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We solve the complex Hessian equation on closed Hermitian manifolds, which generalizes the Kähler case proven by Hou, Ma and Wu and Dinew and Kołodziej. Solving the equation can be reduced to the derivation of a priori second-order estimates. We introduce a new method to prove the C^0 estimate. The C^2 estimate can be derived if we use the auxiliary function which is mainly due to Hou, Ma and Wu and Tosatti and Weinkove.

1. Introduction

Let (M, ω) be a closed Hermitian manifold of complex dimension $n \geq 2$. In this paper, we study the Hessian equation

$$\begin{cases} \binom{n}{k}\omega_u^k \wedge \omega^{n-k} = e^f \omega^n, \\ \sup_M u = 0, \\ \omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u \in \Gamma_k(M), \end{cases} \tag{1-1}$$

where $\binom{n}{k} = n!/(k!(n-k)!)$, $\Gamma_k(M)$ is a convex cone (see (2-2) in Section 2) and $1 \leq k \leq n$.

The complex Hessian equation is an important class of fully nonlinear elliptic equations. It arises naturally from many significant geometric problems. When $k = 1$, it is the classical Laplacian equation. For $k = n$, equation (1-1) is the complex Monge–Ampère equation

$$\omega_u^n = e^f \omega^n, \quad \sup_M u = 0. \tag{1-2}$$

Yau [1978] solved equation (1-2) on compact Kähler manifolds, and his solution is now known as Calabi–Yau theorem. For general Hermitian manifolds, (1-2) has been solved by Cherrier [1987] for dimension 2. Guan and Li [2010] and Zhang [2010] obtained C^1 and C^2 estimates for dimension $n \geq 2$. Finally, Tosatti and Weinkove [2010] derived the C^0 estimate and thus solved (1-2) for arbitrary dimension.

While $1 < k < n$, equation (1-1) has more complicated structure and also is closely related to many important geometric problems. For example, for $k = 2$, it

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relates to Fu and Yau's [2008] generalization of the Strominger system which comes from superstring theory. Several significant results about the Fu–Yau equation have been obtained by Phong, Picard and Zhang [Phong et al. 2016a; 2016b; 2017]. When $k = n-1$, it has similar features to the Monge–Ampère type equation in the study of Gauduchon conjecture by Tosatti and Weinkove [2015; 2017] and Tosatti, Weinkove and Székelyhidi [Székelyhidi et al. 2015].

We now come back to the complex Hessian equation. To solve it, it is crucial to derive the *a priori* estimates up to second-order. If (M, ω) is a Kähler manifold, Hou, Ma and Wu [Hou et al. 2010] proved

$$\max |\partial\bar{\partial}u|_g \le C(1+\max|\nabla u|_g^2), \tag{1-3}$$

where C does not depend on the gradient bound of the solution.

They also pointed out that (1-3) may be adapted to the blow up analysis to get the gradient estimate. Later on, combining (1-3) with a blow up argument, Dinew and Kołodziej [2017] obtained the gradient estimate. Then equation (1-1) can be solved on Kähler manifolds.

In this paper, we solve the complex Hessian equation on closed Hermitian manifolds. More precisely,

Theorem 1.1. *Let (M, g) be a closed Hermitian manifold of complex dimension $n \ge 2$ and f be a smooth real function on M. Then there exist a unique real number b and a unique smooth real function u on M solving*

$$\binom{n}{k}\omega_u^k \wedge \omega^{n-k} = e^{f+b}\omega^n, \quad \omega_u \in \Gamma_k(M), \quad \sup_M u = 0. \tag{1-4}$$

We use the continuity method to solve problem (1-4). The openness follows from implicit function theory. The closeness argument can be reduced to *a priori* estimates up to the second order by the standard Evans–Krylov theory. Actually, we can derive the zero-order estimate and the second-order estimate of solutions of equation (1-1) and thus use the blow up method to obtain the gradient estimate.

For the complex Monge–Ampère equation on closed Hermitian manifolds, Tosatti and Weinkove [2010] derived C^0 estimate by proving a Cherrier-type inequality which was originally proved in [Cherrier 1987]. For the Hessian equation (1-1), we can prove a similar Cherrier-type inequality by a new method which combines an inductive argument with key inequalities for k-th elementary symmetric functions in [Chou and Wang 2001]. For the C^2 estimate, the main difficulty is that there are new terms of the form $T * D^3u$, where T is the torsion of ω. To control these terms, we use the auxiliary function due to Tosatti and Weinkove [2013]. The auxiliary function originally comes from Hou, Ma and Wu [Hou et al. 2010]. For the Hessian equation, the main difference is that for equation (1-1) we need to apply some lemmas for the k-th elementary symmetric functions which were proved by Hou, Ma and Wu [Hou et al. 2010].

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, a Cherrier-type inequality is derived, and then we obtain the C^0 estimate. In Section 4, we prove the C^2 estimate by a similar auxiliary function used in [Tosatti and Weinkove 2013].

Székelyhidi [2015] has also obtained similar results, but our methods are different.

2. Preliminaries

Let (M, g) be a closed Hermitian manifold and let ∇ denote the Chern connection of g. In this section we give some preliminaries about the k-th elementary symmetric function and the commutation formula of covariant derivatives.

Elementary symmetric functions. The k-th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Let $\lambda(a_{i\bar{j}})$ denote the eigenvalues of the Hermitian matrix $\{a_{i\bar{j}}\}$; we define

$$\sigma_k(a_{i\bar{j}}) = \sigma_k(\lambda\{a_{i\bar{j}}\}).$$

The definition of σ_k can be naturally extended to a Hermitian manifold. Indeed, let $A^{1,1}(M, \mathbb{R})$ be the space of smooth real $(1, 1)$-forms on M; for $\chi \in A^{1,1}(M, \mathbb{R})$ we define

$$\sigma_k(\chi) = \binom{n}{k} \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}.$$

Definition 2.1.

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0,\ j = 1, \ldots, k\}. \tag{2-1}$$

Similarly, we define Γ_k on M as follows

$$\Gamma_k(M) := \{\chi \in A^{1,1}(M, \mathbb{R}) : \sigma_j(\chi) > 0,\ j = 1, \ldots, k\}. \tag{2-2}$$

Furthermore, $\sigma_r(\lambda | i_1 \cdots i_l)$, with $i_1, \ldots, i_l$ being distinct, denotes the r-th symmetric function with $\lambda_{i_1} = \cdots = \lambda_{i_l} = 0$. For more details about elementary symmetric functions, one can see the lecture notes [Wang 2009].

To prove the C^0 estimate, we need the following lemma for elementary symmetric functions:

Lemma 2.2. *Suppose that $\lambda \in \Gamma_k$, $3 \le k \le n$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then there exists a positive constant C depending only on k and n, such that for $1 \le i \le k-2$,*

$$|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}| \le (n-3)^{k-2}\sigma_i(\lambda|j), \quad 1 \le j_1 < j_2 < \cdots < j_i \le n, \quad j_l \ne j, 1 \le l \le i, 1 \le j \le n. \tag{2-3}$$

Proof. Since

$$\sum_{p=k}^{n} \lambda_p = \sigma_1(\lambda|12\cdots k-1) > 0, \quad \text{and} \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n,$$

then

$$|\lambda_p| \le (n-k)\lambda_k, k+1 \le p \le n. \tag{2-4}$$

We first prove the lemma for $k=3$. In this case, one needs to prove

$$|\lambda_l| \le C\sigma_1(\lambda|j) \quad \text{for } 1 \le j, l \le n \text{ and } l \ne j.$$

Since $\sigma_1(\lambda|j) = \lambda_l + \sigma_1(\lambda|jl)$, $\lambda_l \le \sigma_1(\lambda|j)$. Now, if $\lambda_l < 0$, then $l \ge 4$. By (2-4),

$$|\lambda_l| \le (n-3)\lambda_3 \le \sigma_1(\lambda|j), 4 \le l \le n.$$

Then the lemma follows for $k=3$.

Next we prove the lemma for the general k, $3 \le k \le n$.

If $j > i$, since $i \le k-2$, $\lambda|j \in \Gamma_{i+1}$, applying [Lin and Trudinger 1994, p. 322, (19)] yields $\sigma_i(\lambda|j) \ge \lambda_1 \cdots \lambda_i$. Since $1 \le l \le i \le k-2$, by (2-4) we have

$$|\lambda_{j_l}| \le \max\{\lambda_l, (n-k)\lambda_k\} \le (n-k)\lambda_l.$$

Then

$$|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}| \le (n-k)^i\lambda_1\cdots\lambda_i \le (n-k)^{k-2}\sigma_i(\lambda|j). \tag{2-5}$$

If $j \le i$, applying [Lin and Trudinger 1994, p. 322, (19)] yields

$$\sigma_i(\lambda|j) \ge \lambda_1\cdots\lambda_{j-1}\lambda_{j+1}\cdots\lambda_{i+1}.$$

Note $j_l \ne j$, so

$$|\lambda_{j_l}| \le \begin{cases} (n-3)\lambda_l, & j_l < j, \\ (n-3)\lambda_{l+1}, & j_l > j. \end{cases}$$

Therefore, we have

$$|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}| \le (n-k)^i\lambda_1\cdots\lambda_{j-1}\lambda_{j+1}\cdots\lambda_{i+1} \le (n-k)^{k-2}\sigma_i(\lambda|j). \tag{2-6}$$

Combining (2-5) and (2-6), we obtain

$$|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}| \le (n-3)^{k-2}\sigma_i(\lambda|j), \quad 1 \le j_1 < j_2 < \cdots < j_i \le n, \quad j_l \ne j, \quad 1 \le l \le i, \quad 1 \le j \le n. \qquad \square$$

By Lemma 2.2, we immediately obtain the following lemma which is a key ingredient in proving Lemma 3.2:

Lemma 2.3. *There exists a positive constant C depending only on (M, ω) and n such that*

$$\left|\frac{\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{i-1}\wedge T_i}{\omega^n}\right|\le C\frac{\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{i}\wedge\omega^{n-i-1}}{\omega^n}, \tag{2-7}$$

where T_i is defined as the combinations of ω, $\partial\omega$, $\partial\bar\partial\omega$; more precisely,

$$T_i=\sum_{0\le 3p+2q\le n-i}\omega^{n-i-3p-2q}\wedge(\sqrt{-1})^p(\partial\omega)^p\wedge(\bar\partial\omega)^p\wedge(\sqrt{-1})^q(\partial\bar\partial\omega)^q$$

for $1\le i\le k-1$.

