POINTWISE CONVERGENCE OF ALMOST PERIODIC FOURIER SERIES AND ASSOCIATED SERIES OF DILATES

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Let $\mathcal{S}^2$ be the Stepanov space with norm $\|f\|_{\mathcal{S}^2} = \sup_{x\in\mathbb{R}}\left(\int_x^{x+1}|f(t)|^2 dt\right)^{1/2}$, $\lambda_n \uparrow \infty$, and let $(a_n)_{n\geq 1}$ satisfy Wiener's condition $\sum_{n\geq 1}\left(\sum_{k:n\leq\lambda_k\leq n+1}|a_k|\right)^2 < \infty$. We establish the following maximal inequality:

$$\left\|\sup_{N\geq 1}\left|\sum_{n=1}^N a_n e^{i\lambda_n t}\right|\right\|_{\mathcal{S}^2} \leq C\left(\sum_{n\geq 1}\left(\sum_{k:n\leq\lambda_k\leq n+1}|a_k|\right)^2\right)^{1/2},$$

where $C > 0$ is a universal constant. Moreover, the series $\sum_{n\geq 1} a_n e^{it\lambda_n}$ converges for λ-a.e. $t \in \mathbb{R}$. We give a simple and direct proof. This contains as a special case Hedenmalm and Saksman's result for Dirichlet series. We also obtain maximal inequalities for corresponding series of dilates. Let $(\lambda_n)_{n\geq 1}$, $(\mu_n)_{n\geq 1}$, be nondecreasing sequences of real numbers greater than 1. We prove the following interpolation theorem. Let $1 \leq p, q \leq 2$ be such that $1/p + 1/q = \frac{3}{2}$. There exists $C > 0$ such that for any sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ of complex numbers such that $\sum_{n\geq 1}\left(\sum_{k:n\leq\lambda_k<n+1}|\alpha_k|\right)^p < \infty$ and $\sum_{n\geq 1}\left(\sum_{k:n\leq\mu_k<n+1}|\beta_k|\right)^q < \infty$, we have

$$\left\|\sup_{N\geq 1}\left|\sum_{n=1}^N \alpha_n D(\lambda_n t)\right|\right\|_{\mathcal{S}^2} \leq C\left(\sum_{n\geq 1}\left(\sum_{k:n\leq\lambda_k<n+1}|\alpha_k|\right)^p\right)^{1/p}\left(\sum_{n\geq 1}\left(\sum_{k:n\leq\mu_k<n+1}|\beta_k|\right)^q\right)^{1/q},$$

where $D(t) = \sum_{n\geq 1}\beta_n e^{i\mu_n t}$ is defined in $\mathcal{S}^2$. Moreover, $\sum_{n\geq 1}\alpha_n D(\lambda_n t)$ converges in $\mathcal{S}^2$ and for λ-a.e. $t \in \mathbb{R}$. We further show that if $\{\lambda_k, k \geq 1\}$ satisfies the condition

$$\sum_{\substack{k\neq\ell,\, k'\neq\ell' \\ (k,\ell)\neq(k',\ell')}} (1 - |(\lambda_k - \lambda_\ell) - (\lambda_{k'} - \lambda_{\ell'})|)_+^2 < \infty,$$

then the series $\sum_k a_k e^{i\lambda_k t}$ converges on a set of positive Lebesgue measure only if the series $\sum_{k=1}^\infty |a_k|^2$ converges. The above condition is in particular fulfilled when $\{\lambda_k, k \geq 1\}$ is a Sidon sequence.

1. Introduction

We study almost everywhere convergence properties of almost periodic Fourier series in the Stepanov space $\mathcal{S}^2$ and of corresponding series of dilates. This space is defined as the subspace of functions f of $L^2_{\text{loc}}(\mathbb{R})$ verifying the following analogue of the Bohr almost periodicity property: *For all $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [x_0, x_0 + K_\varepsilon]$ such that $\|f(\cdot + \tau) - f(\cdot)\|_{\mathcal{S}^2} \le \varepsilon$.* The Stepanov norm in $\mathcal{S}^2$ is defined by

$$\|f\|_{\mathcal{S}^2} = \sup_{x \in \mathbb{R}} \left(\int_x^{x+1} |f(t)|^2 \, dt \right)^{1/2}.$$

Recall some basic facts. By the fundamental theorem on almost periodic functions, see [Besicovitch 1932, p. 88], the Stepanov space $\mathcal{S}^2$ coincides with the closure of the set of generalized trigonometric polynomials $\left\{ \sum_{k=1}^n a_k e^{i\lambda_k t} : \alpha_k \in \mathbb{C}, \ \lambda_k \in \mathbb{R} \right\}$ with respect to this norm. It is clear by considering for instance $f = \chi_{[0,1]}$ that the space $\{ f \in L^2_{\text{loc}}(\mathbb{R}) : \|f\|_{\mathcal{S}^2} < \infty \}$ is strictly larger than $\mathcal{S}^2$. Introduce also the Besicovitch seminorm of order 2 of $f \in L^2_{\text{loc}}(\mathbb{R})$

$$(1\text{-}1) \qquad \|f\|_{\mathcal{B}^2} = \limsup_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt \right)^{1/2}.$$

For every $\lambda \in \mathbb{R}$ and every $f \in L^1_{\text{loc}}(\mathbb{R})$ define the Fourier coefficient $\hat{f}(\lambda)$ of the exponent λ of f by

$$(1\text{-}2) \qquad \hat{f}(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x} \, dx,$$

whenever the limit exists. It is easily seen, by approximating by generalized trigonometric polynomials in the Stepanov norm, that the above limit exists for every $f \in \mathcal{S}^2$ and every $\lambda \in \mathbb{R}$. Moreover, for any finite family $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, we have by the Parseval equation in $\mathcal{B}^2$, see [Bellman 1944, p. 109],

$$\sum_{k=1}^n |\hat{f}(\lambda_k)|^2 \le \|f\|_{\mathcal{B}^2}^2 \le \|f\|_{\mathcal{S}^2}^2.$$

In particular, for $f \in \mathcal{S}^2$, $\Lambda := \{ \lambda \in \mathbb{R} : \hat{f}(\lambda) \ne 0 \}$ is countable. We call Λ the (set of) Fourier exponents of f. Let $f \in \mathcal{S}^2$ have of Fourier exponents Λ. Then

$$(1\text{-}3) \qquad \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 \le \|f\|_{\mathcal{B}^2}^2 \le \|f\|_{\mathcal{S}^2}^2.$$

We then define formally the Fourier series of $f \in \mathcal{S}^2$ as

$$\sum_{\lambda \in \Lambda} \hat{f}(\lambda) e^{i\lambda \cdot}.$$

Notice that the set $\Lambda \cap [-A, A]$ may be infinite for a given $A > 0$.

In this paper we are interested in the convergence of the Fourier series of f (to f) either in the Stepanov sense or in the almost everywhere sense, and the same sort of consideration will motivate us in the study of associated series of dilates. This second question is actually our main objective. See Section 3.

Concerning convergence of the Fourier series, it is necessary to recall Bredihina's extension to S^2 of Kolmogorov's theorem asserting that if $s_n(x)$ are the partial sums of the Fourier series of a function $f \in L^2(\mathbb{T})$, then $s_{m_n}(x)$ converges almost everywhere to f provided that $m_{n+1}/m_n \geq q > 1$. Bredihina [1968] showed that the Fourier series of a function in S^2 with α-separated frequencies ($\alpha > 0$), namely $|\lambda_k - \lambda_\ell| \geq \alpha > 0$ for all $k, \ell, k \neq \ell$, converges almost everywhere along any exponentially increasing subsequence. That is, for every $\rho > 1$, the sequence $\{\sum_{1 \leq k \leq \rho^n} \hat{f}(\lambda_k)e^{i\lambda_k t}, n \geq 1\}$ converges for λ-almost every $t \in \mathbb{R}$. The corresponding maximal inequality has been recently obtained by Bailey [2014] who also considered Stepanov spaces of higher order.

