Pacific Journal of Mathematics

BRANCHING LAWS FOR THE METAPLECTIC COVER OF GL_2

SHIV PRAKASH PATEL

Volume 291 No. 2 December 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 291, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.291.461

BRANCHING LAWS FOR THE METAPLECTIC COVER OF GL_2

SHIV PRAKASH PATEL

Let F be a nonarchimedean local field of characteristic zero and E/F be a quadratic extension. The aim of this article is to study the multiplicity of an irreducible admissible representation of $\mathrm{GL}_2(F)$ occurring in an irreducible admissible genuine representation of the nontrivial two-fold covering $\widetilde{\mathrm{GL}}_2(E)$ of $\mathrm{GL}_2(E)$.

1. Introduction

Let F be a nonarchimedean local field of characteristic zero and let E be a quadratic extension of F. The branching laws for restriction of representations of $\mathrm{SO}_{n+1}(F)$ to $\mathrm{SO}_n(F)$ were formulated as conjectures by B. Gross and D. Prasad [1992], and these are widely known as Gross–Prasad conjectures although they have been completely proved by Mœglin and Waldspurger [2012]. The first case of these conjectures is for the restriction of representations of $\mathrm{GL}_2(F)$ to its maximal tori, which was considered by J. B. Tunnell [1983] and H. Saito [1993]. A metaplectic analog of this result was recently considered by the author in a joint work with Prasad, where the restriction of representations of metaplectic $\mathrm{GL}_2(F)$ to inverse images of the maximal tori was studied [Patel and Prasad 2017]. The results of Tunnell and Saito have, in particular, a multiplicity one result which is then refined in terms of certain ϵ-factors. The metaplectic case of this restriction loses the multiplicity one property, but still one has finite multiplicities which are bounded by some explicit constants. The next case of Gross–Prasad conjectures can be considered to be the restriction of representations of $\mathrm{GL}_2(E)$ to $\mathrm{GL}_2(F)$ which was studied by Prasad [1992]. These cases played an important role in the formulation of Gross–Prasad conjectures. Our aim in this paper is to study an analogous restriction of representations of metaplectic $\mathrm{GL}_2(E)$ to $\mathrm{GL}_2(F)$.

The problem of decomposing a representation of $\mathrm{GL}_2(E)$ restricted to $\mathrm{GL}_2(F)$ was considered and solved by Prasad [1992], proving a multiplicity one theorem, and giving an explicit classification of representations π_1 of $\mathrm{GL}_2(E)$ and π_2 of $\mathrm{GL}_2(F)$ such that there exists a nonzero $\mathrm{GL}_2(F)$ invariant linear form:

$$l : \pi_1 \otimes \pi_2 \to \mathbb{C}.$$

MSC2010: primary 22E35; secondary 22E50.
Keywords: metaplectic group, branching laws.

This problem is closely related to a similar branching law from $\mathrm{GL}_2(E)$ to $D_F^\times$, where D_F is the unique quaternion division algebra which is central over F, and $D_F^\times \hookrightarrow \mathrm{GL}_2(E)$. We recall that the embedding $D_F^\times \hookrightarrow \mathrm{GL}_2(E)$ is given by fixing an isomorphism $D_F \otimes E \cong M_2(E)$, by the Skolem–Noether theorem, which is unique up to conjugation by elements of $\mathrm{GL}_2(E)$. Henceforth, we fix one such embedding of $D_F^\times$ inside $\mathrm{GL}_2(E)$. The restriction problems for the pair $(\mathrm{GL}_2(E), \mathrm{GL}_2(F))$ and $(\mathrm{GL}_2(E), D_F^\times)$ are related by a certain dichotomy. More precisely, the following result was proved in [Prasad 1992]:

Theorem 1.1 (Prasad). *Let π_1 and π_2 be irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(E)$ and $\mathrm{GL}_2(F)$, respectively, such that the central character of π_1 restricted to the center of $\mathrm{GL}_2(F)$ is the same as the central character of π_2. Then*:

(1) *For a principal series representation π_2 of $\mathrm{GL}_2(F)$, we have*

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) = 1.$$

(2) *For a discrete series representation π_2 of $\mathrm{GL}_2(F)$, letting π_2' be the finite-dimensional representation of $D_F^\times$ associated to π_2 by the Jacquet–Langlands correspondence*, *we have*

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) + \dim \mathrm{Hom}_{D_F^\times}(\pi_1, \pi_2') = 1.$$

In this paper, we study the analogous problem in the metaplectic setting. More precisely, instead of considering $\mathrm{GL}_2(E)$ we will consider the group $\widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times}$ which is a topological central extension of $\mathrm{GL}_2(E)$ by $\mathbb{C}^\times$, which is obtained from the two-fold topological central extension $\widetilde{\mathrm{GL}}_2(E)$ described below. We recall that there is unique (up to isomorphism) two-fold cover of $\mathrm{SL}_2(E)$ called the metaplectic cover and denoted by $\widetilde{\mathrm{SL}}_2(E)$ in this paper, but there are many inequivalent two-fold coverings of $\mathrm{GL}_2(E)$ which extend this two-fold covering of $\mathrm{SL}_2(E)$. We fix a covering of $\mathrm{GL}_2(E)$ as follows. Observe that $\mathrm{GL}_2(E)$ is a semidirect product of $\mathrm{SL}_2(E)$ and $E^\times$, where $E^\times$ sits inside $\mathrm{GL}_2(E)$ by $e \mapsto \left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The action of $E^\times$ on $\mathrm{SL}_2(E)$ lifts to an action on $\widetilde{\mathrm{SL}}_2(E)$. Denote $\widetilde{\mathrm{GL}}_2(E)$ to be $\widetilde{\mathrm{SL}}_2(E) \rtimes E^\times$ which we call "the" metaplectic cover of $\mathrm{GL}_2(E)$. This cover can be described by an explicit 2-cocycle on $\mathrm{GL}_2(E)$ with values in $\{\pm 1\}$, see [Kubota 1969]. The group $\widetilde{\mathrm{GL}}_2(E)$ is a topological central extension of $\mathrm{GL}_2(E)$ by $\mu_2 := \{\pm 1\}$, i.e., we have an exact sequence of topological groups:

$$1 \to \mu_2 \to \widetilde{\mathrm{GL}}_2(E) \to \mathrm{GL}_2(E) \to 1.$$

The group $\widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} := \widetilde{\mathrm{GL}}_2(E) \times_{\mu_2} \mathbb{C}^\times$ is called the $\mathbb{C}^\times$-cover of $\mathrm{GL}_2(E)$ obtained from the two-fold cover $\widetilde{\mathrm{GL}}_2(E)$, and is a topological central extension of $\mathrm{GL}_2(E)$

by $\mathbb{C}^\times$, i.e., we have an exact sequence of topological groups:

$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} \to \mathrm{GL}_2(E) \to 1.$$

Now we recall the following result regarding splitting of this cover when restricted to certain subgroups. This makes it possible to consider an analog of the Prasad's restriction problem in the metaplectic case.

Theorem 1.2 [Patel 2016]. *Let E be a quadratic extension of a nonarchimedean local field and $\widetilde{\mathrm{GL}}_2(E)$ be the two-fold metaplectic covering of $\mathrm{GL}_2(E)$. Then*:

(1) *The two-fold metaplectic covering $\widetilde{\mathrm{GL}}_2(E)$ splits over the subgroup $\mathrm{GL}_2(F)$.*

(2) *The $\mathbb{C}^\times$-covering obtained from $\widetilde{\mathrm{GL}}_2(E)$ splits over the subgroup $D_F^\times$.*

Note that the splittings over $\mathrm{GL}_2(F)$ and $D_F^\times$ in Theorem 1.2 are not unique. As there is more than one splitting in each case, to study the problem of decomposing a representation of $\widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times}$ restricted to $\mathrm{GL}_2(F)$ and $D_F^\times$, we must fix one splitting of each of the subgroups $\mathrm{GL}_2(F)$ and $D_F^\times$, which are related to each other. We make the following working hypothesis, which has been formulated by Prasad.

Working Hypothesis 1.3. Let L be a quadratic extension of F. Write R for the restriction of scalars torus $R_{L/F}\mathbb{G}_m$. Thus $R(F) = L^\times$. Fix embeddings of R into GL_2 and $D_F^\times$ (viewed as algebraic groups over F). The sets of splittings

$$\begin{tikzcd} & \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} \arrow[d] \\ \mathrm{GL}_2(F) \arrow[ur, dashed, hook, "s"] \arrow[r] & \mathrm{GL}_2(E) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} \arrow[d] \\ D_F^\times \arrow[ur, dashed, hook, "s'"] \arrow[r] & \mathrm{GL}_2(E) \end{tikzcd}$$

are principal homogeneous spaces over the group $\mathrm{Hom}(F^\times, \mathbb{C}^\times)$. More explicitly, two splittings s_1, s_2 of $\mathrm{GL}_2(F)$ will be related by

$$s_2(g) = \chi(\det g) \cdot s_1(g)$$

for some character $\chi \in \mathrm{Hom}(F^\times, \mathbb{C}^\times)$ (for $D_F^\times$ the det should be replaced by Nm the reduced norm map). A pair (s, s') of splittings, where

$$s : \mathrm{GL}_2(F) \to \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} \quad \text{and} \quad s' : D_F^\times \to \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times},$$

is called a pair of "compatible splittings" if for any quadratic extension L/F with the fixed embedding of R into GL_2 and $D_F^\times$ the restriction of s and s' to $L^\times$ as in the following diagrams

$$\begin{tikzcd} & & \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} \arrow[d] \\ L^\times \arrow[r] \arrow[urr, hook, dashed] & \mathrm{GL}_2(F) \arrow[r] \arrow[ur, hook, dashed, "s"'] & \mathrm{GL}_2(E) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & & \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times} \arrow[d] \\ L^\times \arrow[r] \arrow[urr, hook, dashed] & D_F^\times \arrow[r] \arrow[ur, hook, dashed, "s'"'] & \mathrm{GL}_2(E) \end{tikzcd}$$

are conjugate in $\widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times}$, i.e., there is an element $g \in \widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times}$ such that $s(L^\times) = g \cdot s'(L^\times) \cdot g^{-1}$. Then we assume that

$$\text{there exists a pair } (s, s') \text{ of compatible splittings.}$$

If (s, s') is a pair of compatible splittings and χ is a character of $F^\times$ then the pair of splittings $(\chi(\det(\bullet)s, \chi(\mathrm{Nm}(\bullet)s')$ is also compatible. Thus, given a single pair (s, s') of compatible splittings, we have a $\mathrm{Hom}(F^\times, \mathbb{C}^\times)$-equivariant bijection between the sets of splittings, in such a way that all pairs matched by the bijection are compatible.

Definition 1.4. A representation of $\widetilde{\mathrm{GL}}_2(E)$ (respectively, $\widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times}$) is called genuine if μ_2 acts nontrivially (respectively, $\mathbb{C}^\times$ acts by identity).

