Pacific Journal of Mathematics

ROST INVARIANT OF THE CENTER, REVISITED

SKIP GARIBALDI AND ALEXANDER S. MERKURJEV

Volume 291 No. 2 December 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 291, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.291.369

ROST INVARIANT OF THE CENTER, REVISITED

SKIP GARIBALDI AND ALEXANDER S. MERKURJEV

The Rost invariant of the Galois cohomology of a simple simply connected algebraic group over a field F is defined regardless of the characteristic of F, but certain formulas for it have only been known under a hypothesis on the characteristic. We improve on those formulas by removing the hypothesis on the characteristic and removing an ad hoc pairing that appeared in those formulas. As a preliminary step of independent interest, we also extend the classification of invariants of quasitrivial tori to all fields.

1. Introduction

Cohomological invariants provide an important tool to distinguish elements of Galois cohomology groups such as $H^1(F, G)$ where G is a semisimple algebraic group. In the case where G is simple and simply connected there are no nonconstant invariants with values in $H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$ for $d < 3$. For $d = 3$, modulo constants the group of invariants $H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$ is finite cyclic with a canonical generator known as the Rost invariant and denoted by r_G; this was shown by Markus Rost in the 1990s and full details can be found in [Garibaldi et al. 2003]. Rost's theorem raised the questions: How do we calculate the Rost invariant of a class in $H^1(F, G)$? What is a formula for it?

At least for G of inner type A_n there is an obvious candidate for r_G, which is certainly equal to mr_G for some m relatively prime to $n+1$. The papers [Merkurjev et al. 2002; Garibaldi and Quéguiner-Mathieu 2007] studied the composition

$$H^1(F, C) \to H^1(F, G) \xrightarrow{r_G} H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \tag{1.1}$$

for C the center of G, and under some assumptions on $\operatorname{char}(F)$, computed the composition in terms of the value of m for type A. Eventually the value of m was determined in [Gille and Quéguiner-Mathieu 2011]. The main result of this paper is Theorem 1.2, which gives a formula for (1.1) that does not depend on the type of G nor on $\operatorname{char}(F)$. This improves on the results of [Merkurjev et al. 2002; Garibaldi and Quéguiner-Mathieu 2007] by removing the hypothesis on the characteristic

Merkurjev has been supported by the NSF grant DMS #1160206.
MSC2010: primary 20G15; secondary 11E72.
Keywords: Rost invariant, cohomological invariant, cup product, Tits class.

and avoiding the ad hoc type-by-type arguments used in those papers. We do rely on [Gille and Quéguiner-Mathieu 2011] for the computation of m for type A, but nothing more.

The strategy is to (1) extend the determination of invariants of quasitrivial tori from [Merkurjev et al. 2002] to all fields (see Theorem 3.7), (2) to follow the general outline of [Garibaldi and Quéguiner-Mathieu 2007] to reduce to the case of type A, and (3) to avoid the ad hoc formulas used in previous work by giving a formula independent of the Killing–Cartan type of G.

Specifically, there is a canonically defined element $t_G^\circ \in H^2(F, C^\circ)$, where C° denotes the dual multiplicative group scheme of C in a sense defined below, and a natural cup product $H^1(F, C) \otimes H^2(F, C^\circ) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$. We prove:

Theorem 1.2. *Let G be a semisimple and simply connected algebraic group over a field F, and $C \subset G$ be the center of G. Let t_G° be the image of the Tits class t_G under $\hat{\rho}^* : H^2(F, C) \to H^2(F, C^\circ)$. Then the diagram*

$$\begin{array}{ccc} H^1(F, C) & \xrightarrow{\ i^*\ } & H^1(F, G) \\ & \searrow^{-t_G^\circ \cup} & \downarrow{\scriptstyle r_G} \\ & & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

commutes, where the cup product map is the one defined in (2.9).

The map $\hat{\rho}^*$ is deduced from a natural map ρ defined in terms of the root system, see Section 5C.

Theorem 1.2 gives a general statement, which we state precisely in Theorem 6.4, for all invariants $H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$.

2. Cohomology of groups of multiplicative type

Let F be a field and M a group scheme of multiplicative type over F. Then M is uniquely determined by the Galois module M^* of characters over F_{sep}. In particular, we have

$$M(F_{\text{sep}}) = \text{Hom}(M^*, F_{\text{sep}}^\times).$$

If M is a torus T, then T^* is a Galois lattice and we set $T_* = \text{Hom}(T^*, \mathbb{Z})$. We have

$$T(F_{\text{sep}}) = T_* \otimes F_{\text{sep}}^\times. \tag{2.1}$$

If M is a finite group scheme C of multiplicative type, set $C_* := \text{Hom}(C^*, \mathbb{Q}/\mathbb{Z})$, so we have a perfect pairing of Galois modules

$$C_* \otimes C^* \to \mathbb{Q}/\mathbb{Z}. \tag{2.2}$$

Write C° for the group of multiplicative type over F with the character module C_*. We call C° the group *dual to C*.

Example 2.3. We write μ_n for the sub-group-scheme of $\mathbb{G}_m$ of n-th roots of unity. The restriction of the natural generator of $\mathbb{G}_m^*$ (the identity $\mathbb{G}_m \to \mathbb{G}_m$) generates μ_n^* and thereby identifies μ_n^* with $\mathbb{Z}/n\mathbb{Z}$. Thus $\mu_n^* = \mathbb{Z}/n\mathbb{Z}$ via the pairing (2.2), hence $\mu_n^\circ = \mu_n$.

The change-of-sites map $\alpha : \operatorname{Spec}(F)_{\text{fppf}} \to \operatorname{Spec}(F)_{\text{ét}}$ yields a functor

$$\alpha_* : \operatorname{Sh}_{\text{fppf}}(F) \to \operatorname{Sh}_{\text{ét}}(F)$$

between the categories of sheaves over F and an exact functor

$$R\alpha_* : D^+ \operatorname{Sh}_{\text{fppf}}(F) \to D^+ \operatorname{Sh}_{\text{ét}}(F)$$

between derived categories.

Every group M of multiplicative type can be viewed as a sheaf of abelian groups either in the étale or fppf topology. We have $\alpha_*(M) = M$ for every group M of multiplicative type. If M is smooth, we have $R^i\alpha_*(M) = 0$ for $i > 0$ by [Milne 1980, proof of Theorem 3.9]. It follows that $R\alpha_*(M) = M$, hence

$$H^i_{\text{ét}}(F, M) = H^i_{\text{ét}}(F, R\alpha_*(M)) = H^i_{\text{fppf}}(F, M), \quad \text{for } M \text{ smooth.} \tag{2.4}$$

If

$$1 \to C \to T \to S \to 1$$

is an exact sequence of algebraic groups with C a finite group of multiplicative type and T and S tori, this sequence is exact in the fppf-topology but not in the étale topology (unless C is smooth). Applying $R\alpha_*$ to the exact triangle

$$C \to T \to S \to C[1]$$

in $D^+ \operatorname{Sh}_{\text{fppf}}(F)$, we get an exact triangle,

$$R\alpha_*(C) \to T(F_{\text{sep}}) \to S(F_{\text{sep}}) \to R\alpha_*(C)[1],$$

in $D^+ \operatorname{Sh}_{\text{ét}}(F)$ since $R\alpha_*(T) = T(F_{\text{sep}})$ and $R\alpha_*(S) = S(F_{\text{sep}})$. In other words,

$$R\alpha_*(C) = \operatorname{cone}(T(F_{\text{sep}}) \to S(F_{\text{sep}}))[-1]. \tag{2.5}$$

Recall that $\mathbb{Z}(1)$ is the complex in $D^+ \operatorname{Sh}_{\text{ét}}(F)$ with only one nonzero term $F^\times_{\text{sep}}$ placed in degree 1, i.e., $\mathbb{Z}(1) = F^\times_{\text{sep}}[-1]$. Set

$$C_*(1) := C_* \overset{L}{\otimes} \mathbb{Z}(1), \qquad C^*(1) := C^* \overset{L}{\otimes} \mathbb{Z}(1),$$

where the derived tensor product is taken in the derived category $D^+ \operatorname{Sh}_{\text{ét}}(F)$. If T is an algebraic torus, we write

$$T_*(1) := T_* \overset{L}{\otimes} \mathbb{Z}(1) = T_* \otimes \mathbb{Z}(1) = T(F_{\text{sep}})[-1].$$

Tensoring the exact sequence

$$0 \to T_* \to S_* \to C_* \to 0$$

with $\mathbb{Z}(1)$ and using (2.1), we get an exact triangle

$$C_*(1) \to T(F_{\mathrm{sep}}) \to S(F_{\mathrm{sep}}) \to C_*(1)[1].$$

It follows from (2.5) that

$$C_*(1) = R\alpha_*(C)$$

and therefore,

$$H^i_{\mathrm{fppf}}(F, C) = H^i_{\text{ét}}(F, R\alpha_*(C)) = H^i_{\text{ét}}(F, C_*(1)).$$

Recall that we also have

$$H^i_{\mathrm{fppf}}(F, T) = H^i_{\text{ét}}(F, T) = H^{i+1}_{\text{ét}}(F, T_*(1)).$$

Remark 2.6. There is a canonical isomorphism (see [Merkurjev 2016, §4c])

$$C_*(1) \simeq C(F_{\mathrm{sep}}) \oplus (C_* \otimes F^\times_{\mathrm{sep}})[-1].$$

The second term in the direct sum vanishes if $\mathrm{char}(F)$ does not divide the order of C_* or if F is perfect.

Notation 2.7. To simplify notation we will write $H^i(F, C)$ for $H^i_{\text{ét}}(F, C_*(1)) = H^i_{\mathrm{fppf}}(F, C)$ and $H^i(F, C^\circ)$ for $H^i_{\text{ét}}(F, C^*(1)) = H^i_{\mathrm{fppf}}(F, C^\circ)$.

Every C-torsor E over F has a class $c(E) \in H^1(F, C)$.

Example 2.8. Taking colimits of the connecting homomorphism arising from the sequences $1 \to \mathbb{G}_m \to \mathrm{GL}_d \to \mathrm{PGL}_d \to 1$ or $1 \to \mu_d \to \mathrm{SL}_d \to \mathrm{PGL}_d \to 1$ — which are exact in the fppf topology — gives isomorphisms $H^2(K, \mathbb{G}_m) \simeq \mathrm{Br}(K)$ and $H^2(K, \mu_n) \simeq {}_n\mathrm{Br}(K)$ as in [Gille and Szamuely 2006, 4.4.5][1], which we use.

In view of (2.4) and Notation 2.7, we work in the derived category of étale sheaves as in, for example, [Freitag and Kiehl 1988, Appendix A.II]. We use the motivic complex $\mathbb{Z}(2)$ of étale sheaves over F defined in [Lichtenbaum 1987; 1990]. Set

$$\mathbb{Q}/\mathbb{Z}(2) := \mathbb{Q}/\mathbb{Z} \overset{L}{\otimes} \mathbb{Z}(2).$$

The complex $\mathbb{Q}/\mathbb{Z}(2)$ is the direct sum of two complexes. The first complex is given by the locally constant étale sheaf (placed in degree 0) the colimit over n prime to $\mathrm{char}(F)$ of the Galois modules $\mu_n^{\otimes 2} := \mu_n \otimes \mu_n$. The second complex is nontrivial only in the case $p = \mathrm{char}(F) > 0$ and it is defined as

$$\operatorname*{colim}_n W_n\Omega^2_{\log}[-2]$$

[1] This reference assumes $\mathrm{char}(F)$ does not divide n, since it uses H^1 to denote Galois cohomology. With our notation, their arguments go through with no change.

with $W_n\Omega^2_{\log}$ the sheaf of *logarithmic de Rham–Witt differentials* (see [Kahn 1996]).