Proof. For $x\in M$, we choose the coordinates such that

$$\omega(x)=\sqrt{-1}\sum_{j=1}^n dz^j\wedge d\bar z^j,\quad \omega_u(x)=\sqrt{-1}\sum_{j=1}^n\lambda_j dz^j\wedge d\bar z^j,$$

and $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$. Write T_i as follows:

$$T_i=(\sqrt{-1})^{n-i}(T_i)_{l_1\cdots l_{n-i},\bar m_1,\cdots\bar m_{n-i}}dz^{l_1}\wedge\cdots\wedge dz^{l_{n-i}}\wedge d\bar z^{m_1}\wedge\cdots\wedge d\bar z^{m_{n-i}}.$$

Then

$$\begin{aligned}\left|\frac{\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{i}\wedge T_i}{\omega^n}\right|&\le C\sum_{j,l=1}^n\sum_{1\le j_1<\cdots<j_i\le n,\ne j,l}|u_j||u_{\bar l}||\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}|\\&\le C\sum_{j=1}^n\sum_{\substack{1\le j_1<\cdots<j_i\le n\\ j_l\ne j,1\le l\le i}}|u_j|^2|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}|\\&\le C\sum_{j=1}^n\sigma_i(\lambda|j)|u_j|^2\\&=C\frac{\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{i}\wedge\omega^{n-i-1}}{\omega^n},\end{aligned}\tag{2-8}$$

where we have used Lemma 2.2 in the last inequality and C depends on the bound of the torsion and the curvature of (M, ω). □

Commutation formula of covariant derivatives. We have, in local complex coordinates $z_1, \ldots, z_n$,

$$g_{i\bar j}=g\Big(\frac{\partial}{\partial z^i},\frac{\partial}{\partial\bar z^j}\Big),\quad \{g^{i\bar j}\}=\{g_{i\bar j}\}^{-1} \tag{2-9}$$

For the Chern connection ∇, we denote the covariant derivatives

$$u_i=\nabla_{\partial/\partial z^i}u,\quad u_{i\bar j}=\nabla_{\partial/\partial\bar z^j}\nabla_{\partial/\partial z^i}u,\quad u_{i\bar jk}=\nabla_{\partial/\partial z^k}\nabla_{\partial/\partial\bar z^j}\nabla_{\partial/\partial z^i}u. \tag{2-10}$$

We use commutation formulas for covariant derivatives on Hermitian manifolds which can be found in [Tosatti and Weinkove 2013]:

(2-11) $$u_{i\bar{j}l} = u_{l\bar{j}i} - T_{li}^{p} u_{p\bar{j}}, \quad u_{pi\bar{j}} = u_{p\bar{j}i} + u_q R_{i\bar{j}p}{}^{q}, \quad u_{i\bar{p}\bar{j}} = u_{i\bar{j}\bar{p}} - \overline{T_{jp}^{q}} u_{i\bar{q}}.$$

(2-12) $$u_{i\bar{j}l\bar{m}} = u_{l\bar{m}i\bar{j}} + u_{p\bar{j}} R_{l\bar{m}i}{}^{p} - u_{p\bar{m}} R_{i\bar{j}l}{}^{p} - T_{li}^{p} u_{p\bar{m}\bar{j}} - \overline{T_{mj}^{p}} u_{l\bar{p}i} - T_{li}^{p} \overline{T_{mj}^{q}} u_{p\bar{q}}.$$

3. Zero-order estimate

In this section we derive the zero-order estimate by proving a Cherrier-type inequality and the lemmas in [Tosatti and Weinkove 2010]. Since the constant b in Theorem 1.1 satisfies

$$|b| \leq \sup |f| + C,$$

where C is a positive constant depending only on (M, ω). Thus, we will assume $b = 0$ for convenience.

Theorem 3.1. *Let u be a solution of Theorem 1.1. Then there exists a constant C depending only on (M, ω), n, k and $\sup_M |f|$ such that*

$$\sup_M |u| \leq C.$$

Due to Tosatti and Weinkove's results, finding the zero-order estimate can be reduced to deriving a Cherrier-type inequality which was first proved by Cherrier [1987]. For the Hessian equation, we use a new method which combines an inductive argument with the key Lemma 2.3. Even for the Monge–Ampère equation, our proof is different from that in [Tosatti and Weinkove 2010].

Lemma 3.2. *There exist constants p_0 and C depending only on (M, ω), n, k and $\sup_M |f|$ such that for any $p \geq p_0$*

$$\int_M |\partial e^{-(p/2)u}|_g^2 \omega^n \leq Cp \int_M e^{-pu} \omega^n.$$

Remark 3.3. Recently, applying our key Lemma 2.2, Sun [2017] also proved the lemma above.

Proof. By the equation, we have

$$\omega_u^k \wedge \omega^{n-k} - \omega^n = \left(\frac{e^f}{\binom{n}{k}} - 1 \right) \omega^n \leq C_0 \omega^n,$$

where C_0 is a constant depending only on $\sup f$, n and k. On the other hand,

(3-1) $$\omega_u^k \wedge \omega^{n-k} - \omega^n = (\omega_u^k - \omega^k) \wedge \omega^{n-k} = \sqrt{-1}\partial\bar{\partial} u \wedge \alpha,$$

where $\alpha = \sum_{i=1}^{k} \omega_u^{i-1} \wedge \omega^{n-i}$.

Now multiply both sides in (3-1) by e^{-pu} and integrate by parts:

$$
\begin{aligned}
C_0\int_M e^{-pu}\omega^n &\ge \int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\alpha \\
&= -\int_M \partial e^{-pu}\sqrt{-1}\bar\partial u\wedge\alpha + \int_M e^{-pu}\sqrt{-1}\bar\partial u\wedge\partial\alpha \\
&= p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\alpha - \frac{1}{p}\int_M \sqrt{-1}\bar\partial e^{-pu}\wedge\partial\alpha \\
&= p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\alpha + \frac{1}{p}\int_M e^{-pu}\sqrt{-1}\bar\partial\partial\alpha \\
&:= A+B,
\end{aligned}
\tag{3-2}
$$

where we denote

$$
A = p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\left(\sum_{i=1}^{k}\omega_u^{i-1}\wedge\omega^{n-i}\right),\qquad B=\frac{1}{p}\int_M e^{-pu}\sqrt{-1}\bar\partial\partial\alpha.
$$

Our goal is to use term A to control term B. Direct calculation gives

$$
\partial\alpha = n\sum_{i=1}^{k-1}\omega_u^{i-1}\wedge\omega^{n-i-1}\wedge\partial\omega + (n-k)\omega_u^{k-1}\wedge\omega^{n-k-1}\wedge\partial\omega,
$$

and

$$
\begin{aligned}
\bar\partial\partial\alpha &= (n-k)(n-k-1)\omega_u^{k-1}\wedge\omega^{n-k-2}\wedge\bar\partial\omega\wedge\partial\omega + (n-k)\omega_u^{k-1}\wedge\omega^{n-k-1}\wedge\bar\partial\partial\omega \\
&\quad + (n-k)(n+k-1)\omega_u^{k-2}\wedge\omega^{n-k-1}\wedge\bar\partial\omega\wedge\partial\omega \\
&\quad + n(n-1)\sum_{i=0}^{k-3}\omega_u^{i}\wedge\omega^{n-i-3}\wedge\bar\partial\omega\wedge\partial\omega + n\sum_{i=0}^{k-2}\omega_u^{i}\wedge\omega^{n-i-2}\wedge\bar\partial\partial\omega.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
B = \frac{(n-k)(n-k-1)}{p}&\int_M e^{-pu}\omega_u^{k-1}\wedge\omega^{n-k-2}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
&+\frac{(n-k)}{p}\int_M e^{-pu}\omega_u^{k-1}\wedge\omega^{n-k-1}\wedge\sqrt{-1}\bar\partial\partial\omega \\
&\quad+\frac{(n+k-1)(n-k)}{p}\int_M e^{-pu}\omega_u^{k-2}\wedge\omega^{n-k-1}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
&\quad\quad+\frac{n(n-1)}{p}\sum_{i=0}^{k-3}\int_M e^{-pu}\omega_u^{i}\wedge\omega^{n-i-3}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
&\quad\quad\quad+\frac{n}{p}\sum_{i=0}^{k-2}\int_M e^{-pu}\omega_u^{i}\wedge\omega^{n-i-2}\wedge\sqrt{-1}\bar\partial\partial\omega.
\end{aligned}
$$

When $k=2$, term B just becomes

$$
\begin{aligned}
B = {} & \frac{(n-2)(n-3)}{p}\int_M e^{-pu}\omega_u\wedge\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& + \frac{(n-2)}{p}\int_M e^{-pu}\omega_u\wedge\omega^{n-3}\wedge\sqrt{-1}\bar\partial\partial\omega \\
& + \frac{(n+1)(n-2)}{p}\int_M e^{-pu}\omega^{n-3}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
= {} & \frac{(n-2)(n-3)}{p}\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& + \frac{(n-2)}{p}\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\omega^{n-3}\wedge\sqrt{-1}\bar\partial\partial\omega \\
& + \frac{2(n-1)(n-2)}{p}\int_M e^{-pu}\omega^{n-3}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& + \frac{(n-2)}{p}\int_M e^{-pu}\omega^{n-2}\wedge\sqrt{-1}\bar\partial\partial\omega \\
\geq {} & \frac{(n-2)(n-3)}{p}\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& + \frac{(n-2)}{p}\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\omega^{n-3}\wedge\sqrt{-1}\bar\partial\partial\omega - \frac{C_1}{p}\int_M e^{-pu}\omega^n.
\end{aligned}
\tag{3-3}
$$

We next use integration by parts to deal with the first term and second term on the right-hand side of the above equality. Indeed,

$$
\begin{aligned}
& \int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& = p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& \quad + \int_M e^{-pu}\sqrt{-1}\bar\partial u\wedge\sqrt{-1}\partial(\omega^{n-4}\wedge\bar\partial\omega\wedge\partial\omega) \\
& = p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega \\
& \quad + \frac{1}{p}\int_M e^{-pu}\sqrt{-1}\bar\partial\partial(\omega^{n-4}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial\omega) \\
& \geq -pC_1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^{n-1} - \frac{C_1}{p}\int_M e^{-pu}\omega^n \\
& \geq -C_1A - \frac{C_1}{p}\int_M e^{-pu}\omega^n.
\end{aligned}
\tag{3-4}
$$

A similar calculation gives

$$\int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}u\wedge\omega^{n-3}\wedge\sqrt{-1}\bar{\partial}\partial\omega \geq -C_1 A-\frac{C_1}{p}\int_M e^{-pu}\omega^n. \tag{3-5}$$

Inserting (3-4) and (3-5) into (3-3), we have

$$B\geq -\frac{C_1}{p}A-\frac{C_1}{p}\int_M e^{-pu}\omega^n.$$

By (3-2) and choosing $p_0=2C_1+1$, we obtain for $p\geq p_0$

$$\frac{A}{2}\leq\Big(1-\frac{C_1}{p}\Big)A\leq\Big(\frac{C_1}{p}+C_0\Big)\int_M e^{-pu}\omega^n\leq (C_0+1)\int_M e^{-pu}\omega^n.$$

By (3-7) below, we thus prove the lemma.