Remark 1.1. For a short proof of Kolmogorov's Theorem, see Marcinkiewicz [1933], who showed that this follows from Fejér's Theorem, see [Zygmund 1968, Theorem 3.4-(III)], and the classical fact that if a series $\sum u_n$ with partial sums s_n has infinitely many lacunary gaps and is summable $(C, 1)$ to sum s, then $s_n \to s$. See Theorem 1.27 in Chapter III of [Zygmund 1968].

In view of Carleson's theorem, a natural question is whether the "full" series converges for any $f \in S^2$.

That question has been addressed in the very specific situation of Dirichlet series by Hedenmalm and Saksman [2003]. A simplified proof may be found in [Konyagin and Queffélec 2001/02] (see also below). They proved the following. Let λ denote here and throughout the Lebesgue measure on the real line.

Theorem 1.2. Let $(a_n)_{n \geq 1}$ be complex numbers such that $\sum_{n \geq 1} n|a_n|^2 < \infty$. Then the series $\sum_{n \geq 1} a_n n^{it}$ converges λ-almost everywhere.

Their condition is optimal when $(a_n)_{n \geq 1}$ is nonincreasing. However, if $(a_n)_{n \geq 1}$ is supported say on $\{2^n : n \geq N\}$ the corresponding series is a standard (periodic) trigonometric series and in that case, the optimality is lost, since the condition is much stronger than Carleson's condition.

On the other hand, it follows from [Wiener 1926] that the series $\sum_{n \geq 1} a_n n^{it}$ converges in S^2 provided that

$$(1\text{-}4) \qquad \sum_{n \geq 0} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k| \right)^2 < \infty.$$

More precisely, the sequence of partial sums converges in S^2 to a limit $f \in S^2$. If $a_n > 0$ for every n, the converse is also true; see [Tornehave 1954].

Our first goal (see the next section) is to prove that (1-4) is sufficient for λ-a.e. convergence and to provide the corresponding maximal inequality. Moreover, it will turn out that the problem of the λ-almost everywhere convergence of the series $\sum_{n\geq 1} a_n e^{i\lambda_n t}$ can be reduced to the study of Dirichlet series.

In doing so, we obtain a Carleson-type theorem for almost periodic series and make the link with the study of almost everywhere convergence of the Fourier series associated with Stepanov's almost periodic functions.

Then, in Section 3, we consider associated series of dilates and obtain a sufficient condition for almost everywhere convergence. We further prove an interpolation theorem. Finally, in Section 4, we obtain a general necessary condition for the convergence almost everywhere of series of functions. The condition involves correlations of order 4. As an application, we show for instance that if $\{\lambda_k, k \geq 1\}$ is a Sidon sequence, and the series $\sum_k a_k e^{i\lambda_k t}$ converges on a set of positive λ-measure, then the series $\sum_{k=1}^{\infty} |a_k|^2$ converges.

2. Almost everywhere convergence of almost periodic Fourier series

We start with the proof by Konyagin and Queffélec of Hedenmalm and Saksman's result, to which we add a maximal inequality.

Proposition 2.1. *There exists $C > 0$ such that for any sequence $(a_n)_{n\geq 1}$ of complex numbers such that $\sum_{n\geq 1} n|a_n|^2 < \infty$,*

$$(2\text{-}1) \qquad \left\| \sup_{n\geq 1} \left| \sum_{k=1}^{n} a_k k^{i\cdot} \right| \right\|_{\mathcal{S}^2} \leq C \left(\sum_{n\geq 1} n|a_n|^2 \right)^{1/2}.$$

Before giving the proof, it is necessary to recall some classical but important facts. Let $g \in L^p(\mathbb{T})$, $1 < p < \infty$. Consider the maximal operator

$$T^* g(x) = \sup_{L=0}^{\infty} \left| \sum_{|k|\leq L} \hat{g}(k) e^{2i\pi kx} \right|.$$

For $f \in L^p(\mathbb{R})$ consider analogously the maximal operator

$$C^* f(x) = \sup_{T>0} \left| \int_{-T}^{T} \hat{f}(t) e^{ixt} \, dt \right|.$$

An operator U on L^p is called strong (p, p) if $\|Uf\|_p \leq C_p \|f\|p$ for all $f \in L^p$. The fact that strong (p, p), $1 < p < \infty$, for T^* is equivalent to strong (p, p) for C^* follows from known elementary arguments, see [Auscher and Carro 1992, p. 166]. We refer to [Hunt 1968, Theorem 1] concerning the deep fact that T^* is strong (p, p), $1 < p < \infty$ and we shall call it "the Carleson–Hunt theorem" when $p = 2$. We will freely use the fact the C^* is consequently strong (p, p), $1 < p < \infty$.

Proof. We first notice that it is enough to prove that

$$(2\text{-}2) \qquad \left\| \sup_{n\geq 1} \left| \sum_{k=1}^{n} a_k k^{i\cdot} \right| \right\|_{L^2[0,1]} \leq C \left(\sum_{n\geq 1} n|a_n|^2 \right)^{1/2}.$$

Indeed, then the desired result follows from the fact that

$$\sum_{k=1}^{n} a_k k^{i(t+x)} = \sum_{k=1}^{n} (a_k k^{ix}) k^{it},$$

since we may apply the above estimate to the sequence $(a_n n^{ix})_{n\geq 1}$ whose moduli are the same as the ones of the sequence $(a_n)_{n\geq 1}$.

Let us prove (2-2). Define $h \in L^2(\mathbb{R})$ by setting $h \equiv 0$ on $(-\infty, 1)$ and for every $n \in \mathbb{N}$, $h(x) = a_n$ whenever $x \in [n, n+1)$.

Let $N \geq 1$. We have

$$\sum_{n=1}^{N} a_n n^{it} = \sum_{n=1}^{N} a_n \int_n^{n+1} (e^{it \log n} - e^{it \log x})\, dx + \int_1^{N+1} h(x) e^{it \log x}\, dx$$

$$= \sum_{n=1}^{N} a_n \int_n^{n+1} (e^{it \log n} - e^{it \log x})\, dx + \int_0^{\log(N+1)} e^x h(e^x) e^{itx}\, dx.$$

Now, for every $x \in [n, n+1)$,

$$|e^{it \log n} - e^{it \log x}| \leq \frac{t}{n}.$$

Hence,

$$\sum_{n\geq 1} \left| a_n \int_n^{n+1} (e^{it \log n} - e^{it \log x})\, dx \right| \leq t \left(\sum_{n\geq 1} n|a_n|^2 \right)^{1/2} \left(\sum_{n\geq 1} \frac{1}{n^3} \right)^{1/2}.$$

On the other hand, $\int_0^{+\infty} e^{2x} |h|^2(e^x)\, dx = \int_1^{+\infty} u|h|^2(u)\, du \leq \sum_{n\geq 1}(n+1)|a_n|^2 < \infty$. Hence, since C^* is strong $(2-2)$,

$$\left\| \sup_{N\geq 1} \left| \int_0^{\log(N+1)} e^x h(e^x) e^{itx}\, dx \right| \right\|_{2,dt}^2 \leq C \int_0^{+\infty} e^{2x} |h|^2(e^x)\, dx.$$

Hence (2-1) follows. □

We now derive an improved version of Proposition 2.1.