Note that

$$H^i(F, \mathbb{Q}/\mathbb{Z}(2)) \simeq H^{i+1}(F, \mathbb{Z}(2))$$

for $i \geq 3$.

Tensoring (2.2) with $\mathbb{Z}(2)$, we get the pairings

$$C_*(1) \overset{L}{\otimes} C^*(1) \to \mathbb{Q}/\mathbb{Z}(2) \quad \text{and} \qquad (2.9)$$
$$H^i(F, C) \otimes H^j(F, C^\circ) \to H^{i+j}(F, \mathbb{Q}/\mathbb{Z}(2)).$$

If S is a torus over F, we have $S_*(1) = S_* \otimes \mathbb{G}_m[-1] = S[-1]$ and the pairings

$$S_* \otimes S^* \to \mathbb{Z}, \quad S_*(1)\overset{L}{\otimes}S^*(1) \to \mathbb{Z}(2) \quad \text{and} \qquad (2.10)$$
$$H^i(F, S) \otimes H^j(F, S^\circ) \to H^{i+j+2}(F, \mathbb{Z}(2)) = H^{i+j+1}(F, \mathbb{Q}/\mathbb{Z}(2))$$

if $i + j \geq 2$.

Let

$$1 \to C \to T \to S \to 1$$

be an exact sequence with T and S tori and C finite. Dualizing we get an exact sequence of dual groups

$$1 \to C^\circ \to S^\circ \to T^\circ \to 1. \qquad (2.11)$$

We have the homomorphisms

$$\varphi : S(F) \to H^1(F, C), \qquad \psi : H^2(F, C^\circ) \to H^2(F, S^\circ).$$

Proposition 2.12. *For every $a \in S(F)$ and $b \in H^2(F, C^\circ)$, we have $\varphi(a) \cup b = a \cup \psi(b)$ in $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$. Here the cup products are taken with respect to the pairings (2.9) and (2.10) respectively.*

Proof. The pairing $S_* \otimes S^* \to \mathbb{Z}$ extends uniquely to a pairing $S_* \otimes T^* \to \mathbb{Q}$. We have then a morphism of exact triangles

$$\begin{array}{ccccccc}
S_*(1)\overset{L}{\otimes}S^*(1) & \longrightarrow & S_*(1)\overset{L}{\otimes}T^*(1) & \longrightarrow & S_*(1)\overset{L}{\otimes}C^*(1) & \longrightarrow & S_*(1)\overset{L}{\otimes}S^*(1)[1] \\
\downarrow & & \downarrow & & \downarrow s & & \downarrow \\
\mathbb{Z}(2) & \longrightarrow & \mathbb{Q}(2) & \longrightarrow & \mathbb{Q}/\mathbb{Z}(2) & \longrightarrow & \mathbb{Z}(2)[1]
\end{array}$$

and a commutative diagram

$$\begin{array}{ccc}
H^1(F, S_*(1)) \otimes H^2(F, C^*(1)) & \longrightarrow & H^1(F, S_*(1)) \otimes H^2(F, S^*(1)[1]) \\
\downarrow & & \downarrow \\
H^3(F, \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & H^3(F, \mathbb{Z}(2)[1])
\end{array}$$

and therefore, a commutative diagram:

$$\begin{array}{ccc} H^0(F,S)\otimes H^2(F,C^\circ) & \longrightarrow & H^0(F,S)\otimes H^2(F,S^\circ) \\ \downarrow & & \downarrow \\ H^3(F,\mathbb{Q}/\mathbb{Z}(2)) & = & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

On the other hand, the composition $S_*(1)\overset{L}{\otimes}C^*(1)\to C_*(1)\overset{L}{\otimes}C^*(1)\to\mathbb{Q}/\mathbb{Z}(2)$ coincides with s. Therefore, we have a commutative diagram

$$\begin{array}{ccc} H^1(F,S_*(1))\otimes H^2(F,C^*(1)) & \longrightarrow & H^1(F,C_*(1))\otimes H^2(F,C^*(1)) \\ \downarrow & & \downarrow \\ H^3(F,\mathbb{Q}/\mathbb{Z}(2)) & = & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

and therefore, a diagram:

$$\begin{array}{ccc} H^0(F,S)\otimes H^2(F,C^\circ) & \longrightarrow & H^1(F,C)\otimes H^2(F,C^\circ) \\ \downarrow & & \downarrow \\ H^3(F,\mathbb{Q}/\mathbb{Z}(2)) & = & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

The result follows. □

Remark 2.13. We have used that the diagram

$$\begin{array}{ccc} H^i(A[a])\otimes H^j(B[b]) & \longrightarrow & H^{i+j}(A[a]\otimes B[b]) \\ \| & & \| \\ H^{i+a}(A)\otimes H^{j+b}(B) & \longrightarrow & H^{i+j+a+b}(A\otimes B) \end{array}$$

is $(-1)^{ib}$-commutative for all complexes A and B.

Let A be an étale algebra over F and C a finite group scheme of multiplicative type over A. Then $C' := R_{A/F}(C)$ is a finite group of multiplicative type over F. Moreover, $C'^\circ \simeq R_{A/F}(C^\circ)$ and there are canonical isomorphisms

$$\iota : H^i(A,C) \xrightarrow{\sim} H^i(F,C') \quad\text{and}\quad \iota^\circ : H^i(A,C^\circ) \xrightarrow{\sim} H^i(F,C'^\circ).$$

Lemma 2.14. *We have* $\iota(x)\cup\iota^\circ(y) = N_{A/F}(x\cup y)$ *in* $H^{i+j}(F,\mathbb{Q}/\mathbb{Z}(2))$ *for every* $x\in H^i(A,C)$ *and* $y\in H^j(A,C^\circ)$.

Proof. The group scheme C'_A is naturally isomorphic to the product $C_1\times C_2\times\cdots\times C_s$ of group schemes over A with $C_1 = C$. Let $\pi : C'_A \to C$ be the natural projection.

Similarly, $C'^{\circ} \simeq C_1^{\circ} \times C_2^{\circ} \times \cdots \times C_s^{\circ}$. Write $\varepsilon : C^{\circ} \to {C'_A}^{\circ}$ for the natural embedding. Then the inverse of ι coincides with the composition

$$H^i(F, C') \xrightarrow{\text{res}} H^i(A, C'_A) \xrightarrow{\pi^*} H^i(A, C)$$

and ι° coincides with the composition

$$H^i(A, C^{\circ}) \xrightarrow{\varepsilon^*} H^i(A, C'^{\circ}_A) \xrightarrow{N_{A/F}} H^i(F, C'^{\circ}).$$

Since $\pi^*(\iota(x)) = x$, we have $\text{res}(\iota(x)) = (x, x_2, \ldots, x_s)$ for some x_i. On the other hand, $\varepsilon^*(y) = (y, 0, \ldots, 0)$, hence

$$\text{res}(\iota(x)) \cup \varepsilon^*(y) = x \cup y. \tag{2.15}$$

Finally,

$$\begin{aligned} \iota(x) \cup \iota^{\circ}(y) &= \iota(x) \cup N_{A/F}(\varepsilon^*(y)) \\ &= N_{A/F}(\text{res}(\iota(x)) \cup \varepsilon^*(y)) && \text{by the projection formula} \\ &= N_{A/F}(x \cup y) && \text{by (2.15).} \end{aligned}$$ □

Lemma 2.16 (projection formula). *Let $f : C \to C'$ be a homomorphism of finite group schemes of multiplicative type. For $a \in H^m(F, C)$, the diagram*

$$\begin{array}{ccc} H^k(F, C'^{\circ}) & \xrightarrow{\cup f_*(a)} & H^{k+m}(F, \mathbb{Q}/\mathbb{Z}(2)) \\ {\scriptstyle f^*}\downarrow & & \| \\ H^k(F, C^{\circ}) & \xrightarrow{\cup a} & H^{k+m}(F, \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

commutes.

Proof. The pairings used in the diagram are induced by the pairings $C^* \otimes C_* \to \mathbb{Q}/\mathbb{Z}$ and $C'^* \otimes C'_* \to \mathbb{Q}/\mathbb{Z}$. The (obvious) projection formula for these pairings reads $\langle f^*(x), y\rangle = \langle x, f_*(y)\rangle$ for $x \in C'^*$ and $y \in C_*$. □

3. Invariants of quasitrivial tori

3A. ***Cohomological invariants.*** For a field F write $H^j(F)$ for the cohomology group $H^j(F, \mathbb{Q}/\mathbb{Z}(j-1))$, where $j \geq 1$ (see [Garibaldi et al. 2003]). In particular, $H^1(F)$ is the character group of continuous homomorphisms $\Gamma_F \to \mathbb{Q}/\mathbb{Z}$ and $H^2(F)$ is the Brauer group $\text{Br}(F)$.

The assignment $K \mapsto H^j(K)$ is functorial with respect to arbitrary field extensions. If K'/K is a finite separable field extension, we have a well-defined *norm map* $N_{K'/K} : H^j(K') \to H^j(K)$.

The graded group $H^*(F)$ is a (left) module over the Milnor ring $K_*(F)$.

Definition 3.1. Let $\mathcal{A}$ be a functor from the category of field extensions of F to pointed sets. A degree d *cohomological invariant of* $\mathcal{A}$ is a collection of maps of pointed sets

$$\iota_K : \mathcal{A}(K) \to H^d(K)$$

for all field extensions K/F, functorial in K. The degree d cohomological invariants of $\mathcal{G}$ form an abelian group denoted by $\mathrm{Inv}^d(\mathcal{A})$. If L/F is a field extension, we have a *restriction homomorphism*

$$\mathrm{Inv}^d(\mathcal{A}) \to \mathrm{Inv}^d(\mathcal{A}_L),$$

where $\mathcal{G}_L$ is the restriction of $\mathcal{G}$ to the category of field extensions of L.

If the functor $\mathcal{A}$ factors through the category of groups, we further consider the subgroup $\mathrm{Inv}^d_h(\mathcal{A})$ of $\mathrm{Inv}^d(\mathcal{A})$ consisting of those invariants ι such that ι_K is a group homomorphism for every K.

Example 3.2. If G is an algebraic group over F, we can view G as a functor taking a field extension K to the group $G(K)$ of K-points of G; in this case we consider $\mathrm{Inv}^d_h(G)$. We have also another functor $H^1(G) : K \to H^1(K, G)$ and we consider $\mathrm{Inv}^d(H^1(G))$. If G is commutative, then $H^1(K, G)$ is a group for every K, and we also consider $\mathrm{Inv}^d_h(H^1(G))$.

3B. *Residues.* Our goal is to prove Theorem 3.7 concerning the group $\mathrm{Inv}^d_h(T)$ for T a quasisplit torus. Such invariants of order not divisible by char(F) were determined in [Merkurjev et al. 2002]. We modify the method from [Merkurjev et al. 2002] so that it works in general. The difficulty is that the groups $H^j(K)$ do not form a cycle module, because the residue homomorphisms need not exist.

If K is a field with discrete valuation v and residue field $\kappa(v)$, write $H^j(F)_{nr,v}$ for the subgroup of all elements of $H^j(F)$ that are split by finite separable extensions K/F such that v admits an unramified extension to K. Note that every element in $H^j(F)_{nr,v}$ of order not divisible by char(F) belongs to $H^j(F)_{nr,v}$.

