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Let $\{f_t\}$ be a family of complex polynomial functions with *line* singularities. We show that if $\{f_t\}$ has a *uniform stable radius* (for the corresponding Milnor fibrations), then the Lê numbers of the functions f_t are independent of t for all small t. A similar assertion was proved by M. Oka and D. B. O'Shea in the case of isolated singularities — a case for which the only nonzero Lê number coincides with the Milnor number.

By combining our result with a theorem of J. Fernández de Bobadilla, we conclude that a family of line singularities in $\mathbb{C}^n$, $n \geq 5$, is topologically trivial if it has a uniform stable radius.

As an important example, we show that families of weighted homogeneous line singularities have a uniform stable radius if the nearby fibres $f_t^{-1}(\eta)$, $\eta \neq 0$, are "uniformly" nonsingular with respect to the deformation parameter t.

1. Introduction

Let $(t, z) := (t, z_1, \dots, z_n)$ be linear coordinates for $\mathbb{C} \times \mathbb{C}^n$ ($n \geq 2$), and let

$$f\colon (\mathbb{C} \times \mathbb{C}^n, \mathbb{C} \times \{\mathbf{0}\}) \to (\mathbb{C}, 0), \quad (t, z) \mapsto f(t, z), \tag{1-1}$$

be a polynomial function. As usual, we write $f_t(z) := f(t, z)$, and for any $\eta \in \mathbb{C}$ we denote by $V(f_t - \eta)$ the hypersurface in $\mathbb{C}^n$ defined by the equation $f_t(z) = \eta$. (Note that (1-1) implies $f_t(\mathbf{0}) = f(t, \mathbf{0}) = 0$, so that the origin $\mathbf{0} \in \mathbb{C}^n$ belongs to the hypersurface $V(f_t) = f_t^{-1}(0)$ for all $t \in \mathbb{C}$.)

The purpose of this paper is to show that if the polynomial function f defines a family $\{f_t\}$ of hypersurfaces with *line* singularities and with a *uniform stable radius* (for the corresponding Milnor fibrations), then the Lê numbers

$$\lambda^0_{f_t,z}(\mathbf{0}), \dots, \lambda^{n-1}_{f_t,z}(\mathbf{0})$$

of the polynomial functions f_t at $\mathbf{0}$ with respect to the coordinates z — which do exist in this case — are independent of t for all small t (see Theorem 4.1). In the

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case of hypersurfaces with *isolated* singularities — a case for which the constancy of the Lê numbers means the constancy of the Milnor number — a similar assertion was proved by M. Oka [1973] and D.B. O'Shea [1983a].

By combining Theorem 4.1 with a theorem of J. Fernández de Bobadilla [2013], to the effect that a family of hypersurfaces with line singularities in $\mathbb{C}^n$, $n \geq 5$, is topologically trivial if it has constant Lê numbers, it follows that a family of hypersurfaces with line singularities in $\mathbb{C}^n$, $n \geq 5$, is topologically trivial if it has a uniform stable radius (see Corollary 4.2).

Oka [1973] and O'Shea [1983a] also proved that, if $\{f_t\}$ is a family of *isolated* hypersurface singularities such that each f_t is *weighted homogeneous* with respect to a given system of weights, then $\{f_t\}$ has a uniform stable radius. In Theorem 5.1, we show this still holds true for weighted homogeneous hypersurfaces with *line* singularities provided that the nearby fibres $V(f_t - \eta)$, $\eta \neq 0$, are "uniformly" nonsingular with respect to the deformation parameter t — that is, nonsingular in a small ball the radius of which does not depends on t. (Note that this condition always holds true for isolated singularities.) In particular, by Theorem 4.1 and Corollary 4.2, such families have constant Lê numbers, and for $n \geq 5$, they are topologically trivial.

Finally, let us observe that by combining Corollary 4.2 with a theorem of Oka [1982] — which says that a family $\{f_t\}$ of nondegenerate functions with constant Newton boundary has a uniform stable radius — we get a new proof of a theorem of J. Damon [1983] which says that if $\{f_t\}$ is a family of nondegenerate line singularities in $\mathbb{C}^n$, $n \geq 5$, with constant Newton boundary, then $\{f_t\}$ is topologically trivial.