For the general k, $3\leq k\leq n$, we claim that there exist constants C_{1i} depending only on n, k and (M,ω) such that the following holds for $0\leq i\leq k-1$:

$$\begin{aligned}\int_M e^{-pu}\omega_u^i\wedge T_i\\ \geq -pC_{1i}\sum_{j=0}^{k-2}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega_u^j\wedge\omega^{n-j-1}-C_{1i}\int_M e^{-pu}\omega^n,\end{aligned} \tag{3-6}$$

where T_i is defined as the combinations of ω, $\partial\omega$, $\partial\bar{\partial}\omega$; more precisely

$$T_i=\sum_{0\leq 3p+2q\leq n-i}\omega^{n-i-3p-2q}\wedge(\sqrt{-1})^p(\partial\omega)^p\wedge(\bar{\partial}\omega)^p\wedge(\sqrt{-1})^q(\partial\bar{\partial}\omega)^q.$$

We use the claim (3-6) to prove the lemma:

$$\begin{aligned}B&\geq -C_1\sum_{i=2}^{k}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega_u^{k-i}\wedge\omega^{n+i-k-1}-\frac{C_1}{p}\int_M e^{-pu}\omega^n\\ &\geq -\frac{C_1}{p}A-\frac{C_1}{p}\int_M e^{-pu}\omega^n.\end{aligned}$$

Thus we have

$$\Big(1-\frac{C_1}{p}\Big)A\leq\Big(\frac{C_1}{p}+C_0\Big)\int_M e^{-pu}\omega^n.$$

Now we choose $p_0=2C_1+1$, then for any $p\geq p_0$,

$$p^2\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^{n-1}\leq 2p(C_0+1)\int_M e^{-pu}\omega^n.$$

Therefore we have

$$\int_M |\partial e^{-(p/2)u}|_g^2 \omega^n = \frac{np^2}{4}\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \omega^{n-1} \le \frac{np(C_0+1)}{2}\int_M e^{-pu}\omega^n = pC\int_M e^{-pu}\omega^n. \tag{3-7}$$

Now, we prove the claim (3-6) by an inductive argument. When $i=1$,

$$\begin{aligned}\int_M e^{-pu}\omega_u \wedge T_1 &= \int_M e^{-pu}\omega \wedge T_1 + \int_M e^{-pu}\sqrt{-1}\partial\bar{\partial} u \wedge T_1\\ &= \int_M e^{-pu}\omega \wedge T_1 - \int_M \partial e^{-pu} \wedge \sqrt{-1}\bar{\partial} u \wedge T_1 + \int_M e^{-pu}\sqrt{-1}\bar{\partial} u \wedge \partial T_1\\ &= \int_M e^{-pu}\omega \wedge T_1 + p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge T_1 - \frac{1}{p}\int_M \sqrt{-1}\bar{\partial} e^{-pu} \wedge \partial T_1\\ &= p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge T_1 + \int_M e^{-pu}\omega \wedge T_1 - \frac{1}{p}\int_M e^{-pu} \wedge \sqrt{-1}\partial\bar{\partial} T_1\\ &\ge -C_1 p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \omega^{n-1} - C_1\int_M e^{-pu}\omega^n.\end{aligned}$$

Suppose that the claim is true for $l \le i-1$; we will prove that the claim is also true for $l=i$. Indeed,

$$\begin{aligned}\int_M e^{-pu}\omega_u^i \wedge T_i &= \int_M e^{-pu}\omega_u^{i-1} \wedge \omega \wedge T_i + \int_M e^{-pu}\sqrt{-1}\partial\bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i\\ &= \int_M e^{-pu}\omega_u^{i-1} \wedge \omega \wedge T_i + p\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \omega_u^{i-1} \wedge T_i\\ &\qquad + \int_M e^{-pu}\bar{\partial} u \wedge \sqrt{-1}\partial(\omega_u^{i-1} \wedge T_i)\\ &:= A_{i,1} + A_{i,2} + A_{i,3}.\end{aligned}$$

By the induction,

$$\begin{aligned}A_{i,1} &= \int_M e^{-pu}\omega_u^{i-1} \wedge \omega \wedge T_i\\ &\ge -pC_{1i}(n,k,\omega)\sum_{j=0}^{k-2}\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \omega_u^j \wedge \omega^{n-j-1}\\ &\qquad - C_{1i}(n,k,\omega)\int_M e^{-pu}\omega^n.\end{aligned}$$

By the inequality (2-7) in Lemma 2.3, we have

$$
\begin{aligned}
A_{i,2} &= p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{i-1}\wedge T_i \\
&\ge -pC_{2i}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{i-1}\wedge\omega^{n-i}.
\end{aligned}
\tag{3-8}
$$

Now we deal with the term $A_{i,3}$:

$$
\begin{aligned}
A_{i,3} &= \int_M e^{-pu}\bar\partial u\wedge\sqrt{-1}\partial(\omega_u^{i-1}\wedge T_i) \\
&= \frac{1}{p}\int_M e^{-pu}\sqrt{-1}\bar\partial\partial(\omega_u^{i-1}\wedge T_i) \\
&= \frac{(i-1)(i-2)}{p}\int_M e^{-pu}\sqrt{-1}\omega_u^{i-3}\wedge\bar\partial\omega\wedge\partial\omega\wedge T_i \\
&\quad + \frac{i-1}{p}\int_M e^{-pu}\omega_u^{i-2}\wedge\sqrt{-1}\bar\partial(\partial\omega\wedge T_i) + \frac{i-1}{p}\int_M e^{-pu}\omega_u^{i-2}\wedge\sqrt{-1}\bar\partial\omega\wedge\partial T_i \\
&\quad - \frac{1}{p}\int_M e^{-pu}\omega_u^{i-1}\wedge\sqrt{-1}\partial\bar\partial T_i \\
&= \frac{(i-1)(i-2)}{p}\int_M e^{-pu}\sqrt{-1}\omega_u^{i-3}\wedge\bar\partial\omega\wedge\partial\omega\wedge T_i \\
&\quad + \frac{i-1}{p}\int_M e^{-pu}\omega_u^{i-2}\wedge[\sqrt{-1}\bar\partial(\partial\omega\wedge T_i) + \sqrt{-1}\bar\partial\omega\wedge\partial T_i] \\
&\quad - \frac{1}{p}\int_M e^{-pu}\omega_u^{i-1}\wedge\sqrt{-1}\partial\bar\partial T_i \\
&\ge -pC_{3i}\sum_{j=0}^{k-2}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega_u^{j}\wedge\omega^{n-j-1} - C_{3i}(n,k,\omega)\int_M e^{-pu}\omega^n.
\end{aligned}
$$

For the last inequality, we have used the induction. □

4. Second-order estimate

In this section we use the auxiliary function in [Tosatti and Weinkove 2013] which is modified by the auxiliary function in [Hou et al. 2010] to derive the second-order estimate of the form (1-3). The difficulty arises from the third-order derivatives. Locally the equation is

$$
\sigma_k(\omega_u) = e^f. \tag{4-1}
$$

Theorem 4.1. *There exists a uniform constant C depending only on (M,ω), n, k and f such that*

$$
\max|\partial\bar\partial u|_g \le C(1+\max|\nabla u|_g^2). \tag{4-2}
$$

Proof. Denote $w_{i\bar{j}} = g_{i\bar{j}} + u_{i\bar{j}}$ and let $\xi \in T^{1,0}M$, $|\xi|_g^2 = 1$.

We use the auxiliary function similar to that in [Tosatti and Weinkove 2013]:

$$H(x,\xi) = \log(w_{k\bar{l}}\xi^k\bar{\xi}^l) + c_0 \log(g^{k\bar{l}} w_{p\bar{l}} w_{k\bar{q}} \xi^p \bar{\xi}^q) + \varphi(|\nabla u|_g^2) + \psi(u),$$

where φ, ψ are given by

$$\varphi(s) = -\frac{1}{2}\log\Big(1 - \frac{s}{2K}\Big), \qquad 0 \le s \le K-1,$$
$$\psi(t) = -A\log\Big(1 + \frac{t}{2L}\Big), \quad -L+1 \le t \le 0,$$

for

$$K := \sup_M |\nabla u|_g^2 + 1, \quad L = \sup_M |u| + 1, \quad A := 2L(C_0+1),$$

where A_0 is a constant to be determined later and c_0 is a small positive constant depending only on n and will be determined later. By [Hou et al. 2010], we have

$$\frac{1}{2K} \ge \varphi' \ge \frac{1}{4K} > 0, \qquad \varphi'' = 2(\varphi')^2 > 0. \tag{4-3}$$

$$\frac{A}{L} \ge -\psi' \ge \frac{A}{2L} = C_0 + 1, \quad \psi'' \ge \frac{2\varepsilon_0}{1-\varepsilon_0}(\psi')^2, \quad \text{for } \varepsilon_0 \le \frac{1}{2A+1}. \tag{4-4}$$

These inequalities will be used below.