Theorem 2.2. *There exists $C > 0$ such that for every sequence $(a_n)_{n\geq 1}$ of complex numbers satisfying* (1-4),

$$(2\text{-}3) \qquad \left\| \sup_{n\geq 1} \left| \sum_{k=1}^{n} a_k k^{i\cdot} \right| \right\|_{\mathcal{S}^2} \leq C \left(\sum_{n\geq 0} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k| \right)^2 \right)^{1/2}.$$

Moreover, $\sum_{n\geq 1} a_n n^{it}$ converges for λ-a.e. $t \in \mathbb{R}$.

Remarks 2.3. The proof of Theorem 2.2 makes use of the Carleson–Hunt theorem (T^* is strong $(2-2)$) and of Proposition 2.1. The latter was proved using that C^* is strong $(2-2)$, which is equivalent to the Carleson–Hunt theorem. On the other hand, given any sequence $(b_n)_{n\geq 1} \in \ell^2$, applying Theorem 2.2 with $(a_n)_{n\geq 1}$ such that $a_{2^k} = b_k$ and $a_n = 0$ otherwise, we see that Theorem 2.2 implies the Carleson–Hunt theorem, hence is equivalent to it. We shall see below that Theorem 2.2 allows one to treat almost everywhere convergence of series $\sum_{n\geq 1} b_n e^{it\lambda_n}$ for non-decreasing sequences $(\lambda_n)_{n\geq 1}$. Notice that Theorem 2.2 corresponds to the case where $\lambda_n = \log n$. For more on the Carleson–Hunt theorem we refer to [Lacey 2004]. See also [Jørsboe and Mejlbro 1982].

Proof. As in the previous proof, it is enough to prove a maximal inequality in $L^2([0, 1])$. We shall first work along the subsequence $(2^n - 1)_{n\geq 1}$.

Let $n \geq 1$ and define $S_{k,n} := \sum_{\ell=2^n}^{k} a_k$ for every $2^n \leq k \leq 2^{n+1} - 1$ and $S_{2^n-1,n} = 0$. In particular, for every $2^n \leq k \leq 2^{n+1} - 1$,

$$|S_{k,n}| \leq \sum_{j=2^n}^{2^{n+1}-1} |a_j|,$$

a fact that will be used freely in the sequel.

By Abel summation by parts, we have

$$\sum_{k=2^n}^{2^{n+1}-1} a_k k^{it} = \sum_{k=2^n}^{2^{n+1}-1} (S_{k,n} - S_{k-1,n}) k^{it} = \sum_{k=2^n}^{2^{n+1}-1} S_{k,n}(k^{it} - (k+1)^{it}) + 2^{(n+1)it} S_{2^{n+1}-1,n}.$$

Since $2^{(n+1)it} = e^{i(n+1)t\log 2}$ and by our assumption $\sum_{n\geq 1} |S_{2^{n+1}-1,n}|^2 < \infty$, it follows from Carleson's theorem that

$$\left\| \sup_{N\geq 1} \sum_{n=1}^{N} S_{2^{n+1}-1,n} 2^{(n+1)it} \right\|_{L^2([0,1],dt)} \leq C \left(\sum_{n\geq 1} |S_{2^{n+1}-1,n}|^2 \right)^{1/2}.$$

Hence, we are back to controlling the L^2-norm of

$$\sup_{N\geq 1} \left| \sum_{n=1}^{N} \sum_{k=2^n}^{2^{n+1}-1} S_{k,n}(k^{it} - (k+1)^{it}) \right|.$$

But we have

$$k^{it} - (k+1)^{it} = e^{it\log k} - e^{it\log(k+1)}$$
$$= e^{it\log k}\left(1 - e^{it\log(1+1/k)} + \frac{it}{k}\right) - \frac{it}{k} e^{it\log k} = u_k(t) - \frac{it}{k} e^{it\log k}.$$

Now there exists $C > 0$ such that $|u_k(t)| \leq C(t + t^2)/k^2$. Hence,

$$\sum_{n \geq 1} \sum_{k=2^n}^{2^{n+1}-1} |S_{k,n}||u_k(t)| \leq C(t+t^2) \sum_{n \geq 1} \frac{\sum_{k=2^n}^{2^{n+1}-1} |a_k|}{2^n}$$

$$\leq C(t+t^2) \left(\sum_{n \geq 1} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k| \right)^2 \right)^{1/2}.$$

It remains to control

$$\sup_{N \geq 1} \left| \sum_{n=1}^{N} \sum_{k=2^n}^{2^{n+1}-1} \frac{S_{k,n}}{k} e^{it \log k} \right|.$$

But we are exactly in the situation of [Proposition 2.1](). Hence

$$\left\| \sup_{N \geq 1} \left| \sum_{n=1}^{N} \sum_{k=2^n}^{2^{n+1}-1} \frac{S_{k,n}}{k} e^{it \log k} \right| \right\|_{L^2([0,1],dt)} \leq C \left(\sum_{n \geq 1} \sum_{k=2^n}^{2^{n+1}-1} k \frac{|S_{k,n}|^2}{k^2} \right)^{1/2}$$

$$\leq \left(\sum_{n \geq 1} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k| \right)^2 \right)^{1/2} < \infty.$$

Let $n \geq 1$ and $2^n \leq \ell \leq 2^{n+1} - 1$. We have

$$\left| \sum_{k=1}^{\ell} a_n k^{it} - \sum_{k=1}^{2^n-1} a_n k^{it} \right| \leq \sum_{k=2^n}^{2^{n+1}-1} |a_k|.$$

Hence,

$$\sup_{N \geq 1} \left| \sum_{n=1}^{N} a_n e^{it \log n} \right| \leq \sup_{N \geq 1} \left| \sum_{n=1}^{2^N-1} a_n e^{it \log n} \right| + \left(\sum_{n \geq 1} \left(\sum_{k=2^n}^{2^{n+1}-1} |a_k| \right)^2 \right)^{1/2}.$$

So, [(2-3)]() is proved. The λ-almost everywhere convergence may be proved by a standard procedure thanks to the maximal inequality. Alternatively, following all the steps of the proof of the maximal inequality lets us give a more direct proof. $\square$

As a corollary we deduce:

Theorem 2.4. *Let $(\lambda_n)_{n \geq 1}$ be an increasing sequence of positive real numbers tending to ∞. Let $(a_n)_{n \geq 1}$ be such that*

$$(2\text{-}4) \qquad \sum_{n \geq 1} \left(\sum_{k:n \leq \lambda_k \leq n+1} |a_k| \right)^2 < \infty.$$

There exists a universal constant $C > 0$ such that

$$(2\text{-}5) \qquad \left\| \sup_{N \geq 1} \left| \sum_{n=1}^{N} a_n e^{i\lambda_n t} \right| \right\|_{\mathcal{S}^2} \leq C \left(\sum_{n \geq 1} \left(\sum_{k : n \leq \lambda_k \leq n+1} |a_k| \right)^2 \right)^{1/2}.$$

Moreover, the series $\sum_{n \geq 1} a_n e^{it\lambda_n}$ converges for λ-a.e. $t \in \mathbb{R}$.

Proof. Write $u_n := [2^{\lambda_n}]$. Hence $(u_n)_{n \geq 1}$ is a nondecreasing sequence of integers. That sequence may overlap from time to time. So let $(v_k)_{k \geq 1}$ be a strictly increasing sequence of integers with same range as $(u_n)_{n \geq 1}$.

Define a sequence $(b_n)_{n \geq 1}$ as follows. Let $n \geq 1$ be such that there exists $k \geq 1$ such that $n = v_k$. Then set $b_n := \sum_{\ell : u_\ell = v_k} a_\ell$. If there is no $k \geq 1$ such that $n = v_k$, set $b_n := 0$.

We first control

$$\sup_{N \geq 1} \left| \sum_{n=1}^{N} b_n e^{it \log_2 n} \right|,$$

where $\log_2$ stands for the logarithm in base 2.

