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A SYMMETRIC 2-TENSOR CANONICALLY ASSOCIATED TO Q-CURVATURE AND ITS APPLICATIONS

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We define a symmetric 2-tensor, called the J-tensor, canonically associated to the Q-curvature on any Riemannian manifold with dimension at least three. The relation between the J-tensor and the Q-curvature is like that between the Ricci tensor and the scalar curvature. Thus the J-tensor can be interpreted as a higher-order analogue of the Ricci tensor. This tensor can be used to understand the Chang–Gursky–Yang theorem on 4-dimensional Q-singular metrics. We show that an *almost-Schur lemma* holds for the Q-curvature, yielding an estimate of the Q-curvature on closed manifolds.

1. Introduction

Let M be a smooth manifold and $\mathcal{M}$ be the space of all metrics on M. Consider scalar curvature as a nonlinear map

$$R : \mathcal{M} \to C^\infty(M), \quad g \mapsto R_g.$$

It is well known that the linearization of scalar curvature at a given metric g is

$$\gamma_g h := DR_g \cdot h = -\Delta_g \operatorname{tr}_g h + \delta_g^2 h - \operatorname{Ric}_g \cdot h, \tag{1-1}$$

where $h \in S_2(M)$ is a symmetric 2-tensor and $\delta_g = -\operatorname{div}_g$; see [Besse 1987; Chow et al. 2006; Fischer and Marsden 1975]. Thus, its L^2-formal adjoint is given by

$$\gamma_g^* f = \nabla_g^2 f - g\Delta_g f - f \operatorname{Ric}_g \tag{1-2}$$

for any smooth function $f \in C^\infty(M)$.

An interesting observation is that, if we take f to be constantly 1, we get

$$\operatorname{Ric}_g = -\gamma_g^* 1.$$

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That means we can recover Ricci tensor from γ_g^*. Furthermore, the scalar curvature is given by

$$R_g = -\operatorname{tr}_g \gamma_g^* 1.$$

Now let (M^n, g) be an n-dimensional Riemannian manifold ($n \geq 3$). We can define the Q-curvature to be

$$Q_g = A_n \Delta_g R_g + B_n |\mathrm{Ric}_g|_g^2 + C_n R_g^2, \tag{1-3}$$

where

$$A_n = -\frac{1}{2(n-1)}, \quad B_n = -\frac{2}{(n-2)^2}, \quad C_n = \frac{n^2(n-4)+16(n-1)}{8(n-1)^2(n-2)^2}.$$

In fact, Q-curvature was introduced originally to generalize the classic *Gauss-Bonnet theorem* on surfaces to closed 4-manifolds (M^4, g):

$$\int_{M^4} \big(Q_g + \tfrac{1}{4}|W_g|_g^2\big)\, dv_g = 8\pi^2 \chi(M), \tag{1-4}$$

where W_g is the Weyl tensor.

Paneitz and Branson extended Q-curvature to any dimension $n \geq 3$ (see [Branson 1985; Paneitz 2008]) such that it satisfies certain conformal invariant properties. For more details, please refer to the appendix of [Lin and Yuan 2016].

Like the scalar curvature, we can also view Q-curvature as a nonlinear map

$$Q : \mathcal{M} \to C^\infty(M), \quad g \mapsto Q_g.$$

Let $\Gamma_g : S_2(M) \to C^\infty(M)$ be the linearization of Q-curvature at the metric g and $\Gamma_g^* : C^\infty(M) \to S_2(M)$ be its L^2-formal adjoint.

Now we can define the central notion in this article:

Definition 1.1. Let (M^n, g) be a Riemannian manifold ($n \geq 3$). We define the symmetric 2-tensor

$$J_g := -\tfrac{1}{2}\Gamma_g^* 1.$$

We say (M, g) is *J-Einstein* if $J_g = \Lambda g$ for some smooth function $\Lambda \in C^\infty(M)$. In particular, it is *J-flat* if $\Lambda = 0$.

In [Lin and Yuan 2016], we calculated the explicit expression of Γ_g^* and showed

$$\operatorname{tr}_g \Gamma_g^* f = \frac{1}{2}\Big(P_g - \frac{n+4}{2} Q_g\Big) f, \tag{1-5}$$

for any $f \in C^\infty(M)$. Here P_g is the *Paneitz operator* defined by

$$P_g = \Delta_g^2 - \operatorname{div}_g[(a_n R_g g + b_n \operatorname{Ric}_g) d] + \frac{n-4}{2} Q_g, \tag{1-6}$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)} \quad \text{and} \quad b_n = -\frac{4}{n-2}.$$