Notation 1.1. In this paper, we are only interested in the behaviour of functions (or hypersurfaces) near the origin $\mathbf{0} \in \mathbb{C}^n$. We denote by B_ε the closed ball centred at $\mathbf{0} \in \mathbb{C}^n$ with radius $\varepsilon > 0$, and we write $\mathring{B}_\varepsilon$ and S_ε for its interior and boundary, respectively. As usual, in $\mathbb{C}$, we write D_ε and $\mathring{D}_\varepsilon$ rather than B_ε and $\mathring{B}_\varepsilon$.

2. Uniform stable radius

By [Hamm and Lê 1973, lemme (2.1.4)], we know that for each t there exists a positive number $r_t > 0$ such that for any pair $(\varepsilon_t, \varepsilon'_t)$ with $0 < \varepsilon'_t \leq \varepsilon_t \leq r_t$, there exists $\delta(\varepsilon_t, \varepsilon'_t) > 0$ such that for any nonzero complex number η with $0 < |\eta| \leq \delta(\varepsilon_t, \varepsilon'_t)$, the hypersurface $V(f_t - \eta)$ is nonsingular in $\mathring{B}_{r_t}$ and transversely intersects with the sphere $S_{\varepsilon''}$ for any ε'' with $\varepsilon'_t \leq \varepsilon'' \leq \varepsilon_t$. Any such a number r_t is called a *stable radius* for the Milnor fibration of f_t at $\mathbf{0}$ [Oka 1982, §2].

Definition 2.1 [Oka 1982, §3]. We say that the family $\{f_t\}$ has a *uniform stable radius* (we also say that $\{f_t\}$ is *uniformly stable*) if there exist $\tau > 0$ and $r > 0$ such that for any pair $(\varepsilon, \varepsilon')$ with $0 < \varepsilon' \leq \varepsilon \leq r$, there exists $\delta(\varepsilon, \varepsilon') > 0$ such that for any nonzero complex number η with $0 < |\eta| \leq \delta(\varepsilon, \varepsilon')$, the hypersurface $V(f_t - \eta)$

is nonsingular in $\mathring{B}_r$ and transversely intersects with the sphere $S_{\varepsilon''}$ for any ε'' with $\varepsilon' \leq \varepsilon'' \leq \varepsilon$ and for any t with $0 \leq |t| \leq \tau$. Any such a number r is called a *uniform stable radius* for $\{f_t\}$.

In the special case where the polynomial function f defines a family $\{f_t\}$ of *isolated* hypersurface singularities (i.e., f_t has an isolated singularity at $\mathbf{0}$ for all small t), then, by [Milnor 1968], we also know that for each t there exists $R_t > 0$ such that the hypersurface $V(f_t)$ is nonsingular in $\mathring{B}_{R_t} \setminus \{\mathbf{0}\}$ and transversely intersects the sphere S_ρ for any ρ with $0 < \rho \leq R_t$.

Definition 2.2 [Oka 1973, §2]. Suppose that f defines a family $\{f_t\}$ of *isolated* hypersurface singularities. We say that $\{f_t\}$ satisfies *condition* (A) if there exist $\nu > 0$ and $R > 0$ such that $V(f_t)$ is nonsingular in $\mathring{B}_R \setminus \{\mathbf{0}\}$ and transversely intersects the sphere S_ρ for any ρ with $0 < \rho \leq R$ and for any t with $0 \leq |t| \leq \nu$.

It is easy to see that a family $\{f_t\}$ of isolated hypersurface singularities satisfies condition (A) if and only if it has no *vanishing fold* and no *nontrivial critical arc* in the sense of [O'Shea 1983a]. Also, it is worthwhile to observe that if $\{f_t\}$ satisfies condition (A), then it has a uniform stable radius [Oka 1973; O'Shea 1983a].

3. The Oka–O'Shea theorem for isolated singularities

Throughout this section we assume that the polynomial function f defines a family $\{f_t\}$ of *isolated* hypersurface singularities.

Theorem 3.1 [Oka 1973; O'Shea 1983a]. *Suppose that f defines a family $\{f_t\}$ of isolated hypersurface singularities. If furthermore $\{f_t\}$ satisfies condition (A) or has a uniform stable radius, then it is μ-constant — that is, the Milnor number $\mu_{f_t}(\mathbf{0})$ of f_t at $\mathbf{0}$ is independent of t for all small t.*

Actually Oka showed that if $\{f_t\}$ satisfies condition (A) or if it has a uniform stable radius, then the Milnor fibrations at $\mathbf{0}$ of f_0 and f_t are isomorphic.