By Theorem 2.2, we have

$$\left\| \sup_{N \geq 1} \left| \sum_{n=1}^{N} b_n e^{i \log_2 n \cdot} \right| \right\|_{\mathcal{S}^2}^2 \leq C \sum_{n \geq 0} \left(\sum_{k=2^n}^{2^{n+1}-1} |b_k| \right)^2 = \sum_{n \geq 0} \left(\sum_{\ell : 2^n \leq u_\ell \leq 2^{n+1}-1} |b_\ell| \right)^2.$$

Now, if $2^n \leq u_\ell \leq 2^{n+1} - 1$, then $n \leq \lambda_\ell \leq n+1$ and our first step is proved.

Let $q \geq p$ be integers. There exist integers $q' \geq p'$ such that $v_{p'} = u_p$ and $v_{q'} = u_q$. We have

$$\left| \sum_{k=p}^{q} a_k e^{it\lambda_k} - \sum_{k=v_{p'}}^{v_{q'}} b_k e^{it \log_2 u_k} \right| \leq \sum_{k : u_k = u_p} |a_k| + \sum_{k : u_k = u_q} |a_k| + \sum_{\ell=p'}^{q'} \sum_{k : u_k = v_\ell} |a_k| |e^{it\lambda_k} - e^{it \log_2 u_k}|.$$

Clearly, it suffices to control

$$\sum_{n \geq 0} \sum_{\ell : 2^n \leq v_\ell \leq 2^{n+1}-1} \sum_{k : u_k = v_\ell} |a_k| |e^{it\lambda_k} - e^{it \log_2 u_k}|.$$

Now, for $2^n \leq v_\ell \leq 2^{n+1} - 1$ and $u_k = v_\ell$, using that $u_k \leq 2^{\lambda_k} \leq u_k + 1$, we see that $|\log_2(2^{\lambda_k}) - \log_2 u_k| \leq C/u_k$ and that

$$|e^{it\lambda_k} - e^{it \log_2 u_k}| = |e^{it \log_2(2^{\lambda_k})} - e^{it \log_2 u_k}| \leq \frac{C|t|}{u_k} \leq \frac{C|t|}{2^n}.$$

Hence, using Cauchy–Schwarz,

$$\sum_{n\geq 0}\sum_{\ell:2^n\leq v_\ell\leq 2^{n+1}-1}\sum_{k:u_k=v_\ell}|a_k||e^{it\lambda_k}-e^{it\log u_k}| \leq Ct\sum_{n\geq 0}2^{-n}\sum_{k:2^n\leq u_k\leq 2^{n+1}-1}|a_k|$$

$$\leq Ct\bigg(\sum_{n\geq 0}\bigg(\sum_{k:2^n\leq u_k\leq 2^{n+1}-1}|a_k|\bigg)^2\bigg)^{1/2},$$

which converges by our assumption. □

We shall now derive an almost everywhere convergence result concerning the Fourier series of an almost periodic function in $\mathcal{S}^2$. We shall first recall known results about norm convergence.

Let $(\lambda_n)_{n\geq 1}$ be a (not necessarily increasing) sequence of positive real numbers. As already mentioned (in the case of Dirichlet series), by [Wiener 1926], see also [Tornehave 1954], if

$$(2\text{-}6) \qquad \sum_{n\geq 0}\bigg(\sum_{k\geq 1:n\leq \lambda_k <n+1}|a_k|\bigg)^2 < \infty,$$

then $\sum_{n\geq 1}a_n e^{i\lambda_n t}$ is the Fourier series of an element of $f\in\mathcal{S}^2$.

On the other hand, if $f\in\mathcal{S}^2$ admits a sequence of positive real numbers $(\lambda_n)_{n\geq 1}$ as frequencies and such that $\hat{f}(\lambda_n)\geq 0$ for every $n\geq 1$, then, see [Tornehave 1954],

$$\sum_{n\geq 0}\bigg(\sum_{k\geq 1:n\leq \lambda_k <n+1}|\hat{f}(\lambda_k)|\bigg)^2 \leq C\|f\|_{\mathcal{S}^2}^2.$$

Hence, (2-6) holds.

Condition (2-6) is thus optimal for deciding whether $\sum_{n\geq 1}a_n e^{i\lambda_n t}$ is the Fourier series of an element of $\mathcal{S}^2$ or not. One can not however expect that it is always necessary, so we provide a counterexample in Proposition 2.7 below.

Let $f\in\mathcal{S}^2$ be such that $\Lambda\subset[0,+\infty)$ (that restriction may be obviously removed). Assume that Λ is α-separated for some $\alpha>0$ and write $\Lambda:=\{\lambda_1<\lambda_2\cdots\}$. Then,

$$\frac{\alpha}{C}\sum_{n\geq 0}\bigg(\sum_{k\geq 1:n\leq \lambda_k <n+1}|\hat{f}(\lambda_k)|\bigg)^2 \leq \sum_{n\geq 1}|\hat{f}(\lambda_n)|^2 \leq \|f\|_{\mathcal{S}^2}^2$$

$$\leq C\sum_{n\geq 0}\bigg(\sum_{k\geq 1:n\leq \lambda_k <n+1}|\hat{f}(\lambda_k)|\bigg)^2.$$

In particular, we have the following direct consequence of Theorem 2.2:

Corollary 2.5. *Let $f \in \mathcal{S}^2$ be such that $\Lambda \subset [0, +\infty)$. Assume that Λ is α-separated for some $\alpha > 0$. There exists $C > 0$, independent of f and α such that*

$$\left\| \sup_{N \geq 1} \left| \sum_{n=1}^{N} \hat{f}(\lambda_n) e^{i\lambda_n \cdot} \right| \right\|_{\mathcal{S}^2} \leq C \frac{\|f\|_{\mathcal{S}^2}}{\alpha}.$$

Moreover, the series $\sum_{n \geq 1} \hat{f}(\lambda_n) e^{i\lambda_n \cdot}$ converges for λ-almost every $t \in \mathbb{R}$.

We now give an example of Fourier series converging in $\mathcal{S}^2$ while (2-6) does not hold. Let us first recall the following result of Halász; see [Queffélec 1984].

Lemma 2.6. *There exists $C > 0$ such that for every sequence of iid Rademacher variables $(\varepsilon_n)_{n \geq 1}$*

$$(2\text{-}7) \qquad \mathbb{E}\left(\sup_{t \in \mathbb{R}} \left| \sum_{k=1}^{n} \varepsilon_k k^{it} \right| \right) \leq C \frac{n}{\log(n+1)}.$$

Proposition 2.7. *Let $(\varepsilon_n)_{n \geq 1}$ be iid Rademacher variables on $(\Omega, \mathcal{F}, \mathbb{P})$. For $\mathbb{P}$-almost all $\omega \in \Omega$, $\sum_{n \geq 1} \varepsilon_n(\omega) n^{it} / n\sqrt{\log(n+1)}$ converges in $\mathcal{S}^2$, while (2-4) is not satisfied (with $a_n = \varepsilon_n(\omega)/n\sqrt{\log(n+1)}$).*

Proof. For every $n \geq 1$, every $2^n \leq k \leq 2^{n+1}$ and every $\omega \in \Omega$, we have

$$\left\| \sum_{\ell=2^n}^{k} \frac{\varepsilon_\ell(\omega) \ell^{it}}{\ell \sqrt{\log(\ell+1)}} \right\|_{\mathcal{S}^2} \leq \sum_{\ell=2^n}^{k} \frac{1}{\ell \sqrt{\log(\ell+1)}} \leq \frac{2}{\sqrt{n}} \xrightarrow[n \to +\infty]{} 0.$$