In particular, $\operatorname{tr}_g \Gamma_g^* 1 = -2Q_g$. Thus

$$\operatorname{tr}_g J_g = Q_g. \tag{1-7}$$

On the other hand, for any smooth vector field $X \in \mathscr{X}(M)$ on M,

$$\begin{aligned}\int_M \langle X, \delta_g \Gamma_g^* f\rangle \, dv_g &= \frac{1}{2}\int_M \langle L_X g, \Gamma_g^* f\rangle \, dv_g \\ &= \frac{1}{2}\int_M f\Gamma_g(L_X g)\, dv_g = \frac{1}{2}\int_M \langle f\, dQ_g, X\rangle \, dv_g.\end{aligned}$$

Thus

$$\delta_g \Gamma_g^* f = \tfrac{1}{2} f \, dQ_g$$

on M. Hence,

$$\operatorname{div}_g J_g = \tfrac{1}{2}\delta_g \Gamma_g^* 1 = \tfrac{1}{4}\, dQ_g. \tag{1-8}$$

Recall that for Ricci tensor, we have

$$\operatorname{tr}_g \operatorname{Ric}_g = R_g \quad \text{and} \quad \operatorname{div}_g \operatorname{Ric}_g = \tfrac{1}{2} dR_g.$$

Therefore, if we consider Q-curvature as a higher-order analogue of scalar curvature, we can interpret J_g as a higher-order analogue of Ricci curvature on Riemannian manifolds.

A notion closely related to the J-tensor is the *Q-singular metric*, which refers to a metric satisfying $\ker \Gamma_g^* \neq \{0\}$. Clearly, J-flat metrics are Q-singular, since it is equivalent to $1 \in \ker \Gamma_g^*$.

One of the motivations for us to study J-flat manifolds is to understand the following theorem by Chang, Gursky and Yang:

Theorem 1.2 [Chang et al. 2002]. *Let (M^4, g) be a Q-singular 4-manifold. Then $1 \in \ker \Gamma_g^*$ if and only if (M^4, g) is Bach flat with vanishing Q-curvature.*

To achieve our goal, we need to give an explicit expression of the J-tensor:

Theorem 1.3. *For $n \geq 3$,*

$$J_g = \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g, \tag{1-9}$$

where B_g is the Bach tensor and

$$\begin{aligned}T_g := (n-2)\Big(\nabla^2 \operatorname{tr}_g S_g - \frac{1}{n} g \Delta_g \operatorname{tr}_g S_g\Big) \\ + 4(n-1)\Big(S_g \times S_g - \frac{1}{n}|S_g|^2 g\Big) - n^2 (\operatorname{tr}_g S_g)\mathring{S}_g.\end{aligned}$$

Here $(S \times S)_{jk} = S_j^i S_{ik}$, S_g is the Schouten tensor and $\mathring{S}_g$ is its traceless part.

Remark 1.4. Note that both the Bach tensor and the tensor T are traceless, thus the traceless part of J is given by

$$\mathring{J}_g = J_g - \frac{1}{n} Q_g g = -\frac{1}{n-2}\Big(B_g + \frac{n-4}{4(n-1)} T_g\Big). \tag{1-10}$$

Thus, an equivalent definition for a metric g being J-Einstein is

$$B_g = -\frac{n-4}{4(n-1)} T_g. \tag{1-11}$$

In particular, when $n = 4$, J-Einstein metrics are exactly Bach flat ones. Hence we can also interpret that J-Einstein metric is a generalization of Bach flat metric on 4-dimensional manifolds.

Remark 1.5. Gursky [1997] introduced a similar tensor for 4-manifolds from the viewpoint of functional determinants. In the same article, he also remarked this tensor can be introduced from the perspective of first variations of total Q-curvature when dimension is at least 5 (see [Case 2012] for a detailed calculation).

With the similar perspective, Gover and Ørsted introduced an abstract tensor called *higher Einstein tensor*, which coincides with our J-tensor in one of its special case. We refer interested readers to their article [Gover and Ørsted 2013].

Note that for any Einstein metric g, its Q-curvature is given by

$$Q_g = B_n |\mathrm{Ric}_g|^2 + C_n R_g^2 = \Big(\frac{1}{n} B_n + C_n\Big) R_g^2 = \frac{(n+2)(n-2)}{8n(n-1)^2} R_g^2,$$

which is a nonnegative constant and vanishes if and only if g is Ricci flat.

It is easy to check that $T_g = 0$ for any Einstein metric g. Combining this with the well-known fact that any Einstein metric is Bach flat, we can easily deduce that any nonflat Einstein metrics are also positive J-Einstein and Ricci flat metrics are J-flat as well.