Suppose $H(x,\xi)$ attains its maximum at the point x_0 in the direction ξ_0. Then we choose local coordinates $\{\partial/\partial z^1, \dots, \partial/\partial z^n\}$ near x_0 such that

$$g_{i\bar{j}}(x_0) = \delta_{ij}, \quad u_{i\bar{j}} = u_{i\bar{i}}(x_0)\delta_{ij}, \quad \lambda_i = w_{i\bar{i}}(x_0) = 1 + u_{i\bar{i}}(x_0) \quad \text{with } \lambda_1 \ge \cdots \ge \lambda_n.$$

We want to prove that

$$H(x_0,\xi) \le H\Big(x_0, \frac{\partial}{\partial z^1}\Big) \quad \text{for all } \xi \in T^{1,0}M,\ |\xi|_g^2 = 1, \sum_{i,j} w_{i\bar{j}}(x_0)\xi^i\xi^{\bar{j}} > 0$$

by choosing c_0 small enough. In fact, at x_0 we have

$$\log(w_{k\bar{l}}\xi^k\bar{\xi}^l) + c_0\log(g^{k\bar{l}} w_{p\bar{l}} w_{k\bar{q}} \xi^p\bar{\xi}^q) = \log\Big(\sum_{k=1}^n w_{k\bar{k}}|\xi^k|^2\Big) + c_0\log\Big(\sum_{k=1}^n |w_{k\bar{k}}|^2|\xi^k|^2\Big).$$

If $w_{n\bar{n}} \ge -w_{1\bar{1}}$, which is always satisfied when $n \le 3$, then $w_{i\bar{i}}^2 \le w_{1\bar{1}}$. Thus we have $H(x_0,\xi) \le H(x_0, \partial/\partial z^1)$.

Now we suppose that $w_{n\bar n} < -w_{1\bar 1}$. Thus we have $n \geq 4$. Let i_0 be the smallest integer satisfying $w_{i\bar i} < -w_{1\bar 1}$. Then $i_0 \geq k+1$. By $|w_{i\bar i}| < (n-2)w_{1\bar 1}$, so

$$\begin{aligned}
&\log \sum_{i=1}^{n} w_{i\bar i}|\xi^i|^2 + c_0 \log \sum_{i=1}^{n} |w_{i\bar i}|^2|\xi^i|^2 \\
&\quad\leq \log w_{1\bar 1}\left(\sum_{i=1}^{i_0-1}|\xi^i|^2 - \sum_{i=i_0}^{n}|\xi^i|^2\right) + c_0 \log\left(w_{1\bar 1}^2 \sum_{i=1}^{i_0-1}|\xi^i|^2 + (n-2)^2 w_{1\bar 1}^2 \sum_{i=1}^{i_0-1}|\xi^i|^2\right) \\
&\quad= \log w_{1\bar 1}(1-2t) + c_0 \log w_{1\bar 1}^2(1-t+(n-2)^2 t) := h(t),
\end{aligned}$$

where $t = \sum_{i=i_0}^{n} |\xi^i|^2 \in \left(0, \frac{1}{2}\right)$.

By choosing $c_0 = 2/((n-2)^2-1)$, we have $h'(t) \leq 0$. Then

$$h(t) \leq h(0) = \log(w_{1\bar 1}) + c_0 \log w_{1\bar 1}^2.$$

Consequently, we obtain

$$H(x_0, \xi) \leq H\left(x_0, \frac{\partial}{\partial z^1}\right) \quad \text{for all } \xi \in T^{1,0}M,\ |\xi|_g^2 = 1,\ \sum_{i,j} \eta_{i\bar j}(x_0)\xi^i \xi^{\bar j} > 0,$$

by choosing $c_0 = 2/((n-2)^2-1)$ when $n \geq 4$ and $c_0 = 1$ when $n \leq 3$.

We extend ξ_0 near x_0 by $\xi_0 = (g_{1\bar 1})^{-1/2}(\partial/\partial z^1)$. Consider the function

$$Q(x) = H(x, \xi_0) = \log(g_{1\bar 1}^{-1} w_{1\bar 1}) + c_0 \log(g_{1\bar 1}^{-1} g^{k\bar l} w_{1\bar l} w_{k\bar 1}) + \varphi(|\nabla u|_g^2) + \psi(u).$$

We will calculate $F^{i\bar j} Q_{i\bar j}$ at x_0 to get the estimate; all the calculations are taken at x_0. For simplicity, we denote $\xi = \xi_0$ in the following. By $\langle \xi, \bar\xi \rangle_g = |\xi|_g^2 = 1$, differentiating both sides, we obtain at x_0

$$\begin{aligned}
0 = \frac{\partial}{\partial z^i}\langle \xi, \bar\xi\rangle_g &= \langle \nabla_{\partial/\partial z^i}\xi, \bar\xi\rangle_g + \langle \xi, \nabla_{\partial/\partial z^i}\bar\xi\rangle_g \\
&= \left\langle \xi^k{}_{,i}\frac{\partial}{\partial z^k}, \overline{\xi^l}\frac{\partial}{\partial \bar z^l}\right\rangle_g + \left\langle \xi^k\frac{\partial}{\partial z^k}, \overline{\xi^l}{}_{,i}\frac{\partial}{\partial \bar z^l}\right\rangle_g \\
&= g_{k\bar l}\xi^k{}_{,i}\overline{\xi^l} + g_{k\bar l}\xi^k\overline{\xi^l}{}_{,i} \\
&= \xi^1{}_{,i} + \overline{\xi^1}{}_{,i}.
\end{aligned} \tag{4-5}$$

We also have the basic formula for $\xi \in T^{1,0}M$:

$$\begin{aligned}
\overline{\xi^k}{}_{,i} &= \frac{\partial\overline{\xi^k}}{\partial z^i} = \frac{\overline{\partial\xi^k}}{\partial \overline{z^i}} = \overline{\xi^k{}_{,\bar i}}, \quad \xi^k{}_{,\bar i} = \frac{\partial\xi^k}{\partial \overline{z^i}} = \frac{\overline{\partial\overline{\xi^k}}}{\partial z^i} = \overline{\overline{\xi^k}{}_{,i}} \\
\overline{\xi^k}{}_{,i} &= \frac{\partial\overline{\xi^k}}{\partial z^i} = \frac{\overline{\partial\xi^k}}{\partial \overline{z^i}} = \overline{\xi^k{}_{,\bar i}}, \quad \xi^k{}_{,\bar i} = \frac{\partial\xi^k}{\partial \overline{z^i}} = \frac{\overline{\partial\overline{\xi^k}}}{\partial z^i} = \overline{\overline{\xi^k}{}_{,i}}
\end{aligned} \tag{4-6}$$

Direct calculations give

$$Q_i = \frac{(w_{k\bar l}\xi^k\overline{\xi^l})_i}{w_{k\bar l}\xi^k\overline{\xi^l}} + c_0\frac{(g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l})_i}{g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l}} + \varphi_i + \psi_i,$$

$$\begin{aligned} Q_{i\bar i} &= \frac{(w_{k\bar l}\xi^k\overline{\xi^l})_{i\bar i}}{w_{k\bar l}\xi^k\overline{\xi^l}} - \frac{(w_{k\bar l}\xi^k\overline{\xi^l})_i(w_{k\bar l}\xi^k\overline{\xi^l})_{\bar i}}{(w_{k\bar l}\xi^k\overline{\xi^l})^2} + c_0\frac{(g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l})_{i\bar i}}{g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l}} \\ &\quad - c_0\frac{(g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l})_i(g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l})_{\bar i}}{(g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l})^2} + \varphi_{i\bar i} + \psi_{i\bar i}. \end{aligned}$$

Next, we want to simplify Q_i and $Q_{i\bar i}$. By (4-5), we have

$$(w_{k\bar l}\xi^k\overline{\xi^l})_i = w_{k\bar l,i}\xi^k\overline{\xi^l} + w_{k\bar l}\xi^k{}_{,i}\overline{\xi^l} + w_{k\bar l}\xi^k\overline{\xi^l}{}_{,i} = w_{1\bar 1,i} + w_{1\bar 1}(\xi^1{}_{,i} + \overline{\xi^1}{}_{,i}) = w_{1\bar 1 i},$$

Thus we have

$$\begin{aligned} &(g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l})_i \\ &\quad = g^{p\bar q}w_{k\bar q i}w_{p\bar l}\xi^k\overline{\xi^l} + g^{p\bar q}w_{k\bar q}w_{p\bar l i}\xi^k\overline{\xi^l} + g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k{}_i\overline{\xi^l} + g^{p\bar q}w_{k\bar q}w_{p\bar l}\xi^k\overline{\xi^l}{}_{,i} \\ &\quad = w_{1\bar 1}(w_{1\bar 1 i} + w_{1\bar 1 i}) + w_{1\bar 1}{}^2(\xi^1{}_{,i} + \overline{\xi^1}{}_{,i}) \\ &\quad = 2w_{1\bar 1}w_{1\bar 1 i}. \end{aligned}$$

Therefore, we obtain the simplified formula for Q_i at x_0:

$$Q_i = \frac{w_{1\bar 1 i}}{w_{1\bar 1}} + c_0\frac{2w_{1\bar 1 i}}{w_{1\bar 1}} + \varphi_i + \psi_i = (1+2c_0)\frac{w_{1\bar 1 i}}{w_{1\bar 1}} + \varphi_i + \psi_i = 0 \tag{4-7}$$

Similar calculations give

$$\begin{aligned} (w_{k\bar l}\xi^k\overline{\xi^l})_{i\bar i} &= [w_{k\bar l i}\xi^k\overline{\xi^l} + w_{k\bar l}(\xi^k{}_i\overline{\xi^l} + \xi^k\overline{\xi^l}{}_i)]_{\bar i} \\ &= w_{k\bar l i\bar i}\xi^k\overline{\xi^l} + w_{k\bar l i}(\xi^k{}_{\bar i}\overline{\xi^l} + \xi^k\overline{\xi^l}{}_{\bar i}) + w_{k\bar l\bar i}(\xi^k{}_i\overline{\xi^l} + \xi^k\overline{\xi^l}{}_i) \\ &\qquad + w_{k\bar l}(\xi^k{}_{i\bar i}\overline{\xi^l} + \xi^k{}_{\bar i}\overline{\xi^l}{}_i + \xi^k{}_i\overline{\xi^l}{}_{\bar i} + \xi^k\overline{\xi^l}{}_{i\bar i}) \\ &= w_{1\bar 1 i\bar i} + w_{k\bar 1 i}\xi^k{}_{\bar i} + w_{1\bar l i}\overline{\xi^l}{}_{\bar i} + w_{k\bar 1\bar i}\xi^k{}_i + w_{1\bar l\bar i}\overline{\xi^l}{}_i \\ &\qquad + w_{1\bar 1}(\xi^1{}_{i\bar i} + \overline{\xi^1}{}_{i\bar i}) + w_{k\bar k}(\xi^k{}_{\bar i}\overline{\xi^k}{}_i + \xi^k{}_i\overline{\xi^k}{}_{\bar i}) \\ &= w_{1\bar 1 i\bar i} + 2\sum_{k\neq 1}\mathrm{Re}(w_{k\bar 1 i}\xi^k{}_{\bar i} + w_{1\bar k i}\overline{\xi^k}{}_i) + w_{1\bar 1}(\xi^1{}_{i\bar i} + \overline{\xi^1}{}_{i\bar i}) \\ &\qquad + w_{k\bar k}(|\xi^k{}_{\bar i}|^2 + |\xi^k{}_i|^2). \end{aligned}$$