Lê Dũng Tráng and C. P. Ramanujam [Lê and Ramanujam 1976] showed that for $n \neq 3$ any family of isolated hypersurface singularities with constant Milnor number is topologically $\mathscr{V}$-equisingular. With the same assumption, J. G. Timourian [1977] showed that the family is actually topologically trivial. We recall that a family $\{f_t\}$ is *topologically $\mathscr{V}$-equisingular* (respectively, *topologically trivial*) if there exist open neighbourhoods $D \subseteq \mathbb{C}$ and $U \subseteq \mathbb{C}^n$ of the origins in $\mathbb{C}$ and $\mathbb{C}^n$, together with a continuous map $\varphi\colon (D \times U, D \times \{\mathbf{0}\}) \to (\mathbb{C}^n, \mathbf{0})$ such that for all sufficiently small t, there is an open neighbourhood $U_t \subseteq U$ of $\mathbf{0} \in \mathbb{C}^n$ such that the map

$$\varphi_t\colon (U_t, \mathbf{0}) \to (\varphi(\{t\} \times U_t), \mathbf{0}), \quad z \mapsto \varphi_t(z) := \varphi(t, z),$$

is a homeomorphism satisfying the relation

$$\varphi_t(V(f_0) \cap U_t) = V(f_t) \cap \varphi_t(U_t)$$

(respectively, the relation $f_0 = f_t \circ \varphi_t$ on U_t).

Note that, in general, "μ-constant" does not imply condition (A) [Oka 1989; Briançon].

Finally, observe that the Briançon–Speder famous family shows that condition (A) does not imply the Whitney conditions along the t-axis [Briançon and Speder 1975].

4. Uniformly stable families of line singularities

Setup and statement of the main result. From now on we suppose that the polynomial function f defines a family $\{f_t\}$ of hypersurfaces with *line* singularities. As in [Massey 1988, §4], by such a family we mean a family $\{f_t\}$ such that for each t small enough, the singular locus Σf_t of f_t near the origin $\mathbf{0} \in \mathbb{C}^n$ is given by the z_1-axis, and the restriction of f_t to the hyperplane $V(z_1)$ defined by $z_1 = 0$ has an isolated singularity at the origin. Then, by [Massey 1995, Remark 1.29], the partition of $V(f_t)$ given by

$$\mathscr{S}_t := \big\{V(f_t) \setminus \Sigma f_t,\ \Sigma f_t \setminus \{\mathbf{0}\},\ \{\mathbf{0}\}\big\}$$

is a *good stratification* for f_t at $\mathbf{0}$, and the hyperplane $V(z_1)$ is a *prepolar slice* for f_t at $\mathbf{0}$ with respect to $\mathscr{S}_t$ for all t small enough. In particular, combined with [Massey 1995, Proposition 1.23], this implies that the *Lê numbers*

$$\lambda^0_{f_t,z}(\mathbf{0}) \quad \text{and} \quad \lambda^1_{f_t,z}(\mathbf{0})$$

of f_t at $\mathbf{0}$ with respect to the coordinates z do exist. (For the definitions of good stratifications, prepolarity and Lê numbers, we refer the reader to [Massey 1995].) Note that for line singularities, the only possible nonzero Lê numbers are precisely $\lambda^0_{f_t,z}(\mathbf{0})$ and $\lambda^1_{f_t,z}(\mathbf{0})$. All the other Lê numbers $\lambda^k_{f_t,z}(\mathbf{0})$ for $2 \le k \le n-1$ are defined and equal to zero; see [Massey 1995].

Here is our main observation.

Theorem 4.1. *Suppose that f defines a family $\{f_t\}$ of hypersurfaces with line singularities. If furthermore $\{f_t\}$ has a uniform stable radius, then it is λ_z-constant — that is, the Lê numbers $\lambda^0_{f_t,z}(\mathbf{0})$ and $\lambda^1_{f_t,z}(\mathbf{0})$ are independent of t for all small t.*

Theorem 4.1 extends to line singularities Oka and O'Shea's Theorem 3.1 concerning isolated singularities. Indeed, for isolated singularities, the only possible nonzero Lê number is $\lambda^0_{f_t,z}(\mathbf{0})$ and the latter coincides with the Milnor number $\mu_{f_t}(\mathbf{0})$.