Hence, it suffices to prove that for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\left(\sum_{n=1}^{2^N} \frac{\varepsilon_n(\omega) n^{it}}{n\sqrt{\log(n+1)}} \right)_{N \geq 1} \quad \text{converges in } \mathcal{S}^2.$$

Let $S_n(t) := \sum_{k=1}^{n} \varepsilon_k k^{it}$ $(S_0(t) = 0)$ and $u_n := \left(n\sqrt{\log(n+1)} \right)^{-1}$. We have

$$\sum_{n=1}^{2^N} \frac{\varepsilon_n(\omega) n^{it}}{n\sqrt{\log(n+1)}} = \sum_{n=1}^{2^N} (S_n(t) - S_{n-1}(t)) u_n = \sum_{n=1}^{2^N} S_n(t)(u_n - u_{n+1}) + S_{2^N}(t) u_{2^N+1}.$$

It follows from (2-7) that

$$\mathbb{E}\left(\sum_{n \geq 1} \sup_{t \in \mathbb{R}} |S_n(t)(u_n - u_{n+1})| \right) < \infty, \quad \mathbb{E}\left(\sum_{n \geq 1} \sup_{t \in \mathbb{R}} |S_{2^N}(t) u_{2^N+1}| \right) < \infty,$$

and the result follows. $\qquad \square$

3. Convergence almost everywhere of associated series of dilates

Theorem 3.1. *Let $(\lambda_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ be nondecreasing sequences of real numbers greater than 1. Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers such that*

$$\sum_{n\geq 1}\left(\sum_{k:n\leq\lambda_k<n+1}|\alpha_k|\right)^2 < \infty.$$

Let $(\beta_n)_{n\geq 1}\in\ell^1$. Then $D(t):=\sum_{n\geq 1}\beta_n e^{i\mu_n t}$ defines a continuous function on $\mathbb{R}$ (and in $\mathcal{S}^2$) and there exists a universal constant $C > 0$ such that

$$(3\text{-}1) \quad \left\|\sup_{N\geq 1}\left|\sum_{n=1}^N \alpha_n D(\lambda_n t)\right|\right\|_{\mathcal{S}^2} \leq C\left(\sum_{n\geq 1}|\beta_n|\right)\left(\sum_{n\geq 1}\left(\sum_{k:n\leq\lambda_k<n+1}|\alpha_k|\right)^2\right)^{1/2}.$$

Moreover, the series $\sum_{n\geq 1}\alpha_n D(\lambda_n t)$ converges in $\mathcal{S}^2$ and for λ-a.e. $t\in\mathbb{R}$.

Proof. Let $x\in\mathbb{R}$. The fact that D is a continuous function in $\mathcal{S}^2$ follows easily from the fact that $(\beta_n)_{n\geq 1}\in\ell^1$. We also have, for every $N\geq 1$,

$$\left|\sum_{n=1}^N \alpha_n D(\lambda_n t)\right| \leq \sum_{k\geq 1}|\beta_k|\left|\sum_{n=1}^N \alpha_n e^{it\lambda_n\mu_k}\right|.$$

By Theorem 2.4, we have

$$\int_x^{x+1}\sup_{N\geq 1}\left|\sum_{n=1}^N \alpha_n e^{it\lambda_n\mu_k}\right|^2 dt = \frac{1}{\mu_k}\int_{\mu_k x}^{\mu_k(x+1)}\sup_{N\geq 1}\left|\sum_{n=1}^N \alpha_n e^{it\lambda_n}\right|^2 dt$$

$$\leq \frac{[\mu_k]+1}{\mu_k}\left\|\sup_{N\geq 1}\left|\sum_{n=1}^N \alpha_n e^{it\lambda_n}\right|\right\|_{\mathcal{S}^2}^2,$$

and (3-1) follows.

The convergence almost everywhere and in $\mathcal{S}^2$ follows by standard arguments. $\square$

We also have the following obvious corollary of Theorem 2.4, whose proof is left to the reader:

Proposition 3.2. *Let $(\lambda_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ be nondecreasing sequences of real numbers greater than 1. Let $(\beta_n)_{n\geq 1}$ be a sequence of complex numbers such that*

$$\sum_{n\geq 1}\left(\sum_{k:n\leq\mu_k<n+1}|\beta_k|\right)^2 < \infty.$$

Let $(\alpha_n)_{n\geq 1} \in \ell^1$. Then, $D(t) := \sum_{n\geq 1}\beta_n e^{i\mu_n t}$ converges in $\mathcal{S}^2$ and there exists a universal constant $C > 0$ such that

$$(3\text{-}2) \quad \left\|\sup_{N\geq 1}\left|\sum_{n=1}^{N}\alpha_n D(\lambda_n t)\right|\right\|_{\mathcal{S}^2} \leq C\left(\sum_{n\geq 1}|\alpha_n|\right)\left(\sum_{n\geq 1}\left(\sum_{k:n\leq \mu_k<n+1}|\beta_k|\right)^2\right)^{1/2}.$$

Moreover, the series $\sum_{n\geq 1}\alpha_n D(\lambda_n t)$ converges in $\mathcal{S}^2$ and for λ-a.e. $t \in \mathbb{R}$.

Theorem 3.3. *Let $(\lambda_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ be nondecreasing sequences of real numbers greater than 1. Let $1 \leq p, q \leq 2$ satisfy $1/p + 1/q = \frac{3}{2}$. There exists $C > 0$ such that for any sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ of complex numbers such that*

$$(3\text{-}3) \quad \sum_{n\geq 1}\left(\sum_{k:n\leq \lambda_k<n+1}|\alpha_k|\right)^p < \infty \quad and \quad \sum_{n\geq 1}\left(\sum_{k:n\leq \mu_k<n+1}|\beta_k|\right)^q < \infty,$$

we have

$$(3\text{-}4) \quad \left\|\sup_{N\geq 1}\left|\sum_{n=1}^{N}\alpha_n D(\lambda_n t)\right|\right\|_{\mathcal{S}^2}$$

$$\leq C\left(\sum_{n\geq 1}\left(\sum_{k:n\leq \lambda_k<n+1}|\alpha_k|\right)^p\right)^{1/p}\left(\sum_{n\geq 1}\left(\sum_{k:n\leq \mu_k<n+1}|\beta_k|\right)^q\right)^{1/q},$$

where $D(t) := \sum_{n\geq 1}\beta_n e^{i\mu_n t}$ is defined in $\mathcal{S}^2$. Moreover, the series $\sum_{n\geq 1}\alpha_n D(\lambda_n t)$ converges in $\mathcal{S}^2$ and for λ-a.e. $t \in \mathbb{R}$.

Before doing the proof let us mention the following immediate corollaries. We first apply Theorem 3.3 with the choice $\mu_n = \log n$, $n \geq 1$ and $\lambda_k = k$, $k \geq 1$.

Corollary 3.4. *Assume that*

$$\sum_{k\geq 1}|\alpha_k|^p < \infty \quad and \quad \sum_{n\geq 1}\left(\sum_{k:2^n\leq k<2^{n+1}}|\beta_k|\right)^q < \infty,$$

for some $1 \leq p, q \leq 2$ such that $1/p + 1/q = \frac{3}{2}$. Let $D(t) := \sum_{n\geq 1}\beta_n n^{it}$. Then the series $\sum_{k\geq 1}\alpha_k D(kt)$ converges in $\mathcal{S}^2$ and for λ-a.e. $t \in \mathbb{R}$.

