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MODULI SPACES OF RANK 2 INSTANTON SHEAVES ON THE PROJECTIVE SPACE

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We study the irreducible components of the moduli space of instanton sheaves on $\mathbb{P}^3$, that is, μ-semistable rank 2 torsion-free sheaves E with $c_1(E) = c_3(E) = 0$ satisfying $h^1(E(-2)) = h^2(E(-2)) = 0$. In particular, we classify all instanton sheaves with $c_2(E) \leq 4$, describing all the irreducible components of their moduli space. A key ingredient for our argument is the study of the moduli space $\mathcal{T}(d)$ of stable sheaves on $\mathbb{P}^3$ with Hilbert polynomial $P(t) = d \cdot t$, which contains, as an open subset, the moduli space of rank 0 instanton sheaves of multiplicity d; we describe all the irreducible components of $\mathcal{T}(d)$ for $d \leq 4$.

1. Introduction

Instanton bundles on $\mathbb{CP}^3$ were introduced by Atiyah, Drinfeld, Hitchin and Manin in the late 1970s as the holomorphic counterparts, via twistor theory, to anti-self-dual connections with finite energy (instantons) on the four-dimensional round sphere S^4. To be more precise, an *instanton bundle of charge n* is a μ-stable rank 2 bundle E on $\mathbb{P}^3$ with $c_1(E) = 0$ and $c_2(E) = n$ satisfying the cohomological condition $h^1(E(-2)) = 0$; equivalently, an instanton bundle of charge n is a locally free sheaf which arises as cohomology of a linear monad of the form

$$0 \to n \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \to (2 + 2n) \cdot \mathcal{O}_{\mathbb{P}^3} \to n \cdot \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{1}$$

The moduli space $\mathcal{I}(n)$ of such objects has been thoroughly studied in the past thirty-five years by various authors and it is now known to be an irreducible [Tikhomirov 2012; 2013], nonsingular [Jardim and Verbitsky 2014] affine [Costa and Ottaviani 2003] variety of dimension $8n - 3$.

The closure of $\mathcal{I}(n)$ within the moduli space $\mathcal{M}(n)$ of semistable rank 2 sheaves with Chern classes $c_1 = 0$, $c_2 = n$ and $c_3 = 0$ contains nonlocally free sheaves which also arise as cohomology of monads of the form (1). Such *instanton sheaves* can

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alternatively be defined as rank 2 torsion-free sheaves satisfying the cohomological conditions

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

We prove that such sheaves are always stable (see Theorem 4 below), so they admit a moduli space $\mathcal{L}(n)$ regarded as an open subset of $\mathcal{M}(n)$ which, of course, contains $\mathcal{I}(n)$.

The spaces $\mathcal{L}(1)$ and $\mathcal{L}(2)$ were essentially known to be irreducible, see details in the first few paragraphs of Section 3 below. However, $\mathcal{L}(3)$ was observed to have at least two irreducible components [Jardim et al. 2015, Remark 8.6], while several new components of $\mathcal{L}(n)$ were constructed in [Jardim et al. 2017].

The main goal of this paper is to characterize the irreducible components of $\mathcal{L}(3)$ and $\mathcal{L}(4)$. We prove:

Main Theorem 1. (i) $\mathcal{L}(3)$ *is a connected quasiprojective variety consisting of exactly two irreducible components each of dimension* 21;

(ii) $\mathcal{L}(4)$ *is a connected quasiprojective variety consisting of exactly four irreducible components*, *three of dimension* 29 *and one of dimension* 32.

For every instanton sheaf E, the quotient $E^{\vee\vee}/E$ is a semistable sheaf with Hilbert polynomial $d\cdot(t+2)$, see Section 2 below. Therefore, an essential ingredient for the proof of Main Theorem 1 is the study of the moduli space $\mathcal{T}(d)$ of semistable sheaves on $\mathbb{P}^3$ with Hilbert polynomial $P(t) = d \cdot t$. Since these spaces are also interesting in their own right, we prove:

Main Theorem 2. (i) $\mathcal{T}(1)$ *is an irreducible projective variety of dimension* 5;

(ii) $\mathcal{T}(2)$ *is a connected projective variety consisting of exactly two irreducible components of dimension* 8;

(iii) $\mathcal{T}(3)$ *is a projective variety consisting of exactly four irreducible components*, *two of dimension* 12 *and two of dimension* 13;

(iv) $\mathcal{T}(4)$ *is a projective variety consisting of exactly eight irreducible components*, *four of dimension* 16, *two of dimension* 17, *one of dimension* 18 *and one of dimension* 20.

We also give a precise description of a generic point in each of the irreducible components mentioned in the statement of the theorem, see Section 4.

2. Stability of instanton sheaves

Recall from [Jardim 2006] that a torsion-free sheaf E on $\mathbb{P}^3$ is called an *instanton sheaf* if $c_1(E) = 0$ and the following cohomological conditions hold:

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

The integer $n := -\chi(E(-1))$ is called the charge of E; it is easy to check that $n = h^1(E(-1)) = c_2(E)$, and that $c_3(E) = 0$. The trivial sheaf $r \cdot \mathcal{O}_{\mathbb{P}^3}$ of rank r is considered as an instanton sheaf of charge zero. In this paper, we will only be interested in rank 2 instanton sheaves.

Recall that the singular locus $\operatorname{Sing}(G)$ of a coherent sheaf G on a nonsingular projective variety X is given by

$$\operatorname{Sing}(G) := \{x \in X \mid G_x \text{ is not free over } \mathcal{O}_{X,x}\},$$

where G_x denotes the stalk of G at a point x and $\mathcal{O}_{X,x}$ is its local ring. The following result, proved in [Gargate and Jardim 2016, Main Theorem], provides a key piece of information regarding the singular loci of rank 2 instanton sheaves.

Theorem 1. *If E is a nonlocally free instanton sheaf of rank 2 on $\mathbb{P}^3$, then*

(i) *its singular locus has pure dimension* 1;

(ii) *$E^{\vee\vee}$ is a (possibly trivial) locally free instanton sheaf.*

Remark 2. In fact, the quotient sheaf $Q_E := E^{\vee\vee}/E$ is a *rank 0 instanton sheaf*, in the sense of [Hauzer and Langer 2011, Section 6.1]; see also [Gargate and Jardim 2016, Section 3.2]. More precisely, a *rank* 0 *instanton sheaf* is a coherent sheaf Q on $\mathbb{P}^3$ such that $h^0(Q(-2)) = h^1(Q(-2)) = 0$; the integer $d := h^0(Q(-1))$ is called the *multiplicity* of Q.

The Hilbert polynomial of a rank 2 instanton sheaf E (in fact, of any coherent sheaf on $\mathbb{P}^3$ of rank 2 with $c_1 = 0$, $c_2 = n$ and $c_3 = 0$) is given by

$$P_E(t) = \tfrac{1}{3}(t+3)\cdot(t+2)\cdot(t+1) - n\cdot(t+2) = 2\cdot\chi(\mathcal{O}_{\mathbb{P}^3}(t)) - n\cdot(t+2). \tag{2}$$

Let $n' := c_2(E^{\vee\vee}) \geq 0$, then it follows from the standard sequence

$$0 \to E \to E^{\vee\vee} \to Q_E \to 0 \tag{3}$$

that

$$P_{Q_E}(t) = d\cdot(t+2) \quad \text{where } d := n - n'.$$

Note that $d = n - n'$ is precisely the multiplicity of Q_E as a rank 0 instanton sheaf.

Rank 0 instanton sheaves can be characterized in the following way.

Proposition 3. *Every rank* 0 *instanton sheaf Q admits a resolution of the form*

$$0 \to d\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \to 2d\cdot\mathcal{O}_{\mathbb{P}^3} \to d\cdot\mathcal{O}_{\mathbb{P}^3}(1) \to Q \to 0. \tag{4}$$

Proof. Consider the Beilinson spectral sequence from [Choi et al. 2016, Section 6], applied to the sheaf $Q' := Q(-2)$. We have $H^0(Q') = 0$, and therefore also $H^0(Q' \otimes \Omega^1_{\mathbb{P}^3}(1)) = 0$ and $H^0(Q'(-1)) = 0$. We adopt the notations of [Choi et al. 2016, Section 6]. Since $\operatorname{Ker}(\varphi_5)/\operatorname{Im}(\varphi_4) = 0$, we deduce that $H^0(Q' \otimes \Omega^2_{\mathbb{P}^3}(2)) = 0$. Thus, the bottom row of the E^1-term of the spectral sequence vanishes. Since φ_7 is an isomorphism, we deduce that φ_1 is injective. Since φ_8 is injective, we deduce that $\mathrm{Ker}(\varphi_2) = \mathrm{Im}(\varphi_1)$. The top row of the E^1-term of the spectral sequence yields the resolution

$$0 \to H^1(Q'(-1)) \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{\varphi_1} H^1(Q' \otimes \Omega^2_{\mathbb{P}^3}(2)) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\varphi_2} H^1(Q' \otimes \Omega^1_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to Q' \to 0.$$

We have

$$\chi(Q' \otimes \Omega^1_{\mathbb{P}^3}(1)) = -d, \qquad \chi(Q' \otimes \Omega^2_{\mathbb{P}^3}(2)) = -2d, \qquad \chi(Q'(-1)) = -d,$$

hence

$$h^1(Q' \otimes \Omega^1_{\mathbb{P}^3}(1)) = d, \qquad h^1(Q' \otimes \Omega^2_{\mathbb{P}^3}(2)) = 2d, \qquad h^1(Q'(-1)) = d.$$

The above exact sequence yields (4). □

Let now examine the stability properties of instanton sheaves.

Theorem 4. *Every nontrivial rank 2 instanton sheaf E is stable. In addition, a nontrivial instanton sheaf E is μ-stable if and only if its double dual $E^{\vee\vee}$ is nontrivial.*

Proof. Since rank 2 instanton sheaves have no global sections [Jardim 2006, Proposition 11], every nontrivial locally free rank 2 instanton sheaf is μ-stable; therefore, if $E^{\vee\vee}$ is nontrivial, then E is also μ-stable. Conversely, if E is μ-stable, then so is $E^{\vee\vee}$, hence it must be nontrivial.

Therefore, in order to prove the first claim of the Theorem, it is enough to consider *quasitrivial instanton sheaves*, i.e., rank 2 instanton sheaves E with $E^{\vee\vee} \simeq 2 \cdot \mathcal{O}_{\mathbb{P}^3}$; note that the multiplicity of Q_E is exactly $n = c_2(E)$.

Since E has no global sections, it can only be destabilized by the ideal sheaf I_C of a subscheme $C \subset \mathbb{P}^3$. Moreover, we can assume that the quotient sheaf E/I_C is torsion-free, thus it is also the ideal sheaf I_D of another subscheme $D \subset \mathbb{P}^3$. We obtain two exact sequences

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & I_C & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & E & \longrightarrow & 2 \cdot \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & Q_E & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & I_D & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Taking the double dual of the top vertical morphisms we obtain, also using the Snake Lemma, the following commutative diagram:

(5)
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_C & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & 2\cdot\mathcal{O}_{\mathbb{P}^3} & \longrightarrow & Q_E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_D & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Since $h^0(Q_E(-2)) = 0$, then $h^0(\mathcal{O}_C(-2)) = 0$ also, hence C must have pure dimension 1. Moreover, note also that $h^1(Q_E(-2)) = 0$ implies $h^1(\mathcal{O}_D(-2)) = 0$.

