A = \delta^{-1}\delta A = \delta^{-1}\Big(DA + \frac{1}{t}\operatorname{ad}(\varrho)(A)\Big)$$

is a fixed-point of the operator $K : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^\bullet$ defined by

$$Ka = \delta^{-1}\Big(Da + \frac{1}{t}\operatorname{ad}(\varrho)(a)\Big).$$

Using an analogous argument to the one above, this operator is a contraction with respect to the total degree, and has a unique fixed-point. Finally, since K is linear the fixed-point has to be zero, which means that $A = 0$. □

Remark 2.10. It is important to note that the above construction of the element ϱ, which will be the crucial ingredient in the universal deformation formula below, is a fairly explicit recursion formula. Writing $\varrho = \sum_{r=3}^{\infty} \varrho^{(r)}$ with components $\varrho^{(r)}$ of homogeneous total degree $\operatorname{Deg}\varrho^{(r)} = r\varrho^{(r)}$ we see that $\varrho^{(3)} = \delta^{-1}(R + t\Omega_1)$ and

$$\varrho^{(r+3)} = \delta^{-1}\bigg(D\varrho^{(r+2)} + \frac{1}{t}\sum_{\ell=1}^{r-1} \varrho^{(\ell+2)} \circ_\pi \varrho^{(r+2-\ell)} + \Omega^{(r+2)}\bigg), \tag{2-35}$$

where $\Omega^{(2k)} = t^k\Omega_k$ for $k \in \mathbb{N}$ and $\Omega^{(2k+1)} = 0$. Moreover, if we find a *flat* ∇, i.e., if $R = 0$, then for trivial $\Omega = 0$ we have $\varrho = 0$ as solution.

3. Universal deformation formula

Let us consider a triangular Lie algebra $(\mathfrak{g}, r)$ acting on a generic associative algebra $(\mathscr{A}, \mu_{\mathscr{A}})$ via derivations. We denote by $\triangleright$ the corresponding Hopf algebra action

$\mathscr{U}(\mathfrak{g}) \to \mathsf{End}(\mathscr{A})$. In the following we refer to

$$\mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet = \prod_{k=0}^{\infty} (\mathscr{A} \otimes \mathrm{S}^k\mathfrak{g}^* \otimes \Lambda^\bullet\mathfrak{g}^*)[\![t]\!]$$

as the *enlarged Fedosov algebra*. The operators defined in the previous section are extended to $\mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet$ by acting trivially on the $\mathscr{A}$-factor and as before on the $\mathcal{W} \otimes \Lambda^\bullet$-factor.

The deformed product $\circ_\pi$ on $\mathcal{W} \otimes \Lambda^\bullet$ together with the product $\mu_{\mathscr{A}}$ of $\mathscr{A}$ yields a new (deformed) $\mathsf{R}[\![t]\!]$-bilinear product $m_\pi^{\mathscr{A}}$ for the extended Fedosov algebra. Explicitly, on factorizing tensors we have

$$m_\pi^{\mathscr{A}}(\xi_1 \otimes f_1 \otimes \alpha_1, \xi_2 \otimes f_2 \otimes \alpha_2) = (\xi_1 \cdot \xi_2) \otimes (f_1 \otimes \alpha_1) \circ_\pi (f_2 \otimes \alpha_2), \tag{3-1}$$

where $\xi_1, \xi_2 \in \mathscr{A}$, $f_1, f_2 \in \mathrm{S}^\bullet\mathfrak{g}^*$ and $\alpha_1, \alpha_2 \in \Lambda^\bullet\mathfrak{g}^*$. We simply write $\xi_1 \cdot \xi_2$ for the (undeformed) product $\mu_{\mathscr{A}}$ of $\mathscr{A}$. Clearly, this new product $m_\pi^{\mathscr{A}}$ is again associative.

As new ingredient we use the action $\triangleright$ to define the operator $L_{\mathscr{A}} : \mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet \to \mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet$ by

$$L_{\mathscr{A}}(\xi \otimes f \otimes \alpha) = e_i \triangleright \xi \otimes f \otimes e^i \wedge \alpha \tag{3-2}$$

on factorizing elements and extend it $\mathsf{R}[\![t]\!]$-linearly as usual. Since the action of Lie algebra elements is by derivations, we see that $L_{\mathscr{A}}$ is a derivation of $\mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet$ of antisymmetric degree $+1$. The sum

$$\mathscr{D}_{\mathscr{A}} = L_{\mathscr{A}} + \mathscr{D}_{\mathrm{F}} \tag{3-3}$$

is thus still a derivation of antisymmetric degree $+1$ which we call the *extended Fedosov derivation*. It turns out to be a differential, too:

Lemma 3.1. *The map $\mathscr{D}_{\mathscr{A}} = L_{\mathscr{A}} + \mathscr{D}_{\mathrm{F}}$ squares to zero.*

Proof. First, we observe that $\mathscr{D}_{\mathscr{A}}^2 = L_{\mathscr{A}}^2 + [\mathscr{D}_{\mathrm{F}}, L_{\mathscr{A}}]$, because $\mathscr{D}_{\mathrm{F}}^2 = 0$. Next, since $\triangleright$ is a Lie algebra action, we immediately obtain

$$L_{\mathscr{A}}^2(\xi \otimes f \otimes \alpha) = \tfrac{1}{2} C_{ij}^k e_k \triangleright \xi \otimes f \otimes e^i \wedge e^j \wedge \alpha$$

on factorizing elements. We clearly have $[\delta, L_{\mathscr{A}}] = 0 = [\mathrm{ad}(\varrho), L_{\mathscr{A}}]$ since the maps act on different tensor factors. It remains to compute the only nontrivial term in $[\mathscr{D}_{\mathrm{F}}, L_{\mathscr{A}}] = [D, L_{\mathscr{A}}]$. Using $\delta_{\mathrm{CE}} e^k = -\frac{1}{2} C_{ij}^k e^i \wedge e^j$, this results immediately in $[D, L_{\mathscr{A}}] = -L_{\mathscr{A}}^2$. □

The cohomology of this differential turns out to be almost trivial: we only have a nontrivial contribution in antisymmetric degree 0, the kernel of $\mathscr{D}_{\mathscr{A}}$. In higher antisymmetric degrees, the following homotopy formula shows that the cohomology is trivial:

Proposition 3.2. *The operator*

$$\mathscr{D}_{\mathscr{A}}^{-1} = \delta^{-1} \frac{1}{\mathsf{id} - \left[\delta^{-1}, D + L_{\mathscr{A}} + \frac{1}{t}\operatorname{ad}(\varrho)\right]} \tag{3-4}$$

is a well-defined $\mathsf{R}[[t]]$*-linear endomorphism of* $\mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet$ *and we have*

$$a = \mathscr{D}_{\mathscr{A}} \mathscr{D}_{\mathscr{A}}^{-1} a + \mathscr{D}_{\mathscr{A}}^{-1} \mathscr{D}_{\mathscr{A}} a + \frac{1}{\mathsf{id} - \left[\delta^{-1}, D + L_{\mathscr{A}} + \frac{1}{t}\operatorname{ad}(\varrho)\right]} \sigma(a). \tag{3-5}$$

for all $a \in \mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet$.

Proof. Let us denote by A the operator $[\delta^{-1}, D + L_{\mathscr{A}} + (1/t)\operatorname{ad}(\varrho)]$. Since it increases the total degree by $+1$, the geometric series $(\mathsf{id} - A)^{-1}$ is well defined as a formal series in the total degree. We start with the Poincaré equation (2-12) and get

$$-\mathscr{D}_{\mathscr{A}} \delta^{-1} a - \delta^{-1} \mathscr{D}_{\mathscr{A}} a + \sigma(a) = (\mathsf{id} - A)a, \tag{3-6}$$

since $\mathcal{D}_{\mathcal{A}}$ deforms the differential $-\delta$ by higher order terms in the total degree. The usual homological perturbation argument then gives (3-4) by a standard computation; see, e.g., [Waldmann 2007, Proposition 6.4.17] for this computation. □

Corollary 3.3. *Let* $a \in \mathscr{A} \otimes \mathcal{W} \otimes \Lambda^0$. *Then* $\mathscr{D}_{\mathscr{A}} a = 0$ *if and only if*

$$a = \frac{1}{\mathsf{id} - \left[\delta^{-1}, D + L_{\mathscr{A}} + \frac{1}{t}\operatorname{ad}(\varrho)\right]} \sigma(a). \tag{3-7}$$

Since the element $a \in \mathscr{A} \otimes \mathcal{W} \otimes \Lambda^0$ is completely determined in the symmetric and antisymmetric degree 0, we can use it to define the extended Fedosov Taylor series.

Definition 3.4 (Extended Fedosov Taylor series). Given the extended Fedosov derivation $\mathscr{D}_{\mathscr{A}} = -\delta + D + L_{\mathscr{A}} + (1/t)\operatorname{ad}(\varrho)$, the extended Fedosov Taylor series of $\xi \in \mathscr{A}[[t]]$ is defined by

$$\tau_{\mathscr{A}}(\xi) = \frac{1}{\mathsf{id} - \left[\delta^{-1}, D + L_{\mathscr{A}} + \frac{1}{t}\operatorname{ad}(\varrho)\right]} \xi. \tag{3-8}$$

Lemma 3.5. *For* $\xi \in \mathscr{A}[[t]]$ *we have*

$$\sigma(\tau_{\mathscr{A}}(\xi)) = \xi. \tag{3-9}$$

Moreover, the map $\tau_{\mathscr{A}} : \mathscr{A}[[t]] \to \ker \mathscr{D}_{\mathscr{A}} \cap \ker \deg_{\mathrm{a}}$ *is a* $\mathsf{R}[[t]]$*-linear isomorphism starting with*

$$\tau_{\mathscr{A}}(\xi) = \sum_{k=0}^{\infty} \left[\delta^{-1}, D + L_{\mathscr{A}} + \frac{1}{t}\operatorname{ad}(\varrho)\right]^k (\xi) = \xi \otimes 1 \otimes 1 + e_i \triangleright \xi \otimes e^i \otimes 1 + \cdots \tag{3-10}$$

in zeroth and first order of the total degree.