In particular, a genuine representation does not factor through $\mathrm{GL}_2(E)$. In what follows, we always consider genuine representations of the metaplectic group $\widetilde{\mathrm{GL}}_2(E)$. Let $B(E)$, $A(E)$ and $N(E)$ be the Borel subgroup, maximal torus and maximal unipotent subgroup of $\mathrm{GL}_2(E)$ consisting of all upper triangular matrices, diagonal matrices and upper triangular unipotent matrices respectively. Let $B(F)$, $A(F)$ and $N(F)$ denote the corresponding subgroups of $\mathrm{GL}_2(F)$. Let Z be the center of $\mathrm{GL}_2(E)$ and $\widetilde{Z}$ the inverse image of Z in $\widetilde{\mathrm{GL}}_2(E)$. Note that $\widetilde{Z}$ is an abelian subgroup of $\widetilde{\mathrm{GL}}_2(E)$ but is not the center of $\widetilde{\mathrm{GL}}_2(E)$; the center of $\widetilde{\mathrm{GL}}_2(E)$ is $\widetilde{Z}^2$, the inverse image of $Z^2 := \{z^2 \mid z \in Z\}$.

Let ψ be a nontrivial additive character of E. Note that the metaplectic covering splits when restricted to the subgroup $N(E)$ and hence ψ gives a character of $N(E)$. Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(E)$ and $\pi_{N(E),\psi}$, the ψ-twisted Jacquet module which is a $\widetilde{Z}$-module. Let ω_π be the central character of π. A character of $\widetilde{Z}$ appearing in $\pi_{N(E),\psi}$ agrees with ω_π when restricted to $\widetilde{Z}^2$. Let $\Omega(\omega_\pi)$ be the set of genuine characters of $\widetilde{Z}$ whose restriction to $\widetilde{Z}^2$ agrees with ω_π. We also realize $\Omega(\omega_\pi)$ as a $\widetilde{Z}$-module, i.e., as direct sum of characters in $\Omega(\omega_\pi)$ with multiplicity one. From [Gelbart et al. 1979, Theorem 4.1], one knows that the multiplicity of a character $\mu \in \Omega(\omega_\pi)$ in the $\widetilde{Z}$-module $\pi_{N(E),\psi}$ is at most one. Hence $\pi_{N(E),\psi}$ is a $\widetilde{Z}$-submodule of $\Omega(\omega_\pi)$. Now we state the main result of this paper.

We abuse notation and write $\widetilde{\mathrm{GL}}_2(E)$ for $\widetilde{\mathrm{GL}}_2(E)_{\mathbb{C}^\times}$.

Theorem 1.5. *Let π_1 be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(E)$ and let π_2 be an infinite-dimensional irreducible admissible representation of $\mathrm{GL}_2(F)$. Assume that the central characters ω_{π_1} of π_1 and ω_{π_2} of π_2 agree on $E^{\times 2}\cap F^\times$. Fix a nontrivial additive character ψ of E such that $\psi|_F = 1$. Let $Q = (\pi_1)_{N(E)}$ be the Jacquet module of π_1. Assume that Working Hypothesis 1.3 holds.*

(A) *Let $\pi_2 = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\chi)$ be a principal series representation of $\mathrm{GL}_2(F)$. Assume $\mathrm{Hom}_{A(F)}(Q, \chi\cdot\delta^{1/2}) = 0$. Then*
$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) = \dim \mathrm{Hom}_{Z(F)}((\pi_1)_{N(E),\psi}, \omega_{\pi_2}).$$

(B) *Let $\pi_1 = \mathrm{Ind}_{\widetilde{B}(E)}^{\widetilde{\mathrm{GL}}_2(E)}(\tilde{\tau})$ be a principal series representation of $\widetilde{\mathrm{GL}}_2(E)$ and π_2 a discrete series representation of $\mathrm{GL}_2(F)$. Let π_2' be the finite-dimensional representation of $D_F^\times$ associated to π_2 by the Jacquet–Langlands correspondence. Assume that*
$$\mathrm{Hom}_{\mathrm{GL}_2(F)}\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\tilde{\tau}.\delta^{1/2}), \pi_2\big) = 0.$$
Then
$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) + \dim \mathrm{Hom}_{D_F^\times}(\pi_1, \pi_2') = \big[E^\times : F^\times E^{\times 2}\big].$$

(C) *Let π_1 be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(E)$ and π_2 a supercuspidal representation of $\mathrm{GL}_2(F)$. Let π_1' be a genuine representation of $\widetilde{\mathrm{GL}}_2(E)$ which has the same central character as that of π_1 and as a $\widetilde{Z}$-module $(\pi_1)_{N(E),\psi}\oplus(\pi_1')_{N(E),\psi} = \Omega(\omega_{\pi_1})$. Let π_2' be the finite-dimensional representation of $D_F^\times$ associated to π_2 by the Jacquet–Langlands correspondence. Then*
$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1\oplus\pi_1', \pi_2) + \dim \mathrm{Hom}_{D_F^\times}(\pi_1\oplus\pi_1', \pi_2') = [E^\times : F^\times E^{\times 2}].$$

The strategy to prove this theorem is similar to that in [Prasad 1992]. We recall it briefly. Part (A) of this theorem is proved by looking at the Kirillov model of an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(E)$ and its Jacquet module with respect to $N(F)$. Part (B) makes use of Mackey theory. For the third part (C), we use a trick of Prasad [1992], where we "transfer" the results of a principal series representation (from part (B)) to those which do not belong to principal series. Prasad transfers the results from principal series representations to discrete series representations. This is done by using character theory and an analog of a result of Casselman and Prasad [Prasad 1992, Theorem 2.7] for $\widetilde{\mathrm{GL}}_2(E)$ which we study in Section 4.

2. Part A of Theorem 1.5

Let $\pi_2 = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\chi)$ be a principal series representation of $\mathrm{GL}_2(F)$ where χ is a character of $A(F)$. By Frobenius reciprocity [Bernstein and Zelevinskii 1976, Theorem 2.28], we get

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1,\pi_2) &= \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\chi)) \\ &= \mathrm{Hom}_{A(F)}((\pi_1)_{N(F)}, \chi\,.\,\delta^{1/2}),\end{aligned}$$

where $(\pi_1)_{N(F)}$ is the Jacquet module of π_1 with respect to $N(F)$. We can describe $(\pi_1)_{N(F)}$ by realizing π_1 in the Kirillov model. Now depending on whether π_1 is a supercuspidal representation or not, we consider them separately.

2A. *Kirillov model and Jacquet module.* Now we describe the Kirillov model of an irreducible admissible genuine representation π of $\widetilde{\mathrm{GL}}_2(E)$. Let $l : \pi \to \pi_{N(E),\psi}$ be the canonical map. Let $\mathcal{C}^\infty(E^\times, \pi_{N(E),\psi})$ denote the space of smooth functions on $E^\times$ with values in $\pi_{N(E),\psi}$. Define the Kirillov mapping

$$\mathtt{K} : \pi \to \mathcal{C}^\infty(E^\times, \pi_{N(E),\psi})$$

given by $v \mapsto \xi_v$ where $\xi_v(x) = l\big(\pi\big(\big(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\big), 1\big)v\big)$. More conceptually, $\pi_{N(E),\psi}$ is a representation of $\widetilde{Z}\cdot N(E)$, and by Frobenius reciprocity, there exists a natural map

$$\pi|_{\widetilde{B}(E)} \to \mathrm{Ind}_{\widetilde{Z}\cdot N(E)}^{\widetilde{B}(E)} \pi_{N(E),\psi}.$$

Since $\widetilde{B}(E)/\widetilde{Z}\cdot N(E)$ can be identified with $E^\times$ sitting as $\big\{\big(\begin{smallmatrix} e & 0 \\ 0 & 1\end{smallmatrix}\big) : e \in E^\times\big\}$ in $\widetilde{B}(E)$, we get a map of $\widetilde{B}(E)$-modules:

$$\pi|_{\widetilde{B}(E)} \to \mathcal{C}^\infty(E^\times, \pi_{N(E),\psi}).$$

We summarize some of the properties of the Kirillov mapping in the following proposition.

Proposition 2.1. (1) *If $v' = \pi\big(\big(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\big), 1\big)v$ for $v \in \pi$ then*

$$\xi_{v'}(x) = (x,d)\psi(bd^{-1}x)\pi\left(\begin{pmatrix} d & 0 \\ 0 & d\end{pmatrix}, 1\right)\xi_v(ad^{-1}x).$$

(2) *For $v \in \pi$ the function ξ_v is a locally constant function on $E^\times$ which vanishes outside a compact subset of E.*

(3) *The map $\mathtt{K}$ is an injective linear map.*

(4) *The image $\mathtt{K}(\pi)$ of the map $\mathtt{K}$ contains the space $\mathcal{S}(E^\times, \pi_{N(E),\psi})$ of smooth functions on $E^\times$ with compact support with values in $\pi_{N(E),\psi}$.*

(5) *The Jacquet module $\pi_{N(E)}$ of π is isomorphic to $\mathtt{K}(\pi)/\mathcal{S}(E^\times, \pi_{N(E),\psi})$.*

(6) *The representation π is supercuspidal if and only if $\mathtt{K}(\pi) = \mathcal{S}(E^\times, \pi_{N(E),\psi})$.*

Proof. Part (1) follows from the definition. The proofs of parts (2) and (3) are verbatim those of Lemma 2 and Lemma 3 in [Godement 1970]. The proofs of parts (4), (5) and (6) follow from the proofs of the corresponding statements of [Prasad and Raghuram 2000, Theorem 3.1]. □

Since the map $\mathtt{K}$ is injective, we can transfer the action of $\widetilde{\mathrm{GL}}_2(E)$ on the space of π to $\mathtt{K}(\pi)$ using the map $\mathtt{K}$. The realization of the representation π on the space $\mathtt{K}(\pi)$ is called the Kirillov model, on which the action of $\widetilde{B}(E)$ is explicitly given by part (1) in Proposition 2.1. It is clear that $\mathcal{S}(E^\times, \pi_{N(E),\psi})$ is $\widetilde{B}(E)$-stable, which gives rise to the following short exact sequence of $\widetilde{B}(E)$-modules

$$0 \to \mathcal{S}(E^\times, \pi_{N(E),\psi}) \to \mathtt{K}(\pi) \to \pi_{N(E)} \to 0. \tag{1}$$

2B. *The Jacquet module with respect to $N(F)$.* In this section, we try to understand the restriction of an irreducible admissible genuine representation π of $\widetilde{\mathrm{GL}}_2(E)$ to $B(F)$. For this, we describe the Jacquet module $\pi_{N(F)}$ of π. We utilize the short exact sequence in equation (1) of $\widetilde{B}(E)$-modules arising from the Kirillov model of π, which is also a short exact sequence of $B(F)$-modules. By the exactness of the Jacquet functor with respect to $N(F)$, we get the following short exact sequence from equation (1),

$$0 \to \mathcal{S}(E^\times, \pi_{N(E),\psi})_{N(F)} \to \mathtt{K}(\pi)_{N(F)} \to \pi_{N(E)} \to 0.$$

Let us first describe $\mathcal{S}(E^\times, \pi_{N(E),\psi})_{N(F)}$, the Jacquet module of $\mathcal{S}(E^\times, \pi_{N(E),\psi})$ with respect to $N(F)$.

