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A FEYNMAN–KAC FORMULA FOR DIFFERENTIAL FORMS ON MANIFOLDS WITH BOUNDARY AND GEOMETRIC APPLICATIONS

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We establish a Feynman–Kac-type formula for differential forms satisfying absolute boundary conditions on Riemannian manifolds with boundary and of bounded geometry. We use this to construct L^2-harmonic forms out of bounded ones on the universal cover of a compact Riemannian manifold whose geometry displays a positivity property expressed in terms of a certain stochastic average of the Weitzenböck operator R_p acting on p-forms and the second fundamental form of the boundary. This extends previous work by Elworthy, Li and Rosenberg on closed manifolds to this more general setting. As an application we find a new obstruction to the existence of metrics with positive R_2 (in particular, positive isotropic curvature) and 2-convex boundary. We also discuss a version of the Feynman–Kac formula for spinors under suitable boundary conditions and use this to prove a semigroup domination result for the corresponding Dirac Laplacian under a mean convexity assumption.

1. Introduction

A celebrated result by Gromov [1971] says that an open manifold carries both positively and negatively curved metrics. Thus, in any such manifold there is enough room to interpolate between two rather distinct types of geometries. In contrast, no such flexibility is available in the context of closed manifolds. For instance, it already follows from Hadamard and Bonnet–Myers theorems from basic Riemannian Geometry that a closed manifold which carries a metric with nonpositive sectional curvature does not carry a metric with positive Ricci curvature.

Our interest here lies in another manifestation of this "exclusion principle" for closed manifolds due to Elworthy, Li and Rosenberg [Elworthy et al. 1998]. Relying heavily on stochastic methods, these authors put forward an elegant refinement of the

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famous Bochner technique with far-reaching consequences. For example, they prove that a sufficiently negatively pinched closed manifold does not carry a metric whose Weitzenböck operator acting on 2-forms is even allowed to be negative in a region of small volume, an improvement which definitely makes the obstruction unapproachable by the classical reasoning [Rosenberg 1997]. We focus here on extending this kind of geometric obstruction to compact manifolds with boundary (∂-manifolds, for short). When pursuing this goal we should have in mind that balls carry a huge variety of metrics as illustrated by geodesic balls in an arbitrary Riemannian manifold. These simple examples also show that the boundary can always be chosen convex just by taking the radius sufficiently small. Thus, even if we insist on having the boundary appropriately convex in both metrics, some topological assumption on the underlying manifold must be imposed. Our purpose is to present results in this direction which qualify as natural extensions of those in [Elworthy et al. 1998].

We now introduce the notation needed to state our main results. If N is a Riemannian ∂-manifold of dimension n, the Weitzenböck decomposition reads

$$\Delta_q = \Delta_q^B + R_q,$$

where $\Delta_q = dd^\star + d^\star d$ is the Hodge Laplacian acting on q-forms, $1 \leq q \leq n-1$, $d^\star = \pm \star d \star$ is the codifferential, $\star$ is the Hodge star operator, Δ_q^B is the Bochner Laplacian and R_q, the Weitzenböck curvature operator, depends linearly on the curvature tensor, albeit in a rather complicated way. Recall that $R_1 = \mathrm{Ric}$, and since $\star R_p = R_{n-p}\star$, this also determines R_{n-1}, but in general the structure of R_q, $2 \leq q \leq n-2$, is notoriously hard to grasp. To these invariants we attach the functions $r_{(q)} : N \to \mathbb{R}$, $r_{(q)}(x) = \inf_{|\omega|=1} \langle R_q(x)\omega, \omega \rangle$, the least eigenvalue of $R_q(x)$. We also consider the principal curvatures $\rho_1, \ldots, \rho_{n-1}$ of ∂N computed with respect to the inward unit normal vector field. For each $x \in \partial N$ and $q = 1, \ldots, n-1$, define

$$\rho_{(q)}(x) = \inf_{1 \leq i_1 < \cdots < i_q \leq n-1} \rho_{i_1}(x) + \cdots + \rho_{i_q}(x),$$

the sum of the q smallest principal curvatures at x. We say that ∂N is q-convex if $\underline{\rho}_{(q)} := \inf_{x \in \partial M} \rho_{(q)}(x) > 0$. Note that q-convexity implies $(q+1)$-convexity. Also, N is said to be convex if $\underline{\rho}_{(1)} \geq 0$ everywhere. Finally, recall that a Riemannian metric h on a manifold is κ-negatively pinched if its sectional curvature satisfies $-1 \leq K_{\mathrm{sec}}(h) < -\kappa < 0$.

Stochastic notions make their entrance in the theory by means of the following considerations. Let N be a Riemannian ∂-manifold. In case N is noncompact we always assume that the underlying metric h is complete and the triple $(N, \partial N, h)$ has bounded geometry in the sense of [Schick 1996; 1998; 2001]. We then consider reflecting Brownian motion $\{x^t\}$ on N starting at some $x^0 \in N$; see Section 5 for a (necessarily brief) description of this diffusion process. Let $\alpha : N \to \mathbb{R}$ and $\beta : \partial N \to \mathbb{R}$ be C^1 functions. Adapting a classical definition to our setting, we say that the pair (α, β) is strongly stochastically positive (s.s.p.) if

$$\limsup_{t\to+\infty} \frac{1}{t} \sup_{x^0\in K} \log \mathbb{E}_{x^0}\left(\exp\left(-\frac{1}{2}\int_0^t \alpha(x^s)\,ds - \int_0^t \beta(x^s)\,dl^s\right)\right) < 0,$$

for any $K \subset N$ compact, where l^t is the boundary local time associated to $\{x^t\}$. This is certainly the case if both α and β have strictly positive lower bounds but the point we would like to emphasize here is that, at least if N is compact, it might well happen with the functions being positive except possibly in regions of small volume, given that the definition involves expectation with respect to the underlying diffusion.

Similarly to [Elworthy et al. 1998], our main results provide examples of ∂-manifolds for which there holds an exclusion principle involving the various notions of curvature appearing above. From now on we always assume that $n \geq 4$ and set $\kappa_p = p^2/(n-p-1)^2$.

Theorem 1.1. *Let M be a compact ∂-manifold with infinite fundamental group. Assume also that M satisfies $H^p(M;\mathbb{R}) \neq 0$, where $2 \leq p < (n-1)/2$. If M carries a convex κ_p-negatively pinched metric then it does not carry a metric with both $(r_{(p\pm1)}, \rho_{(p-1)})$ s.s.p.*

Our next result, which handles the least possible value for the degree p, has a somewhat more satisfactory statement.

Theorem 1.2. *Let M be a compact manifold with nonamenable fundamental group. If M carries a convex κ_1-negatively pinched metric then it does not carry a metric with $(r_{(2)}, \rho_{(2)})$ s.s.p.*

Remark 1.1. These results correspond respectively to Corollary 2.1 and Theorem 2.3 in [Elworthy et al. 1998]. We point out that our assumptions on the fundamental group are natural in the sense that they are automatically satisfied there. As mentioned above, balls are obvious counterexamples to our results if the topological assumptions are removed. Also, the manifold $\mathbb{S}^1 \times \mathbb{D}^{n-1}$ shows that merely assuming that the fundamental group is infinite does not suffice in Theorem 1.2; see Remark 1.5 below. On the other hand, it is not clear whether the convexity hypothesis with respect to the negatively curved metric can be relaxed somehow.

Using Theorem 1.2 we can exhibit an interesting family of compact ∂-manifolds for which a natural class of metrics is excluded.

Theorem 1.3. *If X is a closed hyperbolic manifold of dimension, $l \geq 2$ then its product with a disk $\mathbb{D}^m$ does not carry a metric with $(r_{(2)}, \rho_{(2)})$ s.s.p.*

Proof. Write $X = \mathbb{H}^l/\Gamma$ as the quotient of hyperbolic space $\mathbb{H}^l$ by a (necessarily nonamenable) group Γ of hyperbolic motions. Embed $\mathbb{H}^l$ as a totally geodesic submanifold of $\mathbb{H}^{l+m}$ and let $\widetilde{M} \subset \mathbb{H}^{l+m}$ be a tubular neighborhood of $\mathbb{H}^l$ of constant radius. Extend the Γ-action to $\widetilde{M}$ in the obvious manner and observe that, since $\widetilde{M}$

is convex, $M = \widetilde{M}/\Gamma = X \times \mathbb{D}^m$ with the induced hyperbolic metric is convex as well. Thus, Theorem 1.2 applies. $\square$

Remark 1.2. Theorem 1.3 provides a geometric obstruction to the existence of metrics with $(r_{(2)}, \rho_{(2)})$ s.s.p. Notice that if the second Betti number of X vanishes, the obstruction can not be detected by the classical version of the Bochner technique for ∂-manifolds [Yano 1970, Chapter 8] even if we assume strict positivity of $(r_{(2)}, \rho_{(2)})$.

Remark 1.3. A larger class of manifolds for which the conclusion of Theorem 1.3 obviously holds is formed by tubular neighborhoods of closed embedded totally geodesic submanifolds in a given hyperbolic manifold.

Corollary 1.1. *Under the conditions of Theorem 1.3, assume that $n = l + m$ is even. Then $X \times \mathbb{D}^m$ does not carry a metric with positive isotropic curvature and 2-convex boundary.*

Proof. For even-dimensional manifolds it is shown in [Micallef and Wang 1993] that positive isotropic curvature implies $R_2 > 0$. $\square$

Remark 1.4. Since the computation in [Micallef and Wang 1993] expresses R_2 as a sum of isotropic curvatures, in Corollary 1.1 we can even relax the condition on the metric to allow the invariants to be negative in a region of small volume.