Proof. The isomorphism property follows directly from Corollary 3.3. The commutator $[\delta^{-1}, D + L_{\mathscr{A}} + (1/t)\,\mathrm{ad}(\varrho)]$ raises the total degree at least by one, thus the zeroth and first order terms in the total degree come from the terms with $k = 0$ and $k = 1$ in the geometric series in (3-10). Here it is easy to see that the only nontrivial contribution is

$$\Big[\delta^{-1}, D + L_{\mathscr{A}} + \frac{1}{t}\,\mathrm{ad}(\varrho)\Big]\xi = L_{\mathscr{A}}\xi,$$

proving the claim in (3-10). Note that already for $k = 2$ we also get contributions of S and $\mathrm{ad}(\varrho)$. □

Given the $\mathsf{R}[\![t]\!]$-linear isomorphism $\tau_{\mathscr{A}} : \mathscr{A}[\![t]\!] \to \ker \mathscr{D}_{\mathscr{A}} \cap \ker \deg_{\mathrm{a}}$ we can turn $\mathscr{A}[\![t]\!]$ into an algebra by pulling back the deformed product: note that the kernel of a derivation is always a subalgebra and hence the intersection $\ker \mathscr{D}_{\mathscr{A}} \cap \ker \deg_{\mathrm{a}}$ is also a subalgebra. This allows us to obtain a universal deformation formula for any $\mathscr{U}(\mathfrak{g})$-module algebra $\mathscr{A}$:

Theorem 3.6 (Universal deformation formula). *Let $\mathfrak{g}$ be a Lie algebra with non-degenerate r-matrix. Moreover, let $s \in \mathrm{S}^2\mathfrak{g}$ be such that there exists a symplectic torsion-free covariant derivative ∇ with s being covariantly constant. Consider then $\pi = r + s$. Finally, let $\Omega \in t\Lambda^2\mathfrak{g}^*[\![t]\!]$ be a formal series of δ_{CE}-closed two-forms. Then for every associative algebra $\mathscr{A}$ with action of $\mathfrak{g}$ by derivations one obtains an associative deformation $m_\star^{\mathscr{A}} : \mathscr{A}[\![t]\!] \times \mathscr{A}[\![t]\!] \to \mathscr{A}[\![t]\!]$ by*

$$m_\star^{\mathscr{A}}(\xi, \eta) = \sigma(m_\pi^{\mathscr{A}}(\tau_{\mathscr{A}}(\xi), \tau_{\mathscr{A}}(\eta))). \tag{3-11}$$

Writing simply $\star = \star_{\Omega,\nabla,s}$ for this new product, one has

$$\xi \star \eta = \xi \cdot \eta + \frac{t}{2}\pi^{ij}(e_i \triangleright \xi)\cdot(e_j \triangleright \eta) + \mathcal{O}(t^2) \quad \text{for } \xi, \eta \in \mathscr{A}. \tag{3-12}$$

Proof. The product $m_\star^{\mathscr{A}}$ is associative, because $m_\pi^{\mathscr{A}}$ is associative and $\tau_{\mathscr{A}}$ is an isomorphism onto a subalgebra with inverse σ. The second part is a direct consequence of Lemma 3.5. □

Remark 3.7. The above theorem can be further generalized by observing that given a Poisson structure on $\mathscr{A}$ induced by a generic bivector on $\mathfrak{g}$, we can reduce to the quotient $\mathfrak{g}/\ker \triangleright$ and obtain an r-matrix on the quotient, inducing the same Poisson structure.

4. Universal construction for Drinfeld twists

Let us consider the particular case in which $\mathscr{A}$ is the tensor algebra $(\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g})), \otimes)$. In this case, we denote by L the operator $L_{\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))} : \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet \to$

$\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$, which is given by

$$L_{\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))}(\xi \otimes f \otimes \alpha) = L_{e_i}\xi \otimes f \otimes e^i \wedge \alpha. \tag{4-1}$$

Here L_{e_i} is the left multiplication in $\mathscr{U}(\mathfrak{g})$ of the element e_i extended as a derivation of the tensor product. Note that it is independent of the choice of the basis in $\mathfrak{g}$.

Applying the results discussed in the last section, we obtain a star product for the tensor algebra over $\mathscr{U}(\mathfrak{g})$ as a particular case of Theorem 3.6:

Corollary 4.1. *The map $m_\star : \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))[[t]] \times \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))[[t]] \to \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))[[t]]$ given by*

$$m_\star(\xi, \eta) = \xi \star \eta = \sigma(m_\pi(\tau(\xi), \tau(\eta))) \tag{4-2}$$

is an associative product and

$$\xi \star \eta = \xi \otimes \eta + \tfrac{1}{2} t \pi^{ij} L_{e_i}\xi \otimes L_{e_j}\eta + \mathcal{O}(t^2) \quad \textit{for } \xi, \eta \in \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g})). \tag{4-3}$$

In the following we prove that the star product $m_\star$ defined above allows one to construct a formal Drinfeld twist. Let us define, for any linear map

$$\Phi : \mathscr{U}(\mathfrak{g})^{\otimes k} \to \mathscr{U}(\mathfrak{g})^{\otimes \ell}, \tag{4-4}$$

the *lifted* map

$$\Phi^{\mathrm{Lift}} : \mathscr{U}(\mathfrak{g})^{\otimes k} \otimes \mathcal{W} \otimes \Lambda^\bullet \ni \xi \otimes f \otimes \alpha \mapsto \Phi(\xi) \otimes f \otimes \alpha \in \mathscr{U}(\mathfrak{g})^{\otimes \ell} \otimes \mathcal{W} \otimes \Lambda^\bullet, \tag{4-5}$$

obeying the following simple properties:

Lemma 4.2. *Let $\Phi : \mathscr{U}(\mathfrak{g})^{\otimes k} \to \mathscr{U}(\mathfrak{g})^{\otimes \ell}$ and $\Psi : \mathscr{U}(\mathfrak{g})^{\otimes m} \to \mathscr{U}(\mathfrak{g})^{\otimes n}$ be linear maps.*

(i) *The lifted map Φ^{Lift} commutes with δ, δ^{-1}, D, and* $\mathrm{ad}(x)$ *for all $x \in \mathcal{W} \otimes \Lambda^\bullet$.*

(ii) *We have*

$$\Phi \circ \sigma|_{\mathscr{U}(\mathfrak{g})^{\otimes k} \otimes \mathcal{W} \otimes \Lambda^\bullet} = \sigma|_{\mathscr{U}(\mathfrak{g})^{\otimes \ell} \otimes \mathcal{W} \otimes \Lambda^\bullet} \circ \Phi^{\mathrm{Lift}}. \tag{4-6}$$

(iii) *We have*

$$(\Phi \otimes \Psi)^{\mathrm{Lift}} m_\pi(a_1, a_2) = m_\pi(\Phi^{\mathrm{Lift}}(a_1), \Psi^{\mathrm{Lift}}(a_2)), \tag{4-7}$$

for any $a_1 \in \mathscr{U}(\mathfrak{g})^{\otimes k} \otimes \mathcal{W} \otimes \Lambda^\bullet$ and $a_2 \in \mathscr{U}(\mathfrak{g})^{\otimes m} \otimes \mathcal{W} \otimes \Lambda^\bullet$.

Let $\eta \in \mathscr{U}(\mathfrak{g})^{\otimes k}[[t]]$ be given. Then we can consider the right multiplication by η using the algebra structure of $\mathscr{U}(\mathfrak{g})^{\otimes k}[[t]]$ coming from the universal enveloping algebra as a map

$$\cdot\eta : \mathscr{U}(\mathfrak{g})^{\otimes k} \ni \xi \mapsto \xi \cdot \eta \in \mathscr{U}(\mathfrak{g})^{\otimes k}. \tag{4-8}$$

To this map we can apply the above lifting process and extend it this way to a $\mathsf{R}[[t]]$-linear map such that on factorizing elements

$$\cdot\eta : \mathscr{U}(\mathfrak{g})^{\otimes k} \otimes \mathcal{W} \otimes \Lambda^\bullet \ni \xi \otimes f \otimes \alpha \mapsto (\xi \cdot \eta) \otimes f \otimes \alpha \in \mathscr{U}(\mathfrak{g})^{\otimes k}, \tag{4-9}$$

where we simply write $\cdot\eta$ instead of $(\cdot\eta)^{\mathrm{Lift}}$. Note that $a \cdot \eta$ is only defined if the tensor degrees k of $\eta \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))$ and a coincide since we use the algebra structure inherited from the universal enveloping algebra.

In the following we denote by $\mathscr{D}$ the derivation $\mathscr{D}_{\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))}$ as obtained in (3-3). We collect some properties how the lifted right multiplications match with the extended Fedosov derivation:

Lemma 4.3. (i) *For any $a \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$ and $\xi \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$, we have $\mathscr{D}(a \cdot \xi) = \mathscr{D}(a) \cdot \xi$*

(ii) *The extended Fedosov-Taylor series τ preserves the tensor degree of elements in $\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$.*

(iii) *For any $\xi, \eta \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$, we have $\tau(\xi \cdot \eta) = \tau(\xi) \cdot \eta$.*

(iv) *For any $a_1 \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$ and $a_2 \in \mathrm{T}^\ell(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$ as well as $\eta_1 \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$ and $\eta_2 \in \mathrm{T}^\ell(\mathscr{U}(\mathfrak{g}))[[t]]$, we have $m_\pi(a_1 \cdot \eta_1, a_2 \cdot_l \eta_2) = m_\pi(a_1, a_2) \cdot (\eta_1 \otimes \eta_2)$.*

Proof. Let $\xi \otimes a \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$ and $\eta \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))$. Then we have

$$\begin{aligned}
\mathscr{D}((\xi \otimes a) \cdot \eta) &= \mathscr{D}((\xi \cdot \eta) \otimes a) \\
&= L_{e_i}(\xi \cdot \eta) \otimes e^i \wedge a + (\xi \cdot \eta) \otimes \mathscr{D}_{\mathrm{F}}(a) \\
&= (L_{e_i}(\xi) \otimes e^i \wedge a) \cdot \eta + (\xi \otimes \mathscr{D}_{\mathrm{F}}(a)) \cdot \eta = \mathscr{D}(a) \cdot \eta.
\end{aligned}$$

This proves the first claim. The second claim follows immediately from the fact that all operators defining τ do not change the tensor degree. In order to prove the claim (iii), let us consider $\xi, \eta \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$. Then we have

$$\mathscr{D}(\tau(\xi) \cdot \eta) = \mathscr{D}(\tau(\xi)) \cdot \eta = 0,$$

according to (i). Thus, $\tau(\xi) \cdot \eta \in \ker \mathscr{D} \cap \ker \deg_{\mathrm{a}}$ and therefore

$$\tau(\xi) \cdot \eta = \tau(\sigma(\tau(\xi) \cdot \eta)) = \tau(\sigma(\tau(\xi)) \cdot \eta) = \tau(\xi \cdot \eta).$$

Finally, to prove the last claim we choose $\xi_1 \otimes f_1 \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$ and $\xi_2 \otimes f_2 \in \mathrm{T}^\ell(\mathscr{U}(\mathfrak{g})) \otimes \mathcal{W} \otimes \Lambda^\bullet$ as well as $\eta_1 \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$ and $\eta_2 \in \mathrm{T}^\ell(\mathscr{U}(\mathfrak{g}))[[t]]$.