With the aid of this notion, we can recover and generalize Theorem 1.2 to any dimension $n \geq 3$:

Corollary 1.6. *Let (M^n, g) be a Q-singular n-dimensional Riemannian manifold. Then $1 \in \ker \Gamma_g^*$ if and only if (M^n, g) is J-flat or equivalently (M^n, g) satisfies*

$$B_g = -\frac{n-4}{4(n-1)} T_g$$

with vanishing Q-curvature.

Remark 1.7. In [Chang et al. 2002], Bach flatness in Theorem 1.2 is derived using the variational property of the Bach tensor for 4-manifolds.

As another application of J-tensor, we can derive the *Schur lemma for Q-curvature* as follows:

Theorem 1.8 (Schur lemma). *Let (M^n, g) be an n-dimensional J-Einstein manifold with $n \neq 4$ or equivalently,*

$$B_g = -\frac{n-4}{4(n-1)} T_g.$$

Then Q_g is a constant on M.

Moreover, the following *almost-Schur lemma* holds exactly like the case for Ricci tensor and scalar curvature, cf., [Cheng 2013; De Lellis and Topping 2012; Ge and Wang 2012].

Theorem 1.9 (almost-Schur lemma). *For $n \neq 4$, let (M^n, g) be an n-dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$\int_M (Q_g - \bar{Q}_g)^2 \, dv_g \leq \frac{16n(n-1)}{(n-4)^2} \int_M |\mathring{J}_g|^2 \, dv_g, \tag{1-12}$$

where $\bar{Q}_g$ is the average of Q_g. Moreover, the equality holds if and only if (M, g) is J-Einstein.

In order to derive an equivalent form of above inequality, we need to define the *J-Schouten tensor* as follows:

$$S_J = \frac{1}{n-4}\Big(J_g - \frac{3}{4(n-1)} Q_g g\Big). \tag{1-13}$$

Immediately, we have

$$\operatorname{tr}_g S_J = \frac{1}{4(n-1)} Q_g \tag{1-14}$$

and

$$\operatorname{div}_g S_J = \frac{1}{4(n-1)} dQ_g = d \operatorname{tr}_g S_J. \tag{1-15}$$

Remark 1.10. Recall the definition of classic Schouten tensor

$$S_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{1}{2(n-1)} R_g g\Big). \tag{1-16}$$

We have

$$\operatorname{tr}_g S_g = \frac{1}{2(n-1)} R_g \tag{1-17}$$

and

$$\operatorname{div}_g S_g = \frac{1}{2(n-1)} dR_g = d \operatorname{tr}_g S_g. \tag{1-18}$$

We can see the tensor S_J shares similar properties with the classic Schouten tensor.

Following the observation in [Ge and Wang 2012], we get immediately the following result by rewriting Theorem 1.9 with J-Schouten tensor:

Corollary 1.11. *For $n \neq 4$, let (M^n, g) be an n-dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$(\mathrm{Vol}_g\, M)^{-(n-8)/n} \int_M \sigma_2^J(g)\, dv_g \leq \frac{n-1}{2n} Y_Q^2(g), \tag{1-19}$$

where

$$Y_Q(g) := \frac{\int_M \sigma_1^J(g)\, dv_g}{(\mathrm{Vol}_g\, M)^{(n-4)/n}}$$

*is the **Q-Yamabe quotient** and $\sigma_i^J(g) = \sigma_i(S_J(g))$, $i = 1, 2$ are the i-th symmetric polynomial of $S_J(g)$. Moreover, the equality holds if and only if (M, g) is J-Einstein.*

Remark 1.12. Our *almost-Schur lemma* can be generalized to a broader setting by combining it with the work [Gover and Ørsted 2013]. More detailed discussion together with some related topics will be presented in a subsequent article.

This article is organized as follows: In Section 2, we derive an explicit formula for the J-tensor and with it we prove Theorem 1.3 and Corollary 1.6. We then prove Theorem 1.8 (Schur lemma) and Theorem 1.9 (almost-Schur lemma) in Section 3.

2. J-flatness and Q-singular metrics

We begin with some discussion of conformal tensors. Let

$$S_{jk} = \frac{1}{n-2}\Big(R_{jk} - \frac{1}{2(n-1)} R g_{jk}\Big) \tag{2-1}$$

be the Schouten tensor.

For $n \geq 4$, the Bach tensor is defined to be

$$B_{jk} = \frac{1}{n-3} \nabla^i \nabla^l W_{ijkl} + W_{ijkl} S^{il}. \tag{2-2}$$

In order to extend the definition to $n = 3$, we introduce the Cotton tensor

$$C_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik}. \tag{2-3}$$

It is related to Weyl tensor by the equation

$$\nabla^l W_{ijkl} = (n-3) C_{ijk}. \tag{2-4}$$

Therefore, for any $n \geq 3$, we can define the Bach tensor as

$$B_{jk} = \nabla^i C_{ijk} + W_{ijkl} S^{il}. \tag{2-5}$$

The following identity is well known for experts; we include calculations here for the convenience of readers.