Note that if $\{f_t\}$ is a λ_z-constant family of line singularities in $\mathbb{C}^n$ with $n \ge 5$, then, by a theorem of D. B. Massey [1988, Theorem (5.2)], the diffeomorphism type of the Milnor fibration of f_t at $\mathbf{0}$ is independent of t for all small t. Under the same assumption, Fernández de Bobadilla [2013, Theorem 42] showed that $\{f_t\}$ is actually topologically trivial. Combining this result with our Theorem 4.1 gives the following corollary.

Corollary 4.2. *Suppose that f defines a family $\{f_t\}$ of hypersurfaces with line singularities in $\mathbb{C}^n$ with $n \geq 5$. If furthermore $\{f_t\}$ has a uniform stable radius, then it is topologically trivial.*

Application to families of nondegenerate line singularities with constant Newton boundary. Oka [1982, Corollary 1] showed that if $\{f_t\}$ is a family of hypersurface singularities — not necessary line singularities — such that for all small t the polynomial function f_t is *nondegenerate* and the *Newton boundary* of f_t at $\mathbf{0}$ with respect to the coordinates z is independent of t, then $\{f_t\}$ has a uniform stable radius. (For the definitions of nondegeneracy and Newton boundary, see [Kouchnirenko 1976; Oka 1979].) Combined with Oka's result, Corollary 4.2 provides a new proof of the following result, which is a particular case of a more general theorem of Damon.

Theorem 4.3 [Damon 1983]. *Suppose that f defines a family $\{f_t\}$ of hypersurfaces with line singularities in $\mathbb{C}^n$ with $n \geq 5$. If furthermore for any sufficiently small t the polynomial function f_t is nondegenerate and the Newton boundary of f_t at $\mathbf{0}$ with respect to the coordinates z is independent of t, then the family $\{f_t\}$ is topologically trivial.*

Proof of Theorem 4.1. Consider the map $\Phi\colon \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^2$ defined by

$$(t, z) \mapsto \Phi(t, z) := (f(t, z), t),$$

and pick positive numbers τ and r which satisfy the condition of Definition 2.1. Then, in particular, the following property holds:

($\mathscr{P}$) For any ε with $0 < \varepsilon < r$, there exists $\delta(\varepsilon) > 0$ such that for any t with $0 \leq |t| \leq \tau$ and for any η with $0 < |\eta| \leq \delta(\varepsilon)$, the hypersurface $V(f_t - \eta)$ is nonsingular in $\mathring{B}_r$ and transversely intersects the sphere S_ε.

This property implies that the critical set $\Sigma\Phi$ of Φ does not intersect the set

$$U(\mathring{B}_r) := (\mathring{D}_\tau \times \mathring{B}_r) \cap \Phi^{-1}((\mathring{D}_{\delta(\varepsilon)} \setminus \{\mathbf{0}\}) \times \mathring{D}_\tau).$$

Indeed, suppose there is a point $(t_0, z_0) \in \Sigma\Phi \cap U(\mathring{B}_r)$. Then $z_0 \in \Sigma(f_{t_0} - f_{t_0}(z_0))$. But this is not possible, since by ($\mathscr{P}$) the hypersurface $V(f_{t_0} - f_{t_0}(z_0))$ is smooth. (We recall that a complex variety can never be a smooth manifold throughout a neighbourhood of a critical point; see [Milnor 1968, §2].)

It also follows from property ($\mathscr{P}$) that the map

$$\Phi|_{U(S_\varepsilon)}\colon U(S_\varepsilon) \to (\mathring{D}_{\delta(\varepsilon)} \setminus \{\mathbf{0}\}) \times \mathring{D}_\tau$$