Proposition 2.2. *There exists an isomorphism*

$$\mathcal{S}(E^\times, \pi_{N(E),\psi})_{N(F)} \cong \mathcal{S}(F^\times, \pi_{N(E),\psi})$$

of $F^\times$-modules where $F^\times$ acts by its natural action on $\mathcal{S}(F^\times, \pi_{N(E),\psi})$.

Proposition 2.2 follows from the proposition below. The author thanks Professor Prasad for suggesting the proof.

Proposition 2.3. *Let ψ be a nontrivial additive character of E such that $\psi|_F = 1$. Let $\mathcal{S}(E^\times)$ be a representation of E where the action of E on $\mathcal{S}(E^\times)$ is given by*

$$(n \cdot f)(x) = \psi(nx) f(x)$$

for $n \in E$, $f \in \mathcal{S}(E^\times)$ and $x \in E^\times$. Then the restriction map

$$\mathcal{S}(E^\times) \longrightarrow \mathcal{S}(F^\times) \tag{2}$$

realizes $\mathcal{S}(E^\times)_F$ the maximal F-coinvariant quotient of $\mathcal{S}(E^\times)$ as $\mathcal{S}(F^\times)$.

Proof. Note that $\mathcal{S}(E^\times) \hookrightarrow \mathcal{S}(E)$. For a fixed Haar measure dw on E, we define the Fourier transform $\mathcal{F}_\psi : \mathcal{S}(E) \to \mathcal{S}(E)$ with respect to the character ψ by

$$\mathcal{F}_\psi(f)(z) := \int_E f(w)\psi(zw)\, dw.$$

As is well known, $\mathcal{F}_\psi : \mathcal{S}(E) \to \mathcal{S}(E)$ is an isomorphism of vector spaces, and the image of $\mathcal{S}(E^\times)$ can be identified with those functions in $\mathcal{S}(E)$ whose integral on

E is zero. The Fourier transform takes the action of E on $\mathcal{S}(E^\times)$ to the restriction of the action of E on $\mathcal{S}(E)$ given by $(n \cdot f)(x) = f(x+n)$ for $n \in E$, $f \in \mathcal{S}(E)$ and $x \in E$. Write $E = F(\sqrt{d})$ for a suitable $d \in F^\times$. Define $\phi : E \to F$ given by

$$\phi(e) = (e - \bar{e})/(2\sqrt{d}),$$

where $\bar{e}$ is the nontrivial Galois conjugate of $e \in E$, i.e., $\bar{e} = x - \sqrt{d}y$ for $e = x + \sqrt{d}y$ with $x, y \in F$. Clearly $\phi(z_1) = \phi(z_2)$ for $z_1, z_2 \in E$ if and only if $z_1 - z_2 \in F$. We define the integration along the fibers of the map $\phi : E \to F$, to be denoted by $I : \mathcal{S}(E) \to \mathcal{S}(F)$, as follows:

$$I(f)(y) := \int_F f(x + \sqrt{d}y)\, dx \quad \text{for all } y \in F.$$

Clearly $I(f)$ belongs to $\mathcal{S}(F)$. Note that the maximal quotient of $\mathcal{S}(E)$ on which F acts trivially (F acting by translation on $\mathcal{S}(E)$) can be identified with $\mathcal{S}(F)$ by integration along the fibers of the map ϕ. Since $\psi|_F = 1$, the restriction of the character $\psi_{\sqrt{d}}$ (given by $x \mapsto \psi(\sqrt{d}x)$ for $x \in E$) from E to F is a nontrivial character of F. The proposition will follow if we prove the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{S}(E) & \xrightarrow{\mathcal{F}_\psi} & \mathcal{S}(E) \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle I} \\ \mathcal{S}(F) & \xrightarrow{\mathcal{F}_{\psi_{\sqrt{d}}}} & \mathcal{S}(F) \end{array}$$

where $\mathcal{F}_\psi$ is the Fourier transform on $\mathcal{S}(E)$ with respect to the character ψ, $\mathcal{F}_{\psi_{\sqrt{d}}}$ is the Fourier transform on $\mathcal{S}(F)$ with respect to $\psi_{\sqrt{d}} = (\psi_{\sqrt{d}})|_F)$, Res denotes the restriction of functions from E to F, and I denotes the integration along the fibers mentioned above. Recall that $\mathcal{F}_{\psi_{\sqrt{d}}} : \mathcal{S}(F) \to \mathcal{S}(F)$ is defined by

$$\mathcal{F}_{\psi_{\sqrt{d}}}(\phi)(x) := \int_F \phi(y)\psi_{\sqrt{d}}(xy)\, dy = \int_F \phi(y)\psi(\sqrt{d}xy)\, dy \text{ for all } x \in F.$$

We prove that the above diagram is commutative. Let $f \in \mathcal{S}(E)$. We want to show that $I \circ \mathcal{F}_\psi(f)(y) = \mathcal{F}_{\psi_{\sqrt{d}}} \circ \mathrm{Res}(f)(y)$ for all $y \in F$. We write an element of E as $x + \sqrt{d}y$ with $x, y \in F$. We choose a measure dx on F which is self dual with respect to $\psi_{\sqrt{d}}$ in the sense that $\mathcal{F}_{\psi_{\sqrt{d}}}(\mathcal{F}_{\psi_{\sqrt{d}}}(\phi))(x) = \phi(-x)$ for all $\phi \in \mathcal{S}(F)$ and $x \in F$. We identify E with $F \times F$ as a vector space. Consider the product measure $dx\, dy$ on $E = F \times F$. Using Fubini's theorem we have

$$\int_F \int_F \phi(z_2)\psi_{\sqrt{d}}(xz_2)\, dz_2\, dx = \mathcal{F}_{\psi_{\sqrt{d}}}(\mathcal{F}_{\psi_{\sqrt{d}}}(\phi))(0) = \phi(0)$$

for $\phi \in \mathcal{S}(F)$. Therefore,

$$
\begin{aligned}
I\circ\mathcal{F}_\psi(f)(y) &= \int_F \mathcal{F}_\psi(f)\big(x+\sqrt{d}y\big)\,dx\\
&= \int_F\int_{E=F\times F} f\big(z_1+\sqrt{d}z_2\big)\psi\big(\big(x+\sqrt{d}y\big)\big(z_1+\sqrt{d}z_2\big)\big)\,dz_1\,dz_2\,dx\\
&= \int_F\int_F\int_F f\big(z_1+\sqrt{d}z_2\big)\psi_{\sqrt{d}}(yz_1+xz_2)\,dz_1\,dz_2\,dx\\
&= \int_F\bigg(\int_F\int_F f\big(z_1+\sqrt{d}z_2\big)\psi_{\sqrt{d}}(xz_2)\,dz_2\,dx\bigg)\psi_{\sqrt{d}}(yz_1)\,dz_1\\
&= \int_F f(z_1)\psi_{\sqrt{d}}(yz_1)\,dz_1 = \mathcal{F}_{\psi_{\sqrt{d}}}\circ\mathrm{Res}(f)(y).
\end{aligned}
$$

This proves the commutativity of the above diagram. □

2C. ***Completion of the proof of Part (A).*** First we consider the case when π_1 is a supercuspidal representation of $\widetilde{\mathrm{GL}}_2(E)$. Then one knows that the functions in the Kirillov model for π_1 have compact support in $E^\times$ and one has

$$\pi_1 \cong \mathcal{S}(E^\times, (\pi_1)_{N(E),\psi})$$

as $\widetilde{B}(E)$modules by Proposition 2.1. Now using Proposition 2.2, we get

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1,\pi_2) &= \mathrm{Hom}_{A(F)}\big((\pi_1)_{N(F)},\,\chi.\delta^{1/2}\big)\\
&= \mathrm{Hom}_{A(F)}\big(\mathcal{S}(E^\times,(\pi_1)_{N(E),\psi})_{N(F)},\,\chi.\delta^{1/2}\big)\\
&= \mathrm{Hom}_{A(F)}\big(\mathcal{S}(F^\times,(\pi_1)_{N(E),\psi}),\,\chi.\delta^{1/2}\big).
\end{aligned}
$$

Since $\mathcal{S}(F^\times,(\pi_1)_{N,\psi}) \cong \mathrm{ind}_{Z(F)}^{A(F)}(\pi_1)_{N(E),\psi}$ as $A(F)$-modules, by Frobenius reciprocity [Bernstein and Zelevinskii 1976, Proposition 2.29], we get

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1,\pi_2) &= \mathrm{Hom}_{A(F)}\big(\mathrm{ind}_{Z(F)}^{A(F)}(\pi_1)_{N(E),\psi},\,\chi.\delta^{1/2}\big)\\
&= \mathrm{Hom}_{Z(F)}\big((\pi_1)_{N(E),\psi},\,(\chi.\delta^{1/2})|_{Z(F)}\big)\\
&= \mathrm{Hom}_{Z(F)}((\pi_1)_{N(E),\psi},\,\omega_{\pi_2}).
\end{aligned}
$$

This proves part (A) of Theorem 1.5 for π_1 a supercuspidal representation.

Now we consider the case when π_1 is not a supercuspidal representation of $\widetilde{\mathrm{GL}}_2(E)$. Then from equation (1) we get the following short exact sequence of $A(F)$-modules:

$$0\to\mathcal{S}(F^\times,(\pi_1)_{N(E),\psi})\to(\pi_1)_{N(F)}\to Q\longrightarrow 0.$$

Now applying the functor $\mathrm{Hom}_{A(F)}\big(-,\,\chi.\delta^{1/2}\big)$, we get the long exact sequence

$$
\begin{aligned}
0\to\mathrm{Hom}_{A(F)}\big(Q,\chi.\delta^{1/2}\big)&\to\mathrm{Hom}_{A(F)}\big((\pi_1)_{N(F)},\chi.\delta^{1/2}\big)\\
&\to\mathrm{Hom}_{A(F)}\big(\mathcal{S}(F^\times,(\pi_1)_{N(E),\psi}),\chi.\delta^{1/2}\big)\to\mathrm{Ext}^1_{A(F)}\big(Q,\chi.\delta^{1/2}\big)\to\cdots
\end{aligned}
$$

Lemma 2.4. $\mathrm{Hom}_{A(F)}\big(Q, \chi.\delta^{1/2}\big) = 0$ *if and only if* $\mathrm{Ext}^1_{A(F)}\big(Q, \chi.\delta^{1/2}\big) = 0$.