The last equality holds because we use (4-2) and (4-5) and the fact

$$w_{k\bar 1 i}\xi^k{}_{\bar i} + w_{1\bar l\bar i}\overline{\xi^l}{}_i = 2\,\mathrm{Re}(w_{k\bar 1 i}\xi^k{}_{\bar i}), \quad w_{1\bar l i}\overline{\xi^l}{}_{\bar i} + w_{k\bar 1\bar i}\xi^k{}_i = 2\,\mathrm{Re}(w_{1\bar k i}\overline{\xi^k}{}_i).$$

We can also calculate

$$
\begin{aligned}
&(g^{p\bar q} w_{k\bar q} w_{p\bar l} \xi^k \overline{\xi^l})_{i\bar i} \\
&= g^{p\bar q}(w_{k\bar q i} w_{p\bar l} \xi^k \overline{\xi^l} + w_{k\bar q} w_{p\bar l i} \xi^k \overline{\xi^l} + w_{k\bar q} w_{p\bar l} \xi^k{}_i \overline{\xi^l} + w_{k\bar q} w_{p\bar l} \xi^k \overline{\xi^l}{}_i)_{\bar i} \\
&= g^{p\bar q}(w_{k\bar q i\bar i} w_{p\bar l} \xi^k \overline{\xi^l} + w_{k\bar q i} w_{p\bar l\bar i} \xi^k \overline{\xi^l} + w_{k\bar q i} w_{p\bar l} \xi^k{}_{\bar i} \overline{\xi^l} + w_{k\bar q i} w_{p\bar l} \xi^k \overline{\xi^l}{}_{\bar i}) \\
&\quad + g^{p\bar q}(w_{k\bar q\bar i} w_{p\bar l i} \xi^k \overline{\xi^l} + w_{k\bar q} w_{p\bar l i\bar i} \xi^k \overline{\xi^l} + w_{k\bar q} w_{p\bar l i} \xi^k{}_{\bar i} \overline{\xi^l} + w_{k\bar q} w_{p\bar l i} \xi^k \overline{\xi^l}{}_{\bar i}) \\
&\qquad + g^{p\bar q}(w_{k\bar q\bar i} w_{p\bar l} \xi^k{}_i \overline{\xi^l} + w_{k\bar q} w_{p\bar l\bar i} \xi^k{}_i \overline{\xi^l} + w_{k\bar q} w_{p\bar l} \xi^k{}_{i\bar i} \overline{\xi^l} + w_{k\bar q} w_{p\bar l} \xi^k{}_i \overline{\xi^l}{}_{\bar i}) \\
&\qquad\quad + g^{p\bar q}(w_{k\bar q\bar i} w_{p\bar l} \xi^k \overline{\xi^l}{}_i + w_{k\bar q} w_{p\bar l\bar i} \xi^k \overline{\xi^l}{}_i + w_{k\bar q} w_{p\bar l} \xi^k{}_{\bar i} \overline{\xi^l}{}_i + w_{k\bar q} w_{p\bar l} \xi^k \overline{\xi^l}{}_{i\bar i}) \\
&= w_{1\bar 1 i\bar i} w_{1\bar 1} + w_{1\bar p i} w_{p\bar 1\bar i} + w_{k\bar 1 i} w_{1\bar 1} \xi^k{}_{\bar i} + w_{1\bar p i} w_{p\bar p} \overline{\xi^p}{}_{\bar i} \\
&\quad + w_{1\bar p\bar i} w_{p\bar 1 i} + w_{1\bar 1} w_{1\bar 1 i\bar i} + w_{p\bar p} w_{p\bar 1 i} \xi^p{}_{\bar i} + w_{1\bar 1} w_{1\bar l i} \overline{\xi^l}{}_{\bar i} \\
&\qquad + w_{k\bar 1\bar i} w_{1\bar 1} \xi^k{}_i + w_{p\bar p} w_{p\bar 1\bar i} \xi^p{}_i + w_{1\bar 1}{}^2 \xi^1{}_{i\bar i} + w_{p\bar p}{}^2 \xi^p{}_i \overline{\xi^p}{}_{\bar i} \\
&\qquad\quad + w_{1\bar p\bar i} w_{p\bar p} \overline{\xi^p}{}_i + w_{1\bar 1} w_{1\bar l\bar i} \overline{\xi^l}{}_i + w_{p\bar p}{}^2 \xi^p{}_{\bar i} \overline{\xi^p}{}_i + w_{1\bar 1}{}^2 \overline{\xi^1}{}_{i\bar i} \\
&= 2 w_{1\bar 1} w_{1\bar 1 i\bar i} + |w_{1\bar p i}|^2 + |w_{1\bar p\bar i}|^2 + 2 w_{1\bar 1} \operatorname{Re}(w_{p\bar 1 i} \xi^p{}_{\bar i} + w_{p\bar 1\bar i} \xi^p{}_i) \\
&\quad + 2 w_{p\bar p} \operatorname{Re}(w_{1\bar p i} \overline{\xi^p}{}_{\bar i} + w_{p\bar 1 i} \xi^p{}_{\bar i}) + w_{p\bar p}{}^2 (|\xi^p{}_i|^2 + |\xi^p{}_{\bar i}|^2) + w_{1\bar 1}{}^2 (\xi^1{}_{i\bar i} + \overline{\xi^1}{}_{i\bar i})
\end{aligned}
$$

Therefore we simplify $Q_{i\bar i}$ at x_0 as follows

$$
\begin{aligned}
Q_{i\bar i} = (1+2c_0)\frac{w_{1\bar 1 i\bar i}}{w_{1\bar 1}} + \frac{c_0}{w_{1\bar 1}^2} \sum_{p\neq 1} (|w_{1\bar p i}|^2 + |w_{1\bar p\bar i}|^2) \\
-(1+2c_0)\frac{|w_{1\bar 1 i}|^2}{w_{1\bar 1}{}^2} + (**)_{i\bar i} + \varphi_{i\bar i} + \psi_{i\bar i},
\end{aligned}
$$

where $(**)_{i\bar i}$ is given by

$$
\begin{aligned}
(**)_{i\bar i} &= \frac{2}{w_{1\bar 1}} \sum_{k\neq 1} \operatorname{Re}(w_{k\bar 1 i} \xi^k{}_{\bar i} + w_{1\bar k i} \overline{\xi^k}{}_i) + \xi^1{}_{i\bar i} + \overline{\xi^1}{}_{i\bar i} + \frac{w_{k\bar k}}{w_{1\bar 1}} (|\xi^k{}_{\bar i}|^2 + |\xi^k{}_i|^2) \\
&\quad + \frac{2c_0}{w_{1\bar 1}} \sum_{p\neq 1} \operatorname{Re}(w_{p\bar 1 i} \xi^p{}_{\bar i} + w_{p\bar 1\bar i} \xi^p{}_i) + \sum_{p\neq 1} \frac{2c_0 w_{p\bar p}}{w_{1\bar 1}{}^2} \operatorname{Re}(w_{1\bar p i} \overline{\xi^p}{}_{\bar i} + w_{p\bar 1 i} \xi^p{}_{\bar i}) \\
&\quad + \frac{2c_0 w_{p\bar p}{}^2}{w_{1\bar 1}{}^2} (|\xi^p{}_i|^2 + |\xi^p{}_{\bar i}|^2) + c_0(\xi^1{}_{i\bar i} + \overline{\xi^1}{}_{i\bar i}).
\end{aligned}
$$

For this term $(**)_{i\bar i}$, we have the estimate

$$
(**)_{i\bar i} \geq -\frac{c_0}{2 w_{1\bar 1}{}^2} \sum_{p\neq 1} (|w_{1\bar p i}|^2 + |w_{1\bar p\bar i}|^2) - C,
$$

where C is a positive constant depending only on (M, ω).

Let $F(\omega_u)=(\sigma_k(\omega_u))^{1/k}$. We denote by

$$F^{i\bar{j}}=\frac{\partial F}{\partial w_{i\bar{j}}},\qquad F^{i\bar{j},p\bar{q}}=\frac{\partial^2 F}{\partial w_{i\bar{j}}\partial w_{p\bar{q}}},$$

where $(w_u)_{i\bar{j}}=g_{i\bar{j}}+u_{i\bar{j}}$. Then, the positive definite matrix $(F^{i\bar{j}}(\omega_u))$ is diagonalized at the point x_0. More precisely,

$$F^{i\bar{j}}(\omega_u)=\delta_{ij}F^{i\bar{i}}(\omega_u)=\frac{1}{k}[\sigma_k(\lambda)]^{1/k-1}\sigma_{k-1}(\lambda|i)\delta_{ij}. \tag{4-8}$$

Furthermore, at x_0,

$$F^{i\bar{j},p\bar{q}}(\omega_u)=\begin{cases}F^{i\bar{i},p\bar{p}}, & \text{if } i=j,\ p=q;\\ F^{i\bar{p},p\bar{i}}, & \text{if } i=q,\ p=j,\ i\neq p;\\ 0, & \text{otherwise,}\end{cases} \tag{4-9}$$

in which

$$\begin{aligned}F^{i\bar{i},p\bar{p}}&=\frac{1}{k}[\sigma_k(\lambda)]^{1/k-1}(1-\delta_{ip})\sigma_{k-2}(\lambda|ip)\\ &\qquad+\frac{1}{k}\Big(\frac{1}{k}-1\Big)[\sigma_k(\lambda)]^{1/k-2}\sigma_{k-1}(\lambda|i)\sigma_{k-1}(\lambda|p),\\ F^{i\bar{p},p\bar{i}}&=-\frac{1}{k}[\sigma_k(\lambda)]^{1/k-1}\sigma_{k-2}(\lambda|ip).\end{aligned}$$