We show that $\dim D = 0$. Indeed, assume that D has dimension 1. Let U be the maximal zero-dimensional subsheaf of $\mathcal{O}_D$, and set $\mathcal{O}_{D'} := \mathcal{O}_D/U$; clearly, D' has pure dimension 1. Next, let $D'' := D'_{\mathrm{red}}$ be the underlying reduced scheme. We end up with two exact sequences

$$0 \to U \to \mathcal{O}_D \to \mathcal{O}_{D'} \to 0 \quad \text{and} \quad 0 \to T \to \mathcal{O}_{D'} \to \mathcal{O}_{D''} \to 0,$$

so that the vanishing of $h^1(\mathcal{O}_D(-2))$ forces $h^1(\mathcal{O}_{D''}(-2)) = 0$.

Still, D'' may be reducible, so let $D'' := D''_1 \cup \cdots \cup D''_p$ be its decomposition into irreducible components. For each index $j = 1, \ldots, p$ we obtain a sequence,

$$0 \to S_j \to \mathcal{O}_{D''} \to \mathcal{O}_{D''_j} \to 0,$$

thus also $h^1(\mathcal{O}_{D''_j}(-2)) = 0$. Let d_j and p_j denote the degree and arithmetic genus of D''_j, respectively. It follows that

$$0 \le h^0(\mathcal{O}_{D''_j}(-2)) = \chi(\mathcal{O}_{D''_j}(-2)) = -2d_j + 1 - p_j,$$

thus $p_j \le -2d_j + 1 \le -1$, which is impossible for a reduced and irreducible curve.

Now let $\delta = h^0(\mathcal{O}_D)$ be the length of D. Since $\deg(C) = n$, we have

$$P_{I_C}(t) = \chi(\mathcal{O}_{\mathbb{P}^3}(t)) - \chi(\mathcal{O}_C(t)) = \chi(\mathcal{O}_{\mathbb{P}^3}(t)) - nt + (\delta - 2n).$$

Comparing with equation (2), we have

(6)
$$\frac{P_E(t)}{2} - P_{I_C}(t) = \frac{n}{2}t + n - \delta,$$

which is positive for t sufficiently large, and therefore E contains no destabilizing subsheaves. □

As a consequence of the proof above, we obtain the following interesting fact.

Corollary 5. *Every rank 2 quasitrivial instanton sheaf of charge n on $\mathbb{P}^3$ is an extension of the ideal of a zero-dimensional scheme D by the ideal of a pure one-dimensional scheme C of degree n containing D. In addition,*

$$\chi(\mathcal{O}_C) = 2n - h^0(\mathcal{O}_D).$$

On the other hand, it is easy to check that every rank 0 instanton sheaf is semistable.

Lemma 6. *Every rank 0 instanton sheaf is semistable.*

Proof. Let Z be a rank 0 instanton sheaf, and let T be a subsheaf of Z with Hilbert polynomial $P_T(t) = a \cdot t + \chi(T)$. Since $h^0(Z(-2)) = 0$, then $h^0(T(-2)) = 0$ and $-2a + \chi(T) = -h^1(T(-2)) \leq 0$. It follows that

$$\frac{\chi(T)}{a} \leq 2 = \frac{\chi(Z)}{d}. \qquad \square$$

Clearly, not every rank 0 instanton sheaf is stable: if Q_1 and Q_2 are rank 0 instanton sheaves, then so is any extension of Q_1 by Q_2, and this cannot possibly be stable.

Conversely, there are semistable sheaves with Hilbert polynomial $dt + 2d$ which are not rank 0 instanton sheaves: just consider $Q := \mathcal{O}_\Sigma(2)$ for an elliptic curve $\Sigma \hookrightarrow \mathbb{P}^3$, so that $h^0(Q(-2)) \neq 0$.

3. Moduli space of instanton sheaves

Let $\mathcal{L}(n)$ denote the open subscheme of the Maruyama moduli space $\mathcal{M}(n)$ of semistable rank 2 torsion-free sheaves with Chern classes $c_1 = 0$, $c_2 = n$ and $c_3 = 0$ consisting of instanton sheaves of charge n. Let $\mathcal{I}(n)$ denote the open subscheme of $\mathcal{M}(n)$ consisting of locally free instanton sheaves. Finally, let $\overline{\mathcal{L}(n)}$ and $\overline{\mathcal{I}(n)}$ denote the closures within $\mathcal{M}(n)$ of $\mathcal{L}(n)$ and $\mathcal{I}(n)$, respectively. We also consider the set $\mathcal{I}^0(n) := \overline{\mathcal{I}(n)} \cap \mathcal{L}(n)$, which consists of those instanton sheaves which either are locally free, or can be deformed into locally free ones.

It was shown in [Tikhomirov 2012; 2013] that $\mathcal{I}(n)$ is irreducible for every $n > 0$; its closure $\overline{\mathcal{I}(n)}$ is called the *instanton component* of $\mathcal{M}(n)$. However, the same is not true for $\mathcal{L}(n)$ as soon as $n \geq 3$. Indeed, it is well known that

$$\overline{\mathcal{I}(1)} = \mathcal{L}(1) = \mathcal{M}(1) \simeq \mathbb{P}^5,$$

see for instance [Jardim et al. 2017, Section 6].

The case $n = 2$ has also been understood.

Proposition 7. $\overline{\mathcal{L}(2)} = \overline{\mathcal{I}(2)}$.

In particular, $\mathcal{L}(2)$ possesses a single irreducible component of dimension 13.

Proof. Le Potier [1993b] showed that $\mathcal{M}(2)$ has exactly 3 irreducible components; according to the description of these components provided in [Jardim et al. 2017, Section 6], only the instanton component $\overline{\mathcal{I}(2)}$ contains instanton sheaves. □

Let us now describe the irreducible components of $\mathcal{L}(n)$ for $n \geq 3$ introduced in [Jardim et al. 2017, Section 3].

Let Σ be an irreducible, nonsingular, complete intersection curve in $\mathbb{P}^3$, given as the intersection of a surface of degree d_1 with a surface of degree d_2, with $1 \leq d_1 \leq d_2$; denote by $\iota : \Sigma \hookrightarrow \mathbb{P}^3$ the inclusion morphism. Choose $L \in \operatorname{Pic}^{g-1}(\Sigma)$ such that $h^0(\Sigma, L) = h^1(\Sigma, L) = 0$. Given a (possibly trivial) locally free instanton sheaf F of charge $c \geq 0$ and an epimorphism $\varphi : F \twoheadrightarrow (\iota_* L)(2)$, the kernel $E := \operatorname{Ker} \varphi$ is an instanton sheaf of charge $c + d_1 d_2$. Thus we may consider the set

$$\mathcal{C}(d_1, d_2, c) := \left\{ [E] \in \mathcal{M}(c + d_1 d_2) \mid E^{\vee\vee} \in \mathcal{I}(c),\ E^{\vee\vee}/E \simeq (\iota_* L)(2) \right\} \tag{7}$$

as a subvariety of $\mathcal{M}(c + d_1 d_2)$. The following result is proved in [Jardim et al. 2017], see Theorems 15, 17 and 23 of that paper.

Theorem 8. *For each $c \geq 0$ and $1 \leq d_1 \leq d_2$ such that $(d_1, d_2) \neq (1,1), (1,2)$, $\overline{\mathcal{C}(d_1, d_2, c)}$ is an irreducible component of $\mathcal{M}(c + d_1 d_2)$ of dimension*

$$\dim \overline{\mathcal{C}(d_1, d_2, c)} = 8c - 3 + \tfrac{1}{2} d_1 d_2 (d_1 + d_2 + 4) + h, \tag{8}$$

where

$$h = \begin{cases} 2\binom{d_1+3}{3} - 4 & \text{if } d_1 = d_2, \\ \binom{d_1+3}{3} + \binom{d_2+3}{3} - \binom{d_2-d_1+3}{3} - 2 & \text{if } d_1 < d_2. \end{cases}$$

In addition, $\overline{\mathcal{C}(d_1, d_2, c)} \cap \mathcal{I}^0(c + d_1 d_2) \neq \varnothing$.

We do not know whether the families $\mathcal{C}(d_1, d_2, c)$ exhaust all components of $\mathcal{L}(n)$, but we prove that this holds for $n = 3$ and 4 in Sections 5 and 6 below, respectively.

However, we remark that the previous result allows for a partial count of the number of components of $\mathcal{L}(n)$. Indeed, let $\tau(n)$ denote the number of irreducible components of the union

$$\overline{\mathcal{I}(n)} \bigcup \Bigg(\bigcup_{d_1 d_2 + c = n} \overline{\mathcal{C}(d_1, d_2, c)} \Bigg).$$

To estimate $\tau(n)$, we must count the different ways in which an integer $n \geq 3$ can be written as $n = d_1 d_2 + c$ with $c \geq 0$, and $1 \leq d_1 \leq d_2$ excluding the pairs $(d_1, d_2) = (1,1), (1,2)$. Consider the function

$$\delta(p) = \begin{cases} \frac{1}{2}(d(p) + 1) & \text{if } p \text{ is a perfect square}, \\ \frac{1}{2} d(p) & \text{otherwise}, \end{cases}$$

where $d(p)$ is the *divisor function*, i.e., the number of divisors of a positive integer p, including p itself. Note that $\delta(p)$ is the number of different ways in which we can write p as a product $d_1 d_2$ with $1 \le d_1 \le d_2$. Adding the instanton component, we have

$$\tau(n) = 1 + \sum_{p=3}^{n} \delta(p) = \frac{1}{2}\Bigl(\sum_{p=3}^{n} d(p) + \lfloor\sqrt{n}\rfloor + 1\Bigr), \tag{9}$$

since $\lfloor\sqrt{n}\rfloor - 1$ accounts for the number of perfect squares between 3 and n.

Lemma 9. *Let $l(n)$ be the number of irreducible components of the moduli space of instanton sheaves of charge n. Then, for n sufficiently large, $l(n) > \frac{1}{2}n \cdot \log(n)$.*

Proof. Determining the asymptotic behavior of the sum of divisors function is a relevant problem in number theory called the *Dirichlet divisor problem*; indeed, it is known that

$$\sum_{p=1}^{n} d(p) = n \cdot \log(n) + (2\gamma - 1)n + O(n^{\theta}),$$

where γ denotes the Euler–Mascheroni constant, and $1/4 \le \theta \le 131/416$, see [Huxley 2003]. Comparing with equation (9), we obtain the desired estimate. □

Also relevant for us is a class of instanton sheaves studied in [Jardim et al. 2015]; more precisely, for $n > 0$ and each $m = 1, \dots, n$, consider the subset $\mathcal{D}(m, n)$ of $\mathcal{M}(n)$ consisting of the isomorphism classes $[E]$ of the sheaves E obtained in this way:

$$\mathcal{D}(m,n) := \bigl\{[E] \in \mathcal{M}(n) \mid [E^{\vee\vee}] \in \mathcal{I}(n-m),\ \Gamma = \operatorname{supp}(E^{\vee\vee}/E) \in \mathcal{R}_0^*(m)_{E^{\vee\vee}},\\ \text{and } E^{\vee\vee}/E \simeq \mathcal{O}_\Gamma((2m-1)\mathrm{pt})\bigr\},$$

where the space $\mathcal{R}_0^*(m)_{E^{\vee\vee}}$ is described as follows: first, let $\mathcal{R}_0^*(m)$ denote the space of nonsingular rational curves $\Gamma \hookrightarrow \mathbb{P}^3$ of degree m whose normal bundle $N_{\Gamma/\mathbb{P}^3}$ is given by $2 \cdot \mathcal{O}_\Gamma((2m-1)\mathrm{pt})$; then, for any instanton sheaf F we set

$$\mathcal{R}_0^*(m)_F := \{\Gamma \in \mathcal{R}_0^*(m) \mid F|_\Gamma \simeq 2 \cdot \mathcal{O}_\Gamma\}.$$

One can show that for every rank 2 instanton sheaf F, the space $\mathcal{R}_0^*(m)_F$ is a nonempty open subset of $\mathcal{R}_0^*(m)$, see [Jardim et al. 2015, Lemma 6.2].