We obtain

$$\begin{aligned} m_\pi((\xi_1 \otimes f_1)\cdot\eta_1, (\xi_2\otimes f_2)\cdot\eta_2) &= m_\pi((\xi_1\cdot\eta_1)\otimes f_1, (\xi_2\cdot\eta_2)\otimes f_2) \\ &= ((\xi_1\cdot\eta_1)\otimes(\xi_2\cdot\eta_2))\otimes(f_1\circ_\pi f_2) \\ &= ((\xi_1\otimes\xi_2)\cdot(\eta_1\otimes\eta_2))\otimes(f_1\circ_\pi f_2) \\ &= ((\xi_1\otimes\xi_2)\otimes(f_1\circ_\pi f_2))\cdot(\eta_1\otimes\eta_2). \end{aligned}$$

This concludes the proof. □

From the above lemma, we observe that the isomorphism τ can be computed for any element $\xi \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$ via

(4-10) $$\tau(\xi) = \tau(1^{\otimes k}\cdot\xi) = \tau(1^{\otimes k})\cdot\xi,$$

where $1 \in \mathscr{U}(\mathfrak{g})$ is the unit element of the universal enveloping algebra. Moreover, from Lemma 4.2, we have

(4-11) $$\xi\star\eta = \sigma(m_\pi(\tau(\xi)\otimes\tau(\eta))) = (1^{\otimes k}\star 1^{\otimes \ell})\cdot(\xi\otimes\eta)$$

for $\xi \in \mathrm{T}^k(\mathscr{U}(\mathfrak{g}))[[t]]$ and $\eta \in \mathrm{T}^\ell(\mathscr{U}(\mathfrak{g}))[[t]]$. Thus $\star$ is entirely determined by the values on tensor powers of the unit element of the universal enveloping algebra. Note that the unit of $\star$ is the unit element in $\mathsf{R} \subseteq \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$ of the *tensor algebra* but not $1 \in \mathscr{U}(\mathfrak{g})$.

Lemma 4.4. *Let* $\Delta : \mathscr{U}(\mathfrak{g})[[t]] \to \mathscr{U}(\mathfrak{g})^{\otimes 2}[[t]]$ *be the coproduct of* $\mathscr{U}(\mathfrak{g})[[t]]$ *and* $\epsilon : \mathscr{U}(\mathfrak{g}) \to \mathsf{R}[[t]]$ *the counit.*

(i) *We have*

(4-12) $$L|_{\mathscr{U}(\mathfrak{g})^{\otimes 2}\otimes\mathcal{W}\otimes\Lambda^\bullet}\circ\Delta^{\mathrm{Lift}} = \Delta^{\mathrm{Lift}}\circ L|_{\mathscr{U}(\mathfrak{g})\otimes\mathcal{W}\otimes\Lambda^\bullet}.$$

(ii) *For the Fedosov-Taylor series one has*

(4-13) $$\Delta^{\mathrm{Lift}}\circ\tau = \tau\circ\Delta.$$

(iii) *We have*

(4-14) $$\epsilon^{\mathrm{Lift}}\circ L|_{\mathscr{U}(\mathfrak{g})\otimes\mathcal{W}\otimes\Lambda^\bullet} = 0.$$

(iv) *For the Fedosov-Taylor series one has*

(4-15) $$\epsilon^{\mathrm{Lift}}\circ\tau = \epsilon.$$

Proof. Let $\xi \otimes f \otimes \alpha \in \mathscr{U}(\mathfrak{g}) \otimes \mathcal{W} \otimes \Lambda^\bullet$. Then we get

$$\begin{aligned}
\Delta^{\text{Lift}} L(\xi \otimes f \otimes \alpha) &= \Delta^{\text{Lift}}(L_{e_i}(\xi) \otimes f \otimes e^i \wedge \alpha) \\
&= \Delta^{\text{Lift}}(e_i \xi \otimes f \otimes e^i \wedge \alpha) \\
&= \Delta(e_i \xi) \otimes f \otimes e^i \wedge \alpha \\
&= \Delta(e_i) \cdot \Delta(\xi) \otimes f \otimes e^i \wedge \alpha \\
&= (e_i \otimes 1 + 1 \otimes e_i) \cdot \Delta(\xi) \otimes f \otimes e^i \wedge \alpha \\
&= L_{e_i}(\Delta(\xi)) \otimes f \otimes e^i \wedge \alpha \\
&= L \Delta^{\text{Lift}}(\xi \otimes f \otimes \alpha),
\end{aligned}$$

since we extended the left multiplication by e_i as a *derivation* of the tensor product to higher tensor powers. Hence all the operators appearing in τ *commute* with Δ^{Lift} and therefore we get the second part. Similarly, we get

$$\begin{aligned}
\epsilon^{\text{Lift}}(L(\xi \otimes f \otimes \alpha) &= \epsilon^{\text{Lift}}(e_i \xi \otimes f \otimes e^i \wedge \alpha) \\
&= \epsilon(e_i \xi) \otimes f \otimes e^i \wedge \alpha = \epsilon(e_i)\epsilon(\xi) \otimes f \otimes e^i \wedge \alpha = 0,
\end{aligned}$$

where we used that ϵ vanishes on primitive elements of $\mathscr{U}(\mathfrak{g})$. Since ϵ^{Lift} commutes with all other operators δ^{-1}, D and $\operatorname{ad}(\varrho)$ according to Lemma 4.2, we first get

$$\epsilon^{\text{Lift}} \circ \left[\delta^{-1}, D + L + \frac{1}{t} \operatorname{ad}(\varrho)\right] = \left[\delta^{-1}, D + \frac{1}{t} \operatorname{ad}(\varrho)\right] \circ \epsilon^{\text{Lift}}.$$

Hence for $\xi \in \mathscr{U}(\mathfrak{g})[[t]]$ we have

$$\begin{aligned}
\epsilon^{\text{Lift}} \tau(\xi) &= \epsilon^{\text{Lift}}\left(\sum_{k=0}^{\infty} \left[\delta^{-1}, D + L + \frac{1}{t} \operatorname{ad}(\varrho)\right]^k \xi\right) \\
&= \sum_{k=0}^{\infty} \left[\delta^{-1}, D + \frac{1}{t} \operatorname{ad}(\varrho)\right]^k \epsilon^{\text{Lift}}(\xi) \\
&= \epsilon(\xi),
\end{aligned}$$

since $\epsilon^{\text{Lift}}(\xi) = \epsilon(\xi)$ is just a constant and hence unaffected by all the operators in the series. Thus only the zeroth term remains. □

This is now the last ingredient to show that the element $1 \star 1$ is the twist we are looking for:

Theorem 4.5. *The element* $1 \star 1 \in \mathscr{U}(\mathfrak{g})^{\otimes 2}[[t]]$ *is a twist such that*

$$1 \star 1 = 1 \otimes 1 + \frac{t}{2}\pi + \mathcal{O}(t^2). \tag{4-16}$$

Proof. First we see that

$$\begin{aligned}(\Delta \otimes \mathsf{id})(1 \star 1) &= (\Delta \otimes \mathsf{id})\sigma(m_\pi(\tau(1), \tau(1))) \\ &= \sigma((\Delta \otimes \mathsf{id})^{\text{Lift}}(m_\pi(\tau(1), \tau(1)))) \\ &= \sigma(m_\pi(\Delta^{\text{Lift}}\tau(1), \tau(1))) \\ &= \sigma(m_\pi(\tau(\Delta(1)), \tau(1))) \\ &= \sigma(m_\pi(\tau(1 \otimes 1), \tau(1))) \\ &= (1 \otimes 1) \star 1.\end{aligned}$$

Similarly, we get $(\mathsf{id} \otimes \Delta)(1 \star 1) = 1 \star (1 \otimes 1)$. Thus, using the associativity of $\star$ we obtain the first condition (1-2) for a twist as follows,

$$\begin{aligned}(\Delta \otimes \mathsf{id})(1 \star 1) \cdot ((1 \star 1) \otimes 1) &= ((1 \otimes 1) \star 1) \cdot ((1 \star 1) \otimes 1) \\ &= (1 \star 1) \star 1 \\ &= 1 \star (1 \star 1) \\ &= (\mathsf{id} \otimes \Delta)(1 \star 1) \cdot (1 \otimes (1 \star 1)).\end{aligned}$$

To check the normalization condition (1-3) we use Lemma 4.2 and Lemma 4.4 again to get

$$\begin{aligned}(\epsilon \otimes \mathsf{id})(1 \star 1) &= (\epsilon \otimes \mathsf{id})\sigma(m_\pi(\tau(1), \tau(1))) \\ &= \sigma((\epsilon \otimes \mathsf{id})^{\text{Lift}}(m_\pi(\tau(1), \tau(1)))) \\ &= \sigma((m_\pi(\epsilon^{\text{Lift}}\tau(1), \tau(1)))) \\ &= \sigma((m_\pi(\epsilon(1), \tau(1)))) \\ &= \epsilon(1)\sigma(\tau(1)) \\ &= 1,\end{aligned}$$

since $\epsilon(1)$ is the unit element of R and thus the unit element of $\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$, which serves as unit element for m_π as well. Similarly we obtain $(\mathsf{id} \otimes \epsilon)(1 \star 1) = 1$. Finally, the facts that the first term in t of $1 \star 1$ is given by π and that zero term in t is $1 \otimes 1$ follow from Corollary 4.1. □

Remark 4.6. From now on we refer to $1 \star 1$ as the *Fedosov twist*

$$\mathcal{F}_{\Omega,\nabla,s} = 1 \star 1, \tag{4-17}$$

corresponding to the choice of the δ_{CE}-closed form Ω, the choice of the torsion-free symplectic covariant derivative and the choice of the covariantly constant s. In the following we will be mainly interested in the dependence of $\mathcal{F}_{\Omega,\nabla,s}$ on the two-forms Ω and hence we shall write $\mathcal{F}_\Omega$ for simplicity. We also note that for $s = 0$ and $\Omega = 0$ we have a *preferred* choice for ∇, namely the one obtained from the Hess trick out of the half-commutator covariant derivative as described in Proposition 2.6. This gives a *canonical twist* $\mathcal{F}_0$ quantizing r.

The results discussed above allow us to give an alternative proof of the Drinfeld theorem [1983], stating the existence of twists for every r-matrix:

Corollary 4.7 (Drinfeld). *Let* $(\mathfrak{g}, r)$ *be a Lie algebra with r-matrix over a field* $\mathbb{K}$ *with characteristic* 0*. Then there exists a formal twist* $\mathcal{F} \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[\![t]\!]$ *such that*

$$\mathcal{F} = 1 \otimes 1 + \frac{t}{2} r + \mathcal{O}(t^2).$$

To conclude this section we consider the question whether the two approaches of universal deformation formulas actually coincide: on the one hand we know that every twist gives a universal deformation formula by (1-1). On the other hand, we have constructed directly a universal deformation formula (3-11) in Theorem 3.6 based on the Fedosov construction. Since we also get a twist from the Fedosov construction, we are interested in the consistence of the two constructions. In order to answer this question, we need some preparation. Hence let $\mathscr{A}$ be an algebra with action of $\mathfrak{g}$ by derivations as before. Then we define the map

$$\bullet : \mathscr{U}(\mathfrak{g}) \otimes \mathcal{W} \otimes \Lambda^\bullet \times \mathscr{A} \ni (\xi \otimes \alpha, a) \mapsto (\xi \otimes \alpha) \bullet a = \xi \triangleright a \otimes \alpha \in \mathscr{A} \otimes \mathcal{W} \otimes \Lambda^\bullet \tag{4-18}$$

for any $a \in \mathscr{A}$ and $\alpha \in \mathcal{W} \otimes \Lambda^\bullet$. Then the following algebraic properties are obtained by a straightforward computation:

Lemma 4.8. *For any* $\xi \in \mathscr{U}(\mathfrak{g})$, $\alpha \in \mathcal{W} \otimes \Lambda^\bullet$ *and* $a \in \mathscr{A}$ *we have*

(i) $\sigma((\xi \otimes \alpha) \bullet a) = \sigma(\xi \otimes \alpha) \triangleright a$,

(ii) $L_{\mathscr{A}}(\xi \triangleright a \otimes \alpha) = L(\xi \otimes \alpha) \bullet a$,

(iii) $\tau_{\mathscr{A}}(a) = \tau(1) \bullet a$,

(iv) $m_\pi^{\mathscr{A}}(\xi_1 \otimes a_1 \otimes \alpha_1, \xi_2 \otimes a_2 \otimes \alpha_2) = (\mu_{\mathscr{A}} \otimes \mathsf{id} \otimes \mathsf{id})(m_\pi(\xi_1 \otimes \alpha_1, \xi_2 \otimes \alpha_2) \bullet (a_1 \otimes a_2))$.