Proposition 2.1. *The Bach tensor can be written as*

$$B_g = \Delta_g S - \nabla^2 \operatorname{tr} S + 2\mathring{Rm} \cdot S - (n-4) S \times S - |S|^2 g - 2(\operatorname{tr} S) S, \tag{2-6}$$

where $(\mathring{Rm} \cdot S)_{jk} = R_{ijkl} S^{il}$ *and* $(S \times S)_{jk} = S^i_j S_{ik}$. *Equivalently,*

$$B_g = \Delta_L S - \nabla^2 \operatorname{tr} S + n\Big(S \times S - \frac{1}{n}|S|^2 g\Big), \tag{2-7}$$

where Δ_L *is the Lichnerowicz Laplacian.*

Proof. By the *second contracted Bianchi identity*,

$$\begin{aligned}\nabla^i S_{ik} = \frac{1}{n-2}\Big(\nabla^i R_{ik} - \frac{1}{2(n-1)}\nabla_k R\Big) &= \frac{1}{n-2}\Big(\frac{1}{2}\nabla_k R - \frac{1}{2(n-1)}\nabla_k R\Big)\\ &= \frac{1}{2(n-1)}\nabla_k R\\ &= \nabla_k \operatorname{tr} S\end{aligned}$$

and

$$\operatorname{tr} S = \frac{1}{n-2}\Big(R - \frac{n}{2(n-1)} R\Big) = \frac{1}{2(n-1)} R,$$

we have

$$\mathrm{Ric} = (n-2) S + (\operatorname{tr} S) g.$$

Using these facts,

$$\begin{aligned}\nabla^i C_{ijk} &= \nabla^i(\nabla_i S_{jk} - \nabla_j S_{ik})\\ &= \Delta_g S_{jk} - (\nabla_j \nabla_i S^i_k + R^i_{ijp} S^p_k - R^p_{ijk} S^i_p)\\ &= \Delta_g S_{jk} - \nabla_j \nabla_k \operatorname{tr} S - (\mathrm{Ric} \times S)_{jk} + (\mathring{Rm} \cdot S)_{jk}\\ &= \Delta_g S_{jk} - \nabla_j \nabla_k \operatorname{tr} S - (((n-2) S + (\operatorname{tr} S) g) \times S)_{jk} + (\mathring{Rm} \cdot S)_{jk}\\ &= \Delta_g S_{jk} - \nabla_j \nabla_k \operatorname{tr} S - (n-2)(S \times S)_{jk} - (\operatorname{tr} S) S_{jk} + (\mathring{Rm} \cdot S)_{jk}\end{aligned}$$

and

$$\begin{aligned}W_{ijkl} S^{il} &= (Rm - S \owedge g)_{ijkl} S^{il}\\ &= R_{ijkl} S^{il} - (S_{il} g_{jk} + S_{jk} g_{il} - S_{ik} g_{jl} - S_{jl} g_{ik}) S^{il}\\ &= (\mathring{Rm} \cdot S)_{jk} - |S|^2 g_{jk} + 2(S \times S)_{jk} - (\operatorname{tr} S) S_{jk},\end{aligned}$$

where $\owedge$ is the *Kulkarni–Nomizu product*:

$$(\alpha \owedge \beta)_{ijkl} := \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{ik}\beta_{jl} - \alpha_{jl}\beta_{ik}$$

for any symmetric 2-tensor $\alpha, \beta \in S_2(M)$.

Combining them, we get

$$B_{jk} = \Delta_g S_{jk} - \nabla_j \nabla_k \operatorname{tr} S + 2(\mathring{Rm} \cdot S)_{jk} - (n-4)(S \times S)_{jk} - |S|^2 g_{jk} - 2(\operatorname{tr} S) S_{jk}.$$

From this,

$$\begin{aligned} B_{jk} &= \Delta_L S_{jk} + 2(\mathrm{Ric}\times S)_{jk} - \nabla_j\nabla_k \operatorname{tr} S - (n-4)(S\times S)_{jk} - |S|^2 g_{jk} - 2(\operatorname{tr} S)S_{jk} \\ &= \Delta_L S_{jk} + 2((\mathrm{Ric} - (\operatorname{tr} S)g)\times S)_{jk} - \nabla_j\nabla_k \operatorname{tr} S - (n-4)(S\times S)_{jk} - |S|^2 g_{jk} \\ &= \Delta_L S - \nabla^2 \operatorname{tr} S + n(S\times S) - |S|^2 g \\ &= \Delta_L S - \nabla^2 \operatorname{tr} S + n\Big(S\times S - \frac{1}{n}|S|^2 g\Big). \qquad \square \end{aligned}$$