Let $\overline{\mathcal{D}(m,n)}$ denote the closure of $\mathcal{D}(m,n)$ within $\mathcal{M}(n)$. Note that since $E^{\vee\vee}$ is a locally free instanton sheaf of charge $n-m$, and $\mathcal{O}_\Gamma(2m-1)$ is a rank 0 instanton sheaf of degree m, then E is an instanton sheaf of charge n, so that $\mathcal{D}(m,n) \subset \mathcal{L}(n)$. In fact, it is shown in [Jardim et al. 2017, Theorem 7.8] that $\overline{\mathcal{D}(m,n)} \subset \mathcal{I}^0(n)$. In addition, we prove:

Proposition 10. *Let $\Gamma_1, \dots, \Gamma_r$ be disjoint, smooth irreducible rational curves in $\mathbb{P}^3$ of degrees $m_1, \dots, m_r$, respectively; set $Q := \bigoplus_{j=1}^{r} \mathcal{O}_{\Gamma_j}(-\mathrm{pt})$. If F is a locally free instanton sheaf of charge c such that $F|_{\Gamma_j} \simeq 2 \cdot \mathcal{O}_{\Gamma_j}$ for each $j = 1, \dots, r$, and $\varphi : F \twoheadrightarrow Q(2)$ is an epimorphism, then $[\ker\varphi] \in \mathcal{I}^0(c + m_1 + \cdots + m_r)$.*

The proof of the previous proposition requires the following technical lemma, proved in [Jardim et al. 2015, Lemma 7.1].

Lemma 11. *Let C be a smooth irreducible curve with a marked point 0, and set $\boldsymbol{B} := C \times \mathbb{P}^3$. Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be $\mathcal{O}_{\boldsymbol{B}}$-sheaves, flat over C and such that $\boldsymbol{F}$ is locally free along* $\operatorname{supp}(\boldsymbol{G})$. *Denote*

$$\boldsymbol{G}_t := \boldsymbol{G}|_{\{t\}\times\mathbb{P}^3} \quad \textit{and} \quad \boldsymbol{F}_t = \boldsymbol{F}|_{\{t\}\times\mathbb{P}^3} \quad \textit{for } t \in C.$$

Assume that, for each $t \in C$,

$$H^i(\mathcal{H}om(\boldsymbol{F}_t, \boldsymbol{G}_t)) = 0, \quad i \geq 1. \tag{10}$$

If $s : \boldsymbol{F}_0 \to \boldsymbol{G}_0$ is an epimorphism, then, after possibly shrinking C, s extends to an epimorphism $\boldsymbol{s} : \boldsymbol{F} \twoheadrightarrow \boldsymbol{G}$.

Proof of Proposition 10. We argue by induction on r; the case $r = 1$ is just the aforementioned result, namely [Jardim et al. 2017, Theorem 7.8].

Let $Q' := \bigoplus_{j=1}^{r-1} \mathcal{O}_{\Gamma_j}(-\mathrm{pt})$, so that $Q = Q' \oplus \mathcal{O}_{\Gamma_r}(-\mathrm{pt})$. Let $E := \ker\varphi$, and let E' denote the kernel of the composition $F \xrightarrow{\varphi} Q(2) \twoheadrightarrow Q'(2)$. We obtain the following exact sequence:

$$0 \to E \to E' \xrightarrow{\varphi'} \mathcal{O}_{\Gamma_r}((2m_r - 1)\mathrm{pt}) \to 0.$$

By the induction hypothesis, $[E']$ is in $\mathcal{I}^0(c+m_1+\cdots+m_{r-1})$, thus one can find an affine open subset $0 \in U \subset \mathbb{A}^1$ and a coherent sheaf $\boldsymbol{E}$ on $\mathbb{P}^3 \times U$, flat over U, such that $\boldsymbol{E}_0 = E'$ and $\boldsymbol{E}_t$ is a locally free instanton sheaf of charge $c+m_1+\cdots+m_{r-1}$ satisfying $\boldsymbol{E}_t|_{\Gamma_r} \simeq 2\cdot\mathcal{O}_{\Gamma_r}$ for every $t \in U \setminus \{0\}$. Setting $\boldsymbol{G} := \pi^*(Q/Q'(2))$ where $\pi : \mathbb{P}^3 \times U \to \mathbb{P}^3$ is the projection onto the first factor, note that

$$H^i(\mathcal{H}om(\boldsymbol{E}_t, \boldsymbol{G}_t)) = H^i(2\cdot\mathcal{O}_{\Gamma_r}((2m_r - 1)\mathrm{pt})) = 0, \qquad \text{for } i \geq 1 \text{ and } t \in U.$$

This claim is clear for $t \neq 0$; when $t = 0$, simply observe that the sequence $0 \to E' \to F \to Q'(2) \to 0$ implies that $E'|_{\Gamma_r} \simeq F|_{\Gamma_r}$, since the support of Q' is disjoint from Γ_r.

By Lemma 11, there exists an epimorphism $\boldsymbol{s} : \boldsymbol{E} \twoheadrightarrow \boldsymbol{G}$ extending $\varphi' : E' \to \mathcal{O}_{\Gamma_r}((2m_r - 1)\mathrm{pt})$, so that $[\ker \boldsymbol{s}_t] \in \mathcal{D}(m_r, c+m_1+\cdots+m_r)$, by construction. It then follows that $[E] \in \overline{\mathcal{D}(m_r, c+m_1+\cdots+m_r)}$, hence, by [Jardim et al. 2017, Theorem 7.8], $[E] \in \mathcal{I}^0(c+m_1+\cdots+m_r)$, as desired. □

Next, we consider the following situation: let Σ be an irreducible, nonsingular, complete intersection curve in $\mathbb{P}^3$, given as the intersection surfaces of degrees d_1 and d_2, with $1 \leq d_1 \leq d_2$ and $(d_1, d_2) \neq (1, 1), (1, 2)$, and let Γ be a smooth irreducible rational curve in $\mathbb{P}^3$ of degree m disjoint from Σ. Set $Q := L \oplus \mathcal{O}_\Gamma(-\mathrm{pt})$ for some $L \in \operatorname{Pic}^{g-1}(\Sigma)$ such that $h^0(\Sigma, L) = h^1(\Sigma, L) = 0$, where g is the genus of Σ.

Proposition 12. *If F is a locally free instanton sheaf of charge c such that $F|_\Gamma \simeq 2\cdot\mathcal{O}_\Gamma$, and $H^1(F^\vee|_\Sigma \otimes L(2)) = 0$. If $\varphi : F \twoheadrightarrow Q(2)$ is an epimorphism, then*

$$[\ker\varphi] \in \overline{\mathcal{C}(d_1, d_2, c+m)}.$$

Proof. The idea is the same as in the proof of Proposition 10. Let E' be the kernel of the composition $F \xrightarrow{\varphi} Q(2) \twoheadrightarrow \mathcal{O}_\Gamma((2m-1)\,\mathrm{pt})$, so that $E := \ker\varphi$ and E' are related via the following exact sequence:

$$0 \to E \to E' \xrightarrow{\varphi'} L(2) \to 0.$$

By [Jardim et al. 2017, Theorem 7.8], one can find an affine open subset $0 \in U \subset \mathbb{A}^1$ and a coherent sheaf $\boldsymbol{E}$ on $\mathbb{P}^3 \times U$, flat over U, such that $\boldsymbol{E}_0 = E'$ and $\boldsymbol{E}_t$ is a locally free instanton sheaf of charge $c+m$ for every $t \in U \setminus \{0\}$.

Setting $\boldsymbol{G} := \pi^* L(2)$, we must, in order to apply Lemma 11, check that

$$H^i(\mathcal{H}om(E_t, G_t)) = 0, \qquad \text{for } i \geq 1 \text{ and } t \in U.$$

Indeed, since $\dim G_t = 1$, it is enough to show that $H^1(\mathcal{H}om(E_t, G_t)) = 0$. Note

$$\mathcal{H}om(E_0, G_0) = \mathcal{H}om(E', L(2)) \simeq \mathcal{H}om(F, L(2)) \simeq F^\vee|_\Sigma \otimes L(2),$$

where the middle isomorphism follows from applying the functor $\mathcal{H}om(\,\cdot\,, L(2))$ to the sequence

$$0 \to E' \to F \to \mathcal{O}_\Gamma((2m-1)\,\mathrm{pt}) \to 0,$$

also exploring the fact that Σ and Γ are disjoint. It follows that $H^1(\mathcal{H}om(E_0, G_0)) = H^1(F^\vee|_\Sigma \otimes L(2)) = 0$ by hypothesis. By semicontinuity of $h^1(\mathcal{H}om(E_t, G_t))$, we can shrink U to another affine open subset $U' \subset \mathbb{A}^1$, if necessary, to guarantee that $H^1(\mathcal{H}om(E_t, G_t)) = 0$ for every $t \in U'$.

By Lemma 11, there exists an epimorphism $\boldsymbol{s} : \boldsymbol{F} \twoheadrightarrow \boldsymbol{G}$ extending $\varphi' : E' \to L(2)$, so that $[\ker \boldsymbol{s}_t] \in \mathcal{C}(d_1, d_2, c+m)$, by construction. Since $E \simeq \ker \boldsymbol{s}_0$, it follows that $[E] \in \overline{\mathcal{C}(d_1, d_2, c+m)}$. □

4. Moduli of sheaves of dimension one and Euler characteristic zero

Given two integers d and χ, $d \geq 1$, let $\mathcal{T}(d, \chi)$ be the moduli space of semistable coherent sheaves on $\mathbb{P}^3$ with Hilbert polynomial $P(t) = d\cdot t + \chi$. In this section, we focus on the space $\mathcal{T}(d) := \mathcal{T}(d, 0)$.

Apart from its intrinsic interest, the space $\mathcal{T}(d)$ is also relevant for the study of instanton sheaves, and the description of $\mathcal{T}(d)$ for $d \leq 4$ provided in this section will be a key ingredient for the proof of the Main Theorem 1.

In addition, let $\mathcal{Z}(d)$ denote the set of rank 0 instanton sheaves of degree d *modulo S-equivalence* (which makes sense, since, by Lemma 6, every rank 0 instanton sheaf is semistable). After a twist by $\mathcal{O}_{\mathbb{P}^3}(-2)$, $\mathcal{Z}(d)$ can be regarded as an open subscheme of the moduli space $\mathcal{T}(d)$ consisting of those sheaves Q satisfying $h^0(Q) = 0$.

The space $\mathcal{T}(d)$ has several distinguished subsets, which we now describe.

First, let $\mathcal{P}_d \subset \mathcal{T}(d)$ be the subset of planar sheaves; it is a fiber bundle over $(\mathbb{P}^3)^*$ with fiber being the moduli space of semistable coherent sheaves on $\mathbb{P}^2$ with Hilbert polynomial $P = d \cdot t$. In view of [Le Potier 1993a, Theorem 1.1], $\mathcal{P}_d$ is a projective irreducible variety of dimension d^2+4. In particular, $\mathcal{P}_d$ is closed.