For matching parameters Ω, ∇, and s of the Fedosov construction, the two approaches coincide:

Proposition 4.9. *For fixed choices of* Ω, ∇, *and* s *and for any* $a, b \in \mathscr{A}$ *we have*

$$a \star_{\Omega,\nabla,s} b = a \star_{\mathcal{F}_{\Omega,\nabla,s}} b. \tag{4-19}$$

Proof. This is now just a matter of computation. We have

$$\begin{aligned} a \star b &= \sigma(m_\pi^{\mathscr{A}}(\tau_{\mathscr{A}}(a) \otimes \tau_{\mathscr{A}}(b))) \\ &\overset{(a)}{=} \sigma(m_\pi((\tau(1) \otimes \tau(1)) \bullet (a \otimes b))) \\ &\overset{(b)}{=} \mu_{\mathscr{A}}(\sigma(m_\pi(\tau(1) \otimes \tau(1))) \rhd (a \otimes b)) \\ &= \mu_{\mathscr{A}}((1 \star 1) \rhd (a \otimes b)) \\ &= a \star_{\mathcal{F}} b, \end{aligned}$$

where in (a) we use the third claim of the above lemma and in (b) the first and the fourth. □

5. Classification of Drinfeld twists

In this section we discuss the classification of twists on universal enveloping algebras for a given Lie algebra $\mathfrak{g}$ with nondegenerate r-matrix. Recall that two twists $\mathcal{F}$ and $\mathcal{F}'$ are said to be *equivalent* and denoted by $\mathcal{F} \sim \mathcal{F}'$ if there exists an element $S \in \mathscr{U}(\mathfrak{g})[[t]]$, with $S = 1 + \mathcal{O}(t)$ and $\epsilon(S) = 1$ such that

$$\Delta(S)\mathcal{F}' = \mathcal{F}(S \otimes S). \tag{5-1}$$

In the following we prove that the set of equivalence classes of twists $\mathrm{Twist}(\mathscr{U}(\mathfrak{g}), r)$ with fixed r-matrix r is in bijection to the formal series in the second Chevalley–Eilenberg cohomology $\mathrm{H}^2_{\mathrm{CE}}(\mathfrak{g})[[t]]$.

We will fix the choice of ∇ and the symmetric part s in the Fedosov construction. Then the cohomological equivalence of the two-forms in the construction yields equivalent twists. In fact, an equivalence can even be computed recursively:

Lemma 5.1. *Let ϱ and ϱ' be the two elements in $\mathcal{W}_2 \otimes \Lambda^1$ uniquely determined from Proposition 2.9, corresponding to two closed two-forms $\Omega, \Omega' \in t\Lambda^2\mathfrak{g}^*[[t]]$, respectively, and let $\Omega - \Omega' = \delta_{\mathrm{CE}} C$ for a fixed $C \in t\mathfrak{g}^*[[t]]$. Then there is a unique solution $h \in \mathcal{W}_3 \otimes \Lambda^0$ of*

$$h = C \otimes 1 + \delta^{-1}\left(Dh - \frac{1}{t}\,\mathrm{ad}(\varrho)h - \frac{\frac{1}{t}\,\mathrm{ad}(h)}{\exp\left(\frac{1}{t}\,\mathrm{ad}(h)\right) - \mathsf{id}}(\varrho' - \varrho)\right), \quad \sigma(h) = 0. \tag{5-2}$$

For this h we have

$$\mathscr{D}'_{\mathrm{F}} = \mathcal{A}_h \mathscr{D}_{\mathrm{F}} \mathcal{A}_{-h},$$

with $\mathcal{A}_h = \exp((1/t)\,\mathrm{ad}(h))$ being an automorphism of $\circ_\pi$.

Proof. In the context of the Fedosov construction it is well known that cohomologous two-forms yield equivalent star products. The above approach with the explicit formula for h follows the arguments of [Reichert and Waldmann 2016, Lemma 3.5] which is based on [Neumaier 2001, §3.5.1.1]. □

Lemma 5.2. *Let $\Omega, \Omega' \in t\Lambda^2\mathfrak{g}^*[[t]]$ be δ_{CE}-cohomologous. Then the corresponding Fedosov twists are equivalent.*

Proof. By assumption, we can find an element $C \in t\mathfrak{g}^*[[t]]$, such that $\Omega - \Omega' = \delta_{\mathrm{CE}} C$. From Lemma 5.1 we get an element $h \in \mathcal{W}_3 \otimes \Lambda^0$ such that $\mathscr{D}'_F = \mathcal{A}_h \mathscr{D}_{\mathrm{F}} \mathcal{A}_{-h}$. An easy computation shows that $\mathcal{A}_h$ commutes with L, therefore

$$\mathscr{D}' = \mathcal{A}_h \mathscr{D} \mathcal{A}_{-h}.$$

Thus, $\mathcal{A}_h$ is an automorphism of m_π with $\mathcal{A}_h : \ker \mathscr{D} \to \ker \mathscr{D}'$ being a bijection between the two kernels. Let us consider the map

$$S_h : \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))[[t]] \ni \xi \mapsto (\sigma \circ \mathcal{A}_h \circ \tau)(\xi) \in \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))[[t]],$$

which defines an equivalence of star products, i.e.,

$$S_h(\xi \star \eta) = S_h(\xi) \star' S_h(\eta) \tag{5-3}$$

for any $\xi, \eta \in \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))[[t]]$. Let $\xi, \eta \in \mathscr{U}(\mathfrak{g})$. Then using Lemma 4.3,

$$\begin{aligned} S_h(\xi \otimes \eta) &= (\sigma \circ \mathcal{A}_h \circ \tau)(\xi \otimes \eta) \\ &= (\sigma \circ \mathcal{A}_h)(\tau(1 \otimes 1) \cdot (\xi \otimes \eta)) \\ &= \sigma((\mathcal{A}_h(\tau(1 \otimes 1))) \cdot (\xi \otimes \eta)) \\ &= \sigma(\mathcal{A}_h(\tau(1 \otimes 1))) \cdot (\xi \otimes \eta) \\ &= \sigma(\mathcal{A}_h(\Delta^{\mathrm{Lift}} \tau(1))) \cdot (\xi \otimes \eta) \\ &= \Delta(\sigma(\mathcal{A}_h(\tau(1)))) \cdot (\xi \otimes \eta) \\ &= \Delta(S_h(1)) \cdot (\xi \otimes \eta). \end{aligned}$$

From the linearity of S_h we immediately get $S_h(\xi \star \eta) = \Delta(S_h(1))(\xi \star \eta)$. Now, putting $\xi = \eta = 1$ in (5-3) and using (4-11) we obtain

$$\Delta(S_h(1)) \cdot (1 \star 1) = S_h(1 \star 1) = S_h(1) \star' S_h(1) = (1 \star' 1) \cdot (S_h(1) \otimes S_h(1)).$$

Thus, the twists $\mathcal{F}_\Omega = 1 \star 1$ and $\mathcal{F}_{\Omega'} = 1 \star' 1$ are equivalent since

$$\epsilon(S_h(1)) = 1.$$ □

Lemma 5.3. *Let $\Omega \in t\Lambda^2\mathfrak{g}^*$ with $\delta_{\mathrm{CE}}\Omega = 0$, x the element in $\mathcal{W}_2 \otimes \Lambda^1$ uniquely determined from Proposition 2.9 and $\mathcal{F}_\Omega$ the corresponding Fedosov twist.*

(i) *The lowest total degree of ϱ, where Ω_k appears, is $2k+1$, and*

$$\varrho^{(2k+1)} = t^k \delta^{-1} \Omega_k + \textit{terms not containing } \Omega_k. \tag{5-4}$$

(ii) *For* $\xi \in \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$ *the lowest total degree of* $\tau(\xi)$ *where* Ω_k *appears is* $2k+1$, *and*

$$\tau(\xi)^{(2k+1)} = \tfrac{1}{2}t^k(e_i \otimes \mathrm{i_a}((e^i)^\sharp)\Omega_k) + \textit{terms not containing } \Omega_k. \tag{5-5}$$

(iii) *The lowest* t*-degree of* $\mathcal{F}_\Omega$ *where* Ω_k *appears is* $k+1$, *and*

$$(F_\Omega)_{k+1} = -\tfrac{1}{2}(\Omega_k)^\sharp + \textit{terms not containing } \Omega_k.$$

(iv) *The map* $\Omega \mapsto \mathcal{F}_\Omega$ *is injective.*

Proof. The proof uses the recursion formula for ϱ as well as the explicit formulas for τ and $\star$ and consists of a careful counting of degrees. It follows along lines of [Waldmann 2007, Theorem 6.4.29]. □

Lemma 5.4. *Let* $\mathcal{F}_\Omega$ *and* $\mathcal{F}_{\Omega'}$ *be two equivalent Fedosov twists corresponding to the closed two-forms* $\Omega, \Omega' \in t\Lambda^2\mathfrak{g}^*$. *Then there exists an element* $C \in t\mathfrak{g}^*[\![t]\!]$, *such that* $\delta_{\mathrm{CE}}C = \Omega - \Omega'$.

Proof. We can assume that Ω and Ω' coincide up to order $k-1$ for $k \in \mathbb{N}$, since they coincide at order 0. Due to Lemma 5.3,

$$(F_\Omega)_i = (F_{\Omega'})_i \quad \text{for any } i \in \{0, \dots, k\}$$

and

$$(F_\Omega)_{k+1} - (F_{\Omega'})_{k+1} = \tfrac{1}{2}(-\Omega_k^\sharp + \Omega'^\sharp_k).$$

From Lemma B.4, we know that we can find an element $\xi \in \mathfrak{g}^*$, such that

$$([(F_\Omega)_{k+1} - (F_{\Omega'})_{k+1}])^\flat = -\Omega_k^\sharp + \Omega'^\sharp_k = \delta_{\mathrm{CE}}\xi,$$

where $[(F_\Omega)_{k+1} - (F_{\Omega'})_{k+1}]$ denotes the skew-symmetrization of $(F_\Omega)_{k+1} - (F_{\Omega'})_{k+1}$. Let us define $\hat{\Omega} = \Omega - t^k\delta_{\mathrm{CE}}\xi$. From Lemma 5.3 we see that

$$(F_{\hat{\Omega}})_{k+1} - (F_{\Omega'})_{k+1} = 0.$$

Therefore the two twists $\mathcal{F}_{\hat{\Omega}}$ and $\mathcal{F}_{\Omega'}$ coincide up to order $k+1$. Finally, since $\mathcal{F}_{\hat{\Omega}}$ and $\mathcal{F}_\Omega$ are equivalent (from Lemma 5.2) and $\mathcal{F}_\Omega$ and $\mathcal{F}_{\Omega'}$ are equivalent by assumption, the two twists $\mathcal{F}_{\hat{\Omega}}$ and $\mathcal{F}_{\Omega'}$ are also equivalent. By induction, we find an element $C \in t\mathfrak{g}^*[\![t]\!]$ such that

$$\mathcal{F}_{\Omega + \delta_{\mathrm{CE}}C} = \mathcal{F}_{\Omega'},$$

and therefore, from Lemma 5.3, $\Omega + \delta_{\mathrm{CE}}C = \Omega'$. □

Lemma 5.5. *Let* $\mathcal{F} \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[\![t]\!]$ *be a formal twist with* r*-matrix* r. *Then there exists a Fedosov twist* $\mathcal{F}_\Omega$ *such that* $\mathcal{F} \sim \mathcal{F}_\Omega$.