Example 3.5. *Let $\frac{1}{2} < \alpha \leq 1$. Choose $1/\alpha < p \leq 2$ and $q = 2p/(3p-2)$ $(1 \leq q < 2)$. Let $D(t) = \sum_{n\geq 1}\beta_n n^{it}$ and assume that*

$$(3\text{-}5) \quad \sum_{n\geq 1}\left(\sum_{k:2^n\leq k<2^{n+1}}|\beta_k|\right)^q < \infty.$$

Then the series

$$(3\text{-}6) \quad \sum_{k\geq 1}\frac{D(kt)}{k^\alpha}$$

converges almost everywhere. This extends to Dirichlet series Hartman and Wint-
ner's result [1938] showing that the series $\Phi_\alpha(x) = \sum_{k=1}^\infty \psi(kx)/k^\alpha$ converges
almost everywhere. Here $\psi(x) = x - [x] - \frac{1}{2} = \sum_{j=1}^\infty \sin 2\pi j x/j$, and $[x]$ is the
integer part of x. That result is also a special case of (3-6): take $\beta_n = 1/j$ if $n = 2^j$,
$j \geq 1$ and $\beta_n = 0$ elsewhere.

Remark 3.6. To our knowledge [Hartman and Wintner 1938] contains, among other
results on Φ_α, the first convergence result for the series of dilates $\sum_{k=1}^\infty \alpha_k \psi(kx)$.

Then, we apply Theorem 3.3 with the choice $\mu_n = n$, $n \geq 1$ and $\lambda_k = k$, $k \geq 1$.

Corollary 3.7. *Assume that*

$$\sum_{k \geq 1} |\alpha_k|^p < \infty \quad and \quad \sum_{j \geq 1} |b_j|^q < \infty$$

for some $1 \leq p, q \leq 2$ such that $1/p + 1/q = \frac{3}{2}$. Let $D(t) = \sum_{\ell \geq 1} b_\ell e^{i\ell t}$. Then the
series $\sum_{k \geq 1} \alpha_k D(kt)$ converges in $\mathcal{S}^2$ and for λ-a.e. $t \in \mathbb{R}$.

Remark 3.8. Suppose that $b_j = \mathcal{O}(1/j^\alpha)$ for some $\frac{1}{2} < \alpha \leq 1$. Assume that

$$\sum_{k \geq 1} |\alpha_k|^p < \infty,$$

for some $1 \leq p < 2/(3-2\alpha)$. Then $\sum_{j \geq 1} |b_j|^q < \infty$ for q such that $1/p+1/q = \frac{3}{2}$ and
we have $1 \leq p, q \leq 2$. We deduce from Corollary 3.7 that the series $\sum_{k \geq 1} \alpha_k D(kt)$
converges in $\mathcal{S}^2$ and for λ-a.e. $t \in \mathbb{R}$. When $\frac{1}{2} < \alpha < 1$, the nearly optimal sufficient
condition $\sum_{k \geq 1} |c_k|^2 \exp\{K(\log k)^{1-\alpha}/(\log\log k)^\alpha\} < \infty$ in which $K = K(\alpha)$ has
been recently established in [Aistleitner et al. 2015, Theorem 2]. See also [Weber
2016, Theorem 3.1] for conditions of individual type, i.e., depending on the support
of the coefficient sequence. When $\alpha = 1$, the optimal sufficient coefficient condition,
namely that $\sum_{k=1}^\infty |\alpha_k|^2 (\log\log k)^{2+\varepsilon}$ converges for some $\varepsilon > 0$ suffices for the
convergence almost everywhere, has been recently obtained by Lewko and Radziwiłł
[2017, Corollary 3].

These results are clearly better. However, we note that our results are, even in
the trigonometrical case, independent from these ones, and concern a larger class
of trigonometrical series $D(t)$.

Proof of Theorem 3.3. Clearly, we only need to prove (3-4). Let $(\alpha_n)_{n \geq 1}$ and
$(\beta_n)_{n \geq 1}$ be in $\ell^1(\mathbb{N})$, fixed for all the proof. Let $D(t) := \sum_{n \geq 1} \beta_n e^{i\mu_n t}$. It is enough
to prove that for every $N \geq 1$,

$$\left\| \sup_{m=1}^N \left| \sum_{n=1}^m \alpha_n D(\lambda_n t) \right| \right\|_{\mathcal{S}^2} \leq C \left(\sum_{n \geq 1} \left(\sum_{k:n \leq \lambda_k < n+1} |\alpha_k| \right)^p \right)^{1/p} \left(\sum_{n \geq 1} \left(\sum_{k:n \leq \mu_k < n+1} |\beta_k| \right)^q \right)^{1/q},$$

for a constant $C > 0$ not depending on N, $(\alpha_n)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$.

We do that by interpolating (3-1) and (3-2). Define Banach spaces as follows:

$$X_1 := \left\{ (a_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}} : \|(a_n)_{n\geq 1}\|_{X_1} := \sum_{n\geq 1} \sum_{k:n\leq \lambda_k < n+1} |a_k| < \infty \right\},$$

$$X_2 := \left\{ (a_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}} : \|(a_n)_{n\geq 1}\|_{X_2} := \left(\sum_{n\geq 1} \left(\sum_{k:n\leq \lambda_k < n+1} |a_k| \right)^2 \right)^{1/2} < \infty \right\},$$

$$Y_1 := \left\{ (b_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}} : \|(b_n)_{n\geq 1}\|_{Y_1} := \sum_{n\geq 1} \sum_{k:n\leq \mu_k < n+1} |b_k| < \infty \right\},$$

$$Y_2 := \left\{ (b_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}} : \|(b_n)_{n\geq 1}\|_{Y_1} := \left(\sum_{n\geq 1} \left(\sum_{k:n\leq \mu_k < n+1} |b_k| \right)^2 \right)^{1/2} < \infty \right\}.$$

For every $t \in \mathbb{R}$, let

$$J(t) := \min \left\{ j \in \mathbb{N} : 1 \leq j \leq N, \; \left| \sum_{n=1}^{j} \alpha_n D(\lambda_n t) \right| = \sup_{m=1}^{N} \left| \sum_{n=1}^{m} \alpha_n D(\lambda_n t) \right| \right\}.$$

Define a linear operator T on $(X_1 + X_2) \times (Y_2 + Y_1)$ by setting

$$T((a_n)_{n\geq 1}, (b_n)_{n\geq 1}) := \sum_{k=1}^{N} \mathbf{1}_{\{k \leq J(t)\}} a_k \left(\sum_{\ell \geq 1} b_\ell e^{i\lambda_k \mu_\ell t} \right).$$

By Propositions 3.1 and 3.2, T is continuous from $X_1 \times Y_2$ to $\mathcal{S}^2$ and from $X_2 \times Y_1$ to $\mathcal{S}^2$.

It follows from paragraph 10.1 of [Calderón 1964] that for every $s \in [0, 1]$ there exists C_s such that, with their notation,

$$\|T((a_n)_{n\geq 1}, (b_n)_{n\geq 1})\|_{\mathcal{S}^2} \leq C_s \|(a_n)_{n\geq 1}\|_{[X_1, X_2]_s} \|(b_n)_{n\geq 1}\|_{[Y_2, Y_1]_s},$$

where

$$\|(a_n)_{n\geq 1}\|_{[X_1, X_2]_s} = \inf\{\|f\|_{\mathcal{F}} : f \in \mathcal{F}, \; f(s) = (a_n)_{n\geq 1}\},$$

and $\mathcal{F}$ is the Banach space of continuous functions f from $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ to $X_1 + X_2$, analytic on $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ such that for every $t \in \mathbb{R}$, $f(it) \in X_1$ and $f(1+it) \in X_2$ with $\lim_{|t|\to+\infty} f(it) = \lim_{|t|\to+\infty} f(1+it) = 0$, endowed with the norm

$$\|f\|_{\mathcal{F}} := \max \left(\sup_{t\in\mathbb{R}} \|f(it)\|_{X_1}, \; \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_2} \right).$$

The norm $\|(b_n)_{n\geq 1}\|_{[Y_2, Y_1]_s}$ is defined similarly.