Proof. The space Q is finite-dimensional and completely reducible. So it is enough to prove the lemma for the one-dimensional representations, i.e., for characters of $A(F)$. Moreover one can regard these representations as representations of $F^\times$ (after tensoring by a suitable character of $A(F)$ so that it descends to a representation of $A(F)/Z(F) \cong F^\times$). Then our lemma follows from the following lemma due to Prasad. $\square$

Lemma 2.5. *If* χ_1 *and* χ_2 *are two characters of* $F^\times$, *then*

$$\dim \mathrm{Hom}_{F^\times}(\chi_1, \chi_2) = \dim \mathrm{Ext}^1_{F^\times}(\chi_1, \chi_2).$$

Proof. Let $\mathcal{O}$ be the ring of integers of F and ϖ a uniformizer of F. Since $F^\times \cong \mathcal{O}^\times \times \varpi^{\mathbb{Z}}$ and $\mathcal{O}^\times$ is compact, $\mathrm{Ext}^i_{F^\times}(\chi_1, \chi_2) = H^i(\mathbb{Z}, \mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2))$. If $\mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2) = 0$, then the lemma is obvious. Hence suppose $\mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2) \neq 0$. Then $\mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2)$ is a certain one dimensional vector space with an action of $\varpi^{\mathbb{Z}}$. If the action of $\varpi^{\mathbb{Z}}$ on $\mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2)$ is nontrivial then $H^i(\mathbb{Z}, \mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2)) = 0$ for all $i \geq 0$. Whereas if the action of $\varpi^{\mathbb{Z}}$ on $\mathrm{Hom}_{\mathcal{O}^\times}(\chi_1, \chi_2)$ is trivial, then $H^0(\mathbb{Z}, \mathbb{C}) \cong H^1(\mathbb{Z}, \mathbb{C}) \cong \mathbb{C}$. $\square$

We have made an assumption that $\mathrm{Hom}_{A(F)}(Q, \chi.\delta^{1/2}) = 0$ and hence by the lemma above, $\mathrm{Ext}^1_{A(F)}(Q, \chi.\delta^{1/2}) = 0$. So in this case

$$\begin{aligned}\mathrm{Hom}_{A(F)}\big((\pi_1)_{N(F)}, \chi.\delta^{1/2}\big) &\cong \mathrm{Hom}_{A(F)}\big(\mathcal{S}(F^\times, (\pi_1)_{N(E),\psi}), \chi.\delta^{1/2}\big) \\ &= \mathrm{Hom}_{Z(F)}((\pi_1)_{N(E),\psi}, \omega_{\pi_2}).\end{aligned}$$

Hence

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) = \dim \mathrm{Hom}_{Z(F)}((\pi_1)_{N(E),\psi}, \omega_{\pi_2}).$$

Remark 2.6. Recall that $Q := (\pi_1)_{N(E)}$ is a finite-dimensional representation of $\tilde{A}(E)$ and we have assumed that $\mathrm{Hom}_{A(F)}(Q, \chi.\delta^{1/2}) = 0$. The number of characters χ of $A(F)$ for which $\mathrm{Hom}_{A(F)}(Q, \chi.\delta^{1/2}) \neq 0$ is at most the dimension of Q. The maximum possible dimension of Q is $2[E^\times : E^{\times 2}]$ (the maximum occurs only if π_1 is a principal series representation). Therefore for a given π_1 we leave out finitely many ($\leq 2[E^\times : E^{\times 2}]$) representations π_2 in our analysis.

3. Part B of Theorem 1.5

In this section, we consider the case when π_1 is a principal series representation of $\widetilde{\mathrm{GL}}_2(E)$ and π_2 a discrete series representation of $\mathrm{GL}_2(F)$.

Let $\pi_1 = \mathrm{Ind}_{\widetilde{B}(E)}^{\widetilde{\mathrm{GL}}_2(E)}(\tilde{\tau})$, where $(\tilde{\tau}, V)$ is a genuine irreducible representation of $\tilde{A} = \tilde{A}(E)$. The group $\tilde{A}$ sits in the central extension

$$1 \to A^2 \times \{\pm 1\} \to \tilde{A} \xrightarrow{p} A/A^2 \to 1,$$

where A/A^2 equals $E^\times/E^{\times 2} \times E^\times/E^{\times 2}$, and the commutator of two elements $\tilde{a}_1$ and $\tilde{a}_2$ of $\tilde{A}$ whose images in A/A^2 are $a_1 = (e_1, f_1)$ and $a_2 = (e_2, f_2)$, is

$$[\tilde{a}_1, \tilde{a}_2] = (e_1, f_2)(e_2, f_1) \in \{\pm 1\} \subset A^2 \times \{\pm 1\},$$

which is the product of Hilbert symbols (e_i, f_j) of E. Since the Hilbert symbol is a nondegenerate bilinear form on $E^\times/E^{\times 2}$, it follows that

$$[\tilde{a}_1, \tilde{a}_2] : A/A^2 \times A/A^2 \to \{\pm 1\}$$

is also a nondegenerate (skew-symmetric) bilinear form. Thus $\tilde{A}$ is closely related to the "usual Heisenberg" groups, and its representation theory is closely related to the representation theory of the "usual Heisenberg" groups. In particular, given a character $\chi : A^2 \times \{\pm 1\} \to \mathbb{C}^\times$ which is nontrivial on $\{\pm 1\}$, there exists a unique irreducible representation of $\tilde{A}$ which contains χ. Further, for any subgroup $A_0 \subset A/A^2$ for which the commutator map $[\tilde{a}_1, \tilde{a}_2]$, $a_i \in A_0$, is identically trivial, and for which A_0 is maximal for this property, $\tilde{A}_0 = p^{-1}(A_0)$ is a maximal abelian subgroup of $\tilde{A}$, and the restriction of an irreducible genuine representation $\tilde{\tau}$ of $\tilde{A}$ to $\tilde{A}_0$ contains all characters of $\tilde{A}_0$ with multiplicity one whose restriction to the center $A^2 \times \{\pm 1\}$ is the central character of $\tilde{\tau}$. Further, $\tilde{\tau} = \mathrm{Ind}_{\tilde{A}_0}^{\tilde{A}} \chi$ where χ is any character of $\tilde{A}_0$ appearing in $\tilde{\tau}$. All the assertions here are consequences of the fact that the inner conjugation action of $\tilde{A}$ on $\tilde{A}_0$ is transitive on the set of characters of $\tilde{A}_0$ with a given restriction on $A^2 \times \{\pm 1\}$; this itself is a consequence of the nondegeneracy of the Hilbert symbol.

It follows that the set of equivalence classes of irreducible genuine representations $\tilde{\tau}$ of $\tilde{A}$ is parametrized by the set of characters of A^2, i.e., a pair of characters of $E^{\times 2}$.

Lemma 3.1. *The subgroup $\widetilde{Z} \cdot A^2$ of $\tilde{A}$ is a maximal abelian subgroup. Let $\tilde{\tau}$ be an irreducible genuine representation of $\tilde{A}$. Then $\tilde{\tau}|_{\widetilde{Z}}$ contains all the genuine characters of $\widetilde{Z}$ which agree with the central character of τ when restricted to $\widetilde{Z}^2$.*

Proof. By explicit description of the commutation relation recalled above it is easy to see that $\widetilde{Z} \cdot A^2$ is a maximal abelian subgroup of $\tilde{A}$. The rest of the statements follow from preceding discussion. □

Proposition 3.2 [Gelbart and Piatetski-Shapiro 1980, Theorem 2.4]. *For some irreducible genuine representation $\tilde{\tau}$ of $\tilde{A}$, let $\pi_1 = \mathrm{Ind}_{\widetilde{B}(E)}^{\widetilde{\mathrm{GL}}_2(E)}(\tilde{\tau})$. Then*

$$(\pi_1)_{N,\psi} \cong \Omega(\pi_1) \cong \tilde{\tau}|_{\widetilde{Z}}.$$

Now as in [Prasad 1992], we use Mackey theory to understand its restriction to $\mathrm{GL}_2(F)$. We have $\widetilde{\mathrm{GL}}_2(E)/\widetilde{B}(E) \cong \mathbb{P}^1_E$ and this has two orbits under the left action of $\mathrm{GL}_2(F)$. One of the orbits is closed, and naturally identified with $\mathbb{P}^1_F \cong \mathrm{GL}_2(F)/B(F)$. The other orbit is open, and can be identified with $\mathbb{P}^1_E - \mathbb{P}^1_F \cong \mathrm{GL}_2(F)/E^\times$. By Mackey theory, we get this exact sequence of $\mathrm{GL}_2(F)$-modules:

$$0 \to \mathrm{ind}_{E^\times}^{\mathrm{GL}_2(F)}(\tilde{\tau}'|_{E^\times}) \to \pi_1 \to \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\tilde{\tau}|_{B(F)}\delta^{1/2}) \to 0, \tag{3}$$

where $\tilde{\tau}'|_{E^\times}$ is the representation of $E^\times$ obtained from the embedding $E^\times \hookrightarrow \tilde{A}$ which comes from conjugating the embedding $E^\times \hookrightarrow \mathrm{GL}_2(F) \hookrightarrow \widetilde{\mathrm{GL}}_2(E)$. We now identify $E^\times$ with its image inside $\tilde{A}$ which is given by $x \mapsto \left(\left(\begin{smallmatrix} x & 0 \\ 0 & \bar{x} \end{smallmatrix}\right), \epsilon(x)\right)$ where $\bar{x}$ is the nontrivial $\mathrm{Gal}(E/F)$-conjugate of x and $\epsilon(x) \in \{\pm 1\}$. Now let π_2 be any irreducible admissible representation of $\mathrm{GL}_2(F)$. By applying the functor $\mathrm{Hom}_{\mathrm{GL}_2(F)}(-, \pi_2)$ to the short exact sequence (3), we get the long exact sequence

$$
\begin{aligned}
&0 \to \mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\tilde{\tau}|_{B(F)}\delta^{1/2}\right), \pi_2\right) \\
&\quad\to \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{ind}_{E^\times}^{\mathrm{GL}_2(F)}(\tilde{\tau}'|_{E^\times}), \pi_2\right) \\
&\quad\to \mathrm{Ext}^1_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\tilde{\tau}|_{B(F)}\delta^{1/2}\right), \pi_2\right) \to \cdots
\end{aligned} \tag{4}
$$

From [Prasad 1990, Corollary 5.9], we know that

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\chi.\delta^{1/2}\right), \pi_2\right) = 0 \Leftrightarrow \mathrm{Ext}^1_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\chi.\delta^{1/2}\right), \pi_2\right) = 0.$$

Since $\tilde{\tau}|_{B(F)}$ factors through $T(F)$, which is direct sum of $[E^\times : E^{\times 2}]$ characters of $T(F)$, we can use the above result of Prasad with χ replaced by $\tilde{\tau}|_{B(F)}$. Then from the exactness of (4), it follows that

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) = 0$$

if and only if

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\tilde{\tau}|_{B(F)}\delta^{1/2}\right), \pi_2\right) = 0$$

and

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{ind}_{E^\times}^{\mathrm{GL}_2(F)}(\tilde{\tau}'|_{E^\times}), \pi_2\right) = 0.$$

Note that the representation $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\tilde{\tau}|_{B(F)}\delta^{1/2}\right)$ consists of exactly $[E^\times : E^{\times 2}]$ principal series representations of $\mathrm{GL}_2(F)$. Since we have made the assumption that $\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\tilde{\tau}.\delta^{1/2}\right), \pi_2\right) = 0$, it follows that

$$\mathrm{Ext}^1_{\mathrm{GL}_2(F)}\left(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}\left(\tilde{\tau}.\delta^{1/2}\right), \pi_2\right) = 0.$$

This gives

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) &\cong \mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\mathrm{ind}_{E^\times}^{\mathrm{GL}_2(F)}(\tilde{\tau}'|_{E^\times}), \pi_2\right) \\
&\cong \mathrm{Hom}_{E^\times}(\tilde{\tau}'|_{E^\times}, \pi_2|_{E^\times}).
\end{aligned}
$$

The following lemma describes $\tilde{\tau}'|_{E^\times}$.