Proof. Let $\mathcal{F} \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$ be a given twist. We can assume that there is a Fedosov twist $\mathcal{F}_\Omega$, which is equivalent to $\mathcal{F}$ up to order k. Therefore we find a $\hat{\mathcal{F}}$ such that $\hat{\mathcal{F}}$ is equivalent to $\mathcal{F}$ and coincides with $\mathcal{F}_\Omega$ up to order k. Due to Lemma B.4, we can find an element $\xi \in \mathfrak{g}^*$ such that

$$[(F_\Omega)_{k+1} - \hat{F}_{k+1})] = (\delta_{\mathrm{CE}}\xi)^\sharp.$$

From Lemma 5.2, the twist $\mathcal{F}_{\Omega'}$ corresponding to $\Omega' = \Omega - t^k \delta_{\mathrm{CE}}\xi$ is equivalent to $\mathcal{F}_\Omega$. Moreover, $\mathcal{F}_{\Omega'}$ coincides with $\hat{\mathcal{F}}$ up to order k, since $\mathcal{F}_{\Omega'}$ coincides with $\mathcal{F}_\Omega$ and

$$(F_{\Omega'})_{k+1} = (F_\Omega)_{k+1} + \tfrac{1}{2}\delta_{\mathrm{CE}}\xi.$$

Therefore the skew-symmetric part of $(F_{\Omega'})_{k+1} - \hat{F}_{k+1}$ is vanishing and this difference is exact with respect to the differential defined in (A-1). Applying Lemma B.2, we can see that $\mathcal{F}_{\Omega'}$ is equivalent to $\hat{\mathcal{F}}$ up to order $k+1$. The claim follows by induction. □

Summing up all the above lemmas we obtain the following characterization of the equivalence classes of twists:

Theorem 5.6 (Classification of twists). *Let $\mathfrak{g}$ be a Lie algebra over R such that $\mathfrak{g}$ is free and finite-dimensional and let $r \in \Lambda^2\mathfrak{g}$ be a classical r-matrix such that $\sharp$ is bijective. Then the set of equivalence classes of twists* $\mathrm{Twist}(\mathscr{U}(\mathfrak{g}), r)$ *with r-matrix r is in bijection to* $\mathrm{H}^2_{\mathrm{CE}}(\mathfrak{g})[[t]]$ *via $\Omega \mapsto \mathcal{F}_\Omega$.*

It is important to remark that even for an abelian Lie algebra $\mathfrak{g}$ the second Chevalley–Eilenberg cohomology $\mathrm{H}^2_{\mathrm{CE}}(\mathfrak{g})[[t]]$ is different from zero. Thus, not all twists are equivalent. An example of a Lie algebra with trivial $\mathrm{H}^2_{\mathrm{CE}}(\mathfrak{g})[[t]]$ is the two-dimensional nonabelian Lie algebra:

Example 5.7 ($ax+b$). Let us consider the two-dimensional Lie algebra given by the R-span of the elements $X, Y \in \mathfrak{g}$ fulfilling

$$[X, Y] = Y, \tag{5-6}$$

with r-matrix $r = X \wedge Y$. We denote the dual basis of $\mathfrak{g}^*$ by $\{X^*, Y^*\}$. Since $\mathfrak{g}$ is two-dimensional, all elements of $\Lambda^2\mathfrak{g}^*$ are a multiple of $X^* \wedge Y^*$, which is closed for dimensional reasons. For Y^* we have

$$(\delta_{\mathrm{CE}}Y^*)(X, Y) = -Y^*([X, Y]) = -Y^*(Y) = -1. \tag{5-7}$$

Therefore $\delta_{\mathrm{CE}}Y^* = -X^* \wedge Y^*$ and $\mathrm{H}^2_{\mathrm{CE}}(\mathfrak{g}) = \{0\}$. From Theorem 5.6 we can therefore conclude that all twists with r-matrix r of $\mathfrak{g}$ are equivalent.

Remark 5.8 (Original construction of Drinfeld). Let us briefly recall the original construction of Drinfeld [1983, Theorem 6]: as a first step he uses the inverse

$B \in \Lambda^2\mathfrak{g}^*$ of r as a 2-cocycle to extend $\mathfrak{g}$ to $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ by considering the new bracket

$$[(X, \lambda), (X', \lambda')]_{\tilde{\mathfrak{g}}} = ([X, X']_{\mathfrak{g}}, B(X, X')), \tag{5-8}$$

where $X, X' \in \mathfrak{g}$ and $\lambda, \lambda' \in \mathbb{R}$. On $\tilde{\mathfrak{g}}^*$ one has the canonical star product quantizing the linear Poisson structure $\star_{DG}$ according to Drinfeld and Gutt [Gutt 1983]. Inside $\tilde{\mathfrak{g}}^*$ one has an affine subspace defined by $H = \mathfrak{g}^* + \ell_0$ where ℓ_0 is the linear functional $\ell_0 : \tilde{\mathfrak{g}} \ni (X, \lambda) \mapsto \lambda$. Since the extension is central, $\star_{DG}$ turns out to be tangential to H, therefore it restricts to an associative star product on H. In a final step, Drinfeld then uses a local diffeomorphism $G \to H$ by mapping g to $\mathrm{Ad}^*_{g^{-1}} \ell_0$ to pull back the star product to G, which turns out to be left-invariant. By [Drinfeld 1983, Theorem 1] this gives a twist. Without major modification it should be possible to include also closed higher order terms $\Omega \in t\Lambda^2\mathfrak{g}^*[[t]]$ by considering $B + \Omega$ instead. We conjecture that

(i) this gives all possible classes of Drinfeld twists by modifying his construction including Ω,

(ii) the resulting classification matches the classification by our Fedosov construction.

Note that a direct comparison of the two approaches will be nontrivial due to the presence of the combinatorics in the BCH formula inside $\star_{DG}$ in the Drinfeld construction on the one hand and the recursion in our Fedosov approach on the other hand. We will come back to this in a future project.

6. Hermitian and completely positive deformations

In this section we bring aspects of positivity into the picture. In addition, let R be now an ordered ring and set $\mathsf{C} = \mathsf{R}(\mathrm{i})$ where $\mathrm{i}^2 = -1$. In C we have a complex conjugation as usual, denoted by $z \mapsto \bar{z}$. The Lie algebra $\mathfrak{g}$ will now be a Lie algebra over R, still being free as a R-module with finite dimension.

The formal power series $\mathsf{R}[[t]]$ are then again an ordered ring in the usual way and we have $\mathsf{C}[[t]] = (\mathsf{R}[[t]])(\mathrm{i})$. Moreover, we consider a *-algebra $\mathscr{A}$ over C which we would like to deform. Here we are interested in *Hermitian* deformations $\star$, where we require

$$(a \star b)^* = b^* \star a^* \quad \text{for all } a, b \in \mathscr{A}[[t]]. \tag{6-1}$$

Instead of the universal enveloping algebra directly, we consider now the complexified universal enveloping algebra $\mathscr{U}_{\mathsf{C}}(\mathfrak{g}) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathsf{R}} \mathsf{C} = \mathscr{U}(\mathfrak{g}_{\mathsf{C}})$, where $\mathfrak{g}_{\mathsf{C}} = \mathfrak{g} \otimes_{\mathsf{R}} \mathsf{C}$ is the complexified Lie algebra. Then this is a *-Hopf algebra, where the *-involution is determined by the requirement

$$X^* = -X \tag{6-2}$$

for $X \in \mathfrak{g}$, i.e., the elements of $\mathfrak{g}$ are *anti-Hermitian*. The needed compatibility of the action of $\mathfrak{g}$ on $\mathscr{A}$ with the *-involution is then

$$(\xi \triangleright a)^* = S(\xi)^* \triangleright a^* \tag{6-3}$$

for all $\xi \in \mathscr{U}_{\mathbb{C}}(\mathfrak{g})$ and $a \in \mathscr{A}$. This is equivalent to $(X \triangleright a)^* = X \triangleright a^*$ for $X \in \mathfrak{g}$. We also set the elements of $\mathfrak{g}^* \subseteq \mathfrak{g}^*_{\mathbb{C}}$ to be *anti-Hermitian*.

In a first step we extend the complex conjugation to tensor powers of $\mathfrak{g}^*_{\mathbb{C}}$ and hence to the complexified Fedosov algebra

$$\mathcal{W}_{\mathbb{C}} \otimes \Lambda^\bullet_{\mathbb{C}} = \left(\prod_{k=0}^{\infty} \mathrm{S}^k \mathfrak{g}^*_{\mathbb{C}} \otimes \Lambda^\bullet \mathfrak{g}^*_{\mathbb{C}} \right) [\![t]\!] \tag{6-4}$$

and obtain a (graded) *-involution, i.e.,

$$((f \otimes \alpha) \cdot (g \otimes \beta))^* = (-1)^{ab} (g \otimes \beta)^* \cdot (f \otimes \alpha)^*, \tag{6-5}$$

where a and b are the antisymmetric degrees of α and β, respectively.

Let $\pi \in \mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}}$ have antisymmetric and symmetric parts $\pi_- \in \Lambda^2 \mathfrak{g}_{\mathbb{C}}$ and $\pi_+ \in \Lambda^2 \mathfrak{g}_{\mathbb{C}}$, respectively. Then for the corresponding operator $\mathcal{P}_\pi$ as in (2-13),

$$\mathsf{T} \circ \overline{\mathcal{P}_\pi (a \otimes b)} = \mathcal{P}_{\tilde{\pi}} \circ \mathsf{T}(\bar{a} \otimes \bar{b}), \tag{6-6}$$

where $\tilde{\pi} = \bar{\pi}_+ - \bar{\pi}_-$. In particular, we have $\tilde{\pi} = \pi$ if and only if π_+ is Hermitian and π_- is anti-Hermitian. We set $t = \mathrm{i}t$ for the formal parameter as in the previous sections, i.e., we want to treat t as imaginary. Then we arrive at the following statement:

Lemma 6.1. *Let $\pi = \pi_+ + \pi_- \in \mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}}$. Then the fiberwise product*

$$a \circ_\pi b = \mu \circ \mathrm{e}^{\frac{1}{2}\mathrm{i}t\mathcal{P}_\pi} (a \otimes b) \tag{6-7}$$

satisfies $(a \circ_\pi b)^* = (-1)^{ab} b^* \circ a^*$ if and only if π_+ is anti-Hermitian and π_- is Hermitian.