We shall now give an upper bound for $\|(a_n)_{n\geq 1}\|_{[X_1,X_2]_s}$. By homogeneity, we may assume that

$$\sum_{n\geq 1}\left(\sum_{n\leq \lambda_k <n+1}|a_k|\right)^{2/(2-s)}=1.$$

Let $\varepsilon > 0$. Define an element f_ε of $\mathcal{F}$ by setting for every $z \in \mathbb{C}$ such that $0\leq \mathrm{Re}\, z\leq 1$, $f_\varepsilon(z)=(c_n(z))_{n\geq 1}$ where, for every $n,k\geq 1$ such that $n\leq \lambda_k <n+1$,

$$c_k(z) = e^{\varepsilon(z^2-s^2)}a_k\left(\sum_{n\leq \lambda_\ell <n+1}|a_\ell|\right)^{(2-z)/(2-s)-1},$$

if $\sum_{n\leq \lambda_\ell <n+1}|a_\ell|\neq 0$ and $c_k(z)=0$ otherwise.

The introduction of ε here is a standard trick to ensure the assumptions

$$\lim_{|t|\to+\infty}f_\varepsilon(it)=\lim_{|t|\to+\infty}f_\varepsilon(1+it)=0.$$

Notice that $f_\varepsilon(s)=(a_n)_{n\geq 1}$. For every $t\in\mathbb{R}$,

$$\|f_\varepsilon(it)\|_{X_1}\leq \sum_{n\geq 1}\left(\sum_{n\leq \lambda_k <n+1}|a_k|\right)^{\frac{2}{(2-s)}}=1.$$

Similarly, for every $t\in\mathbb{R}$,

$$\|f_\varepsilon(1+it)\|_{X_2}\leq e^\varepsilon \sum_{n\geq 1}\left(\sum_{n\leq \lambda_k <n+1}|a_k|\right)^{\frac{2}{(2-s)}}=e^\varepsilon.$$

Letting $\varepsilon\to 0$, we infer that

$$\|(a_n)_{n\geq 1}\|_{[X_1,X_2]_s}\leq 1=\left(\sum_{n\geq 1}\left(\sum_{n\leq \lambda_k <n+1}|a_k|\right)^{\frac{2}{(2-s)}}\right)^{\frac{2-s}{2}}.$$

Similarly, one can prove that

$$\|(b_n)_{n\geq 1}\|_{[X_1,X_2]_s}\leq \left(\sum_{n\geq 1}\left(\sum_{n\leq \lambda_k <n+1}|b_k|\right)^{\frac{2}{(1+s)}}\right)^{\frac{1+s}{2}}.$$

Taking $s=2(1-1/p)$ yields the desired result. $\qquad\square$

4. A necessary condition for convergence almost everywhere

Hartman [1942] has proved the following result:

Theorem 4.1. *Assume that*

(4-1)
$$\frac{\lambda_k}{\lambda_{k-1}}\geq q > 1,\quad k\geq 1.$$

Assume that the series $\sum_{k=1}^\infty a_k e^{i\lambda_k t}$ converges for almost all real t. Then the series $\sum_{k=1}^\infty |a_k|^2$ converges.

The proof is similar to Zygmund's [1968, Proof of Lemma 6.5, Chapter V] (see also p. 120–122 of the 1935 edition).

Remark 4.2. The converse of Theorem 4.1 is due to Kac [1941]. If $\sum_{k=1}^{\infty} |a_k|^2$ converges, then the series $\sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$ with $(\lambda_k)_{k\geq 1}$ verifying (4-1), converges for almost all real t. Kac's proof is a modification of Marcinkiewicz's. See Remark 1.1. In place of Fejér's theorem, another summation method is used. See Theorem 13 and pages 84–85 in [Titchmarsh 1948], and Theorem 21 in [Hardy and Riesz 1915].

Theorem 4.1 can be extended in the following way:

Theorem 4.3. *Let $\{\lambda_k, k \geq 1\}$ be a increasing sequence of positive reals satisfying the condition*

$$(4\text{-}2) \qquad M := \sum_{\substack{k \neq \ell, \, k' \neq \ell' \\ (k,\ell) \neq (k',\ell')}} (1 - |(\lambda_k - \lambda_\ell) - (\lambda_{k'} - \lambda_{\ell'})|)_+^2 < \infty.$$

Assume that

$$(4\text{-}3) \qquad \lambda \left\{ \sum_k a_k e^{i\lambda_k t} \text{ converges} \right\} > 0.$$

Then the series $\sum_{k=1}^{\infty} |a_k|^2$ converges.

Remark 4.4. By considering integers k such that $n \leq \lambda_k < n + \frac{1}{2}$, and next those such that $n + \frac{1}{2} \leq \lambda_k \leq n + 1$, we observe that condition (4-2) implies that

$$\sup_n \# \{k : n \leq \lambda_k < n + 1\} < \infty.$$

We give an application. Recall that a Sidon sequence is a set of integers with the property that the pairwise sums of elements are all distinct. As a corollary we get

Corollary 4.5. *Let $\{\lambda_k, k \geq 1\}$ be a Sidon sequence. Assume that (4-3) is satisfied. Then the series $\sum_{k=1}^{\infty} |a_k|^2$ converges.*

Remark 4.6. In contrast with Hadamard gap sequences, Sidon sequences may grow at most polynomially. See [Ruzsa 2001] where it is for instance proved that the sequence $\{n^5 + [\xi n^4], n \geq n_0\}$ is for some real number $\xi \in [0, 1]$ and n_0 large, a Sidon sequence.

Proof of Corollary 4.5. Let $(k, \ell) \neq (k', \ell')$ with $k \neq \ell$ and $k' \neq \ell'$. As the equation $\lambda_k - \lambda_\ell = \lambda_{k'} - \lambda_{\ell'}$ means $\lambda_k + \lambda_{\ell'} = \lambda_\ell + \lambda_{k'}$, the fact that $\{\lambda_k, k \geq 1\}$ is a Sidon sequence implies that the only possible solutions are $k = k'$, $\ell' = \ell$ or $k = \ell$, $\ell' = k'$. The last one is impossible by assumption, and the first would mean that $(k, \ell) = (k', \ell')$ which is excluded. Consequently, $\lambda_k - \lambda_\ell \neq \lambda_{k'} - \lambda_{\ell'}$. Hence the sum in (4-2) is always zero. $\qquad \square$

Remark 4.7. It follows from Hartman's proof that under condition (4-1), the sequence of differences $\lambda_k - \lambda_\ell$, $k \neq \ell$ is a finite union of subsequences such that the difference of any two numbers of the same subsequence exceeds 1. These subsequences fulfill assumption (4-2) of Theorem 4.3, and thus Theorem 4.1 follows from Theorem 4.3.

Theorem 4.3 is a consequence of the following general necessary condition for almost everywhere convergence of series of functions.

Theorem 4.8. *Let $(X, \mathcal{B}, \tau)$ be a probability space. Let $\{g_k, k \geq 1\} \subset L^4(\tau)$ be a sequence of functions with $\|g_k\|_{2,\tau} = 1$, $\|g_k\|_{4,\tau} \leq K$ and satisfying the condition*

$$(4\text{-}4) \qquad M := \sum_{\substack{k \neq \ell, \, k' \neq \ell' \\ (k,\ell) \neq (k',\ell')}} |\langle g_k \overline{g_\ell}, g_{k'} \overline{g_{\ell'}} \rangle_\tau|^2 < \infty.$$

Assume that

$$(4\text{-}5) \qquad \tau \left\{ \sum_k a_k g_k(t) \text{ converges} \right\} > 0.$$

Then the series $\sum_{k=1}^\infty |a_k|^2$ converges.