Next, consider the subsets $\mathcal{R}_d^o, \mathcal{E}_d^o \subset \mathcal{T}(d)$ of sheaves supported on smooth rational curves of degree d, respectively, on smooth elliptic curves of degree d. Let $\mathcal{R}_d$ and $\mathcal{E}_d$ denote their closures.

Given a partition $(d_1, \dots, d_s)$ of d such that $d_1 \geq \cdots \geq d_s$, we denote by

$$\mathcal{T}_{d_1,\dots,d_s} \subset \mathcal{T}(d)$$

the locally closed subset of points of the form

$$[Q_1 \oplus \cdots \oplus Q_s], \tag{11}$$

where Q_i gives a stable point in $\mathcal{T}(d_i)$; in particular, $\mathcal{T}_d$ is the open subset of stable points in $\mathcal{T}(d)$. Let $\mathcal{T}^o_{d_1,\dots,d_s} \subset \mathcal{T}_{d_1,\dots,d_s}$ be the open dense subset given by the condition that $\mathrm{supp}(Q_i)$ be mutually disjoint. Clearly, each irreducible component of $\mathcal{T}^o_{d_1,\dots,d_s}$ is an open dense subset of an irreducible component of $\mathcal{T}(d)$. Hence the irreducible components of the closure of $\mathcal{T}_{d_1,\dots,d_s}$ within $\mathcal{T}(d)$, henceforth denoted by $\overline{\mathcal{T}}_{d_1,\dots,d_s}$, are also irreducible components of $\mathcal{T}(d)$. On the other hand, each point of $\mathcal{T}(d)$ is an S-equivalence class of a polystable (e.g., stable) sheaf of the form (11). Hence, Lemma 13 follows.

Lemma 13. (i) *All irreducible components of $\mathcal{T}(d)$ are exhausted by the irreducible components of the union*

$$\bigcup_{(d_1,\dots,d_s)} \overline{\mathcal{T}}_{d_1,\dots,d_s}, \tag{12}$$

this union being taken over all the partitions $(d_1, \dots, d_s)$ of d.

(ii) *For a given partition $(d_1, \dots, d_s)$ of d, each irreducible component of $\overline{\mathcal{T}}_{d_1,\dots,d_s}$ is birational to a symmetric product,*

$$(\mathcal{X}_1 \times \cdots \times \mathcal{X}_s)/\Sigma,$$

of irreducible components $\mathcal{X}_i$ of $\mathcal{T}_{d_i}$, where Σ is the subgroup of the full symmetric group Σ_s of degree s generated by the transpositions (i, j) for which $d_i = d_j$ and $\mathcal{X}_i = \mathcal{X}_j$.

Proof. We have only to prove statement (ii). Indeed, let $\Sigma' \subset \Sigma_s$ be the subgroup generated by the transpositions $(i, i+1)$ for which $d_i = d_{i+1}$. We have a bijective morphism

$$(\mathcal{T}_{d_1} \times \cdots \times \mathcal{T}_{d_s})/\Sigma' \to \mathcal{T}_{d_1,\dots,d_s}, \qquad ([Q_1], \dots, [Q_s]) \longmapsto [Q_1 \oplus \cdots \oplus Q_s],$$

which is an isomorphism over $\mathcal{T}^o_{d_1,\dots,d_s}$, because over this set we can construct local inverse maps. Whence, the statement (ii) follows. □

Remark 14. Lemma 13 implies that the problem of finding the irreducible components of $\mathcal{T}(d)$ is reduced to the problem of finding the irreducible components of $\mathcal{T}_2, \dots, \mathcal{T}_d$.

Remark 15. It also follows from Lemma 13 that the number of irreducible components of $\mathcal{T}(d)$ is at least as large as the number of partitions of d, usually denoted $p(d)$. A well-known formula by Hardy and Ramanujan gives the following asymptotic expression

$$p(d) \sim \frac{1}{4\sqrt{3}\cdot d} \exp\Bigl(\pi\sqrt{\frac{2d}{3}}\Bigr).$$

Therefore, the number of irreducible components of $\mathcal{T}(d)$ grows at least exponentially on $\sqrt{d}$. However, as we shall see below in the cases $d = 3$ and $d = 4$, $p(d)$ is just a rough underestimate of the number of irreducible components of $\mathcal{T}(d)$.

Given a coherent sheaf Q on $\mathbb{P}^3$, we define

$$Q^{\mathrm{D}} := \mathcal{E}xt^c(Q, \omega_{\mathbb{P}^3}),$$

where $c = \operatorname{codim}(Q)$. We shall later use the following general result regarding stable sheaves in $\mathcal{T}(d)$.

Lemma 16. *Assume that $\mathcal{F}$ gives a stable point in $\mathcal{T}(d)$ and that $P \in \operatorname{supp}(\mathcal{F})$ is a closed point. Then there are exact sequences*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathbb{C}_P \to 0 \tag{13}$$

and

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathbb{C}_P \to 0 \tag{14}$$

for some sheaves $\mathcal{E} \in \mathcal{T}(d, -1)$ and $\mathcal{G} \in \mathcal{T}(d, +1)$.

Proof. Choose a surjective morphism $\mathcal{F} \to \mathbb{C}_P$ and denote its kernel by $\mathcal{E}$. Since $\mathcal{F}$ is stable, $\mathcal{E}$ is semistable, so we have sequence (13). According to [Maican 2010, Theorem 13], the dual sheaf $\mathcal{F}^{\mathrm{D}}$ gives a stable point in $\mathcal{T}(d)$. Thus, we have an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{F}^{\mathrm{D}} \to \mathbb{C}_P \to 0$$

with $\mathcal{E}_1 \in \mathcal{T}(d, -1)$. According to [Maican 2010, Remark 4], $\mathcal{F}$ is reflexive. According to [Maican 2010, Theorem 13], the sheaf $\mathcal{G} = \mathcal{E}_1^{\mathrm{D}}$ gives a point in $\mathcal{T}(d, 1)$. Since $\mathcal{F}^{\mathrm{D}}$ is pure, we can apply [Huybrechts and Lehn 1997, Proposition 1.1.10] to deduce that

$$\mathcal{E}xt^3(\mathcal{F}^{\mathrm{D}}, \omega_{\mathbb{P}^3}) = 0.$$

The long exact sequence of $\mathcal{E}xt$-sheaves associated to the above exact sequence yields (14). □

The goal of this section is to describe the irreducible components of $\mathcal{T}(d)$ for $d \le 4$. According to [Drézet and Maican 2011], for $\mathcal{F} \in \mathcal{T}(d)$ we have the following cohomological conditions

$$h^0(\mathcal{F}) = 0 \qquad \text{if } d = 1 \text{ or } 2,$$
$$h^0(\mathcal{F}) \le 1 \qquad \text{if } d = 3 \text{ or } 4.$$

4.1. ***Moduli of sheaves of degree* 1 *and* 2.** The case $d = 1$ is straightforward: clearly, $\mathcal{T}(1) \simeq \mathcal{R}_1$, being isomorphic to the Grassmannian of lines in $\mathbb{P}^3$.

Proposition 17. *The moduli space* $\mathcal{T}(1)$ *is an irreducible projective variety of dimension* 4.

In addition, it is easy to see that $\mathcal{Z}(1) = \mathcal{T}(1)$.

Proposition 18. *The moduli space* $\mathcal{T}(2)$ *is connected, and has two irreducible components, each of dimension* 8: $\mathcal{P}_2$ (*which coincides with* $\mathcal{R}_2$) *and* $\overline{\mathcal{T}}_{1,1}$.

Proof. If $\mathcal{F} \in \mathcal{T}_2$, then we have the exact sequence (14) in which $\mathcal{G} \in \mathcal{T}(2, 1)$. Thus, $\mathcal{G}$ is the structure sheaf of a conic curve, hence $\mathcal{G}$ is planar, and hence $\mathcal{F}$ is planar. We conclude that $\mathcal{T}(2) = \mathcal{P}_2 \cup \overline{\mathcal{T}}_{1,1}$. The intersection $\mathcal{P}_2 \cap \overline{\mathcal{T}}_{1,1}$ consists of those points of the form $[\mathcal{O}_{\ell_1}(-1) \oplus \mathcal{O}_{\ell_2}(-1)]$ where ℓ_1 and ℓ_2 are two intersecting (and possibly coincident) lines. □

Note also that $\mathcal{Z}(2) = \mathcal{T}(2)$; the fact that $\mathcal{Z}(2)$ consists of two irreducible components of dimension 8 should be compared with [Hauzer and Langer 2011, Corollary 6.12], where the authors prove that the moduli space of *framed* rank 0 instanton sheaves of multiplicity 2 also consists of two irreducible components of dimension 8.

4.2. ***Moduli of sheaves of degree 3.***

Proposition 19. *The moduli space* $\mathcal{T}(3)$ *has four irreducible components* $\mathcal{P}_3$, $\mathcal{R}_3$, $\overline{\mathcal{T}}_{2,1}$ *and* $\overline{\mathcal{T}}_{1,1,1}$, *of dimension* 13, 13, 12, *and* 12, *respectively.*

Proof. By Proposition 18 we have $\overline{\mathcal{T}}_2 = \mathcal{P}_2$, so that in view of Lemma 13 we already obtain the irreducible components $\overline{\mathcal{T}}_{2,1}$ and $\overline{\mathcal{T}}_{1,1,1}$ of $\mathcal{T}(3)$. Therefore, by Remark 14, we only have to find the irreducible components of $\mathcal{T}_3$.

Thus, given $\mathcal{F} \in \mathcal{T}_3$, take a point $P \in \operatorname{supp}(\mathcal{F})$. We then have the exact sequence (14) for $\mathcal{G} \in \mathcal{T}(3, 1)$. According to [Freiermuth and Trautmann 2004, Theorem 1.1], $\mathcal{T}(3, 1)$ has two irreducible components: the subset $\mathcal{P}$ of planar sheaves and the subset $\mathcal{R}$ that is the closure of the set of structure sheaves of twisted cubics. Moreover, all sheaves in $\mathcal{R} \setminus \mathcal{P}$ are structure sheaves of curves $R \subset \mathbb{P}^3$ of degree 3 and arithmetic genus zero. If $\mathcal{G}$ is planar, then $\mathcal{F}$ is planar. If $\mathcal{G} = \mathcal{O}_R$, then $R = \operatorname{supp}(\mathcal{F})$, where the scheme-theoretic support $\operatorname{supp}(\mathcal{F})$ of the sheaf $\mathcal{F}$ is defined by the 0-th Fitting ideal $\operatorname{Fitt}^0(\mathcal{F}) : \mathcal{I}_{R/\mathbb{P}^3} = \operatorname{Fitt}^0(\mathcal{F})$. The morphism

$$\rho \colon \mathcal{T}_3 \setminus \mathcal{P}_3 \to \mathcal{R} \setminus \mathcal{P}, \qquad \rho([\mathcal{F}]) = [\mathcal{O}_{\operatorname{supp}(\mathcal{F})}],$$

is injective. Indeed, if $\rho([\mathcal{F}_1]) = \rho([\mathcal{F}_2])$, then $\mathrm{supp}(\mathcal{F}_1) = \mathrm{supp}(\mathcal{F}_2) = R$. Choose a point $P \in R$. We have exact sequences

$$0 \to \mathcal{F}_1 \to \mathcal{G}_1 \to \mathbb{C}_P \to 0, \qquad 0 \to \mathcal{F}_2 \to \mathcal{G}_2 \to \mathbb{C}_P \to 0,$$

with $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{T}(3, 1)$. Clearly, $\mathcal{G}_1$ and $\mathcal{G}_2$ are both isomorphic to $\mathcal{O}_R$, hence $\mathcal{F}_1$ and $\mathcal{F}_2$ are both isomorphic to the ideal sheaf $\mathcal{I}_{P,R}$ of P in R. The image of ρ is a constructible set of the irreducible variety $\mathcal{R} \setminus \mathcal{P}$ and contains an open subset of $\mathcal{R} \setminus \mathcal{P}$, namely the subset given by the condition that R be irreducible. Indeed, if R is irreducible, then it is easy to check that $\mathcal{I}_{P,R}$ is stable; we have $\rho([\mathcal{I}_{P,R}]) = [\mathcal{O}_R]$. We deduce that $\mathcal{T}_3 \setminus \mathcal{P}_3$ is irreducible. It follows that $\mathcal{R}_3^o$ is dense in $\mathcal{T}_3 \setminus \mathcal{P}_3$. Thus, $\mathcal{T}_3$ has two irreducible components, hence $\mathcal{T}(3)$ has the four irreducible components announced in the proposition. □