Lemma 3.3. *If we identify $E^\times$ with its image $\left\{\left(\left(\begin{smallmatrix} x & 0 \\ 0 & \bar{x} \end{smallmatrix}\right), \epsilon(x)\right) \mid x \in E^\times\right\}$ inside $\tilde{A}$ as above then the subgroup $E^\times \cdot \tilde{A}^2$ inside $\tilde{A}$ is a maximal abelian subgroup. Moreover, $\tilde{\tau}'|_{E^\times}$ contains all the characters of $E^\times$ which are same as $\omega_{\tilde{\tau}}|_{E^{\times 2}}$ when restricted to $E^{\times 2}$, where $\omega_{\tilde{\tau}}$ is the central character of $\tilde{\tau}$.*

Proof. From the explicit cocycle description and the nondegeneracy of the quadratic Hilbert symbol, it is easy to verify that $E^\times \cdot \tilde{A}^2$ is a maximal abelian subgroup of $\tilde{A}$. The rest follows from the discussion preceding Lemma 3.1. $\square$

As π_2 is a discrete series representation, it is not always true (unlike what happens in case of a principal series representation) that any character of $E^\times$, whose restriction to $F^\times$ is the same as the central character of π_2, appears in π_2. Let π_2' be the finite dimensional representation of $D_F^\times$ associated to π_2 by the Jacquet–Langlands correspondence. Considering the left action of $D_F^\times$ on

$$\mathbb{P}_E^1 \cong \widetilde{\mathrm{GL}}_2(E)/\widetilde{B}(E)$$

induced by $D_F^\times \hookrightarrow \widetilde{\mathrm{GL}}_2(E)$ it is easy to verify that $\mathbb{P}_E^1 \cong D_F^\times/E^\times$. Then by Mackey theory, when restricted to $D_F^\times$, the principal series representation π_1 becomes isomorphic to $\mathrm{ind}_{E^\times}^{D_F^\times}(\tilde{\tau}'|_{E^\times})$. Therefore,

$$\begin{aligned}\mathrm{Hom}_{D_{F^\times}}(\pi_1, \pi_2') &\cong \mathrm{Hom}_{D_{F^\times}}\left(\mathrm{ind}_{E^\times}^{D_F^\times}(\tilde{\tau}'|_{E^\times}), \pi_2'\right)\\ &\cong \mathrm{Hom}_{E^\times}(\tilde{\tau}'|_{E^\times}, \pi_2'|_{E^\times}).\end{aligned}$$

In order to prove

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1, \pi_2) + \dim \mathrm{Hom}_{D_F^\times}(\pi_1, \pi_2') = [E^\times : F^\times E^{\times 2}] \tag{5}$$

we shall prove

$$\dim \mathrm{Hom}_{E^\times}(\tilde{\tau}'|_{E^\times}, \pi_2|_{E^\times}) + \dim \mathrm{Hom}_{E^\times}(\tilde{\tau}'|_{E^\times}, \pi_2'|_{E^\times}) = [E^\times : F^\times E^{\times 2}]. \tag{6}$$

By Remark 2.9 in [Prasad 1992], a character of $E^\times$ whose restriction to $F^\times$ is the same as the central character of π_2 appears either in π_2 with multiplicity one or in π_2' with multiplicity one, and exactly one of the two possibilities hold. Note that we are assuming that the two embeddings of $E^\times$, one via $\mathrm{GL}_2(F)$ and the other via $D_F^\times$ are conjugate in $\widetilde{\mathrm{GL}}_2(E)$. Then the left-hand side of equation (6) is the same as the number of characters of $E^\times$ appearing in $(\tilde{\tau}, V)$ which upon restriction to $F^\times$ coincide with the central character of π_2, which equals $\dim \mathrm{Hom}_{F^\times}(\tilde{\tau}|_{F^\times}, \omega_{\pi_2})$. We are reduced to the following lemma.

Lemma 3.4. *Let $(\tilde{\tau}, V)$ be an irreducible genuine representation of $\tilde{A}$ and let χ be a character of $Z(F) = F^\times$ such that $\chi|_{E^{\times 2} \cap F^\times} = \tilde{\tau}|_{E^{\times 2} \cap F^\times}$. Then*

$$\dim \mathrm{Hom}_{F^\times}(\tilde{\tau}, \chi) = [E^\times : F^\times E^{\times 2}].$$

Proof. Note that $E^{\times 2} \cap F^\times = Z^{\times 2} \cap F^\times$. From Proposition 3.2, $\tilde{\tau}|_{\widetilde{Z}} \cong \Omega(\omega_{\pi_1})$. If a character $\mu \in \Omega(\omega_{\pi_1})$ is specified on $F^\times$ then it is specified on $F^\times E^{\times 2}$. Therefore the number of characters in $\Omega(\omega_{\pi_1})$ which agree with χ when restricted to $F^\times$ is equal to $[E^\times : F^\times E^{\times 2}]$. $\square$

4. A theorem of Casselman and Prasad

As mentioned in the introduction, we use results of part (B) involving principal series representation and "transfer" these to the other cases, as stated in part (C) which involves restriction of the two representations. To make such a transfer possible Prasad used a result which says that if two irreducible representations of $\mathrm{GL}_2(E)$ have the same central characters then the difference of their characters is a smooth function on $\mathrm{GL}_2(E)$. We will need a similar theorem for $\widetilde{\mathrm{GL}}_2(E)$, which we prove in this section. In order to do this, we recall a variant of a theorem of Rodier which is true for covering groups in general; this variant is proved in [Patel 2015]. Let us first recall some facts about germ expansions, restricted only to $\widetilde{\mathrm{SL}}_2(E)$.

For any nonzero nilpotent orbit in $\mathfrak{sl}_2(E)$ there is a lower triangular nilpotent matrix $Y_a = \left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right)$ such that Y_a belongs to the nilpotent orbit. For a given nonzero nilpotent orbit, the element a is uniquely determined modulo $E^{\times 2}$. We write $\mathcal{N}_a$ for the nilpotent orbit which contains Y_a. Thus the set of all nonzero nilpotent orbits is $\{\mathcal{N}_a \mid a \in E^\times/E^{\times 2}\}$.

Let τ be an irreducible admissible genuine representation of $\widetilde{\mathrm{SL}}_2(E)$. Recall that for an irreducible admissible genuine representation τ of $\widetilde{\mathrm{SL}}_2(E)$, the character distribution Θ_τ is a smooth function on the set of regular semisimple elements. The Harish-Chandra–Howe character expansion of Θ_τ in a neighborhood of the identity is given as follows:

$$\Theta_\tau \circ \exp = c_0(\tau) + \sum_{a \in E^\times/E^{\times 2}} c_a(\tau) \cdot \hat{\mu}_{\mathcal{N}_a}$$

where $c_0(\tau), c_a(\tau)$ are constants and $\hat{\mu}_{\mathcal{N}_a}$ is the Fourier transform of a suitably chosen $\mathrm{SL}_2(E)$-invariant (under the adjoint action) measure on $\mathcal{N}_a$.

Fix a nontrivial additive character ψ of E. Define a character χ of N by $\chi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \psi(x)$. For $a \in E^\times$ we write ψ_a for the character of E given by $\psi_a(x) = \psi(ax)$. We write (N, ψ) for the nondegenerate Whittaker datum (N, χ). It can be seen that the set of conjugacy classes of nondegenerate Whittaker data has a set of representatives $\{(N, \psi_a) \mid a \in E^\times/E^{\times 2}\}$.

From the proof of the main theorem in [Patel 2015], the bijection between $\{\mathcal{N}_a \mid a \in E^\times/E^{\times 2}\}$ and $\{(N, \psi_a) \mid a \in E^\times/E^{\times 2}\}$ given by $\mathcal{N}_a \leftrightarrow (N, \psi_a)$ satisfies the following property: $c_a \neq 0$ if and only if the representation τ of $\widetilde{\mathrm{SL}}_2(E)$ admits a nonzero (N, ψ_a)-Whittaker functional.

It follows from [Gelbart et al. 1979, Theorem 4.1] that for any nontrivial additive character ψ' of N, the dimension of the space of (N, ψ')-Whittaker functionals for τ is at most one. Therefore, from the theorem of Rodier, as extended in [Patel 2015], each $c_a(\tau)$ is either 1 or 0 depending on whether τ admits a nonzero Whittaker functional corresponding to the nondegenerate Whittaker datum (N, ψ_a) or not.

Remark 4.1. Let $\widetilde{G}$ be a topological central extension of a connected reductive group G by μ_r, a cyclic group of order r. For $g \in \widetilde{G}$ there exists a semisimple element $g_s \in \widetilde{G}$ such that g belongs to any conjugation invariant neighborhood of $g_s \in \widetilde{G}$.

Let τ_1 and τ_2 be two irreducible admissible genuine representations of $\widetilde{\mathrm{SL}}_2(E)$. Note that $\widetilde{\{\pm 1\}}$ is the center of $\widetilde{\mathrm{SL}}_2(E)$ and these are the only nonregular semisimple elements of $\widetilde{\mathrm{SL}}_2(E)$. It is known that the character distributions Θ_{τ_1} and Θ_{τ_2} are given by smooth functions at regular semisimple elements. Therefore $\Theta_{\tau_1} - \Theta_{\tau_2}$ is also a smooth function at regular semisimple elements. For $i = 1, 2$, and any element $z \in \widetilde{\{\pm 1\}}$, the character expansion of τ_i in a neighborhood of z is given by the $\omega_{\tau_i}(z)$ multiplied by the character expansion of τ_i in a neighborhood of the identity. Therefore, if we know that $\Theta_{\tau_1} - \Theta_{\tau_2}$ is also smooth in a neighborhood of the identity and both the representations τ_1 and τ_2 have the same central characters then $\Theta_{\tau_1} - \Theta_{\tau_2}$ is a smooth function on the whole of $\widetilde{\mathrm{SL}}_2(E)$.

For any nontrivial additive character ψ' of E, let us assume that τ_1 admits a nonzero Whittaker functional for (N, ψ') if and only if τ_2 does so too. Under this assumption $c_a(\tau_1) = c_a(\tau_2)$ for all $a \in E^\times/E^{\times 2}$. Then we have the following result.

Theorem 4.2. *Let τ_1, τ_2 be two irreducible admissible genuine representations of $\widetilde{\mathrm{SL}}_2(E)$ with the same central characters. For a nontrivial additive character ψ' of E, assume that τ_1 admits a nonzero Whittaker functional with respect to (N, ψ') if and only if τ_2 admits a nonzero Whittaker functional with respect to (N, ψ'). Then $\Theta_{\tau_1} - \Theta_{\tau_2}$ is constant in a neighborhood of identity and hence extends to a smooth function on all of $\widetilde{\mathrm{SL}}_2(E)$.*

Using Theorem 4.2, we prove an extension of a theorem of Casselman and Prasad [Prasad 1992, Theorem 2.7]. From [Patel and Prasad 2016], let us recall the following lemma.