Proof of Theorem 4.8. We use Hartman's method and the below classical generalization of Bessel's inequality.

Lemma 4.9 (Bellman–Boas inequality). *Let $x, y_1, \ldots, y_n$ be elements of an inner product space $(H, \langle \cdot, \cdot \rangle)$. Then*

$$\sum_{i=1}^n |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|^2 \right)^{1/2} \right\}.$$

See [Bellman 1944] for instance. As

$$\left\{ t : \sum_k a_k g_k(t) \text{ converges} \right\} = \bigcap_{\varepsilon > 0} \bigcup_V \bigcap_{u > v > V} \left\{ t : \left| \sum_{k=v}^u a_k g_k(t) \right| \leq \varepsilon \right\},$$

by assumption it follows that for any $\varepsilon > 0$, there exists an integer V such that if

$$A := \bigcap_{u > v > V} \left\{ \left| \sum_{k=v}^u a_k g_k(t) \right| \leq \varepsilon \right\},$$

then

$$(4\text{-}6) \qquad \tau(A) > 0.$$

Assume the series $\sum_{k \geq 1} |a_k|^2$ is divergent. We are going to prove that this will contradict (4-6).

By squaring out,

$$(4\text{-}7) \quad \int_A \left| \sum_{k=n}^m a_k g_k(t) \right|^2 \tau(dt) = \tau(A) \sum_{k=n}^m |a_k|^2 + \sum_{\substack{k,\ell=n \\ k \neq \ell}}^m a_k \bar{a}_\ell \int_A g_k(t) \overline{g_\ell}(t) \tau(dt).$$

By using the Cauchy–Schwarz inequality,

$$\left| \sum_{\substack{k,\ell=n \\ k \neq \ell}}^m a_k \bar{a}_\ell \int_A g_k(t) \overline{g_\ell}(t) \tau(dt) \right| \leq \left(\sum_{\substack{k,\ell=n \\ k \neq \ell}}^m |a_k|^2 |a_\ell|^2 \right)^{1/2} \left(\sum_{\substack{k,\ell=n \\ k \neq \ell}}^m \left| \int_A g_k(t) \overline{g_\ell}(t) \tau(dt) \right|^2 \right)^{1/2}.$$

Applying Lemma 4.9 to the system of vectors of $L_\tau^2(\mathbb{R})$, $\chi(A)$, $g_k(t) \overline{g_\ell}(t)$, $n \leq k, \ell \leq m$ gives, in view of the assumption made,

$$\sum_{\substack{k,\ell=n \\ k \neq \ell}}^m \left| \int_A g_k(t) \overline{g_\ell}(t) \tau(dt) \right|^2 \leq \tau(A)^2 \left\{ K^2 + \left(\sum_{\substack{(k,\ell) \neq (k',\ell') \\ n \leq k \neq \ell \leq m \\ n \leq k' \neq \ell' \leq m}} |\langle g_k \overline{g_\ell}, g_{k'} \overline{g_{\ell'}} \rangle_\tau|^2 \right)^{1/2} \right\}$$

$$\leq \tau(A)^2 \{ K^2 + M^{1/2} \}.$$

Letting n, m tend to infinity, it follows that the series $\sum_{k \neq \ell} \left| \int_A g_k(t) \overline{g_\ell}(t) \tau(dt) \right|^2$ converges. Consequently, for all $m > n$, $n > N$, N depending on A

$$\sum_{\substack{k,\ell=n \\ k \neq \ell}}^m \left| \int_A g_k(t) \overline{g_\ell}(t) \tau(dt) \right|^2 \leq \tau(A)^2/4.$$

There is no loss in assuming $N > V$, which we do. Therefore

$$\left| \sum_{\substack{k,\ell=n \\ k \neq \ell}}^m a_k \bar{a}_\ell \int_A g_k(t) \overline{g_\ell}(t) \tau(dt) \right| \leq \left(\sum_{\substack{k,\ell=n \\ k \neq \ell}}^m |a_k|^2 |a_\ell|^2 \right)^{1/2} \left(\frac{\tau(A)}{2} \right).$$

This along with (4-7) implies

$$(4\text{-}8) \quad \int_A \left| \sum_{k=n}^m a_k g_k(t) \right|^2 \tau(dt) \geq \left(\frac{\tau(A)}{2} \right) \sum_{k=n}^m |a_k|^2,$$

for all $m > n > N$. We get

$$(4\text{-}9) \quad \left(\frac{\tau(A)}{2} \right) \sum_{k=n}^m |a_k|^2 \leq \int_A \left| \sum_{k=n}^m a_k g_k(t) \right|^2 \tau(dt) \leq \varepsilon^2 \tau(A),$$

where for the last inequality we have used the fact $N > V$ and the definition of A.

We are now free to let m tend to infinity in (4-9), which we do. We deduce that necessarily $\tau(A) = 0$, a contradiction with (4-6). This finishes the proof. □

Proof of Theorem 4.3. Choose $\tau(dt)$ as the density function on the real line associated to $\tau(t) = (1 - \cos t)/\pi t^2$. Then

$$\int_{\mathbb{R}} \tau(dt) = 1, \quad \int_{\mathbb{R}} e^{ixt} \tau(dt) = (1 - |x|)_+.$$

Since τ is absolutely continuous with respect to the Lebesgue measure, (4-3) holds with τ in place of λ. Next choose $g_k(t) = e^{i\lambda_k t}$. We have

$$\langle g_k \overline{g_\ell}, g_{k'} \overline{g_{\ell'}} \rangle_\tau = (1 - |(\lambda_k - \lambda_\ell) - (\lambda_{k'} - \lambda_{\ell'})|)_+.$$

Condition (4-4) is thus fulfilled. Theorem 4.8 applies and we deduce that the series $\sum_{k=1}^{\infty} |a_k|^2$ converges. $\qquad\square$

Final note

While finishing this paper, we discovered that Theorem 2.4 was proved by Guniya [1985] using a completely different method from ours. Guniya's proof makes use of Wiener's result [1926] (previously mentioned) and does not seem to provide directly a maximal inequality. Our proof is somewhat more elementary. Moreover it allows one to recover Wiener's result and provides at the same time a maximal inequality. It seems that Guniya's paper has been completely overlooked among the mathematical community. We observe in particular that Theorem 2.4 notably includes obviously Hedenmalm and Saksman's result [2003] published nearly twenty years after [Guniya 1985].

We now briefly explain Guniya's approach (see Theorem 1.2, (8) and Lemmas after and paragraph 2.10). The proof follows from the combination of several different results proved in the paper, and is based on Riemann theory of trigonometric series [Zygmund 1968, Chapter XVI-8]. Assume that the coefficients are positive. Then the series $\sum_n c_n e^{i\lambda_n x}$ converges in $\mathcal{S}^2$ to some f. Let I, J be two intervals with $|I| < 2\pi$, $|J| = 2\pi$ and $I \not\subseteq J$. Let F be represented by the term-by-term integrated Fourier series of f, and let L be a bump function of class C^5 equal to 1 on I and to 0 on $J \backslash I'$ where $I \subset I' \not\subseteq J$. Then by a theorem due to Zygmund [1968, Theorem 9.19], the partial sums of the Fourier series of f are uniformly equiconvergent on I with the partial sum of a trigonometric series $\sum_m a_m e^{imx}$. Next, if FL admits a second order derivative in the sense of distributions, say g, then the above trigonometric series is the one of g. And the a.e. convergence on I follows from Carleson's theorem. It remains to be proven that under condition (2-4), F has indeed second order Schwarz derivatives, controlled by the L^2 norm of f, which should follow from Theorem 2.2 in [Guniya 1985].

Acknowledgements

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