There are *residue homomorphisms* (see [Garibaldi et al. 2003] or [Kato 1982])

$$\partial_v : H^j(K)_{nr,v} \to H^{j-1}(\kappa(v)).$$

Example 3.3. Let $K = F(t)$ and let v be the discrete valuation associated with t. Then $\kappa(v) = F$ and $\partial_v(t \cdot h_K) = h$ for all $h \in H^{j-1}(F)$.

Lemma 3.4. *Let K'/K be a field extension and let v' be a discrete valuation on K' unramified over its restriction v on K. Then the diagram*

$$\begin{array}{ccc} H^j(K)_{nr,v} & \xrightarrow{\partial_v} & H^{j-1}(\kappa(v)) \\ \downarrow & & \downarrow \\ H^j(K')_{nr,v'} & \xrightarrow{\partial_{v'}} & H^{j-1}(\kappa(v')) \end{array}$$

commutes.

3C. ***Invariants of tori.*** Let A be an étale F-algebra and T^A the corresponding quasisplit torus, i.e.,

$$T^A(K) = (A \otimes_F K)^\times$$

for every field extension K/F. If B is another étale F-algebra, then

$$T^{A\times B} = T^A \times T^B$$

and

$$\mathrm{Inv}_h^d(T^{A\times B}) \simeq \mathrm{Inv}_h^d(T^A) \oplus \mathrm{Inv}_h^d(T^B).$$

Write A as a product of fields: $A = L_1 \times L_2 \times \cdots \times L_s$. Set

$$H^i(A) := H^i(L_1) \oplus H^i(L_2) \oplus \cdots \oplus H^i(L_s).$$

For $d \geq 2$ define a homomorphism

$$\alpha^A : H^{d-1}(A) \to \mathrm{Inv}_h^d(T^A)$$

as follows. If $h \in H^{d-1}(A)$, then the invariant $\alpha^A(h)$ is defined by

$$\alpha^A(h)(t) = N_{A\otimes K/K}(t \cdot h_{A\otimes K}) \in H^d(K)$$

for a field extension K/F and $t \in T^A(K) = (A \otimes_F K)^\times$.

Remark 3.5. In the notation of the previous section, $(T^A)^\circ \simeq T^A$, and we have

$$H^{d-1}(F, (T^A)^\circ) = H^{d-1}(F, T^A) = H^{d-1}(A, \mathbb{G}_m) = H^{d-1}(A).$$

The pairing (2.10) for the torus T^A, $i = 0$, and $j = 2$,

$$A^\times \otimes H^2(A) = T^A(F) \otimes H^2(F, (T^A)^\circ) \to H^3(F),$$

takes $t \otimes h$ to $N_{A/F}(t \cup h_A) = \alpha^A(h)(t)$. In other words, the map α^A coincides with the map

$$H^2(F, (T^A)^\circ) \to \mathrm{Inv}_h^3(T^A)$$

given by the cup product.

Note that every element $h \in H^{d-1}(A)$ is split by an étale extension of A, hence the invariant $\alpha^A(h)$ vanishes when restricted to F_{sep}.

Question 3.6. Do all invariants in $\mathrm{Inv}_h^d(T^A)$ vanish when restricted to F_{sep}?

The answer is “yes” when $\mathrm{char}(F) = 0$. For any prime $p \neq \mathrm{char}(F)$ and for F separably closed, the zero map is the only invariant $T^A(*) \to H^d(*, \mathbb{Q}_p/\mathbb{Z}_p(d-1))$ that is a homomorphism of groups [Merkurjev 1999, Proposition 2.5].

The main result of this section is:

Theorem 3.7. *The sequence*

$$0 \to H^{d-1}(A) \xrightarrow{\alpha^A} \operatorname{Inv}_h^d(T^A) \xrightarrow{\text{res}} \operatorname{Inv}_h^d(T^A_{\text{sep}})$$

is exact.

That is, defining $\widetilde{\operatorname{Inv}}_h^d(T^A) := \ker \operatorname{res}$, we claim that $\alpha^A : H^{d-1}(A) \xrightarrow{\sim} \widetilde{\operatorname{Inv}}_h^d(T^A)$.

The torus T^A is embedded into the affine space $\mathbb{A}(A)$ as an open set. Let Z^A be the closed complement $\mathbb{A}(A) \setminus T^A$ and let S^A be the smooth locus of Z^A (see [Merkurjev et al. 2002]). Then S^A is a smooth scheme over A. In fact, S^A is a quasisplit torus over A of the A-algebra A' such that $A \times A' \simeq A \otimes_F A$. We have $A = L_1 \times L_2 \times \cdots \times L_s$, where the L_i are finite separable field extensions of F, and the connected components of S^A (as well as the irreducible components of Z^A) are in one-to-one correspondence with the factors L_i. Let v_i for $i = 1, 2, \ldots, s$ be the discrete valuation of the function field $F(T^A)$ corresponding to the i-th connected component S_i of S^A, or equivalently, to the i-th irreducible component Z_i of Z^A. The residue field of v_i is equal to the function field $F(Z_i) = F(S_i)$. We then have the residue homomorphisms

$$\partial_i : H^d(F(T^A))_{nr,v_i} \to H^{d-1}(F(Z_i)) = H^{d-1}(F(S_i)).$$

Write $\widetilde{H}^d(F(T^A))$ for the kernel of the natural homomorphism $H^d(F(T^A)) \to H^d(F_{\text{sep}}(T^A))$. Since every extension of the valuation v_i to $F_{\text{sep}}(T^A)$ is unramified, we have $\widetilde{H}^d(F(T^A)) \subset H^d(F(T^A))_{nr,v_i}$ for all i. Write $F(S^A)$ for the product of $F(S_i)$ over all i. The sum of the restrictions of the maps ∂_i on $\widetilde{H}^d(F(T^A))$ yields a homomorphism

$$\partial^A : \widetilde{H}^d(F(T^A)) \to H^{d-1}(F(S^A)).$$

Applying $u \in \widetilde{\operatorname{Inv}}_h^d(T^A)$ to the generic element g_{gen} of T^A over the function field $F(T^A)$, we get a cohomology class $u(g_{\text{gen}}) \in H^d(F(T^A))$. By assumption on u, we have $u(g_{\text{gen}}) \in \widetilde{H}^d(F(T^A))$. Applying ∂^A, we get a homomorphism

$$\beta^A : \widetilde{\operatorname{Inv}}_h^d(T^A) \to H^{d-1}(F(S^A)), \qquad u \mapsto \partial^A(u(g_{\text{gen}})).$$

If B is another étale F-algebra, we have (see [Merkurjev et al. 2002])

$$S^{A\times B} = S^A \times T^B + T^A \times S^B.$$

In particular, $F(S^A) \subset F(S^{A\times B}) \supset F(S^B)$. Lemma 3.4 then gives:

Lemma 3.8. *The diagram*

$$\begin{array}{ccc}
\widetilde{\operatorname{Inv}}_h^d(T^A) \oplus \widetilde{\operatorname{Inv}}_h^d(T^B) & \xrightarrow{\beta^A \oplus \beta^B} & H^{d-1}(F(S^A)) \oplus H^{d-1}(F(S^B)) \\
\| & & \downarrow \\
\widetilde{\operatorname{Inv}}_h^d(T^{A\times B}) & \xrightarrow{\beta^{A\times B}} & H^{d-1}(F(S^{A\times B}))
\end{array}$$

commutes.

Recall that S^A is a smooth scheme over A with an A-point. It follows that $A \subset F(S^A)$ and the natural homomorphism

$$H^j(A) \to H^j(F(S^A))$$

is injective by [Garibaldi et al. 2003, Proposition A.10]. We shall view $H^j(A)$ as a subgroup of $H^j(F(S^A))$.

Let $A = L_1 \times L_2 \times \cdots \times L_s$ be the decomposition of an étale F-algebra A into a product of fields. The *height* of A is the maximum of the degrees $[L_i : F]$. The height of A is 1 if and only if A is split. The following proposition will be proved by induction on the height of A.

Proposition 3.9. *The image of the homomorphism β^A is contained in $H^{d-1}(A)$.*

Proof. By Lemma 3.8 we may assume that $A = L$ is a field. If $L = F$, we have $S^A = \operatorname{Spec} F$, so $A = F(S^A)$ and the statement is clear.

Suppose $L \neq F$. The algebra L is a canonical direct factor of $L \otimes_F L$. It follows that the homomorphism β^L is a direct summand of $\beta^{L\otimes L}$. Since the height of the L-algebra $L \otimes_F L$ is less than the height of A, by the induction hypothesis, $\operatorname{Im}(\beta^{L\otimes L}) \subset H^{d-1}(L \otimes L)$. It follows that $\operatorname{Im}(\beta^L) \subset H^{d-1}(L)$. □

It follows from Proposition 3.9 that we can view β^A as a homomorphism

$$\beta^A : \widetilde{\operatorname{Inv}}^d_h(T^A) \to H^{d-1}(A).$$

We will show that α^A and β^A are isomorphisms inverse to each other. First consider the simplest case.

Lemma 3.10. *The maps α^A and β^A are isomorphisms inverse to each other in the case $A = F$.*

Proof. If $A = F$, then we have $T^A = \mathbb{G}_m$. The generic element g_{gen} is equal to $t \in F(t)^\times = F(\mathbb{G}_m)$. Let $h \in H^{d-1}(A) = H^{d-1}(F)$. Then the invariant $\alpha^F(h)$ takes t to $t \cdot h \in \widetilde{H}^d(F(t))$. By Example 3.3, $\beta^F(\alpha^F(h)) = \partial_v(t \cdot h) = h$, i.e., the composition $\beta^F \circ \alpha^F$ is the identity. It suffices to show that α^F is surjective.

Take $u \in \widetilde{\operatorname{Inv}}^d_h(\mathbb{G}_m)$. We consider t as an element of the complete field $L := F((t))$ and let $x = u_L(t) \in H^d(L)$. By assumption, x is split by the maximal unramified extension $L' := F_{\text{sep}}((t))$ of L. By a theorem of Kato [1982],

$$x \in \operatorname{Ker}\big(H^d(L) \to H^d(L')\big) = H^d(F) \oplus t \cdot H^{d-1}(F),$$

i.e., $x = h'_L + t \cdot h_L$ for some $h' \in H^d(F)$ and $h \in H^{d-1}(F)$.

Let K/F be a field extension. We want to compute $u_K(a) \in H^d(K)$ for an element $a \in K^\times$. Consider the field homomorphism $\varphi : L \to M := K((t))$ taking a power series $f(t)$ to $f(at)$. By functoriality,

$$u_M(at) = u_M(\varphi(t)) = \varphi_*(u_L(t)) = \varphi_*(x) = \varphi_*(h'_L + t \cdot h_L) = h'_M + (at) \cdot h_M,$$

therefore,

$$u_M(a) = u_M(at) - u_M(t) = (h'_M + (at) \cdot h_M) - (h'_M + t \cdot h_M) = a \cdot h_M.$$

It follows that $u(a) = a \cdot h_K$ since the homomorphism $H^d(K) \to H^d(M)$ is injective by [Garibaldi et al. 2003, Proposition A.9]. We have proved that $u = \alpha^A(h)$, i.e., α^A is surjective. □

Lemma 3.11. *The homomorphism β^A is injective.*

Proof. The proof is similar to the proof of Proposition 3.9. We induct on the height of A. The right vertical homomorphism in Lemma 3.8 is isomorphic to the direct sum of the two homomorphisms $H^{d-1}(F(S^A)) \to H^{d-1}(F(S^A \times T^B))$ and $H^{d-1}(F(S^B)) \to H^{d-1}(F(T^A \times S^B))$. Both homomorphisms are injective by [Garibaldi et al. 2003, Proposition A.10]. It follows from Lemma 3.8 that we may assume that $A = L$ is a field.