Lemma 4.3. *Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(E)$. Write $\widetilde{\mathrm{GL}}_2(E)_+ = \widetilde{Z} \cdot \widetilde{\mathrm{SL}}_2(E)$. Then there exists an irreducible admissible genuine representation τ of $\widetilde{\mathrm{SL}}_2(E)$ and a genuine character μ of $\widetilde{Z}$ with $\mu|_{\widetilde{\{\pm 1\}}} = \omega_\tau$ and*

$$\pi \cong \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)} \mu\tau.$$

Moreover, we have

$$\pi|_{\widetilde{\mathrm{GL}}_2(E)_+} \cong \bigoplus_{a \in \widetilde{\mathrm{GL}}_2(E)/\widetilde{\mathrm{GL}}_2(E)_+} \mu^a \tau^a.$$

Now we prove the theorem of Casselman and Prasad for the $\widetilde{\mathrm{GL}}_2(E)$.

Theorem 4.4. *Let ψ be a nontrivial character of E. Let π_1 and π_2 be two irreducible admissible genuine representations of $\widetilde{\mathrm{GL}}_2(E)$ with the same central* *characters such that* $(\pi_1)_{N,\psi} \cong (\pi_2)_{N,\psi}$ *as* $\widetilde{Z}$*-modules. Then* $\Theta_{\pi_1} - \Theta_{\pi_2}$*, initially defined on regular semisimple elements of* $\widetilde{\mathrm{GL}}_2(E)$*, extends to a smooth function on all of* $\widetilde{\mathrm{GL}}_2(E)$*.*

Proof. We know that Θ_{π_1} and Θ_{π_2} are smooth on the set of regular semisimple elements, so is $\Theta_{\pi_1} - \Theta_{\pi_2}$. To prove the smoothness of $\Theta_{\pi_1} - \Theta_{\pi_2}$ on all of $\widetilde{\mathrm{GL}}_2(E)$, we need to prove the smoothness at every point in $\widetilde{Z}$. As $\widetilde{Z}$ is not the center of $\widetilde{\mathrm{GL}}_2(E)$, the smoothness at the identity is not enough to imply the smoothness at every point in $\widetilde{Z}$. Note that $\widetilde{Z}$ is the center of $\widetilde{\mathrm{GL}}_2(E)_+ := \widetilde{Z} \cdot \widetilde{\mathrm{SL}}_2(E)$ and $\widetilde{\mathrm{GL}}_2(E)_+$ is an open and normal subgroup of $\widetilde{\mathrm{GL}}_2(E)$ of index $[E^\times : E^{\times 2}]$.

Using Lemma 4.3, choose irreducible admissible genuine representations τ_1 and τ_2 of $\widetilde{\mathrm{SL}}_2(E)$ and genuine characters μ_1, μ_2 of $\widetilde{Z}$ such that

$$\pi_1 = \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)}(\mu_1\tau_1) \quad \text{and} \quad \pi_2 = \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)}(\mu_2\tau_2). \tag{7}$$

From Lemma 4.3, we have

$$\pi_1|_{\widetilde{\mathrm{GL}}_2(E)_+} = \bigoplus_{a \in E^\times/E^{\times 2}} (\mu_1\tau_1)^a \quad \text{and} \quad \pi_2|_{\widetilde{\mathrm{GL}}_2(E)_+} = \bigoplus_{a \in E^\times/E^{\times 2}} (\mu_2\tau_2)^a. \tag{8}$$

We also know that all the characters μ_1^a for $a \in E^\times/E^{\times 2}$ are distinct. From the identity (8) we find that

$$(\pi_1)_{N(E),\psi} = \bigoplus_{a \in E^\times/E^{\times 2}} \mu_1^a(\tau_1^a)_{N(E),\psi} \text{ and } (\pi_2)_{N(E),\psi} = \bigoplus_{a \in E^\times/E^{\times 2}} \mu_2^a(\tau_2^a)_{N(E),\psi}. \tag{9}$$

Since $(\pi_1)_{N,\psi} \cong (\pi_2)_{N,\psi}$ as $\widetilde{Z}$-modules, in particular, the parts corresponding to μ^a-eigenspaces are isomorphic for all $a \in E^\times/E^{\times 2}$. Therefore $\mu_1 = \mu_2^b$ for some $b \in E^\times/E^{\times 2}$. Since

$$\pi_2 = \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)}(\mu_2\tau_2) = \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)}(\mu_2^b\tau_2^b),$$

by changing τ_2 by τ_2^b, we can assume $\pi_1 = \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)}(\mu\tau_1)$, and $\pi_2 = \mathrm{ind}_{\widetilde{\mathrm{GL}}_2(E)_+}^{\widetilde{\mathrm{GL}}_2(E)}(\mu\tau_2)$. Now $(\pi_1)_{N(E),\psi} \cong (\pi_2)_{N(E),\psi}$ as $\widetilde{Z}$-modules translates into $(\tau_1^a)_{N(E),\psi} \cong (\tau_2^a)_{N(E),\psi}$ for all $a \in E^\times/E^{\times 2}$. Therefore, by Theorem 4.2, $\Theta_{\tau_1^a} - \Theta_{\tau_2^a}$ is constant in a neighborhood of the identity for all $a \in E^\times/E^{\times 2}$.

Let $\Theta_{\rho,g}$ denote the character expansion of an irreducible admissible representation ρ in a neighborhood of the point g. Then

$$\Theta_{\pi_1,\tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \Theta_{(\mu\tau_1)^a,\,\tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \mu^a(\tilde{z})\Theta_{\tau_1^a,1}$$

and

$$\Theta_{\pi_2,\tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \Theta_{(\mu\tau_2)^a,\,\tilde{z}} = \sum_{a \in E^\times/E^{\times 2}} \mu^a(\tilde{z})\Theta_{\tau_2^a,1}.$$

This proves that $\Theta_{\pi_1} - \Theta_{\pi_2}$ is a constant function on regular semisimple points in some neighborhood of $\tilde{z}$ for all $\tilde{z} \in \widetilde{Z} \subset \widetilde{\mathrm{GL}}_2(E)$, and therefore it extends to a smooth function in that neighborhood of $\tilde{z}$. Thus $\Theta_{\pi_1} - \Theta_{\pi_2}$, which is initially defined on regular semisimple elements of $\widetilde{\mathrm{GL}}_2(E)$, extends to a smooth function on all of $\widetilde{\mathrm{GL}}_2(E)$. □

Corollary 4.5. *Let π_1, π_2 be two irreducible admissible genuine representations of $\widetilde{\mathrm{GL}}_2(E)$ with the same central character such that $(\pi_1)_{N,\psi} \cong (\pi_2)_{N,\psi}$ as $\widetilde{Z}$-modules. Let H be a subgroup of $\widetilde{\mathrm{GL}}_2(E)$ that is compact modulo center. Then there exist finite-dimensional representations σ_1, σ_2 of H such that*

$$\pi_1|_H \oplus \sigma_1 \cong \pi_2|_H \oplus \sigma_2.$$

In other words, this corollary says that the virtual representation $(\pi_1 - \pi_2)|_H$ is finite-dimensional and hence the multiplicity of an irreducible representation of H in $(\pi_1 - \pi_2)|_H$ will be finite.

5. Part C of Theorem 1.5

Let π_1 be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}}_2(E)$. We take another admissible genuine representation π_1' having the same central character as that of π_1 and satisfying $(\pi_1)_{N(E),\psi} \oplus (\pi_1')_{N(E),\psi} \cong \Omega(\omega_{\pi_1})$ as $\widetilde{Z}$-modules. From Proposition 3.2, if π_1 is a principal series representation then we can take $\pi_1' = 0$. It can be seen that if π_1 is not a principal series representation then $(\pi_1)_{N(E),\psi}$ is a proper $\widetilde{Z}$-submodule of $\Omega(\omega_{\pi_1})$ forcing $\pi_1' \neq 0$. In particular, if π_1 is one of the Jordan–Hölder factors of a reducible principal series representation then one can take π_1' to be the other Jordan–Hölder factor of the principal series representation. It should be noted that for a supercuspidal representation π_1 we do not have any obvious choice for π_1'.

Let π_2 be a supercuspidal representation of $\mathrm{GL}_2(F)$. To prove Theorem 1.5 in this case, we use character theory and deduce the result by using the result of restriction of a principal series representation of $\widetilde{\mathrm{GL}}_2(E)$ which has already been proved in Section 3. We can assume, if necessary after twisting by a character of $F^\times$, that π_2 is a minimal representation. Recall that an irreducible representation π_2 of $\mathrm{GL}_2(F)$ is called minimal if the conductor of π_2 is less than or equal to the conductor of $\pi_2 \otimes \chi$ for any character χ of $F^\times$. By a theorem of Kutzko [1978], a minimal supercuspidal representation π_2 of $\mathrm{GL}_2(F)$ is of the form $\mathrm{ind}_{\mathcal{K}}^{\mathrm{GL}_2(F)}(W_2)$, where W_2 is a representation of a maximal compact modulo center subgroup $\mathcal{K}$ of $\mathrm{GL}_2(F)$. By Frobenius reciprocity,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1 \oplus \pi_1', \pi_2) &= \mathrm{Hom}_{\mathrm{GL}_2(F)}\big(\pi_1 \oplus \pi_1', \mathrm{ind}_{\mathcal{K}}^{\mathrm{GL}_2(F)}(W_2)\big)\\ &= \mathrm{Hom}_{\mathcal{K}}\big((\pi_1 \oplus \pi_1')|_{\mathcal{K}}, W_2\big).\end{aligned}$$

To prove Theorem 1.5, it suffices to prove that

$$\dim \mathrm{Hom}_{\mathcal{K}}((\pi_1 \oplus \pi_1')|_{\mathcal{K}}, W_2) + \dim \mathrm{Hom}_{D_F^\times}(\pi_1 \oplus \pi_1', \pi_2') = [E^\times : F^\times E^{\times 2}].$$

For any (virtual) representation π of $\widetilde{\mathrm{GL}}_2(E)$, let $m(\pi, W_2) = \dim \mathrm{Hom}_{\mathcal{K}}[\pi|_{\mathcal{K}}, W_2]$ and $m(\pi, \pi_2') = \dim \mathrm{Hom}_{D_F^\times}[\pi, \pi_2']$. With these notations we will prove

$$m(\pi_1 \oplus \pi_1', W_2) + m(\pi_1 \oplus \pi_1', \pi_2') = [E^\times : F^\times E^{\times 2}]. \tag{10}$$