4.3. *Moduli of sheaves of degree 4.*

Proposition 20. *The moduli space $\mathcal{T}(4)$ has eight irreducible components*: $\mathcal{P}_4$, $\mathcal{E}_4$, $\mathcal{R}_4$, $\overline{\mathcal{T}}_{2,2}$, $\overline{\mathcal{T}}_{2,1,1}$, $\overline{\mathcal{T}}_{1,1,1,1}$ *and two irreducible components of $\mathcal{T}_{3,1}$ that are birational to $\mathcal{P}_3 \times \mathcal{T}_1$ and to $\mathcal{R}_3 \times \mathcal{T}_1$, respectively. Their dimensions are*, *respectively*, 20, 18, 16, 16, 16, 16, 17, 17.

Proof. By Propositions 18 and 19 and Lemma 13 we already have 5 irreducible components of $\mathcal{T}(4)$ which are $\overline{\mathcal{T}}_{2,2}$, $\overline{\mathcal{T}}_{2,1,1}$, $\overline{\mathcal{T}}_{1,1,1,1}$ and two irreducible components of $\mathcal{T}_{3,1}$ that are birational to $\mathcal{P}_3 \times \mathcal{T}_1$ and to $\mathcal{R}_3 \times \mathcal{T}_1$, respectively. Therefore by Remark 14 we have only to find the irreducible components of $\mathcal{T}_4$. Thus, given $\mathcal{F} \in \mathcal{T}_4$, take a point $P \in \mathrm{supp}(\mathcal{F})$. We then have the exact sequence (14) for $\mathcal{G} \in \mathcal{T}(4, 1)$. According to [Choi et al. 2016, Theorem 4.12], $\mathcal{T}(4, 1)$ has three irreducible components: the subset $\mathcal{P}$ of planar sheaves, the subset $\mathcal{R}$ that is the closure of the set of structure sheaves of rational quartic curves, and the set $\mathcal{E}$ that is the closure of the set of sheaves of the form $\mathcal{O}_E(P')$, where E is a smooth elliptic quartic curve and $P' \in E$. If $\mathcal{G} \in \mathcal{P}$, then $\mathcal{F} \in \mathcal{P}_4$. The sheaves in $\mathcal{R} \setminus (\mathcal{P} \cup \mathcal{E})$ are structure sheaves of quartic curves of arithmetic genus zero. The sheaves in $\mathcal{E} \setminus \mathcal{P}$ are supported on quartic curves of arithmetic genus 1. Let $\mathcal{T}_{4,\mathrm{rat}} \subset \mathcal{T}_4$ be the subset of sheaves whose support is a quartic curve of arithmetic genus zero. As in Proposition 19, we can construct an injective dominant morphism

$$\rho\colon \mathcal{T}_{4,\mathrm{rat}} \to \mathcal{R} \setminus (\mathcal{P} \cup \mathcal{E}), \qquad \rho([\mathcal{F}]) = [\mathcal{O}_{\mathrm{supp}(\mathcal{F})}].$$

It follows that $\mathcal{T}_{4,\mathrm{rat}}$ is irreducible, hence $\mathcal{T}_{4,\mathrm{rat}} \subset \mathcal{R}_4$. To finish the proof of the proposition we need to show that $\mathcal{T}_4 \setminus (\mathcal{P}_4 \cup \mathcal{T}_{4,\mathrm{rat}})$ is contained in $\mathcal{E}_4$.

According to [Maican 2017, Section 3], the sheaves $\mathcal{G}$ in $\mathcal{E} \setminus \mathcal{P}$ are of two kinds:

(i) $\mathcal{O}_E(P')$ for a curve E of arithmetic genus 1 given by an ideal of the form (q_1, q_2), where q_1, q_2 are quadratic forms, and $P' \in E$. Notice that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_{P'}, \mathcal{O}_E) \xrightarrow{\sim} \mathbb{C},$$

so the notation $\mathcal{O}_E(P')$ is justified. Also note that $\mathcal{O}_E$ is stable.

(ii) Nonplanar extensions of the form

$$0 \to \mathcal{O}_L(-1) \to \mathcal{G} \to \mathcal{C} \to 0,$$

where L is a line and $\mathcal{C}$ gives a point in $\mathcal{T}_H(3, 1)$ for a plane H possibly containing L. (Here and below we use the notation $\mathcal{T}_S(d, \chi)$ for the moduli space of one-dimensional sheaves on a given surface S in $\mathbb{P}^3$ with Hilbert polynomial $P(t) = dt + \chi$. We also set $\mathcal{T}_S(d) := \mathcal{T}_S(d, 0)$.)

Claim 1: Case (ii) is unfeasible.

Assume, firstly, that $P \in H$. Tensoring (14) with $\mathcal{O}_H$, we get the exact sequence

$$\mathcal{F}|_H \to \mathcal{G}|_H \xrightarrow{\alpha} \mathbb{C}_P \to 0.$$

Thus, $\mathrm{Ker}(\alpha)$ is a quotient sheaf of $\mathcal{F}$ of slope zero. This contradicts the stability of $\mathcal{F}$. Assume, secondly, that $P \notin H$. According to [Maican 2017, Proposition 3.5], we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{O}_L \to 0$$

for some sheaf $\mathcal{E} \in \mathcal{T}_H(3)$. The composite map $\mathcal{E} \to \mathcal{G} \to \mathbb{C}_P$ is zero, hence $\mathcal{E}$ is a subsheaf of $\mathcal{F}$. This contradicts the stability of $\mathcal{F}$ and proves Claim 1.

It remains to deal with the sheaves from (i). We have one of the following possibilities:

(a) E is contained in a smooth quadric surface S.

(b) E is contained in an irreducible cone Σ but not in a smooth quadric surface.

(c) $\mathrm{span}\{q_1, q_2\}$ contains only reducible quadratic forms and q_1 and q_2 have no common factor.

Claim 2: In case (a), $\mathcal{F}$ belongs to $\mathcal{E}_4$.

Notice that $\mathcal{F} \in \mathcal{T}_S(4)$. According to [Ballico and Huh 2014, Proposition 7], $\mathcal{T}_S(4)$ has five disjoint irreducible components $\mathcal{T}_S(p, q, 4)$, where (p, q) is the type of the support of the one-dimensional sheaf with respect to $\mathrm{Pic}(S)$. Clearly, $\mathcal{F} \in \mathcal{T}_S(2, 2, 4)$. Thus, $\mathcal{F}$ is a limit of sheaves in $\mathcal{T}_S(2, 2, 4)$ supported on smooth curves of type $(2, 2)$, hence $\mathcal{F} \in \mathcal{E}_4$.

It remains to deal with cases (b) and (c). Next we reduce further to the case when $P = P'$. Notice that, if $P \neq P'$, then $\mathcal{F} \simeq \mathcal{O}_E(P') \otimes (\mathcal{O}_E(P))^{\mathrm{D}}$, hence the notation $\mathcal{F} = \mathcal{O}_E(P' - P)$ is justified.

Claim 3: Assume that $\mathcal{F} = \mathcal{O}_E(P' - P)$ for an elliptic quartic curve E and distinct closed points $P', P \in E$. Then $\mathcal{F}$ belongs to $\mathcal{E}_4$.

Let $Z_1, \dots, Z_m$ denote the irreducible components of E. Fix $i, j \in \{1, \dots, m\}$. Consider the locally closed subset $\mathcal{X} \subset \mathcal{E} \times \mathcal{E}$ of pairs $([\mathcal{O}_{E'}(P_1)], [\mathcal{O}_{E'}(P_2)])$, where E' is a quartic curve of arithmetic genus 1 whose ideal is generated by two quadratic polynomials, and P_1 and P_2 are distinct points on E' such that $P_1 \notin \cup_{k \neq i} Z_k$ and $P_2 \notin \cup_{k \neq j} Z_k$. Consider the morphisms

$$\xi : \mathcal{X} \to \mathcal{T}(4), \qquad ([\mathcal{O}_{E'}(P_1)], [\mathcal{O}_{E'}(P_2)]) \longmapsto [\mathcal{O}_{E'}(P_1 - P_2)],$$
$$\sigma : \mathcal{X} \to \mathrm{Hilb}_{\mathbb{P}^3}(4t), \qquad ([\mathcal{O}_{E'}(P_1)], [\mathcal{O}_{E'}(P_2)]) \longmapsto E',$$

where $\mathrm{Hilb}_{\mathbb{P}^3}(4t)$ is the Hilbert scheme of subschemes of $\mathbb{P}^3$ with Hilbert polynomial $P(t) = 4t$. According to [Chen and Nollet 2012, Examples 2.8 and 4.8], $\mathrm{Hilb}_{\mathbb{P}^3}(4t)$ consists of two irreducible components, denoted $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. The generic member of $\boldsymbol{H}_1$ is a smooth elliptic quartic curve. The generic member of $\boldsymbol{H}_2$ is the disjoint union of a planar quartic curve and two isolated points. Note that $\boldsymbol{H}_2$ lies in the closed subset $\{E' \mid h^0(\mathcal{O}_{E'}) \geq 3\}$. Since E lies in the complement of this subset, we deduce that $E \in \boldsymbol{H}_1$. It follows that there exists an irreducible quasiprojective curve $\Gamma \subset \mathrm{Hilb}_{\mathbb{P}^3}(4t)$ containing E, such that $\Gamma \setminus \{E\}$ consists of smooth elliptic quartic curves (see the proof of [Maican 2017, Proposition 4.2]). The fibers of the map $\sigma^{-1}(\Gamma) \to \Gamma$ are irreducible of dimension 2. By [Shafarevich 1994, Theorem 8, p. 77], we deduce that $\sigma^{-1}(\Gamma)$ is irreducible. Thus, $\xi(\sigma^{-1}(\Gamma))$ is irreducible. This set contains $[\mathcal{O}_E(P' - P)]$ for $P' \in Z_i \setminus \cup_{k \neq i} Z_k$ and $P \in Z_j \setminus \cup_{k \neq j} Z_k$. The generic member of $\xi(\sigma^{-1}(\Gamma))$ is a sheaf supported on a smooth elliptic quartic curve. We conclude that $[\mathcal{O}_E(P' - P)] \in \mathcal{E}_4$. Since i and j are arbitrary, the result is true for all P' and P closed points on E.