The case $L = F$ follows from Lemma 3.10, so we may assume that $L \neq F$. The homomorphism β^L is a direct summand of $\beta^{L \otimes L}$. The latter is injective by the induction hypothesis, hence so is β^L. □

Lemma 3.12. *The composition $\beta^A \circ \alpha^A$ is the identity.*

Proof. We again induct by the height of A. By Lemma 3.8 that we may assume that $A = L$ is a field.

The case $L = F$ follows from Lemma 3.10, so we may assume that $L \neq F$. The homomorphisms α^L and β^L are direct summands of $\alpha^{L \otimes L}$ and $\beta^{L \otimes L}$, respectively. The composition $\beta^{L \otimes L} \circ \alpha^{L \otimes L}$ is the identity by the induction hypothesis, hence $\beta^A \circ \alpha^A$ is also the identity. □

It follows from Lemma 3.11 and Lemma 3.12 that α^A and β^A are isomorphisms inverse to each other. This completes the proof of Theorem 3.7.

4. Invariants of groups of multiplicative type

In this section, C denotes a group of multiplicative type over F such that there exists an exact sequence

$$1 \to C \to T \to S \to 1$$

such that S and T are quasitrivial tori. For example, this holds if C is the center of a simply connected semisimple group G over F, such as μ_n. In that case, C is isomorphic to the center of the quasisplit inner form G^q of G, and we take T to be any quasitrivial maximal torus in G^q. Then T^* is the weight lattice Λ_w and $S^* \simeq \Lambda_r$, where the Galois action permutes the fundamental weights and simple roots, respectively.

Proposition 4.1. *Every invariant in $\widetilde{\mathrm{Inv}}^3_h(H^1(C))$ is given by the cup product via the pairing (2.9) with a unique element in $H^2(F, C^\circ)$.*

Proof. Since $H^1(K, T) = 1$ for every K, the connecting homomorphism $S(K) \to H^1(K, C)$ is surjective for every K and therefore the natural homomorphism

$$\mathrm{Inv}^3_h(H^1(C)) \to \mathrm{Inv}^3_h(S)$$

is injective.

Consider the diagram

$$\begin{array}{ccccc} H^2(F, C^\circ) & \hookrightarrow & H^2(F, S^\circ) & \longrightarrow & H^2(F, T^\circ) \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ \widetilde{\mathrm{Inv}}^3_h(H^1(C)) & \hookrightarrow & \widetilde{\mathrm{Inv}}^3_h(S) & \longrightarrow & \widetilde{\mathrm{Inv}}^3_h(T) \end{array}$$

where the vertical homomorphisms are given by cup products and the top row comes from the exact sequence (2.11); it is exact since $H^1(K, T^\circ) = 1$ for every field extension K/F. The bottom row comes from applying $\widetilde{\mathrm{Inv}}^3_h$ to the sequence $T(K) \to S(K) \to H^1(K, C)$; it is a complex. The vertical arrows are cup products, and the middle and right ones are isomorphisms by Theorem 3.7 and Remark 3.5. The diagram commutes by Proposition 2.12. By diagram chase, the left vertical map is an isomorphism. □

Note that the group $H^2(F, T)$ is a direct sum of the Brauer groups of finite extensions of F. Therefore, we have the following, a coarser version of [Garibaldi 2012, Proposition 7]:

Lemma 4.2. *The homomorphism $H^2(F, C) \to \coprod \mathrm{Br}(K)$, where the direct sum is taken over all field extensions K/F and all characters of C over K, is injective.*

Remark 4.3. The group G becomes quasisplit over the function field $F(X)$ of the variety X of Borel subgroups of G, so $F(X)$ kills t_G. But the kernel of $H^2(F, C) \to H^2(F(X), C)$ need not be generated by t_G, as can be seen by taking G of inner type D_n for n divisible by 4.

5. Root system preliminaries

5A. ***Notation.*** Let V be a real vector space and $R \subset V$ a root system (which we assume is reduced). Write $\Lambda_r \subset \Lambda_w$ for the *root* and *weight lattices*, respectively. For every root $\alpha \in R$, the reflection s_α with respect to α is given by the formula

$$s_\alpha(x) = x - \alpha^\vee(x) \cdot \alpha, \tag{5.1}$$

for every $x \in V$, where $\alpha^\vee \in V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is the *coroot* dual to α. Write $R^\vee \subset V^*$ for the dual root system and $\Lambda_r^\vee \subset \Lambda_w^\vee$ for the corresponding lattices.

We have

$$\Lambda_r^\vee = (\Lambda_w)^* := \operatorname{Hom}(\Lambda_w, \mathbb{Z}) \quad \text{and} \quad \Lambda_w^\vee = (\Lambda_r)^*.$$

The *Weyl group* W of R is a normal subgroup of the automorphism group $\operatorname{Aut}(R)$ of the root system R. The factor group $\operatorname{Aut}(R)/W$ is isomorphic to the automorphism group $\operatorname{Aut}(\operatorname{Dyn}(R))$ of the Dynkin diagram of R. There is a unique $\operatorname{Aut}(R)$-invariant scalar product $(\, , \,)$ on V normalized so that square-length $d_{\alpha^\vee} := (\alpha, \alpha)$ of short roots in every irreducible component of R is equal to 1. The formula (5.1) yields an equality

$$\alpha^\vee(x) = \frac{2(\alpha, x)}{(\alpha, \alpha)}$$

for all $x \in V$ and $\alpha \in R$.

We may repeat this construction with the dual root system $R^\vee$, defining $(\, , \,)^\vee$ on V^* so that the square-length $d_\alpha := (\alpha^\vee, \alpha^\vee)^\vee$ is 1 for short coroots $\alpha^\vee$ (equivalently, long roots α).

5B. ***The map φ.***

Proposition 5.2. *There is a unique $\mathbb{R}$-linear map $\varphi : V^* \to V$ such that $\varphi(\alpha^\vee) = \alpha$ for all short $\alpha^\vee$. Furthermore, φ is $\operatorname{Aut}(R)$-invariant, $\varphi(\alpha^\vee) = d_\alpha \alpha$ for all $\alpha^\vee \in R^\vee$, $\varphi(\Lambda_w^\vee) \subseteq \Lambda_w$, and $\varphi(\Lambda_r^\vee) \subseteq \Lambda_r$. Analogous statements hold for $\varphi^\vee : V \to V^*$. If R is irreducible, then $\varphi\varphi^\vee : V^* \to V^*$ and $\varphi^\vee\varphi : V \to V$ are multiplication by d_α for α a short root.*

Proof. Define $\varphi^\vee$ by $\langle \varphi^\vee(x), y \rangle = 2(x, y)$ for $x, y \in V$ and φ by $\langle x', \varphi(y') \rangle = 2(x', y')^\vee$ for $x', y' \in V^*$. We have

$$\langle \varphi^\vee(\alpha), x \rangle = 2(\alpha, x) = (\alpha, \alpha) \cdot \alpha^\vee(x) = d_{\alpha^\vee} \cdot \alpha^\vee(x),$$

hence $\varphi^\vee(\alpha) = d_{\alpha^\vee} \cdot \alpha^\vee$, and similarly for φ. For uniqueness of φ and $\varphi^\vee$, it suffices to note that the short roots generate V^*, which is obvious because they generate a subspace that is invariant under the Weyl group.

Let $x \in \Lambda_w$. By definition,

$$\mathbb{Z} \ni \alpha^\vee(x) = \frac{2(x, \alpha)}{(\alpha, \alpha)}$$

for all $\alpha \in R$. It follows that $\langle \varphi^\vee(x), \alpha \rangle = 2(x, \alpha) \in \mathbb{Z}$ since $(\alpha, \alpha) \in \mathbb{Z}$. Therefore,

$$\varphi(x) \in \Lambda_w^\vee.$$

For each root $\beta \in R$, $\varphi^\vee\varphi(\beta^\vee) = d_\beta d_{\beta^\vee} \beta^\vee$ and similarly for $\varphi\varphi^\vee$. As R is irreducible, either all roots have the same length (in which case $d_\beta d_{\beta^\vee} = 1$) or there are two lengths and β and $\beta^\vee$ have different lengths (in which case $d_\beta d_{\beta^\vee}$ is the square-length of a long root); in either case the product equals d_α as claimed. □

Remark 5.3. If the root system R is simply laced, then φ gives isomorphisms from V^*, $\Lambda_w^\vee$, and $\Lambda_r^\vee$ to V, Λ_w, and Λ_r, respectively, that agree with the canonical bijection $R^\vee \to R$ defined by $\alpha^\vee \leftrightarrow \alpha$.

Example 5.4. For $\alpha^\vee$ a simple coroot, we write $f_\alpha^\vee$ for the corresponding fundamental dominant weight of $R^\vee$. Consider an element $x' = \sum x_\beta \beta^\vee$ where β ranges over the simple roots. As $(f_\alpha^\vee, \beta^\vee)^\vee = \frac{1}{2}\langle f_\alpha^\vee, \beta\rangle(\beta^\vee, \beta^\vee)^\vee$, we have $(f_\alpha^\vee, x')^\vee = x_\alpha (f_\alpha^\vee, \alpha^\vee)^\vee = \frac{1}{2} d_\alpha x_\alpha$. That is, $\langle \varphi(f_\alpha^\vee), x'\rangle = d_\alpha x_\alpha = \langle d_\alpha f_\alpha, x'\rangle$ for all x', and we conclude that $\varphi(f_\alpha^\vee) = d_\alpha f_\alpha$.

Remark 5.5. Let $q \in S^2(\Lambda_w)^W$ be the only quadratic form on $\Lambda_r^\vee$ that is equal to 1 on every short coroot in every component of $R^\vee$. It is shown in [Merkurjev 2016, Lemma 2.1] that the polar form p of q in $\Lambda_w \otimes \Lambda_w$ in fact belongs to $\Lambda_r \otimes \Lambda_w$. Then the restriction of φ on $\Lambda_w^\vee$ coincides with the composition

$$\Lambda_w^\vee \xrightarrow{\mathrm{id}\otimes p} \Lambda_w^\vee \otimes (\Lambda_r \otimes \Lambda_w) = (\Lambda_w^\vee \otimes \Lambda_r) \otimes \Lambda_w \to \Lambda_w.$$

5C. ***The map ρ.*** Write $\Delta := \Lambda_w/\Lambda_r$ and $\Delta^\vee := \Lambda_w^\vee/\Lambda_r^\vee$. Note that Δ and $\Delta^\vee$ are dual to each other with respect to the pairing

$$\Delta \otimes \Delta^\vee \to \mathbb{Q}/\mathbb{Z}.$$

The group W acts trivially on Δ and $\Delta^\vee$, hence Δ and $\Delta^\vee$ are $\mathrm{Aut}(\mathrm{Dyn}(R))$-modules. The homomorphism φ yields an $\mathrm{Aut}(R)$-equivariant homomorphism

$$\rho : \Delta^\vee \to \Delta.$$

The map ρ is an isomorphism if R is simply laced (because φ is an isomorphism) or if $\Lambda_w = \Lambda_r$. Similarly, $\rho = 0$ if and only if $\varphi(\Lambda_w^\vee) \subseteq \Lambda_r$, if and only if $p \in \Lambda_r \otimes \Lambda_r$.