Let Ps be an irreducible principal series representation of $\widetilde{\mathrm{GL}}_2(E)$ whose central character ω_{Ps} is the same as the central character ω_{π_1} of π_1 (it is clear that one exists). By Proposition 3.2, we know that $(Ps)_{N(E),\psi} \cong \Omega(\omega_{Ps})$ as a $\widetilde{Z}$-module. On the other hand, the representation π_1' has been chosen in such a way that $(\pi_1)_{N(E),\psi} \oplus (\pi_1')_{N(E),\psi} = \Omega(\omega_{\pi_1})$ as a $\widetilde{Z}$-module. Then, as a $\widetilde{Z}$-module we have

$$(\pi_1 \oplus \pi_1')_{N(E),\psi} = (\pi_1)_{N(E),\psi} \oplus (\pi_1')_{N(E),\psi} = \Omega(\omega_{\pi_1}) = \Omega(\omega_{Ps}) = (Ps)_{N(E),\psi}.$$

We have already proved in Section 3 that

$$m(Ps, W_2) + m(Ps, \pi_2') = [E^\times : F^\times E^{\times 2}].$$

In order to prove equation (10), we prove

$$m(\pi_1 \oplus \pi_1' - Ps, W_2) + m(\pi_1 \oplus \pi_1' - Ps, \pi_2') = 0. \tag{11}$$

The relation in equation (11) follows from the following theorem:

Theorem 5.1. *Let Π_1, Π_2 be two genuine representations of $\widetilde{\mathrm{GL}}_2(E)$ of finite length, having the same central characters, and such that $(\Pi_1)_{N(E),\psi} \cong (\Pi_2)_{N(E),\psi}$ as $\widetilde{Z}$-modules. Let π_2 be an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ such that the central characters ω_{Π_1} of Π_1 and ω_{π_2} of π_2 agree on $F^\times \cap E^{\times 2}$. Let π_2' be the finite-dimensional representation of $D_F^\times$ associated to π_2 by the Jacquet–Langlands correspondence. Then*

$$m(\Pi_1 - \Pi_2, \pi_2) + m(\Pi_1 - \Pi_2, \pi_2') = 0.$$

We will use character theory to prove this relation, following [Prasad 1992] very closely. First of all, by Theorem 4.4, $\Theta_{\Pi_1 - \Pi_2}$ is given by a smooth function on $\widetilde{\mathrm{GL}}_2(E)$. Now we recall the Weyl integration formula for $\mathrm{GL}_2(F)$.

5A. *Weyl integration formula.*

Lemma 5.2 [Jacquet and Langlands 1970, Formula 7.2.2]. *For a smooth and compactly supported function f on $\mathrm{GL}_2(F)$ we have*

$$\int_{\mathrm{GL}_2(F)} f(y)dy = \sum_{E_i} \int_{E_i} \triangle(x) \left(\frac{1}{2} \int_{E_i \backslash \mathrm{GL}_2(F)} f(\bar{g}^{-1} x \bar{g})\, d\bar{g} \right) dx, \tag{12}$$

where the E_i are representatives for the distinct conjugacy classes of maximal tori in $\mathrm{GL}_2(F)$ and

$$\triangle(x) = \left\| \frac{(x_1 - x_2)^2}{x_1 x_2} \right\|_F$$

where x_1 and x_2 are the eigenvalues of x.

We will use this formula to integrate the function $f(x) = \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{W_2}(x)$ on $\mathcal{K}$ which is extended to $\mathrm{GL}_2(F)$ by setting it to be zero outside $\mathcal{K}$. In addition, we also need the following result of Harish-Chandra, cf. [Prasad 1992, Proposition 4.3.2].

Lemma 5.3 (Harish-Chandra). *Let $F(g) = (gv, v)$ be a matrix coefficient of a supercuspidal representation π of a reductive p-adic group G with center Z. Then the orbital integrals of F at regular nonelliptic elements vanish. Moreover, the orbital integral of F at a regular elliptic element x contained in a torus T is given by the formula*

$$\int_{T\backslash G} F(\bar{g}^{-1}x\bar{g})d\bar{g} = \frac{(v, v)\cdot\Theta_\pi(x)}{d(\pi)\cdot \mathrm{vol}(T/Z)}, \tag{13}$$

where $d(\pi)$ denotes the formal degree of the representation π (which depends on a choice of Haar measure on $T\backslash G$).

Since π_2 is obtained by induction from W_2, a matrix coefficient of W_2 (extended to $\mathrm{GL}_2(F)$ by setting it to be zero outside $\mathcal{K}$) is also a matrix coefficient of π_2. It follows that

(1) for the choice of Haar measure on $\mathrm{GL}_2(F)/F^\times$ giving $\mathcal{K}/F^\times$ measure 1,
$$\dim W_2 = d(\pi_2),$$

(2) for a separable quadratic field extension E_i of F and a regular elliptic element x of $\mathrm{GL}_2(E)$ which generates E_i, and for the above Haar measure $d\bar{g}$,

$$\int_{E_i^\times\backslash \mathrm{GL}_2(F)} \Theta_{W_2}(\bar{g}^{-1}x\bar{g})d\bar{g} = \frac{\Theta_{\pi_2}(x)}{\mathrm{vol}(E_i^\times/F^\times)}. \tag{14}$$

Equation (14) can be explained as follows. Let the dimension of W_2 be n and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of W_2. For $g \in \mathcal{K}$ the map $g \mapsto F_i(g) := \langle ge_i, e_i\rangle$ defines a matrix coefficient of W_2 for all $i = 1, \dots, n$. Then $\Theta_{W_2}(g) = \sum_{i=1}^n F_i(g)$. Now consider all these F_i as matrix coefficients of π_2. Apply Lemma 5.3 for $F = F_i$ and sum up over all $i = 1, \dots, n$ then we get equation (14), since $d(\pi_2) = \dim W_2 = n$.

5B. ***Completion of the proof of Theorem 1.5.*** We recall the following important observations from Section 5A and Theorem 4.4:

(1) The virtual representation $(\Pi_1 - \Pi_2)|_{\mathcal{K}}$ is finite-dimensional.

(2) Θ_{W_2} is a matrix coefficient of π_2 (extended to $\mathrm{GL}_2(F)$ by zero outside $\mathcal{K}$).

(3) There is Haar measure on $\mathrm{GL}_2(F)/F^\times$ giving $\mathrm{vol}(\mathcal{K}/F^\times) = 1$ such that the (14) is satisfied.

(4) The orbital integral in equation (13) vanishes if T is the maximal split torus.

Let the E_i's be the quadratic extensions of F. Then these observations together with Lemma 5.3 imply the following

$$\begin{aligned}
&m(\Pi_1 - \Pi_2, W_2) \\
&\qquad = \frac{1}{\mathrm{vol}(\mathcal{K}/F^\times)} \int\limits_{\mathcal{K}/F^\times} \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{W_2}(x)\, dx \\
&\qquad = \frac{1}{\mathrm{vol}(\mathcal{K}/F^\times)} \int\limits_{\mathrm{GL}_2(F)/F^\times} \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{W_2}(x)\, dx \\
&\qquad = \frac{1}{\mathrm{vol}(\mathcal{K}/F^\times)} \sum_{E_i} \int\limits_{E_i^\times/F^\times} \triangle(x) \Bigg[\frac{1}{2} \int\limits_{E_i^\times \backslash \mathrm{GL}_2(F)} \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{W_2}(\bar{g}^{-1} x \bar{g})\, d\bar{g} \Bigg] dx \\
&\qquad = \sum_{E_i} \frac{1}{2\,\mathrm{vol}(E_i^\times/F^\times)} \int\limits_{E_i^\times/F^\times} (\triangle \cdot \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{\pi_2})(x)\, dx.
\end{aligned}$$

Similarly, we have the equality

$$m(\Pi_1 - \Pi_2, \pi_2') = \sum_{E_i} \frac{1}{2\,\mathrm{vol}(E_i^\times/F^\times)} \int\limits_{E_i^\times/F^\times} (\triangle \cdot \Theta_{\Pi_1-\Pi_2} \cdot \Theta_{\pi_2'})(x) dx.$$

Note that the E_i's correspond to quadratic extensions of F and the embeddings of $\mathrm{GL}_2(F)$ and $D_F^\times$ have been fixed so that Working Hypothesis 1.3 (as stated in the introduction) is satisfied, i.e., the embeddings of the E_i in $\mathrm{GL}_2(F)$ and in $D_F^\times$ are conjugate in $\widetilde{\mathrm{GL}}_2(E)$. Then the value of $\Theta_{\Pi_1-\Pi_2}(x)$ for $x \in E_i$, does not depend on the inclusion of E_i inside $\widetilde{\mathrm{GL}}_2(E)$, i.e., on whether the inclusion is via $\mathrm{GL}_2(F)$ or via $D_F^\times$. Now using the relation $\Theta_{\pi_2}(x) = -\Theta_{\pi_2'}(x)$ on regular elliptic elements x [Jacquet and Langlands 1970, Proposition 15.5], we conclude the following, which proves equation (11):

$$m(\Pi_1 - \Pi_2, W_2) + m(\Pi_1 - \Pi_2, \pi_2') = 0.$$

6. A remark on higher multiplicity

We have shown that the restriction of an irreducible admissible representation of $\widetilde{\mathrm{GL}}_2(E)$, for example a principal series representation, to the subgroup $\mathrm{GL}_2(F)$ has multiplicity more than one. Given the important role multiplicity one theorems play, it would be desirable to modify the situation so that multiplicity one might be true. One natural way to do this is to decrease the larger group, and increase the smaller group. In this section we discuss some natural subgroups of the group $\widetilde{\mathrm{GL}}_2(E)$ which can be used, but unfortunately, it still does not help one to achieve multiplicity one situation. We discuss this modification in this section in some detail.

Let us take the subgroup of $\widetilde{\mathrm{GL}}_2(E)$ which is generated by $\mathrm{GL}_2(F)$ and $\widetilde{Z}$. We will prove that this subgroup also fails to achieve multiplicity one for the restriction problem from $\widetilde{\mathrm{GL}}_2(E)$ to $\mathrm{GL}_2(F)\cdot\widetilde{Z}$. Let

$$H=\mathrm{GL}_2(F)\subset H_+=Z\cdot\mathrm{GL}_2(F)\subset\mathrm{GL}_2(E).$$

We will show that the restriction of an irreducible admissible representation of $\widetilde{\mathrm{GL}}_2(E)$ to the subgroup $\widetilde{H}_+$ has higher multiplicity. Note that the subgroups $\widetilde{Z}$ and $\mathrm{GL}_2(F)$ do not commute but $\widetilde{Z}^2$ commutes with $\mathrm{GL}_2(F)$. In fact, the commutator relation is given by

$$[\tilde{e},\tilde{g}]=(e,\det g)_E\in\{\pm1\}\subset\widetilde{\mathrm{GL}}_2(E),\tag{15}$$

where $\tilde{e}\in\widetilde{Z}$ and $\tilde{g}\in\widetilde{\mathrm{GL}}_2(F)$ lie over elements $e\in Z$ and $g\in\mathrm{GL}_2(F)$ respectively, and $(-,-)_E$ denotes the Hilbert symbol for the field E. The lemma below proves that the center of $\widetilde{H}_+$ is $\widetilde{Z^2F^\times}$.