Claim 4: In case (c), E is a quadruple line supported on a line L. More precisely, there are three distinct planes H, H', H'' containing L, such that

$$E = (H \cup H') \cap (2H'').$$

The claim will follow if we can show that there are linearly independent one-forms u, v such that $q_1, q_2 \in \mathbb{C}[u, v]$. Indeed, in this case (q_1, q_2) has the normal form $(uv, (u+v)^2)$. We argue by contradiction. Assume that $q_1 = XY$ and $q_2 = Zl$. Consider first the case when $l = aX + bY + cZ$. We will find $\lambda \in \mathbb{C}$ such that $f = XY + \lambda Zl$ is irreducible, which is equivalent to saying that

$$\frac{\partial f}{\partial X} = Y + a\lambda Z, \qquad \frac{\partial f}{\partial Y} = X + b\lambda Z, \qquad \frac{\partial f}{\partial Z} = \lambda(aX + bY + 2cZ)$$

have no common zero, or, equivalently,

$$\begin{vmatrix} 0 & 1 & a\lambda \\ 1 & 0 & b\lambda \\ a\lambda & b\lambda & 2c\lambda \end{vmatrix} \neq 0.$$

We have reduced to the inequality $2ab\lambda^2 - 2c\lambda \neq 0$. If $c \neq 0$ we can find a solution. If $c = 0$, then $ab \neq 0$, otherwise q_1 and q_2 would have a common factor, and we can choose any $\lambda \in \mathbb{C}^*$. Assume now that $l = aX + bY + cZ + dW$ with $d \neq 0$. Note that $f = XY + \lambda Zl$ is irreducible if its image in

$$\mathbb{C}[X, Y, Z, W]/\langle (c-1)Z + dW \rangle \simeq \mathbb{C}[X, Y, Z]$$

is irreducible. The above isomorphism sends f to $XY + \lambda Z(aX + bY + Z)$ which brings us to the case examined above.

Claim 5: In case (c), $\mathcal{F}$ belongs to $\mathcal{E}_4$.

We have $\mathcal{O}_E|_H \simeq \mathcal{O}_C$ and $\mathcal{O}_E|_{H'} \simeq \mathcal{O}_{C'}$ for conic curves C and C' supported on L. The kernel of the map $\mathcal{O}_E \to \mathcal{O}_C$ has Hilbert polynomial $2t - 1$ and is stable, because $\mathcal{O}_E$ is stable, hence it is isomorphic to $\mathcal{O}_{C'}(-1)$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_E & \longrightarrow & \mathcal{O}_E(P') & \longrightarrow & \mathbb{C}_{P'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_E(P')|_H & \longrightarrow & \mathbb{C}_{P'} & \longrightarrow & 0 \end{array}$$

in which the second row is obtained by restricting the first row to H. Applying the snake lemma, we obtain the first row of the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{C'}(-1) & \longrightarrow & \mathcal{O}_E(P') & \longrightarrow & \mathcal{O}_E(P')|_H & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ & & & & \mathbb{C}_P & = & \mathbb{C}_P & & \end{array}$$

Applying the snake lemma to this diagram, we get the exact sequence

$$0 \to \mathcal{O}_{C'}(-1) \to \mathcal{F} \to \mathrm{Ker}(\alpha) \to 0.$$

Note that $\mathrm{Ker}(\alpha)$ has Hilbert polynomial $2t + 1$ and is semistable, being a quotient of the stable sheaf $\mathcal{F}$. It follows that $\mathrm{Ker}(\alpha) \simeq \mathcal{O}_C$. Thus, $\mathcal{F}$ gives a point in the set $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_{C'}(-1)))^{\mathrm{s}}$ of stable nonsplit extensions of $\mathcal{O}_C$ by $\mathcal{O}_{C'}(-1)$.

Consider the family of planes H''_t, $t \in \mathbb{P}^1 \setminus \{0, \infty\}$, containing L and different from H and H'. Denote $E_t = (H \cup H') \cap (2H''_t)$. We have a two-dimensional family of semistable sheaves

$$\left\{ \mathcal{O}_{E_t}(P' - P'') \mid t \in \mathbb{P}^1 \setminus \{0, \infty\},\ P'' \in L \setminus \{P'\} \right\} \subset \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_{C'}(-1))).$$

This family is dense in the right-hand side because $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{C'}(-1)) \simeq \mathbb{C}^3$. To prove this we use the standard exact sequence obtained from Thomas' spectral sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{O}_{H'}}(\mathcal{O}_C|_{H'}, \mathcal{O}_{C'}(-1)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{C'}(-1)) \\ \to \mathrm{Hom}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{H'}), \mathcal{O}_{C'}(-1)) \to \mathrm{Ext}^2_{\mathcal{O}_{H'}}(\mathcal{O}_C|_{H'}, \mathcal{O}_{C'}(-1)),$$

see also [Choi et al. 2016, Lemma 4.2]. Note that $\mathcal{O}_C|_{H'} \simeq \mathcal{O}_L$. Using Serre duality we obtain the isomorphisms

$$\mathrm{Ext}^2_{\mathcal{O}_{H'}}(\mathcal{O}_L, \mathcal{O}_{C'}(-1)) \simeq \mathrm{Hom}_{\mathcal{O}_{H'}}(\mathcal{O}_{C'}(-1), \mathcal{O}_L(-3))^* = 0,$$
$$\mathrm{Ext}^1_{\mathcal{O}_{H'}}(\mathcal{O}_L, \mathcal{O}_{C'}(-1)) \simeq \mathrm{Ext}^1_{\mathcal{O}_{H'}}(\mathcal{O}_{C'}(-1), \mathcal{O}_L(-3))^* \simeq \mathbb{C}^2.$$

The last isomorphism follows from the long exact sequence of extension sheaves

$$0 = \mathrm{Hom}\big(\mathcal{O}_{H'}(-1), \mathcal{O}_L(-3)\big) \to \mathrm{Hom}\big(\mathcal{O}_{H'}(-3), \mathcal{O}_L(-3)\big) \simeq H^0(\mathcal{O}_L) \simeq \mathbb{C} \\ \to \mathrm{Ext}^1_{\mathcal{O}_{H'}}\big(\mathcal{O}_{C'}(-1), \mathcal{O}_L(-3)\big) \to \mathrm{Ext}^1_{\mathcal{O}_{H'}}\big(\mathcal{O}_{H'}(-1), \mathcal{O}_L(-3)\big) \simeq H^1\big(\mathcal{O}_L(-2)\big) \simeq \mathbb{C} \\ \to \mathrm{Ext}^1_{\mathcal{O}_{H'}}\big(\mathcal{O}_{H'}(-3), \mathcal{O}_L(-3)\big) = 0$$

derived from the short exact sequence

$$0 \to \mathcal{O}_{H'}(-3) \to \mathcal{O}_{H'}(-1) \to \mathcal{O}_{C'}(-1) \to 0.$$

Choose linear forms u and u' defining H and H'. Restricting the standard resolution

$$0 \to \mathcal{O}(-3) \xrightarrow{\begin{bmatrix} -u \\ (u')^2 \end{bmatrix}} \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\begin{bmatrix} (u')^2 & u \end{bmatrix}} \mathcal{O} \to \mathcal{O}_C \to 0$$

to H', we see that $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{H'})$ is isomorphic to the cohomology of the complex

$$\mathcal{O}_{H'}(-3) \xrightarrow{\begin{bmatrix} -u|_{H'} \\ 0 \end{bmatrix}} \mathcal{O}_{H'}(-2) \oplus \mathcal{O}_{H'}(-1) \xrightarrow{\begin{bmatrix} 0 & u|_{H'} \end{bmatrix}} \mathcal{O}_{H'}$$

that is, to $\mathcal{O}_L(-2)$. Using the fact that $\mathcal{O}_{C'}(-1)$ and $\mathcal{O}_L(-2)$ are reflexive, we have the isomorphisms

$$\mathrm{Hom}\big(\mathcal{O}_L(-2), \mathcal{O}_{C'}(-1)\big) \simeq \mathrm{Hom}\big(\mathcal{O}_{C'}(-1)^{\mathrm{D}}, \mathcal{O}_L(-2)^{\mathrm{D}}\big) \simeq \mathrm{Hom}(\mathcal{O}_{C'}, \mathcal{O}_L) \simeq \mathbb{C}.$$

The above discussion shows that $[\mathcal{F}]$ is a limit of points in $\mathcal{T}(4)$ of the form $[\mathcal{O}_{E_t}(P' - P'')]$, with $P' \neq P''$. Claim 5 now follows from Claim 3.

It remains to consider sheaves $\mathcal{F}$ given by sequence (14) in which $\mathcal{G} = \mathcal{O}_E(P)$ and E is as at (b). We reduce further to the case when E has no regular points.

Claim 6: Assume E has a regular point. Then $\mathcal{F} \simeq \mathcal{O}_E$, hence $\mathcal{F}$ belongs to $\mathcal{E}_4$.

The proof of the claim is obvious because P in sequence (14) can be chosen arbitrarily on E. We choose $P \in \mathrm{reg}(E)$. The kernel of the map $\mathcal{O}_E(P) \to \mathbb{C}_P$ is $\mathcal{O}_E$. Note that E belongs to the irreducible component $\boldsymbol{H}_1$ of $\mathrm{Hilb}_{\mathbb{P}^3}(4t)$, hence it is the limit of smooth elliptic quartic curves.

Claim 7: Let $E \subset \mathbb{P}^3$ be a quartic curve of arithmetic genus 1 which is contained in an irreducible cone Σ, but not in a smooth quadric surface. Assume that E has no regular points. Then we have one of the following two possibilities:

(b1) $E = \Sigma \cap (H \cup H')$, where $H,\ H'$ are distinct planes each intersecting Σ along a double line.

(b2) $E = \Sigma \cap (2H)$, where H is a plane intersecting Σ along a double line.

To fix notations assume that Σ has vertex O and base a conic curve Γ contained in a plane Π. Assume first that $E = \Sigma \cap \Sigma'$ for Σ' another irreducible cone. If Σ and Σ' have distinct vertices, then E has regular points. Thus, Σ' has vertex O and base an irreducible conic curve Γ' contained in Π. Since E has no regular points, $\Gamma \cap \Gamma'$ is the union of two double points Q_1 and Q_2. Now E is the cone with vertex O and base $Q_1 \cup Q_2$, so E is as at (b1).

Assume next that $E = \Sigma \cap (H \cup H')$ for distinct planes H and H'. If H or H' does not contain O, then E has regular points. If H or H' is not tangent to Γ, then E has regular points. We deduce that E is as in (b1).

Assume, finally, that $E = \Sigma \cap (2H)$ for a double plane $2H$. If $O \notin H$, then it can be shown that E is contained in a smooth quadric surface. Indeed, assume that Σ has equation $X^2 + Y^2 + Z^2 = 0$ and H has equation $W = 0$. Then E is contained in the smooth quadric surface with equation

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

Thus, $O \in H$. If $\Gamma \cap H$ is the union of two distinct points, then $\Gamma \cap (2H)$ is the union of two double points Q_1 and Q_2 and E is as in (b1). If $\Gamma \cap H$ is a double point, then E is as in (b2).

Claim 8: In case (b1), $\mathcal{F}$ belongs to $\mathcal{E}_4$.