We have, in addition, at x_0

$$\sum_{i=1}^n F^{i\bar{i}}w_{i\bar{i}}=\sum_{i=1}^n F^{i\bar{i}}\lambda_i=\sigma_k^{1/k}=e^{f/k}. \tag{4-10}$$

By the maximum principal, we have

$$\begin{aligned}0\geq F^{i\bar{j}}Q_{i\bar{j}}&=F^{i\bar{i}}Q_{i\bar{i}}\\ &\geq(1+2c_0)\sum_{i=1}^n\frac{F^{i\bar{i}}u_{1\bar{1}i\bar{i}}}{w_{1\bar{1}}}+\frac{c_0}{2}\sum_{i=1}^n\sum_{p\neq 1}\frac{F^{i\bar{i}}|u_{1\bar{p}i}|^2}{w_{1\bar{1}}^2}\\ &\qquad-(1+2c_0)\sum_{i=1}^n\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2}+\psi'\sum_{i=1}^n F^{i\bar{i}}u_{i\bar{i}}+\psi''\sum_{i=1}^n F^{i\bar{i}}|u_i|^2\\ &\qquad+\varphi''\sum_{i=1}^n F^{i\bar{i}}|\nabla u|_i^2|\nabla u|_{\bar{i}}^2+\varphi'\sum_{i,p=1}^n F^{i\bar{i}}(|u_{p\bar{i}}|^2+|u_{pi}|^2)\\ &\qquad+\varphi'\sum_{i,p=1}^n F^{i\bar{i}}(u_{p\bar{i}i}u_{\bar{p}}+u_{\bar{p}i\bar{i}}u_p)-C_1\sum_{i=1}^n F^{i\bar{i}}\\ &:=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8-C_1\sum_{i=1}^n F^{i\bar{i}}\end{aligned} \tag{4-11}$$

The equation can be written as $F(\omega_u) = e^{f/k} := h$. Differentiating this, we get

$$\sum_{i,j=1}^{n} F^{i\bar{j}} u_{i\bar{j}l} = \nabla_l F = h_l, \quad \sum_{i,j=1}^{n} F^{i\bar{j}} u_{i\bar{j}l\bar{m}} + \sum_{i,j,p,q=1}^{n} F^{i\bar{j},p\bar{q}} u_{i\bar{j}l} u_{p\bar{q}\bar{m}} = h_{l\bar{m}}.$$

and

$$\sum_{i=1}^{n} F^{i\bar{i}} u_{i\bar{i}1\bar{1}} = h_{1\bar{1}} - \sum_{i,j,p,q=1}^{n} F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}\bar{1}}.$$

By commuting the covariant derivatives formula (2-12), we have

$$\begin{aligned}(4\text{-}12)\quad \sum_{i=1}^{n} F^{i\bar{i}} u_{1\bar{1}i\bar{i}} &= \sum_{i=1}^{n} F^{i\bar{i}} u_{i\bar{i}1\bar{1}} + \sum_{i=1}^{n} F^{i\bar{i}} \Big(u_{1\bar{1}} - \sum_{i=1}^{n} u_{i\bar{i}}\Big) R_{i\bar{i}1\bar{1}} \\ &\quad + \sum_{i=1}^{n} F^{i\bar{i}} \Big(\sum_{p=1}^{n} T_{1i}^{p} u_{p\bar{1}\bar{i}} + \sum_{q=1}^{n} \overline{T_{1i}^{q}} u_{1\bar{q}i} - \sum_{p=1}^{n} |T_{1i}^{p}|^2 u_{p\bar{p}} \Big).\end{aligned}$$

Inserting (4-12) into the term I_1, we have

$$\begin{aligned}(4\text{-}13)\quad I_1 &= (1+2c_0) \sum_{i=1}^{n} \frac{F^{i\bar{i}} u_{1\bar{1}i\bar{i}}}{w_{1\bar{1}}} \\ &= (1+2c_0) \sum_{i=1}^{n} \frac{F^{i\bar{i}} u_{i\bar{i}1\bar{1}}}{w_{1\bar{1}}} + (1+2c_0) \sum_{i=1}^{n} \frac{F^{i\bar{i}} (u_{1\bar{1}} - u_{i\bar{i}}) R_{i\bar{i}1\bar{1}}}{w_{1\bar{1}}} \\ &\quad + 2(1+2c_0) \sum_{i,p=1}^{n} F^{i\bar{i}} \operatorname{Re}\Big(\frac{T_{1i}^{p} u_{p\bar{1}\bar{i}}}{w_{1\bar{1}}} \Big) - (1+2c_0) \sum_{i,p=1}^{n} F^{i\bar{i}} \frac{|T_{1i}^{p}|^2 u_{p\bar{p}}}{w_{1\bar{1}}} \\ &= (1+2c_0) \frac{h_{1\bar{1}}}{w_{1\bar{1}}} - (1+2c_0) \sum_{i,j,p,q=1}^{n} \frac{F^{i\bar{j},p\bar{q}} u_{i\bar{j}1} u_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} \\ &\quad + (1+2c_0) \sum_{i=1}^{n} \frac{F^{i\bar{i}} (u_{1\bar{1}} - u_{i\bar{i}}) R_{i\bar{i}1\bar{1}}}{w_{1\bar{1}}} + 2(1+2c_0) \sum_{i} F^{i\bar{i}} \operatorname{Re}\Big(\frac{T_{1i}^{1} u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}} \Big) \\ &\quad + 2(1+2c_0) \sum_{i=1}^{n} F^{i\bar{i}} \operatorname{Re}\Big(\sum_{p\neq 1} \frac{T_{1i}^{p} u_{p\bar{1}\bar{i}}}{w_{1\bar{1}}} \Big) - (1+2c_0) \sum_{i,p=1}^{n} F^{i\bar{i}} \frac{|T_{1i}^{p}|^2 u_{p\bar{p}}}{w_{1\bar{1}}} \\ &:= I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16}.\end{aligned}$$

We estimate each term in this sum. First we have

$$I_{11} + I_{13} + I_{16} \geq -C_1 - 3(nC_2 + C_3) \sum_{i=1}^{n} F^{i\bar{i}},$$

where we have supposed that $\sup_M |T|_g^2 \leq C_2$, $\sup_M |R| \leq C_3$.

We claim $I_{15}+I_2 \geq -18n^2C_2\sum_{i=1}^n F^{i\bar{i}}$. Indeed, since $\frac{1}{n^2}\leq c_0\leq 1$, we have

$$\begin{aligned} I_{15}+I_2 &= \frac{c_0}{2}\sum_{i=1}^n\sum_{p\neq 1}\frac{F^{i\bar{i}}|u_{1\bar{p}i}|^2}{w_{1\bar{1}}^2}+2(1+2c_0)\sum_{i=1}^n F^{i\bar{i}}\operatorname{Re}\Big(\sum_{p\neq 1}\frac{T_{1i}^p u_{p\bar{1}\bar{i}}}{w_{1\bar{1}}}\Big)\\ &= \frac{c_0}{2}\sum_{i=1}^n F^{i\bar{i}}\sum_{p\neq 1}\Big|\frac{u_{1\bar{p}i}}{w_{1\bar{1}}}+\frac{2(1+2c_0)}{c_0}T_{1i}^p\Big|^2-\frac{2(1+2c_0)^2}{c_0}\sum_{i=1}^n\sum_{p\neq 1}F^{i\bar{i}}|T_{1i}^p|^2\\ &\geq -\frac{2(1+2c_0)^2}{c_0}\sum_{i=1}^n\sum_{p\neq 1}F^{i\bar{i}}|T_{1i}^p|^2\\ &\geq -18n^2C_2\sum_{i=1}^n F^{i\bar{i}}. \end{aligned}$$

Then we obtain

$$I_1+I_2 \geq -(1+2c_0)\sum_{i,j,p,q=1}^n\frac{F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}}}{w_{1\bar{1}}}+2(1+2c_0)\sum_{i=1}^n F^{i\bar{i}}\operatorname{Re}\Big(\frac{T_{1i}^1 u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\Big) - (21n^2C_2+3C_3)\sum_{i=1}^n F^{i\bar{i}}-C_1. \tag{4-14}$$

For terms I_7+I_8, we claim

$$I_7+I_8 \geq \frac{1}{2}\varphi'\sum_{i=1}^n F^{i\bar{i}}|u_{i\bar{i}}|_1^2-(C_2+C_3)\sum_{i=1}^n F^{i\bar{i}}-C_1. \tag{4-15}$$

Indeed, by the commutation formula for covariant derivatives (2-11) in Section 2,

$$u_{pi\bar{i}}=u_{i\bar{i}p}+T_{pi}^i u_{i\bar{i}}+u_q R_{i\bar{i}p\bar{q}},\qquad u_{\bar{p}i\bar{i}}=u_{i\bar{p}\bar{i}}=u_{i\bar{i}\bar{p}}-\overline{T_{ip}^i}u_{i\bar{i}}.$$

Then

$$\begin{aligned} \sum_{i=1}^n F^{i\bar{i}}u_{pi\bar{i}} &= \sum_{i=1}^n F^{i\bar{i}}u_{i\bar{i}p}+\sum_{i=1}^n F^{i\bar{i}}(T_{pi}^i u_{i\bar{i}}+u_q R_{i\bar{i}p\bar{q}})\\ &= h_p+\sum_{i=1}^n F^{i\bar{i}}(T_{pi}^i u_{i\bar{i}}+u_q R_{i\bar{i}p\bar{q}})\\ \sum_{i=1}^n F^{i\bar{i}}u_{\bar{p}i\bar{i}} &= \sum_{i=1}^n F^{i\bar{i}}u_{i\bar{i}\bar{p}}+\sum_{i=1}^n F^{i\bar{i}}\overline{T_{ip}^i}u_{i\bar{i}}=h_{\bar{p}}+\sum_{i=1}^n F^{i\bar{i}}\overline{T_{ip}^i}u_{i\bar{i}} \end{aligned}$$