This lemma is now the motivation to take a *real* classical r-matrix $r \in \Lambda^2 \mathfrak{g} \subseteq \Lambda^2 \mathfrak{g}_{\mathbb{C}}$. Moreover, writing the symmetric part of π as $\pi_+ = \mathrm{i}s$, then $s = \bar{s} \in \mathrm{S}^2 \mathfrak{g}$ is Hermitian as well. In the following we shall assume that these reality conditions are satisfied.

It is now not very surprising that with such a Poisson tensor π on $\mathfrak{g}$ we can achieve a Hermitian deformation of a *-algebra $\mathscr{A}$ by the Fedosov construction. We summarize the relevant properties in the following proposition:

Proposition 6.2. *Let $\pi = r + \mathrm{i}s$ with a real strongly nondegenerate r-matrix $r \in \Lambda^2 \mathfrak{g}$ and a real symmetric $s \in \mathrm{S}^2 \mathfrak{g}$ such that there exists a symplectic torsion-free covariant derivative ∇ for $\mathfrak{g}$ with $\nabla s = 0$.*

(i) *The operators* δ, δ^{-1}, *and* σ *are real.*

(ii) *The operator* D *is real and* $D^2 = (1/\mathrm{i}t)\,\mathrm{ad}(R)$ *with a Hermitian curvature* $R = R^*$.

(iii) *Suppose that* $\Omega = \Omega^* \in \Lambda^2 \mathfrak{g}^*_{\mathsf{C}}[[t]]$ *is a formal series of Hermitian* δ_{CE}*-closed two-forms. Then the unique* $\varrho \in \mathcal{W}_2 \otimes \Lambda^1$ *with*

$$\delta\varrho = R + D\varrho + \frac{1}{\mathrm{i}t}\varrho \circ_\pi \varrho + \Omega \tag{6-8}$$

and $\delta^{-1}\varrho = 0$ *is Hermitian, too. In this case, the Fedosov derivative* $\mathscr{D}_{\mathrm{F}} = -\delta + D + 1/(\mathrm{i}t)\,\mathrm{ad}(\varrho)$ *is real.*

Suppose now in addition that $\mathscr{A}$ *is a* **-algebra over* C *with a* **-action of* $\mathfrak{g}$, *i.e.*, (6-3).

(iv) *The operator* $L_{\mathscr{A}}$ *as well as the extended Fedosov derivation* $\mathscr{D}_{\mathscr{A}}$ *are real.*

(v) *The Fedosov–Taylor series* $\tau_{\mathscr{A}}$ *is real.*

(vi) *The formal deformation* $\star$ *from Theorem 3.6 is a Hermitian deformation.*

When we apply this to the twist itself we first have to clarify which *-involution we take on the tensor algebra $\mathrm{T}^\bullet(\mathscr{U}_{\mathsf{C}}(\mathfrak{g}))$: by the universal property of the tensor algebra, there is a unique way to extend the *-involution of $\mathscr{U}_{\mathsf{C}}(\mathfrak{g})$ as a *-involution. With respect to this *-involution we have $r^* = -r$ since r is not only real as an element of $\mathfrak{g}_{\mathsf{C}} \otimes \mathfrak{g}_{\mathsf{C}}$ but also antisymmetric, causing an additional sign with respect to the *-involution of $\mathrm{T}^\bullet(\mathscr{U}_{\mathsf{C}}(\mathfrak{g}))$. Analogously, we have $s^* = s$ for the real and symmetric part of π.

Corollary 6.3. *The Fedosov twist* $\mathcal{F}$ *is Hermitian.*

Proof. Indeed, $1 \in \mathscr{U}_{\mathsf{C}}(\mathfrak{g})$ is Hermitian and hence $(1 \star 1)^* = 1^* \star 1^* = 1 \star 1$. □

Up to now we have not yet used the fact that R is ordered but only that we have a *-involution. The ordering of R allows one to transfer concepts of positivity from R to every *-algebra over C. Recall that a linear functional $\omega : \mathscr{A} \to \mathsf{C}$ is called *positive* if

$$\omega(a^* a) \geq 0 \tag{6-9}$$

for all $a \in \mathscr{A}$. This allows one to define an algebra element $a \in \mathscr{A}$ to be *positive* if $\omega(a) \geq 0$ for all positive ω. Note that the positive elements denoted by $\mathscr{A}^+$, form a convex cone in $\mathscr{A}$ and $a \in \mathscr{A}^+$ implies $b^* a b \in \mathscr{A}^+$ for all $b \in \mathscr{A}$. Moreover, elements of the form $a = b^* b$ are clearly positive: their convex combinations are denoted by $\mathscr{A}^{++}$ and are called *algebraically positive*. More details on these notions of positivity can be found in [Bursztyn and Waldmann 2001; 2005a; Waldmann 2005].

Since with R also $\mathsf{R}[[t]]$ is ordered, one can compare the positive elements of $\mathscr{A}$ and the ones of $(\mathscr{A}[[t]], \star)$, where $\star$ is a Hermitian deformation. The first trivial observation is that for a positive linear functional $\boldsymbol{\omega} = \omega_0 + t\omega_1 + \cdots$ of the deformed algebra, i.e., $\boldsymbol{\omega}(a^* \star a) \geq 0$ for all $a \in \mathscr{A}[\![t]\!]$ the classical limit ω_0 of $\boldsymbol{\omega}$ is a positive functional of the undeformed algebra. The converse need not be true: one has examples where a positive ω_0 is not directly positive for the deformed algebras, i.e., one needs higher order corrections, and one has examples where one simply can not find such higher order corrections at all; see [Bursztyn and Waldmann 2000; 2005b]. One calls the deformation $\star$ a *positive deformation* if every positive linear functional ω_0 of the undeformed algebra $\mathscr{A}$ can be deformed into a positive functional $\boldsymbol{\omega} = \omega_0 + t\omega_1 + \cdots$ of the deformed algebra $(\mathscr{A}[\![t]\!], \star)$. Moreover, since also $\mathrm{M}_n(\mathscr{A})$ is a *-algebra in a natural way we call $\star$ a *completely positive deformation* if for all n the canonical extension of $\star$ to $\mathrm{M}_n(\mathscr{A})[\![t]\!]$ is a positive deformation of $\mathrm{M}_n(\mathscr{A})$; see [Bursztyn and Waldmann 2005b]. Finally, if no higher order corrections are needed, then $\star$ is called a *strongly positive deformation*; see [Bursztyn and Waldmann 2000, Definition 4.1]

In a next step we want to use a Kähler structure for $\mathfrak{g}$. In general, this will not exist so we have to require it explicitly. In detail, we want to be able to find a basis $e_1, \ldots, e_n, f_1, \ldots, f_n \in \mathfrak{g}$ with the property that the r-matrix decomposes into

$$(e^k \otimes f^\ell)(r) = A^{k\ell} = -(f^\ell \otimes e^k)(r), \quad (e^k \otimes e^\ell)(r) = B^{k\ell} = -(f^k \otimes f^\ell)(r) \tag{6-10}$$

with a symmetric matrix $A = A^{\mathrm{T}} \in \mathrm{M}_n(\mathsf{R})$ and an antisymmetric matrix $B = -B^{\mathrm{T}} \in \mathrm{M}_n(\mathsf{R})$. We set

$$s = A^{k\ell}(e_k \otimes e_\ell + f_k \otimes f_\ell) + B^{k\ell} e_k \otimes f_\ell + B^{k\ell} f_\ell \otimes e_k. \tag{6-11}$$

The requirement of being *Kähler* is now that first we find a symplectic covariant derivative ∇ with $\nabla s = 0$. Second, we require the symmetric two-tensor s to be positive in the sense that for all $x \in \mathfrak{g}^*$ we have $(x \otimes x)(s) \geq 0$. In this case we call s (and the compatible ∇) a Kähler structure for r. We have chosen this more coordinate-based formulation over the invariant one since in the case of an ordered ring R instead of the reals $\mathbb{R}$ it is more convenient to start directly with the nice basis we need later on.

As usual we consider now $\mathfrak{g}_{\mathsf{C}}$ with the vectors

$$Z_k = \tfrac{1}{2}(e_k - \mathrm{i} f_k) \quad \text{and} \quad \bar{Z}_\ell = \tfrac{1}{2}(e_\ell + \mathrm{i} f_\ell), \tag{6-12}$$

which together constitute a basis of the complexified Lie algebra. Finally, we have the complex matrix

$$g = A + \mathrm{i} B \in \mathrm{M}_n(\mathsf{C}), \tag{6-13}$$

which now satisfies the positivity requirement

$$\bar{z}_k g^{k\ell} z_\ell \geq 0 \quad \text{for all } z_1, \ldots, z_n \in \mathsf{C}. \tag{6-14}$$

If our ring R has sufficiently many inverses and square roots, one can even find a basis $e_1, \dots, e_n, f_1, \dots, f_n$ such that g becomes the unit matrix. However, since we want to stay with an arbitrary ordered ring R we do not assume this.

We now use $\pi = r + \mathrm{i}s$ to obtain a fiberwise Hermitian product $\circ_{\mathrm{Wick}}$, called the fiberwise Wick product. Important now is the following explicit form of $\circ_{\mathrm{Wick}}$, which is a routine verification:

Lemma 6.4. *For the fiberwise Wick product $\circ_{\mathrm{Wick}}$ built out of $\pi = r + \mathrm{i}s$ with a Kähler structure s one has*

$$a \circ_{\mathrm{Wick}} b = \mu \circ \mathrm{e}^{2tg^{k\ell}\, \mathrm{i_s}(Z_k) \otimes \mathrm{i_s}(\bar{Z}_\ell)}(a \otimes b), \tag{6-15}$$

where g is the matrix from (6-13).