Remark 1.5. The standard product metric on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ is known to have positive isotropic curvature. It is easy to check that if $r < \pi/2$ the boundary of the tubular neighborhood $U_r \subset \mathbb{S}^1 \times \mathbb{S}^{n-1}$ of radius r of the circle factor is 2-convex. Thus, the conclusion of Corollary 1.1 does not hold for $U_r = \mathbb{S}^1 \times \mathbb{D}^{n-1}$. Notice that U_r carries a convex hyperbolic metric since its universal cover $\widetilde{U}_r = \mathbb{R} \times \mathbb{D}^{n-1}$ is diffeomorphic to a tubular neighborhood of a geodesic in $\mathbb{H}^n$. The problem here is that the fundamental group is abelian, hence amenable, and the argument leading to Theorem 1.2 breaks down. This also can be understood in stochastic terms. In effect, the proof of Theorem 1.2 shows that, under the given conditions, Brownian motion on the universal cover is transient, while recurrence certainly occurs in $\widetilde{U}_r$; see Remark 5.1. In this respect it would be interesting to investigate if the conclusion of Theorem 1.3 holds in case X is flat or, more generally, has nonpositive sectional curvature.

Remark 1.6. Compact ∂-manifolds with positive isotropic curvature have deserved a lot of attention in recent years. An important result by Fraser [2002] says that such a ∂-manifold is contractible if it is simply connected and its boundary is connected and 2-convex. The proof combines index estimates for minimal surfaces and a variant of the Sachs–Uhlenbeck theory adapted to this setting. However, as the examples in Remark 1.5 attest, this geometric condition is compatible with an infinite fundamental group. With no assumption on the fundamental group or on the topology of the boundary, the techniques in [Fraser 2002] still imply that all the (absolute and relative) homotopy groups vanish in the range $2 \leq i \leq n/2$. Moreover, it is shown in [Chen and Fraser 2010] that the fundamental group of the boundary injects into the fundamental group of the manifold. However, if we take $m \geq l+2$ it is easy to check that none of these homotopical obstructions rules out the metrics in Corollary 1.1. We point out that a conjecture in [Fraser 2002] asserts that a closed, embedded 2-convex hypersurface in a manifold with positive isotropic curvature is either $\mathbb{S}^n$ or a connected sum of finitely many copies of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Since the fundamental group of a closed hyperbolic manifold is neither infinite cyclic nor a free product, Corollary 1.1 provides further support to the conjecture.

This paper is partly inspired by the beautiful work by Elworthy, Rosenberg and Li [Elworthy et al. 1998]. Their ideas are used in Section 2 to construct L^2 harmonic forms on the universal cover of certain compact ∂-manifolds starting from bounded ones. This is precisely where stochastic techniques come into play and a crucial ingredient at this point is a Feynman–Kac-type formula for differential forms in higher degree meeting absolute boundary conditions. In order not to interrupt the exposition, this technical result is established in the final Section 5 following ideas in [Hsu 2002a], where the case of 1-forms is treated; see also [Airault 1976; Ikeda and Watanabe 1989] for previous contributions. To illustrate the flexibility of the method we also discuss a similar formula for spinors evolving under the heat semigroup generated by the Dirac Laplacian on a spin^c ∂-manifold under suitable boundary conditions. Another important ingredient in the argument is a Donnelly–Xavier-type eigenvalue estimate described in Section 3, whose proof uses both the convexity and the assumption that the fundamental group is infinite. Combined with Schick's [1996; 1998] L^2 Hodge–de Rham theory this allows us to prove a vanishing result for the relevant L^2 cohomology group. Finally, the proofs of the main applications (Theorem 1.1 and 1.2 above) are presented in Section 4.

2. From bounded to L^2-harmonic forms

We consider a complete Riemannian ∂-manifold N with volume element dN and boundary ∂N oriented by an inward unit normal vector field ν. As always we assume that the triple $(N, \partial N, h)$ has bounded geometry in the sense of [Schick 1996; 1998; 2001]. For us the case of interest occurs when $N = \widetilde{M}$, the universal cover of a compact ∂-manifold (M, g) and $h = \tilde{g}$, the lifted metric. Recall that a q-form ω on N satisfies absolute boundary conditions if

$$\nu \lrcorner \omega = 0, \quad \nu \lrcorner d\omega = 0 \tag{2-1}$$

along ∂N. Equivalently,

$$\omega_{\mathrm{nor}} = 0, \quad (d\omega)_{\mathrm{nor}} = 0, \tag{2-2}$$

where $\omega = \omega_{\tan} + \nu \wedge \omega_{\mathrm{nor}}$ is the natural decomposition of ω in its tangential and normal components. Here, we identify ν to its dual 1-form in the standard manner. For simplicity we say that ω is absolute if any of these conditions is satisfied. Notice that for $q = 0$ this means that the given function satisfies Neumann boundary condition.

For $t > 0$ let $P_t = e^{-t\Delta_q^{\mathrm{abs}}/2}$ be the corresponding heat kernel acting on forms. Thus, for any absolute q-form $\omega_0 \in L^2 \cap L^\infty$, $\omega_t = P_t\omega_0$ is a solution to the initial-boundary value problem

$$\frac{\partial \omega_t}{\partial t} + \frac{1}{2}\Delta_q^{\mathrm{abs}}\omega_t = 0, \quad \lim_{t\to 0}\omega_t = \omega_0, \quad \nu \lrcorner \omega_t = 0, \quad \nu \lrcorner d\omega_t = 0. \tag{2-3}$$

Moreover, the long term behavior of the flow is determined by the space of absolute L^2-harmonic q-forms on (N, h) in the sense that

$$P = \lim_{t\to+\infty} P_t \tag{2-4}$$

exists and defines the orthogonal projection onto this space. Proofs of these facts follow from standard spectral theory and the elliptic machinery developed in [Schick 1996; 1998].

A key ingredient in our approach is a Feynman–Kac-type representation of any solution ω_t as above in terms of Brownian motion in N. This is well known to hold in the boundaryless case [Elworthy 1988; Hsu 2002b; Güneysu 2010; Malliavin 1974; Stroock 2000]. However, as pointed out in [Hsu 2002a], where the case $q = 1$ is discussed in detail, extra difficulties appear when trying to establish a similar result in the presence of a boundary. In Section 5 we explain how the method in [Hsu 2002a] can be adapted to establish a Feynman–Kac formula for solutions of (2-3), regardless of the value of q; see Theorem 5.2. For the moment we need an immediate consequence of this formula, namely, the useful estimate

$$|\omega_t(x^0)| \le \mathbb{E}_{x^0}\left(|\omega_0(x^t)| \exp\left(-\frac{1}{2}\int_0^t r_{(q)}(x^s)\, ds - \int_0^t \rho_{(q)}(x^s)\, dl^s\right)\right), \tag{2-5}$$

where $\{x^t\}$ is reflecting Brownian motion on N starting at x^0 and l^t is the associated boundary local time. The remarkable feature of (2-5) is that the geometric quantities $r_{(q)}$ and $\rho_{(q)}$ play entirely similar roles in stochastically controlling the solution in the long run. Now we put this estimate to good use and establish a central result in this work; compare to [Elworthy et al. 1998, Lemma 2.1].

Proposition 2.1. *Let $P = \lim_{t\to+\infty} P_t$,*

$$\theta_q(x^0) = \int_0^{+\infty} \mathbb{E}_{x^0}\left(\exp\left(-\frac{1}{2}\int_0^t r_{(q)}(x^s)\, ds - \int_0^t \rho_{(q)}(x^s)\, dl^s\right)\right) dt,$$

and take compactly supported p-forms ϕ and ψ with $\psi_{\mathrm{nor}} = 0$ along ∂N and $\phi = 0$ in a neighborhood of ∂N. If $2 \le p \le n-2$,

$$\left|\int_N \langle P\phi - \phi, \psi\rangle\, dN\right| \le \tfrac{1}{2}\big(\sup\nolimits_{x^0 \in \operatorname{supp}\phi} \theta_{p+1}(x^0)\big)|d\psi|_\infty |d\phi|_1 + \tfrac{1}{2}\big(\sup\nolimits_{x^0 \in \operatorname{supp}\phi} \theta_{p-1}(x^0)\big)|d^\star\psi|_\infty |d^\star\phi|_1.$$

If $p = 1$ we have instead

$$\left|\int_N \langle P\phi - \phi, \psi\rangle\, dN\right| \le \tfrac{1}{2}\big(\sup\nolimits_{x^0 \in \operatorname{supp}\phi} \theta_2(x^0)\big)|d\psi|_\infty |d\phi|_1 + \tfrac{1}{2}\sup\nolimits_{x^0 \in \operatorname{supp}\phi}\left|\int_0^{+\infty} (P_\tau d^\star\phi)(x^0)\, d\tau\right| |d^\star\psi|_\infty.$$

Proof. We have

$$\begin{aligned}\int_N \langle P\phi - \phi, \psi\rangle\, dN &= \lim_{t\to+\infty}\int_N \langle P_t\phi - P_0\phi, \psi\rangle\, dN \\ &= \lim_{t\to+\infty}\int_0^t \int_N \langle \partial_\tau P_\tau\phi, \psi\rangle\, dN\, d\tau \\ &= -\frac{1}{2}\int_0^{+\infty}\int_N \langle \Delta_p^{\mathrm{abs}} P_\tau\phi, \psi\rangle\, dN\, d\tau \\ &= -\frac{1}{2}\int_0^{+\infty}\int_N \langle d P_\tau\, d^\star\phi, \psi\rangle\, dN\, d\tau - \frac{1}{2}\int_0^{+\infty}\int_N \langle d^\star P_\tau\, d\phi, \psi\rangle\, dN\, d\tau.\end{aligned}$$

We now recall Green's formula: if $\alpha \wedge \star\beta$ is compactly supported then

$$\int_N \langle d\alpha, \beta\rangle\, dN = \int_N \langle \alpha, d^\star\beta\rangle\, dN + \int_{\partial N} \alpha_{\mathrm{tan}} \wedge \star\beta_{\mathrm{nor}}.$$

Since $(P_\tau d\phi)_{\mathrm{nor}} = 0$ this leads to

$$\int_N \langle P\phi - \phi, \psi\rangle\, dN = -\frac{1}{2}\int_0^{+\infty}\int_N \langle P_\tau\, d^\star\phi, d^\star\psi\rangle\, dN\, d\tau - \frac{1}{2}\int_0^{+\infty}\int_N \langle P_\tau\, d\phi, d\psi\rangle\, dN\, d\tau.$$

The result now follows by applying (2-5) to $\omega_\tau = P_\tau\, d^\star\psi$ and $\omega_\tau = P_\tau\, d\psi$. □

From this we derive the existence of absolute L^2-harmonic p-forms from bounded ones under appropriate positivity assumptions; compare to [Elworthy et al. 1998, Theorem 2.1]. In the following we denote by $H^q_{(2),\mathrm{abs}}(N, h)$ the q-th L^2 absolute cohomology group of (N, h). We refer to [Schick 1996; 1998] for the definition and basic properties of these invariants, including the corresponding L^2 Hodge–de Rham theory.