The Q-curvature can also be rewritten using Schouten tensor:

Lemma 2.2. $$Q_g = -\Delta_g \operatorname{tr} S - 2|S|^2 + \frac{n}{2}(\operatorname{tr} S)^2.$$

Proof. Using the equalities $\mathrm{Ric} = (n-2)S + (\operatorname{tr} S)g$ and $R = 2(n-1)\operatorname{tr} S$,

$$\begin{aligned} Q_g &= A_n \Delta_g R + B_n |\mathrm{Ric}|^2 + C_n R^2 \\ &= 2(n-1)A_n \Delta_g \operatorname{tr} S + B_n |(n-2)S + (\operatorname{tr} S)g|^2 + 4(n-1)^2 C_n (\operatorname{tr} S)^2 \\ &= -\Delta_g \operatorname{tr} S - 2|S|^2 + ((3n-4)B_n + 4(n-1)^2 C_n)(\operatorname{tr} S)^2 \\ &= -\Delta_g \operatorname{tr} S - 2|S|^2 + \frac{n}{2}(\operatorname{tr} S)^2. \qquad \square \end{aligned}$$

We recall the expression of Γ_g^* in [Lin and Yuan 2016] as follows:

Lemma 2.3.

$$\begin{aligned} \Gamma_g^* f := A_n\big(&-g\Delta^2 f + \nabla^2\Delta f - \mathrm{Ric}\,\Delta f + \tfrac{1}{2} g\delta(f\,dR) + \nabla(f\,dR) - f\nabla^2 R\big) \\ &- B_n(\Delta(f\,\mathrm{Ric}) + 2f\,\mathring{Rm}\cdot\mathrm{Ric} + g\delta^2(f\,\mathrm{Ric}) + 2\nabla\delta(f\,\mathrm{Ric})) \\ &- 2C_n(g\Delta(fR) - \nabla^2(fR) + fR\,\mathrm{Ric}). \end{aligned} \tag{2-8}$$

Now we can calculate an explicit expression of J_g:

Theorem 2.4. *For $n \geq 3$,*

$$J_g = \frac{1}{n}Q_g g - \frac{1}{n-2}B_g - \frac{n-4}{4(n-1)(n-2)}T_g, \tag{2-9}$$

where

$$\begin{aligned} T_g := (n-2)&\Big(\nabla^2 \operatorname{tr}_g S_g - \frac{1}{n} g\Delta_g \operatorname{tr}_g S_g\Big) \\ &+ 4(n-1)\Big(S_g\times S_g - \frac{1}{n}|S_g|^2 g\Big) - n^2(\operatorname{tr}_g S_g)\mathring{S}_g. \end{aligned}$$

Here $\mathring{S}_g = S_g - (1/n)\operatorname{tr}_g S_g g$ is the traceless part of Schouten tensor.

Proof. By Lemma 2.3,

$$\begin{aligned} \Gamma_g^* 1 = -\big(\tfrac{1}{2}A_n + \tfrac{1}{2}B_n + 2C_n\big)g\Delta R + (B_n + 2C_n)\nabla^2 R& \\ - B_n(\Delta\,\mathrm{Ric} + 2\mathring{Rm}\cdot\mathrm{Ric}) - 2C_n R\,\mathrm{Ric}&. \end{aligned}$$

Applying equalities $\mathrm{Ric} = (n-2)S + (\operatorname{tr} S)g$ and $R = 2(n-1)\operatorname{tr} S$,

$$\begin{aligned}\Gamma_g^* 1 &= -((n-1)A_n + nB_n + 4(n-1)C_n)g\Delta \operatorname{tr} S + 2(n-1)(B_n+2C_n)\nabla^2 \operatorname{tr} S \\ &\qquad - (n-2)B_n(\Delta S + 2\mathring{Rm}\cdot S) - 2(n-2)(B_n + 2(n-1)C_n)(\operatorname{tr} S)S \\ &\qquad - 2(B_n+2(n-1)C_n)(\operatorname{tr} S)^2 g \\ &= \frac{3}{2(n-1)} g\Delta \operatorname{tr} S + \frac{2}{n-2}(\Delta S + 2\mathring{Rm}\cdot S) + \frac{n^2-10n+12}{2(n-1)(n-2)}\nabla^2 \operatorname{tr} S \\ &\qquad - \frac{n^2-2n+4}{2(n-1)}(\operatorname{tr} S)S - \frac{n^2-2n+4}{2(n-1)(n-2)}(\operatorname{tr} S)^2 g.\end{aligned}$$