Inserting the above formula into I_8, we obtain

$$
\begin{aligned}
(4\text{-}16)\quad I_8 &= \varphi' \sum_{i,p=1}^{n} F^{i\bar{i}} (u_{p\bar{i}i} u_{\bar{p}} + u_{\bar{p}i\bar{i}} u_p) \\
&= \varphi' \sum_{p=1}^{n} u_{\bar{p}} \Big[h_p + \sum_{i=1}^{n} F^{i\bar{i}} (T^i_{pi} u_{i\bar{i}} + u_q R_{i\bar{i}p\bar{q}}) \Big] + \varphi' \sum_{p=1}^{n} u_p \Big[h_{\bar{p}} - \sum_{i=1}^{n} F^{i\bar{i}} \overline{T^i_{ip}} u_{i\bar{i}} \Big] \\
&= 2\varphi' \sum_{i,p=1}^{n} F^{i\bar{i}} u_{i\bar{i}} \operatorname{Re}(u_{\bar{p}} T^i_{pi}) + \varphi' \sum_{p=1}^{n} \Big[2 \operatorname{Re}(u_{\bar{p}} h_p) + \sum_{i,q=1}^{n} u_{\bar{p}} u_q F^{i\bar{i}} R_{i\bar{i}p\bar{q}} \Big] \\
&= I_{81} + I_{82}.
\end{aligned}
$$

For the term I_{82}, we have

$$
I_{82} \ge -C_3 \sum_{i=1}^{n} F^{i\bar{i}} - C_1.
$$

For the term I_{81}, we obtain

$$
\begin{aligned}
I_{81} + I_7 &= 2\varphi' \sum_{i,p=1}^{n} F^{i\bar{i}} u_{i\bar{i}} \operatorname{Re}(u_{\bar{p}} T^i_{pi}) + \varphi' \sum_{i,p=1}^{n} F^{i\bar{i}} (|u_{p\bar{i}}|^2 + |u_{pi}|^2) \\
&\ge \varphi' \sum_{i=1}^{n} F^{i\bar{i}} \Big[|u_{i\bar{i}}|^2 + 2u_{i\bar{i}} \operatorname{Re}\Big(\sum_{p=1}^{n} u_{\bar{p}} T^i_{pi} \Big) \Big] \\
&= \varphi' \sum_{i=1}^{n} F^{i\bar{i}} \Big| \frac{u_{i\bar{i}}}{2} + 2 \sum_{p=1}^{n} u_p \overline{T^i_{pi}} \Big|^2 + \frac{3}{4} \varphi' \sum_{i=1}^{n} F^{i\bar{i}} |u_{i\bar{i}}|^2 - 4\varphi' \sum_{i=1}^{n} F^{i\bar{i}} \Big| \sum_{p=1}^{n} u_p \overline{T^i_{pi}} \Big|^2 \\
&\ge \frac{1}{2} \varphi' \sum_{i=1}^{n} F^{i\bar{i}} |u_{i\bar{i}}|^2 - C_2 \sum_{i=1}^{n} F^{i\bar{i}}.
\end{aligned}
$$

Thus we have proved the above claim (4-15). Moreover, applying (4-10) yields

$$
\begin{aligned}
\psi' \sum_{i=1}^{n} F^{i\bar{i}} u_{i\bar{i}} &= \psi' \sum_{i=1}^{n} F^{i\bar{i}} (\lambda_i - 1) \\
&= \psi' h - \psi' \sum_{i=1}^{n} F^{i\bar{i}} \ge -2(C_0 + 1) \sup_M e^{f/k} - \psi' \sum_{i=1}^{n} F^{i\bar{i}}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}|u_{i\bar i}|^2&=\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}(\lambda_i-1)^2\\&=\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}\lambda_i^2-\varphi'\sum_{i=1}^{n}F^{i\bar i}\lambda_i+\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}\\&=\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}\lambda_i^2-\varphi'h+\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}\\&\geq\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}\lambda_i^2-\frac{1}{2}\sup_M e^{f/k}+\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}.\end{aligned}$$

Inserting these terms into (4-11), we obtain

$$\begin{aligned}(4\text{-}17)\quad 0&\geq F^{i\bar i}Q_{i\bar i}\\&\geq-(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar j,p\bar q}u_{i\bar j1}u_{p\bar q\bar 1}}{w_{1\bar 1}}+2(1+2c_0)\sum_{i=1}^{n}F^{i\bar i}\operatorname{Re}\Big(\frac{T_{1i}^{1}u_{1\bar 1\bar i}}{w_{1\bar 1}}\Big)\\&\qquad-(1+2c_0)\sum_{i=1}^{n}\frac{F^{i\bar i}|u_{1\bar 1 i}|^2}{w_{1\bar 1}^2}\\&\qquad+\varphi''\sum_{i=1}^{n}F^{i\bar i}|\nabla u|_i^2|\nabla u|_{\bar i}^2+\psi''\sum_{i=1}^{n}F^{i\bar i}|u_i|^2+\frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar i}\lambda_i^2\\&\qquad+\Big(-\psi'+\frac{1}{2}\varphi'-22n^2C_2-4C_3\Big)\sum_{i=1}^{n}F^{i\bar i}-C_1\\&=A_1+A_2+A_3+A_4+A_5+A_6\\&\qquad\qquad+\Big(-\psi'+\frac{1}{2}\varphi'-22n^2C_2-4C_3\Big)\sum_{i=1}^{n}F^{i\bar i}-C_1,\end{aligned}$$

where C_1 is a positive constant depending only on C_0, $\sup e^{f/k}$, $\sup|\nabla(e^{f/k})|^2$ and $\sup|\partial\bar\partial(e^{f/k})|$.

Let $\varepsilon=\frac{1}{4}\delta\leq\frac{1}{16}$ and $\delta=1/(2A+1)$, where $A=2L(C_0+1)$ and $C_0=31n^2C_2+4C_3$. We divide into two cases to derive the estimate, which is similar to [Hou et al. 2010].

Case 1: $\lambda_n<-\varepsilon\lambda_1$.

By condition (4-7), for $1\leq i\leq n$, we have

$$\begin{aligned}-(1+2c_0)^2\Big|\frac{u_{1\bar 1 i}}{w_{1\bar 1}}\Big|^2&=-|\varphi'|\nabla u|_i^2+\psi'u_i|^2\geq-2(\varphi')^2|\nabla u|_i^2|\nabla u|_{\bar i}^2-2(\psi')^2|u_i|^2\\&=-\varphi''|\nabla u|_i^2|\nabla u|_{\bar i}^2-2(\psi')^2|u_i|^2.\end{aligned}$$

This gives

$$\begin{aligned}A_2 &= 2(1+2c_0)\sum_{i\neq 1} F^{i\bar i}\,\mathrm{Re}\Big(\frac{T^1_{1i}u_{1\bar 1\bar i}}{w_{1\bar 1}}\Big)\\ &\geq -c_0\sum_{i\neq 1} F^{i\bar i}\Big|\frac{u_{1\bar 1 i}}{w_{1\bar 1}}\Big|^2 - \frac{(1+2c_0)^2}{c_0}\sum_{i\neq 1}F^{i\bar i}|T^1_{1i}|^2\\ &\geq -c_0\sum_{i\neq 1} F^{i\bar i}\Big|\frac{u_{1\bar 1 i}}{w_{1\bar 1}}\Big|^2 - 9n^2C_2\sum_{i\neq 1}F^{i\bar i}|T^1_{1i}|^2\end{aligned}$$

Thus

$$\begin{aligned}A_2+A_3 &\geq -(1+3c_0)\sum_{i=1}^n \frac{F^{i\bar i}|u_{1\bar 1 i}|^2}{w^2_{1\bar 1}} - 9n^2C_2\sum_{i\neq 1}F^{i\bar i}|T^1_{1i}|^2\\ &\geq -(1+2c_0)^2\sum_{i=1}^n \frac{F^{i\bar i}|u_{1\bar 1 i}|^2}{w^2_{1\bar 1}} - 9n^2C_2\sum_{i=1}^n F^{i\bar i}\\ &= -A_4 - 2(\psi')^2\sum_{i=1}^n F^{i\bar i}|u_i|^2 - 9n^2C_2\sum_{i=1}^n F^{i\bar i}.\end{aligned}$$

We therefore obtain

$$A_2+A_3+A_4 \geq -2(\psi')^2\sum_{i=1}^n F^{i\bar i}|u_i|^2 - 9n^2C_2\sum_{i=1}^n F^{i\bar i}. \tag{4-18}$$

Using the inequality

$$\sum_{i=1}^n F^{i\bar i}\lambda_i^2 \geq F^{n\bar n}\lambda_n^2 > \varepsilon^2 F^{n\bar n}\lambda_1^2 \geq \frac{\varepsilon^2}{n}\sum_{i=1}^n F^{i\bar i}\lambda_1^2,$$

we have

$$A_6 = \frac{1}{2}\varphi'\sum_{i=1}^n F^{i\bar i}\lambda_i^2 \geq \frac{\varepsilon^2}{2n}\varphi'\sum_{i=1}^n F^{i\bar i}\lambda_1^2. \tag{4-19}$$

Combining (4-17) and (4-18) (4-19), we obtain

$$\begin{aligned}0 \geq \sum_{i=1}^n F^{i\bar i}Q_{i\bar i} &\geq \frac{\varepsilon^2}{2n}\varphi'\sum_{i=1}^n F^{i\bar i}\lambda_1^2 - 2(\psi')^2\sum_{i=1}^n F^{i\bar i}|u_i|^2\\ &\qquad + \Big(-\psi' + \frac{1}{2}\varphi' - 31n^2C_2 - 4C_3\Big)\sum_{i=1}^n F^{i\bar i} - C_1\\ &\geq \Big(\frac{\varepsilon^2}{8nK}\lambda_1^2 - 8K(C_0+1)^2\Big)\sum_{i=1}^n F^{i\bar i} - C_1\\ &\geq \frac{\varepsilon^2}{8nK}\lambda_1^2 - 8K(C_0+1)^2 - C_1,\end{aligned}$$

where we use the fact that $\sum_{i=1}^{n} F^{i\bar{i}} \geq 1$, which follows from the definition of F^{ii} and the Newton–Maclaurin inequality.

Hence, we obtain

$$\lambda_1 \leq 8\sqrt{2}(2A+1)\sqrt{nK(8K(C_0+1)^2+C_1)} \leq CK.$$

Case 2: $\lambda_n > -\varepsilon\lambda_1$.