Proposition 2.2. *Let (N, h) and p be as above. Assume that both $\sup_{x^0 \in K} \theta_{p+1}(x^0)$ and $\sup_{x^0 \in K} \theta_{p-1}(x^0)$ are finite if $2 \leq p \leq n-2$ and that both $\sup_{x^0 \in K} \theta_2(x^0)$ and $\sup_{x^0 \in K} \left| \int_0^{+\infty} (P_\tau \, d^\star \phi)(x^0) \, d\tau \right|$ are finite if $p = 1$, where $K \subset N$ is any compact. Then N carries a nontrivial absolute L^2-harmonic p-form whenever it carries a nontrivial absolute bounded harmonic p-form. In particular, $H^p_{(2),\mathrm{abs}}(N, h)$ is nontrivial.*

Proof. Let ψ be a nontrivial absolute bounded harmonic p-form. Consider a Gaffney-type cutoff sequence $\{h_n\}$, i.e., each function h_n satisfies $0 \leq h_n \leq 1$, $|\nabla h_n| \leq 1/n$, $h_n \to 1$ and $\partial h_n / \partial \nu = 0$ [Gaffney 1959] and set $\psi_n = h_n \psi$, so that each ψ_n is a compactly supported absolute form. Also, $\psi_n \to \psi$ and since $d\psi_n = dh_n \wedge \psi$ and $d^* \psi_n = -\nabla h_n \lrcorner \psi$ we see that $|d\psi_n|_\infty + |d^\star \psi_n|_\infty \to 0$ as $n \to +\infty$. Applying Proposition 2.1 with ψ replaced by ψ_n and sending $n \to +\infty$ we see that

$$\int_N \langle P\phi - \phi, \psi \rangle \, dN = 0.$$

If no nontrivial absolute L^2-harmonic p-form exists then $P\phi = 0$ for any ϕ and hence $\psi = 0$, a contradiction. The last assertion follows from the L^2 Hodge–de Rham theory in [Schick 1996; 1998]. □

Remark 2.1. Implicit in the discussion above is the well-known fact that the bounded geometry assumption implies that reflecting Brownian motion x^t is nonexplosive. For the sake of completeness we include here the well-known argument. We first observe that the geometric assumption implies that both $\underline{r}_{(1)}$ and $\underline{\rho}_{(1)}$ are finite. Let ξ and η be compactly supported functions on N with $\partial \xi / \partial \nu = 0$ along ∂N and $\eta = 0$ in a neighborhood of ∂N. Proceeding as above we find that

$$\int_N (P_t \xi - \xi) \eta \, dN = -\frac{1}{2} \int_0^t \int_N \langle P_\tau \, d\xi, d\eta \rangle \, dN \, d\tau, \qquad t > 0.$$

Using (2-5) with $\omega = d\xi$ we get

$$\left| \int_N (P_t \xi - \xi) \eta \, dN \right| \leq \frac{1}{2} |d\xi|_\infty |d\eta|_1 \sup_{0 \leq \tau \leq t} e^{-\tau \underline{r}_{(1)}/2 - \underline{\rho}_{(1)} \int_0^\tau dl^s}.$$

Again applying Gaffney's trick, i.e., replacing ξ by ξ_n approaching $\mathbf{1}$, the function identically equal to 1, and satisfying $|d\xi_n|_\infty \to 0$ as $n \to +\infty$, we conclude that $P_t \mathbf{1} = \mathbf{1}$. The result follows.

3. A Donnelly–Xavier-type estimate for ∂-manifolds

In this section we present a Donnelly–Xavier-type estimate for the universal cover of κ-negatively pinched ∂-manifolds which implies the vanishing of certain absolute L^2 cohomology groups. This extends to this setting a sharp result for boundaryless manifolds obtained in [Elworthy and Rosenberg 1993], which by its turn improves on the original result in [Donnelly and Xavier 1984]. The exact analogue for ∂-manifolds of the estimate in that work, hence with a tighter pinching, appears in [Schick 1996]; see Remark 3.1 below. Our proof adapts a computation in [Ballmann and Brüning 2001, Section 5], where the sharp result for boundaryless manifolds is also achieved, and relies on a rather general integral formula.

Proposition 3.1. *Let (N,h) be a ∂-manifold, $f:N\to\mathbb{R}$ a C^2 function and $\{\mu_i\}_{i=1}^n$ the eigenvalues of the Hessian operator of f. If $p\geq 1$ then for any compactly supported p-form ω in N,*

$$\begin{aligned}\int_N(\langle d\omega,\nabla f\wedge\omega\rangle+\langle d^\star\omega,\nabla f\lrcorner\omega\rangle)\,dN\\ =\int_N\Bigl(\sum_i\mu_i|e_i\lrcorner\omega|^2+\tfrac12|\omega|^2\Delta_0 f\Bigr)\,dN-\int_{\partial N}\langle\nabla f\lrcorner\omega,\nu\lrcorner\omega\rangle\,d\partial N\\ -\frac12\int_{\partial N}|\omega|^2\langle\nabla f,\nu\rangle\,d\partial N,\end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame diagonalizing the Hessian of f and ν is the inward unit normal vector field along ∂N.

Proof. Consider the vector field V defined by $\langle V,W\rangle=\langle\nabla f\lrcorner\omega,W\lrcorner\omega\rangle$, for any W. A computation in [Ballmann and Brüning 2001, Section 5] gives

$$\operatorname{div}V=\sum_i\mu_i|e_i\lrcorner\omega|^2-\langle d\omega,\nabla f\wedge\omega\rangle-\langle d^\star\omega,\nabla f\lrcorner\omega\rangle+\langle\nabla_{\nabla f}\omega,\omega\rangle.$$

Integrating by parts we obtain

$$\begin{aligned}\int_N(\langle d\omega,\nabla f\wedge\omega\rangle+\langle d^\star\omega,\nabla f\lrcorner\omega\rangle)\,dN\\ =\int_N\Bigl(\sum_i\mu_i|e_i\lrcorner\omega|^2+\langle\nabla_{\nabla f}\omega,\omega\rangle\Bigr)\,dN-\int_{\partial N}\langle\nabla f\lrcorner\omega,\nu\lrcorner\omega\rangle\,d\partial N.\end{aligned}$$

We thus obtain, as required:

$$\begin{aligned}\int_N\Bigl(\langle\nabla_{\nabla f}\omega,\omega\rangle-\tfrac12|\omega|^2\Delta_0 f\Bigr)\,dN\\ =\frac12\int_N(\langle\nabla f,\nabla|\omega|^2\rangle-|\omega|^2\Delta_0 f)\,dN\\ =\frac12\int_N\operatorname{div}(|\omega|^2\nabla f)\,dN=-\frac12\int_{\partial N}|\omega|^2\langle\nabla f,\nu\rangle\,d\partial N.\quad\square\end{aligned}$$

We can now present a version of the Donnelly–Xavier-type estimate that suffices for our purposes.

Proposition 3.2. *Let (M, g) be a compact and convex ∂-manifold with infinite fundamental group and assume that g satisfies $-1 \le K_{\sec}(g) \le -\kappa < 0$. If $p \ge 1$ then for any compactly supported p-form ω in $\widetilde{M}$ satisfying $\nu \lrcorner \omega = 0$ along $\partial \widetilde{M}$,*

$$|d\omega|_2 + |d^\star \omega|_2 \ge \tfrac{1}{2}((n-p-1)\sqrt{\kappa} - p)|\omega|_2. \tag{3-1}$$

Proof. Convexity implies that any $x \in M \setminus \partial M$ and $y \in M$ can be joined by a minimizing geodesic segment lying in the interior of M (except possibly for y). The same holds in $\widetilde{M}$ with the segment now being unique. Thus, for any $x \in \widetilde{M} \setminus \partial \widetilde{M}$ the Riemannian distance d_x to x is well-defined. Notice that $\langle \nabla d_x, \nu \rangle \le 0$ along $\partial \widetilde{N}$, $|\nabla d_x| = 1$ and $\Delta_0 d_x = -\sum_i \mu_i$, where we may assume that $\mu_1 = 0$. Thus, using the boundary condition $\nu \lrcorner \omega = 0$ and Proposition 3.1 with $f = d_x$ we obtain

$$|\omega|_2(|d\omega|_2 + |d^\star\omega|_2) \ge \int_N \left(\sum_i \mu_i |e_i \lrcorner \omega|^2 - \frac{1}{2}|\omega|^2 \sum_i \mu_i \right) d\widetilde{M}.$$

Expand $\omega = \sum_I \omega_I e_I$, where $I = \{i_1 < \cdots < i_p\}$ and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$. Since $\sum_i \mu_i |e_i \lrcorner e_I|^2 = \sum_{i \in I} \mu_i$, the right-hand side equals

$$\frac{1}{2} \int_N \sum_{i,I} \left(\sum_{i \notin I} \eta_i - \sum_{i \in I} \eta_i \right) |\omega_I|^2 \, d\widetilde{M},$$

where $\eta_i = -\mu_i$ are the principal curvatures of the geodesic ball centered at x. Thus, by standard comparison theory this is bounded from below by

$$\frac{1}{2} \int_N ((n-p-1)\sqrt{\kappa} \coth \sqrt{\kappa}\, d_x - p \coth d_x) |\omega|^2 \, d\widetilde{M}.$$