We have $\mathcal{O}_{E|H} \simeq \mathcal{O}_C$ and $\mathcal{O}_{E|H'} \simeq \mathcal{O}_{C'}$ for conic curves C and C' supported on lines L and L', respectively. Assume that $P \in L$ and choose a point $P' \in L$ not necessarily distinct from P. Let $\mathcal{F}' \in \mathcal{T}_4$ be given by the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{O}_E(P') \to \mathbb{C}_P \to 0.$$

As in the first paragraph in the proof of Claim 5, we see that $\mathcal{F}'$ gives a point in the set $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_{C'}(-1)))^{\mathrm{s}}$. We have $\dim \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{C'}(-1)) \le 2$. Indeed, start with the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{O}_{H'}}\big(\mathcal{O}_{C|H'}, \mathcal{O}_{C'}(-1)\big) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}\big(\mathcal{O}_C, \mathcal{O}_{C'}(-1)\big) \\ \to \mathrm{Hom}\big(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{H'}), \mathcal{O}_{C'}(-1)\big).$$

The group on the second line vanishes because $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{H'})$ is supported on O while $\mathcal{O}_{C'}(-1)$ has no zero-dimensional torsion. It follows that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_{C'}(-1)) \simeq \mathrm{Ext}^1_{\mathcal{O}_{H'}}(\mathcal{O}_{C|H'}, \mathcal{O}_{C'}(-1)).$$

The sheaf $\mathcal{O}_{C|H'}$ is the structure sheaf of a double point supported on O, hence we have the exact sequence

$$\mathbb{C}\simeq\operatorname{Ext}^1_{\mathcal{O}_{H'}}(\mathbb{C}_O,\mathcal{O}_{C'}(-1))\to\operatorname{Ext}^1_{\mathcal{O}_{H'}}(\mathcal{O}_{C|H'},\mathcal{O}_{C'}(-1))\to\operatorname{Ext}^1_{\mathcal{O}_{H'}}(\mathbb{C}_O,\mathcal{O}_{C'}(-1))\simeq\mathbb{C}$$

from which we get our estimate on the dimension of the middle vector space.

The one-dimensional family $\mathcal{O}_E(P'-P)$, $P'\in L\setminus\{P\}$, is therefore dense in $\mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_C,\mathcal{O}_{C'}(-1)))^{\mathrm{s}}$, hence, in view of Claim 3, $\mathcal{F}$ is a limit of sheaves in $\mathcal{E}_4$. We conclude that $\mathcal{F}\in\mathcal{E}_4$.

Claim 9: In case (b2), $\mathcal{F}$ belongs to $\mathcal{E}_4$.

Let L be the reduced support of $\Sigma\cap H$. We have $\mathcal{O}_{E|H}\simeq\mathcal{O}_C$ for a conic curve supported on L. Choose a point $P'\in L$ not necessarily distinct from P and let $\mathcal{F}'\in\mathcal{T}_4$ be given by the exact sequence

$$0\to\mathcal{F}'\to\mathcal{O}_E(P')\to\mathbb{C}_P\to 0.$$

As in the first paragraph of the proof of Claim 5, we see that $\mathcal{F}'$ gives a point in the set $\mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_C,\mathcal{O}_C(-1)))^{\mathrm{s}}$. We have $\dim\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C,\mathcal{O}_C(-1))=5$. This follows from the exact sequence

$$\begin{aligned}0\to\operatorname{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_C,\mathcal{O}_C(-1))&\to\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C,\mathcal{O}_C(-1))\\&\to\operatorname{Hom}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C,\mathcal{O}_H),\mathcal{O}_C(-1))\to\operatorname{Ext}^2_{\mathcal{O}_H}(\mathcal{O}_C,\mathcal{O}_C(-1)).\end{aligned}$$

From Serre duality we get

$$\operatorname{Ext}^2_{\mathcal{O}_H}(\mathcal{O}_C,\mathcal{O}_C(-1))\simeq\operatorname{Hom}_{\mathcal{O}_H}(\mathcal{O}_C(-1),\mathcal{O}_C(-3))^*\simeq H^0(\mathcal{O}_C(-2))^*=0.$$

We have $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C,\mathcal{O}_H)\simeq\mathcal{O}_C(-1)$ hence $\operatorname{Hom}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C,\mathcal{O}_H),\mathcal{O}_C(-1))\simeq\mathbb{C}$. Applying the functor $\operatorname{Hom}(-,\mathcal{O}_C(-1))$ to the short exact sequence

$$0\to\mathcal{O}_H(-2)\to\mathcal{O}_H\to\mathcal{O}_C\to 0,$$

we obtain the exact sequence,

$$\begin{aligned}0\to\operatorname{Hom}(\mathcal{O}_H(-2),\mathcal{O}_C(-1))\simeq H^0(\mathcal{O}_C(1))\simeq\mathbb{C}^3&\to\operatorname{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_C,\mathcal{O}_C(-1))\\&\to\operatorname{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_H,\mathcal{O}_C(-1))\simeq H^1(\mathcal{O}_C(-1))\simeq\mathbb{C}\to 0,\end{aligned}$$

since $\operatorname{Hom}(\mathcal{O}_H,\mathcal{O}_C(-1))\simeq H^0(\mathcal{O}_C(-1))=0$, and $\operatorname{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_H(-2),\mathcal{O}_C(-1))\simeq H^1(\mathcal{O}_C(1))=0$.

Denote $Q=L\cap\Pi$. We have a three-dimensional family Γ_t of irreducible conic curves in Π that contain Q and are tangent to H. Let Σ_t be the cone with vertex O and base Γ_t. Put $E_t=\Sigma_t\cap(2H)$. The four-dimensional family $\mathcal{O}_{E_t}(P'-P)$, $P'\in L\setminus\{P\}$ is dense in $\mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_C,\mathcal{O}_C(-1)))^{\mathrm{s}}$, hence, in view of Claim 3, $\mathcal{F}$ is the limit of sheaves in $\mathcal{E}_4$. We conclude that $\mathcal{F}\in\mathcal{E}_4$. □

The proof of Main Theorem 2 is finally complete.

5. Components and connectedness of $\mathcal{L}(3)$

We are now ready to prove that the moduli space of rank 2 instanton sheaves of charge 3 on $\mathbb{P}^3$ is connected and has precisely two irreducible components. Indeed, two components of $\overline{\mathcal{L}(3)}$ have already been identified above:

(I) $\overline{\mathcal{I}(3)}$, whose generic point corresponds to a locally free instanton sheaf.

(II) $\overline{\mathcal{C}(1,3,0)}$, whose generic point corresponds to an instanton sheaf E fitting into an exact sequence of the form

$$0 \to E \to 2\cdot\mathcal{O}_{\mathbb{P}^3} \to \iota_* L(2) \to 0, \tag{15}$$

where $\iota : \Sigma \hookrightarrow \mathbb{P}^3$ is the inclusion of a nonsingular plane cubic Σ, and $L \in \mathrm{Pic}^0(\Sigma)$ is such that $h^0(\Sigma, L) = 0$.

Both components have dimension 21; this is a classical result for the component $\mathcal{I}^0(3)$, while the dimension of $\mathcal{C}(1,3,0)$ is given by Theorem 8. In addition, this same result also guarantees that the union $\mathcal{I}^0(3) \cup \mathcal{C}(1,3,0)$ is connected.

Therefore, our task is to prove that $\overline{\mathcal{L}(3)}$ has no other irreducible components, i.e., that every instanton sheaf of charge 3 can be deformed either into a locally free instanton sheaf, or into an instanton sheaf given by a sequence of the form (15).

So let E be a nonlocally free instanton sheaf of charge 3, and let $Q_E := E^{\vee\vee}/E$ be the corresponding rank 0 instanton sheaf; let d_E denote the degree of Q_E. There are three possibilities to consider: $d_E = 1$, $d_E = 2$ and $d_E = 3$.

The first possibility is easy to deal with: if $d_E = 1$, then $Q_E = \mathcal{O}_\ell(1)$, where $\ell \hookrightarrow \mathbb{P}^3$ is a line in $\mathbb{P}^3$. It follows that E fits into an exact sequence of the form

$$0 \to E \to F \to \mathcal{O}_\ell(1) \to 0,$$

where F is a locally free instanton sheaf of charge 2. However, [Jardim et al. 2015, Proposition 7.2] ensures that E can be deformed in a ('t Hooft) locally free instanton sheaf of charge 3. In other words, if $d_E = 1$, then E lies within $\mathcal{I}^0(3)$.

Now, if $d_E = 2$, then, since Q_E is semistable and by Proposition 18 above, one can find an affine open subset $0 \in U \subset \mathbb{A}^1$ and a coherent sheaf $\boldsymbol{G}$ on $\mathbb{P}^3 \times U$ such that $G_0 = Q_E$ and, for $u \neq 0$, either

(i) $G_u = \mathcal{O}_\Gamma(3\mathrm{pt})$, where Γ is a nonsingular conic in $\mathbb{P}^3$; or

(ii) $G_u = \mathcal{O}_{\ell_1}(1) \oplus \mathcal{O}_{\ell_2}(1)$ where ℓ_1 and ℓ_2 are skew lines in $\mathbb{P}^3$.

Since $d_E = 2$, $E^{\vee\vee}$ is a locally free instanton sheaf of charge 1 (also known as a null-correlation bundle), we set $N := E^{\vee\vee}$. Take $\boldsymbol{F} := \pi^* N$, where $\pi : \mathbb{P}^3 \times U \to \mathbb{P}^3$ is the projection onto the first factor. Let $s : N \twoheadrightarrow Q_E$ be the epimorphism given by the standard sequence (3). For every $u \in U$, the sheaf $\mathcal{H}om(F_u, G_u) \simeq N \otimes G_u$ is supported in dimension 1, thus clearly $H^i(\mathcal{H}om(F_u, G_u)) = 0$ for $i = 2, 3$. For $u \neq 0$ we can, after possibly shrinking U, assume that either $N|_\Gamma \simeq 2\cdot\mathcal{O}_\Gamma$ or $N|_{\ell_1} \simeq 2\cdot\mathcal{O}_{\ell_1}$

and $N|_{\ell_2} \simeq 2\cdot\mathcal{O}_{\ell_2}$; in both situations, it is easy to check that $H^1(\mathcal{H}om(F_u, G_u)) = 0$. Finally, for $u = 0$, we twist the resolution of Q_E

$$0 \to 2\cdot\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} 4\cdot\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} 2\cdot\mathcal{O}_{\mathbb{P}^3}(1) \to Q_E \to 0$$

by N and check that $H^1(\mathcal{H}om(F_0, G_0)) = H^1(N\otimes Q_E) \simeq H^2(N\otimes \operatorname{im}\beta) = 0$.

Therefore, it follows from Lemma 11 that there exists an epimorphism $\boldsymbol{s} : \boldsymbol{F} \twoheadrightarrow \boldsymbol{G}$ on $U\times\mathbb{P}^3$ extending $s : N \twoheadrightarrow Q_E$. Let $\boldsymbol{E} := \ker \boldsymbol{s}$; clearly, $E_0 := \boldsymbol{E}|_{\{0\}\times\mathbb{P}^3} = E$. For $0 \neq u \in U$, E_u fits into the exact sequence

$$0 \to E_u \to N \to G_u \to 0.$$

In the case (i) described above, E_u lies within $\mathcal{D}(2, 3)$ for $u \neq 0$, hence $E = E_0$ lies within $\overline{\mathcal{D}(2, 3)}$, which is contained in $\overline{\mathcal{I}(3)}$ by [Jardim et al. 2015, Theorem 7.8]. In other words, E can be deformed into a locally free instanton sheaf of charge 3, thus it lies within $\mathcal{I}^0(3)$.

In the case (ii), Proposition 10 also implies that $[E_0] \in \mathcal{I}^0(3)$.

An argument similar to the one used in the proof of [Jardim et al. 2015, Proposition 7.2] works to show that E can be deformed into a locally free ('t Hooft) instanton sheaf.

Summing up, we conclude that if $d_E = 2$, then E lies within $\mathcal{I}^0(3)$.