Let $I = \{i \in \{1, \dots, n\} | \sigma_{k-1}(\lambda|i) \geq \varepsilon^{-1}\sigma_{k-1}(\lambda|1)\}$. Obviously, $1 \notin I$ and $i \in I$ if and only if $F^{i\bar{i}} > \varepsilon^{-1}F^{1\bar{1}}$. We first treat those indices which are not in I. By (4-7), we have

$$\begin{aligned}
-(1+2c_0)\sum_{i\notin I}\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} &+ 2(1+2c_0)\sum_{i\notin I}F^{i\bar{i}}\operatorname{Re}\frac{T_{1i}^1 u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}} \\
&\geq -(1+2c_0)^2\sum_{i\notin I}\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} - \frac{(1+2c_0)^2}{c_0}\sum_{i\notin I}F^{i\bar{i}}|T_{1i}^1|^2 \\
&= -\varphi''\sum_{i\notin I}F^{i\bar{i}}|\nabla u|^2{}_i|\nabla u|^2{}_{\bar{i}} - 2(\psi')^2\sum_{i\notin I}F^{i\bar{i}}|u_i|^2 - 9n^2C_2\sum_{i\notin I}F^{i\bar{i}}|T_{1i}^1|^2 \\
&\geq -\varphi''\sum_{i\notin I}F^{i\bar{i}}|\nabla u|^2{}_i|\nabla u|^2{}_{\bar{i}} - 2\varepsilon^{-1}K(\psi')^2F^{1\bar{1}} - 9n^2C_2\sum_{i=1}^{n}F^{i\bar{i}}.
\end{aligned}$$

Substituting the above inequality into (4-17) yields

$$\begin{aligned}
0 \geq F^{i\bar{i}}Q_{i\bar{i}} &\geq -(1+2c_0)\sum_{i,j,p,q=1}^{n}\frac{F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}}}{w_{1\bar{1}}} + 2(1+2c_0)\sum_{i\in I}F^{i\bar{i}}\operatorname{Re}\Big(\frac{T_{1i}^1 u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}}\Big) \\
&\quad - (1+2c_0)\sum_{i\in I}\frac{F^{i\bar{i}}|u_{1\bar{1}i}|^2}{w_{1\bar{1}}^2} + \varphi''\sum_{i\in I}F^{i\bar{i}}|\nabla u|_i^2|\nabla u|_{\bar{i}}^2 + \psi''\sum_{i=1}^{n}F^{i\bar{i}}|u_i|^2 \\
&\quad + \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - 2\varepsilon^{-1}K(\psi')^2F^{1\bar{1}} \\
&\quad + \Big(-\psi' + \frac{1}{2}\varphi' - 31n^2C_2 - 4C_3\Big)\sum_{i=1}^{n}F^{i\bar{i}} - C_1 \\
&= B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8.
\end{aligned} \tag{4-20}$$

Firstly, we have

$$B_6 + B_7 = \frac{1}{2}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2 - 2\varepsilon^{-1}K(\psi')^2F^{1\bar{1}} \geq \frac{1}{4}\varphi'\sum_{i=1}^{n}F^{i\bar{i}}\lambda_i^2,$$

where we assume that $\frac{1}{4}\varphi' F^{1\bar{1}}\lambda_1^2 \geq 2\varepsilon^{-1}K(\psi')^2 F^{1\bar{1}}$ (for otherwise $\frac{1}{4}\varphi' F^{1\bar{1}}\lambda_1^2 \leq 2\varepsilon^{-1}K(\psi')^2 F^{1\bar{1}}$, i.e., $\lambda_1 \leq CK$).

We next use B_1 to cancel the other terms containing the third derivatives of u. As the proof in [Hou et al. 2010, p. 559], we have

$$\lambda_1\sigma_{k-2}(\lambda|1i) \geq (1-2\varepsilon)\sigma_{k-1}(\lambda|i) \quad \text{for } i \in I.$$

Then

$$-\lambda_1 F^{i\bar{1},1\bar{i}} = \frac{F^{1-k}}{k}\lambda_1\sigma_{k-2}(\lambda|1i) \geq \frac{F^{1-k}}{k}(1-2\varepsilon)\sigma_{k-1}(\lambda|i) = (1-2\varepsilon)F^{i\bar{i}}.$$

Since $u_{i\bar{1}1} = u_{1\bar{1}i} - T^1_{1i}(\lambda_1 - 1)$, we get

$$\begin{aligned}
B_1 &= -\frac{1+2c_0}{\lambda_1}\sum_{i,j,p,q=1}^{n} F^{i\bar{j},p\bar{q}}u_{i\bar{j}1}u_{p\bar{q}\bar{1}} \\
&\geq -\frac{1+2c_0}{\lambda_1^2}\sum_{i\in I}\lambda_1 F^{i\bar{1},1\bar{i}}u_{i\bar{1}1}u_{1\bar{i}\bar{1}} \\
&\geq \frac{1+2c_0}{\lambda_1^2}(1-2\varepsilon)\sum_{i\in I}F^{i\bar{i}}|u_{1\bar{1}i} - T^1_{1i}(\lambda_1-1)|^2,
\end{aligned}$$

and

$$B_2 = \frac{2(1+2c_0)}{\lambda_1}\sum_{i\in I}F^{i\bar{i}}\,\mathrm{Re}(T^1_{1i}u_{1\bar{1}\bar{i}}).$$

From (4-7), we have

$$\begin{aligned}
B_4 &= \varphi''\sum_{i\in I}F^{i\bar{i}}|\nabla u|^2_i|\nabla u|^2_{\bar{i}} \\
&= 2\sum_{i\in I}F^{i\bar{i}}\left|(1+2c_0)\frac{u_{1\bar{1}\bar{i}}}{w_{1\bar{1}}} + \psi' u_i\right|^2 \\
&\geq 2(1+2c_0)^2\delta\sum_{i\in I}F^{i\bar{i}}\frac{|u_{1\bar{1}\bar{i}}|^2}{w^2_{1\bar{1}}} - \frac{2\delta}{1-\delta}(\psi')^2\sum_{i\in I}F^{i\bar{i}}|u_i|^2 \\
&\geq 2(1+2c_0)^2\delta\sum_{i\in I}F^{i\bar{i}}\frac{|u_{1\bar{1}\bar{i}}|^2}{w^2_{1\bar{1}}} - B_5,
\end{aligned}$$

where we use $(2\delta/(1-\delta))(\psi')^2 = \psi''$ by choosing $\delta = 1/(2A+1)$.

So we get

$$B_3 + B_4 + B_5 \geq -(1+2c_0)\frac{[1-2(1+2c_0)\delta]}{\lambda_1^2}\sum_{i\in I}F^{i\bar{i}}|u_{1\bar{1}\bar{i}}|^2.$$

Then we conclude

$$\begin{aligned}
&B_1+B_2+B_3+B_4+B_5\\
&\qquad\ge \frac{1+2c_0}{\lambda_1^2}(1-2\varepsilon)\sum_{i\in I}F^{i\bar{i}}|u_{1\bar{1}i}-T^1_{1i}(\lambda_1-1)|^2\\
&\qquad\quad-(1+2c_0)\frac{[1-2(1+2c_0)\delta]}{\lambda_1^2}\sum_{i\in I}F^{i\bar{i}}|u_{1\bar{1}\bar{i}}|^2\\
&\qquad\quad+\frac{2(1+2c_0)}{\lambda_1^2}\sum_{i\in I}F^{i\bar{i}}\operatorname{Re}(\lambda_1 T^1_{1i}u_{1\bar{1}\bar{i}})\\
&\qquad=\frac{1+2c_0}{\lambda_1^2}\sum_{i\in I}F^{i\bar{i}}\{(1-2\varepsilon)|u_{1\bar{1}i}-T^1_{1i}(\lambda_1-1)|^2\\
&\qquad\quad-(1-2(1+2c_0)\delta)|u_{1\bar{1}i}|^2+2\operatorname{Re}(\lambda_1 T^1_{1i}u_{1\bar{1}\bar{i}})\}\\
&\qquad=\frac{1+2c_0}{\lambda_1^2}\sum_{i\in I}F^{i\bar{i}}\{(2(1+2c_0)\delta-2\varepsilon)|u_{1\bar{1}i}|^2\\
&\qquad\quad+2[2\varepsilon(\lambda_1-1)+1]\operatorname{Re}(T^1_{1i}u_{1\bar{1}\bar{i}})+(1-2\varepsilon)(\lambda_1-1)^2|T^1_{1i}|^2\}\\
&\qquad\ge 0,
\end{aligned}$$

where the last inequality holds if we choose $\varepsilon=\frac{1}{4}\delta\le\frac{1}{16}$. In fact,

$$\begin{aligned}
\Delta=B^2-4AC&=4[2\varepsilon(\lambda_1-1)+1]^2-4(1-2\varepsilon)(\lambda_1-1)^2(2(1+2c_0)\delta-2\varepsilon)\\
&\le 36\varepsilon^2(\lambda_1-1)^2-4(1-2\varepsilon)(\lambda_1-1)^2(2(1+2c_0)\delta-2\varepsilon)\\
&\le 4(\lambda_1-1)^2(9\varepsilon^2-2(1-2\varepsilon)((1+2c_0)\delta)+2\varepsilon(1-2\varepsilon))\\
&\le 4(\lambda_1-1)^2(5\varepsilon^2+2\varepsilon-\delta)\\
&\le 4(\lambda_1-1)^2(4\varepsilon-\delta)\\
&=0.
\end{aligned}$$

Then we finally obtain

$$\begin{aligned}
0&\ge\frac{1}{4}\varphi'\sum_{i=1}^n F^{i\bar{i}}|u_{i\bar{i}}|^2+\Big(-\psi'+\frac{1}{2}\varphi'-C_2-C_3\Big)\sum_{i=1}^n F^{i\bar{i}}-C_1\\
&=\Big(-\psi'+\frac{1}{2}\varphi'-C_2-C_3\Big)\sum_{i=1}^n F^{i\bar{i}}+\frac{1}{4}\varphi'\sum_{i=1}^n F^{i\bar{i}}|u_{i\bar{i}}|^2-C_1\\
&\ge\sum_{i=1}^n F^{i\bar{i}}+\frac{1}{16K}\sum_{i=1}^n F^{i\bar{i}}{\lambda_i}^2-C_1,
\end{aligned}$$

where we use $-\psi'\ge C_0+1$ and $C_0=31n^2C_2+4C_3$.

In particular, $\sum_{i=1}^{n} F^{i\bar{i}} \le C$. By Lemma 2.2 in [Hou et al. 2010], we have $F^{1\bar{1}} \ge c(n,k)/C_1^{k-1}$, where $c(n,k)$ is a positive constant depending only on n and k. Then we get the desired estimate

$$\lambda_1 \le \frac{4C_1^{k/2}}{c(n,k)^{1/2}}\sqrt{K},$$

where C_1 is given in (4-17). □

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