Since $\operatorname{tr}\Gamma_g^* 1 = -2Q_g$, by Lemma 2.2,

$$\begin{aligned}&\Gamma_g^* 1 + \frac{2}{n}Q_g g \\ &\quad = \Big(\frac{3}{2(n-1)} - \frac{2}{n}\Big) g\Delta \operatorname{tr} S + \frac{2}{n-2}(\Delta S + 2\mathring{Rm}\cdot S) + \frac{n^2-10n+12}{2(n-1)(n-2)}\nabla^2 \operatorname{tr} S \\ &\qquad\qquad - \frac{4}{n}|S|^2 g - \frac{n^2-2n+4}{2(n-1)}(\operatorname{tr} S)S + \Big(1 - \frac{n^2-2n+4}{2(n-1)(n-2)}\Big)(\operatorname{tr} S)^2 g \\ &\quad = -\frac{n-4}{2n(n-1)} g\Delta \operatorname{tr} S + \frac{2}{n-2}(\Delta S + 2\mathring{Rm}\cdot S) + \frac{n^2-10n+12}{2(n-1)(n-2)}\nabla^2 \operatorname{tr} S \\ &\qquad\qquad - \frac{4}{n}|S|^2 g - \frac{n^2-2n+4}{2(n-1)}(\operatorname{tr} S)S + \frac{n(n-4)}{2(n-1)(n-2)}(\operatorname{tr} S)^2 g.\end{aligned}$$

Applying Proposition 2.1,

$$\begin{aligned}\Gamma_g^* 1 + \frac{2}{n}Q_g g &= \frac{2}{n-2}B_g - \frac{n-4}{2n(n-1)} g\Delta \operatorname{tr} S + \Big(\frac{2}{n-2} + \frac{n^2-10n+12}{2(n-1)(n-2)}\Big)\nabla^2 \operatorname{tr} S \\ &\qquad + \frac{2(n-4)}{n-2} S\times S + \Big(\frac{2}{n-2} - \frac{4}{n}\Big)|S|^2 g \\ &\qquad + \Big(\frac{4}{n-2} - \frac{n^2-2n+4}{2(n-1)}\Big)(\operatorname{tr} S)S + \frac{n(n-4)}{2(n-1)(n-2)}(\operatorname{tr} S)^2 g.\end{aligned}$$

That is,

$$\begin{aligned}\Gamma_g^* 1 + \frac{2}{n}Q_g g &= \frac{2}{n-2}B_g - \frac{n-4}{2n(n-1)} g\Delta \operatorname{tr} S + \frac{n-4}{2(n-1)}\nabla^2 \operatorname{tr} S + \frac{2(n-4)}{n-2} S\times S \\ &\qquad - \frac{2(n-4)}{n(n-2)}|S|^2 g - \frac{n^2(n-4)}{2(n-1)(n-2)}(\operatorname{tr} S)S + \frac{n(n-4)}{2(n-1)(n-2)}(\operatorname{tr} S)^2 g \\ &= \frac{2}{n-2}B_g + \frac{n-4}{2(n-1)}\Big(\nabla^2 \operatorname{tr} S - \frac{1}{n} g\Delta \operatorname{tr} S\Big) + \frac{2(n-4)}{n-2}\Big(S\times S - \frac{1}{n}|S|^2 g\Big) \\ &\qquad - \frac{n^2(n-4)}{2(n-1)(n-2)}(\operatorname{tr} S)\Big(S - \frac{1}{n}(\operatorname{tr} S)g\Big) \\ &= \frac{2}{n-2}B_g + \frac{n-4}{2(n-1)(n-2)}T_g,\end{aligned}$$

where

$$T_g := (n-2)\Big(\nabla^2 \operatorname{tr}_g S_g - \frac{1}{n} g \Delta_g \operatorname{tr}_g S_g\Big) + 4(n-1)\Big(S_g \times S_g - \frac{1}{n}|S_g|^2 g\Big) - n^2(\operatorname{tr}_g S_g)\mathring{S}_g.$$

Therefore,

$$J_g = -\frac{1}{2}\Gamma_g^* 1 = \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g. \qquad \square$$

Immediately, we have the following generalization of Theorem 1.2:

Corollary 2.5. *Let (M^n, g) be a Q-singular n-dimensional Riemannian manifold. Then $1 \in \ker \Gamma_g^*$ if and only if (M^n, g) is J-flat or equivalently (M^n, g) satisfies*

$$B_g = -\frac{n-4}{4(n-1)} T_g \tag{2-10}$$

with vanishing Q-curvature.

Remark 2.6. A similar result holds for Ricci curvature: a vacuum static space admits a constant static potential if and only if it is Ricci flat, cf., [Fischer and Marsden 1975].