Example 5.6. Suppose R has type C_n for some $n \geq 3$. Consulting the tables in [Bourbaki 2002], $f_n^\vee$, the fundamental weight of $R^\vee$ dual to the unique long simple root α_n, is the only fundamental weight of $R^\vee$ not in the root lattice. As α_n is long, $d_{\alpha_n} = 1$, so $\varphi(f_n^\vee) = f_n$, which belongs to Λ_r if and only if n is even. That is, $\rho = 0$ if and only if n is even; for n odd, ρ is an isomorphism.

Example 5.7. Suppose R has type B_n for some $n \geq 2$. For the unique short simple root α_n, $d_{\alpha_n} = 2$, and $\varphi(f_n^\vee) = 2f_n$ is in Λ_r. For $1 \leq i < n$, $\varphi(f_i^\vee) = f_i \in \Lambda_r$. We find that $\rho = 0$ regardless of n.

Thus we have determined $\ker \rho$ for every irreducible root system.

Example 5.8. Suppose R is irreducible and $\mathrm{char}(F) = d_\alpha$ for some short root α. Then for G, $G^\vee$ simple simply connected with root system R, $R^\vee$ respectively, there is a "very special" isogeny $\pi : G \to G^\vee$. The restriction of π to a maximal torus in G induces a $\mathbb{Z}$-linear map on the cocharacter lattices $\pi_* : \Lambda_r^\vee \to \Lambda_r$, which, by [Conrad et al. 2015, Proposition 7.1.5] or [Steinberg 1963, 10.1], equals φ.

In the case $R = B_n$, π is the composition of the natural map $G = \mathrm{Spin}_{2n+1} \to \mathrm{SO}_{2n+1}$ with the natural (characteristic 2 only) map $\mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$. As π vanishes on the center of G, it follows that $\rho = 0$ as in Example 5.7. Similarly, in case $R = C_n$, one can recover Example 5.6 by noting that the composition $\pi : G = \mathrm{Sp}_{2n} \to \mathrm{Spin}_{2n+1}$ with the spin representation $\mathrm{Spin}_{2n+1} \hookrightarrow \mathrm{GL}_{2^n}$ is the irreducible representation of G with highest weight f_n by [Steinberg 1963, §11].

Example 5.9. For $R = A_{n-1}$, define $\tau : \Delta \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ via $\tau(f_1) = 1/n \in \mathbb{Q}/\mathbb{Z}$. As $\langle f_1, f_1^\vee \rangle = (n-1)/n \in \mathbb{Q}$, defining $\tau^\vee : \Delta^\vee \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ via $\tau^\vee(f_1^\vee) = -1/n \in \mathbb{Q}/\mathbb{Z}$ gives a commutative diagram

$$\begin{array}{ccc} \Delta \otimes \Delta^\vee & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle \tau\otimes\tau^\vee}\downarrow\wr & \nearrow{\scriptstyle \text{natural}} & \\ \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} & & \end{array}$$

i.e., $\tau^\vee$ is the isomorphism induced by τ and the natural pairings. Furthermore, although ρ is induced by the canonical isomorphism $R^\vee \simeq R$, the previous discussion shows that the diagram

$$\begin{array}{ccc} \Delta^\vee & \xrightarrow{\rho} & \Delta \\ {\scriptstyle \tau^\vee}\downarrow\wr & & {\scriptstyle \tau}\downarrow\wr \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{-1} & \mathbb{Z}/n\mathbb{Z} \end{array} \tag{5.10}$$

commutes, where the bottom arrow is multiplication by -1.

(The action of $\mathrm{Aut}(R)$ on Δ interchanges f_1 and f_{n-1}. Defining instead $\tau(-f_1) = \tau(f_{n-1}) = 1/n$ also gives the same commutative diagram (5.10). That is, the commutativity of (5.10) is invariant under $\mathrm{Aut}(R)$.)

6. Statement of the main result

6A. *The map ρ.* Let G be a simply connected semisimple group with root system R. Let C be the center of G. Then $C^* = \Lambda_w/\Lambda_r = \Delta$ and $C_* = \Lambda_w^\vee/\Lambda_r^\vee = \Delta^\vee$, and we get from Section 5C a homomorphism

$$\rho = \rho_G : C_* \to C^*$$

of Galois modules. Therefore, we have a group homomorphism

$$\hat{\rho} = \hat{\rho}_G : C \to C^\circ.$$

Note that $\hat{\rho}$ is an isomorphism if R is simply laced.

6B. *The Tits class.* Let G be a simply connected group over F with center C. Write t_G for the *Tits class* $t_G \in H^2(F, C)$. By definition, $t_G = -\partial(\xi_G)$, where

$$\partial : H^1(F, G/C) \to H^2(F, C)$$

is the connecting map for the exact sequence $1 \to C \to G \to G/C \to 1$ and $\xi_G \in H^1(F, G/C)$ is the unique class such that the twisted group ${}^{\xi}G$ is quasisplit.

6C. *Rost invariant for an absolutely simple group.* Let G be a simply connected group over F. Recall (see [Garibaldi et al. 2003]) that, for G absolutely simple, Rost defined an invariant $r_G \in \mathrm{Inv}^3(H^1(G))$ called the *Rost invariant*, i.e., a map

$$r_G : H^1(F, G) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

that is functorial in F.

Lemma 6.1. *If G is an absolutely simple and simply connected algebraic group, then $o(r_G) \cdot t_G = 0$.*

Proof. The order $o(r_G)$ of r_G is calculated in [Garibaldi et al. 2003], and in each case it is a multiple of the order of t_G. □

As mentioned in [Gille 2000, §2.3], there are several definitions of the Rost invariant that may differ by a sign. Gille and Quéguiner [2011] proved that for the definition of the Rost invariant r_G they chose, in the case $G = \mathbf{SL}_1(A)$ for a central simple algebra A of degree n over F, the value of r_G on the image of the class $aF^{\times n} \in F^\times/F^{\times n} = H^1(F, \mu_n)$ in $H^1(F, G)$ is equal to $(a) \cup [A]$ if n is not divisible by $\mathrm{char}(F)$ and to $-(a) \cup [A]$ if n is a power of $p = \mathrm{char}(F) > 0$. Therefore, *we normalize the Rost invariant by multiplying the p-primary component of the Rost invariant (of all groups) by -1 in the case $p = \mathrm{char}(F) > 0$.*

6D. *The main theorem.* For G semisimple and simply connected over F, there is an isomorphism

$$\psi : G \xrightarrow{\sim} \prod_{i=1}^{n} R_{F_i/F}(G_i), \tag{6.2}$$

where the F_i are finite separable extensions of F, and G_i is an absolutely simple and simply connected F_i-group. The product of the corestrictions of the r_{G_i} (in the sense of [Garibaldi et al. 2003, page 34]) is then an invariant of $H^1(G)$, which we also denote by r_G and call the Rost invariant of G. The map ψ identifies the center C of G with $\prod_i R_{F_i/F}(C_i)$ for C_i the center of G_i, and the Tits class $t_G \in H^2(F, C)$ with $\sum t_{G_i} \in \sum H^2(F_i, C_i)$.

The composition $r_G \circ i^*$ is a group homomorphism by [Merkurjev et al. 2002, Corollary 1.8] or [Garibaldi 2001, Lemma 7.1]. That is, the composition $r_G \circ i^*$ in Theorem 1.2 taken over all field extensions of F can be viewed not only as an invariant of $H^1(C)$, but as an element of $\mathrm{Inv}^3_h(H^1(C)$ as in Definition 3.1. Over a separable closure of F, the inclusion of C into G factors through a maximal split torus and hence this invariant is trivial by Hilbert's Theorem 90. By Proposition 4.1 the composition is given by the cup product with a unique element in $H^2(F, C^\circ)$. We will prove Theorem 1.2, which says that this element is equal to $-t_G^\circ$.

6E. ***Alternative formulation.*** Alternatively, we could formulate the main theorem as follows. The group of invariants $\mathrm{Inv}^3(H^1(G))$ is a sum of n cyclic groups with generators (the corestrictions of) the r_{G_i}, and in view of Lemma 6.1 we may define a homomorphism

$$\mathrm{Inv}^3(H^1(G)) \to H^2(F, C) \quad \text{via} \sum n_i r_{G_i} \mapsto \sum -n_i t_{G_i}. \tag{6.3}$$

Theorem 6.4. *For every invariant $s : H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$, the composition*

$$H^1(*, C) \to H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$$

equals the cup product with the image of s under the composition

$$\mathrm{Inv}^3(H^1(G)) \to H^2(F, C) \to H^2(F, C^\circ).$$

This will follow immediately from the main theorem, which we will prove over the course of the next few sections.

7. Rost invariant of transfers

The following statement is straightforward.

Lemma 7.1. *Let A be an étale F-algebra and G a simply connected semisimple group scheme over A, with C the center of G. Then $C' := R_{A/F}(C)$ is the center of $G' := R_{A/F}(G)$ and $C'^\circ \simeq R_{A/F}(C^\circ)$, and the diagram*

$$\begin{array}{ccc} H^i(A, C) & \xrightarrow{\hat\rho_G^*} & H^i(A, C^\circ) \\ \downarrow \wr & & \downarrow \wr \\ H^i(F, C') & \xrightarrow{\hat\rho_{G'}^*} & H^i(F, C'^\circ) \end{array}$$

commutes.

Lemma 7.2. *Set $C' := R_{A/F}(C)$ and $G' := R_{A/F}(G)$. Then the image of t_G under the isomorphism $H^2(A, C) \xrightarrow{\sim} H^2(F, C')$ is equal to $t_{G'}$.*

Proof. The corestriction of a quasisplit group is quasisplit. □

Lemma 7.3. *Let G be a simply connected semisimple algebraic group scheme over an étale F-algebra A. If Theorem 1.2 holds for G, then it also holds for $R_{A/F}(G)$.*

Proof. Let C be the center of G and $G' := R_{A/F}(G)$. The group $C' := R_{A/F}(C)$ is the center of G'. Let $x \in H^1(A, C)$ and let $x' \in H^1(F, C')$ be the image of x under the isomorphism $\nu : H^1(A, C) \xrightarrow{\sim} H^1(F, C')$. We have

$$\begin{aligned} r_{G'}(i'^*(x')) &= r_{G'}(\nu(i^*(x))) \\ &= N_{A/F}(r_G(i^*(x))) && \text{by [Garibaldi et al. 2003, Proposition 9.8]} \\ &= N_{A/F}(-t_G^\circ \cup x) && \text{by Theorem 1.2 for } x \\ &= -t_{G'}^\circ \cup x' && \text{by Lemmas 2.14, 7.1 and 7.2.} \end{aligned}$$

□

If Theorem 1.2 holds for semisimple groups G_1 and G_2, then it also holds for the group $G_1 \times G_2$. Combining this with Lemma 7.3 reduces the proof of Theorem 1.2 to the case where G is absolutely almost simple.

8. Rost invariant for groups of type A

In this section, we will prove Theorem 1.2 for G absolutely simple of type A_{n-1} for each $n \geq 2$.

8A. ***Inner type.*** Suppose G has inner type. Then there is an isomorphism $\psi : \mathbf{SL}_1(A) \xrightarrow{\sim} G$, where A is a central simple algebra of degree n over F. The map ψ restricts to an isomorphism $\mu_n \xrightarrow{\sim} C$, identifying C^* with $\mathbb{Z}/n\mathbb{Z}$, and induces $\psi^\circ : C^\circ \xrightarrow{\sim} \mu_n$. We find a commutative diagram

$$\begin{array}{ccc} H^2(F, C^\circ) \otimes H^1(F, C) & \longrightarrow & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow{\scriptstyle \psi^\circ \otimes \psi^{-1}} & & \| \\ H^2(F, \mu_n) \otimes H^1(F, \mu_n) & \longrightarrow & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \end{array} \tag{8.1}$$

where the top and bottom arrows are the cup product from (2.9).