Lemma 6.1. *For an element $e\in E^\times$, the map $F^\times\to\{\pm1\}$ defined by $f\mapsto(e,f)_E$ is trivial if and only if $e\in F^\times E^{\times2}$.*

Proof. Let $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_F$ denote the Hilbert symbol of the field E and F respectively. For $e\in E^\times$ and $f\in F^\times$, the following is well known [Bender 1973]:

$$(e,f)_E=(N_{E/F}(e),f)_F,$$

where $N_{E/F}$ is the norm map of the extension E/F. Therefore, if $(e,f)_E=1$ is true for all $f\in F^\times$, then by the nondegeneracy of the Hilbert symbol $(\cdot,\cdot)_F$ one will have $N_{E/F}(e)\in F^{\times2}$. The inverse image of $F^{\times2}$ under the norm map $N_{E/F}$ is now seen to be $E^{\times2}F^\times$ since this subgroup surjects onto $F^{\times2}$ under the norm mapping, and contains the kernel $\{z/\bar{z}=z^2/z\bar{z}:z\in E^\times\}$ of $N_{E/F}$. □

Let σ be an irreducible admissible representation of $\mathrm{GL}_2(F)$. For any character χ of $F^\times$ let us abuse the notation and simply write $\sigma\otimes\chi$ for $\sigma\otimes(\chi\circ\det)$. By the commutator relation (15), for $a\in Z$ and $g\in\mathrm{GL}_2(F)$ we have

$$a(g,\epsilon)a^{-1}=(g,\chi_a(\det g)\epsilon),$$

where χ_a is given by $x\mapsto(x,a)_E$ for all $x\in E^\times$. Therefore, the conjugation action by $a\in Z$ takes σ to the quadratic twist $\sigma\otimes\chi_a$. We have the following lemma which easily follows from Clifford theory.

Lemma 6.2. *Let $\widetilde{H}_0=\widetilde{Z}^2\cdot\mathrm{GL}_2(F)$. Let σ be an irreducible admissible representation of $\mathrm{GL}_2(F)$. Assume that $\sigma\otimes\chi_a\not\cong\sigma$ for any nontrivial element $a\in E^\times/F^\times E^{\times2}$. Fix a genuine character η of $\widetilde{Z}^2$ such that $\eta|_{F^\times\cap\widetilde{Z}^2}=\omega_\sigma|_{F^\times\cap\widetilde{Z}^2}$. Then $\rho=\mathrm{Ind}_{\widetilde{H}_0}^{\widetilde{H}_+}(\eta\sigma)$ is an irreducible representation of $\widetilde{H}_+$. The representation ρ is the only irreducible representation of $\widetilde{H}_+$ whose central character restricted* *to* $\widetilde{Z}^2$ *is* η *and also contains* σ. *Moreover,* $\rho|_{\widetilde{H}_0} \cong \bigoplus_{a\in E^\times/F^\times E^{\times 2}} \eta(\sigma\otimes\chi_a)$. *In particular, from Lemma 6.1, the restriction of* ρ *to* $\widetilde{H}_0$ *is multiplicity free.*

Note that if σ is a principal series representation of $\mathrm{GL}_2(F)$ which is not of the form $Ps(\chi_1,\chi_2)$ with χ_1/χ_2 a quadratic character, then such principal series representation of $\mathrm{GL}_2(F)$ have no nontrivial self twist, i.e., for any character χ of $F^\times$ the relation $Ps(\chi_1,\chi_2)\otimes(\chi\circ\det)\cong Ps(\chi_1,\chi_2)$ implies that χ is trivial. Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}_2(E)}$ such that $\dim \mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi,\sigma)\geq 2$. Let η be the central character of π. Note that the central character of any irreducible representation of $\widetilde{H}_+$, which is contained in π, agrees with η when restricted to $\widetilde{Z}^2$. As in the previous lemma, we let $\rho=\mathrm{Ind}_{\widetilde{H}_0}^{\widetilde{H}_+}(\eta\sigma)$. The representation ρ is the only representation of $\widetilde{H}_+$ which appears in π and contains σ. So the multiplicity of such a principal series representation σ of $\mathrm{GL}_2(F)$ in the restriction of an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}_2(E)}$ is the same as the multiplicity of the corresponding irreducible representation of $\widetilde{H}_+$, i.e., $\dim\mathrm{Hom}_{\widetilde{H}_+}(\pi,\rho)=\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi,\sigma)\geq 2$. Thus we conclude that the restriction of representations of $\widetilde{\mathrm{GL}_2(E)}$ to $\widetilde{H}_+$ has higher multiplicity.

On the other hand, let us take the group $G=\{g\in\mathrm{GL}_2(E):\det g\in F^\times E^{\times 2}\}$. Note that this subgroup G contains $\mathrm{GL}_2(E)_+=Z\cdot\mathrm{SL}_2(E)$. We will prove that the pair $(\widetilde{G},\mathrm{GL}_2(F))$ also fails to achieve multiplicity one for the restriction problem from $\widetilde{G}$ to $\mathrm{GL}_2(F)$. From the commutation relation (15), it follows that the center of the group $\widetilde{G}$ is $\widetilde{F^\times Z^2}$. Recall that the restriction from $\widetilde{\mathrm{GL}_2(E)}$ to $\widetilde{\mathrm{GL}_2(E)}_+$ is multiplicity free and $\widetilde{G}\supset\widetilde{\mathrm{GL}_2(E)}_+$, thus the restriction from $\widetilde{\mathrm{GL}_2(E)}$ to $\widetilde{G}$ is also multiplicity free. Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}_2(E)}$ and ρ be an irreducible admissible genuine representation of $\widetilde{G}$ such that $\rho\hookrightarrow\pi|_{\widetilde{G}}$. Then we have

$$\pi|_{\widetilde{G}}=\bigoplus_{a\in E^\times/F^\times E^{\times 2}}\rho^a.$$

For $a_1\neq a_2$ in $E^\times/F^\times E^{\times 2}$, $\rho^{a_1}\not\cong\rho^{a_2}$. In fact, the central characters of ρ^{a_1} and ρ^{a_2} are different when restricted to $F^\times$.

Let π be an irreducible admissible genuine representation of $\widetilde{\mathrm{GL}_2(E)}$ and σ an irreducible admissible representation of $\mathrm{GL}_2(F)$ such that

$$\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi,\sigma)\geq 2.$$

If $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\rho^{a_1},\sigma)\neq 0$ then $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\rho^{a_2},\sigma)=0$ for $a_2\neq a_1$ in $E^\times/F^\times E^{\times 2}$, since the central character of ρ^{a_2} restricted to $F^\times$ will be different from the central character of σ. Thus there exists only one $a\in E^\times/F^\times E^{\times 2}$ such that $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\rho^a,\sigma)\neq 0$. We can assume that $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\rho,\sigma)\neq 0$. We have

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(\rho,\sigma)=\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi,\sigma)$$

and hence $\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\rho,\sigma)\geq 2$.

Acknowledgements. The author would like to express his indebtedness to Professor Dipendra Prasad. Without his continuous encouragement and help this paper would not have appeared in the present form. The author is thankful to Professor Sandeep Varma for several helpful discussions, and the anonymous referee for several helpful comments.

References

[Bender 1973] E. A. Bender, "A lifting formula for the Hilbert symbol", *Proc. Amer. Math. Soc.* **40** (1973), 63–65. MR Zbl

[Bernstein and Zelevinskii 1976] I. N. Bernstein and A. V. Zelevinskii, "Representations of the group $GL(n, F)$, where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3 (1976), 5–70. In Russian; translated in *Russian Math. Surv.* **31**:3 (1976), 1–68. MR Zbl

[Gelbart and Piatetski-Shapiro 1980] S. Gelbart and I. I. Piatetski-Shapiro, "Distinguished representations and modular forms of half-integral weight", *Invent. Math.* **59**:2 (1980), 145–188. MR Zbl

[Gelbart et al. 1979] S. Gelbart, R. Howe, and I. Piatetski-Shapiro, "Uniqueness and existence of Whittaker models for the metaplectic group", *Israel J. Math.* **34**:1-2 (1979), 21–37. MR Zbl

[Godement 1970] R. Godement, *Notes on Jacquet–Langlands' theory*, Inst. Advanced Study, Princeton, 1970.

[Gross and Prasad 1992] B. H. Gross and D. Prasad, "On the decomposition of a representation of SO_n when restricted to SO_{n-1}", *Canad. J. Math.* **44**:5 (1992), 974–1002. MR Zbl

[Jacquet and Langlands 1970] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math. **114**, Springer, Berlin, 1970. MR Zbl

[Kubota 1969] T. Kubota, *On automorphic functions and the reciprocity law in a number field*, Lectures in Math., Kyoto Univ. **2**, Kinokuniya, Tokyo, 1969. MR Zbl

[Kutzko 1978] P. C. Kutzko, "On the supercuspidal representations of GL_2", *Amer. J. Math.* **100**:1 (1978), 43–60. MR Zbl

[Mœglin and Waldspurger 2012] C. Mœglin and J.-L. Waldspurger, "La conjecture locale de Gross–Prasad pour les groupes spéciaux orthogonaux: le cas général", pp. 167–216 in *Sur les conjectures de Gross et Prasad, II*, Astérisque **347**, Société Mathématique de France, Paris, 2012. MR Zbl

[Patel 2015] S. P. Patel, "A theorem of Mœglin and Waldspurger for covering groups", *Pacific J. Math.* **273**:1 (2015), 225–239. MR Zbl

[Patel 2016] S. P. Patel, "A question on splitting of metaplectic covers", *Comm. Algebra* **44**:12 (2016), 5095–5104. MR Zbl

[Patel and Prasad 2016] S. P. Patel and D. Prasad, "Multiplicity formula for restriction of representations of $\widetilde{GL_2}(F)$ to $\widetilde{SL_2}(F)$", *Proc. Amer. Math. Soc.* **144**:2 (2016), 903–908. MR Zbl

[Patel and Prasad 2017] S. P. Patel and D. Prasad, "Restriction of representations of metaplectic GL_2 to tori", preprint, 2017. To appear in *Israel J. Math.* arXiv

[Prasad 1990] D. Prasad, "Trilinear forms for representations of GL(2) and local ϵ-factors", *Compositio Math.* **75**:1 (1990), 1–46. MR Zbl

[Prasad 1992] D. Prasad, "Invariant forms for representations of GL_2 over a local field", *Amer. J. Math.* **114**:6 (1992), 1317–1363. MR Zbl

[Prasad and Raghuram 2000] D. Prasad and A. Raghuram, "Kirillov theory for $GL_2(\mathfrak{D})$ where $\mathfrak{D}$ is a division algebra over a non-Archimedean local field", *Duke Math. J.* **104**:1 (2000), 19–44. MR Zbl

[Saito 1993] H. Saito, "On Tunnell's formula for characters of GL(2)", *Compositio Math.* **85**:1 (1993), 99–108. MR Zbl

[Tunnell 1983] J. B. Tunnell, "Local ϵ-factors and characters of GL(2)", *Amer. J. Math.* **105**:6 (1983), 1277–1307. MR Zbl

Received September 21, 2016. Revised May 9, 2017.

SHIV PRAKASH PATEL
DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY OF THE NEGEV
P.O.B. 653
8410501
ISRAEL
shivprakashpatel@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 291 No. 2 December 2017