3. An almost-Schur lemma for Q-curvature

Since the tensor J_g can be interpreted as a higher-order analogue of Ricci tensor, we can also derive the Schur lemma for J_g as follows:

Theorem 3.1 (Schur lemma). *Let (M^n, g) be an n-dimensional J-Einstein manifold with $n \neq 4$ or equivalently,*

$$B_g = -\frac{n-4}{4(n-1)} T_g.$$

Then Q_g is a constant on M.

Proof. By the assumption, $J_g = \Lambda g$ for some smooth function Λ on M. Then

$$\Lambda = \frac{1}{n} \operatorname{tr}_g J_g = \frac{1}{n} Q_g \quad \text{and} \quad d\Lambda = \operatorname{div}_g J_g = \frac{1}{4} dQ_g.$$

Therefore,

$$\frac{n-4}{4n} dQ_g = 0$$

on M, which implies that Q_g is a constant on M provided $n \neq 4$. $\square$

Remark 3.2. When $n = 4$, J-Einstein metrics are exactly Bach flat ones. Due to the conformal invariance of Bach flatness in dimension 4, we can easily see that the constancy of Q-curvature can not always be achieved. Thus the above Schur

Lemma does not hold for 4-dimensional manifolds, which is exactly like the classic Schur lemma for surfaces.

In fact, a more general result can be derived:

Theorem 3.3 (almost-Schur lemma). *For $n \neq 4$, let (M^n, g) be an n-dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$\int_M (Q_g - \bar{Q}_g)^2 \, dv_g \leq \frac{16n(n-1)}{(n-4)^2} \int_M |\mathring{J}_g|^2 \, dv_g, \tag{3-1}$$

where $\bar{Q}_g$ is the average of Q_g. Moreover, the equality holds if and only if (M^n, g) is J-Einstein.

The proof is along the same lines as in [De Lellis and Topping 2012]. For completeness, we include it here. For more details, please refer to that work.

Proof. Let u be the unique solution to

$$\begin{cases} \Delta_g u &= Q_g - \bar{Q}_g, \\ \int_M u \, dv_g &= 0. \end{cases}$$

Then

$$\begin{aligned} \int_M (Q_g - \bar{Q}_g)^2 \, dv_g = \int_M (Q_g - \bar{Q}_g) \Delta_g u \, dv_g &= -\int_M \langle \nabla Q_g, \nabla u \rangle \, dv_g \\ &= -\frac{4n}{n-4} \int_M \langle \operatorname{div}_g \mathring{J}_g, \nabla u \rangle, \end{aligned}$$

where for the last step we use the fact

$$\operatorname{div}_g \mathring{J}_g = \operatorname{div}_g \Big(J_g - \frac{1}{n} Q_g g \Big) = \frac{1}{4} \, dQ_g - \frac{1}{n} \, dQ_g = \frac{n-4}{4n} \, dQ_g.$$

Integrating by parts,

$$\begin{aligned} -\frac{4n}{n-4} \int_M \langle \operatorname{div}_g \mathring{J}_g, \nabla u \rangle \, dv_g &= \frac{4n}{n-4} \int_M \langle \mathring{J}_g, \nabla^2 u \rangle \, dv_g \\ &= \frac{4n}{n-4} \int_M \Big\langle \mathring{J}_g, \nabla^2 u - \frac{1}{n} g \Delta_g u \Big\rangle \, dv_g \\ &\leq \frac{4n}{n-4} \bigg(\int_M |\mathring{J}_g|^2 \, dv_g \bigg)^{1/2} \bigg(\int_M \Big| \nabla^2 u - \frac{1}{n} g \Delta_g u \Big|^2 \, dv_g \bigg)^{1/2} \\ &= \frac{4n}{n-4} \bigg(\int_M |\mathring{J}_g|^2 dv_g \bigg)^{1/2} \bigg(\int_M |\nabla^2 u|^2 - \frac{1}{n} (\Delta_g u)^2 \, dv_g \bigg)^{1/2}. \end{aligned}$$

From the *Bochner formula* and the assumption $\operatorname{Ric}_g > 0$,

$$\int_M |\nabla^2 u|^2 \, dv_g = \int_M (\Delta_g u)^2 \, dv_g - \int_M \operatorname{Ric}_g(\nabla u, \nabla u) \, dv_g \leq \int_M (\Delta_g u)^2 \, dv_g.$$

Thus,

$$\int_M (Q_g - \bar{Q}_g)^2\, dv_g \le \frac{4n}{n-4}\left(\int_M |\mathring{J}_g|^2\, dv_g\right)^{1/2}\left(\frac{n-1}{n}(\Delta_g u)^2\, dv_g\right)^{1/2}$$
$$= \frac{4n}{n-4}\left(\int_M |\mathring{J}_g|^2\, dv_g\right)^{1/2}\left(\frac{n-1}{n}(Q_g - \bar{Q}_g)^2\, dv_g\right)^{1/2}.$$

That is,

$$\int_M (Q_g - \bar{Q}_g)^2\, dv_g \le \frac{16n(n-1)}{(n-4)^2}\int_M |\mathring{J}_g|^2\, dv_g.$$

Now we consider the equality case.