Now observe that $\widetilde{M}$ has infinite diameter because $\pi_1(M)$ is infinite. Hence, we can find a sequence $\{x_i\} \subset \widetilde{M}$ so that $d_{x_i}(y) \to +\infty$ uniformly in $y \in \operatorname{supp} \omega$. By taking $x = x_i$ and passing to the limit we obtain the desired result. □

Remark 3.1. Notice (3-1) is meaningful only if $\kappa > \kappa_p$, which forces $\kappa_p < 1$, that is, $2p < n-1$. We note that Schick [1996] proved that under the conditions above

$$|d\omega|_2 + |d^\star\omega|_2 \ge \tfrac{1}{2}((n-1)\sqrt{\kappa} - 2p)|\omega|_2.$$

This only makes sense if $\kappa > \kappa'_p := 4p^2/(n-1)^2$, which again forces $2p < n-1$, but notice that (3-1) gives a better pinching constant if $1 \le p < (n-1)/2$. It is observed in the same work that

$$|d\eta|_2 + |d^\star\eta|_2 \ge \tfrac{1}{2}((n-1)\sqrt{\kappa} - 2(n-p))|\eta|_2,$$

for any p-form η satisfying $\nu \wedge \eta = 0$ along $\partial \widetilde{M}$. Taking $p = n$ and using Hodge duality, we find that

$$|d\varphi|_2 \ge \tfrac{1}{2}(n-1)\sqrt{\kappa}|\varphi|_2, \tag{3-2}$$

for any compactly supported function φ satisfying the Neumann boundary condition. In other words, (3-1) holds for $p = 0$ as well. This transplants to our setting a famous estimate by McKean [1970]. Observe however that the assumption on the fundamental group is essential in (3-2) as the first Neumann eigenvalue of geodesic balls in hyperbolic space converges to zero as the radius goes to infinity [Chavel 1984]. Thus, (3-2) illustrates a situation where a topological condition on a compact ∂-manifold poses spectral constraints on its universal cover.

With these estimates at hand it is rather straightforward to establish vanishing theorems for L^2-harmonic forms. For this we consider (M, g) as in Proposition 3.2 and define the absolute Hodge Laplacian Δ_p^{abs} on $\widetilde{M}$ with domain $\mathcal{D}(\Delta_p^{\mathrm{abs}}) = \{\omega \in H^2(\wedge^p T^*\widetilde{M});\, \omega_{\mathrm{nor}} = 0,\, (d\omega)_{\mathrm{nor}} = 0\}$. Let $\lambda_p^{\mathrm{abs}}(\tilde{g}) = \inf \operatorname{Spec}(\Delta_p^{\mathrm{abs}})$. The spectral argument in [Schick 1996, Section 6] then provides, under the conditions of Proposition 3.2, the lower bound

$$\lambda_p^{\mathrm{abs}}(\tilde{g}) \geq \tfrac{1}{4}((n-p-1)\sqrt{\kappa} - p)^2. \tag{3-3}$$

We remark that the proof in [Schick 1996] uses induction in p starting at $p = 0$, which corresponds to (3-2). Here we use this to prove the following vanishing result.

Proposition 3.3. *Let (M, g) be a compact and convex ∂-manifold with infinite fundamental group and assume that g is κ_p-negatively pinched where $2 \leq 2p < n-1$. Then $\lambda_p^{\mathrm{abs}}(\tilde{g}) > 0$ and $(\widetilde{M}, \tilde{g})$ carries no nontrivial absolute L^2-harmonic p-form. Hence, $H^p_{(2),\mathrm{abs}}(\widetilde{M}, \tilde{g})$ vanishes.*

Proof. The assumptions imply that $\kappa_p < 1$, so we can find $\kappa_p < \kappa < 1$ such that $-1 \leq K_{\mathrm{sec}}(\tilde{g}) \leq -\kappa$. The result follows from (3-3) and the L^2 Hodge–de Rham theory in [Schick 1996; 1998]. □

4. The proofs of Theorems 1.1 and 1.2

Here we prove the main results of this work. Notice that if $(r_{(q)}, \rho_{(q)})$ is s.s.p. then

$$\sup_{x^0 \in K} \theta_q(x^0) < +\infty \quad \text{for any } K. \tag{4-1}$$

Also, if (α, β) is s.s.p. then $(\bar{\alpha}, \bar{\beta})$ is s.s.p. as well for any $\bar{\alpha} \geq \alpha$ and $\bar{\beta} \geq \beta$.

Proof of Theorem 1.1. If M is convex with respect to a κ_p-negatively pinched metric g_- then $H^p_{(2),\mathrm{abs}}(\widetilde{M}, \tilde{g}_-)$ vanishes by Proposition 3.3. On the other hand, by standard Hodge theory for compact ∂-manifolds [Taylor 2011], any nontrivial class in $H^p(M; \mathbb{R})$ can be represented by a nontrivial absolute harmonic p-form with respect to any metric g_+ on M. The lift of this form to $(\widetilde{M}, \tilde{g}_+)$ defines a nontrivial absolute harmonic p-form which is uniformly bounded. Now, if g_+ has both $(r_{(p\pm 1)}, \rho_{(p-1)})$ s.s.p. then the corresponding invariants of $\tilde{g}_+$ are s.s.p. as well, since the property is preserved by passage to covers; see Remark 5.1. In particular, (4-1) holds with $q = p \pm 1$. Thus we may apply Proposition 2.2 to conclude that $H^p_{(2),\mathrm{abs}}(\widetilde{M}, \tilde{g}_+) \neq \{0\}$. Since $H^p_{(2),\mathrm{abs}}(\widetilde{M}, \cdot)$ is a quasi-isometric invariant of the metric [Schick 1996] we obtain a contradiction which completes the proof. $\square$

We now consider Theorem 1.2. For its proof we need an extension of a well-known result in [Lyons and Sullivan 1984] to our setting.

Proposition 4.1. *If (M, g) is a compact ∂-manifold and $\pi_1(M)$ is nonamenable then $(\widetilde{M}, \tilde{g})$ carries a nonconstant bounded absolute harmonic function.*

Proof. The argument in [Lyons and Sullivan 1984, Section 5] carries over to our case. More precisely, using the Neumann heat kernel we construct a natural $\pi_1(M)$-equivariant projection from $L^\infty_{\mathrm{abs}}(\widetilde{M})$, the space of absolute bounded functions, onto $\mathcal{H}^\infty_{\mathrm{abs}}(\widetilde{M}, \tilde{g})$, the space of bounded absolute harmonic functions. Also, there exists a $\pi_1(M)$-equivariant injection $l^\infty(\pi_1(M)) \hookrightarrow L^\infty_{\mathrm{abs}}(\widetilde{M})$. Hence, if $\mathcal{H}^\infty_{\mathrm{abs}}(\widetilde{M}, \tilde{g}) = \mathbb{R}$ the composition $l^\infty(\pi_1(M)) \to \mathbb{R}$ defines an invariant mean. $\square$

Proof of Theorem 1.2. If M carries a metric g_- which is κ_1-negatively curved, then $H^1_{(2),\mathrm{abs}}(\widetilde{M}, \tilde{g}_-)$ vanishes. On the other hand, by Proposition 4.1, for any metric g_+ on M, $(\widetilde{M}, \tilde{g}_+)$ carries a nonconstant bounded absolute harmonic function, say f. This implies that reflecting Brownian motion in $(\widetilde{M}, \tilde{g}_+)$ is transient and in particular

$$\sup_{x^0 \in K} \int_0^{+\infty} (P_t\, d^\star \phi)(x^0)\, dt < +\infty,$$

for any $K \subset \widetilde{M}$ and compactly supported 1-form ϕ as in Proposition 2.1; see [Grigor'yan 1999, Theorem 5.1]. Assuming that g_- is such that the corresponding pair $(r_{(2)}, \rho_{(2)})$ is s.s.p. we can apply Proposition 2.2 because $\psi = df$ is a bounded absolute harmonic 1-form; see Lemma 4.1 below. Hence, $H^1_{(2),\mathrm{abs}}(\widetilde{M}, \tilde{g}_+) \neq \{0\}$ and we get a contradiction. Thus, the proof of Theorem 1.2 is complete as soon as the next lemma is established. $\square$

Lemma 4.1. *If f is a uniformly bounded absolute function as above then the absolute harmonic 1-form $\phi = df$ is uniformly bounded as well.*

Proof. Assume that $|f| \leq K$. The Bismut–Elworthy–Li formula in [Elworthy and Li 1994, Theorem 3.1] holds for our reflecting Brownian motion x^t. Hence, if $v^0 \in T_{x^0}\widetilde{M}$ and $P_t = e^{-t\Delta_0^{\mathrm{abs}}/2}$ then

$$d(P_t f)_{x^0}(v^0) = \frac{1}{t}\mathbb{E}_{x^0}\biggl(f(x^t) \int_0^t \langle v^s, dx^s\rangle_{x^s} \biggr), \quad t > 0,$$

where v^t is defined by (5-4) below. Since f is harmonic, $P_t f = f$. Thus,

$$|df_{x^0}(v^0)| \leq \frac{|v^0|}{t} \sup\nolimits_{\widetilde{M}} f \int_0^t ds \leq K|v^0|,$$

as desired. $\square$

5. A Feynman–Kac formula on ∂-manifolds

In this final section we explain how the method put forward in [Airault 1976; Hsu 2002a] can be adapted to prove a Feynman–Kac-type formula for q-forms on ∂-manifolds. As an illustration of the flexibility of the method we also include a similar formula for spinors evolving by the heat semigroup of the Dirac Laplacian on spin^c ∂-manifolds. These results are presented in the second and third subsections, respectively, after some preparatory material in the first subsection.