The connecting homomorphism arising from the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

gives an isomorphism $H^1(K, \mu_n) \simeq K^\times / K^{\times n}$ for every extension K/F. For each field extension K/F, the isomorphism ψ identifies the map $H^1(K, C) \to H^1(K, G)$ with the obvious map $K^\times / K^{\times n} = H^1(K, \mu_n) \to H^1(K, \mathbf{SL}_1(A)) = K^\times / \operatorname{Nrd}(A_K^\times)$. Further, $\psi^{-1}(t_G) \in H^2(K, \mu_n)$ is the Brauer class $[A]$ of A as in [Knus et al. 1998, pages 378 and 426]. By Example 5.9, the composition

$$H^1(F, \mu_n) \xrightarrow{\psi} H^1(F, C) \xrightarrow{\hat{\rho}^*} H^1(F, C^\circ) \xrightarrow{\psi^\circ} H^1(F, \mu_n)$$

is multiplication by -1 and in particular $[A] \mapsto t_G \mapsto t_G^\circ \mapsto -[A]$. That is, Theorem 1.2 claims that the diagram

$$\begin{array}{ccccc} H^1(K, \mu_n) & \xrightarrow{\psi^{-1}} & H^1(K, C) & \longrightarrow & H^1(K, G) \\ \downarrow{\scriptstyle [A]\otimes} & & & & \downarrow{\scriptstyle r_G} \\ H^2(K, \mu_n) \otimes H^1(K, \mu_n) & & \longrightarrow & & H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \end{array} \tag{8.2}$$

commutes, where the bottom arrow is the same as in (8.1).

Let p be a prime integer and write m for the largest power of p dividing n. Both maps $H^1(K, \mu_n) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ in (8.2) are group homomorphisms, so it suffices to verify Theorem 1.2 on each p-primary component $r_G(x)_p$ of the Rost invariant with values in $\mathbb{Q}_p/\mathbb{Z}_p(2)$. In the case where p does not divide $\operatorname{char}(F)$, the commutativity of (8.2) is part of [Gille and Quéguiner-Mathieu 2011, Theorem 1.1]. (Note that the definition of cup product used in [Gille and Quéguiner-Mathieu 2011], the one from [Gille and Szamuely 2006, §3.4], is the same as (8.1), cf. [Freitag and Kiehl 1988, pages 302–303].)

Now let $p = \mathrm{char}(F) > 0$. Consider the sheaf $\nu_m(j)$ in the étale topology over F defined by $\nu_m(j)(L) = K_j(L)/p^m K_j(L)$. The natural morphisms $\mathbb{Z}(j) \to \nu_m(j)[-j]$ for $j \le 2$ are consistent with the products, hence we have a commutative diagram:

$$\begin{array}{ccc} (\mathbb{Z}/m\mathbb{Z})(1)\overset{L}{\otimes}(\mathbb{Z}/m\mathbb{Z})(1) & \longrightarrow & (\mathbb{Z}/m\mathbb{Z})(2) \\ \downarrow \wr & & \downarrow \wr \\ \nu_m(1)[-1] \otimes \nu_m(1)[-1] & \longrightarrow & \nu_m(2)[-2] \end{array}$$

Therefore, we have a commutative diagram

$$\begin{array}{ccc} H^2(F, \mu_m) \otimes H^1(F, \mu_m) & \longrightarrow & H^3(F, \mathbb{Z}/p^m\mathbb{Z}(2)) \\ \downarrow \wr & & \downarrow \wr \\ H^1(F, \nu_m(1)) \otimes H^0(F, \nu_m(1)) & \longrightarrow & H^1(F, \nu_m(2)) \end{array}$$

(see Remark 2.13 after Proposition 2.12). The bottom arrow is given by the cup product map

$${}_m\mathrm{Br}(F) \otimes (F^\times/F^{\times m}) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

(see [Gille and Quéguiner-Mathieu 2011, 4D]). It is shown in [Gille and Quéguiner-Mathieu 2011, Theorem 1.1] that the p-component of the Rost invariant of G is given by the formula

$$r_G(x)_p = [A]_p \cup (x) \in H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

for every $x \in K^\times$. (The formula in [Gille and Quéguiner-Mathieu 2011] contains an additional minus sign, but it does not appear here due to the adjustment in the definition of r_G in Section 6C.) This completes the proof of Theorem 1.2 for groups of inner type A.

8B. ***Outer type.*** Now suppose that G has outer type A_{n-1}. There is an isomorphism $\psi : G \xrightarrow{\sim} \mathbf{SU}(B, \tau)$, where B is a central simple algebra of degree n over a separable quadratic field extension K/F with an involution τ of the second kind (τ restricts to a nontrivial automorphism of K/F). The map ψ identifies C with $\mu_{n,[K]}$, and $C = C^\circ$.

Suppose first that n is odd. Since $G_K \simeq \mathbf{SL}_1(B)$, the theorem holds over K. As K has degree 2 over F and C has odd exponent, the natural map $H^1(F, C) \to H^1(K, C)$ is injective, hence the theorem holds over F by the following general lemma.

Lemma 8.3. *Let $L_1, L_2, \dots, L_s$ be field extensions of F such that the natural homomorphism $H^2(F, C) \to \prod_i H^2(L_i, C)$ is injective. If Theorem 1.2 holds for G over all fields L_i, then it also holds over F.*

Proof. The left vertical map in Theorem 1.2 is multiplication by some element $h \in H^2(F, C^\circ)$. We need to show that $h = -t_G^\circ$. This equality holds over all fields L_i, hence it holds over F by the injectivity assumption. □

So we may assume that n is even. Then $H^1(F, C)$ is isomorphic to a factor group of the group of pairs $(a, z) \in F^\times \times K^\times$ such that $N_{K/F}(z) = a^n$ and $H^2(F, C)$ is isomorphic to a subgroup of $\mathrm{Br}(F) \oplus \mathrm{Br}(K)$ of all pairs (v, u) such that $v_K = mu$ and $N_{K/F}(u) = 0$, see [Merkurjev et al. 2002, pages 795–796].

Suppose that B is split; we follow the argument in [Knus et al. 1998, 31.44]. Then $\mathbf{SU}(B, \tau) = \mathbf{SU}(h)$, where h is a hermitian form of trivial discriminant on a vector K-space of dimension n for the quadratic extension K/F. Let $q(v) := h(v, v)$ be the associated quadratic form on V viewed as a $2n$-dimensional F-space. The quadratic form q is nondegenerate, and we can view $\mathbf{SU}(h)$ as a subgroup of $H := \mathbf{Spin}(V, q)$. The Dynkin index of G in H is 1, hence the composition $H^1(K, G) \to H^1(K, H) \xrightarrow{r_H} H^3(K)$ equals the Rost invariant of G. Then r_H is given by the Arason invariant, which has order 2. A computation shows that the image of the pair (a, z) representing an element $x \in H^1(F, C)$ under the composition

$$H^1(F, C) \to H^1(F, G) \xrightarrow{r_G} H^3(F)$$

coincides with $[D] \cup x$, where D is the class of the discriminant algebra of h. On the other hand, $[D] \cup x$ coincides with the cup product of x with the Tits class $t_G = -t_G^\circ$ represented by the pair $([D], 0)$ in $H^2(F, C^\circ)$, proving Theorem 1.2 in this case.

Now drop the assumption that B is split. As for the n odd case, the theorem holds over K. Note that there is an injective map $H^2(F, C) \to \mathrm{Br}(F) \oplus \mathrm{Br}(K)$. Let $X = R_{K/F}(SB(B))$. By [Merkurjev and Tignol 1995, 2.4.6], the map $\mathrm{Br}(F) \to \mathrm{Br}(F(X))$ is injective, hence the natural homomorphism

$$H^2(F, C) \to H^2(F, C_{F(X)}) \oplus H^2(F, C_K)$$

is injective. The theorem holds over K and by the preceding paragraph the theorem holds over $F(X)$. Therefore, by Lemma 8.3, the theorem holds over F.

9. Conclusion of the proof of Theorem 1.2

Choose a system of simple roots Π of G. Write Π_r for the subset of Π consisting of all simple roots whose fundamental weight belongs to Λ_r and let $\Pi' := \Pi \setminus \Pi_r$. Inspection of the Dynkin diagram shows that all connected components of Π' have type A.

Every element of Π_r is fixed by every automorphism of the Dynkin diagram, hence is fixed by the $*$-action of the absolute Galois group of F on Π. It follows that the variety X of parabolic subgroups of G_{sep} of type Π' is defined over F. By [Merkurjev and Tignol 1995], the kernel of the restriction map $\mathrm{Br}(K) \to \mathrm{Br}(K(X))$ for every field extension K/F is generated by the Tits algebras associated with the classes in C^* of the fundamental weights f_α corresponding to the simple roots $\alpha \in \Pi_r$. But $f_\alpha \in \Lambda_r$, so these Tits algebras are split [Tits 1971], hence the restriction map $\mathrm{Br}(K) \to \mathrm{Br}(K(X))$ is injective and, by Lemma 4.2, the natural homomorphism $H^2(F, C) \to H^2(F(X), C)$ is injective. In view of Lemma 8.3, it suffices to prove Theorem 1.2 over the field $F(X)$, i.e., we may assume that G has a parabolic subgroup of type Π'. The Levi subgroup G' of that parabolic is simply connected with Dynkin diagram Π', and its center C' contains C [Garibaldi and Quéguiner-Mathieu 2007, Proposition 5.5]. Write j for the embedding homomorphism $C \to C'$ and j° for the dual $C'^\circ \to C^\circ$.

Let $G' = \prod_i G'_i$ with G_i simply connected simple groups, $C = \prod C_i$, where C_i is the center of G_i, and $\Pi'_i \subset \Pi$ is the system of simple roots of G_i. Write j_i° for the composition $C_i'^\circ \to C'^\circ \to C^\circ$.

Lemma 9.1. *The map $j_i^* : H^2(F, C) \to H^2(F, C'_i)$ takes the Tits class t_G to $t_{G'_i}$.*

Proof #1. It suffices to check that $j^*(t_G) = t_{G'}$, for the projection

$$H^2(F, C') \to H^2(F, C'_i)$$

sends $t_{G'} \mapsto t_{G'_i}$.

There is a rank $|\Pi_r|$ split torus S in G whose centralizer is $S \cdot G'$. Arguing as in Tits' Witt-type theorem [Tits 1966, 2.7.1, 2.7.2(d)], one sees that the quasisplit inner form of G is obtained by twisting G by a 1-cocycle γ with values in $C_G(S)/C$, equivalently, in G'/C. Clearly, twisting G' by γ gives the quasisplit inner form of G'. The Tits class t_G is defined to be $-\partial_G(\gamma)$ where ∂_G is the connecting homomorphism $H^1(F, G/C) \to H^2(F, C)$ induced by the exact sequence $1 \to C \to G \to G/C \to 1$ and similarly for G' and C'. The diagram

$$\begin{array}{ccccc} H^1(F, G'/C) & \longrightarrow & H^1(F, G/C) & \xrightarrow{\partial_G} & H^2(F, C) \\ \downarrow & & & & \downarrow{\scriptstyle j^*} \\ H^1(F, G'/C') & & \xrightarrow{\quad\partial_{G'}\quad} & & H^2(F, C') \end{array}$$

commutes trivially, so $j^*(t_G) = j^*(-\partial_G(\gamma)) = -\partial_{G'}(\gamma) = t_{G'}$ as claimed. □

Proof #2. For each $\chi \in T^*$, define $F(\chi)$ to be the subfield of F_{sep} of elements fixed by the stabilizer of χ under the Galois action. Note that because G is absolutely almost simple, the $*$-action fixes Π_r elementwise, and $F(\chi)$ equals the field extension

$F(\chi|_{T'})$ defined analogously for $\chi \in (T')^*$. The diagram

$$\begin{array}{ccc} H^2(F, C) & \xrightarrow{j^*} & H^2(F, C') \\ & \searrow^{\chi|_C} & \downarrow{\scriptstyle \chi|_{C'}} \\ & & H^2(F(\chi), \mathbb{G}_m) \end{array}$$

commutes. Now $\chi|_{C'}(t_{G'} - j^*(t_G)) = \chi|_{C'}(t_{G'}) - \chi|_C(t_G)$, which is zero for all $\chi \in T^*$ by [Tits 1971, §5.5]. As $\prod_{\chi \in (T')^*} \chi|_{C'}$ is injective by [Garibaldi 2012, Proposition 7], $t_{G'} = j^*(t_G)$ as claimed. □

The diagram Π'_i is simply laced. Write d_i for the square-length of $\alpha^\vee \in R^\vee$ for $\alpha \in \Pi'_i$.