If g is J-Einstein, then Q_g is a constant by the *Schur lemma* (Theorem 1.8). Thus both sides of inequality (3-1) vanish and equality is achieved.

On the contrary, assume in (3-1) equality is achieved:

$$\int_M (Q_g - \bar{Q}_g)^2\, dv_g = \frac{16n(n-1)}{(n-4)^2}\int_M |\mathring{J}_g|^2\, dv_g.$$

Then in particular we have $\mathrm{Ric}(\nabla u, \nabla u) = 0$, which implies that $\nabla u = 0$ and hence u is a constant on M, since we assume $\mathrm{Ric}_g > 0$.

Thus $Q \equiv \bar{Q}$ on M and

$$\int_M |\mathring{J}_g|^2\, dv_g = \frac{(n-4)^2}{16n(n-1)}\int_M (Q_g - \bar{Q}_g)^2\, dv_g = 0.$$

Therefore, $\mathring{J}_g \equiv 0$ on M, i.e., (M, g) is J-Einstein. □

Remark 3.4. By assuming $\mathrm{Ric} \ge -(n-1)Kg$ for some constant $K \ge 0$ and following the proof in [Cheng 2013], the inequality (3-1) can be improved to

$$\int_M (Q_g - \bar{Q}_g)^2\, dv_g \le \frac{16n(n-1)}{(n-4)^2}\left(1 + \frac{nK}{\lambda_1}\right)\int_M |\mathring{J}_g|^2\, dv_g, \tag{3-2}$$

where $\lambda_1 > 0$ is the first nonzero eigenvalue of $(-\Delta_g)$.

Now we can derive an equivalent form of inequality (3-1):

Corollary 3.5. *For $n \neq 4$, let (M^n, g) be an n-dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$(\mathrm{Vol}_g\, M)^{-(n-8)/n}\int_M \sigma_2^J(g)\, dv_g \le \frac{n-1}{2n} Y_Q^2(g). \tag{3-3}$$

Moreover, the equality holds if and only if (M^n, g) is J-Einstein.

Proof. Note that

$$\sigma_1^J(g) = \mathrm{tr}_g\, S_J = \frac{1}{4(n-1)} Q_g$$

and

$$\sigma_2^J(g) = \frac{1}{2}((\sigma_1^J)^2 - |S_J|^2) = \frac{n-1}{2n}(\sigma_1^J)^2 - \frac{1}{2(n-4)^2}|\mathring{J}_g|^2,$$

where we use the fact

$$|S_J|^2 = \left|\mathring{S}_J + \frac{1}{n}(\mathrm{tr}_g\, S_J)g\right|^2 = \left|\frac{1}{n-4}\mathring{J}_g + \frac{1}{n}(\sigma_1^J)g\right|^2 = \frac{1}{(n-4)^2}|\mathring{J}_g|^2 + \frac{1}{n}(\sigma_1^J)^2.$$

By substituting these terms in the inequality (3-1), we get

$$\left(\int_M \sigma_1^J(g)\, dv_g\right)^2 \geq \frac{2n}{n-1}\,\mathrm{Vol}_g(M)\int_M \sigma_2^J(g)\, dv_g.$$

Therefore,

$$\begin{aligned}\int_M \sigma_2^J(g)\, dv_g &\leq \frac{n-1}{2n}(\mathrm{Vol}_g\, M)^{-1}\left(\int_M \sigma_1^J(g)\, dv_g\right)^2 \\ &= \frac{n-1}{2n}(\mathrm{Vol}_g\, M)^{(n-8)/n}\left(\frac{\int_M \sigma_1^J(g)\, dv_g}{(\mathrm{Vol}_g\, M)^{(n-4)/n}}\right)^2 \\ &= \frac{n-1}{2n}(\mathrm{Vol}_g\, M)^{(n-8)/n}Y_Q^2(g).\end{aligned}$$

□

Remark 3.6. Note that the *Q-Yamabe quotient*

$$Y_Q(g) := \frac{\int_M \sigma_1^J(g)\, dv_g}{(\mathrm{Vol}_g\, M)^{(n-4)/n}}$$

is scaling invariant and in particular, when $n = 8$,

$$\int_M \sigma_2^J(g)\, dv_g \leq \frac{7}{16}Y_Q^2(g),$$

provided that $\mathrm{Ric}_g > 0$, where the equality holds if and only if (M, g) is J-Einstein.

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