The Eells–Elworthy–Malliavin approach. Let (N, h) be a Riemannian ∂-manifold of dimension n. As in Section 2 we assume that $(N, \partial N, h)$ has bounded geometry. Let $\pi : P_{\mathrm{O}_n}(N) \to N$ be the orthonormal frame bundle of N. This is a principal bundle with structural group O_n, the orthogonal group in dimension n. Any orthogonal representation $\zeta : \mathrm{O}_n \to \mathrm{End}(V)$ gives rise to the associated vector bundle $\mathcal{E}_\zeta = P_{\mathrm{O}_n}(N) \times_\zeta V$, which comes endowed with a natural metric and compatible connection derived from h and its Levi-Civita connection ∇. Moreover, any section $\sigma \in \Gamma(\mathcal{E}_\zeta)$ can be identified to its lift $\sigma^\dagger : P_{\mathrm{O}_n}(N) \to V$, which is ζ-equivariant in the sense that $\sigma^\dagger(ug) = \zeta(g^{-1})(\sigma^\dagger(u)),\ u \in P_{\mathrm{O}_n}(N),\ g \in \mathrm{O}_n$. Also, we recall that in terms of lifts, covariant derivation essentially corresponds to Lie differentiation along horizontal tangent vectors.

Any bundle $\mathcal{E}_\zeta$ as above comes equipped with a second order elliptic operator $\Delta^B = -\operatorname{tr}_h \nabla^2 : \Gamma(\mathcal{E}_\zeta) \to \Gamma(\mathcal{E}_\zeta)$, the Bochner Laplacian. Here, ∇^2 is the standard Hessian operator acting on sections. Given an algebraic (zero order) self-adjoint map $\mathcal{R} \in \Gamma(\mathrm{End}(\mathcal{E}_\zeta))$ we can form the elliptic operator

$$\Delta = \Delta^B + \mathcal{R}$$

acting on $\Gamma(\mathcal{E}_\zeta)$. Standard results [Eichhorn 2007; Schick 1996; 1998] imply that the heat semigroup $P_t = e^{-t\Delta/2}$ has the property that, for any $\sigma_0 \in L^2 \cap L^\infty$, $\sigma_t = P_t\sigma_0$ solves the heat equation

$$\frac{\partial \sigma_t}{\partial t} + \frac{1}{2}\Delta\sigma_t = 0, \qquad \lim_{t\to 0} \sigma_t = \sigma_0, \tag{5-1}$$

where we eventually impose elliptic boundary conditions in case $\partial N \neq \varnothing$.

An important question concerning us here is whether the solutions of (5-1) admit a stochastic representation in terms of Brownian motion on N. If $\partial N = \varnothing$ this problem admits a very elegant solution in great generality and a Feynman–Kac formula is available [Elworthy 1988; Güneysu 2010; Hsu 1999; 2002b; Malliavin 1974; Stroock 2000]. Moreover, this representation permits us to estimate the solutions in terms of the overall expectation of $\mathcal{R}$ with respect to the diffusion process; see (5-5)–(5-6) below. However, in the presence of a boundary it is well known that the problem is much harder to handle; see [Hsu 2002a].

Let us assume that N has a nonempty boundary endowed with an inward unit normal field ν. We first briefly recall how reflecting Brownian motion is defined on N. We take for granted that Brownian motion $\{b^t\}$ on $\mathbb{R}^n$ is defined. This is the diffusion process which has half the standard Laplacian $\sum_i \partial_i^2$ as generator. To transplant this to N we make use of the so-called Eells–Elworthy–Malliavin approach [Elworthy 1988; Eells and Elworthy 1971; Hsu 1999; 2002b; Stroock 2000]. Note that any $u \in P_{\mathrm{O}_n}(N)$ defines an isometry $u : \mathbb{R}^n \to T_x N,\ x = \pi(u)$. Also, the Levi-Civita connection on TN lifts to an Ehresmann connection on $P_{\mathrm{O}_n}(M)$ which determines fundamental horizontal vector fields H_i, $i = 1, \dots, n$. As explained in [Hsu 2002b, Chapter 2], these elementary remarks naturally lead to an identification of semimartingales on $\mathbb{R}^n$, horizontal semimartingales on $P_{\mathrm{O}_n}(M)$ and semimartingales on M. Thus, on $P_{\mathrm{O}_n}(N)$ we may consider the stochastic differential equation

$$du^t = \sum_{i=1}^{n} H_i(u^t) \circ db_i^t + \nu^{\dagger}(u^t)\, dl^t, \tag{5-2}$$

which has a unique solution $\{u^t\}$ starting at any initial frame u^0. This is a horizontal reflecting Brownian motion on $P_{\mathrm{O}_n}(N)$ and its projection $x^t = \pi u^t$ defines reflecting Brownian motion on N starting at $x^0 = \pi u^0$. Moreover, l^t is the associated boundary local time. Notice that x^t satisfies

$$dx^t = \sum_{i=1}^{n} X_i(x^t) \circ db_i^t + \nu(x^t)\, dl^t, \qquad X_i = \pi_* H_i, \tag{5-3}$$

so that if F^t is the corresponding stochastic flow, i.e., $x^t = F^t(x^0)$, then $v^t = dF^t_{x^0}(v^0)$, $v^0 \in T_{x^0}N$, satisfies the derivative equation

$$dv^t = \sum_{i=1}^{n} (\nabla X_i)(v^t) \circ db_i^t + (\nabla \nu)(v^t)\, dl^t. \tag{5-4}$$

Remark 5.1. Due to the obvious functorial character of this construction it is not hard to obtain highly desirable properties of Brownian motion. For instance, if the manifold splits as an isometric product of two other manifolds then its Brownian motion is the product of Brownian motions on the factors. In particular, if $N = X \times Y$, where Y is a compact ∂-manifold, then Brownian motion in N is transient if and only if the same happens to X. Also, if $\widetilde{N} \to N$ is a normal Riemannian covering then Brownian motion in $\widetilde{N}$ projects down to Brownian motion in N. From this it is obvious that a pair (α, β) on $(N, \partial N)$ is s.s.p. if and only if its lift $(\tilde{\alpha}, \tilde{\beta})$ on $(\widetilde{N}, \partial \widetilde{N})$ is s.s.p. as well.

We now describe how this formalism leads to an elegant approach to Feynman–Kac-type formulas. Let $\mathcal{A} \in \Gamma(\mathrm{End}(\mathcal{E}_\zeta|_{\partial N}))$ be a pointwise self-adjoint map.

In practice, $\mathcal{A}$ relates to the zero order piece of the given boundary conditions. In analogy with the boundaryless case, Itô's calculus suggests considering the multiplicative functional $M^t \in \operatorname{End}(V)$ satisfying

$$dM^t + M^t\big(\tfrac{1}{2}\mathcal{R}^\dagger\, dt + \mathcal{A}^\dagger\, dl^t\big) = 0, \quad M^0 = I.$$

Standard results imply that a solution exists along each path u^t. We now apply Itô's formula to the process $M^t\sigma^\dagger(T-t, u^t)$, $0 \le t \le T$, where σ is a (time-dependent) section of $\mathcal{E}_\zeta$. With the help of (5-2) we obtain

$$\begin{aligned} dM^t\sigma^\dagger(T-t, u^t) = [M^t\mathcal{L}_H\sigma^\dagger(T-t, u^t), db^t] - M^t L^\dagger\sigma^\dagger(T-t, u^t)\, dt \\ + M^t(\mathcal{L}_{\nu^\dagger} - \mathcal{A}^\dagger)\sigma^\dagger(T-t, u^t)\, dl^t, \end{aligned}$$

where $\mathcal{L}$ is the Lie derivative,

$$[M^t\mathcal{L}_H\sigma^\dagger(T-t, u^t), db_t]_i = \sum_{j=1}^{\dim V}\sum_{k=1}^{n} M^t_{ij}\mathcal{L}_{H_k}\sigma^\dagger_j(T-t, u^t)\, db^t_k,$$

and

$$L^\dagger = \frac{\partial}{\partial t} + \frac{1}{2}(\Delta^\dagger_B + \mathcal{R}^\dagger)$$

is the lifted heat operator, with $\Delta^\dagger_B = -\sum_k \mathcal{L}^2_{H_k}$ being the horizontal Bochner Laplacian. Notice that in case $\partial N = \varnothing$ and σ satisfies (5-1) the computation gives

$$dM^t\sigma^\dagger(T-t, u^t) = [M^t\mathcal{L}_H\sigma^\dagger(T-t, u^t), db^t],$$

which characterizes $M^t\sigma^\dagger(T-t, u^t)$ as a martingale. Equating the expectations of this process at $t = 0$ and $t = T$ yields the celebrated Feynman–Kac formula

$$\sigma^\dagger(t, u^0) = \mathbb{E}_{u^0}(M^t\sigma^\dagger(0, u^t)), \tag{5-5}$$

where $dM^t = -M^t\mathcal{R}^\dagger\, dt/2$ [Elworthy 1988; Hsu 1999; 2002b; Güneysu 2010; Stroock 2000]. From this we easily obtain the well-known estimate

$$|\sigma(t, x^0)| \le \mathbb{E}_{x^0}\left(|\sigma(0, x^t)| \exp\left(-\frac{1}{2}\int_0^t \underline{\mathcal{R}}(x^s)\, ds\right)\right), \tag{5-6}$$

where $\underline{\mathcal{R}}(x)$ is the least eigenvalue of $\mathcal{R}(x)$. However, if $\partial N \ne \varnothing$ the calculation merely says that $\{M^t\}$ is the multiplicative functional associated with the operator L under boundary conditions

$$(\nabla_\nu - \mathcal{A})\sigma = 0. \tag{5-7}$$

As we shall see below through examples, (5-7) is too stringent to encompass boundary conditions commonly occurring in applications, which usually are of mixed type.

The Feynman–Kac formula for absolute differential forms. It turns out that natural elliptic boundary conditions do not quite fit into the prescription in (5-7). Hence, the formalism in the previous subsection does not apply as presented. We illustrate this issue by considering the case $\zeta = \wedge^q \mu_n^*$, where μ_n is the standard representation of O_n on $\mathbb{R}^n$, so that $\mathcal{E}_\zeta$ is the bundle of q-forms over N. In this case, $\mathcal{A}$ is explicitly described in terms of the second fundamental form of ∂N but degeneracies occur due to the splitting of forms into tangential and normal components which is inherent to absolute boundary conditions.