Lemma 9.2. *The homomorphism $\hat{\rho}_G : C \to C^\circ$ coincides with the composition*

$$C \xrightarrow{j} C' \xrightarrow{\hat{\rho}_{G'}} C'^\circ = \prod_i C'^\circ_i \xrightarrow{\prod_i (j_i{}^\circ)^{d_i}} C^\circ,$$

where j_i is the composition $C \to C' \to C'_i$.

Proof. For every simple root $\alpha \in \Pi$ write f_α for the corresponding fundamental weight. Write Λ'_r and Λ'_w for the root and weight lattices, respectively, of the root system R' of G'. Let

$$\Phi := \{ f_\alpha \mid \alpha \in \Pi_r \}.$$

Then Φ is a $\mathbb{Z}$-basis for the kernel of the natural surjection $\Lambda_w \to \Lambda'_w$. If $\alpha \in \Pi'$, we write α' for the image of α and f'_α for the image of f_α under this surjection. All α' (respectively, f'_α) form the system of simple roots (respectively, fundamental weights) of R'. If $\alpha \in \Pi'$, the image $\alpha'^\vee$ of $\alpha^\vee$ under the inclusion $\Lambda'_r{}^\vee \hookrightarrow \Lambda_r{}^\vee$ is a simple coroot of R'.

If V is the real vector space of R, then $R' \subset V' := V/\operatorname{span}(\Phi)$ and $R'^\vee \subset V'^* \subset V^*$. Let $x \in \Lambda_w^\vee$, i.e., $\langle x, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Pi$. Since $\Phi \subset \Lambda_r$, we have $a_\alpha := \langle x, f_\alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Pi_r$. Then the linear form $x' := x - \sum_{\alpha \in \Pi_r} a_\alpha \alpha^\vee$ vanishes on the subspace of V spanned by Φ, hence $x' \in \Lambda'_w{}^\vee$. We then have a well-defined homomorphism

$$s : \Lambda_w^\vee \to \Lambda'_w{}^\vee, \qquad x \mapsto x'. \tag{9.3}$$

If $\alpha \in \Pi'$, then $\langle x', \alpha \rangle = \langle x', \alpha' \rangle$. It follows that if $x' = \sum_{\alpha \in \Pi} b_\alpha f_\alpha^\vee$ in $\Lambda_w^\vee$ for $b_\alpha = \langle x', \alpha \rangle \in \mathbb{Z}$, then $x' = \sum_{\alpha \in \Pi'} b_\alpha f'^\vee_\alpha$ in $\Lambda'_w{}^\vee$.

Since $\Phi \subset \Lambda_r$, we have a surjective homomorphism

$$C'^* = \Lambda'_w/\Lambda'_r = \Lambda_w/\operatorname{span}(\Phi, \Pi') \to \Lambda_w/\Lambda_r = C^*$$

dual to the inclusion of C into C'. The dual homomorphism

$$C_* = \Lambda_w^\vee/\Lambda_r^\vee \to \Lambda'_w{}^\vee/\Lambda'_r{}^\vee = C'_*$$

is induced by s.

Consider the diagram

$$\begin{array}{ccc} \Lambda_w^\vee & \xrightarrow{\varphi} & \Lambda_w \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle t} \\ \Lambda_w'^\vee & \xrightarrow{\varphi'} & \Lambda_w' \end{array}$$

where the map t is defined by $t(f'_\alpha) = d_\alpha f_\alpha$ for all $\alpha \in \Pi'$ and the maps φ and φ' are defined in Proposition 5.2.

It suffices to prove that $\operatorname{Im}(t \circ \varphi' \circ s - \varphi) \subset \Lambda_r$.

Consider the other diagram

$$\begin{array}{ccc} \Lambda_w^\vee & \xrightarrow{\rho} & \Lambda_w \\ {\scriptstyle t^\vee}\uparrow & & \uparrow{\scriptstyle t} \\ \Lambda_w'^\vee & \xrightarrow{\rho'} & \Lambda_w' \end{array}$$

where $t^\vee(f_\alpha'^\vee) = f_\alpha^\vee$ for all $\alpha \in \Pi'$. This diagram is commutative. Indeed,

$$(\rho \circ t^\vee)(f_\alpha'^\vee) = \rho(f_\alpha^\vee) = d_\alpha f_\alpha = t(f'_\alpha) = (t \circ \rho')(f_\alpha'^\vee),$$

where the second equality is by Example 5.4. (Recall that the root system R' of G' is simply laced, hence $\rho'(f_\alpha'^\vee) = f'_\alpha$.)

We claim that

$$(t^\vee \circ s)(x) - x \in \operatorname{span}(\Phi^\vee) + \Lambda_r^\vee$$

for every $x \in \Lambda_w^\vee$, where $\Phi^\vee := \{f_\alpha^\vee \mid \alpha \in \Pi_r\}$. Indeed, in the notation of (9.3) we have

$$\begin{aligned}(t^\vee \circ s)(x) - x &= t^\vee(x') - x = t^\vee(x') - x' - \sum_{\alpha\in\Pi_r} a_\alpha \alpha^\vee \\ &= t^\vee\Big(\sum_{\alpha\in\Pi'} b_\alpha f_\alpha'^\vee\Big) - \sum_{\alpha\in\Pi} b_\alpha f_\alpha^\vee - \sum_{\alpha\in\Pi_r} a_\alpha \alpha^\vee \\ &= -\sum_{\alpha\in\Pi_r} b_\alpha f_\alpha^\vee - \sum_{\alpha\in\Pi_r} a_\alpha \alpha^\vee \in \operatorname{span}(\Phi^\vee) + \Lambda_r^\vee.\end{aligned}$$

It follows from the claim that

$$(t\circ\rho'\circ s)(x) - \rho(x) = (\rho\circ t^\vee\circ s)(x) - \rho(x) = \rho((t^\vee\circ s)(x) - x) \in \rho(\operatorname{span}(\Phi^\vee) + \Lambda_r^\vee).$$

As $\rho(f_\alpha^\vee) = d_\alpha f_\alpha \in \Lambda_r$ for all $f_\alpha \in \Phi$, this is contained in Λ_r, proving the claim. □

Lemmas 9.1 and 9.2 yield:

Corollary 9.4. *The element t_G° is equal to $\sum_i d_i \cdot j_i^{\circ *}(t_{G'_i}^\circ)$.*

Lemma 9.5. *The diagram*

$$\begin{array}{ccc} H^1(F,G') & = & \prod_i H^1(F,G'_i) \\ \downarrow & & \downarrow {\scriptstyle \sum d_i \cdot r_{G'_i}} \\ H^1(F,G) & \xrightarrow{r_G} & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

commutes.

Proof. The composition

$$H^1(F,G'_i) \to H^1(F,G) \xrightarrow{r_G} H^3(F,\mathbb{Q}/\mathbb{Z}(2))$$

coincides with the k-th multiple of the Rost invariant $r_{G'_i}$, where k is the order of the cokernel of the map $Q(G) \to Q(G'_i)$ of infinite cyclic groups generated by positive definite quadratic forms q_G and $q_{G'_i}$ on the lattices of coroots normalized so that the forms have value 1 on the short coroots (see [Garibaldi et al. 2003]). Recall that all coroots of G'_i have the same length, hence $q_{G'_i}$ has value 1 on all coroots of G'_i. Therefore, k coincides with d_i, the square-length of all coroots of G'_i viewed as coroots of G. □

Write each $G'_i = R_{L_i/F}(H_i)$ for L_i a separable field extension of F and H_i a simply connected absolutely simple algebraic group of type A over L_i. Theorem 1.2 is proved for such groups in Section 8. By Lemma 7.3, Theorem 1.2 holds for the group G'_i and hence for G'.

Let $x \in H^1(F,C)$ and let $y \in H^1(F,G)$, $\prod x'_i \in H^1(F,C') = \prod H^1(F,C'_i)$ and $\prod y'_i \in \prod H^1(F,G'_i)$ denote its images under the natural maps. We find

$$\begin{aligned} r_G(y) &= \sum_i d_i \cdot r_{G'_i}(y_i) && \text{by Lemma 9.5} \\ &= \sum_i d_i \cdot (-t^\circ_{G'_i} \cup x'_i) && \text{by the main theorem for all } G'_i \\ &= \sum_i d_i \cdot j_i^{\circ *}(-t^\circ_{G'_i}) \cup x && \text{by Lemma 2.16} \\ &= -t^\circ_G \cup x && \text{by Corollary 9.4.} \end{aligned}$$

This completes the proof of Theorem 1.2.

10. Concrete formulas

The explicit formulas for the restriction of the Rost invariant to the center given in [Merkurjev et al. 2002; Garibaldi and Quéguiner-Mathieu 2007] (for F of good characteristic) relied on an ad hoc formula for a pairing $C \otimes C \to \mathbb{Q}/\mathbb{Z}(2)$ depending on the type of G. In this section, we deduce those formulas from Theorem 1.2; as a consequence we find that those formulas hold regardless of char(F).

10A. ***The pairing induced by ρ.*** The map ρ defines a bilinear pairing $\Delta^\vee \otimes \Delta^\vee \to \mathbb{Q}/\mathbb{Z}$ via

$$\Delta^\vee \otimes \Delta^\vee \xrightarrow{\mathrm{id}\otimes\rho} \Delta^\vee \otimes \Delta \to \mathbb{Q}/\mathbb{Z}. \tag{10.1}$$

We now determine this pairing for each simple root system R.

For R with different root lengths, ρ is zero and hence (10.1) is zero unless $R = C_n$ for odd $n \geq 3$. In that case (and also for $R = E_7$), $\Delta \simeq \mathbb{Z}/2 \simeq \Delta^\vee$ and ρ is the unique isomorphism, so (10.1) amounts to the product map $x \otimes y \mapsto xy$. Therefore we may assume that R has only one root length.

If $\Delta^\vee$ is cyclic, we pick a fundamental dominant weight $f_i^\vee$ that generates $\Delta^\vee$ and the pairing (10.1) is determined by the image of $f_i^\vee \otimes f_i^\vee$. The image of this under $\mathrm{id}\otimes\rho$ is $f_i^\vee \otimes f_i$ as in Example 5.4, so the image in $\mathbb{Q}/\mathbb{Z}$ is the same as that of the coefficient of the simple root α_i appearing in the expression for f_i in terms of simple roots, for which we refer to [Bourbaki 2002].