(restriction of Φ to $U(S_\varepsilon) := (\mathring{D}_\tau \times S_\varepsilon) \cap \Phi^{-1}((\mathring{D}_{\delta(\varepsilon)} \setminus \{\mathbf{0}\}) \times \mathring{D}_\tau)$) is a submersion. Indeed, as $\Sigma\Phi \cap U(\mathring{B}_r) = \varnothing$ and $U(\mathring{B}_r)$ is an open subset of $\mathbb{C} \times \mathbb{C}^n$, the map

$$\Phi|_{U(\mathring{B}_r)}\colon U(\mathring{B}_r) \to (\mathring{D}_{\delta(\varepsilon)} \setminus \{\mathbf{0}\}) \times \mathring{D}_\tau$$

is a submersion. Thus, to show that $\Phi|_{U(S_\varepsilon)}$ is a submersion, it suffices to observe that the inclusion $U(S_\varepsilon) \hookrightarrow U(\mathring{B}_r)$ is transverse to the submanifold $\Phi|^{-1}_{U(\mathring{B}_r)}(f(t,z),t)$ for any point $(t,z) \in U(S_\varepsilon)$ — or equivalently that the submanifolds

$$\Phi|^{-1}_{U(\mathring{B}_r)}(f(t,z),t) \quad \text{and} \quad (\{t\} \times S_\varepsilon) \cap U(\mathring{B}_r)$$

are transverse to each other. This is exactly the content of $(\mathscr{P})$.

Now, as $\Phi|_{U(S_\varepsilon)}$ is also a proper map, a result of Massey and D. Siersma [1992, Proposition 1.10] shows that the Milnor number of a generic hyperplane slice of f_t at a point on Σf_t sufficiently close to the origin (which coincides with the Lê number $\lambda^1_{f_t,z}(\mathbf{0})$ for line singularities; see [Lê 1980; Massey 1988]) is independent of t for all small t.

Finally, since the family $\{f_t\}$ has a uniform stable radius — the full strength of this assumption is used here — it follows from [Oka 1982, Lemma 2] that the diffeomorphism type of the Milnor fibration of f_t at the origin is independent of t for all small t. In particular, the reduced Euler characteristic $\tilde{\chi}(F_{f_t,\mathbf{0}})$ of the Milnor fibre $F_{f_t,\mathbf{0}}$ of f_t at $\mathbf{0}$, which by [Massey 1995, Theorem 3.3] equals

$$(-1)^{n-1}\lambda^0_{f_t,z}(\mathbf{0}) + (-1)^{n-2}\lambda^1_{f_t,z}(\mathbf{0}),$$

is independent of t for all small t. The constancy of $\lambda^0_{f_t,z}(\mathbf{0})$ now follows from that of $\lambda^1_{f_t,z}(\mathbf{0})$.

5. Uniform stable radius and weighted homogeneous line singularities

By a result of Oka [1973] and O'Shea [1983a], we know that if $\{f_t\}$ is a family of *isolated* hypersurface singularities such that each f_t is *weighted homogeneous* with respect to a given system of weights, then $\{f_t\}$ satisfies condition (A), and hence, is uniformly stable. Our next observation says this still holds true for weighted homogeneous *line* singularities provided that the nearby fibres $V(f_t - \eta)$, $\eta \neq 0$, of the functions f_t are "uniformly" nonsingular with respect to the deformation parameter t — that is, nonsingular in a small ball the radius of which does *not* depends on t. (We recall that by [Hamm and Lê 1973] the nearby fibres are "individually" nonsingular — that is, nonsingular in a small ball the radius of which depends on t.)

Theorem 5.1. *Suppose that f defines a family $\{f_t\}$ of hypersurfaces with line singularities such that each f_t is weighted homogeneous with respect to a given system of weights $\boldsymbol{w} = (w_1, \dots, w_n)$ on the variables $(z_1, \dots, z_n)$, with $w_i \in \mathbb{N} \setminus \{0\}$. Also, assume that the nearby fibres $V(f_t - \eta)$, $\eta \neq 0$, of the functions f_t are uniformly nonsingular with respect to the deformation parameter t — that is, there exist positive numbers τ, r, δ such that for any $0 < |\eta| \leq \delta$ and $0 \leq |t| \leq \tau$, the hypersurface $V(f_t - \eta)$ is nonsingular in $\mathring{B}_r$. Under these assumptions, the family*

$\{f_t\}$ has a uniform stable radius. (*In particular*, *$\{f_t\}$ is λ_z-constant*, *and for $n \geq 5$*, *it is topologically trivial.*)