The splitting is determined by the "fermionic relation" $\nu \lrcorner \nu \wedge + \nu \wedge \nu \lrcorner = I$, which induces an orthogonal decomposition

$$\wedge^q T^*N|_{\partial N} = \mathrm{Ran}(\nu \lrcorner \nu \wedge) \oplus \mathrm{Ran}(\nu \wedge \nu \lrcorner),$$

and we denote by $\Pi_{\tan}$ and Π_{nor} the orthogonal projections onto the factors. As is clear from the notation, these maps project onto the space of tangential and normal q-forms, respectively.

Let

$$A : T\partial N \to T\partial N, \quad AX = -\nabla_X \nu,$$

be the second fundamental form of ∂N, which we extend to $TN|_{\partial N}$ by declaring that $A\nu = 0$. This induces the pointwise self-adjoint map $\mathcal{A}_q \in \mathrm{End}(\wedge^q T^*N|_{\partial N})$,

$$(\mathcal{A}_q \omega)(X_1, \dots, X_q) = \sum_i \omega(X_1, \dots, AX_i, \dots, X_q).$$

Notice that $\Pi_{\mathrm{nor}} \mathcal{A}_q \omega = 0$, that is, $\mathcal{A}_q \omega$ only has tangential components. In order to determine the tangential coefficients of $\mathcal{A}_q \omega$ we fix an orthonormal frame $\{e_1, \dots, e_{n-1}\}$ in $T\partial N$ which is principal at $x \in \partial N$ in the sense that $Ae_i = \rho_i e_i$. We then find that, at x,

$$(\mathcal{A}_q \omega)(e_{i_1}, \dots, e_{i_q}) = \left(\sum_{j=1}^q \rho_{i_j} \right) \omega(e_{i_1}, \dots, e_{i_q}). \tag{5-8}$$

The next result is inspired by [Hsu 2002a, Lemma 4.1]; see also [Yano 1970; Donnelly and Li 1982] for similar computations.

Proposition 5.1. *A q-form ω is absolute if and only if its lift $\omega^\dagger$ satisfies*

$$\Pi^\dagger_{\mathrm{nor}} \omega^\dagger = 0 \quad \text{and} \quad \Pi^\dagger_{\tan}(\mathcal{L}_{\nu^\dagger} - \mathcal{A}^\dagger_q)\omega^\dagger = 0 \qquad \textit{on } \partial P_{\mathrm{O}_n}(N). \tag{5-9}$$

Proof. We work downstairs on ∂N and drop the dagger from the notation. First, $\omega_{\mathrm{nor}} = 0$ means that $\omega = \omega_{\tan} + \nu \wedge \omega_{\mathrm{nor}} = \omega_{\tan}$, that is, $\Pi^\dagger_{\mathrm{nor}} \omega^\dagger = 0$. On the other hand, in terms of the principal frame $\{e_i\}$ above,

$$
\begin{aligned}
\nu \lrcorner d\omega(e_{i_1},\dots,e_{i_q}) &= d\omega(\nu,e_{i_1},\dots,e_{i_q}) \\
&= \nu(\omega(e_{i_1},\dots,e_{i_q})) + \sum_j (-1)^j e_{i_j}(\omega(\nu,e_{i_1},\dots,\widehat{e_{i_j}},\dots,e_{i_q})) \\
&\qquad + \sum_j (-1)^j \omega([\nu,e_{i_j}],e_{i_1},\dots,\widehat{e_{i_j}},\dots,e_{i_q}) \\
&\qquad\quad + \sum_{1\le j<k} (-1)^{j+k}\omega([e_{i_j},e_{i_k}],\nu,e_{i_1},\dots,\widehat{e_{i_j}},\dots,\widehat{e_{i_k}},\dots,e_{i_q}) \\
&= \nu(\omega(e_{i_1},\dots,e_{i_q})) + \sum_j (-1)^j e_{i_j}((\nu\lrcorner\omega)(e_{i_1},\dots,\widehat{e_{i_j}},\dots,e_{i_q})) \\
&\qquad - \sum_j \omega(e_{i_1},\dots,[\nu,e_{i_j}],\dots,e_{i_q}) \\
&= \nu(\omega(e_{i_1},\dots,e_{i_q})) - \sum_j \omega(e_{i_1},\dots,\nabla_\nu e_{i_j},\dots,e_{i_q}) \\
&\qquad - \Bigl(\sum_j \rho_{i_j}\Bigr)\omega(e_{i_1},\dots,e_{i_q}),
\end{aligned}
$$

where we used that $[e_{i_j}, e_{i_k}] = 0$, certainly a justifiable assumption, and $\nu\lrcorner\omega = 0$. But

$$\nu(\omega(e_{i_1}, \dots, e_{i_q})) = (\nabla_\nu\omega)(e_{i_1}, \dots, e_{i_q}) + \sum_j \omega(e_{i_1}, \dots, \nabla_\nu e_{i_j}, \dots, e_{i_q}),$$

so we obtain

$$\nu\lrcorner d\omega(e_{i_1}, \dots, e_{i_q}) = \Bigl(\nabla_\nu - \sum_j \rho_{i_j}\Bigr)\omega(e_{i_1}, \dots, e_{i_q}).$$

The results follow in view of (5-8). □

This proposition shows that absolute boundary conditions are of mixed type, namely, they are Dirichlet in normal directions and Neumann in tangential directions. This should be compared with (5-7), which is of pure Neumann type. This confirms that Itô's calculus is insensitive to the projections defining absolute boundary conditions. To remedy this we proceed as in [Hsu 2002a]. We can write the boundary condition as the superposition of two independent components, namely,

$$\Pi^\dagger_{\tan}(\mathcal{L}_{\nu^\dagger} - \mathcal{A}^\dagger_q)\omega^\dagger - \Pi^\dagger_{\mathrm{nor}}\omega^\dagger = 0.$$

The key idea, which goes back to [Airault 1976], is to fix $\epsilon > 0$ and replace $\Pi^\dagger_{\tan}$ by $\Pi^\dagger_{\tan} + \epsilon I$ above, so the condition becomes

$$(\mathcal{L}_{\nu^\dagger} - (\mathcal{A}^\dagger_q + \epsilon^{-1}\Pi^\dagger_{\mathrm{nor}}))\omega^\dagger = 0,$$

which in a sense is the best we can reach in terms of resemblance to (5-7). The next step is to solve for $\mathcal{M}^t_\epsilon \in \mathrm{End}(\wedge^q \mathbb{R}^n)$ in

$$d\mathcal{M}^t_\epsilon + \mathcal{M}^t_\epsilon\big(\tfrac{1}{2}R^\dagger_q(u^t)\,dt + (\mathcal{A}^\dagger_q(u^t) + \epsilon^{-1}\Pi^\dagger_{\mathrm{nor}}(u^t))\,dl^t\big) = 0, \quad \mathcal{M}^0_\epsilon = I. \tag{5-10}$$

Proposition 5.2. *For all $\epsilon > 0$ such that $\epsilon^{-1} \geq \underline{\rho}_{(q)}$ we have*

$$|\mathcal{M}^t_\epsilon| \leq \exp\bigg(-\frac{1}{2}\int_0^t r_{(q)}(x^s)\,ds - \int_0^t \rho_{(q)}(x^s)\,dl^s\bigg), \quad t > 0. \tag{5-11}$$

Proof. The same as in [Hsu 2002a, Lemma 3.1], once we take into account that, as is clear from (5-8), the sums $\sum_{j=1}^q \rho_{i_j}$ are the eigenvalues of $\Pi_{\tan}\mathcal{A}_q$. □

The following convergence result provides the crucial input in the argument.

Theorem 5.1. *As $\epsilon \to 0$, $\mathcal{M}^t_\epsilon$ converges to a multiplicative functional $\mathcal{M}^t$ in the sense that $\lim_{\epsilon\to 0}\mathbb{E}|\mathcal{M}^t_\epsilon - \mathcal{M}^t|^2 = 0$. Moreover, $\mathcal{M}^t\Pi^\dagger_{\mathrm{nor}}(u) = 0$ whenever $u \in \partial P_{O_n}(N)$.*

Proof. The rather technical proof of this result for $q = 1$ is presented in detail in [Hsu 2002a]. Fortunately, with the formalism above in place, it is not hard to check that the proof of the general case follows along the lines of the original argument. More precisely, in that work the letters P and Q denote normal and tangential projection, respectively. If we replace these symbols by Π_{nor} and $\Pi_{\tan}$, the proof there works here with only minor modifications. Therefore, it is omitted. □

We now have all the ingredients needed to prove the Feynman–Kac-type formula for differential forms.

Theorem 5.2. *Let ω_0 be an absolute L^2 q-form on N as above. If $P_t = e^{-t\Delta^{\mathrm{abs}}_q/2}$ is the corresponding heat semigroup, so that $\omega_t = P_t\omega_0$ provides the solution to*

$$\frac{\partial\omega_t}{\partial t} + \frac{1}{2}\Delta^{\mathrm{abs}}_q\omega_t = 0, \quad \lim_{t\to 0}\omega_t = \omega_0, \quad \nu\lrcorner\omega_t = 0, \quad \nu\lrcorner d\omega_t = 0, \tag{5-12}$$

then the following Feynman–Kac formula holds:

$$\omega^\dagger_t(u^0) = \mathbb{E}_{u^0}(\mathcal{M}^t\omega^\dagger_0(u^t)), \tag{5-13}$$

where u_t is the horizontal reflecting Brownian motion starting at u_0. Consequently,

$$|\omega_t(x^0)| \leq \mathbb{E}_{x^0}\bigg(|\omega_0(x^t)|\exp\bigg(-\frac{1}{2}\int_0^t r_{(q)}(x^s)\,ds - \int_0^t \rho_{(q)}(x^s)\,dl^s\bigg)\bigg), \tag{5-14}$$

where $x^t = \pi u^t$.