The first important observation is that the scalar matrix g can be viewed as an element of $\mathrm{M}_n(\mathscr{A})$ for any unital *-algebra. Then we have the following positivity property:

Lemma 6.5. *Let $\mathscr{A}$ be a unital *-algebra over C. Then for all $m \in \mathbb{N}$ and for all $a_{k_1 \cdots k_m} \in \mathscr{A}$ with $k_1, \dots, k_m = 1, \dots, n$*

$$\sum_{k_1, \ell_1, \dots, k_m, \ell_m = 1}^{n} g^{k_1\ell_1} \cdots g^{k_m\ell_m} a^*_{k_1 \cdots k_m} a_{\ell_1 \cdots \ell_m} \in \mathscr{A}^+. \tag{6-16}$$

Proof. First we note that $g^{\otimes m} = g \otimes \cdots \otimes g \in \mathrm{M}_n(\mathsf{C}) \otimes \cdots \otimes \mathrm{M}_n(\mathsf{C}) = \mathrm{M}_{n^m}(\mathsf{C})$ still satisfies the positivity property

$$\sum_{k_1, \ell_1, \dots, k_m, \ell_m = 1}^{n} g^{k_1\ell_1} \cdots g^{k_m\ell_m} \overline{z^{(1)}_{k_1}} \cdots \overline{z^{(m)}_{k_m}} z^{(1)}_{\ell_1} \cdots z^{(m)}_{\ell_m} \geq 0 \quad \text{for all } z^{(1)}, \dots, z^{(m)} \in \mathsf{C}^n$$

as the left-hand side clearly factorizes into m copies of the left hand side of (6-14). Hence $g^{\otimes m} \in \mathrm{M}_{n^m}(\mathsf{C})$ is a positive element. For a given positive linear functional $\omega : \mathscr{A} \to \mathsf{C}$ and $b_1, \dots, b_N \in \mathscr{A}$ we consider the matrix $(\omega(b_i^* b_j)) \in \mathrm{M}_N(\mathsf{C})$. We claim that this matrix is positive, too. Indeed, with the criterion from [Bursztyn and Waldmann 2001, App. A], for all $z_1, \dots, z_N \in \mathsf{C}$,

$$\sum_{i,j=1}^{N} \bar{z}_i \omega(b_i^* b_j) z_j = \omega\Bigg(\bigg(\sum_{i=1}^{N} z_i b_i\bigg)^* \bigg(\sum_{j=1}^{N} z_j b_j\bigg)\Bigg) \geq 0,$$

hence $(\omega(b_i^* b_j))$ is positive. Putting these statements together we see, for every positive linear functional $\omega : \mathscr{A} \to \mathsf{C}$, for the matrix $\Omega = (\omega(a^*_{k_1\cdots k_m} a_{\ell_1\cdots\ell_m})) \in \mathrm{M}_{n^m}(\mathsf{C})$,

$$\omega\Bigg(\sum_{k_1,\ell_1,\dots,k_m,\ell_m=1}^{n} g^{k_1\ell_1}\cdots g^{k_m\ell_m} a^*_{k_1\cdots k_m} a_{\ell_1\cdots\ell_m}\Bigg) = \sum_{k_1,\ell_1,\dots,k_m,\ell_m=1}^{n} g^{k_1\ell_1}\cdots g^{k_m\ell_m}\,\omega(a^*_{k_1\cdots k_m} a_{\ell_1\cdots\ell_m}) = \mathsf{tr}(g^{\otimes m}\Omega) \geq 0,$$

since the trace of the product of two positive matrices is positive by [Bursztyn and Waldmann 2001, Appendix A]. Note that for a *ring* R one has to use this slightly more complicated argumentation: for a field one could use the diagonalization of g instead. By definition of $\mathscr{A}^+$, this shows the positivity of (6-16). □

Remark 6.6. Suppose that in addition $g = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is diagonal with positive $\lambda_1, \dots, \lambda_n > 0$. In this case one can directly see that the left-hand side of (6-16) is a convex combination of squares and hence in $\mathscr{A}^{++}$. This situation can often be achieved, e.g., for $\mathsf{R} = \mathbb{R}$.

We come now to the main theorem of this section: unlike the Weyl-type deformation, using the fiberwise Wick product yields a positive deformation in a universal way:

Theorem 6.7. *Let $\mathscr{A}$ be a unital *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with a *-action of $\mathfrak{g}$ and let $\Omega = \Omega^* \in \Lambda^2\mathfrak{g}^*_{\mathsf{C}}$ be a formal series of Hermitian δ_{CE}-closed two-forms. Moreover, let s be a Kähler structure for the nondegenerate r-matrix $r \in \mathfrak{g}$ and consider the fiberwise Wick product $\circ_{\mathrm{Wick}}$ yielding the Hermitian deformation $\star_{\mathrm{Wick}}$ as in Proposition 6.2.*

(i) *For all $a \in \mathscr{A}$,*

$$a^* \star_{\mathrm{Wick}} a = \sum_{m=0}^{\infty} \frac{(2t)^m}{m!} \sum_{k_1,\dots,k_m,\ell_1,\dots,\ell_m=1}^{n} g^{k_1\ell_1}\cdots g^{k_m\ell_m} a^*_{k_1\cdots k_m} a_{\ell_1\cdots\ell_m}, \tag{6-17}$$

where $a_{k_1\cdots k_m} = \sigma(\mathrm{i}_{\mathrm{s}}(\bar{Z}_{k_1})\cdots \mathrm{i}_{\mathrm{s}}(\bar{Z}_{k_m})\tau_{\mathrm{Wick}}(a))$.

(ii) *The deformation $\star_{\mathrm{Wick}}$ is strongly positive.*

Proof. From Lemma 6.4 we immediately get (6-17). Now let $\omega : \mathscr{A} \to \mathsf{C}$ be positive. Then also the $\mathsf{C}[[t]]$-linear extension $\omega : \mathscr{A}[[t]] \to \mathsf{C}[[t]]$ is positive with respect to the undeformed product: this is a simple consequence of the Cauchy–Schwarz inequality for ω. Then we apply Lemma 6.5 to conclude that $\omega(a^* \star a) \geq 0$. □

Corollary 6.8. *The Wick-type twist $\mathcal{F}_{\mathrm{Wick}}$ in the Kähler situation is a convex series of positive elements.*

Remark 6.9 (Positive twist). Note that already for a Hermitian deformation, the twist $\mathcal{F} = 1 \star 1 = 1^* \star 1$ constructed as above is a *positive* element of the deformed algebra $\mathrm{T}^\bullet(\mathscr{U}_\mathbb{C}(\mathfrak{g}))[[t]]$. However, this seems to be not yet very significant: it is the statement of Corollary 6.8 and Theorem 6.7 which gives the additional and important feature of the corresponding universal deformation formula.

Appendix A: Hochschild-Kostant-Rosenberg theorem

Let us define the map

$$\partial : \mathscr{U}(\mathfrak{g}) \ni \xi \mapsto \xi \otimes 1 + 1 \otimes \xi - \Delta(\xi) \in \mathscr{U}(\mathfrak{g})^{\otimes 2}, \tag{A-1}$$

and extend it as a graded derivation of degree $+1$ of the tensor product to $\mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$. We recall that the map $\partial : \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g})) \to \mathrm{T}^\bullet(\mathscr{U}(\mathfrak{g}))$ is a differential. Its cohomology is described as follows:

Theorem A.1 (Hochschild–Kostant–Rosenberg). *Let $C \in \mathrm{T}^p(\mathscr{U}(\mathfrak{g}))$ such that $\partial C = 0$. Then there is a $X \in \Lambda^k \mathfrak{g}$ and a $S \in \mathrm{T}^{p-1}(\mathscr{U}(\mathfrak{g}))$ with*

$$C = X + \partial S \tag{A-2}$$

with $X = \mathrm{Alt}(C)$.

We do not prove the above theorem in full generality, since we need only the case $p = 2$. In this case the proof consists of the following two lemmas:

Lemma A.2. *Let $C \in \mathrm{T}^2(\mathscr{U}(\mathfrak{g}))$ with $\partial C = 0$. Then*

(i) $\partial \mathtt{T}(C) = 0$.

(ii) *The antisymmetric part satisfies $C - \mathtt{T}(C) \in \mathfrak{g} \wedge \mathfrak{g} \subseteq \mathrm{T}^2(\mathscr{U}(\mathfrak{g}))$.*

Proof. We have

$$\partial C = 0 \Longleftrightarrow C \otimes 1 + (\Delta \otimes \mathsf{id})(C) = 1 \otimes C + (\mathsf{id} \otimes \Delta)(C).$$

Thus,

$$\begin{aligned} \mathtt{T}(C) \otimes 1 &= (\mathtt{T} \otimes \mathsf{id})(C \otimes 1) \\ &= (\mathtt{T} \otimes \mathsf{id})(1 \otimes C + (\mathsf{id} \otimes \Delta)(C) - (\Delta \otimes \mathsf{id})(C)) \\ &= C_{13} + (\mathtt{T} \otimes \mathsf{id})(\mathsf{id} \otimes \Delta)(C) - (\Delta \otimes \mathsf{id})(C). \end{aligned}$$

Now we apply the cyclic permutation to this equation and get

$$1 \otimes \mathtt{T}(C) = \mathtt{T}(C) \otimes 1 + (\Delta \otimes \mathsf{id})(\mathtt{T}(C)) - (\mathsf{id} \otimes \Delta)(\mathtt{T}(C)),$$

which is equivalent to $\partial \mathtt{T}(C) = 0$. Since ∂ is linear, we get $\partial(T - \mathtt{T}(C)) = 0$ and denote $A = T - \mathtt{T}(C)$, which is now skew-symmetric. We define

$$Q = (\Delta \otimes \mathsf{id})A - A_{23} - A_{13}$$

and get that $Q = -\mathtt{Alt}(Q)$ with the fact that A is ∂-closed. Therefore we have $Q = \mathtt{Alt}^3 Q = (-1)^3 Q = -Q$ and we can conclude $Q = 0$. Thus, A has to be primitive in the first argument and with the skew-symmetry we get the same statement for the second argument. □

Lemma A.3. *Let $C \in \mathrm{T}^2(\mathscr{U}(\mathfrak{g}))$ with $\partial C = 0$. Then there exists a $S \in \mathscr{U}(\mathfrak{g})$ and a $X \in \mathfrak{g} \wedge \mathfrak{g}$, such that*

$$C = X + \partial S, \tag{A-3}$$

where $X = \frac{1}{2}(C - \mathtt{T}(C))$.

Proof. It is clear from Lemma A.2, that X is well defined and we have to prove that symmetric C are ∂-exact. So we assume that $C \in \mathrm{T}^2(\mathscr{U}(\mathfrak{g}))$ is ∂-closed and symmetric. Let k be the highest order appearing in C and assume the claim is true for all $r < k$ (in the sense of the filtration of $\mathscr{U}(\mathfrak{g}) = \bigcup_{n \in \mathbb{N}_0} \mathscr{U}(\mathfrak{g})_n$). Then we can write for a given basis $\{e_i\}_{i \in \{1,\ldots,n\}}$

$$C = \sum_{|\boldsymbol{i}|=k} e_{\boldsymbol{i}} \otimes D^{\boldsymbol{i}} + \text{l.o.t.}.$$

We mean lower order terms with respect to the filtration in the first tensor degree and the $\boldsymbol{i}$ are multi-indices such that $e_{\boldsymbol{i}} = e_{i_1} \cdots e_{i_k}$. We can assume that $D_{\boldsymbol{i}}$ is symmetric in the multi-index, because we can compensate for asymmetry by lower order terms. Since $\partial(\mathscr{U}(\mathfrak{g})_m) \subseteq \mathscr{U}(\mathfrak{g})_{m-1} \otimes \mathscr{U}(\mathfrak{g})_{m-1}$, we see that $\partial C = 0$ implies that $\partial D^{\boldsymbol{i}} = 0$, which is equivalent to $D^{\boldsymbol{i}} \in \mathfrak{g}$. Therefore, we can write

$$C = \sum_{|\boldsymbol{i}|=k} D^{\boldsymbol{i},j} e_{\boldsymbol{i}} \otimes e_j + H,$$

where $H \in \mathscr{U}(\mathfrak{g})_{k-1} \otimes \mathscr{U}(\mathfrak{g})$ is now of order strictly less then k in the first argument. Now we expand $H = \sum_{|\boldsymbol{i}_1|,|\boldsymbol{i}_2| \leq k-1} H_{\boldsymbol{i}_1,\boldsymbol{i}_2} e_{\boldsymbol{i}_1} \otimes e_{\boldsymbol{i}_2}$ and see, by using

$$\begin{aligned} 0 = \partial C &= \sum_{|\boldsymbol{i}|=k} D^{\boldsymbol{i},j} \partial(e_{\boldsymbol{i}}) \otimes e_j + \partial H \\ &= -D^{i_1,\ldots,i_k,j} \sum_r e_{i_1} \cdots \widehat{e_{i_r}} \cdots e_{i_k} \otimes e_{i_r} \otimes e_j + \partial H + \text{l.o.t.}, \end{aligned}$$

that H has to be of the form

$$H = \sum_{|\boldsymbol{i}_1|=k-1, |\boldsymbol{i}_2|=2} H_{\boldsymbol{i}_1,\boldsymbol{i}_2} e_{\boldsymbol{i}_1} \otimes e_{\boldsymbol{i}_2} + \text{l.o.t.},$$

and hence

$$\partial H = \sum_{|\boldsymbol{i}_1|=k-1, j_1, j_2} H_{\boldsymbol{i}_1, j_1, j_2} e_{\boldsymbol{i}_1} \otimes e_{j_1} \otimes e_{j_2} + \text{l.o.t.}.$$