Finally, consider $d_E = 3$, so that $E^{\vee\vee} = 2\cdot\mathcal{O}_{\mathbb{P}^3}$. Since Q_E is semistable, it follows from Proposition 19 that one can find an affine open subset $0 \in U \subset \mathbb{A}^1$ and a coherent sheaf $\boldsymbol{G}$ on $\mathbb{P}^3\times U$ such that $G_0 = Q_E$ and, for $u \neq 0$, either

(i) $G_u = \mathcal{O}_\Delta(5\text{pt})$, where Δ is a nonsingular twisted cubic in $\mathbb{P}^3$; or

(ii) $G_u = \mathcal{O}_\Gamma(3\text{pt}) \oplus \mathcal{O}_\ell(1)$, where Γ is a nonsingular conic and ℓ is a line disjoint from Γ; or

(iii) $G_u = \mathcal{O}_{\ell_1}(1) \oplus \mathcal{O}_{\ell_2}(1) \oplus \mathcal{O}_{\ell_3}(1)$ where ℓ_j are 3 skew lines in $\mathbb{P}^3$; or

(iv) $G_u = L(2)$, where $L \in \operatorname{Pic}^0(\Sigma)$, for some nonsingular plane cubic Σ in $\mathbb{P}^3$.

Now set $\boldsymbol{F} := 2\cdot\pi^*\mathcal{O}_{\mathbb{P}^3}$. Note that $H^i(\mathcal{H}om(F_u, G_u)) = H^i(2\cdot G_u)$, and this vanishes for $i = 1, 2, 3$ in all of the four cases outlined above for $u \neq 0$. For $u = 0$, $H^i(G_0) = H^i(Q_E)$ and this vanishes by dimension of Q_E when $i = 2, 3$, and by the vanishing of $h^1(Q_E(-2))$ when $i = 1$.

We complete the argument as before; again, it follows from Lemma 11 that there exists an epimorphism $\boldsymbol{s} : \boldsymbol{F} \twoheadrightarrow \boldsymbol{G}$ extending the epimorphism $s : 2\cdot\mathcal{O}_{\mathbb{P}^3} \twoheadrightarrow Q_E$ obtained from the standard sequence (3) for E. Let $\boldsymbol{E} := \ker \boldsymbol{s}$; then clearly, $E_0 := \boldsymbol{E}|_{\{0\}\times\mathbb{P}^3} = E$. For $u \neq 0$, E_u fits into the exact sequence

$$0 \to E_u \to 2\cdot\mathcal{O}_{\mathbb{P}^3} \to G_u \to 0.$$

In the cases (i) through (iii), we know from [Jardim et al. 2015, Theorem 7.8] and Proposition 10 above that $[E_0] \in \overline{\mathcal{D}(3, 3)}$, thus also $[E] \in \mathcal{I}^0(3)$.

In the case (iv), E_u lies within $\mathcal{C}(1,3,0)$ for $u \neq 0$, by definition. It follows that $[E] \in \overline{\mathcal{C}(1,3,0)}$.

This completes the proof of the first part of Main Theorem 1.

6. Components and connectedness of $\mathcal{L}(4)$

In this section we prove the second part of Main Theorem 1, i.e., we enumerate the irreducible components of $\mathcal{L}(4)$, and show that $\mathcal{L}(4)$ is connected. Note that, from Theorem 8, we already know four irreducible components of $\overline{\mathcal{L}(4)}$:

(I) $\overline{\mathcal{I}(4)}$, whose generic point corresponds to a locally free instanton sheaf.

(II) $\overline{\mathcal{C}(1,3,1)}$, whose generic point corresponds to an instanton sheaf E fitting into an exact sequence of the form

$$0 \to E \to N \to \iota_* L(2) \to 0, \tag{16}$$

where N is a null-correlation bundle, $\iota : \Sigma \hookrightarrow \mathbb{P}^3$ is the inclusion of a nonsingular plane cubic Σ, and $L \in \mathrm{Pic}^0(\Sigma)$ is such that $h^0(\Sigma, L) = 0$.

(III) $\overline{\mathcal{C}(2,2,0)}$, whose generic point corresponds to an instanton sheaf E fitting into an exact sequence of the form

$$0 \to E \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to \iota_* L(2) \to 0, \tag{17}$$

where $\iota : \Sigma \hookrightarrow \mathbb{P}^3$ is the inclusion of a nonsingular elliptic space quartic Σ, and $L \in \mathrm{Pic}^0(\Sigma)$ is such that $h^0(\Sigma, L) = 0$.

(IV) $\overline{\mathcal{C}(1,4,0)}$, whose generic point corresponds to an instanton sheaf E fitting into an exact sequence of the form

$$0 \to E \to 2 \cdot \mathcal{O}_{\mathbb{P}^3} \to \iota_* L(2) \to 0, \tag{18}$$

where $\iota : \Sigma \hookrightarrow \mathbb{P}^3$ is the inclusion of a nonsingular plane quartic Σ, and $L \in \mathrm{Pic}^2(\Sigma)$ is such that $h^0(\Sigma, L) = 0$.

The first three components have dimension 29, and the last one has dimension 32; this is a classical result for the component $\mathcal{I}^0(4)$, while the dimensions of $\mathcal{C}(1,3,1)$, $\mathcal{C}(2,2,0)$ and $\mathcal{C}(1,4,0)$ are given by Theorem 8 above. Furthermore, [Jardim et al. 2017, Theorem 23] implies that each of the last three components intersects $\mathcal{I}^0(4)$. Thus the union of these four components is connected.

To finish the proof of the second part of Main Theorem 1, it is again enough to show that there are no other irreducible components in $\mathcal{L}(4)$, except for those described above. The argument here is the same as before, exploring Theorem 1, Remark 2 and Proposition 20.

Take any $[E] \in \mathcal{L}(4)$ and consider the triple (3). Then, in view of Theorem 1 and Remark 2, Q_E is a rank 0 instanton sheaf of multiplicity $1 \leq d_E \leq 4$, and $E^{\vee\vee}$ is an instanton bundle of charge $4 - d_E$. Consider the possible cases for d_E.

The case $d_E = 1$. As in the similar case in Section 5, $Q_E = \mathcal{O}_\ell(1)$ where l is a line in $\mathbb{P}^3$. Respectively, $[E^{\vee\vee}] \in \mathcal{I}(3)$. Deforming ℓ in $\mathbb{P}^3$ we may assume that $E^{\vee\vee}|_\ell \simeq 2\cdot\mathcal{O}_\ell$, so that $[E] \in \mathcal{D}(1,4)$. Therefore, $[E] \in \mathcal{I}^0(4)$.

The case $d_E = 2$. As in the similar case in Section 5, Q_E can be deformed in a flat family either into a sheaf $\mathcal{O}_\Gamma(3\mathrm{pt})$, where Γ is a nonsingular conic in $\mathbb{P}^3$, or into a sheaf $\mathcal{O}_{\ell_1}(1)\oplus\mathcal{O}_{\ell_2}(1)$ where ℓ_1 and ℓ_2 are skew lines in $\mathbb{P}^3$. Respectively, $[E^{\vee\vee}] \in \mathcal{I}(2)$. Now the same argument as in Section 5 shows that $[E] \in \mathcal{I}^0(4)$.

The case $d_E = 3$. Then $E^{\vee\vee}$ is a null-correlation bundle and, as in the case $d_E = 3$ of Section 5, the sheaf Q_E deforms in a flat family to one of the sheaves:

(i) $L(2)$, where $L \in \mathrm{Pic}^0(\Sigma)$, for some nonsingular plane cubic Σ in $\mathbb{P}^3$.

(ii) $\mathcal{O}_\Delta(5\mathrm{pt})$, where Δ is a nonsingular twisted cubic in $\mathbb{P}^3$.

(iii) $\mathcal{O}_\Gamma(3\mathrm{pt})\oplus\mathcal{O}_\ell(1)$, where Γ is a nonsingular conic and ℓ is a line disjoint from Γ.

(iv) $\mathcal{O}_{\ell_1}(1)\oplus\mathcal{O}_{\ell_2}(1)\oplus\mathcal{O}_{\ell_3}(1)$ where ℓ_j are 3 skew lines in $\mathbb{P}^3$.

By definition, $[E] \in \overline{\mathcal{C}(1,3,1)}$ in the case (i). The same argument as in Section 5, based on [Jardim et al. 2015, Theorem 7.8] and Proposition 10, shows that $[E] \in \overline{\mathcal{I}(4)}$ in the cases (ii) through (iv).

The case $d_E = 4$. Then $E^{\vee\vee} \simeq 2\cdot\mathcal{O}_{\mathbb{P}^3}$ and, according to Proposition 20, the sheaf Q_E deforms in a flat family to one of the sheaves:

(i) $L(2)$, where $L \in \mathrm{Pic}^2(\Sigma)$, for some nonsingular plane quartic Σ in $\mathbb{P}^3$, and L satisfies an open condition $h^1(L) = 0$.

(ii) $L(2)$, where $0 \neq L \in \mathrm{Pic}^0(\Delta)$, for some nonsingular space elliptic quartic Δ in $\mathbb{P}^3$.

(iii) $\mathcal{O}_\Delta(7\mathrm{pt})$ for some nonsingular rational space quartic Δ in $\mathbb{P}^3$.

(iv) $L(2)\oplus\mathcal{O}_\ell(1)$, where $L \in \mathrm{Pic}^0(\Sigma)$, for some nonsingular plane cubic Σ in $\mathbb{P}^3$ and a line ℓ disjoint from Σ.

(v) $\mathcal{O}_\Delta(5\mathrm{pt})\oplus\mathcal{O}_\ell(1)$, where Δ is a nonsingular twisted cubic and ℓ is a line disjoint from Δ.

(vi) $\mathcal{O}_{\Gamma_1}(3\mathrm{pt})\oplus\mathcal{O}_{\Gamma_2}(3\mathrm{pt})$, where Γ_1 and Γ_2 are nonsingular, disjoint conics.

(vii) $\mathcal{O}_\Gamma(3\mathrm{pt})\oplus\mathcal{O}_{\ell_1}(1)\oplus\mathcal{O}_{\ell_2}(1)$, where Γ is a nonsingular conic, and ℓ_1 and ℓ_2 are two skew lines disjoint from Γ.

(viii) $\mathcal{O}_\ell(1)\oplus\mathcal{O}_{\ell_2}(1)\oplus\mathcal{O}_{\ell_3}(1)\oplus\mathcal{O}_{\ell_4}(1)$, where ℓ_1, ℓ_2, ℓ_3, ℓ_4 are four disjoint lines in $\mathbb{P}^3$.

In the case (i), since, in the notation of Lemma 11, $F_0 = 2\cdot\mathcal{O}_{\mathbb{P}^3}$ and $G_0 = L$, $H^i(\mathcal{H}om(F_0, G_0)) = 0$, where $i \geq 1$, and therefore the condition (10) is satisfied by the semicontinuity, so that the deformation argument as above shows that $E \in \overline{\mathcal{C}(1,4,0)}$.

In case (ii), by the same reason, $[E] \in \overline{\mathcal{C}(2,2,0)}$.

In case (iii), a similar argument shows that $[E] \in \overline{\mathcal{D}(4,4)}$, and thus $[E] \in \overline{\mathcal{I}(4)}$.

In case (iv), Proposition 12 guarantees that $[E] \in \overline{\mathcal{C}(1,3,1)}$.

In the cases remaining, (v) through (viii), as in cases (ii) and (iii) for $d_E = 3$ above, we again obtain $[E] \in \overline{\mathcal{I}(4)}$.

Main Theorem 1 is finally proved.

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