Proof. Itô's formula and (5-2) yield

$$\begin{aligned} d\mathcal{M}^t_\epsilon\omega^\dagger_{T-t}(u^t) = [\mathcal{M}^t_\epsilon\mathcal{L}_H\omega^\dagger_{T-t}(u^t), db^t] - \mathcal{M}^t_\epsilon L^\dagger\omega^\dagger_{T-t}(u^t)\,dt \\ + \mathcal{M}^t_\epsilon(\mathcal{L}_{\nu^\dagger} - \mathcal{A}^\dagger - \epsilon^{-1}\Pi^\dagger_{\mathrm{nor}})\omega^\dagger_{T-t}(u^t)\,dl^t. \end{aligned}$$

If ω_t is a solution of (5-12) then the second term on the right-hand side drops out. Moreover, by Proposition 5.1 the same happens to the term involving ϵ^{-1}. Sending $\epsilon \to 0$ we end up with

$$d\mathcal{M}^t\omega^\dagger_{T-t}(u^t) = [\mathcal{M}^t\mathcal{L}_H\omega^\dagger_{T-t}(u^t), db^t] + \mathcal{M}^t\Pi^\dagger_{\tan}(\mathcal{L}_{\nu^\dagger} - \mathcal{A}^\dagger)\omega^\dagger_{T-t}(u^t)\, dl^t,$$

where the insertion of $\Pi^\dagger_{\tan}$ in the last term is legitimate due to the last assertion in Theorem 5.1. By Proposition 5.1 this actually reduces to

$$d\mathcal{M}^t\omega^\dagger_{T-t}(u^t) = [\mathcal{M}^t\mathcal{L}_H\omega^\dagger_{T-t}(u^t), db^t],$$

which shows that $\mathcal{M}^t\omega^\dagger_{T-t}(u^t)$ is a martingale. Thus, (5-13) follows by equating the expectations at $t = 0$ and $t = T$. Finally, (5-14) follows from (5-11). $\square$

The estimate (5-14) has many interesting consequences. We illustrate its usefulness by mentioning a semigroup domination result which can be proved as in [Elworthy and Rosenberg 1988, Theorem 3A]; see also [Bérard 1990; Donnelly and Li 1982; Elworthy 1988; Hsu 1999; 2002b] for similar results.

Theorem 5.3. *Let $(N, \partial N, h)$ be as above and assume that $\rho_{(q)} \geq 0$ for some $1 \leq q \leq n-1$. Then there holds*

$$|e^{-t\Delta_q^{\mathrm{abs}}/2}(x, y)| \leq \binom{n}{q} e^{-\underline{r}_{(q)}t/2} e^{-t\Delta_0^{\mathrm{abs}}/2}(x, y), \quad x, y \in N,\ t > 0,$$

where $\underline{r}_{(q)} = \inf_{x\in N} r_{(q)}(x)$. In particular, if $\lambda_0^{\mathrm{abs}}(h) + \underline{r}_{(q)} \geq 0$ and $r_{(q)} > \underline{r}_{(q)}$ somewhere then N carries no nontrivial absolute L^2-harmonic q-form.

The Feynman–Kac formula for spinors. Let N be a spin^c ∂-manifold [Friedrich 2000]. As usual we assume that $(N, \partial N, h)$ has bounded geometry. Let $\mathbb{S}N = P_{\mathrm{Spin}^c_n}(N) \times_\zeta V$ be the spin^c bundle of N, where ζ is the complex spin representation. Recall that $P_{\mathrm{Spin}^c_n}(N)$ is a Spin^c principal bundle double covering $P_{\mathrm{SO}_n}(N) \times P_{\mathrm{U}_1}(N)$, where $P_{\mathrm{U}_1}(N)$ is the U_1 principal bundle associated to the auxiliary complex line bundle $\mathcal{F}$. After fixing a unitary connection C on $\mathcal{F}$, the Levi-Civita connection on TN induces a metric connection on $\mathbb{S}N$, still denoted ∇. The corresponding Dirac operator $D : \Gamma(\mathbb{S}N) \to \Gamma(\mathbb{S}N)$ is locally given by

$$D\psi = \sum_{i=1}^n \gamma(e_i)\nabla_{e_i}\psi, \quad \psi \in \Gamma(\mathbb{S}N),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame and $\gamma : \mathrm{Cl}(TN) \to \mathrm{End}(\mathbb{S}N)$ is the Clifford product. The Dirac Laplacian operator is

$$D^2\psi = \Delta_B\psi + \mathfrak{R}\psi, \tag{5-15}$$

where
$$\mathfrak{R}\psi = \frac{R}{4}\psi + \frac{1}{2}\gamma(i\Omega).$$
Here, R is the scalar curvature of h and $i\Omega$ is the curvature 2-form of C.

The spin^c bundle $\mathbb{S}N|_{\partial N}$, obtained by restricting $\mathbb{S}N$ to ∂N, becomes a Dirac bundle if its Clifford product is
$$\gamma^{\intercal}(X)\psi = \gamma(X)\gamma(\nu)\psi, \quad X \in \Gamma(T\partial N), \quad \psi \in \Gamma(\mathbb{S}N|_{\partial N}),$$
and its connection is

$$\nabla^{\intercal}_X\psi = \nabla_X\psi - \tfrac{1}{2}\gamma^{\intercal}(AX)\psi, \tag{5-16}$$

where as usual $A = -\nabla\nu$ is the second fundamental form of ∂N; see [Nakad and Roth 2013]. The associated Dirac operator $D^{\intercal} : \Gamma(\mathbb{S}N|_{\partial N}) \to \Gamma(\mathbb{S}N|_{\partial N})$ is
$$D^{\intercal}\psi = \sum_{j=1}^{n-1}\gamma^{\intercal}(e_j)\nabla^{\intercal}_{e_j}\psi,$$
where the frame has been adapted so that $e_n = \nu$. Imposing that $Ae_j = \rho_j e_j$, where ρ_j are the principal curvatures of ∂N, a direct computation shows that
$$D^{\intercal}\psi = \frac{K}{2}\psi + \sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi,$$
where $K = \operatorname{tr} A$ is the mean curvature. It follows that this tangential Dirac operator enters into the boundary decomposition of D, namely,

$$-\gamma(\nu)D = \nabla_\nu + D^{\intercal} - \frac{K}{2}, \tag{5-17}$$

which by its turn appears in Green's formula for the Dirac Laplacian

$$\int_N \langle D^2\psi, \xi\rangle\, dN = \int_N \langle D\psi, D\xi\rangle\, dN - \int_{\partial N}\langle \gamma(\nu)D\psi, \xi\rangle\, d\partial N, \tag{5-18}$$

where ψ and ξ are compactly supported. Also, since $\gamma^{\intercal}(e_j)\gamma(\nu) = -\gamma(\nu)\gamma^{\intercal}(e_j)$ and $\nabla^{\intercal}_{e_j}\gamma(\nu) = \gamma(\nu)\nabla^{\intercal}_{e_j}$, we see that

$$D^{\intercal}\gamma(\nu) = -\gamma(\nu)D^{\intercal}. \tag{5-19}$$

Now fix a nontrivial orthogonal projection $\Pi \in \Gamma(\operatorname{End}(\mathbb{S}N|_{\partial N}))$ and set $\Pi_+ = \Pi$ and $\Pi_- = I - \Pi$. It is clear from (5-17) – (5-18) that any of the boundary conditions

$$\Pi_\pm\psi = 0, \quad \Pi_\mp\Big(\nabla_\nu + D^{\intercal} - \frac{K}{2}\Big)\psi = 0, \tag{5-20}$$

turns the Dirac Laplacian D^2 into a formally self-adjoint operator. The next definition isolates a notion of compatibility between the tangential Dirac operator and the projections which will allow us to get rid of the middle term in the second condition above.

Definition 5.1. We say that the tangential Dirac operator $D^{\intercal}$ intertwines the projections if $\Pi_{\pm} D^{\intercal} = D^{\intercal}\Pi_{\mp}$.

Remark 5.2. If $D^{\intercal}$ intertwines the projections then $\Pi_{\pm} D^{\intercal}\Pi_{\pm} = D^{\intercal}\Pi_{\mp}\Pi_{\pm} = 0$. Equivalently, $\langle D^{\intercal}\Pi_{\pm}\psi, \Pi_{\pm}\xi\rangle = 0$ for any spinors ψ and ξ.

Proposition 5.3. *Under the conditions above assume further that $D^{\intercal}$ intertwines the projections as in Definition 5.1. Then a spinor $\psi \in \Gamma(\mathbb{S}N|_{\partial N})$ satisfies the boundary conditions* (5-20) *if and only if its lift $\psi^{\dagger} : P_{\mathrm{Spin}_n^c}(N) \to V$ satisfies*

$$\Pi^{\dagger}_{\pm}\psi^{\dagger} = 0 \quad \text{and} \quad \Pi^{\dagger}_{\mp}\Big(\mathcal{L}_{\nu^{\dagger}} - \frac{K^{\dagger}}{2}\Big)\psi^{\dagger} = 0 \quad \text{on} \quad \partial P_{\mathrm{Spin}_n^c}(N). \tag{5-21}$$

Proof. Obvious in view of (5-20) and Remark 5.2. □

We can now proceed exactly as in the previous subsection. We assume that (5-21) gives rise to a self-adjoint elliptic realization of D^2 and we denote by $e^{-tD^2/2}$ the corresponding heat semigroup [Grubb 2003]. We lift everything in sight to $P_{\mathrm{Spin}_n^c}(N)$ and consider there the functional $\mathcal{M}^t_{\epsilon}$ defined by

$$d\mathcal{M}^t_{\epsilon} + \mathcal{M}^t_{\epsilon}\Big(\frac{1}{2}\mathfrak{R}^{\dagger}(u^t)dt + \Big(\frac{1}{2}K^{\dagger}(u^t) + \epsilon^{-1}\Pi^{\dagger}_{+}(u^t)\Big)\,dl^t\Big) = 0, \quad \mathcal{M}^0_{\epsilon} = I.$$

The limiting functional $\mathcal{M}^t$, whose existence is guaranteed by the analogue of Theorem 5.1, appears in the corresponding Feynman–Kac formula.