This implies that $D^{i_1,\dots,i_k,j}$ is symmetric in all indices, since $\partial C = 0$ and $H_{\boldsymbol{i}_1,j_1,j_2} = H_{\boldsymbol{i}_1,j_2,j_1}$. Thus for

$$G = \frac{1}{k+1} D^{i_1,\dots,i_{k+1}} e_{i_1} \cdots e_{i_{k+1}},$$

we have

$$\partial G = -\sum_{|\boldsymbol{i}|=k} D^{\boldsymbol{i},j}(e_{\boldsymbol{i}} \otimes e_j + e_j \otimes e_{\boldsymbol{i}}) + \text{l.o.t.}.$$

Note that here the lower order terms are meant in both tensor arguments. Using the symmetry of C, we obtain

$$C = \sum_{|\boldsymbol{i}|=k} D^{\boldsymbol{i},j}(e_{\boldsymbol{i}} \otimes e_j + e_j \otimes e_{\boldsymbol{i}}) + \text{l.o.t.},$$

again the lower order terms are in both tensor factors. Thus,

$$C + \partial G \in \mathscr{U}(\mathfrak{g})_{k-1} \otimes \mathscr{U}(\mathfrak{g})_{k-1}.$$

This implies the lemma, because for $k = 0$ the statement is trivial. □

Corollary A.4. *Let $C \in \mathrm{T}^2(\mathscr{U}(\mathfrak{g}))$ with $\partial C = 0$ and $(\epsilon \otimes \mathsf{id})C = (\mathsf{id} \otimes \epsilon)C = 0$. Then we can find $S \in \mathscr{U}(\mathfrak{g})$ and $X \in \Lambda^2\mathfrak{g}$ such that $C = X + \partial S$ with $\epsilon(S) = 0$.*

Proof. The statement is clear from the construction of Lemma A.2. □

Appendix B: Technical Lemmas

In this section we prove several technical results necessary for the proofs is Section 5.

Lemma B.1. *Let $\mathcal{F}, \mathcal{F}' \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[\![t]\!]$ be two twists coinciding up to order k. Then*

$$\partial(F_{k+1} - F'_{k+1}) = 0. \tag{B-1}$$

Proof. We have

$$\begin{aligned}
\partial(F_{k+1}) &= 1 \otimes F_{k+1} - F_{k+1} \otimes 1 + (\mathsf{id} \otimes \Delta)(F_{k+1}) - (\Delta \otimes \mathsf{id})(F_{k+1}) \\
&= \sum_{i=0}^{k+1} (1 \otimes F_i)(\mathsf{id} \otimes \Delta)(F_{k+1-i}) - \sum_{i=1}^{k} (1 \otimes F_i)(\mathsf{id} \otimes \Delta)(F_{k+1-i}) \\
&\quad + \sum_{i=1}^{k} (F_i \otimes 1)(\Delta \otimes \mathsf{id})(F_{k+1-i}) - \sum_{i=0}^{k+1} (F_i \otimes 1)(\Delta \otimes \mathsf{id})(F_{k+1-i}) \\
&= -\sum_{i=1}^{k} (1 \otimes F_i)(\mathsf{id} \otimes \Delta)(F_{k+1-i}) + \sum_{i=1}^{k} (F_i \otimes 1)(\Delta \otimes \mathsf{id})(F_{k+1-i}) \\
&= -\sum_{i=1}^{k} (1 \otimes F'_i)(\mathsf{id} \otimes \Delta)(F'_{k+1-i}) + \sum_{i=1}^{k} (F'_i \otimes 1)(\Delta \otimes \mathsf{id})(F'_{k+1-i}) \\
&= \partial(F'_{k+1}).
\end{aligned}$$ □

Lemma B.2. *Let $\mathcal{F}, \mathcal{F}' \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$ be two twists coinciding up to order k such that*

$$F_{k+1} - F'_{k+1} = \partial T_{k+1}. \tag{B-2}$$

Then they are equivalent up to order $k+1$.

Proof. Consider $\exp(t^{k+1}T_{k+1}) = 1 + t^{k+1}T_{k+1} + \mathcal{O}(t^{k+2})$. Then we have

$$(\Delta(\exp(t^{k+1}T_{k+1}))\mathcal{F})_i = (\mathcal{F}'(\exp(t^{k+1}T_{k+1}) \otimes \exp(t^{k+1}T_{k+1})))_i$$

for any $i \leq k+1$. Note that, because

$$(\epsilon \otimes \mathsf{id})(F_{k+1} - F'_{k+1}) = (\mathsf{id} \otimes \epsilon)(F_{k+1} - F'_{k+1}) = 0,$$

we can choose T_{k+1} such that $\epsilon(T_{k+1}) = 0$ and therefore $\epsilon(\exp(t^{k+1}T_{k+1})) = 1$. □

Lemma B.3. *Let $\mathcal{F}, \mathcal{F}' \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$ be two equivalent twists coinciding up to order k. Then there exists a $T = 1 + t^k T_k + \mathcal{O}(t^{k+1}) \in \mathscr{U}(\mathfrak{g})[[t]]$ such that*

$$\Delta(T)\mathcal{F}' = \mathcal{F}(T \otimes T). \tag{B-3}$$

Proof. Since the twists $\mathcal{F}$ and $\mathcal{F}'$ are equivalent, there is a $\tilde{T} = 1 + t^\ell \tilde{T}_\ell + \mathcal{O}(t^{\ell+1})$ such that

$$\Delta(\tilde{T})F' = F(\tilde{T} \otimes \tilde{T}).$$

Let us consider $\ell \leq k$. The above equation at order ℓ reads

$$\Delta(\tilde{T}_\ell) + F'_\ell = F_\ell + \tilde{T}_\ell \otimes 1 + 1 \otimes \tilde{T}_\ell.$$

Therefore, since $\mathcal{F}$ and $\mathcal{F}'$ coincide up to order k,

$$\Delta(\tilde{T}_\ell) = \tilde{T}_\ell \otimes 1 + 1 \otimes \tilde{T}_\ell,$$

and $\tilde{T}_\ell \in \mathfrak{g} \subseteq \mathscr{U}(\mathfrak{g})$. For $\ell < k$ we get at order $\ell + 1$

$$\Delta(\tilde{T}_{\ell+1}) + \Delta(\tilde{T}_\ell)F_1' + F_{\ell+1}' = F_{\ell+1} + F_1(\tilde{T}_\ell \otimes 1 + 1 \otimes \tilde{T}_\ell) + \tilde{T}_{\ell+1} \otimes 1 + 1 \otimes \tilde{T}_{\ell+1}.$$

The skew-symmetrization of the above equation gives

$$(\tilde{T}_\ell \otimes 1 + 1 \otimes \tilde{T}_\ell)r = r(\tilde{T}_\ell \otimes 1 + 1 \otimes \tilde{T}_\ell).$$

An easy computation shows that this property is equivalent to $\delta_{\mathrm{CE}}\tilde{T}_\ell^\flat = 0$. Thus, we can define the map $S : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$ by defining it on primitive elements via

$$\mathfrak{g} \ni \xi \mapsto \tilde{T}_\ell^\flat(\xi) \cdot 1 \in \mathscr{U}(\mathfrak{g})$$

and extend it as a derivation of the product of $\mathscr{U}(\mathfrak{g})$. This map allows us to define an element

$$A = \frac{1}{t}(\epsilon \circ S \otimes \mathsf{id})[\mathcal{F}] = -\tilde{T}_\ell + \mathcal{O}(t),$$

which fulfills $\Delta(A)\mathcal{F} = \mathcal{F}(A \otimes 1 + 1 \otimes A)$ and $\epsilon(A) = 0$. Thus we get

$$\exp(t^\ell A)\mathcal{F} = \mathcal{F}(\exp(t^\ell A) \otimes \exp(t^\ell A)) \quad \text{as well as} \quad \epsilon(\exp(t^\ell A)) = 1.$$

We define $T = \exp(t^\ell A)\tilde{T}$ and obtain $\Delta(T)\mathcal{F}' = \mathcal{F}(T \otimes T)$ and

$$T = 1 + t^{\ell+1}T_{\ell+1} + \mathcal{O}(t^{\ell+2}).$$

Repeating this method $k - \ell$ times, we get an equivalence starting at order k. □

Lemma B.4. *Let $\mathcal{F}, \mathcal{F}' \in (\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))[[t]]$ be two equivalent twists coinciding up to order k. Then there exists an element $\xi \in \mathfrak{g}^*$ such that*

$$([F_{k+1} - F_{k+1}'])^\flat = \delta_{\mathrm{CE}}\xi. \tag{B-4}$$

Proof. First, $[F_{k+1} - F_{k+1}'] \in \Lambda^2\mathfrak{g}$, because of Theorem A.1 and since, as in Lemma B.1, $\partial(F_{k+1} - F_{k+1}') = 0$. From Lemma B.3 we know that we can find an element $T = 1 + t^k T_k + \mathcal{O}(t^{k+1})$ in $\mathscr{U}(\mathfrak{g})$ such that $\Delta(T)\mathcal{F}' = \mathcal{F}(T \otimes T)$. At order k this reads

$$\Delta(T_k) + F_k' = F_k + T_k \otimes 1 + 1 \otimes T_k,$$

which is equivalent to $T_k \in \mathfrak{g}$, because $F_k' = F_k$. At order $k + 1$, we can see that

$$\Delta(T_{k+1}) + \Delta(T_k)F_1' + F_{k+1}' = F_{k+1} + F_1(T_k \otimes 1 + 1 \otimes T_k) + T_{k+1} \otimes 1 + 1 \otimes T_{k+1}.$$

For the skew-symmetric part we have

$$[F_{k+1} - F_{k+1}'] = (T_k \otimes 1 + 1 \otimes T_k)r - r(T_k \otimes 1 + 1 \otimes T_k) = [T_k \otimes 1 + 1 \otimes T_k, r],$$

which is equivalent to $([F_{k+1} - F'_{k+1}])^\flat = -\delta_{\mathrm{CE}} T_k^\flat$. □

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