For $R = A_n$, we have $\Delta^\vee \simeq \mathbb{Z}/(n+1)$ generated by $f_1^\vee$ and $f_1^\vee \otimes f_1^\vee \mapsto n/(n+1)$, cf. Example 5.9.

For $R = D_n$ for odd $n > 4$, $\Delta^\vee \simeq \mathbb{Z}/4$ generated by $f_n^\vee$ and $f_n^\vee \otimes f_n^\vee \mapsto n/4$.

For $R = E_6$, we have $\Delta^\vee \simeq \mathbb{Z}/3$ generated by $f_1^\vee$ and $f_1^\vee \otimes f_1^\vee \mapsto 1/3$.

For $R = D_n$ for even $n \geq 4$, $\Delta^\vee$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $f_{n-1}^\vee$, $f_n^\vee$. The tables show that $f_{n-1}^\vee \otimes f_{n-1}^\vee$ and $f_n^\vee \otimes f_n^\vee$ map to $n/4$ whereas $f_n^\vee \otimes f_{n-1}^\vee$ and $f_{n-1}^\vee \otimes f_n^\vee$ map to $(n-2)/4$. That is, viewing (10.1) as a bilinear form on $\mathbb{F}_2 \oplus \mathbb{F}_2$, for $n \equiv 0 \bmod 4$ it is the wedge product (which is hyperbolic) and for $n \equiv 2 \bmod 4$ it is the unique (up to isomorphism) metabolic form that is not hyperbolic.

10B. ***The cup product on C.*** Let G be a simple simply connected algebraic group over F with center C. The pairing (10.1) reads as follows:

$$C_* \otimes C_* \xrightarrow{\mathrm{id}\otimes\rho} C_* \otimes C^* \to \mathbb{Q}/\mathbb{Z}.$$

Twisting (tensoring with $\mathbb{Z}(1)\overset{L}{\otimes}\mathbb{Z}(1)$) we get a composition

$$C_*(1) \overset{L}{\otimes} C_*(1) \to C_*(1) \overset{L}{\otimes} C^*(1) \to \mathbb{Q}/\mathbb{Z}(2),$$

where the second map was already defined in (2.9). Therefore, we have a pairing

$$H^1(F, C) \otimes H^2(F, C) \to H^1(F, C) \otimes H^2(F, C^\circ) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \tag{10.2}$$

which we denote by $\bullet$. In this language, Theorem 1.2 says that

$$r_G i^*(x) = -x \bullet t_G \quad \text{for } x \in H^1(F, C). \tag{10.3}$$

Combining this observation with the computation of (10.1) recovers the formulas given in [Merkurjev et al. 2002; Garibaldi and Quéguiner-Mathieu 2007], with no restriction on char(F).

Example 10.4. Suppose G has inner type D_n for some $n \geq 4$. Then G is isomorphic to $\mathbf{Spin}(A, \sigma, f)$ for some central simple algebra A with quadratic pair (σ, f) such that the (even) Clifford algebra of (A, σ, f) is a product $C_+ \times C_-$, see [Knus et al. 1998, 26.15]. Put μ_2 for the kernel of the map $\mathbf{Spin}(A, \sigma, f) \to \mathbf{SO}(A, \sigma, f)$ and write i_2 for the inclusion $\mu_2 \hookrightarrow G$. (The highest weights of the representations $\mathbf{Spin}(A, \sigma, f) \to \mathbf{GL}_1(C_\varepsilon)$ for $\varepsilon = \pm$ both restrict to the nonzero character on $i_2(\mu_2)$.)

We claim that, *for* $z \in H^1(F, \mu_2)$, *the equalities*

$$r_G i_2^*(z) = \begin{cases} z \cup [A] & \text{if } n \text{ even,} \\ z \cup [C_+] & \text{if } n \text{ odd,} \end{cases} \tag{10.5}$$

hold in $H^3(F, \mathbb{Z}/2\mathbb{Z}(2))$. This can be seen by combining (10.3) with the calculations in Section 10A. Alternatively, arguing as in the beginning of Section 9, it suffices to verify (10.5) in case the variety X has an F-point, where we may check the equality via Lemma 9.5 on the subgroup G'. Then Equation 12.2 of [Garibaldi and Quéguiner-Mathieu 2007] settles the n even case, and an analogous computation handles n odd. Note that for n odd, one could also write $z \cup [C_-]$ in (10.5), as $[C_-] = 3[C_+]$ and $2z = 0$.

Example 10.6. The exact sequence $1 \to C \xrightarrow{i} G \to G/C \to 1$ gives a connecting homomorphism $\partial : (G/C)(F) \to H^1(F, C)$. It follows from (10.3) that, for $y \in (G/C)(F)$, $\partial(y) \bullet t_G = r_G i^* \partial(y) = 0$, i.e.,

$$(\operatorname{im} \partial) \bullet t_G = 0 \quad \text{in } H^3(F, \mathbb{Q}/\mathbb{Z}(2)). \tag{10.7}$$

For G of inner type A_{n-1}, G is isomorphic to $\mathbf{SL}_1(A)$ for a central simple algebra A and we may identify $\operatorname{im} \partial$ with $\operatorname{Nrd}(A^\times) \subseteq H^1(F, \mu_n)$. In this case, (10.7) says: *If* $x \in \operatorname{Nrd}(A^\times)$, *then* $(x) \cup [A] = 0$.

For G of type C_n, G is isomorphic to $\mathbf{Sp}(A, \sigma)$ for a central simple algebra A with symplectic involution σ and we may identify $\operatorname{im} \partial$ with the group $G(A, \sigma)$ of multipliers of similitudes of (A, σ). If n is even, (10.7) is an empty claim because $\bullet$ is identically zero. If n is odd, (10.7) says that $G(A, \sigma) \cup [A] = 0$, i.e., since A is Brauer-equivalent to a quaternion algebra, $G(A, \sigma) \subseteq \operatorname{Nrd}(A^\times)$; this is proved in [Knus et al. 1998, 12.22].

References

[Bourbaki 2002] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Springer, Berlin, 2002. MR Zbl

[Conrad et al. 2015] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, 2nd ed., New Mathematical Monographs **26**, Cambridge Univ. Press, 2015. MR Zbl

[Freitag and Kiehl 1988] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjecture*, Ergebnisse der Mathematik (3) **13**, Springer, Berlin, 1988. MR Zbl

[Garibaldi 2001] S. Garibaldi, "The Rost invariant has trivial kernel for quasi-split groups of low rank", *Comment. Math. Helv.* **76**:4 (2001), 684–711. MR Zbl

[Garibaldi 2012] S. Garibaldi, "Outer automorphisms of algebraic groups and determining groups by their maximal tori", *Michigan Math. J.* **61**:2 (2012), 227–237. MR Zbl

[Garibaldi and Quéguiner-Mathieu 2007] S. Garibaldi and A. Quéguiner-Mathieu, "Restricting the Rost invariant to the center", *Algebra i Analiz* **19**:2 (2007), 52–73. Reprinted in *St. Petersburg Math. J.* **19**:2 (2008), 197–213. MR Zbl

[Garibaldi et al. 2003] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series **28**, American Mathematical Society, Providence, RI, 2003. MR Zbl

[Gille 2000] P. Gille, "Invariants cohomologiques de Rost en caractéristique positive", *K-Theory* **21**:1 (2000), 57–100. MR Zbl

[Gille and Quéguiner-Mathieu 2011] P. Gille and A. Quéguiner-Mathieu, "Formules pour l'invariant de Rost", *Algebra Number Theory* **5**:1 (2011), 1–35. MR Zbl

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge Univ. Press, 2006. MR Zbl

[Kahn 1996] B. Kahn, "Applications of weight-two motivic cohomology", *Doc. Math.* **1** (1996), 395–416. MR Zbl

[Kato 1982] K. Kato, "Galois cohomology of complete discrete valuation fields", pp. 215–238 in *Algebraic K-theory, II* (Oberwolfach, 1980), edited by R. K. Dennis, Lecture Notes in Math. **967**, Springer, Berlin, 1982. MR Zbl

[Knus et al. 1998] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Amer. Math. Soc. Colloquium Publications **44**, American Mathematical Society, Providence, RI, 1998. MR Zbl

[Lichtenbaum 1987] S. Lichtenbaum, "The construction of weight-two arithmetic cohomology", *Invent. Math.* **88**:1 (1987), 183–215. MR Zbl

[Lichtenbaum 1990] S. Lichtenbaum, "New results on weight-two motivic cohomology", pp. 35–55 in *The Grothendieck Festschrift, III*, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Merkurjev 1999] A. Merkurjev, "Invariants of algebraic groups", *J. Reine Angew. Math.* **508** (1999), 127–156. MR Zbl

[Merkurjev 2016] A. Merkurjev, "Weight two motivic cohomology of classifying spaces for semisimple groups", *Amer. J. Math.* **138**:3 (2016), 763–792. MR Zbl

[Merkurjev and Tignol 1995] A. S. Merkurjev and J.-P. Tignol, "The multipliers of similitudes and the Brauer group of homogeneous varieties", *J. Reine Angew. Math.* **461** (1995), 13–47. MR Zbl

[Merkurjev et al. 2002] A. S. Merkurjev, R. Parimala, and J.-P. Tignol, "Invariants of quasitrivial tori and the Rost invariant", *Algebra i Analiz* **14**:5 (2002), 110–151. Reprinted in *St. Petersburg Math. J.* **14**:5 (2003), 791–821. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton Univ. Press, 1980. MR Zbl

[Steinberg 1963] R. Steinberg, "Representations of algebraic groups", *Nagoya Math. J.* **22** (1963), 33–56. MR Zbl

[Tits 1966] J. Tits, "Classification of algebraic semisimple groups", pp. 33–62 in *Algebraic groups and discontinuous subgroups* (Boulder, CO, 1965), edited by A. Borel and G. D. Mostow, Proc. Sympos. Pure Math. **9**, American Mathematical Society, Providence, RI, 1966. MR Zbl

[Tits 1971] J. Tits, "Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque", *J. Reine Angew. Math.* **247** (1971), 196–220. MR Zbl

Received September 26, 2016. Revised February 8, 2017.

SKIP GARIBALDI
IDA CENTER FOR COMMUNICATIONS RESEARCH
SAN DIEGO, CA 92121
UNITED STATES
skip@member.ams.org

ALEXANDER S. MERKURJEV
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90095-1555
UNITED STATES
merkurev@math.ucla.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 291 No. 2 December 2017

Torsion pairs in silting theory 257
LIDIA ANGELERI HÜGEL, FREDERIK MARKS and JORGE VITÓRIA

Transfinite diameter on complex algebraic varieties 279
DAVID A. COX and SIONE MA'U

A universal construction of universal deformation formulas, Drinfeld twists and their positivity 319
CHIARA ESPOSITO, JONAS SCHNITZER and STEFAN WALDMANN

Uniform stable radius, Lê numbers and topological triviality for line singularities 359
CHRISTOPHE EYRAL

Rost invariant of the center, revisited 369
SKIP GARIBALDI and ALEXANDER S. MERKURJEV

Moduli spaces of rank 2 instanton sheaves on the projective space 399
MARCOS JARDIM, MARIO MAICAN and ALEXANDER S. TIKHOMIROV

A symmetric 2-tensor canonically associated to Q-curvature and its applications 425
YUEH-JU LIN and WEI YUAN

Gauge invariants from the powers of antipodes 439
CRIS NEGRON and SIU-HUNG NG

Branching laws for the metaplectic cover of GL_2 461
SHIV PRAKASH PATEL

Hessian equations on closed Hermitian manifolds 485
DEKAI ZHANG