Theorem 5.4. *Let $\psi_0 \in \Gamma(\mathbb{S}N)$ be a spinor satisfying any of the boundary conditions* (5-20), *where we assume $D^{\intercal}$ intertwines the projections as in Definition 5.1. If $\psi_t = e^{-tD^2/2}\psi_0$ is the solution to*

$$\frac{\partial\psi_t}{\partial t} + \frac{1}{2}D^2\psi_t = 0, \quad \lim_{t\to 0}\psi_t = \psi_0, \quad \Pi_{\pm}\psi_t = 0, \quad \Pi_{\mp}\Big(\nabla_{\nu} - \frac{K}{2}\Big)\psi_t = 0, \tag{5-22}$$

then the following Feynman–Kac formula holds:

$$\psi^{\dagger}_t(u^0) = \mathbb{E}_{u^0}(\mathcal{M}_t\psi^{\dagger}_0(u^t)), \tag{5-23}$$

where u^t is the horizontal reflecting Brownian motion on $P_{\mathrm{Spin}_n^c}(N)$ starting at u^0. As a consequence,

$$|\psi_t(x^0)| \le \mathbb{E}_{x^0}\Big(|\psi_0(x^t)| \exp\Big(-\frac{1}{2}\int_0^t \mathfrak{r}(x^s)\,ds - \frac{1}{2}\int_0^t K(x^s)\,dl^s\Big)\Big), \tag{5-24}$$

where $\mathfrak{r}(x) = \inf_{|\psi|=1}\langle\mathfrak{R}(x)\psi, \psi\rangle$.

Proof. The same as in Theorem 5.2. □

It is worthwhile to state the analogue of Theorem 5.3 for spinors.

Theorem 5.5. *Let (N, h) be a* spin^c *∂-manifold as above and assume that $K \geq 0$ along ∂N. Let $e^{-tD^2/2}$ be the heat semigroup of the Dirac Laplacian acting on spinors subject to boundary conditions as in Theorem 5.4. Then*

$$|e^{-tD^2/2}(x, y)| \leq 2^{[n/2]+1} e^{-\underline{\mathfrak{r}}t/2} e^{-t\Delta_0^{\mathrm{abs}}/2}(x, y), \quad x, y \in N, t > 0,$$

where $\underline{\mathfrak{r}} = \inf_{x \in N} \mathfrak{r}(x)$. In particular, if $\lambda_0^{\mathrm{abs}}(h) + \underline{\mathfrak{r}} \geq 0$ and $\mathfrak{r} > \underline{\mathfrak{r}}$ somewhere then N carries no nontrivial L^2-harmonic spinor satisfying the given boundary conditions.

We now discuss a couple of examples of local boundary conditions for spinors to which Theorem 5.4 applies.

Example 5.1. (Chirality boundary condition) A chirality operator on a spin^c ∂-manifold $(N, \partial N)$ is an orthogonal and parallel involution $Q \in \Gamma(\mathrm{End}(\mathbb{S}N))$ which anticommutes with the Clifford product with any tangent vector. Examples include the Clifford product with the complex volume element in an even-dimensional spin manifold and with the timelike unit normal to an immersed spacelike hypersurface in a Lorentzian spin manifold. It is easy to check that $D^{\intercal} Q = Q D^{\intercal}$ and $D^{\intercal}\gamma(\nu) = -\gamma(\nu) D^{\intercal}$. Given any such Q define the boundary chirality operator $\hat{Q} = \gamma(\nu) Q \in \Gamma(\mathrm{End}(\mathbb{S}N)|_{\partial N})$, which is still an orthogonal and parallel involution with associated projections given by

$$\Pi_{\pm} = \tfrac{1}{2}(I \mp \hat{Q}). \tag{5-25}$$

Since $D^{\intercal}\hat{Q} = D^{\intercal}\gamma(\nu)Q = -\gamma(\nu)QD^{\intercal} = -\hat{Q}D^{\intercal}$, we conclude that $D^{\intercal}\Pi_{\pm} = \Pi_{\mp}D^{\intercal}$, that is, $D^{\intercal}$ intertwines the projections. Thus, Theorem 5.4 applies to the self-adjoint elliptic realization of D^2 under this boundary condition.

Example 5.2 (MIT bag boundary condition). This time we choose $\hat{Q} = i\gamma(\nu)$, an involution which clearly satisfies $D^{\intercal}\hat{Q} = -\hat{Q}D^{\intercal}$. Thus, $D^{\intercal}$ intertwines the projections exactly as in the previous example and Theorem 5.4 again applies to the self-adjoint elliptic realization of D^2 under this boundary condition.

Remark 5.3. For the sake of comparison, it is instructive to examine how absolute and relative boundary conditions for differential forms fit into the framework developed in this subsection. In particular, this helps to clarify the role played by Proposition 5.1 and its analogue for relative forms. Recall that $\wedge^{\bullet}T^*N$ has the structure of a Clifford module if we define the Clifford product by tangent vectors as $\gamma(v) = v \wedge - v\lrcorner$. The corresponding Dirac operator is $D = d + d^{\star}$, so that $D^2 = \Delta$, the Hodge Laplacian. If ω is a q-form then we know that along ∂N,

$$\omega = \omega_{\tan} + \nu \wedge \omega_{\mathrm{nor}} = \Pi_{\tan}\omega + \Pi_{\mathrm{nor}}\omega.$$

Instead of (5-16) we now have

$$\nabla_X^{\intercal} = \nabla_X^{\partial N} + \nu \wedge A(X)\lrcorner.$$

A direct computation then shows that, with respect to the splitting above, the boundary decomposition of D is

$$-\gamma(\nu)D\begin{pmatrix}\omega_{\tan}\\ \omega_{\mathrm{nor}}\end{pmatrix}=\begin{pmatrix}\nabla_\nu\omega_{\tan}\\ \nabla_\nu\omega_{\mathrm{nor}}\end{pmatrix}-\begin{pmatrix}\mathcal{A}_q^{\tan} & D_{\partial N}\\ D_{\partial N} & \mathcal{A}_{q-1}^{\mathrm{nor}}\end{pmatrix}\begin{pmatrix}\omega_{\tan}\\ \omega_{\mathrm{nor}}\end{pmatrix},$$

where $D_{\partial N}=d_{\partial N}+d^*_{\partial N}$ and in terms of a principal frame,

$$\mathcal{A}_q^{\tan,\mathrm{nor}}=\sum_j \rho_j\,\Pi_{e_j}^{\tan,\mathrm{nor}},$$

with $\Pi_v^{\tan}=v\wedge v\lrcorner$ and $\Pi_v^{\mathrm{nor}}=v\lrcorner v\wedge$. If $\omega_{\mathrm{nor}}=0$ then $\mathcal{A}_q^{\tan}\omega=\mathcal{A}_q\omega$ and the boundary integral in Green's formula for the Hodge Laplacian is

$$\int_{\partial N}\big(\langle\nabla_\nu\omega_{\tan},\omega_{\tan}\rangle-\langle\mathcal{A}_q\omega_{\tan},\omega_{\tan}\rangle-\langle D_{\partial N}\omega_{\tan},\omega_{\tan}\rangle\big)\,d\partial N.$$

However, the last term vanishes because the forms involved in the inner product have different parities. Thus, the right boundary conditions are

$$\Pi_{\mathrm{nor}}\omega=0,\quad \Pi_{\tan}(\nabla_\nu-\mathcal{A}_q)\omega=0. \tag{5-26}$$

Proposition 5.1 then shows that (5-26) defines absolute boundary conditions for the Hodge Laplacian. Similarly, if $\omega_{\tan}=0$ then $\omega=\nu\wedge\omega_{\mathrm{nor}}$ and $\mathcal{A}_{q-1}^{\mathrm{nor}}\omega_{\mathrm{nor}}=\star\mathcal{A}_{n-q}\star\omega_{\mathrm{nor}}$, where here $\star$ is the Hodge star operator of ∂N. This time the boundary integral is

$$\int_{\partial N}\big(\langle\nabla_\nu\omega_{\mathrm{nor}},\omega_{\mathrm{nor}}\rangle-\langle\star\mathcal{A}_{n-q}\star\omega_{\mathrm{nor}},\omega_{\mathrm{nor}}\rangle-\langle D_{\partial N}\omega_{\mathrm{nor}},\omega_{\mathrm{nor}}\rangle\big)\,d\partial N.$$

Again, the last term drops out and the correct boundary conditions are

$$\Pi_{\tan}\omega=0,\quad \Pi_{\mathrm{nor}}(\nabla_\nu-\star\mathcal{A}_{n-q}\star)\omega=0. \tag{5-27}$$

As in Proposition 5.1 we compute that

$$\begin{aligned}(\nabla_\nu-\star\mathcal{A}_{n-q}\star)\omega(\nu,e_{i_1},\dots,e_{i_{q-1}})&=(\nu\wedge d^\star\omega)(\nu,e_{i_1},\dots,e_{i_{q-1}})\\&=(\nu\lrcorner\nu\wedge d^\star\omega)(e_{i_1},\dots,e_{i_{q-1}})\\&=(\Pi_{\tan}\,d^\star\omega)(e_{i_1},\dots,e_{i_{q-1}})\end{aligned}$$

so that (5-27) can be rewritten as

$$\omega_{\tan}=0,\quad (d^\star\omega)_{\tan}=0.$$

This is exactly how relative boundary conditions for the Hodge Laplacian are defined [Taylor 2011]. We thus see that for differential forms the cancellations leading to the correct boundary conditions are caused by the fact that $D_{\partial N}$ clearly intertwines the projections onto the spaces of even and odd degree forms; compare to Definition 5.1.

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