Proof. The argument is similar to those used in [Oka 1973; O'Shea 1983a]. Suppose that the family $\{f_t\}$ does not have a uniform stable radius. Then, as the nearby fibres of the functions f_t are uniformly nonsingular with respect to the deformation parameter t, for all $\tau > 0$ and all $r > 0$ small enough, there exist $0 < \varepsilon' \leq \varepsilon \leq r$ such that for all sufficiently small $\delta > 0$ there exist η_δ, ε_δ and t_δ, with $0 < |\eta_\delta| \leq \delta$, $\varepsilon' \leq \varepsilon_\delta \leq \varepsilon$ and $|t_\delta| \leq \tau$, such that $V(f_{t_\delta} - \eta_\delta)$ is nonsingular in $\mathring{B}_r$ and does not transversely intersect the sphere S_{ε_δ}. It follows that there is a point $z_\delta \in V(f_{t_\delta} - \eta_\delta) \cap S_{\varepsilon_\delta}$ which is a critical point of the restriction to $V(f_{t_\delta} - \eta_\delta) \cap B_r$ of the squared distance function:

$$z \in V(f_{t_\delta} - \eta_\delta) \cap B_r \mapsto \|z\|^2 = \sum_{1 \leq i \leq n} |z_i|^2.$$

In other words, the point (t_δ, z_δ) lies in the intersection of $D_\tau \times (B_\varepsilon \setminus \mathring{B}_{\varepsilon'})$ with the *real* algebraic set C consisting of the points (t, z) such that

$$\left(\frac{\partial f_t}{\partial z_1}(z), \ldots, \frac{\partial f_t}{\partial z_n}(z)\right) = \lambda \bar{z} \tag{5-1}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$, where $\bar{z} := (\bar{z}_1, \ldots, \bar{z}_n)$ and $\bar{z}_i$ denotes the complex conjugate of z_i (see e.g., [O'Shea 1983b, Lemma 1]). Let $C_{\tau,r} := C \cap (D_\tau \times (B_\varepsilon \setminus \mathring{B}_{\varepsilon'}))$. Take $\delta := \delta(m) := 1/m$ (where $m \in \mathbb{N} \setminus \{0\}$ is sufficiently large), and consider the corresponding sequence of points $(t_{\delta(m)}, z_{\delta(m)})$ in $C_{\tau,r}$. As $C_{\tau,r}$ is compact, taking a subsequence if necessary, we may assume that $(t_{\delta(m)}, z_{\delta(m)})$ converges to a point $(t_{\tau,r}, z_{\tau,r}) \in C_{\tau,r}$, and hence $\eta_{\delta(m)} := f(t_{\delta(m)}, z_{\delta(m)})$ tends to $f(t_{\tau,r}, z_{\tau,r})$ as $m \to \infty$. Since $0 < |\eta_{\delta(m)}| \leq \delta(m) = 1/m \to 0$ as $m \to \infty$, we have $f(t_{\tau,r}, z_{\tau,r}) = 0$. Thus $(t_{\tau,r}, z_{\tau,r}) \in V(f) \cap C_{\tau,r}$.

Now, since $f_{t_{\tau,r}}$ is weighted homogeneous with respect to the weights $\boldsymbol{w} = (w_1, \ldots, w_n)$, the *Euler identity* implies the following contradiction:

$$d_{\boldsymbol{w}} \cdot \underbrace{f_{t_{\tau,r}}(z_{\tau,r})}_{=0} \overset{\text{Euler}}{=} \sum_{1 \leq i \leq n} w_i (z_{\tau,r})_i \frac{\partial f_{t_{\tau,r}}}{\partial z_i}(z_{\tau,r}) \overset{(5\text{-}1)}{=} \lambda \sum_{1 \leq i \leq n} w_i |(z_{\tau,r})_i|^2 \neq 0,$$

where $d_{\boldsymbol{w}}$ is the weighted degree of $f_{t_{\tau,r}}$ with respect to the weights $\boldsymbol{w}$ and $(z_{\tau,r})_i$ is the i-th component of $z_{\tau,r}$. □

Remark 5.2. Actually, the proof shows that if f defines a family $\{f_t\}$ of hypersurfaces — not necessarily with line singularities — such that each f_t is weighted homogeneous with respect to a given system of weights $\boldsymbol{w}$, and if furthermore, the nearby fibres $V(f_t - \eta)$, $\eta \neq 0$, of the functions f_t are uniformly nonsingular with respect to the deformation parameter t, then the family $\{f_t\}$ has a uniform stable radius.

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